Lecture 11

Quantitative Political Science

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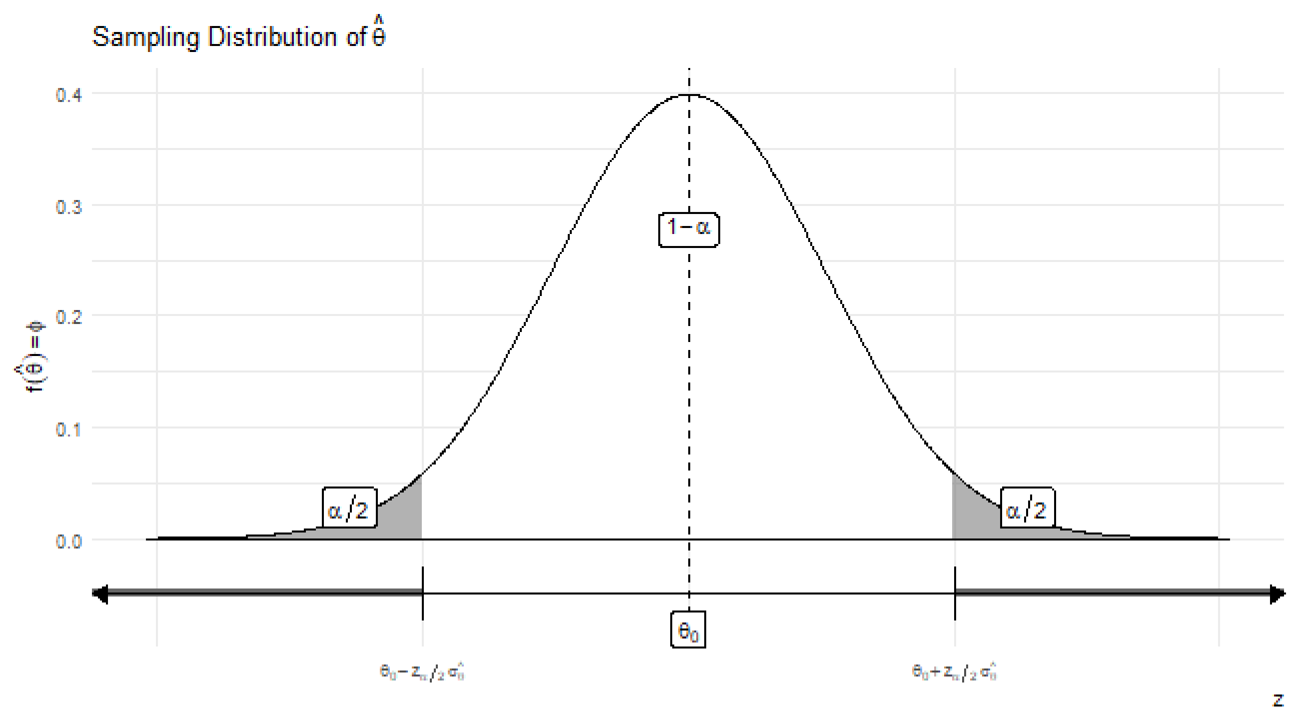
Agenda

