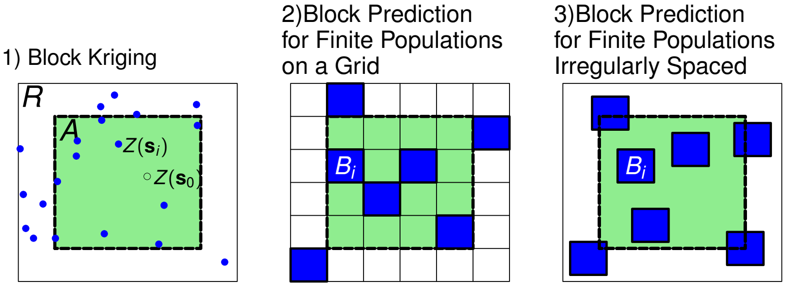
Estimating Abundance from Counts in Large Data Sets of Irregularly-Spaced Plots using Spatial Basis Functions

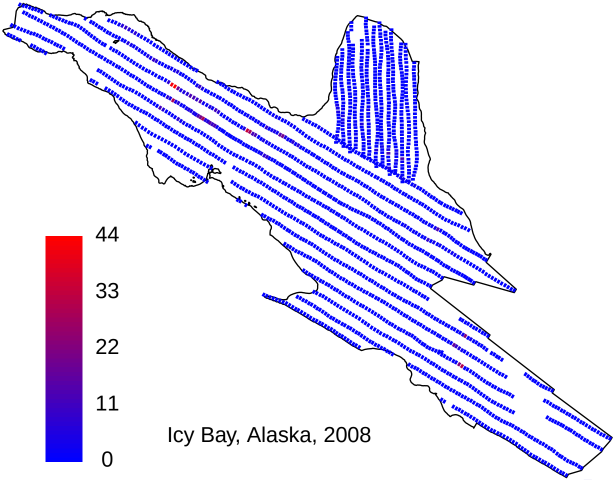
Jay Ver Hoef

NOAA National Marine Mammal Lab
NOAA Fisheries
International Arctic Research Center
Fairbanks, Alaska, USA

Introduction



Motivating Example



Goals

An estimator that is:

- fast to compute, robust, and requires few modeling decisions, similar to classical survey methods,
- based only on counts within plots; actual spatial locations of animals are unknown,
- for the actual number of seals, not the mean of some assumed process that generated the data,
- have a variance estimator with a population correction factor that shrinks to zero as the proportion of the study area that gets sampled goes to one,
- unbiased with valid confidence intervals,
- able to accommodate nonstationary variance throughout the area

Inhomogeneous Spatial Point Processes

$T(V)$ is the total number of points in planar region V

$$\lambda(\mathbf{s}) = \lim_{|dx| \to 0} \frac{E(T(dx))}{|dx|}$$

Expected abundance in $A \subseteq R$:

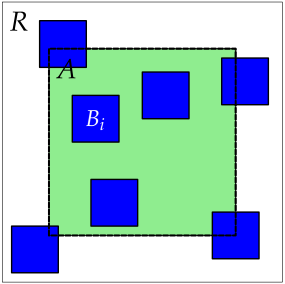
$$\mu(A) = \int_A \lambda(\mathbf{u}|\boldsymbol{\theta}) d\mathbf{u}$$

Abundance is assumed random

$$T(A) \sim \text{Poi}(\mu(A))$$

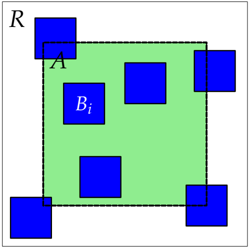
Resulting in an observed pattern $\mathcal{S}^+ = (\mathbf{s}_1, \ldots, \mathbf{s}_N)$

Outline of an Estimator



- $\mathcal{B} = \cup_{i=1}^{n}(B_i \cap A)$
- $\mathcal{U} \equiv \overline{\mathcal{B}} \cap A$
- $T(A) = T(\mathcal{B}) + T(\mathcal{U})$
- $T(\mathcal{U}) \sim \mathrm{Poi}(\mu(\mathcal{U}))$
- $\mu(\mathcal{U}) = \int_{\mathcal{U}} \lambda(\mathbf{u}|\boldsymbol{\theta})d\mathbf{u}$
- $\widehat{T}(A) = T(\mathcal{B}) + \widehat{T}(\mathcal{U})$
- $T(\mathcal{B}) \to T(A) \Rightarrow \widehat{T}(A) \to T(A)$

