My maths

Preamble

We define the following:

Variable	Definition	Description
W	$= \{1, \ldots, n\}$	set of voters
A	$\subseteq W$	a coalition, $\varnothing$ is an empty coalition, N is a grand coalition
$\#A$	$\in \mathbb{N}$	cardinality of a set
$\mathcal{P}(W)$	$= \{A \subseteq W\}$	power set, set of all coalitions, $\#\mathcal{P}(W) = 2^n$
$\mathfrak{G}$	$\subseteq \mathcal{P}(W)$	set of winning coalitions, $A \notin \mathfrak{G}$ is a losing coalition
$\mathfrak{M}(\mathfrak{G})$	$:= \{A \in \mathfrak{G} : B \notin \mathfrak{G}, \forall B \subset A\}$	minimal winning coalitions
$\mathfrak{W}$	$:= (W, \mathfrak{G})$	a simple game where $W \in \mathfrak{G}$ and $\varnothing \notin \mathfrak{G}$. If $A \in \mathfrak{G}$, then $B \in \mathfrak{G}, \forall B \supset A$ (monotonicity)
g	$: W \to [0, \infty)$	function returning the voting weight of a coalition
q	$\in [0, \infty)$	quota that has to be reached
g and q exists s.t.	$\sum_{w \in A} g(w) \geq q, \forall A \in \mathfrak{G}$, $\sum_{w \in B} g(w) < q, \forall B \in (\mathcal{P}(W) \setminus \mathfrak{G})$	a weighted simple game

Minimal Winning Coalitions

$\mathfrak{G}$ can also be defined in correlation to its minimal winning coalitions,

$$\mathfrak{G} = \{A \in \mathcal{P}(W) : \exists B \in \mathfrak{M}(\mathfrak{G}) : B \subseteq A\}.$$

This is because minimal winning coalitions are just the minimal elements in $\mathfrak{G}$ with respect to the partial order $\subseteq$. $\mathfrak{G}$ always meets the *upset* or *filter* condition (Anderson 1987; Engel 1997). This MWC-set is called a *basis* as well.

MWCs are *antichains* in $\mathcal{P}(W)$, also known as *Sperner families*. An antichain $\tilde{\mathfrak{M}}$ is a non-empty set of subsets of W such that $A \nsubseteq B$ and $B \nsubseteq A$ for all $A, B \in \tilde{\mathfrak{M}}$.

The cardinality of $\mathfrak{M}(\mathfrak{G})$ is known to satisfy $1 \leq \#\mathfrak{M}(\mathfrak{G}) \leq \binom{n}{\lfloor n/2 \rfloor}$. The number of voting systems based on the number of voters is equal to the Dedekind number - 2.

Power Indices

Penrose-Banzhaf Index

$$\text{Banzhaf score} \qquad \mathrm{BS}_w := \#\{A \in \mathfrak{G} : w \in A, (A \setminus \{w\}) \notin \mathfrak{G}\}$$

$$\text{Penrose-Banzhaf power} \qquad \mathrm{PBP}_w := \frac{\mathrm{BS}_w}{2^{n-1}}$$

$$\text{Penrose-Banzhaf index} \qquad \mathrm{PBI}_w := \frac{\mathrm{BS}_w}{\sum_{i=1}^{n} \mathrm{BS}_i}$$

Note how $0 \leq \mathrm{PBI}_w \leq 1$ and $\sum_{i=1}^{n} \mathrm{PBI}_i = 1$.

As an example, take $W = \{1, 2, 3, 4\}$ and $\mathfrak{M}(\mathfrak{G}) = \{\{1, 2\}, \{1, 3\}\}$. We have the following:

$$\mathfrak{G} = \{\{1, 2, 3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2\}, \{1, 3\}\}$$

$$\begin{array}{cccc} \mathrm{BS}_1 = 6 & \mathrm{BS}_2 = 2 & \mathrm{BS}_3 = 2 & \mathrm{BS}_4 = 0 \\ \mathrm{PBI}_1 = 0.6 & \mathrm{PBI}_2 = 0.2 & \mathrm{PBI}_3 = 0.2 & \mathrm{PBI}_4 = 0 \end{array}$$

Shapley-Shubik Index

$$\mathrm{SSI}_w := \sum_{\substack{A \in \mathfrak{G} \\ \text{where } w \text{ is decisive for } A}} \frac{(n - \#A)!(\#A - 1)!}{n!}$$

From the example before we can determine that

$$\mathrm{SSI}_1 = 16/24 \quad \mathrm{SSI}_2 = 4/24 \quad \mathrm{SSI}_3 = 4/24$$

Note again that $0 \leq \mathrm{SSI}_w \leq 1$ and $\sum_{i=1}^{n} \mathrm{SSI}_i = 1$. Both Banzhaf and Shapley measure the influence of voters in different ways. The choice of the index depends on the behavior of the voters.

