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A flexible rule for evidential combination in Dempster–Shafer theory of evidence

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H I G H L I G H T S

- Identify some properties that an evidential combination rule should follow.
- Propose a novel evidential combination rule to satisfy the properties identified, which can (1) resolve well-known Zadeh's counter-example, and (2) handle different intuitive cases that other existing rules cannot.

A R T I C L E   I N F O

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A B S T R A C T

Dempster's combination rule in Dempster–Shafer theory of evidence is widely used to combine multiple pieces of evidence. However, when the evidence is severely conflicting, the result could be counter-intuitive. Thus, many alternative combination rules have been proposed to address this issue. Nevertheless, the existing ones sometimes behave not very well. This may be because they do not hold some essential properties. To this end, this paper firstly identifies some of the important properties. Then, following the cues from these properties, we propose a novel evidential combination rule as a remediation of Dempster's combination rule in Dempster–Shafer theory. Our new rule is based on the concept of *complete conflict* (we introduced in this paper), Dempster's combination rule, and the concept of evidence weight. Moreover, we illustrate the effectiveness of our new rule by using it to successfully resolve well-known Zadeh's counter-example, which is against Dempster's combination rule. Finally, we confirm the advantages of our method over the existing methods through some examples.

1. Introduction

Dempster–Shafer theory of evidence (D–S theory) [1,2] is widely used for modelling and reasoning with ambiguous information in applications [3–6]. In this theory, when multiple pieces of evidence for an assumption are accumulated from multiple distinct sources, they need to be combined to see how strongly they support the assumption together [2,7–9]. In D–S theory, Dempster's combination rule is widely employed to do this. Nevertheless, many researchers challenge its validity and consistency when it is used to combine highly conflicting evidence [10–12], which makes the use of Dempster's combination rule questionable. Thus, they analyse the cause of its counter-intuitive behaviours, and accordingly proposed a number of new evidential combination rules to remove the defect (*e.g.*, [7,13–15]). More specifically, most of them aim mainly at handling a situation that violates human intuitions: the combination of a lowly supportive state by all relevant evidence could cause the fully supportive state. Usually these rules exhibit a number of good behaviours when combining particular sorts of evidence, but to some extent these rules are actually ad hoc because of lacking a set of essential properties for justifying them. As Smets [16] pointed out, the pragmatic fact "our rule works fine" of course is not a proper justification (at most a necessary condition).

To tackle the above problems, firstly in this paper we identify a number of basic properties that an evidential combination rule may manifest. Then, we analyse the existing models and show most of them do not satisfy some of these basic properties. Hence, after analysing the cause why Dempster's combination rule does not work properly, we introduce the concept of *complete conflict*, and further propose a flexible combination rule, which is based on Dempster's combination rule [1,2] and the concept of evidence weight (obtained from the distances between different pieces of evidence [17]). Moreover, we prove our combination rule does hold the properties. In other words, we use these properties to

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justify our rule. We also show how to use our rule to resolve well-known Zadeh's example [12] against Dempster's combination rule. Finally, we further confirm the advantages of our method over the existing models in some examples.

This paper advances the state of the art in the area of D–S theory in the following aspects: (i) identify a number of properties that reflect human intuitions of combining multiple pieces of evidence from different, reliable sources; (ii) design a flexible evidential combination rule that holds these properties; (iii) use our rule to resolve Zadeh's counter-example [12], and (iv) show that our rule can handle different intuitive cases that other existing rules cannot.

The rest of this paper is organised as follows. Section 2 recaps the background knowledge that we will use in this paper. Section 3 identifies a number of properties that a process of combining multi-evidence from independent sources may exhibit. Section 4 analyses whether or not some existing methods hold these properties. Section 5 proposes a flexible evidence combination rule, and proves that our rule holds the properties we identified. Section 6 uses our rule to resolve Zadeh's counter-example and further check our rule with another example. Finally, Section 7 concludes the paper with future work.

2. Preliminaries

This section recaps some basic concepts and notations in D–S theory [1,2], Zadeh's counter-example [12], and the measure of evidential distance [17].

Definition 1 (*Mass Function [1,2]*)**.** Let Θ be a set of exhaustive and mutually exclusive elements, called a frame of discernment (or simple a frame). Function $m : 2^{\Theta} \to [0, 1]$ is a mass function if

$$m(\emptyset) = 0, \tag{1}$$

$$\sum_{A\subseteq\Theta} m(A) = 1, \tag{2}$$

where the mass value of $m(A)$ ($A \subseteq \Theta$) represents the degree to which the corresponding evidence supports A. In particular, a mass function is *completely ignorable* if and only if $m(\Theta) = 1$, and F is the *focal element* set of the mass function m_i if for any $B \in F$, $m(B) > 0$.

D–S theory provides a method (*i.e.*, Dempster's combination rule) to combine evidence accumulated from multiple independent sources as follows:

Definition 2 (*Dempster's Combination Rule [1,2]*)**.** Let m_1 and m_2 be two mass functions over a frame of discernment Θ, corresponding to two pieces of evidence from independent and totally reliable sources. Then their combined mass function, denoted as $m_{\oplus\{m_1,m_2\}} = m_1 \oplus m_2$, is given by:

$$m_{\oplus\{m_1,m_2\}}(X) = \begin{cases} 0 & \text{if } X = \emptyset, \\ \dfrac{\sum_{A\bigcap B=X} m_1(A)m_2(B)}{1-\sum_{A\bigcap B=\emptyset} m_1(A)m_2(B)} & \text{otherwise.} \end{cases} \tag{3}$$

Dempster's combination rule has been used to combine multiple pieces of evidence from different independent sources in many applications, such as information fusion [18–21], pattern recognition [22–24], and decision making [3,25–27], and it has some desired properties such as commutativity and associativity [1,2]. However, it has been criticised and debated upon some of its counterintuitive combined results [9,12,28,29]. Perhaps the most famous one among them is as follows:

Example 1 (*Zadeh's Counter-example [12]*)**.** Let m_1 and m_2 be two mass functions defined on a frame of discernment $\Theta = \{a, b, c\}$ with:

$$m_1(\{a\}) = 0.99,\ m_1(\{b\}) = 0.01,\ m_1(\{c\}) = 0; \tag{4}$$

$$m_2(\{a\}) = 0,\ m_2(\{b\}) = 0.01,\ m_2(\{c\}) = 0.99. \tag{5}$$

They mean that assumption a is strongly supported by the first piece of evidence but absolutely denied by the second one, assumption b is weekly supported by both, and assumption c is strongly supported by the second but absolutely denied by the first. So, for assumptions a and c, the two pieces of evidence are highly conflicting. Now let us see what will happen if we use Dempster's combination rule to combine the two highly conflicting mass functions. By the rule, we can obtain the combined mass value of assumption b as follows:

$$\begin{aligned} m_{\oplus\{m_1,m_2\}}(\{b\}) =& \frac{\sum_{A\bigcap B=\{b\}} m_1(A)m_2(B)}{1-\sum_{A_i\bigcap B_j=\emptyset} m_1(A)m_2(B)} \\ =& \frac{m_1(\{b\})m_2(\{b\})}{1-m_1(\{a\})m_2(\{b\}+m_1(\{a\})m_2(\{c\})+m_1(\{b\})m_2(\{c\}))} \\ =& \frac{0.01\times 0.01}{1-(0.99\times 0.99+0.99\times 0.01+0.01\times 0.99)} \\ =& 1. \end{aligned} \tag{6}$$

That is, assumption b, hardly supported by each piece of evidence, turns out to be fully supported after the combination of the two pieces of evidence. Therefore, Zadeh argued that such a result highly violates our intuition about the evidence combination.

Finally, we recall a distance measure between two mass functions representing two pieces of evidence:

Definition 3 (*Evidential Distance [17]*)**.** Let m_1 and m_2 be any two mass functions over a frame of discernment Θ. The distance between m_1 and m_2, denoted as $d(m_1, m_2)$, is given by:

$$d(m_1, m_2) = \sqrt{\frac{1}{2}\sum_{\substack{\emptyset\neq A\subseteq\Theta \\ \emptyset\neq B\subseteq\Theta}} \frac{|A\cap B|}{|A\cup B|}(m_1(A)-m_2(A))(m_1(B)-m_2(B))}. \tag{7}$$

Importantly, for any mass function m, by the above definition, we have that $d(m, m) = 0$. Moreover, the evidential distance can effectively indicate the overall influence of different subsets and their mass values. Therefore, it can well characterise the evidential discrepancy. Finally, based on the above definition of distance, the similarity of mass function m_1 to mass function m_2 can be defined as

$$s(m_1, m_2) = 1 - d(m_1, m_2). \tag{8}$$

That is, the smaller the distance between two mass functions, the more similar the two. In particular, since $d(m_1, m_2) = d(m_2, m_1)$ and $d(m, m) = 0$, we have:

$$s(m_1, m_2) = s(m_2, m_1), \tag{9}$$

$$s(m, m) = 1. \tag{10}$$

3. Properties of evidence combination

In order to design more flexible rule for combining multiple pieces of evidence from multiple independent sources, this section will identify some intuitive properties that a specific, appropriate evidential combination rule may need to hold.