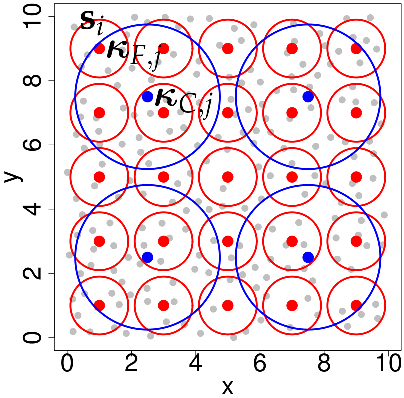
From IPP to Poisson Regression



- $Y(B_i) \sim \text{Poi}(\mu(B_i))$
- $\mu(B_i) = \int_{B_i} \lambda(\mathbf{u}|\boldsymbol{\theta}) d\mathbf{u}$
- Let $\mathbf{s}_i$ be centroid of B_i
- $\mu(B_i) \approx |B_i| \lambda(\mathbf{s}_i|\boldsymbol{\theta})$
- $\log(\mu(B_i)) = \log(|B_i|) + \log(\lambda(\mathbf{s}_i|\boldsymbol{\theta}))$
- $\log(\lambda(\mathbf{s}_i|\boldsymbol{\theta})) = \mathbf{x}(\mathbf{s}_i)'\boldsymbol{\beta}$

Now us spatial basis functions to generate $\mathbf{x}(\mathbf{s}_i)$

Spatial Basis Functions



- $C(h; \rho) = \exp(-h^2/\rho)$
- $\mathbf{X}_{i,j} = C(\|\mathbf{s}_i - \boldsymbol{\kappa}_{F,j})\|; \rho_F)$; $j = 2, \ldots, K_F + 1$
- $\mathbf{X}_{i,j} = C(\|\mathbf{s}_i - \boldsymbol{\kappa}_{C,j})\|; \rho_C)$; $j = K_F + 2, \ldots, K_F + K_C + 1$

knot location: k-means clustering of dense grid of spatial coordinates

Fitting the Model

minimize minus the log-likelihood:

$$-\ell(\boldsymbol{\rho}, \boldsymbol{\beta}; \mathbf{y}) \propto \sum_{i=1}^{n} |B_i| \exp(\mathbf{x}_{\boldsymbol{\rho}}(\mathbf{s}_i)'\boldsymbol{\beta}) - y_i \log|B_i| - y_i \mathbf{x}_{\boldsymbol{\rho}}(\mathbf{s}_i)'\boldsymbol{\beta}$$

Two-part algorithm:

- Condition on ρ and use IWLS to estimate β (with offset for $|B_i|$, ala GLMs)
- optimize for ρ numerically

Back to the Estimator

- $\widehat{T}(A) = T(\mathcal{B}) + \widehat{T}(\mathcal{U})$
- $\widehat{T}(\mathcal{U}) = \mu(\mathcal{U}) = \int_{\mathcal{U}} \lambda(\mathbf{u}|\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\beta}}) d\mathbf{u}$
- $\lambda(\mathbf{u}|\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\beta}}) = \exp(\mathbf{x}_{\hat{\boldsymbol{\rho}}}(\mathbf{u})'\hat{\boldsymbol{\beta}})$

Approximate integral with dense grid of n_p points within $\mathbf{u}_j \in \mathcal{U}$.

$$\widehat{T}(A) = T(\mathcal{B}) + \sum_{j=1}^{n_p} |U_i| \exp(\mathbf{x}_{\hat{\boldsymbol{\rho}}}(\mathbf{u}_j)'\hat{\boldsymbol{\beta}})$$

where $|U_i|$ is a small area around each $\mathbf{u}_j$

Variance

$\text{MSPE}(\hat{T}(A)) = E[(\hat{T}(A) - T(A))^2; \boldsymbol{\beta}] = E[(\hat{T}(\mathcal{U}) - T(\mathcal{U}))^2; \boldsymbol{\beta}]$
Note: as $\mathcal{U} \cap A \to \varnothing \Rightarrow \text{MSPE}(\widehat{T}(A)) \to 0$
From IPP assumption: $\hat{T}(\mathcal{U})$ independent from $T(\mathcal{U})$.
Assuming unbiasedness, $E[(\hat{T}(\mathcal{U})] = E[T(\mathcal{U})]$,

$$\begin{aligned}\text{MSPE} &= \text{var}[T(\mathcal{U}); \boldsymbol{\beta}] + \text{var}[\hat{T}(\mathcal{U}); \boldsymbol{\beta}] \\ &= \mu(\mathcal{U}; \boldsymbol{\beta}) + \text{var}[\hat{T}(\mathcal{U}); \boldsymbol{\beta}]\end{aligned}$$

Now, what about $\text{var}[\hat{T}(\mathcal{U}); \boldsymbol{\beta}]$?

