Block Prediction

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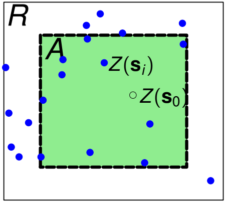
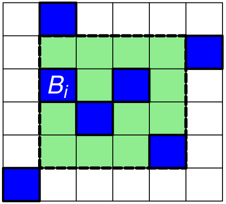
Introduction

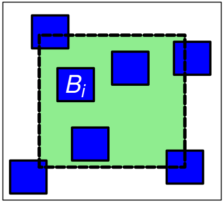
1) Block Kriging



2)Block Prediction for Finite Populations on a Grid

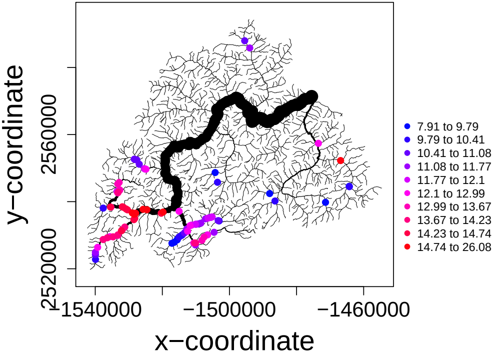


3)Block Prediction for Finite Populations Irregularly Spaced

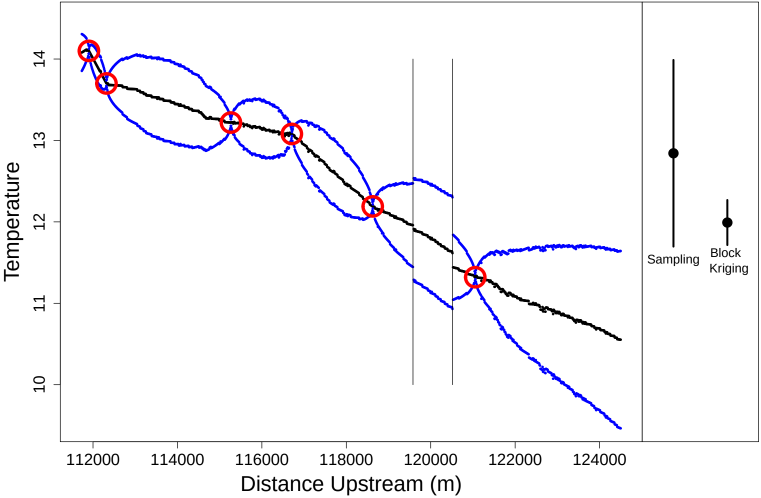


Block Prediction on a Stream Network

Peterson, E.E., Ver Hoef, J.M., Isaak, D.J., Falke, J.A., Fortin, M-J, Jordan, C., McNyset, K., Monestiez, P., Ruesch, A.S., Sengupta, A., Som, N., Steel, A., Theobald, D.M., Torgersen, C.E., Wenger, S.J. 2013. Stream networks in space: concepts, models, and synthesis. *Ecology Letters.* doi: 10.1111/ele.12084.



Block Prediction on a Stream Network



Review of BLUP and Point Prediction

$$\begin{pmatrix} \mathbf{z} \\ Z(\mathbf{s}_0) \end{pmatrix} = \begin{pmatrix} \mathbf{X} \\ \mathbf{x}(\mathbf{s}_0)' \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} \boldsymbol{\epsilon} \\ \epsilon(\mathbf{s}_0) \end{pmatrix}$$

$$\operatorname{cov}\begin{pmatrix} \boldsymbol{\epsilon} \\ \epsilon(\mathbf{s}_0) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{c} \\ \mathbf{c}' & \sigma_0^2 \end{pmatrix}$$