In the following we give the formal definition of these properties first and then explain their meaning.

Definition 4 (*Basic Properties of Evidential Combination*)**.** Let Z be a set of all possible mass functions over a frame of discernment $\Theta = \{s_1, \ldots, s_n\}$, where $\{s_1, \ldots, s_n\}$ is a set of exhaustive and mutually exclusive elements, $M \subset Z$ be a set of mass functions over the frame of discernment Θ, and m_M be the combined result of all mass functions on set M. Then an operator $\odot : Z \times Z \to Z$ is a combination operator if it satisfies:

(i) **Local computation availability**. (a) Commutativity: $m_1 \odot m_2 = m_2 \odot m_1$. (b) Quasi-associativity: $m_1 \odot \cdots \odot m_n = T(m_1 \circ \cdots \circ m_n)$, where $\circ$ is an associativity operator over Z,[1] and T is a mapping from Z to Z.
(ii) **Neutral element commitment**. If $m_i(\Theta) = 1$, then $m_M \odot m_i = m_M$ for any mass function set M.
(iii) **Possibility reservation**. If $\exists\, m_i \in M$ and $a \in \Theta$ such that $m_i(\{a\}) > 0$, then $m_M(\{a\}) > 0$.
(iv) **Convergence toward certainty**. If $\exists\, m_i \in M$ and $m_i(\{a\}) > m_i(X)$ for any $X \subset \Theta \backslash \{a\}$, then

$$m_i^n(\{a\}) > m_i^{n-1}(\{a\}),$$

where m_i^n means using combination operator $\odot$ to combine the mass function m_i n times.[2]
(v) **Invariance of iterated indifference evidence**. If $m_k(\{s_i\}) = 1/n$ for any $s_i \in \Theta$ and any $m_k \in M$, then $m_M = m_k$.
(vi) **Weak specialisation**. If $m_M(A) > 0$, then there exists $m_i \in M$ such that $m_i(B) > 0$ and $A \subseteq B$.

Let us explain the properties in the above definition one by one. The *first* property makes it easy to add more evidence to an already combined body of evidence. It involves the computation complexity of a combination rule when many multivariate belief functions are given (*e.g.*, hundreds or thousands). So with the local computation, we can tremendously reduce the complexity of the computations from NP-complete with respect to the number of mass functions involved down to NP-complete with respect to the largest domain of the underlying join tree [30]. Actually, in this property, we relax the requirement of associativity to allow more appropriate operators for evidence combination without raising too much computation complexity.

The *second* property means that the completely ignorable evidence is a neutral element. That is, the overall evaluation for a set of evidence should not be influenced by the combination with a piece of completely ignorable evidence. This is because the property reflects that a piece of completely ignorable evidence cannot provide us with any information about the real state. In other words, we cannot benefit from a piece of completely ignorable evidence in finding out the true answer to a question under consideration. Therefore, the completely ignorable evidence should work as a neutral element that neither enhances nor weakens the existing belief. The property can be exemplified by an airport security surveillance scenario as follows:

Example 2. Suppose in the control centre of the airport, at 20:55, a person uses a backup staff card of the flight desk services assistant to enter the front door of the control centre. Now assume the backup staff card is kept by one of three persons in a frame $\Theta = \{A, B, C\}$, where A stands for Aaron, B stands for Belly, and C stands for Clifford. Thus, the intelligent surveillance system makes a suggestion: one of these three suspects may be illegally infiltrating the control centre. After that, camera 23 in the middle of the corridor to the control centre at 21:03 captures the person's back image. Now suppose the surveillance system gets a new piece of evidence: the gender of the suspect is male. However, since all the persons in the frame are male, the evidence is completely ignorable, so the suggestion of the intelligent surveillance system should not be changed after considering the new evidence.

The *third* property means that a combination process should not exclude the possibility of a state that the relevant evidence supports. The following is a real-world application example of this property:

Example 3 (*Example 2 Continued*)**.** Suppose the security system checks the working schedule of the airport and finds that Aaron should be at work today. Thus, one piece of evidence collected by the surveillance system supports that Aaron may be the criminal. In this case, after considering all the relevant pieces of evidence, the suggestion of the security system should not exclude the possibility that Aaron is a criminal.

Moreover, since Zadeh's counter-example is caused by excluding the possibility of a supportive state in the combined result, this property implies that the combination rule should be able to resolve Zadeh's counter-example. Thus, it can be regarded as a criterion to judge whether or not a combination rule is reasonable, because it is regarded as the main defect of Dempster's combination rule. Since the main criticism and debate on Dempster's combination rule is its failure of resolving the Zadeh's counter-example. Thus, we need a new combination rule that can resolve the counter-example, which we will present in Section 5.

The *fourth* property means that the combination process should converge toward the dominant opinion. It reflects a desired property that the mass value of a really *true* state should be supported more and more strongly when more and more supportive evidence is accumulated. A real-world application example of this property is as follows:

Example 4 (*Example 2 Continued*)**.** Suppose the surveillance system checks the surveillance record in the database and finds that camera 29 captures Aaron has loitered near the front door of the control centre ten times at different times of today from 18:23 to 20:50. In other words, the surveillance receives ten pieces of new evidence, and all of them point out that there exists a chance that Aaron has some potential threat. In this case, even for each piece of evidence, the chance that Aaron has some potential threat is very small but higher than other suspects (it is acceptable that a flight desk services assistant loiters near the front door of the control centre occasionally), the surveillance system should suggest that Aaron has a high chance to be the criminal after considering these ten pieces of evidence.

The *fifth* property reflects that the result of combining multiple pieces of indifferent evidence is identical to each piece of evidence. The property is relevant to the property of indifference [31] in probability theory with evidence combination in D–S theory. Namely, if any piece of available evidence supports that all the states are equally possible, then their combination result implies the same conclusion. The following is an example:

Example 5 (*Example 2 Continued*)**.** Suppose the security system makes a judgement that the chance of each possible suspect is equal. After that, the security system checks the work record and finds all three people have make a copy of the backup key. In this case, after considering such new evidence, the security system should still suggest the criminal is equally possible over three suspects A, B, and C. That is, there is 1/3 chance for that each of Aaron, Belly, and Clifford is the criminal.

1 That is, $(m_1 \circ m_2) \circ m_3 = m_1 \circ (m_2 \circ m_3)$ for any mass functions m_1, m_2, and m_3.

2 For example, $m_i^2 = m_i \odot m_i$, $m_i^3 = m_i \odot m_i \odot m_i$, and so on.

Finally, the *sixth* property reflects that for any focal element of the combined mass function is a subset of a focal element of a piece of evidence. That is, a combined result should at least give us equally precise information about the question concerned than each piece of evidence. Here the precise of the combined result is determined by it specifics from the relevant evidence. For example, if a piece of evidence supports that sets $\{A, B, C\}$ and $\{A, B\}$ are possible and the combined result indicates sets $\{A\}$ and $\{B\}$ is possible, then we can say the combined result gives us more precise information than this evidence (*i.e.*, the combined result reduce our ambiguities about $\{A, B, C\}$ and $\{A, B\}$ to $\{A\}$ and $\{B\}$).

In other words, the focal elements of the combined result are the more specific subsets of the focal elements of the mass function corresponding to the relevant evidence. Hence, it means that the judgement of the combined result should not conflict with any piece of evidence (*i.e.*, it should not support a state that all the pieces of relevant evidence are against and the decision maker should be less ambiguous after the combination). Actually, the *sixth* property is a weak version of specialisation in [16] without the drawback of violating the *fourth* property about convergence toward certainty: for a mass function with $m(A) = 0.8$ and $m(B) = 0.2$ ($A, B \in \Theta$ and $A \cap B = \emptyset$), no matter how many pieces of evidence with the same mass function are available, the combined result that satisfies specialisation should not assign A a value that is higher than 0.8. So, the concept of specialisation may be too strong to handle the combination of conflictive mass functions because it could exclude some desired properties in the combination process. The intuition of the property can be reflected in the following example:

Example 6 (*Example 2 Continued*)**.** Suppose at 21:08 camera 25 in the control centre captures the person's side image and find that the person has a beard. Assume that by the previous surveillance record, the security system has the following information: Aaron has a beard, Belly has no beard, and Clifford has a moustache stubble. Therefore, after combining this information, the security system suggests that very likely Aaron is the criminal but Clifford is the criminal with a low chance (i.e., Belly should not be the criminal according to the available evidence). Meanwhile, the classification programme of age implies that the age of the person in camera 25 is higher than 45. Since both Aaron and Clifford is higher than 45 and Belly is 25. Then, after considering such a new piece of evidence, the security system unlikely suggests that the criminal is Belly.

In summary, these properties can be regarded as some constraints on some aspects of designing a combination rule. The first property is on the computation complexity of the combination rule, the second one is on the neutral element existence, the third and sixth are on the focal elements of combined result, and the fourth and fifth are on the mass value of combined result.

4. Analysis of existing methods

In general, there are three main viewpoints on improving Dempster's combination rule to resolve Zadeh's counter-example. In this section, we will discuss these existing improvements to see whether they satisfy the properties we proposed in the previous section.