Variance

Recall delta method result: $\mathrm{var}(f(\mathbf{y})) \approx \mathbf{d}'\boldsymbol{\Sigma}\mathbf{d}$
Jay M. Ver Hoef (2012) Who Invented the Delta Method? The American Statistician, 66:2, 124-127

where $\mathrm{var}(\mathbf{y}) = \boldsymbol{\Sigma}$ and $d_i = \partial f(\mathbf{y})/\partial y_i$

$$d_i = \frac{\partial \hat{T}(\mathcal{U})}{\partial \beta_i} = \int_{\mathcal{U}} x_i(\mathbf{u}) \exp(\mathbf{x}(\mathbf{u})'\hat{\boldsymbol{\beta}}) d\mathbf{u} \approx \frac{|\mathcal{U}|}{n_p} \sum_{i=1}^{n_p} x_i(\mathbf{s}_i) \exp(\mathbf{x}(\mathbf{s}_i)'\hat{\boldsymbol{\beta}})$$

From Rathbun and Cressie, (1994), if $\hat{\boldsymbol{\beta}}$ is MLE,

$$\hat{\boldsymbol{\Sigma}} = \left[\sum_{i=1}^{n} \int_{B_i} \mathbf{x}(\mathbf{s})\mathbf{x}(\mathbf{s})' \exp(\mathbf{x}(\mathbf{s})'\hat{\boldsymbol{\beta}}) d\mathbf{s}\right]^{-1} \approx \left[|B| \sum_{i=1}^{n} \mathbf{x}(\mathbf{s}_i)\mathbf{x}(\mathbf{s}_i)' \exp(\mathbf{x}(\mathbf{s}_i)'\hat{\boldsymbol{\beta}})\right]^{-1}$$

if $|B_i| = |B| \; \forall \; i$.

Rathbun, S. L. and Cressie, N. (1994), "Asymptotic Properties of Estimators for the Parameters of Spatial Inhomogeneous Poisson Point Processes," Advances in Applied Probability, 26, 122-154.

Summary

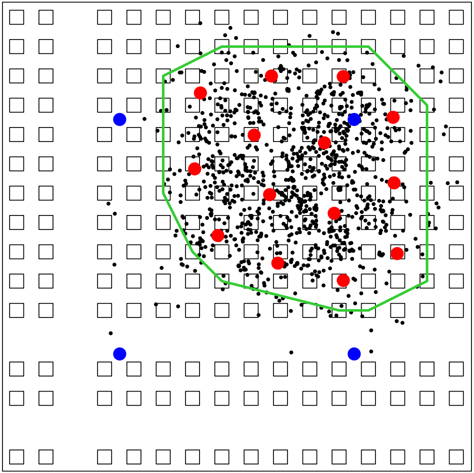
$$\widehat{T}(A) = T(\mathcal{B}) + \frac{|\mathcal{U}|}{n_p} \sum_{j=1}^{n_p} \exp(\mathbf{x}_{\hat{\boldsymbol{\rho}}}(\mathbf{u}_j)'\hat{\boldsymbol{\beta}})$$

$$\begin{aligned}\widetilde{\mathrm{var}}(\widehat{T}(A)) = \quad & \frac{|\mathcal{U}|}{n_p} \sum_{j=1}^{n_p} \exp(\mathbf{x}_{\hat{\boldsymbol{\rho}}}(\mathbf{u}_j)'\hat{\boldsymbol{\beta}}) + \\ & \mathbf{d}' \left[|B| \sum_{i=1}^{n} \mathbf{x}(\mathbf{s}_i)\mathbf{x}(\mathbf{s}_i)' \exp(\mathbf{x}(\mathbf{s}_i)'\hat{\boldsymbol{\beta}}) \right]^{-1} \mathbf{d}\end{aligned}$$

