Lecture 15 notes

Tuesday, October 31, 2023 9:31 AM

$y_i = \beta_0 + \beta_1 x_i + u_i \leftarrow$

$y_i = \hat{\beta}_0 + \hat{\beta}_1 x_i + \hat{u}_i$

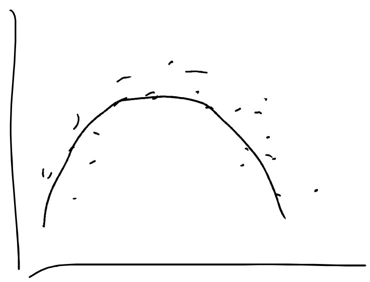
$\hat{y}_i = \underline{\hat{\beta}_0 + \hat{\beta}_1} x_i$

Ass 1: relationship b/t x & y is "linear in parameters"

$y = \underline{\beta_0 + \beta_1 x_1 + \beta_2 x_2} \cdots \beta_k x_k$

$y = x^{\beta_1}$

$y = \beta_0 + \beta_1 x + \beta_2 x^2 + u$



Ass 2: x & y are i.i.d.

Ass 3: $VAR(x) \neq 0$

Ass 4: ZCM "zero conditional mean"

$E(u|x) = 0$

Unbiasedness of $\hat{\beta}_1$

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = \frac{cov(x,y)}{var(x)} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$$

$$\sum (x_i - \bar{x})(y_i - \bar{y}) = \sum (x_i - \bar{x}) y_i - \sum (x_i - \bar{x}) \bar{y}$$

$\bar{x} = \frac{1}{n} \sum x_i$

$n\bar{x} = \sum x_i$

$$= \sum (x_i - \bar{x}) y_i - \left[\sum x_i \bar{y} - \underbrace{\sum \bar{x} \bar{y}}_{n\bar{x}\bar{y}} \right]$$

$$= \sum (x_i - \bar{x}) y_i - \left[\underbrace{n\bar{x}\bar{y} - n\bar{x}\bar{y}}_{0} \right]$$

$$\hat{\beta}_1 = \underline{\sum (x_i - \bar{x}) y_i}$$

$y_i = \beta_0 + \beta_1 x_i + u_i$

$y_i \quad \beta_0 \ \beta_1 \ldots$

$$\hat{\beta}_1 = \frac{\sum (x_i - \bar{x}) y_i}{\sum (x_i - \bar{x})^2}$$

$$= \frac{\sum (x_i - \bar{x})(\beta_0 + \beta_1 x_i + u_i)}{\sum (x_i - \bar{x})^2}$$

$$= \frac{\cancel{\beta_0 \sum \overset{(1)}{(x_i - \bar{x})}} + \beta_1 \sum \overset{(2)}{(x_i - \bar{x})} x_i + \sum \overset{(3)}{(x_i - \bar{x})} u_i}{\sum (x_i - \bar{x})^2}$$

$$\sum (x_i - \bar{x}) = 0$$

$$\sum x_i - \sum \bar{x} = \sum x_i - n\bar{x}$$

$$= n\bar{x} - n\bar{x} = 0$$

$$\hat{\beta}_1 = \frac{\beta_1 \sum \overset{(1)}{(x_i - \bar{x})} x_i + \sum (x_i - \bar{x}) u_i}{\sum (x_i - \bar{x})^2}$$

(1) $\sum (x_i - \bar{x}) x_i = \sum (x_i^2 - x_i \bar{x})$

$$= \sum x_i^2 - \bar{x} \underbrace{\sum x_i}_{n\bar{x}}$$

$$= \sum x_i^2 - n\bar{x}^2$$

$\boxed{\text{Trick: add } 0 = n\bar{x}^2 - n\bar{x}^2}$

$$= \sum x_i^2 - 2n\bar{x}^2 + n\bar{x}^2 \quad \because \ n\bar{x} = \sum x_i, \ n\bar{x}^2 = \sum \bar{x}^2$$

$$= \sum x_i^2 - 2n\bar{x}^2 + n\bar{x}^2 \quad \because \; n\bar{x} = \sum x_i, \; n\bar{x}^2 = \sum \bar{x}^2$$

$$= \sum x_i^2 - 2\bar{x} \sum x_i + \sum \bar{x}^2$$

$$= \sum \left(x_i^2 - 2\bar{x} x_i + \bar{x}^2 \right)$$

$$= \sum (x_i - \bar{x})^2$$

$$\hat{\beta}_1 = \frac{\beta_1 \sum (x_i - \bar{x})^2 + \sum (x_i - \bar{x}) u_i}{\sum (x_i - \bar{x})^2}$$

$$= \beta_1 + \frac{\sum (x_i - \bar{x}) u_i}{\sum (x_i - \bar{x})^2} \Longleftrightarrow SST_x$$

