High Order Farfield Expansion ABC coupled with IGA and Finite Differences Applied to Acoustic Multiple Scattering

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Abstract

We have constructed and successfully applied *high order local Farfield Expansions* absorbing boundary conditions (FEABC) for time-harmonic single acoustic scattering in two– and three–dimensions in previous works [1, 2]. We have also extended the formulation of FEABC to two and three dimensional acoustic multiple scattering in previous papers. In this work, we present some numerical results for two-dimensional multiple scattering from obstacles of arbitrary shape. We will also discuss weak formulations of these multiple scattering problems as our first step to implement general curvilinear finite element methods in the context of *Isogeometric Analysis* (IGA) for multiple scattering.

Keywords: Acoustic multiple scattering, High order local absorbing boundary conditions

1 The Local FEABC for multiple scattering

For brevity, we specialize our discussion to the two dimensional case but its extension to three dimensions follows a similar procedure [1]. We consider M disjoint obstacles each occupying a bounded domain with boundary Γ_m for $m = 1, \dots M$. The unbounded region in the exterior of Γ_m is denoted by Ω_m. The obstacles are sufficiently separated from each other as to enclose each one with disjoint circular artificial boundaries $\mathcal{B}_m$. The computational region Ω_m^- is bounded internally by the obstacle boundary Γ_m and externally by the artificial boundary $\mathcal{B}_m$. The unbounded region in the exterior of $\mathcal{B}_m$ is denoted by Ω_m^+ so that $\mathcal{B}_m$ is precisely the interface between Ω_m^- and Ω_m^+. We also consider the following definitions:

$$\Omega = \bigcap_{m=1}^M \Omega_m, \quad \Omega^- = \bigcup_{m=1}^M \Omega_m^-,$$
$$\Omega^+ = \bigcap_{m=1}^M \Omega_m^+ \quad \text{and} \quad \Gamma = \bigcup_{m=1}^M \Gamma_m.$$

The scattering problem that we are considering consists of the scattering of a plane incident wave, u_{inc}, from multiple soft (Dirichlet) or hard (Neumann) obstacles embedded in the unbounded two-dimensional region Ω. As stated in our previous work, the construction of the FEABC is based on a decomposition of the scattered field u into purely-outgoing wave fields u^m, such that $u = \sum_{m=1}^M u^m$ in Ω^+, where each u^m is an outgoing wave radiating from the artificial boundary B^m. The fundamental idea of this work is the use of a truncated expansion introduced by Karp in 1961 to represent each u^m in Ω_m^+ as

$$u^{\mathrm{m}}(r^m, \theta^m) = H_0(kr^m) \sum_{l=0}^{L-1} \frac{F_l^m(\theta^m)}{(kr^m)^l} + H_1(kr^m) \sum_{l=0}^{L-1} \frac{G_l^m(\theta^m)}{(kr^m)^l}. \qquad (1)$$

The angular functions $F_l^m(\theta^m)$ and $G_l^m(\theta^m)$ are additional unknowns. They depend on the geometry of the scatterers and the properties of the domains Ω_m^-. An improved version of the formulation for the scattered field u is given by

$$\Delta u + k^2 u = 0, \qquad \text{in} \quad \Omega^-, \qquad (2)$$

$$u = -u_{inc}, \ \text{or} \quad \partial_r u = -\partial_r u_{inc}, \qquad \text{in} \quad \Gamma, \qquad (3)$$

$$u = \sum_{m=1}^M u^m \qquad \text{on } \mathcal{B}_m, \qquad (4)$$

$$\frac{\partial u}{\partial \nu^m} = \sum_{m=1}^M \frac{\partial u^m}{\partial \nu^m} \qquad \text{on } \mathcal{B}_m, \qquad (5)$$

$$\mathscr{H}^m[u] = \sum_{m=1}^M \mathscr{H}^m[u^m] \qquad \text{on } \mathcal{B}_m, \qquad (6)$$

for $m = 1, \dots M$ in equations (4)-(6).

In (5), ν^m denotes the normal derivative on B_m. The symbol $\mathscr{H}^m$ is the Helmholtz operator in terms of the local polar coordinate system in Ω_m. The Eqs. (4)-(5) are the usual continuity of u and its normal derivative at the interface $\mathcal{B}_m$. The condition (6) establishes the continuity of the Helmholtz operator at the interface. The system is completed by adding the recurrence formulas for the angular

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functions for $l = 1 \dots L-1$, defined on B^m,

$$2l\, G_l^m(\theta) = (l-1)^2 F_{l-1}^m(\theta) + d_\theta^2 F_{l-1}^m(\theta) \quad (7)$$
$$2l\, F_l^m(\theta) = -l^2 G_{l-1}^m(\theta) - d_\theta^2\, G_{l-1}^m(\theta). \quad (8)$$