Since most of the existing combination rules are based on the conjunctive rule proposed by [1,2], we recap it first:

Definition 5 (*The Conjunctive Rule [1,2]*)**.** Let m_1 and m_2 be two mass functions over a frame of discernment Θ. Function $f_{1,2}$ that is resulted from the application of the conjunctive rule is given by

$$f_{1,2}(X) = \sum_{A \bigcap B = X} m_1(A)m_2(B), \tag{11}$$

where $X \in \Theta$.

In the following, we will further show the three main viewpoints on improving Dempster's combination rule.

4.1. The first viewpoint

From the first viewpoint, the counter-intuitive behaviours are imputed to the way to deal with conflicting mass assignments. It relates to two issues about Dempster's combination rule.

The first one is how to assign the conflicting mass assignments. Yager [15] suggested the conflicting mass assignments should be assigned to the universal set of the frame of discernment, *i.e.*,

$$m^{Y}_{1,2}(X) = \begin{cases} 0 & \text{if } X = \emptyset, \\ f_{1,2}(\Theta) + f_{1,2}(\emptyset) & \text{if } X = \Theta, \\ f_{1,2}(X) & \text{otherwise,} \end{cases} \tag{12}$$

where $f_{1,2}$ is the combined result of the conjunctive rule (*i.e.*, formula (11)). However, since this method assigns the entire value of the normalisation constant in Dempster's combination rule to the frame of discernment, it does not satisfy the property of convergence toward certainty. For example, for two pieces of evidence with the same mass function with $m_i(\{a\}) = 0.9$ and $m_i(\{b\}) = 0.1$ $(i = 1, 2)$, by Yager's method (*i.e.*, formula (12)), the combined result is: $m_{1,2}(\{a\}) = 0.81$, $m_{1,2}(\{b\}) = 0.01$, and $m_{1,2}(\Theta) = 0.18$. In other words, by Yager's method, the more evidence a decision maker has, the more confusing the decision maker is about the real state. Clearly this conclusion violates our intuition. Also several types of proportional conflict redistribution rules (PCRR) based on different criterions were reviewed in [32], such as:

$$m^{PCRR}_{1,2}(X) = \begin{cases} 0 & \text{if } X = \emptyset, \\ f_{1,2}(X) + k_X f_{1,2}(\emptyset) & \text{otherwise,} \end{cases} \tag{13}$$

where $f_{1,2}$ is the combined result of the conjunctive rule (*i.e.*, formula (11)) and k_X is a parameter to redistribution the mass value of $f_{1,2}(\emptyset)$ to each non-empty subset of the combined result and it satisfies $\sum_{X \subset \Theta} k_X = 1$. Hence, Lefevre et al. [33] proposed a unified model to handle conflicting evidence by a weighting factor to determine how conflicting masses are to be distributed over the subsets of the frame of discernment. However, as Smets [16] and Haenni [30] pointed out, most proportional conflict redistribution operators are not associative and quasi-associative. Since such two properties mean that the order in which the combination is done is irrelevant, the lack of the properties means the method cannot be used in a framework of local computations.

The second one is whether or not it is reasonable to apply conjunctive rule as Dempster's combination rule for information fusion. As an answer to this question, Dubois et al. [14] proposed a combination rule based on the disjunctive rule, as follows:

$$m^{DP}_{1,2}(X) = \begin{cases} 0 & \text{if } x = \emptyset, \\ f_{1,2}(X) + \displaystyle\sum_{\substack{A \bigcap B = \emptyset, \\ A \bigcup B = X}} m_1(A)m_2(B) & \text{otherwise,} \end{cases} \tag{14}$$

where $f_{1,2}$ is the combined result of the conjunctive rule (*i.e.*, formula (11)). However, the disjunctive rule does not satisfy the property of convergence toward certainty, either. For example, in the case that two pieces of evidence have the same mass function, *i.e.*, $m_i(\{a\}) = 0.9$ and $m_i(\{b\}) = 0.1$ $(i = 1, 2)$, the combined result is: $m_{1,2}(\{a\}) = 0.81$, $m_{1,2}(\{b\}) = 0.01$, and $m_{1,2}(\{a, b\}) = 0.18$. Therefore, it has the same problem as

Yager's method. Also Deng et al. [34] and Elouedi and Mercier [35] proposed some other rule of combining evidence with adapted conflict. Nevertheless, they do not satisfy the property of associative and quasi-associative [34,35]. In addition, Smarandache and Dezert [32] tried to extend evidence theory, by proposing Dezert–Smarandache theory, to solve the problem of high conflicting evidence. Nonetheless, its computation complexity is higher, and in the case of low conflicting evidence, the result is inferior to D–S theory [36].

4.2. The second viewpoint

From the second viewpoint, the counter-intuitive behaviour is imputed to the questionable evidence reliability caused by any disturbance or the maintaining condition of the information sources. As a result, the methods of this viewpoint suggest that the original mass functions should be reconstructed to handle the counter-intuitive behaviour properly. Thus, researchers propose a number of methods to construct the discounted mass function for the original evidence, such as the classical discounted mass function [2,37], discounted one by a convex combination of sources [38], and discounted one by inconsistent measurements [39]. However, these methods have at least one of the following limitations: (i) most of them (*e.g.*, [2,37,38]) require an additional parameter to solve the problem; and (ii) it just evades the criticism of Dempster's combination rule by making an additional assumption that conflicting evidence cannot be all fully reliable. Unfortunately, in reality different pieces of evidence may describe different aspects of the same issue, so it is possible that all the pieces of relevant evidence is fully reliable. For example, in a multiple-sensor surveillance scenario, a camera may reveal that a suspect is a female according to the image information that the suspect has a long hair, while a sound recorder (audio information) may reveal that a suspect is a female. In this case, since the two pieces of evidence describe the possible gender of the suspect from two different viewpoints, the evidence from both resources is fully reliable. Therefore, the method of constructing the discounted mass function just evades the criticism rather than solves the criticism.

On the other hand, some researchers investigate how to reconstruct a mass function based on the relevant evidence and then combine the reconstructed mass function several times by Dempster's combination rule to obtain the combined result. For example, Murphy [40] uses the averaging mass function. That is, for mass function set $M = \{m_1, \ldots, m_n\}$, their combined mass function is

$$m_M^M = \oplus \overline{m}^{n-1}, \tag{15}$$

where $\overline{m}(A) = (\sum_{i=1}^{n} m_i(A))/n$, and $\oplus\overline{m}^{n-1}$ means combined mass function $\overline{m}$ using Dempster's combination rule $n-1$ times.

Then other researchers [22,41–43] propose to reconstruct mass function based on the weighted averaging, where the weighted function obtained by the distance between evidence [17], Deng's entropy [36], or an ambiguity measurement. However, all of these methods could cause another counter-intuitive behaviour, because the original mass function can be changed after it is combined with a completely ignorable mass function in these methods, as shown in the sixth and seventh rows of Table 1. Also, the examples about the violation of intuition for the methods in [22,41–43] can be obtained by the similar examples as those of Murphy's method (*i.e.*, formula (15)) in Table 1. All of these methods violate the second property (see Table 2).

4.3. The third viewpoint

The previous two viewpoints both are under the closed-world assumption (*i.e.*, the frame of discernment is exhaustive). However, in some real-world applications, sometimes a system has to be run in an open world where the knowledge the system holds is incomplete. Therefore, Smets [38] and Smets and Kennes [44] proposed a modified method that assigns the conflicting mass assignments to empty set under the assumption that the real state might not be any of the alternatives listed in the frame of discernment (*i.e.*, an open world assumption). Actually, their method is the conjunctive rule in formula (11). Unfortunately, the open world assumption has some problems as well: (i) it still does not give a reasonable solution to the evidence combination problem in the case that the decision maker assures the frame of discernment is exhaustive; and (ii) the idea of assigning the entire conflicting mass value to the unknown state may violate our intuition. For example, if two distinct and totally reliable sources imply two mass functions m_1 and m_2 as $m_1(s_i) = m_2(s_i) = 0.2$ ($i = 1, \ldots, 5$), then by using the conjunctive rule (*i.e.*, formula (11)) to combine them, we can got that the mass value of empty set is 0.8. In other words, the model violates the fifth property.

Recently, a generalised evidence theory [36] has been proposed by extending D–S theory to a more general form with the open world assumption. In this theory $m(\emptyset) = 0$ is unnecessary, and $\emptyset$ is the focal elements outside of a frame with the meaning that the possibility of the true state turns out to be outside a frame. Moreover, its combination rule is:

$$m_{1,2}(A) = \begin{cases} \dfrac{(1-m(\emptyset))\sum_{B\cap C=A} m_1(B)m_2(C)}{1-f_{1,2}(\emptyset)} & \text{if } A \neq \emptyset \text{ and } f_{1,2}(\emptyset) \neq 1, \\ 0 & \text{if } A \neq \emptyset \text{ and } f_{1,2}(\emptyset) = 1, \\ 1 & \text{if } A = \emptyset \text{ and } f_{1,2}(\emptyset) = 1, \\ m_1(\emptyset)m_2(\emptyset) & \text{if } A = \emptyset \text{ and } f_{1,2}(\emptyset) \neq 1, \end{cases} \tag{16}$$

where $f_{1,2}$ is the combined result of the conjunctive rule (*i.e.*, formula (11)). Later on, Jiang and Zhan [7] proposed a modified generalised combination rule based on the generalised evidence theory. However, since when $m(\emptyset) = 0$ these two methods (*i.e.*, Deng's method and the method of Jiang and Zhan both degenerate to conventional D–S theory [7,36], it does not give a solution to Zadeh's counter-example, either).

