SOME NOTES ON THE KEPLER TRIANGLE AND THE MAXIMUM GENERALIZED GOLDEN RIGHT TRIANGLE

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ABSTRACT. An interesting conic section problem is solved, then a sequence of golden right triangles T_n is derived from an identity of Fibonacci numbers, last, some geometric properties of T_1 which is named Kepler triangle are discussed, and some constructions of T_2 which is just the maximum generalized golden right triangle of T_n are obtained.

1. INTRODUCTION

As the great astronomer Johannes Kepler stated,"Geometry has two great treasures: one is the theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel"[1, p. 160].

2. A CONIC SECTION PROBLEM

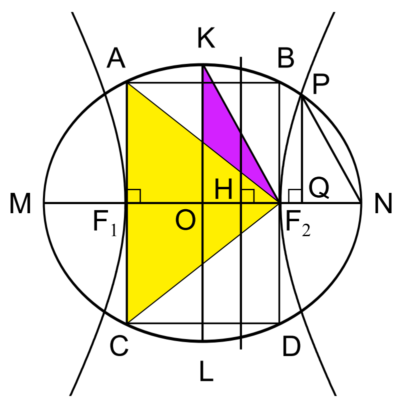

FIGURE 1. A golden ellipse and a golden hyperbola

Let's consider an interesting problem involving an ellipse and a hyperbola in Figure 1. First, we construct an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and a hyperbola $\frac{x^2}{c^2} - \frac{y^2}{b^2} = 1$, where a is the semi-major axis, b is the semi-minor axis of the ellipse, and $a^2 = b^2 + c^2$, such that, the eccentricity e_1 of the ellipse and e_2 of the hyperbola satisfy the condition $e_1 e_2 = 1$, and the foci of the ellipse becomes the corresponding vertex of the hyperbola, next, let F_1 and F_2 denote the foci of the ellipse, KL the minor axis, MN the major axis, and O the origin, without loss of generality, we set $c = OF_2 = 1$.

Then, let P be the top-right intersection point of the ellipse and the hyperbola, construct a segment PQ perpendicular to ON and intersecting ON at the foot Q, let H be the intersection point of ON and the right directrix $x = \frac{1}{a}$ of the hyperbola, now, our problem is:

If $PN \parallel KF_2$, what will e_1, e_2, $\frac{ON}{OQ}$ and $\frac{OQ}{HQ}$ be?

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Proposition 2.1. *If $PN \parallel KF_2$, then $e_1 = \frac{1}{\phi\sqrt{\phi}}$, $e_2 = \phi\sqrt{\phi}$ and $\frac{ON}{OQ} = \frac{OQ}{HQ} = \phi$, where $\phi = \frac{1+\sqrt{5}}{2}$, in other words, Q divides ON into the golden ratio, and H divides OQ into the golden ratio.*

Proof. First, solve the equation set (2.1)

$$\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\ x^2 - \frac{y^2}{b^2} = 1 \end{cases} \tag{2.1}$$

to get the coordinates of P, we get $x_P = OQ = \sqrt{\frac{2a^2}{a^2+1}}$, $y_P = PQ = \sqrt{\frac{(a^2-1)b^2}{a^2+1}}$, and by $PN \parallel KF_2$, we have $\frac{PQ}{QN} = \frac{KO}{OF_2}$, and get

$$\sqrt{\frac{(a^2-1)b^2}{a^2+1}} = (a - \sqrt{\frac{2a^2}{a^2+1}})b$$

and the final form

$$(a^2-1)(a^4-4a^2-1) = 0 \tag{2.2}$$

Since $c = 1$ and $a > c$, then we obtain the unique solution $a = \phi\sqrt{\phi}$ from (2.2), hence $e_1 = \frac{c}{a} = \frac{1}{\phi\sqrt{\phi}}$, $e_2 = \phi\sqrt{\phi}$, then $\frac{ON}{OQ} = \frac{\phi\sqrt{\phi}}{\sqrt{\phi}} = \phi$, $OH = \frac{1}{a} = \frac{1}{\phi\sqrt{\phi}}$, $HQ = OQ - OH = \frac{1}{\sqrt{\phi}}$, thus, $\frac{OQ}{HQ} = \phi$, and we also get $QN = ON - OQ = \frac{1}{\sqrt{\phi}} = HQ$, which means Q is the midpoint of HN. □