Best Linear Unbiased Prediction (BLUP) (or [Universal] Kriging)
minimize: $E(\boldsymbol{\lambda}'\mathbf{z} - Z(\mathbf{s}_0))^2$ subject to $E[\boldsymbol{\lambda}'\mathbf{z}] = E[Z(\mathbf{s}_0)] \; \forall \; \boldsymbol{\beta}$.
Unbiasedness $\Rightarrow \mathbf{X}'\boldsymbol{\lambda} = \mathbf{x}(\mathbf{s}_0)$

$$E[\boldsymbol{\lambda}'\boldsymbol{\epsilon}\boldsymbol{\epsilon}'\boldsymbol{\lambda} - 2\boldsymbol{\lambda}'\boldsymbol{\epsilon}\epsilon(\mathbf{s}_0) + \epsilon(\mathbf{s}_0)^2] = \boldsymbol{\lambda}'\boldsymbol{\Sigma}\boldsymbol{\lambda} - 2\boldsymbol{\lambda}'\mathbf{c} + \sigma_0^2$$

$$\frac{\partial}{\boldsymbol{\lambda}'}[\boldsymbol{\lambda}'\boldsymbol{\Sigma}\boldsymbol{\lambda} - 2\boldsymbol{\lambda}'\mathbf{c} + \sigma_0^2 + 2\mathbf{m}'(\mathbf{X}'\boldsymbol{\lambda} - \mathbf{x}(\mathbf{s}_0))] = 0$$

$$\frac{\partial}{\mathbf{m}'}[\boldsymbol{\lambda}'\boldsymbol{\Sigma}\boldsymbol{\lambda} - 2\boldsymbol{\lambda}'\mathbf{c} + \sigma_0^2 + 2\mathbf{m}'(\mathbf{X}'\boldsymbol{\lambda} - \mathbf{x}(\mathbf{s}_0))] = 0$$

$$\Rightarrow 2\boldsymbol{\Sigma}\boldsymbol{\lambda} - 2\mathbf{c} + 2\mathbf{X}\mathbf{m} = 0, \qquad \mathbf{X}'\boldsymbol{\lambda} - \mathbf{x}(\mathbf{s}_0) = 0$$

Review of BLUP and Point Prediction

solve

$$\begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{X} \\ \mathbf{X}' & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \mathbf{m} \end{pmatrix} = \begin{pmatrix} \mathbf{c} \\ \mathbf{x}(\mathbf{s}_0) \end{pmatrix}$$

$$\begin{aligned}
&\boldsymbol{\Sigma}\boldsymbol{\lambda} - \mathbf{c} + \mathbf{X}\mathbf{m} = 0 \Rightarrow \boldsymbol{\lambda} = \boldsymbol{\Sigma}^{-1}(\mathbf{c} - \mathbf{X}\mathbf{m}) \\
&\mathbf{X}'\boldsymbol{\lambda} - \mathbf{x}(\mathbf{s}_0) = 0 \Rightarrow \mathbf{X}'\boldsymbol{\Sigma}^{-1}(\mathbf{c} - \mathbf{X}\mathbf{m}) = \mathbf{x}(\mathbf{s}_0) \\
&\Rightarrow \mathbf{m} = (\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{c} - \mathbf{x}(\mathbf{s}_0)) \\
&\Rightarrow \boldsymbol{\lambda} = \boldsymbol{\Sigma}^{-1}(\mathbf{c} + \mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}(\mathbf{x}(\mathbf{s}_0) - \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{c}))
\end{aligned}$$

$$\hat{Z}(\mathbf{s}_0) = \boldsymbol{\lambda}'\mathbf{z}, \quad \mathrm{var}(\hat{Z}(\mathbf{s}_0)) = \boldsymbol{\lambda}'\boldsymbol{\Sigma}\boldsymbol{\lambda} - 2\boldsymbol{\lambda}'\mathbf{c} + \sigma_0^2$$