4.4. Violation of basic properties

Actually, via some examples as shown in Table 3, we can show how the each existing rules we discuss violates the basic properties we identified in the previous section. In general, Table 2 summarises the situations of whether or not a number of combination rules hold the basic properties when the evidence is distinct and totally reliable.[3] In the first row, the ordinal numbers mean the first property to the sixth property. In the second column, we can see that PCRR method [32], unified weight model [33], disjunctive rule (*i.e.*, formula (14)) Murphy's method (*i.e.*, formula (15)), and its similar methods [22,41–43] violate the property of local computation availability. Therefore, it means that all these models cannot combine a set of mass functions in any order. The third column, we find that only Murphy's method (*i.e.*, formula (15)) and other similar methods violate

[3] Here, since we only concern whether or not other combination rules can satisfy all basic properties we identify in this paper, we only list the properties that unsatisfied by other rules. Also, since we just focus on the case of distinct and totally reliable evidence, we omit the methods with the discounted mass function in Table 2.

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Table 1
The examples of different combination rules violating intuitions.

	Formula No.	Original mass function	Combined result
Dempster's rule	(3)	$m_1(\{a\}) = 0.99$, $m_1(\{b\}) = 0.01$; $m_2(\{c\}) = 0.99$, $m_2(\{b\}) = 0.01$.	$m_{1,2}(\{b\}) = 1$.
Yager's rule	(12)	$m_1(s_i) = m_2(s_i) = 0.2$. $(i = 1, 2, \ldots, 5)$, $\Theta = \{s_i\}$.	$m_{1,2}(s_i) = 0.04$, $m_{1,2}(\Theta) = 0.8$.
PCRR or Unified weight model	(13)	Not Satisfies Quasi-Associativity	
Disjunctive rule	(14)	$m_i(\{a\}) = 0.9$, $m_i\{b\} = 0.1$. $(i = 1, 2, \ldots, 4)$, $M = \{m_i\}$.	$m_{1,2,3}(\{a\}) = 0.891$; $m_M(\{a\}) = 0.8829$.
Murphy's method	(15)	$m_1(\{a\}) = 0.9$, $m_1(\Theta) = 0.1$; $m_2(\Theta) = m_3(\Theta) = 1$; $(\Theta = \{a, b\})$.	$m_{1,2,3}(\{a\}) = 0.657$; $m_{1,2,3}(\Theta) = 0.343$.
Smets' method	(11)	$m_1(s_i) = m_2(s_i) = 0.2$. $(i = 1, 2, \ldots, 5)$.	$m_{1,2}(s_i) = 0.04$, $m_{1,2}(\emptyset) = 0.8$.
Generalised evidence theory	(16)	$m_1(\{a\}) = 0.9$, $m_1(\{b\}) = 0.1$; $m_2(\{c\}) = 0.9$, $m_2(\{b\}) = 0.1$.	$m_{1,2}(\{b\}) = 1$, $m_{1,2}(\emptyset) = 0$.

Table 2
Different combination rule's Inconsistence with the properties identified.

	Property (i)	Property (ii)	Property (iii)	Property (iv)	Property (v)	Property (vi)
Dempster's rule	Yes	Yes	No	Yes	Yes	Yes
Yager's rule	Yes	Yes	No	No	No	No
PCRR	No	Yes	Yes	Yes	Yes	Yes
Unified weight model	No	Yes	Yes	Yes	Yes	Yes
Disjunctive rule	No	Yes	No	No	No	No
Murphy's method and all similar methods	No	No	Yes	Yes	Yes	Yes
Smet's method	Yes	Yes	No	No	No	No
Generalised evidence theory	Yes	Yes	No	Yes	Yes	Yes

the second property about neutral element commitment. In the fourth column, we can see that Dempster's rule, Yager's rule, Disjunctive rule (*i.e.*, formula (14)), Smet's method [38,44], and generalised evidence theory [7,36] violate the third property of possibility reservation. That is, either these methods may exclude the possibility of a state when the relevant evidence supports that it is possible, or their combined results are more ambiguous about the judgement of a state than relevance evidence. From the fifth column to the seventh column, we can see that Yager's rule, Disjunctive rule (*i.e.*, formula (14)), and Smet's method do not satisfy the properties of convergence toward certainty, invariance of iterated indifference evidence and weak specialisation.

The examples about the violation of intuitions are summarised in Table 1. For Dempster's combination rule and generalised evidence theory, they violate Zadeh's counter-example. And Yager's rule and Smets' method (*i.e.*, formula (11)) cannot solve the iteration combination of indifference evidence well, because they assign most of the mass values to the empty set or the frame. Murphy's method (*i.e.*, formula (15)) violate the intuition that the completely ignorable evidence should not influence the combined result. Finally, disjunctive rule (*i.e.*, formula (14)) cannot guarantee the property of convergence toward certainty. In summary, none of these existing methods can satisfy all the basic properties that we have identified in this paper, and so all of these methods can cause that some combined results violate human intuitions.

As a result, to satisfy all properties we identified in Section 3, in the next section we will propose a new flexible rule of evidence combination for the D–S theory.

5. Flexible evidential combination rule

This section will present a novel evidential combination rule, called flexible evidential combination rule, which can hold the properties we identified in Section 3. More specially, we first argue that the counter-intuitive behaviours of Dempster's combination rule are caused by the *complete conflict* of some focal elements of the original mass functions. Then, we view the combination process as a two-stage process: first combine all the pieces of available evidence by Dempster's combination rule; and then redistribute the combination mass value based on the weight values obtained from evidential distances [17] among the complete conflict elements. Since the rule gives the same result as Dempster's combination rule if there does not exist any complete conflict element, we call it *flexible combination rule*.

Firstly, we introduce the concept of *complete conflict*. It means that for a set of mass functions, there exists a focal element of a mass function, such that its intersection with any focal element of another mass function is an empty set.

Example 7. Let $m_1(\{a\}) = m_2(\{a\}) = 0.9$, $m_1(\{b, c\}) = m_2(\{b, c\}) = 0.1$, and $m_3(\{b\}) = m_3(\{c\}) = 0.5$. Then we have

- the focal element set of m_1, denoted as F_1, is $\{\{a\}, \{b, c\}\}$;
- the focal element set of m_2, denoted as F_2, is $\{\{a\}, \{b, c\}\}$; and
- the focal element set of m_3, denoted as F_3, is $\{\{b\}, \{c\}\}$.

Therefore, we have $\{a\} \in F_1$, such that for any element $X \in F_3$, we have $\{a\} \cap X = \emptyset$. It means set state a is completely conflicting for these mass functions, and the combined mass function obtained by using Dempster's combination rule implies a result inconsistent with our intuition: a high support to state a in mass functions m_1 and m_2 is completely eliminated in the combined result.

In other words, for a set of evidence, if one piece of evidence supports that some states are possible but another piece of evidence supports that these states are impossible, then completely conflicting judgements are made. Thus, we have:

Table 3
The evidence of different combination rules violating the properties.

