Estimating Abundance from Counts in Large Data Sets of Irregularly-Spaced Plots using Spatial Basis Functions

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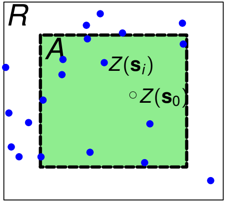
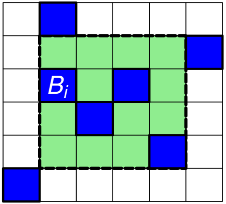
Introduction

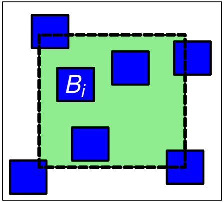
1) Block Kriging



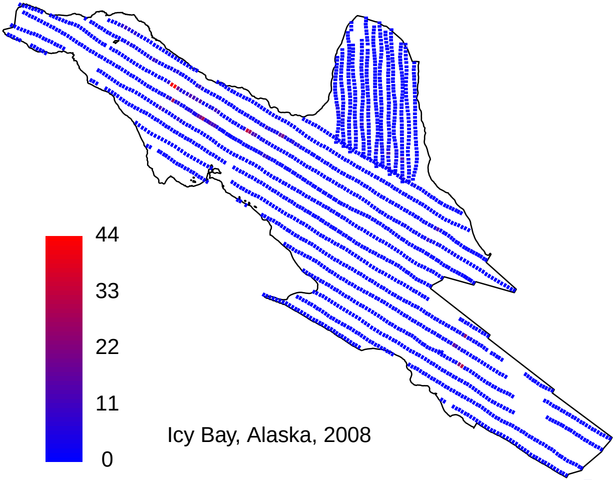
2)Block Prediction for Finite Populations on a Grid



3)Block Prediction for Finite Populations Irregularly Spaced



Motivating Example



Goals

An estimator that is:

- fast to compute, robust, and requires few modeling decisions, similar to classical survey methods,
- based only on counts within plots; actual spatial locations of animals are unknown,
- for the actual number of seals, not the mean of some assumed process that generated the data,
- have a variance estimator with a population correction factor that shrinks to zero as the proportion of the study area that gets sampled goes to one,
- unbiased with valid confidence intervals,
- able to accommodate nonstationary variance throughout the area

Inhomogeneous Spatial Point Processes

$T(V)$ is the total number of points in planar region V

$$\lambda(\mathbf{s}) = \lim_{|dx| \to 0} \frac{E(T(dx))}{|dx|}$$

Expected abundance in $A \subseteq R$:

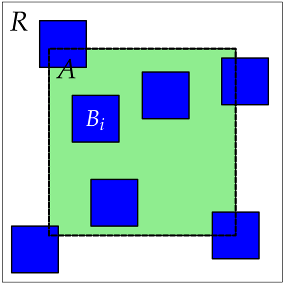
$$\mu(A) = \int_A \lambda(\mathbf{u}|\boldsymbol{\theta}) d\mathbf{u}$$

Abundance is assumed random

$$T(A) \sim \mathrm{Poi}(\mu(A))$$

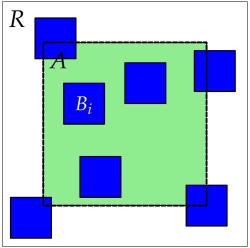
Resulting in an observed pattern $\mathcal{S}^+ = (\mathbf{s}_1, \ldots, \mathbf{s}_N)$

Outline of an Estimator



- $\mathcal{B} = \cup_{i=1}^{n}(B_i \cap A)$
- $\mathcal{U} \equiv \overline{\mathcal{B}} \cap A$
- $T(A) = T(\mathcal{B}) + T(\mathcal{U})$
- $T(\mathcal{U}) \sim \text{Poi}(\mu(\mathcal{U}))$
- $\mu(\mathcal{U}) = \int_{\mathcal{U}} \lambda(\mathbf{u}|\boldsymbol{\theta})d\mathbf{u}$
- $\widehat{T}(A) = T(\mathcal{B}) + \widehat{T}(\mathcal{U})$
- $T(\mathcal{B}) \rightarrow T(A) \Rightarrow \widehat{T}(A) \rightarrow T(A)$