Alternative Formulas

$$\hat{Z}(\mathbf{s}_0) = \boldsymbol{\lambda}'\mathbf{z} = \mathbf{c}'\boldsymbol{\Sigma}^{-1}(\mathbf{z} - \hat{\boldsymbol{\mu}}) + \hat{\boldsymbol{\mu}}_0$$

where

$$\hat{\boldsymbol{\mu}} = \mathbf{X}\hat{\boldsymbol{\beta}}_{GLS} \quad \hat{\boldsymbol{\mu}}_0 = \mathbf{x}(\mathbf{s}_0)'\hat{\boldsymbol{\beta}}_{GLS} \quad \hat{\boldsymbol{\beta}}_{GLS} = (\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{z}$$

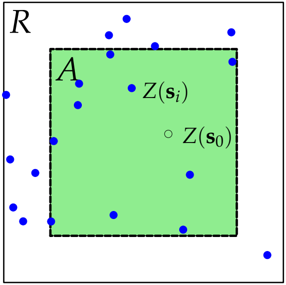
and

$$E(\boldsymbol{\lambda}'\mathbf{z} - Z(\mathbf{s}_0))^2 = \sigma_0^2 - \mathbf{c}'\boldsymbol{\Sigma}^{-1}\mathbf{c} + \mathbf{d}'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{d}$$

where

$$\mathbf{d} = \mathbf{x}(\mathbf{s}_0) - \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{c}$$

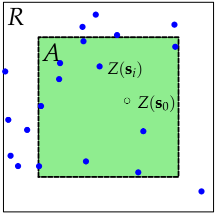
Block Prediction



- $Z(A) \equiv \int_A Z(\mathbf{u})d\mathbf{u}/|A|$
- $x_i(A) \equiv \int_A x_i(\mathbf{u})d\mathbf{u}/|A|$
- $\mathbf{x}_A \equiv [x_1(A), \ldots, x_p(A)]'$
- $\epsilon(A) \equiv \int_A \epsilon(\mathbf{u})d\mathbf{u}/|A|$

$$\begin{pmatrix} \mathbf{z} \\ Z(A) \end{pmatrix} = \begin{pmatrix} \mathbf{X} \\ \mathbf{x}'_A \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} \boldsymbol{\epsilon} \\ \epsilon(A) \end{pmatrix}$$

Block Prediction



- $c_i(A) \equiv \text{cov}(\epsilon(\mathbf{s}_i), \epsilon(A)) = \int_A \text{cov}(\epsilon(\mathbf{s}_i), \epsilon(\mathbf{u}))d\mathbf{u}/|A|$
- $\mathbf{c}_A \equiv [c_1(A), \ldots, c_n(A)]$
- $\sigma_A^2 \equiv \int_A \int_A \text{cov}(\epsilon(\mathbf{u}), \epsilon(\mathbf{v}))d\mathbf{u}d\mathbf{v}/|A|^2$

recall: $\text{cov}(Y_1, Y_2 + Y_3) = \text{cov}(Y_1, Y_2) + \text{cov}(Y_1, Y_3)$

$$\text{cov}\begin{pmatrix} \boldsymbol{\epsilon} \\ \epsilon(A) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{c}_A \\ \mathbf{c}_A' & \sigma_A^2 \end{pmatrix}$$

Block Prediction

$$\begin{pmatrix} \mathbf{z} \\ Z(A) \end{pmatrix} = \begin{pmatrix} \mathbf{X} \\ \mathbf{x}'_A \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} \boldsymbol{\epsilon} \\ \epsilon(A) \end{pmatrix}$$

$$\mathrm{cov} \begin{pmatrix} \boldsymbol{\epsilon} \\ \epsilon(A) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{c}_A \\ \mathbf{c}'_A & \sigma^2_A \end{pmatrix}$$