	Original mass function	Combined result	Violated property
Dempster's rule	$m_1(\{a\})=0.9$, $m_1(\{b\})=0.1$; $m_2(\{c\})=0.9$, $m_2(\{b\})=0.1$.	$m_{1,2}(\{b\})=1$.	Violate the 3rd property: $m_1(\{a\})>0$, but $m_{1,2}(\{a\})=0$.
Yager's rule	$m_1(\{a\})=0.9$, $m_1(\{b\})=0.1$; $m_2(\{c\})=0.9$, $m_2(\{b\})=0.1$.	$m_{1,2}(\{b\})=0.01$ $m_{1,2}(\{\Theta\})=0.99$.	Violate the 3rd property: $m_1(\{a\})>0$, but $m_{1,2}(\{a\})=0$.
	$m_1(\{a\})=0.9, m_1(\Theta)=0.1$	$m_1^2(\{a\})=0.81$ $m_1^2(\Theta)=0.19$	Violate the 4th property: $m_1^2(\{a\})<m_1(\{a\})$.
	$m_1(s_i)=m_2(s_i)=0.2$. $(i=1,2,\ldots,5)$, $\Theta=\{s_i\}$.	$m_{1,2}(s_i)=0.04$, $m_{1,2}(\Theta)=0.8$.	Violate the 5th property: $m_1(s_i)=m_2(s_i)\neq m_{12}(\{s_i\})$. Violate the 6th property: $m_{1,2}(\Theta)>0$, but $m_1(\Theta)=m_2(\Theta)=0$.
PCRR	Smets [16] and Haenni [30] pointed out that it violate the 1st property.		
Unified weight model	Smets [16] and Haenni [30] pointed out that it violate the 1st property.		
Disjunctive rule	$m_1(\{a\})=m_2(\{c\})=m_3(\{a\})=0.9$; $m_1(\{b\})=m_2(\{b\})=m_3(\{b\})=0.1$.	$m_{(1,2),3}(\{a\})=0.81$. $m_{(1,3),2}(\{a\})=0$.	Violate the 1st property: $m_{(1,2),3}\neq m_{(1,3),2}$.
	$m_1(\{a\})=0.9$, $m_1(\{b\})=0.1$; $m_2(\{b\})=1$.	$m_{1,2}(\{a,b\})=0.9$, $m_{1,2}(\{b\})=0.1$	Violate the 3rd property: $m_1(\{a\})>0$, but $m_{1,2}(\{a\})=0$.
	$m_1(\{a\})=0.9, m_1(b)=0.1$	$m_1^2(\{a\})=0.81$ $m_1^2(\{a,b\})=0.19$ $m_1^2(\{b\})=0.01$	Violate the 4th property: $m_1^2(\{a\})<m_1(\{a\})$. Violate the 6th property: $m_{1,2}(\{a,b\})>0$, but $\forall A\supseteq\{a,b\}$, we have $m_1(A)=m_2(A)=0$.
	$m_1(\{s_i\})=m_2(\{s_i\})=0.5$. $(i=1,2)$, $\Theta=\{s_i\}$.	$m_{1,2}(\{s_i\})=0.25$, $m_{1,2}(\{s_1,s_2\})=0.5$.	Violate the 5th property: $m_1(\{s_i\})=m_2(\{s_i\})\neq m_{12}(\{s_i\})$.
Murphy's method	$m_1(\{a\})=0.9, m_1(\Theta)=0.1$; $m_2(\Theta)=m_3(\Theta)=1$; $(\Theta=\{a,b\})$.	$m_{1,2,3}(\{a\})=0.657$; $m_{1,2,3}(\Theta)=0.343$.	Violate the 2nd property: $m_2(\Theta)=m_3(\Theta)=1$ but $m_1\neq m_{1,2,3}$.
Smets' method	$m_1(\{a\})=0.9$, $m_1(\{b\})=0.1$; $m_2(\{c\})=0.9$, $m_2(\{b\})=0.1$.	$m_{1,2}(\{b\})=0.01$ $m_{1,2}(\{\emptyset\})=0.99$.	Violate the 3rd property: $m_1(\{a\})>0$, but $m_{1,2}(\{a\})=0$.
	$m_1(\{a\})=0.9, m_1(\Theta)=0.1$	$m_1^2(\{a\})=0.81$ $m_1^2(\Theta)=0.1$, $m_1^2(\emptyset)=0.18$.	Violate the 4th property: $m_1^2(\{a\})<m_1(\{a\})$.
	$m_1(\{s_i\})=m_2(\{s_i\})=0.2$. $(i=1,2,\ldots,5)$, $\Theta=\{s_i\}$.	$m_{1,2}(\{s_i\})=0.04$, $m_{1,2}(\emptyset)=0.8$.	Violate the 5th property: $m_1(\{s_i\})=m_2(\{s_i\})\neq m_{12}(\{s_i\})$. Violate the 6th property: $m_{1,2}(\Theta)>0$, but $m_1(\Theta)=m_2(\Theta)=0$.
Generalised evidence theory	$m_1(\{a\})=0.9$, $m_1(\{b\})=0.1$; $m_2(\{c\})=0.9$, $m_2(\{b\})=0.1$.	$m_{1,2}(\{b\})=1$, $m_{1,2}(\{\emptyset\})=0$.	Violate the 3rd property: $m_1(\{a\})>0$, but $m_{1,2}(\{a\})=0$.

Definition 6 (*Complete Conflict Set*)**.** Let $M=\{m_k \mid k=1,\ldots,n\}$ be a mass function set over a frame Θ, F_k be the set of all the focal elements of a mass function $m_k\in M$. Then $\Upsilon_M\subseteq 2^{\Theta}$ is a complete conflict set with respect to mass function set M if and only if for any $A\in\Upsilon_M$, there exist two focal element sets F_i and F_j for two mass functions $m_i, m_j\in M$, such that $A\in F_i$ and $\forall B\in F_j, A\cap B=\emptyset$.

We can find that for mass function set M, if $A\in F_i$ and $\forall B\in F_j, A\cap B=\emptyset$, then for any $C\subset A$, we have:

$$C\cap B=\emptyset,$$

$$\sum_{X\bigcap Y=C} m_1(X)m_2(Y)=0.$$

In other words, the combined mass value of any subset of any $A\in\Upsilon_M$ is zero by applying Dempster's combination rule (*i.e.*, formula (3)). So, it means no matter how high mass value $m_i(A)$ is and how many mass functions in M consider A as a focal element, if subset A is completely conflictive (*i.e.*, $A\in\Upsilon_M$), the result of Dempster's combination rule still rules it and its more special subsets out of the set of the possible states. This is the reason why Dempster's combination rule could cause some counter-intuitive behaviours. Thus, since the decision maker cannot tell which state is true in the case of complete conflict, it is reasonable to remain the conflictive elements based on the support degree of them in all available evidence. To represent the support degree, we give a formal method to obtain the weights over a frame, inspired by the method of [41], as follows:

Definition 7 (*Weight of a State Set*)**.** For a set of mass functions $M=\{m_k \mid k=1,\ldots,n\}$ over a frame Θ, let $s^*(m_i,m_j)$ be the similarity measure between two mass functions m_i and m_j with $m_j(\Theta)\neq 1$. Then the weight value of a subset $A\subseteq\Theta$ over all the relevant mass function in M for combination redistribution, denoted by $\omega_M(A)$, is given by:

$$\omega_M(A)=\sum_{i=1}^{n} m_i(A)\eta(m_i), \tag{17}$$

where $\eta(m_i)$, called the credibility degree of mass function m_i, is defined as:

$$\eta(m_i)=\begin{cases}0 & \text{if } m_i(\Theta)=1,\\ \dfrac{\sum_{j=1}^{n} s^*(m_i,m_j)}{\sum_{k=1}^{n}\sum_{j=1}^{n} s^*(m_k,m_j)} & \text{otherwise.}\end{cases} \tag{18}$$

In the above definition, actually $\sum_{j=1}^{n} s^*(m_i, m_j)$ is the overall similarity measure for a given mass function m_i compared with all mass functions in M except the mass function that is completely ignorable. So, since by the *second* property, the completely ignorable mass function is a neutral element in the combination process, $\sum_{j=1}^{n} s^*(m_i, m_j)$ shows the support degree of the piece of evidence represented by m_i in M. That is, the higher the value of $\sum_{j=1}^{n} s^*(m_i, m_j)$ is, the more mass functions in M support the claim of mass function m_i about the mass value distribution. Hence, by $\eta(m_i)$, we can obtain the credibility of a mass function m_i over a set of evidence M. Here, the credibility degree shows the relative importance of the available evidence. Clearly, since the completely ignorable mass function should not influence the combined result, the relative importance of such a mass function should be zero. Thus, we can define the credibility degree of the completely ignorable mass function as zero. Finally, by combining the credibility degree of each mass function and the support degree of any subset $A \subseteq \Theta$ in each mass function $m_i \in M$, we can obtain the weight value of each subset $A \subseteq \Theta$ over all the relevant mass functions (*i.e.*, $\omega_M(A)$). Here, $\omega_M(A)$ means that after considering the credibility of each mass function as well as the support degree of A in each mass function, we can give a composite evaluation of the weight of a set $A \subseteq \Theta$ supported by all available evidence in M. Finally, by Definition 1, we can obtain

$$\begin{aligned}\omega_M(\emptyset) &= \sum_{i=1}^{n} m_i(\emptyset)\eta(m_i)\\ &= \sum_{i=1}^{n} 0 \times \eta(m_i)\\ &= 0.\end{aligned} \tag{19}$$

Moreover, by formulae (17) and (18), since $\sum_{A\subseteq\Theta} m_i(A) = 1$, we have

$$\begin{aligned}\sum_{A\subseteq\Theta}\omega_M(A) &= \sum_{A\subseteq\Theta}\sum_{i=1}^{n} m_i(A)\frac{\sum_{j=1}^{n} s^*(m_i,m_j)}{\sum_{k=1}^{n}\sum_{j=1}^{n} s^*(m_k,m_j)}\\ &= \sum_{i=1}^{n}\left(\frac{\sum_{j=1}^{n} s^*(m_i,m_j)}{\sum_{k=1}^{n}\sum_{j=1}^{n} s^*(m_k,m_j)}\sum_{A\subseteq\Theta} m_i(A)\right)\\ &= \sum_{i=1}^{n}\left(\frac{\sum_{j=1}^{n} s^*(m_i,m_j)}{\sum_{k=1}^{n}\sum_{j=1}^{n} s^*(m_k,m_j)}\times 1\right)\\ &= 1.\end{aligned} \tag{20}$$