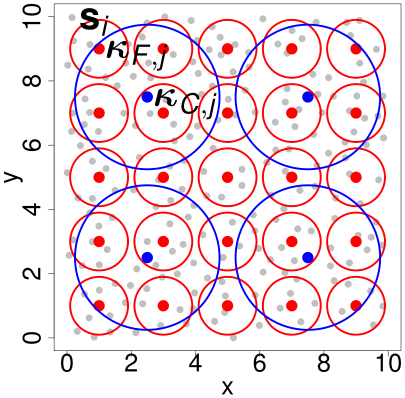
From IPP to Poisson Regression



- $Y(B_i) \sim \text{Poi}(\mu(B_i))$
- $\mu(B_i) = \int_{B_i} \lambda(\mathbf{u}|\boldsymbol{\theta})d\mathbf{u}$
- Let $\mathbf{s}_i$ be centroid of B_i
- $\mu(B_i) \approx |B_i|\lambda(\mathbf{s}_i|\boldsymbol{\theta})$
- $\log(\mu(B_i)) = \log(|B_i|) + \log(\lambda(\mathbf{s}_i|\boldsymbol{\theta}))$
- $\log(\lambda(\mathbf{s}_i|\boldsymbol{\theta})) = \mathbf{x}(\mathbf{s}_i)'\boldsymbol{\beta}$

Now us spatial basis functions to generate $\mathbf{x}(\mathbf{s}_i)$

Spatial Basis Functions



- $C(h;\rho) = \exp(-h^2/\rho)$
- $\mathbf{X}_{i,j} = C(\|\mathbf{s}_i - \boldsymbol{\kappa}_{F,j})\|;\rho_F)$; $j = 2,\ldots,K_F+1$
- $\mathbf{X}_{i,j} = C(\|\mathbf{s}_i - \boldsymbol{\kappa}_{C,j})\|;\rho_C)$; $j = K_F+2,\ldots,K_F+K_C+1$

knot location: k-means clustering of dense grid of spatial coordinates

Fitting the Model

minimize minus the log-likelihood:

$$-\ell(\boldsymbol{\rho}, \boldsymbol{\beta}; \mathbf{y}) \propto \sum_{i=1}^{n} |B_i| \exp(\mathbf{x}_{\boldsymbol{\rho}}(\mathbf{s}_i)'\boldsymbol{\beta}) - y_i \log|B_i| - y_i \mathbf{x}_{\boldsymbol{\rho}}(\mathbf{s}_i)'\boldsymbol{\beta}$$

Two-part algorithm:

- Condition on ρ and use IWLS to estimate β (with offset for $|B_i|$, ala GLMs)
- optimize for ρ numerically

Back to the Estimator

- $\widehat{T}(A) = T(\mathcal{B}) + \widehat{T}(\mathcal{U})$
- $\widehat{T}(\mathcal{U}) = \mu(\mathcal{U}) = \int_{\mathcal{U}} \lambda(\mathbf{u}|\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\beta}}) d\mathbf{u}$
- $\lambda(\mathbf{u}|\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\beta}}) = \exp(\mathbf{x}_{\hat{\boldsymbol{\rho}}}(\mathbf{u})'\hat{\boldsymbol{\beta}})$

Approximate integral with dense grid of n_p points within $\mathbf{u}_j \in \mathcal{U}$.

$$\widehat{T}(A) = T(\mathcal{B}) + \sum_{j=1}^{n_p} |U_i| \exp(\mathbf{x}_{\hat{\boldsymbol{\rho}}}(\mathbf{u}_j)'\hat{\boldsymbol{\beta}})$$

where $|U_i|$ is a small area around each $\mathbf{u}_j$

Variance

$$\mathrm{MSPE}(\hat{T}(A)) = E[(\hat{T}(A) - T(A))^2; \boldsymbol{\beta}] = E[(\hat{T}(\mathcal{U}) - T(\mathcal{U}))^2; \boldsymbol{\beta}]$$

Note: as $\mathcal{U} \cap A \to \varnothing \Rightarrow \mathrm{MSPE}(\widehat{T}(A)) \to 0$

From IPP assumption: $\hat{T}(\mathcal{U})$ independent from $T(\mathcal{U})$.