Best Linear Unbiased Prediction (BLUP)
minimize: $E(\boldsymbol{\lambda}'\mathbf{z} - Z(A))^2$ subject to $E[\boldsymbol{\lambda}'\mathbf{z}] = E[Z(A)] \ \forall \ \boldsymbol{\beta}$
Unbiasedness $\Rightarrow \mathbf{X}'\boldsymbol{\lambda} = \mathbf{x}_A$
$E[\boldsymbol{\lambda}'\boldsymbol{\epsilon}\boldsymbol{\epsilon}'\boldsymbol{\lambda} - 2\boldsymbol{\lambda}'\boldsymbol{\epsilon}\epsilon(A) + \epsilon(A)^2] = \boldsymbol{\lambda}'\boldsymbol{\Sigma}\boldsymbol{\lambda} - 2\boldsymbol{\lambda}'\mathbf{c}_A + \sigma^2_A$
$\boldsymbol{\lambda} = \boldsymbol{\Sigma}^{-1}(\mathbf{c}_A + \mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}(\mathbf{x}_A - \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{c}_A))$
$\hat{Z}(A) = \boldsymbol{\lambda}'\mathbf{z}, \quad \mathrm{var}(\hat{Z}(A)) = \boldsymbol{\lambda}'\boldsymbol{\Sigma}\boldsymbol{\lambda} - 2\boldsymbol{\lambda}'\mathbf{c}_A + \sigma^2_A$

Alternative Formulas

$$\hat{Z}(A) = \boldsymbol{\lambda}'\mathbf{z} = \mathbf{c}_A'\boldsymbol{\Sigma}^{-1}(\mathbf{z} - \hat{\boldsymbol{\mu}}) + \hat{\boldsymbol{\mu}}_A$$

where

$$\hat{\boldsymbol{\mu}} = \mathbf{X}\hat{\boldsymbol{\beta}}_{GLS} \quad \hat{\boldsymbol{\mu}}_A = \mathbf{x}_A'\hat{\boldsymbol{\beta}}_{GLS} \quad \hat{\boldsymbol{\beta}}_{GLS} = (\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{z}$$

and

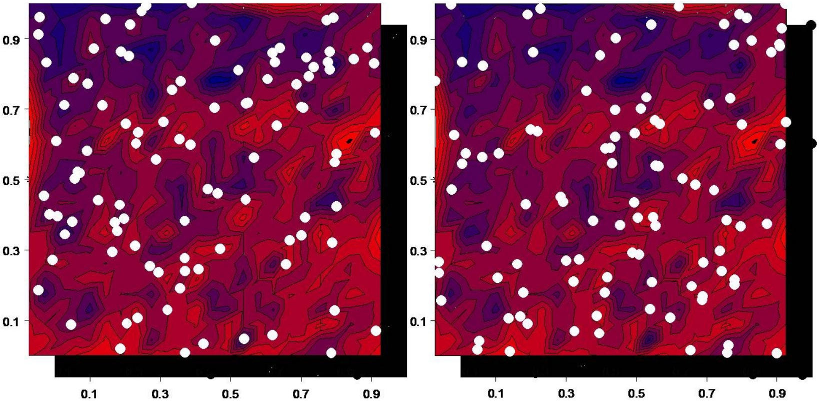
$$E(\boldsymbol{\lambda}'\mathbf{z} - Z(A))^2 = \sigma_A^2 - \mathbf{c}_A'\boldsymbol{\Sigma}^{-1}\mathbf{c}_A + \mathbf{d}_A'(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{d}_A$$

where

$$\mathbf{d}_A = \mathbf{x}_A - \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{c}_A$$

Fixed Pattern, Random Samples

Ver Hoef, J.M. 2002. Sampling and geostatistics for spatial data. *Ecoscience* **9**: 152 - 161.

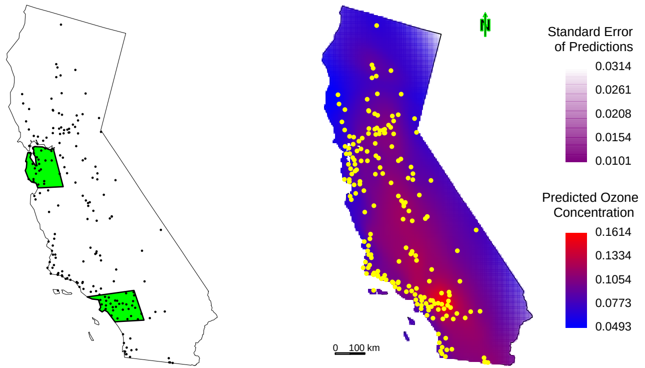


Fixed Pattern, Random Samples

1000 random samples of size 100

Validation Statistics	Simple Random Sample	Block Prediction
Bias	0.002	-0.020
RMSPE	1.28	1.02
RAEV	1.29	1.00
80%CI	0.813	0.806