In Definition 7, since the weight value shows the support degree of a subset $A \subseteq \Theta$ after considering all the available mass functions, somehow it means that a judgement (*i.e.*, a focal element of a mass function) with some support degree should not be assigned zero in the combined result without any weak specialisation. Therefore, it is reasonable to remain the support degrees by redistributing the Dempster combination mass value in the case that a complete conflict exists. Formally, we have:

Definition 8 (*Flexible Evidential Combination Rule*)**.** For a mass function set

$$M = \{m_k \mid k = 1, \ldots, n\}$$

over a frame Θ, let $m_{\bigoplus M}$ be the value of combining all the mass functions in M by formula (3) (*i.e.*, Dempster's combination rule), A be a subset of frame Θ (*i.e.*, $A \subseteq \Theta$), $m_{\bigoplus M}(A)$ be the weight value of a subset $A \subseteq \Theta$ over M, and Υ_M be the complete conflict set of M. Then the mass function m_M that global-completely combines a mass functions set M is given by:

$$m_M(X) = \begin{cases} 0 & \text{if } X = \emptyset,\\ (1-\delta_M)m_{\bigoplus M}(X) & \text{if } X \subseteq \Theta,\ X \notin \Upsilon_M, \text{ and } X \neq \emptyset,\\ \sum\limits_{A\supseteq X}\omega_M(A)\dfrac{|X|}{|A|} & \text{if } X \subseteq \Theta,\ X \in \Upsilon_M \text{ and } X \neq \emptyset, \end{cases} \tag{21}$$

where δ_M is the compatible redistribution value given by

$$\delta_M = \begin{cases} 0 & \text{if } \Upsilon_M = \emptyset,\\ \sum\limits_{X\in\Upsilon_M}\sum\limits_{A\supseteq X}\omega_M(A)\dfrac{|X|}{|A|} & \text{otherwise.}\end{cases} \tag{22}$$

Here, since $\omega_M(\emptyset) = 0$ and $\sum_{A\subseteq\Theta}\omega_M(A) = 1$ (see formulae (19) and (20)), by formulae (21) and (22), we have $\sum_{A\subseteq\Theta} m_M(A) = 1$ and $m_M(\emptyset) = 0$. In other words, the function defined in Definition 8 indeed satisfies the requirement of a mass function in Definition 1. And formula (21) means that when the complete conflict exists, the result of Dempster's combination rule should be discounted according to the complete conflict set's weight δ, and the mass value of the complete conflict subset should be redistributed based on the weight of the subsets that include it. Hence, $\omega_M(A)\frac{|X|}{|A|}$ means that we need to consider the number of elements in a set A to redistribute the mass value. Finally, the intuition behind this definition is that in the combination process, we keep the completely conflicting judgement, but update the partially consistent judgements of the relevant evidence to more special judgements according to their supporting weights. Therefore, for the combined result, it should include two parts: (i) the value of partial consistent judgement and (ii) the flexible mass value of conflicting judgement.

Finally, one of the main reasons why we instantiate the basic properties with the flexible combination rule is that this rule has a desired feature as follows: *If a judgement is supported by most of the available evidence greatly, it should have more effect upon the final combined result.* This feature captures the essence of a combination rule is to aggregate different pieces of evidence into a combined conclusion that accommodates the opinions of all the relevant evidence as much as possible. Therefore, if a set of states is highly supported by the majority of pieces of evidence, at least one subset of the state set is highly supported in the combination conclusion.

The following theorem states that our new rule exhibits the basic properties in Definition 4.

Theorem 1. *Flexible evidential combination rule holds the properties in Definition 4.*

Proof. We check to the properties listed in Definition 4 one by one.

(i) By the fact that Dempster's combination rule satisfies commutativity [1,2], and Definitions 7 and 8, obviously our rule satisfies commutativity. Moreover, since Dempster's combination rule is associative [1,2] and our combined result is a redistribution process of the value of combining all the mass functions in M by formula (3) with a weight value in Definition 8, it satisfies the property of quasi-associativity. So, our new rule holds the first property in Definition 4.

(ii) Let $\odot$ be the operator given in Definition 8 and $\oplus$ be the operator given in Definition 2. Then by Definition 2, for any mass function $m_i \in M$, if $m_i(\Theta) = 1$, we have $m_{\oplus M} = m_{\oplus M} \oplus m_i$. Hence, by Definition 6, we have $C_M = C_{M\cup\{m_i\}}$. Thus, by Definitions 6 and

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7, we have $\omega_M(A) = \omega_{M\cup\{m_i\}}(A)$ and $C_M = C_{M\cup\{m_i\}}$. Further, by Definition 8 we have $m_M \odot m_i = m_M$. Therefore, our new rule holds the second property in Definition 4.

(iii) Assume that there exist $m_i \in M$ and $a \in \Theta$ such that $m_i(\{a\}) > 0$ and $m_{\oplus M}(\{a\}) = 0$. Then, by Definition 8 we have $\{a\} \in \Upsilon_M$ and $\sum_{A\supseteq X} \omega_M(A)\frac{|X|}{|A|} = 0$, or $\{a\} \notin \Upsilon_M$ and $(1-\delta_M)m_{\oplus M}(X) = 0$. In the *first case*, for any $A \supseteq \{a\}$, since $\frac{|\{a\}|}{|A|} > 0$, $\omega_M(A) = 0$. Moreover, since $\{a\} \supseteq \{a\}$, by Definition 7 we have $\omega_M(\{a\}) = \sum_{k=1}^{n} m_k(\{a\})C(m_k) = 0$. Thus, we have $m_i(\{a\})\eta(m_i) = 0$. By Definition 7, we have $\eta(m_i) > 0$. So, $m_i(\{a\}) = 0$, which is contrary to the assumption of $m_i(\{a\}) > 0$. In the *second case*, since $\{a\} \notin \Upsilon_M$, by Definition 6 and $m_i(\{a\}) > 0$, we have that for any set of focal elements F_j of mass function $m_j \in M$, there exists $B \in F_j$ such that $\{a\} \cap B \neq \emptyset$. So, by formula (3) we have $m_{\oplus M}(\{a\}) > 0$. Therefore, we have $\delta_M = 1$, which means that all the focal elements of each mass function in M belong to Υ_M. That is, $\{a\} \in \Upsilon_M$. However, $\{a\} \notin \Upsilon_M$ and $\{a\} \in \Upsilon_M$ are self-contradiction. Thus, since both the cases of our assumption cause contradiction, we can conclude that if there exist $m_i \in M$ and $a \in \Theta$ such that $m_i(\{a\}) > 0$ then $m_M(\{a\}) > 0$. Thus, our new rule holds property (iii) in Definition 4.

(iv) Since combining any mass function m_1 by using our new rule n times, the complete conflict set is empty. Thus, by Definition 8 our combined result is the same as that of Dempster's combination rule in this case. Moreover, since Dempster's rule has the property of convergence toward certainty [1,2], it is straightforward that ours holds property (iv) in Definition 4.

(v) Similar to the above discussion about property (iv), when the complete conflict set is empty, our combined result is the same as that of Dempster's one. Hence, by using Dempster's one, the iteration of indifferent evidence outputs the same result as each piece of evidence. So, ours holds property (v) in Definition 4.

(vi) Our combined result has two parts: (a) one is obtained by the discounted value from the result of Dempster's combination rule, and (b) the other is obtained by the weight values. So, by Definitions 2 and 7, it is straightforward that our rule holds property (vi) in Definition 4. □

6. Illustration

In this section, we will first use our rule to resolve Zadeh's counter-example, and then uses another example to check whether or not our rule holds the properties we identified in Section 3. Finally, we will show the comparison with the existing methods in some examples.

First, let us show how our new rule resolves Zadeh's counter-example (*i.e.*, Example 1). Recall there are two mass functions m_1 and m_2 that are defined on a frame of discernment $\Theta = \{a, b, c\}$, satisfying (4) and (5), and by Dempster's combination rule (3), we have $m_{\oplus\{m_1,m_2\}}(\{b\}) = 1$ (see (6)). Hence, by Definition 6, after checking the focal elements of each mass function (*i.e.*, $\{a\}$ and $\{b\}$ for m_1, and $\{b\}$ and $\{c\}$ for m_2) in Zadeh's example, we can find that $m_1(\{a\}) > 0$ and for any $A \cap \{a\} \neq \emptyset$, we have $m_2(A) = 0$, and thus we have $\Upsilon_M = \{\{a\}, \{c\}\}$. Moreover, by formulae (9) and (10), and by Definition 7, we have:

$$
\begin{aligned}
\omega_M(\{a\}) &= \sum_{i=1}^{2} m_i(\{a\}) \frac{\sum_{j=1}^{2} s^*(m_i, m_j)}{\sum_{k=1}^{2}\sum_{j=1}^{2} s^*(m_k, m_j)} \\
&= m_1(\{a\}) \frac{s^*(m_1,m_1)+s^*(m_1,m_2)}{s^*(m_1,m_1)+s^*(m_1,m_2)+s^*(m_2,m_1)+s^*(m_2,m_2)} + \\
&\quad m_2(\{a\}) \frac{s^*(m_2,m_1)+s^*(m_2,m_2)}{s^*(m_1,m_1)+s^*(m_1,m_2)+s^*(m_2,m_1)+s^*(m_2,m_2)} \\
&= 0.99 \times \frac{1+s^*(m_1,m_2)}{1+s^*(m_1,m_2)+s^*(m_2,m_1)+1} + 0 \\
&\quad \times \frac{s^*(m_2,m_1)+1}{1+s^*(m_1,m_2)+s^*(m_2,m_1)+1} \\
&= 0.99 \times \frac{1+s^*(m_1,m_2)}{1+s^*(m_1,m_2)+s^*(m_1,m_2)+1} \\
&= 0.99 \times \frac{1}{2} \\
&= 0.495.
\end{aligned}
$$