3. A SEQUENCE OF GOLDEN RIGHT TRIANGLES T_n

There is an identity (3.1) of the golden ratio[2] and Fibonacci numbers (see, e.g., [3, p. 78]),

$$\phi^{n+1} = F_{n+1}\phi + F_n, \quad (n = 0, 1, 2, \dots) \tag{3.1}$$

and we notice that $\triangle KOF_2$ in Figure 1 is a right triangle having sides $(1, \sqrt{2\phi}, \phi\sqrt{\phi})$, interestingly, if we rewrite (3.1) in the form of (3.2),

$$1 + \left(\sqrt{\frac{\phi F_{n+1}}{F_n}}\right)^2 = \left(\sqrt{\frac{\phi^{n+1}}{F_n}}\right)^2, \quad (n = 1, 2, 3, \dots) \tag{3.2}$$

we will find that it's just the second triangle, denoted as T_2, which is also the maximum right triangle of a golden right triangles sequence T_n with sides (1, $\sqrt{\frac{\phi F_{n+1}}{F_n}}$, $\sqrt{\frac{\phi^{n+1}}{F_n}}$) by (3.2), see Figure 2, and the first triangle T_1, of sides (1, $\sqrt{\phi}$, ϕ), is just the Kepler triangle[2, p. 149][4] whose side lengths are in geometric progression, and the Kepler triangle is also of minimum area in the sequence T_n, as shown in the area inequality (3.3) of T_n.

$$\triangle T_1 \le \triangle T_n \le \triangle T_2 \tag{3.3}$$

It is also interesting that, let $n \to +\infty$, the limiting triangle of T_n, of sides (1, ϕ,$\sqrt{1+\phi^2}$)[5], is just a right triangle which forms half of a well-known golden rectangle[6][3, p. 274][7, p. 115].

4. SOME GEOMETRIC PROPERTIES OF THE KEPLER TRIANGLE

Next, we'll discuss some geometric properties of the Kepler triangle. In Figure 3, construct a Kepler triangle $\triangle ABC$ with $BC = 1$, $AB = \sqrt{\phi}$, then, let D, E, and F be the golden section points of BC, AB, and AC respectively, and $\frac{BD}{DC} = \frac{AE}{EB} = \frac{CF}{FA} = \phi$.

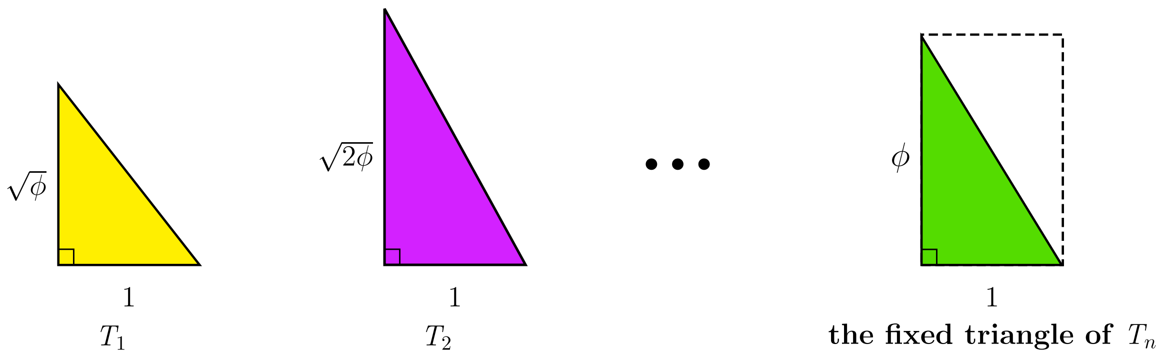

FIGURE 2. A sequence of golden right triangles T_n

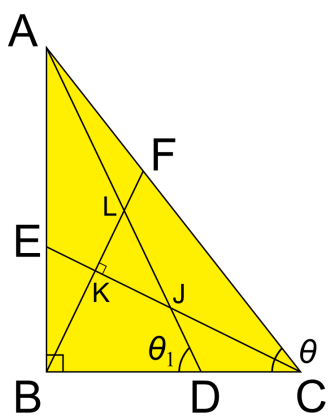

FIGURE 3. Some properties of the Kepler triangle

Lemma 4.1. *Let $\angle ACB = \theta$, $\angle ADB = \theta_1$, then we have*

$$\theta + 2\theta_1 = \pi \tag{4.1}$$

$$\angle BCE = \angle ECF = \frac{\theta}{2} \tag{4.2}$$

Proof. It's known that $\tan\theta = \frac{AB}{BC} = \sqrt{\phi}$ and $\tan\theta_1 = \frac{AB}{BD} = \phi\sqrt{\phi}$, then we get

$$\tan 2\theta_1 = \frac{2\tan\theta_1}{1-\tan^2\theta_1} = -\sqrt{\phi}$$

and $\tan 2\theta_1 + \tan\theta = 0$, hence (4.1) proved. Then, $\tan\angle BCE = \frac{BE}{BC} = \frac{1}{\phi\sqrt{\phi}}$, we get $\tan(2\angle BCE) = \sqrt{\phi} = \tan\angle ACB = \tan\theta$, hence (4.2) proved. □