<u>LIE:</u> $E[X] = E[E(X|Y)]$

$$E(\hat{\beta}_1 | x) = E\left[\beta_1 + \frac{\sum (x_i - \bar{x}) u_i}{SST_x} \;\middle|\; X \right]$$

$$= E(\beta_1 | x) + E\left[\frac{\sum (x_i - \bar{x}) u_i}{SST_x} \;\middle|\; X \right]$$

$$= \beta_1 + \frac{1}{SST_x} E\left[\sum (x_i - \bar{x}) u_i \,|\, X \right]$$

$= \beta_1 + \frac{1}{SST_x} \ldots$

$$= \beta_1 + \underbrace{\frac{1}{SST_x} \underbrace{\sum (x_i - \bar{x}) E(u_i|x)}_{0}}_{0}$$

$$E(\hat{\beta}_1|x) = \beta_1 + 0$$

$$E(\hat{\beta}_1) = E[E(\hat{\beta}_1|x)]$$
$$= \beta_1$$

<u>Unbiasedness $\hat{\beta}_0$</u>

$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$; remember $y_i = \beta_0 + \beta_1 x_i + u_i$

$\bar{y} = \beta_0 + \beta_1 \bar{x} + \bar{u}$

$$= \beta_0 + \underline{\beta_1 \bar{x}} + \bar{u} - \underline{\hat{\beta}_1 \bar{x}}$$

$$= \beta_0 + (\beta_1 - \hat{\beta}_1)\bar{x} + \bar{u}$$

$$E(\hat{\beta}_0|x) = E(\beta_0 + (\beta_1 - \hat{\beta}_1)\bar{x} + \bar{u} | x)$$
$$= E(\beta_0|x) + E((\beta_1 - \hat{\beta}_1)\bar{x}|x) + E(\bar{u}|x)$$

$$= E(\beta_0|x) + E(\beta_1-\beta_1)\bar{x}|x + E(u|x)$$

$$= \beta_0 + \left(E(\beta_1|x) - E(\hat{\beta}_1|x)\bar{x}\right] + E(\bar{u}|x)$$

$$= \beta_0 + \underbrace{(\beta_1 - \beta_1)\bar{x}}_{0} + 0$$

$$= \beta_0$$

<u>OVB</u>

$$y_i = \beta_0 + \beta_1 x_i + \beta_2 z_i + v_i$$

$$y_i = \beta_0 + \beta_1 x_i + u_i$$

$$= \beta_0 + \beta_1 x_i + \underbrace{(\beta_2 z_i + v_i)}_{u_i}$$

$$\hat{\beta}_1 = \beta_1 + \frac{\sum(x_i - \bar{x})u_i}{SST_x}$$

$$= \beta_1 + \frac{\sum(x_i - \bar{x})(\beta_2 z_i + v_i)}{SST_x}$$

$$E(\hat{\beta}_1|x) = E\left[\beta_1 + \frac{\boxed{\sum(x_i - \bar{x})}(\beta_2 z_i + v_i)}{\boxed{SST_x}} \middle| x\right]$$

$$= \beta_1 + \frac{\sum (x_i - \bar{x}) E[(\beta_2 z_i + v_i) | x]}{SST_x}$$

$$= \beta_1 + \beta_2 \left[z_i \frac{\sum (x_i - \bar{x})}{SST_x} \right]$$

$$z_i \frac{\sum (x_i - \bar{x})}{SST_x} = \frac{cov(z, x)}{var(x)}$$

$$= \beta_1 + \beta_2 \boxed{\frac{cov(z, x)}{var(x)}} \iff z_i = \alpha_0 + \alpha_1 x_i + \varepsilon_i$$