Similarly, we can obtain $\omega_M(\{b\}) = 0.01$ and $\omega_M(\{c\}) = 0.495$. Finally, by Definition 8 we have

$$m_M(\{a\}) = \omega_M(\{a\}) \frac{|\{a\}|}{|\{a\}|} = 0.495 \times 1 = 0.495.$$

Similarly, we can obtain $m_M(\{b\}) = 0.01$ and $m_M(\{c\}) = 0.495$. Moreover, since m_1 implies that c is impossible and m_2 implies that a is impossible in Example 1 (*i.e.*, Zadeh's counter-example), this result means that without further evidence, it is reasonable to remain the contradiction judgements of two pieces of evidence based on their weight values.

Now we turn to examine our rule by another example.

Example 8. In a multi-sensor target recognition system, suppose there are six pieces of evidence from six different sensors and the set of possible targets is $\Theta = \{a, b, c\}$, and the mass function set $M = \{m_1, \ldots, m_6\}$ that represents these pieces of evidence are defined as follows:

$$
\begin{aligned}
&m_1(\{a\}) = 0.6, m_1(\{b\}) = 0.1, m_1(\{c\}) = 0.3; \\
&m_2(\{b\}) = 0.8, m_2(\{c\}) = 0.2; \\
&m_3(\{a\}) = 0.55, m_3(\{b\}) = 0.1, m_3(\{c\}) = 0.35; \\
&m_4(\{a\}) = 0.7, m_4(\{b\}) = 0.1, m_4(\{c\}) = 0.2; \\
&m_5(\{a, c\}) = 0.9, m_5(\{b\}) = 0.1; \\
&m_6(\Theta) = 1.
\end{aligned}
$$

Now we use our rule to combine all the above mass functions. First of all, by Dempster's combination rule (3) we have:

$$
\begin{aligned}
m_{\oplus M}(\{b\}) &= \frac{\prod_{i=1}^{5} m_i(\{b\})m_6(\Theta)}{\prod_{i=1}^{5} m_i(\{b\})m_6(\Theta) + \prod_{i=1}^{4} m_i(\{c\})m_5(\{a,c\})m_6(\Theta)} \\
&= \frac{0.1 \times 0.8 \times 0.1 \times 0.1 \times 0.1 \times 1}{0.1 \times 0.8 \times 0.1 \times 0.1 \times 0.1 \times 1 + 0.3 \times 0.2 \times 0.35 \times 0.2 \times 0.9 \times 1} \\
&= 0.0207.
\end{aligned}
$$

Similarly, we can obtain $m_{\oplus M}(\{c\}) = 0.9793$. Sequentially, we check the complete conflict of mass function set M according to Definition 6. It is easy to find that $\{a\} \in F_1$ (where F_1 is the focal element set of mass function m_1) and for any $A \in F_2$ (where F_2 is the focal element set of mass function m_2), we have $\{a\} \cap A = \emptyset$. By checking each focal element of any mass function in M, we can find that only $\{a\}$ belongs to the complete conflict set by Definition 6. Therefore, we have $\Upsilon_M = \{\{a\}\}$.

Since the complete conflict set is not empty, we need to consider the redistribution of the combined result by the weight value of each subset of the frame. The similarity measure between any two different mass functions, which both are not completely ignorable mass functions, can be obtained by formulae (7) and (8) as follows. Thus, we have s*(m1, m2) given in Box I.

Similarly, we can obtain:

$$
\begin{aligned}
&s^*(m_1, m_3) = 0.95, s^*(m_1, m_4) = 0.9, s^*(m_1, m_5) = 0.526; \\
&s^*(m_2, m_1) = 0.344, s^*(m_2, m_3) = 0.362, \\
&s^*(m_2, m_4) = 0.3, s^*(m_2, m_5) = 0.238; \\
&s^*(m_3, m_1) = 0.95, s^*(m_3, m_2) = 0.362, s^*(m_3, m_4) = 0.85, \\
&s^*(m_3, m_5) = 0.539; \\
&s^*(m_4, m_1) = 0.9, s^*(m_4, m_2) = 0.3, s^*(m_4, m_3) = 0.85, \\
&s^*(m_4, m_5) = 0.485; \\
&s^*(m_5, m_1) = 0.526, s^*(m_5, m_2) = 0.238, \\
&s^*(m_5, m_3) = 0.539, s^*(m_5, m_4) = 0.485.
\end{aligned}
$$

$$
\begin{aligned}
s^*(m_1,m_2) =& 1-\sqrt{\frac{1}{2}\sum_{\substack{\emptyset\neq A\subseteq\Theta\\ \emptyset\neq B\subseteq\Theta}}\frac{|A\cap B|}{|A\cup B|}(m_1(A)-m_2(A))(m_1(B)-m_2(B))}\\
=& 1-\sqrt{\frac{1}{2}\left(\frac{|a\cap a|}{|a\cup a|}(m_1(a)-m_2(a))^2+\frac{|b\cap b|}{|b\cup b|}(m_1(b)-m_2(b))^2+\frac{|c\cap c|}{|c\cup c|}(m_1(c)-m_2(c))^2\right)}\\
=& 1-\sqrt{\frac{1}{2}\left(0.6^2+(-0.7)^2+0.1^2\right)}\\
=& 0.344.
\end{aligned}
$$

Box I.

Then, with $s^*(m_i,m_i)=1$ (see formula (10)) for $i=1,\ldots,5$, we can easily obtain

$$
\sum_{i=1}^{5}\sum_{j=1}^{5}s^*(m_i,m_j)=15.988,\ \sum_{j=1}^{5}s^*(m_1,m_j)=3.72,
$$
$$
\sum_{j=1}^{5}s^*(m_3,m_j)=3.701,\ \sum_{j=1}^{5}s^*(m_4,m_j)=3.535.
$$

Thus, by Definition 7, we have

$$
\begin{aligned}
&\omega_M(\{a\})\\
=&\sum_{i=1}^{5}m_i(\{a\})\frac{\sum_{j=1}^{5}s^*(m_i,m_j)}{\sum_{i=1}^{5}\sum_{j=1}^{5}s^*(m_i,m_j)}\\
=&\frac{1}{\sum_{i=1}^{5}\sum_{j=1}^{5}s^*(m_i,m_j)}\sum_{i=1}^{5}m_i(\{a\})\sum_{j=1}^{5}s^*(m_i,m_j)\\
=&\frac{1}{15.988}\Bigg(m_1(\{a\})\sum_{j=1}^{5}s^*(m_1,m_j)+m_2(\{a\})\sum_{j=1}^{5}s^*(m_2,m_j)\\
&+m_3(\{a\})\sum_{j=1}^{5}s^*(m_3,m_j)+\\
&m_4(\{a\})\sum_{j=1}^{5}s^*(m_4,m_j)+m_5(\{a\})\sum_{j=1}^{5}s^*(m_5,m_j)\Bigg)\\
=&\frac{1}{15.988}\Bigg(0.6\sum_{j=1}^{5}s^*(m_1,m_j)+0\sum_{j=1}^{5}s^*(m_2,m_j)\\
&+0.55\sum_{j=1}^{5}s^*(m_3,m_j)+\\
&0.7\sum_{j=1}^{5}s^*(m_4,m_j)+0\sum_{j=1}^{5}s^*(m_5,m_j)\Bigg)\\
=&\frac{1}{15.988}\Bigg(0.6\sum_{j=1}^{5}s^*(m_1,m_j)+0.55\sum_{j=1}^{5}s^*(m_3,m_j)\\
&+0.7\sum_{j=1}^{5}s^*(m_4,m_j)\Bigg)\\
=&\frac{1}{15.988}(0.6\times 3.72+0.55\times 3.701+0.7\times 3.535)\\
=&0.4217.
\end{aligned}
$$

Similarly, we can obtain $\omega_M(\{b\})=0.157$, $\omega_M(\{c\})=0.1982$, and $\omega_M(\{a,c\})=0.2231$.