Ozone Example



Ozone Example

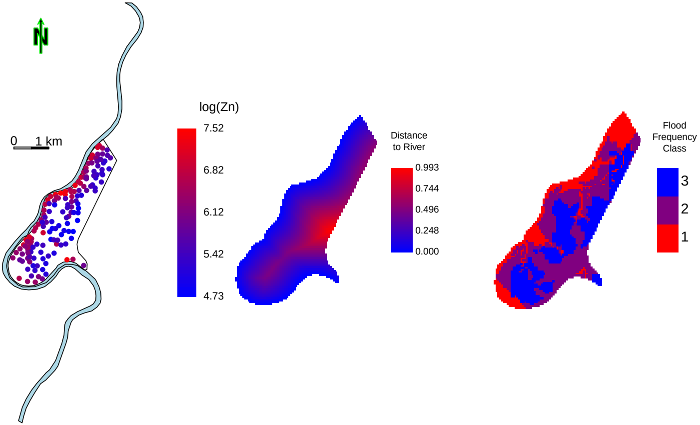
```
library(spPlotSampCourse)
library(maptools)
path <- system.file("rawdata/airPolluteCA", package = "spPlotSampCourse")
outlineFile <- paste(path, "/", "ca_outline", sep = "")
otl <- readShapePoly(outlineFile)
pointsFile <- paste(path, "/", "ca_ozone_pts", sep = "")
pts <- readShapePoints(pointsFile)
polyLAFile <- paste(path, "/", "polyLA", sep = "")
polyLA <- readShapePoly(polyLAFile)
polySFFile <- paste(path, "/", "polySF", sep = "")
polySF <- readShapePoly(polySFFile)
ozFit1 <- splmm(OZONE ~ 1, spdata = pts, estMeth = "REML", varComps = "circular",
    useAnisotropy = TRUE)
blockPredGridLA <- createBlockPredGrid(polyLA)
predictBlock(ozFit1, blockPredGridLA)


##    OZONE BlockPredSE
## 1 0.1256     0.00178


blockPredGridSF <- createBlockPredGrid(polySF)
predictBlock(ozFit1, blockPredGridSF)


##     OZONE BlockPredSE
## 1 0.08856    0.002618
```

Meuse Example



Meuse Example

```
data(meuse)
coordinates(meuse) <- ~x + y
proj4string(meuse) <- CRS("+init=epsg:28992")
data(meuse.grid)
coordinates(meuse.grid) <- ~x + y
proj4string(meuse.grid) <- CRS("+init=epsg:28992")
znFit1 <- splmm(zinc ~ dist + ffreq, spdata = meuse, varComps = "besselK")
predictBlock(znFit1, meuse.grid)

##    zinc BlockPredSE
## 1 351.9       18.73
```

Meuse Example

```
summary(znFit1)$coefficients
```

```
##              Estimate Std. Error t value  Pr(>|t|)
## (Intercept)    959.9      93.87  10.225 5.620e-19
## dist         -1189.6     234.71  -5.068 1.159e-06
## ffreq2        -288.7      37.26  -7.748 1.262e-12
## ffreq3        -276.1      57.92  -4.766 4.371e-06
```

```
summary(znFit1)$covparms
```

```
##   Variance Component Parameter Type  Estimate
## 1             nugget         nugget 7.858e+03
## 2            besselK         parsil 8.486e+04
## 3            besselK          range 8.250e+02
## 4            besselK         extrap 7.807e-01
```

```
summary(znFit1)$R2g
```

```
## [1] 0.3965
```