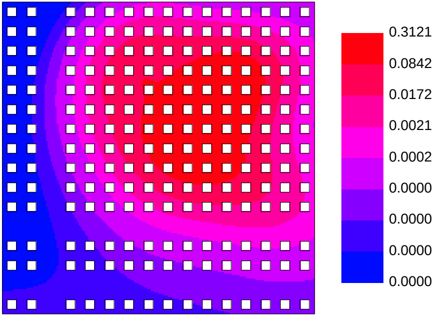
where

$$d_i = \frac{|\mathcal{U}|}{n_p} \sum_{i=1}^{n_p} x_i(\mathbf{s}_i) \exp(\mathbf{x}(\mathbf{s}_i)'\hat{\boldsymbol{\beta}})$$

Simulated Example

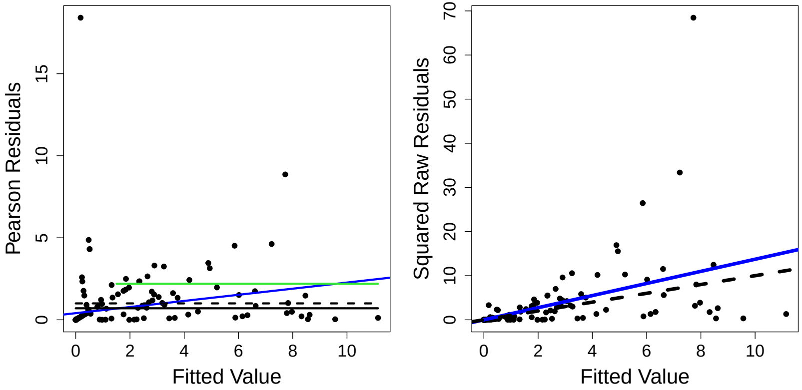


Simulated Example



True abundance was 1079
Estimated abundance was 1143 with standard error 62

Residuals Plots



Overdispersion Estimators

- The traditional estimator:

$$\omega_{OD} = \max\left(1, \frac{1}{n-r}\sum_{i=1}^{n}\frac{(y_i-\phi_i)^2}{\phi_i}\right)$$

where r is the rank of $\mathbf{X}$.

- Weighted regression estimator:

$$\omega_{WR} = \max\left(1, \arg\min_{\omega}\sum_{i=1}^{n}\sqrt{\phi_i}[(y_i-\phi_i)^2-\omega\phi_i]^2\right),$$

where $\sqrt{\phi_i}$ were the weights

Overdispersion Estimators

- Trimmed Mean:

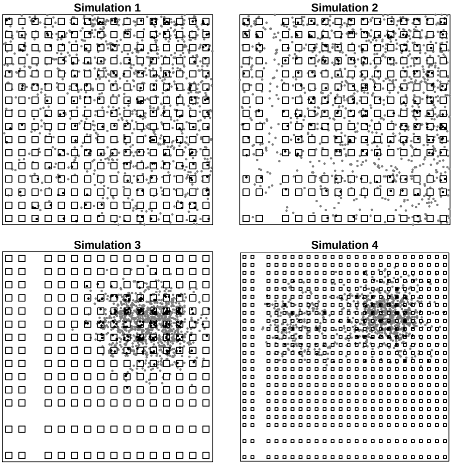
$$\omega_{TG}(p) = \max\left(1, \frac{1}{n - \lfloor np \rfloor - r} \sum_{i=\lfloor np \rfloor + 1}^{n} \frac{(y_{(i)} - \phi_{(i)})^2}{\phi_{(i)}}\right)$$

where $0 \leq p \leq 1$, $y_{(i)}$ and $\phi_{(i)}$ are ordered values, and $\lfloor x \rfloor$ rounds x down to the nearest integer.

Adjusted Variance Estimators

- $\widehat{\text{var}}_{OD}(\widehat{T}(A)) = \omega_{OD}\widetilde{\text{var}}(\widehat{T}(A))$
- $\widehat{\text{var}}_{WR}(\widehat{T}(A)) = \omega_{WR}\widetilde{\text{var}}(\widehat{T}(A))$
- $\widehat{\text{var}}_{TG}(\widehat{T}(A); p) = \omega_{TG}p\widetilde{\text{var}}(\widehat{T}(A))$
- $\widehat{\text{var}}_{TL}(\widehat{T}(A); p) = \frac{|\mathcal{U}|}{n_p}\sum_{j=1}^{n_p}\exp(\mathbf{x}_{\hat{\boldsymbol{\rho}}}(\mathbf{u}_j)'\hat{\boldsymbol{\beta}})\times$
  $\max(1, \omega_{TG}(p)I(\exp(\mathbf{x}(\mathbf{s}_j)'\hat{\boldsymbol{\beta}}) \geq \phi_{(\lfloor np \rfloor)})+$
  $\mathbf{d}'\left[|B|\sum_{i=1}^{n}\mathbf{x}(\mathbf{s}_i)\mathbf{x}(\mathbf{s}_i)'\exp(\mathbf{x}(\mathbf{s}_i)'\hat{\boldsymbol{\beta}})\right]^{-1}\mathbf{d}$
  where $I(\cdot)$ is the indicator function