- Voters are completely independent from each other: Penrose-Banzhaf Index
- A common belief has influence on the choice of all voters: Shapley-Shubik Index

Note that the solutions are dependent on the *decisiveness* of voters, not necessarily the MWCs alone. For those, the Deegan-Packel index and the Holler-Packel index can be taken a look at.

We are also interested in *power profiles* concerning the power index under consideration. What are the sets and values of of potential power profiles of voters?

Not every constellation of voting power is possible. For two voters, two power distributions are possible: either one has all and the other none of the power, or both have half of the power.

How do we calculate the PBI_w and SSI_w only using the MWC-set?

Calculations

New Banzhaf score formula

Given a voting system $\mathfrak{W}$ with $\mathfrak{M}(\mathfrak{G}) = \{V_1, \ldots, V_m\}$ minimal winning coalitions and $\#\mathfrak{M}(\mathfrak{G}) = m$ we define

$$\mathrm{BS}_w = \sum_{r=1}^{m} (-1)^{r-1} \sum_{1 \leq i_1 < \cdots < i_r \leq m} t_{i_1, \ldots, i_r}(w)$$

$$\text{with } t_{i_1,\dots,i_r}(w) := \begin{cases} 2^{n-\#\bigcup_{j=1}^{r} V_{i_j}}, & \text{for } w \in \bigcup_{j=1}^{r} V_{i_j}, \\ 0, & \text{for } w \notin \bigcup_{j=1}^{r} V_{i_j}. \end{cases}$$

Recall the example with $W = \{1, 2, 3, 4\}$ and $\mathfrak{M}(\mathfrak{G}) = \{\{1, 2\}, \{1, 3\}\}$. So, $m = 2$. This gives us:

$$\begin{aligned}
\mathrm{BS}_w &= \sum_{r=1}^{2} (-1)^{r-1} \sum_{1 \le i_1 < \cdots < i_r \le 2} t_{i_1,\dots,i_r}(w) \\
&= \sum_{1 \le i_1 \le 2} t_{i_1}(w) - \sum_{1 \le i_1 < i_2 \le 2} t_{i_1,i_2}(w) \\
&= (t_1(w) + t_2(w)) - (t_{1,2}(w))
\end{aligned}$$

The V_is are defined as follows:

$$\begin{aligned}
V_1 &= \{1, 2\} \\
V_2 &= \{1, 3\}
\end{aligned}$$

Considering each player, we get the following.

Player 1

$$\begin{aligned}
\mathrm{BS}_1 &= (2^{4-\#\{1,2\}} + 2^{4-\#\{1,3\}}) - (2^{4-\#\{1,2,3\}}) \\
&= (4 + 4) - (2) = 6
\end{aligned}$$

Player 2

$$\begin{aligned}
\mathrm{BS}_2 &= (2^{4-\#\{1,2\}} + 0) - (2^{4-\#\{1,2,3\}}) \\
&= (4 + 0) - (2) = 2
\end{aligned}$$

Player 3

$$\begin{aligned}
\mathrm{BS}_3 &= (0 + 2^{4-\#\{1,3\}}) - (2^{4-\#\{1,2,3\}}) \\
&= (0 + 4) - (2) = 2
\end{aligned}$$

Player 4

$$\begin{aligned}
\mathrm{BS}_4 &= (0 + 0) - (0) \\
&= (0 + 0) - (0) = 0
\end{aligned}$$

Anderson, Ian. 1987. *Combinatorics of Finite Sets*. Clarendon Press/Oxford Science Publications, Oxford.
Engel, Konrad. 1997. *Sperner Theory*. 65. Cambridge University Press.