Theorem 4.2. *Let J, K, L be the intersection points of EC, AD; EC, BF; and BF, AD respectively, then $\triangle JKL$ is just a right triangle which is similar to the Kepler triangle $\triangle ABC$.*

Proof. It's easy to find that $\triangle BCF$ is an isosceles triangle with $BC = CF = 1$, and it's known $\angle BCE = \angle ECF$ in (4.2) of Lemma 4.1, thus, $BF \perp EC$, which means $\triangle JKL$ is a right triangle, then, $\angle DJC = \angle ADB - \angle BCE = \theta_1 - \frac{\theta}{2}$ and by (4.1), we get $\angle DJC = \theta_1 - \frac{\theta}{2} = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \angle ACB = \angle BAC$, therefore, $\angle KJL = \angle DJC = \angle BAC$, hence proved. □

5. SOME CONSTRUCTIONS AND PROPERTIES OF THE MAXIMUM GENERALIZED GOLDEN RIGHT TRIANGLE

Probably, the second golden right triangle T_2 of sides $(1, \sqrt{2\phi}, \phi\sqrt{\phi})$ has not been discussed before, we only know it's the maximum triangle of T_n in (3.3), here, we have an interesting construction 5.1 in Figure 4.

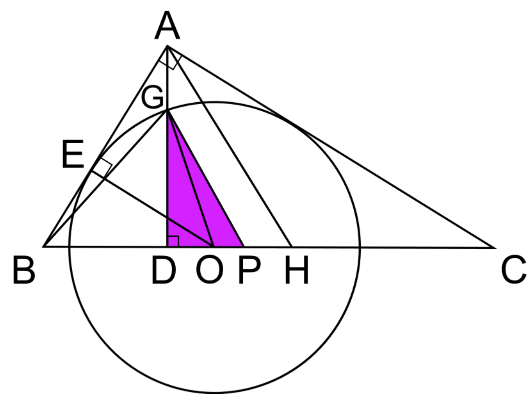

FIGURE 4. An interesting construction of a triangle similar to T_2

Construction 5.1. *A 5-step construction of a right triangle similar to T_2:*

(1) construct a limiting triangle $\triangle BAC$ of T_n with $AB = 1$, $AC = \phi$
(2) construct the height AD intersecting BC at the foot D
(3) construct E dividing AB into the golden ratio and $\frac{AE}{EB} = \phi$, through E, construct a perpendicular segment EO to AB, intersecting BC at point O
(4) draw a circle with the center at O and the radius OE, cutting AD at G
(5) construct DH on BC that $DH = BD$, construct P dividing DH into the golden ratio and $\frac{DP}{PH} = \phi$

Then $\triangle GDP$ is just a right triangle having sides proportional to $(1, \sqrt{2\phi}, \phi\sqrt{\phi})$, and interestingly, the center O of the circle also divides DP into the golden ratio and $\frac{DP}{DO} = \phi$.

Proof. In Figure 4, it's known that $AB = 1$, $AC = \phi$, then we get $DH = BD = \frac{1}{\sqrt{1+\phi^2}}$, $DP = \frac{DH}{\phi} = \frac{1}{\phi\sqrt{1+\phi^2}}$, $DO = BO - BD = BE\sqrt{1+\phi^2} - \frac{1}{\sqrt{1+\phi^2}} = \frac{1}{\phi^2\sqrt{1+\phi^2}}$, $OG = OE = AE = \frac{1}{\phi}$, $GD = \sqrt{OG^2 - DO^2} = \frac{\sqrt{2}}{\sqrt{\phi+\phi^3}}$, then $\frac{GD}{DP} = \sqrt{2\phi}$, $\frac{DP}{DO} = \phi$, and we are done. $\square$

We've proved O divides DP into the golden ratio, easily, we get a simple property in T_2,

Proposition 5.2.

$$\sin \angle DGO = \frac{DO}{GO} = \frac{1}{\phi\sqrt{1+\phi^2}} = \tan\frac{\pi}{10} \tag{5.1}$$

and In Figure 4, we have

Proposition 5.3.

$$\angle DGO + \angle BGP = \frac{\pi}{2} \tag{5.2}$$

Proof. Apply the law of cosines to $\triangle BGP$, we get

$$\cos\angle BGP = \frac{BG^2 + GP^2 - BP^2}{2BG \cdot GP} = \frac{1}{\phi\sqrt{1+\phi^2}} = \sin\angle DGO$$

hence (5.2) proved, and by (5.2), we notice that $\angle BGP = \angle DOG$, hence we get a corollary that $\triangle BGP$ is similar to $\triangle BOG$. □

In fact, by observing the above construction 5.1, we can obtain a simpler construction 5.4 of T_2 by first creating a golden rectangle, see Figure 5.