Simulations



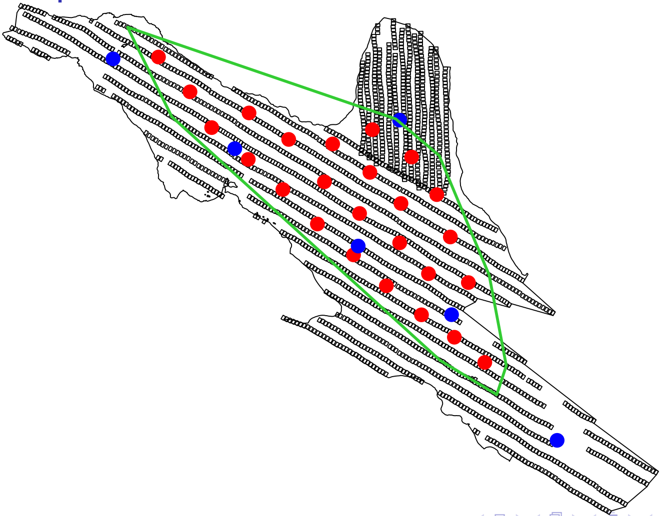
Simulation Experiment 3

	knots1	knots2	knots3	knots4
bias	-2.389	-4.365	-2.919	-1.637
RMSPE	79.207	79.250	79.285	80.175
CI90	0.775	0.772	0.781	0.777
CI90OD	0.801	0.783	0.789	0.782
CI90WR	0.918	0.906	0.865	0.837
CI90TG	0.923	0.930	0.929	0.946
CI90TL	0.871	0.883	0.878	0.903
fail	0.000	0.000	0.000	0.018

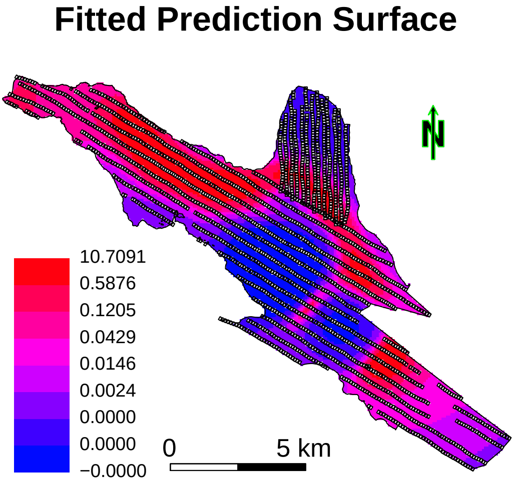
Simulation Experiment 4

	knots1	knots2	knots3	knots4
bias	5.179	3.440	7.139	14.536
RMSPE	60.403	61.021	61.808	67.601
CI90	0.831	0.825	0.826	0.836
CI90OD	0.844	0.831		
CI90WR	0.939	0.928	0.924	0.917
CI90TG	0.929	0.919	0.909	0.918
CI90TL	0.893	0.892	0.885	0.888
fail	0.000	0.000	0.000	0.013

Real Example



Real Example



Real Example

```
sealDensity <- sum(plots@data[, "counts"])/(sCSout$propSurveyed * totalArea)
sealDensity * totalArea


## [1] 3960


summary(sCSout)


##
## Estimates:
##
## Total:
## [1] 4017
## Standard Errors:
##      SE SE.ODTrad SE.ODTrimGlobal SE.ODTrimLocal SE.ODRegr
## 1 112.2      9721           326.2            230     392.4
##
##
## Range Parameters:
##   coarseScale fineScale
## 1        8567      1055
##
##
## Proportion Surveyed:
## [1] 0.253
```

Goals

An estimator that is:

- fast to compute, robust, and requires few modeling decisions, similar to classical survey methods,
- based only on counts within plots; actual spatial locations of animals are unknown,
- for the actual number of seals, not the mean of some assumed process that generated the data,
- have a variance estimator with a population correction factor that shrinks to zero as the proportion of the study area that gets sampled goes to one,
- unbiased with valid confidence intervals,
- able to accommodate nonstationary variance throughout the area