Assuming unbiasedness, $E[(\hat{T}(\mathcal{U})] = E[T(\mathcal{U})]$,

$$\begin{aligned}\mathrm{MSPE} &= \mathrm{var}[T(\mathcal{U}); \boldsymbol{\beta}] + \mathrm{var}[\hat{T}(\mathcal{U}); \boldsymbol{\beta}] \\ &= \mu(\mathcal{U}; \boldsymbol{\beta}) + \mathrm{var}[\hat{T}(\mathcal{U}); \boldsymbol{\beta}]\end{aligned}$$

Now, what about $\mathrm{var}[\hat{T}(\mathcal{U}); \boldsymbol{\beta}]$?

Variance

Recall delta method result: $\mathrm{var}(f(\mathbf{y})) \approx \mathbf{d}'\boldsymbol{\Sigma}\mathbf{d}$

Jay M. Ver Hoef (2012) Who Invented the Delta Method? The American Statistician, 66:2, 124-127

where $\mathrm{var}(\mathbf{y}) = \boldsymbol{\Sigma}$ and $d_i = \partial f(\mathbf{y})/\partial y_i$

$$d_i = \frac{\partial \hat{T}(\mathcal{U})}{\partial \beta_i} = \int_{\mathcal{U}} x_i(\mathbf{u}) \exp(\mathbf{x}(\mathbf{u})'\hat{\boldsymbol{\beta}})d\mathbf{u} \approx \frac{|\mathcal{U}|}{n_p} \sum_{i=1}^{n_p} x_i(\mathbf{s}_i) \exp(\mathbf{x}(\mathbf{s}_i)'\hat{\boldsymbol{\beta}})$$

From Rathbun and Cressie, (1994), if $\hat{\beta}$ is MLE,

$$\hat{\boldsymbol{\Sigma}} = \left[\sum_{i=1}^{n} \int_{B_i} \mathbf{x}(\mathbf{s})\mathbf{x}(\mathbf{s})' \exp(\mathbf{x}(\mathbf{s})'\hat{\boldsymbol{\beta}})d\mathbf{s}\right]^{-1} \approx \left[|B| \sum_{i=1}^{n} \mathbf{x}(\mathbf{s}_i)\mathbf{x}(\mathbf{s}_i)' \exp(\mathbf{x}(\mathbf{s}_i)'\hat{\boldsymbol{\beta}})\right]^{-1}$$

if $|B_i| = |B| \; \forall \; i$.

Rathbun, S. L. and Cressie, N. (1994), “Asymptotic Properties of Estimators for the Parameters of Spatial Inhomogeneous Poisson Point Processes,” Advances in Applied Probability, 26, 122-154.

Summary

$$\widehat{T}(A) = T(\mathcal{B}) + \frac{|\mathcal{U}|}{n_p} \sum_{j=1}^{n_p} \exp(\mathbf{x}_{\hat{\boldsymbol{\rho}}}(\mathbf{u}_j)'\hat{\boldsymbol{\beta}})$$

$$\begin{aligned}\widehat{\mathrm{var}}(\widehat{T}(A)) = \quad & \frac{|\mathcal{U}|}{n_p} \sum_{j=1}^{n_p} \exp(\mathbf{x}_{\hat{\boldsymbol{\rho}}}(\mathbf{u}_j)'\hat{\boldsymbol{\beta}}) + \\ & \mathbf{d}' \left[|B| \sum_{i=1}^{n} \mathbf{x}(\mathbf{s}_i)\mathbf{x}(\mathbf{s}_i)' \exp(\mathbf{x}(\mathbf{s}_i)'\hat{\boldsymbol{\beta}}) \right]^{-1} \mathbf{d}\end{aligned}$$

where

$$d_i = \frac{|\mathcal{U}|}{n_p} \sum_{i=1}^{n_p} x_i(\mathbf{s}_i) \exp(\mathbf{x}(\mathbf{s}_i)'\hat{\boldsymbol{\beta}})$$

Simulation