Thus, by Definition 8, we have the redistributed mass function as follows:

$$
\begin{aligned}
m'_M(\{a\}) &= \omega_M(\{a\})\frac{|\{a\}|}{|\{a\}|}+\omega_M(\{a,c\})\frac{|\{a\}|}{|\{a,c\}|}\\
&=0.4217+\frac{0.2231}{2}=0.5333,\\
m'_M(\{b\}) &= (1-\delta_M)m_{\bigoplus M}(\{b\})=(1-0.5333)\\
&\quad\times 0.0207=0.0097,\\
m'_M(\{c\}) &= (1-\delta_M)m_{\bigoplus M}(\{c\})=(1-0.5333)\times 0.9793=0.457.
\end{aligned}
$$

Now we discuss whether or not the redistributed mass function exhibits the desired properties. Let $sim_i=\sum_{j=1}^{n}s^*(m_i,m_j)$, for any $m_j(\Theta)\neq 1$. Then we have:

$$
\begin{aligned}
sim_1 =& s^*(m_1,m_1)+s^*(m_1,m_2)+s^*(m_1,m_3)\\
&+s^*(m_1,m_4)+s^*(m_1,m_5)\\
=&1+0.344+0.95+0.9+0.526\\
=&3.72.
\end{aligned}
$$

Similarly, we can obtain $sim_2=2.24$, $sim_3=3.7$, $sim_4=3.54$, and $sim_5=2.79$. That is, since mass function m_1 has the highest value of the similarity measure to the other mass functions, we can say that the result of m_1 is supported by most of evidence. Thus, we can see that the combined result of our rule indeed gives a conclusion similar to m_1: state a is most possible and state b is least possible. So, we also show the feature of our rule: if a judgement is supported by most of the available evidence greatly, it should has more effect upon the final combined result.

Also, in this example, we can find that the result is the same no matter which order we use for combining the set of mass function. Thus, it satisfies the *first* property about the local computation available. Moreover, we can find that the result is the same as the combined result without m_6 by our flexible combination rule. So, it exhibits the *second* property: the overall support situation of a set of evidence should not be influenced by the combination with a completely ignorable evidence. At the same time, since $m_1(\{x\})>0$ and $m_M(\{x\})>0$ for $x\in\{a,b,c\}$, our combined result indeed satisfies the *third* property. Hence, if we just consider $M'=\{m_1,m_3,m_4,m_5\}$ in this example, which means that all the pieces of evidence support that state a or the state set including state a is most possible (*i.e.*, the dominant opinion of each evidence is the same), we can obtain the combined result by the similar method as follows: $m_{M'}(\{a\})=0.9163$, $m_{M'}(\{b\})=0.0004$, and $m_{M'}(\{c\})=0.0833$. That is, our combined result converges toward the dominant opinion, satisfying the *fourth* property. At the same time, let $\Theta=\{a,b,c\}$ be a frame, $M=\{m_1,m_2\}$, and $m_1(\{x\})=m_2(\{x\})=1/3$

Table 4
The examples of different combination rules violating intuitions.

	Original mass function	Combination result	Result of our rule
Dempster's rule	$m_1(\{a\}) = 0.99$, $m_1(\{b\}) = 0.01$; $m_2(\{c\}) = 0.99$, $m_2(\{b\}) = 0.01$.	$m_{1,2}(\{b\}) = 1$.	$m_{1,2}(\{a\}) = 0.495$, $m_{1,2}(\{b\}) = 0.01$, $m_{1,2}(\{c\}) = 0.495$.
Yager's rule	$m_1(s_i) = m_2(s_i) = 0.2$. $(i = 1, 2, \ldots, 5)$, $\Theta = \{s_i\}$.	$m_{1,2}(s_i) = 0.04$, $m_{1,2}(\Theta) = 0.8$.	$m_{1,2}(s_i) = 0.2$.
PCRR or Unified weight model	Not Satisfies Quasi-Associativity		
Disjunctive rule	$m_i(\{a\}) = 0.9$, $m_i\{b\} = 0.1$. $(i = 1, 2, \ldots, 4)$, $M = \{m_i\}$.	$m_{1,2,3}(\{a\}) = 0.891$; $m_M(\{a\}) = 0.8829$.	$m_{1,2,3}(\{a\}) = 0.9987$; $m_M(\{a\}) = 0.9999$.
Murphy's method	$m_1(\{a\}) = 0.9$, $m_1(\Theta) = 0.1$; $m_2(\Theta) = m_3(\Theta) = 1$; $(\Theta = \{a, b\})$.	$m_{1,2,3}(\{a\}) = 0.657$; $m_{1,2,3}(\Theta) = 0.343$	$m_{1,2,3}(\{a\}) = 0.9$, $m_{1,2,3}(\Theta) = 0.1$.
Smets' method	$m_1(s_i) = m_2(s_i) = 0.2$. $(i = 1, 2, \ldots, 5)$.	$m_{1,2}(s_i) = 0.04$, $m_{1,2}(\emptyset) = 0.8$.	$m_{1,2}(s_i) = 0.2$.
Generalised evidence theory	$m_1(\{a\}) = 0.9$, $m_1(\{b\}) = 0.1$; $m_2(\{c\}) = 0.9$, $m_2(\{b\}) = 0.1$.	$m_{1,2}(\{b\}) = 1$, $m_{1,2}(\emptyset) = 0$	$m_{1,2}(\{a\}) = 0.495$, $m_{1,2}(\{b\}) = 0.01$, $m_{1,2}(\{c\}) = 0.495$.

$(x \in \{a, b, c\})$ be two mass functions over Θ. Then, by Dempster's combination rule (3), for any $x \in \{a, b, c\}$, we have

$$\begin{aligned} m_{\oplus M}(\{x\}) =& \frac{m_1(\{x\})m_2(\{x\})}{1 - \sum_{A \bigcap B \emptyset} m_1(A)m_2(B)} \\ =& \frac{\frac{1}{3} \times \frac{1}{3}}{1 - (\frac{1}{3} \times \frac{1}{3} \times 6)} \\ =& \frac{1}{3}. \end{aligned}$$

Sequentially, we check the complete conflict of mass function set M according to Definition 6. Then, it is easy to find that for any element A that belongs to focal element set F_i of mass function m_i $(i \in \{1, 2\})$, we can always find an element B that belongs to focal element set F_j $(i \neq j)$ such that $A \cap B \neq \emptyset$. Thus, $\Upsilon_M = \emptyset$. In other words, by Definition 8, for any $x \in \{a, b, c\}$, we have $\delta_M = 0$ for the compatible redistribution value. Therefore, by formula (21), we have

$$m_M(\{x\}) = (1 - \delta_M)m_{\oplus M}(\{x\}) = (1 - 0) \times \frac{1}{3} = \frac{1}{3}.$$

Thus, our combination rule satisfies the *fifth* property in this example.

Finally, since the focal elements of m_M (or $m_{M'}$) is $\{\{a\}, \{b\}, \{c\}\}$ and $m_1 \in M$ (or $m_1 \in M'$), by the fact that $m_1(\{a\}) > 0$, $m_1(\{b\}) > 0$, and $m_1(\{c\}) > 0$, and by Definition 4, we can easily see that the combined result of all the mass functions in M (or M') satisfies the *sixth* property. Moreover, since our combination rule satisfies all the properties we have mentioned in Section 3, we can find that it can give an acceptable solution for the examples in Table 1 about the violation of human intuitions for the existing methods as shown in Table 4.

7. Conclusion and future works

D–S theory is widely used for modelling and reasoning with ambiguous information, in which Dempster's rule for combining multiple pieces of evidence from different sources plays a critical role. However, the rule has been challenged by many researchers. To take these challenges, in this paper, we first identify a number of basic properties that can well reflect human intuitions in combining multiple pieces of evidence from different sources, including local computation availability, neutral element commitment, possibility reservation, convergence toward certainty, invariance of iterated indifference evidence, and weak specialisation. Moreover, in order to make such properties easier to understand, we also give a real-world based application about intelligence security surveillance system. Further, after analysing the existing combination rules, we find that none of these rules can exhibit all the basic properties identified in this paper, and so all of these methods can cause some combined results that violate human intuitions. Thus, based on evidential weights and our concept of complete conflict, we propose a flexible combination rule that can satisfy all properties we proposed. In addition, we use our rule to resolve well-known Zadeh's counter-example, give an example to examine our rule for the properties we identified, and give a model comparison with the existing work. The comparisons show clearly that our combination rule indeed has its advantage in the combination of uncertain, imprecise, and even high conflicting heterogeneous evidence.

There are many possible extensions to our work. Maybe the most interesting one is to axiomatise our combination rule. Another tempting avenue is to apply our combination rule into some applications, such as multiple sensor surveillance system [20], wireless sensor network [45], and automated e-business negotiation [46–49]. Also, since D–S theory can be used to deal with multiple criteria decision making problem [3,50] (*i.e.*, it can describe both quantitative and qualitative criteria under various uncertainties including ignorance and randomness and the combination rule can be used to combine the ranking of each choice with respect to each criterion), it is interesting to extend our method with criteria weights for multi-criteria decision making. Finally, it is significant to analyse more properties and the rationalities about the idea of complete conflict in information fusion as well as the theoretical comparison with other proposed conflict concepts in [2,9,36].

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Declaration of competing interest

No author associated with this paper has disclosed any potential or pertinent conflicts which may be perceived to have impending conflict with this work. For full disclosure statements refer to https://doi.org/10.1016/j.asoc.2019.105512.

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