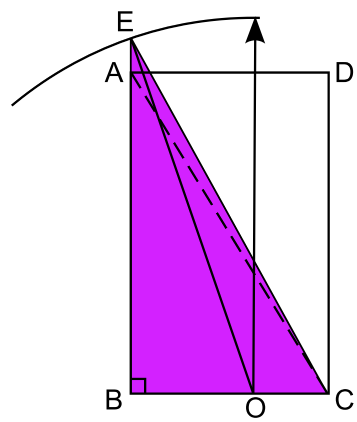

FIGURE 5. A simple construction of T_2

Construction 5.4. *A simple 3-step construction of T_2:*

(1) construct a golden rectangle $ABCD$ with $AB = \phi$, $BC = 1$ (see, e.g., [7, p. 118]*)*
(2) construct O dividing BC into the golden ratio and $\frac{BO}{OC} = \phi$
(3) draw an arc with the center at O and the radius of length AC, cutting the extension of BA at E, and join E to C

Then $\triangle EBC$ is just T_2 having sides $(1, \sqrt{2\phi}, \phi\sqrt{\phi})$.

Proof. $BO = \frac{BC}{\phi} = \frac{1}{\phi}$, $EO = AC = \sqrt{1+\phi^2}$, thus, $BE = \sqrt{EO^2 - BO^2} = \sqrt{2\phi}$. □

We also have another simple construction 5.5 of T_2 by first creating a Kepler triangle, see Figure 6.

Construction 5.5. *Another 3-step construction of T_2:*

(1) construct a Kepler triangle $\triangle ABC$ with $AB = \sqrt{\phi}$, $BC = 1$ (see, e.g., [4]*)*
(2) construct a square $ABDE$ externally on the side AB
(3) draw an arc with the center at B and the radius BE, cutting the extension of BA at F, and join F to C

Then $\triangle FBC$ is just T_2.

Proof. $BF = BE = \sqrt{2}AB = \sqrt{2\phi}$. □

The last construction 5.6 of T_2 we are showing here in Figure 7 is by using Thales' theorem of a circle.

Construction 5.6. *A 3-step construction of T_2:*

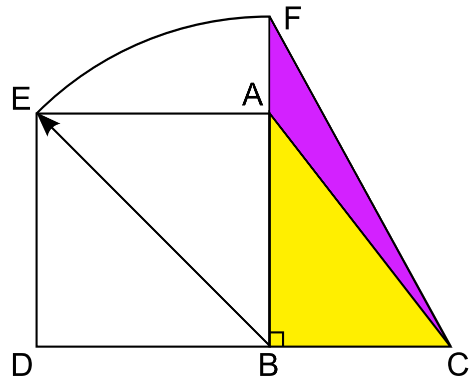

FIGURE 6. Another simple construction of T_2

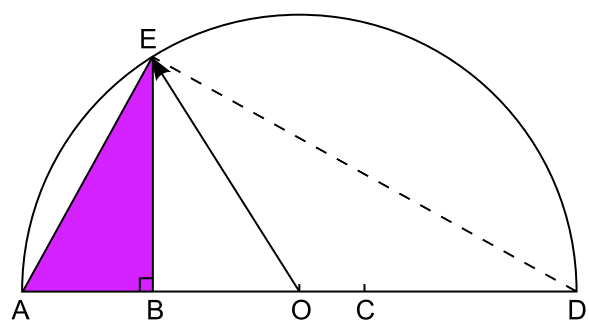

FIGURE 7. The last construction of T_2

(1) construct a segment $AB = 1$, construct point C on the extension of AB that $\frac{BC}{BA} = \phi$, and extend BC to D that $CD = BC$
(2) draw a semicircle with its center at the midpoint O of AD, and the radius OA
(3) through B, construct a perpendicular segment BE to AB, and intersecting the semicircle at point E, and join E to A

Then $\triangle ABE$ is just T_2.

Proof. According to Thales' theorem, $\triangle AED$ is just a right triangle, then we get $BE = \sqrt{AB \cdot BD} = \sqrt{2\phi}$. □

Last, back to Figure 1 again, we show a property in the golden ellipse of eccentricity $\frac{1}{\phi\sqrt{\phi}}$, let AC and BD denote the latus rectum of the ellipse, then we have

Proposition 5.7. *The rectangle $ACDB$ is made up of 4 congruent right triangles similar to the Kepler triangle and $\frac{F_1F_2}{AF_1} = \sqrt{\phi}$. Also, it is shown in [8] that, $\triangle AF_2C$ is just the kind of isosceles triangle of smallest perimeter which circumscribes a semicircle.*

Proof. $F_1F_2 = 2$, $AF_1 = \frac{2}{\sqrt{\phi}}$, then $\frac{F_1F_2}{AF_1} = \sqrt{\phi}$, hence proved. □

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