Explaining Curved-Fold Behavior through Normalized Coordinate Equations and Energy Methods

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Abstract

While it is possible to model and design curved-fold origami, it remains challenging to predict the natural configurations that physical models will assume. In this paper we simplify curved-fold calculations through normalized coordinate equations—equations that relate properties of a curved fold at a point to the normalized distance that the point lies from the edge of regression—and use these equations to develop an energy method to find natural (or lowest energy) configurations for a general curved fold.

Curved folds can be modeled as a pair of developable surfaces (surfaces that are singly curved and can be unfolded to a plane) connected along a 3-dimensional curve. Developable surfaces are a class of ruled surfaces, and the generalized curve that the set of rulings is tangent to is known as the edge of regression. Using a unit speed curve fold parametrization, we show that at any point u along the crease, the distance between the crease and the edge of regression is given by

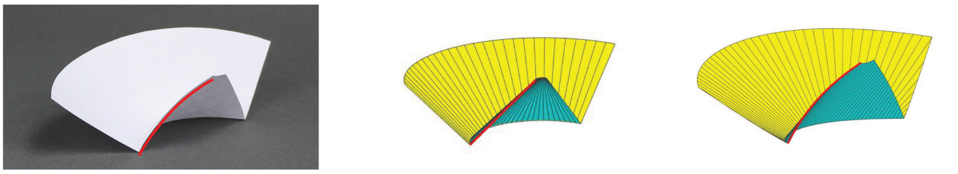
$$v_0(u) = \frac{\sin\beta(u)}{\kappa_g(u) + \beta'(u)} \tag{1}$$

where κ_g denotes the geodesic curvature of the crease and β denotes the ruling angle, or angle between the tangent and ruling directions. We then define the normalized coordinate Υ of a surface point (u, v) as the ratio of the distance along a ruling line from a point to the edge of regression over the distance along the same ruling line from the crease to the edge of regression:

$$\Upsilon(u, v) = \frac{v_0(u) - v}{v_0(u)} \tag{2}$$

Many curved fold relationships such as the geodesic equation can be simplified through the use of normalized coordinates. We derive simplified equations to express the principle curvature and bending energies of fold surfaces in terms of normalized coordinates.

To quantify the energy associated with a specific fold configuration, we define the *fold energy* as the sum of the left and right surface *bending energies*, and the *crease energy*. To calculate the fold energy for an arbitrary fold configuration it is necessary to express the fold boundary in terms of fold surfaces. The process of embedding boundary curves in fold surfaces can be complicated and computationally expensive. In this paper we present a novel method in which boundary curves can be expressed in embedded form through a simple root-finding process. We then use a calculus of variations approach to identify minimal fold energy—or



(a) Physical Model (b) Planar-crease Simulated (c) Minimum-Energy Simulated

Figure 1: Quarter circular crease of width 0.9 units

natural—configurations of a curved crease. We apply this energy method to identify natural configurations for various curved creases and make some surprising observations.

Various authors have observed that folds tend toward zero torsion (planar), constant fold angle (uniform) configurations. We show that the lowest energy state of a quarter circular curved crease, while close to planar, is not a planar uniform fold. This result is shown in Figure 1, where a physical model is compared to both a planar uniform crease, and an optimized uniform crease. In the case of the fold depicted in Figure 1, introducing torsion to the uniform crease reduced energy by approximately 64%. In addition, we show that this result holds for a quarter circular crease where the width of the fold surfaces is small compared to the crease length.

Given that the natural configuration of a quarter circular crease is not a planar, uniform fold, we calculate various natural configurations by optimizing torsion and fold angle functions simultaneously. One such simulated natural configuration is shown in Figure 2(a), and an intriguing result is observed—the natural ruling field remains almost completely fixed under a variety of initial fold angles and material stiffness factors. This leads us to believe that although a natural fold is neither planar nor uniform, it may remain ruling-rigid foldable throughout its motion.

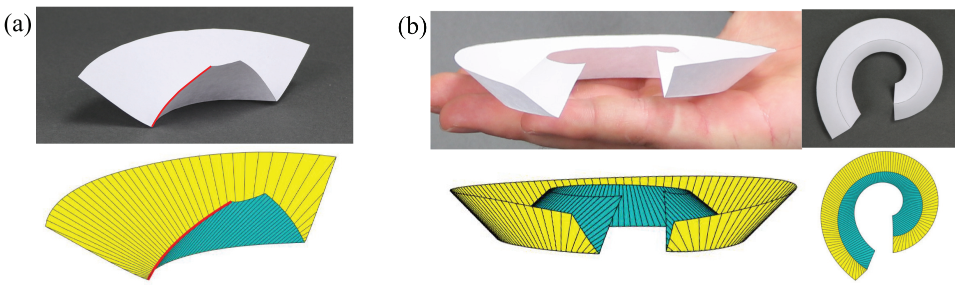


Figure 2: Natural fold configurations for **(a)** a circular crease and **(b)** a non-circular crease

The proposed energy method is valid for general curved creases, as shown in Figure 2(b) where physical and simulated models of a non-circular crease are compared.

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