The weak formulation for this BVP is an extension of the one found in [2] to several obstacles. For the IGA application to the BVP (3)-(8) with Dirichlet BC, we define the function spaces

$$\mathscr{S} = \{(u, F_0^1, G_0^1, \dots F_0^M, G_0^M, \dots F_{L-1}^M, G_{L-1}^M) | u = \\ = -u_{inc} \text{ on } \Gamma, u \in H^1(\Omega^-), F_l^m, G_l^m \in H^1(\mathcal{B}_m)\}$$
$$\mathscr{S}_0^m = \{v^m \in H^1(\Omega_m^-) \,|\, v^m = 0, \text{ on } \Gamma_m\},$$

for $m = 1, \dots M$. Then, the weak formulation consists of finding $(u, F_0^1, G_0^1, \dots F_{L-1}^M, G_{L-1}^M) \in \mathscr{S}$ such that the following equations are satisfied:

$$a(u, v^m) - \sum_{\bar{m}=1}^{M} (\partial u_{\nu^m}^{\bar{m}}, v^m)_{\mathcal{B}_m} = 0, \; v^m \in \mathscr{S}_0^m,$$
$$a(u, v^m) = \int_{\Omega_m^-} \left(\nabla u \cdot \nabla v^m - k^2 u v^m\right) d\Omega_m^-,$$
$$(u, \hat{v}^m)_{\mathcal{B}_m} - \sum_{\bar{m}=1}^{M} (u^{\bar{m}}, \hat{v}^m)_{\mathcal{B}_m} = 0, \text{ for } \hat{v}^m \in H^1(\mathcal{B}_m)$$
$$(\mathscr{H}^m[u], \hat{v}^m)_{\mathcal{B}_m} = 0, \;\text{ for } \hat{v}^m \in H^1(\mathcal{B}_m)$$
$$2l(F_l^m, \hat{v}^m)_{\mathcal{B}_m} + l^2(G_{l-1}^m, \hat{v}^m)_{\mathcal{B}_m} - \left((G_{l-1}^m)', (\hat{v}^m)'\right)_{\mathcal{B}_m}.$$

The angular function G_l^m satisfies a similar equation to this last equation.

Some of our numerical results obtained by numerically solving (2)-(8) with a second order finite difference approximation in generalized curvilinear coordinates are illustrated by Figs. 1- 3. We will present numerical results from the IGA technique at the conference.

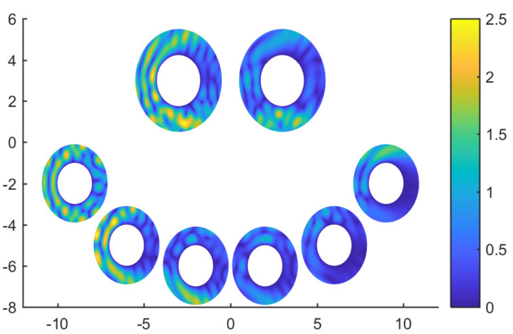

Figure 1: Total pressure field with $k = 2\pi$, for eight soft cylinders using $L = 8$ terms in the FEABC

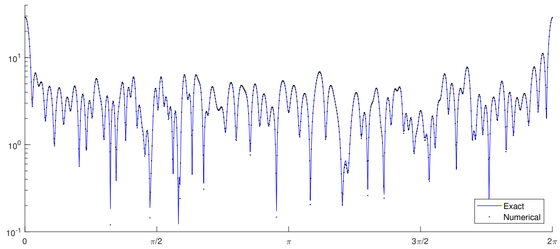

Figure 2: Comparison of exact and numerical far-field pattern for the eight cylinders of Fig. 1

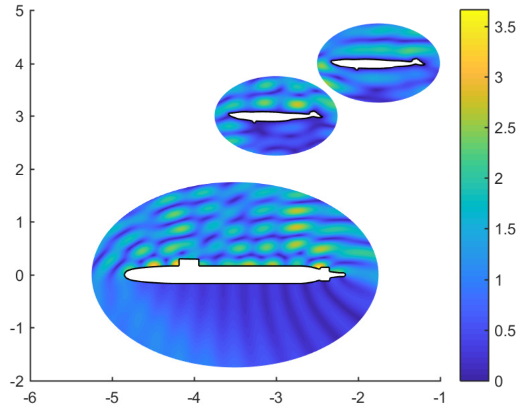

Figure 3: Scattering from a hard submarine and two soft whales with $k = 4\pi$ using $L = 12$ terms

2 Concluding remarks

In the experiments shown above, the numbers of terms, L in the FEABC was increased to achieve the best possible approximation. The overall order of convergence of the combined method for the cylindrical scatterers was two due to the second order of convergence of the numerical method used in the interior. The local nature of the FEABC is of great advantage when compared to alternative ABCs such as Dirichlet to Neumann.

References

[1] V. Villamizar, S. Acosta, and B. Dastrup. High order local absorbing boundary conditions for acoustic waves in terms of farfield expansions. *J. Comput. Phys.*, 333:331–351, 2017.

[2] T. Khajah and V. Villamizar. Highly accurate acoustic scattering: Isogeometric analysis coupled with local high order farfield expansion ABC. *Comput. Methods Appl. Mech. Engrg.*, 349:477–498, 2019.

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