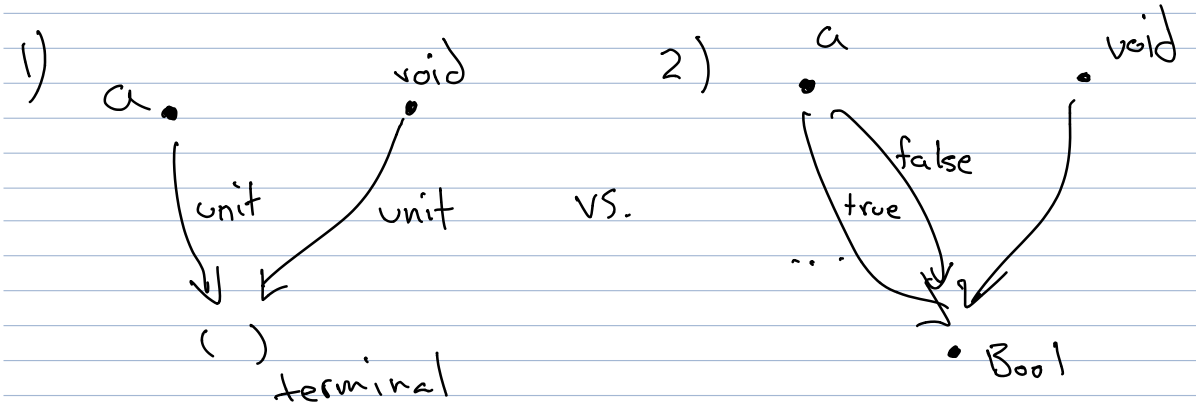
A **terminal object** is an object in a category in which there is exactly one arrow from all other objects in the category to it.



This example is of type sets.
In (1), the only arrow that maps to the singleton set (empty tuple) is the function that ignores its argument and returns an empty tuple.

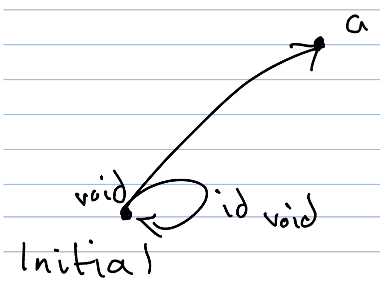
Note this is how the singleton set can be defined in category theory.

More rigorously;

$\forall a \quad \exists f :: a \to ()$

$\forall f :: a \to (),\ g :: a \to () \implies f = g$ (uniqueness)

An **Initial Object** is an object in a category that has a unique arrow going to every other object in the category.



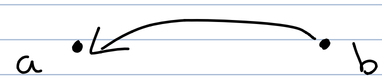
Essentially the inverse of the terminal object.
It is equated to the empty set in set theory.

Is there more than singleton set?

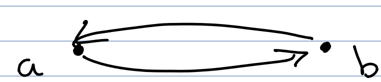
Well there can be multiple terminal objects. There is always a single isomorphism between them, The proof is simple:

For terminal objects a + b

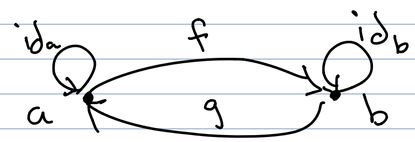
There is a <u>unique</u> arrow from each other object to 'a'



Same for 'b'



Since there is a unique arrow from each object, there must be 1 arrow from 'a' to 'a' and 'b' to 'b', which must be the identity



Therefore $g \circ f = id_a$

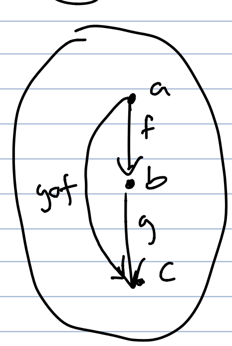
$f \circ g = id_b$

Theorem

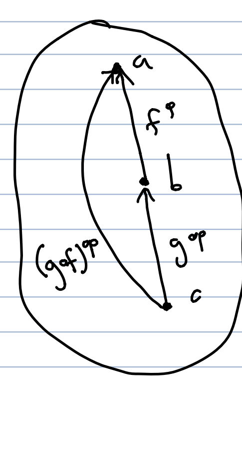
Given any category C, a category like C can be constructed where all arrows are reversed, called C^{op}

Where composition is preserved

C



C^{op}



A cartesian product $A \times B$ can be viewed generically in terms of arrows as

$$
\begin{tikzcd}
 & A \times B \arrow[dl, "first"'] \arrow[dr, "second"] & \\
a & & b
\end{tikzcd}
$$

where 'a' is A and 'b' is B

But if there are multiple objects that match this pattern, the "best" object to be viewed as a cartesian product can be distinguished as the object with the most <u>fundamental</u> morphisms to 'a' & 'b'

$$
\begin{tikzcd}
 & c' \arrow[d, dashed, "m"] \arrow[ddl, bend right, "p'"'] \arrow[ddr, bend left, "q'"] & \\
 & c \arrow[dl, "p"] \arrow[dr, "q"'] & \\
a & & b
\end{tikzcd}
$$

Since the morphism 'm' allows p' & q' to be factorized in terms of p & q

$$p \circ m = p'$$
$$q \circ m = q'$$

Morphisms from 'c' are more fundamental, making 'c' the best choice.

More specifically, a categorical **product** is an object with two morphisms

c

$p : c \to a$
$q : c \to b$

With the universal property that for any

c'
$p' : c' \to a$
$q' : c' \to b$

There is a unique morphism

$m : c' \to c$

so that

$p' = p \circ m$
$q' = q \circ m$