1. Type 1 and Type II Error
2. Calculating Power
3. p-values

Type I Error

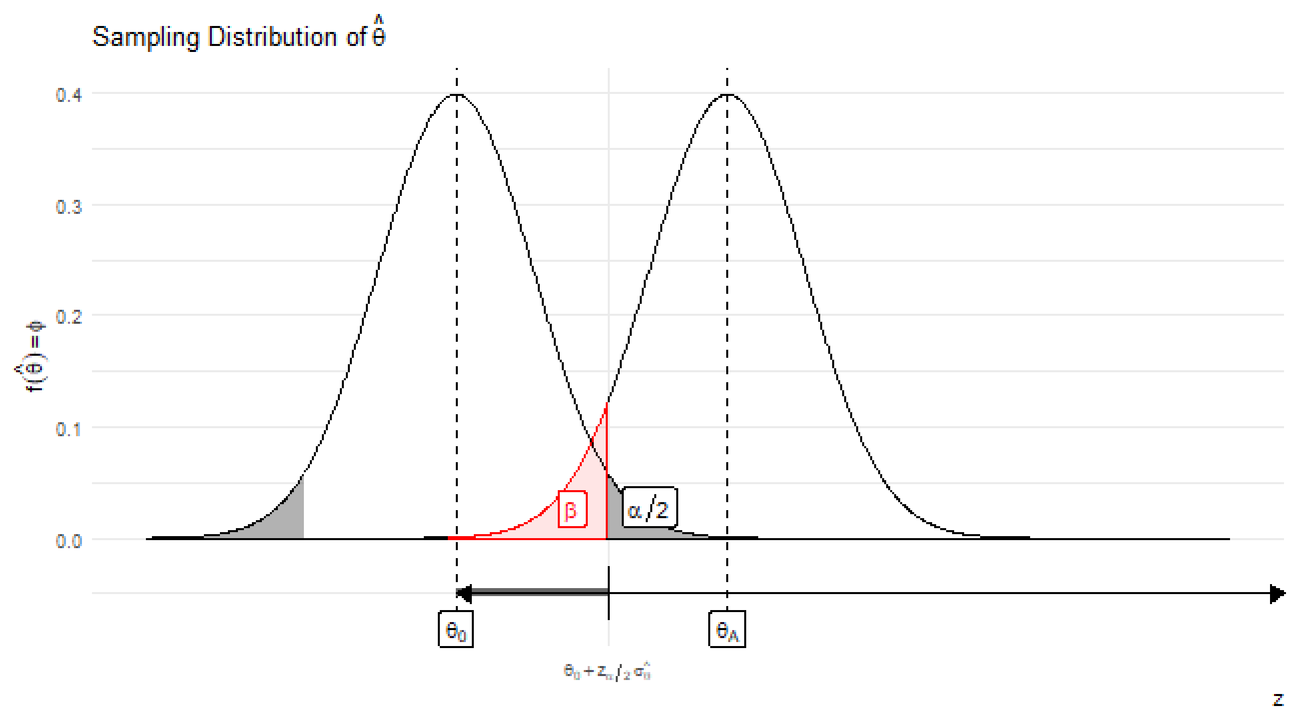
- **Type I error**: reject H_0 when it is actually true
  - What does this look like?



Type I error

- We will (purely by chance):
  - Observe an estimated $\hat{\theta}$ in the *RR* $100 * \alpha$% of the time
  - Thus falsely reject the null even though it's true
- This is Type I error!

Type II error



Type II error

- Suppose that the alternative hypothesis is true
- But we always conduct our hypothesis test **under the assumption that the null is true**
- If the sampling distribution of our estimator $\hat{\theta} \sim \mathcal{N}(\theta_A, \sigma_{\hat{\theta}})$, we will mistakenly accept the null $100 * \beta$ % of the time
- Define **power** as $1 - \beta$

$$\begin{aligned}
\text{Power} &= 1 - \beta \\
&= 1 - Pr(\text{Accept } H_0 | H_A \text{ true}) \\
&= 1 - Pr(\hat{\theta} < \theta_0 + z_{\alpha/2}\sigma_{\hat{\theta}} | \theta = \theta_A)
\end{aligned}$$

Power

- We can do this!

$$
\begin{aligned}
\beta &= Pr(\hat{\theta} < \theta_0 + z_\alpha \sigma_{\hat{\theta}} | \theta = \theta_A) \\
&= Pr\left(\frac{\hat{\theta} - \theta_A}{\sigma_{\hat{\theta}}} < \frac{\theta_0 + z_\alpha \sigma_{\hat{\theta}}}{\sigma_{\hat{\theta}}} | \theta = \theta_A\right) \\
&= \Phi\left(\frac{\theta_0 + z_\alpha \sigma_{\hat{\theta}} - \theta_A}{\sigma_{\hat{\theta}}}\right) \\
&= \Phi\left(\frac{\theta_0 - \theta_A}{\sigma_{\hat{\theta}}} + z_\alpha\right)
\end{aligned}
$$

Power

- We know θ_0 and θ_A (or we can specify them)
- We have also specified α and therefore z_α

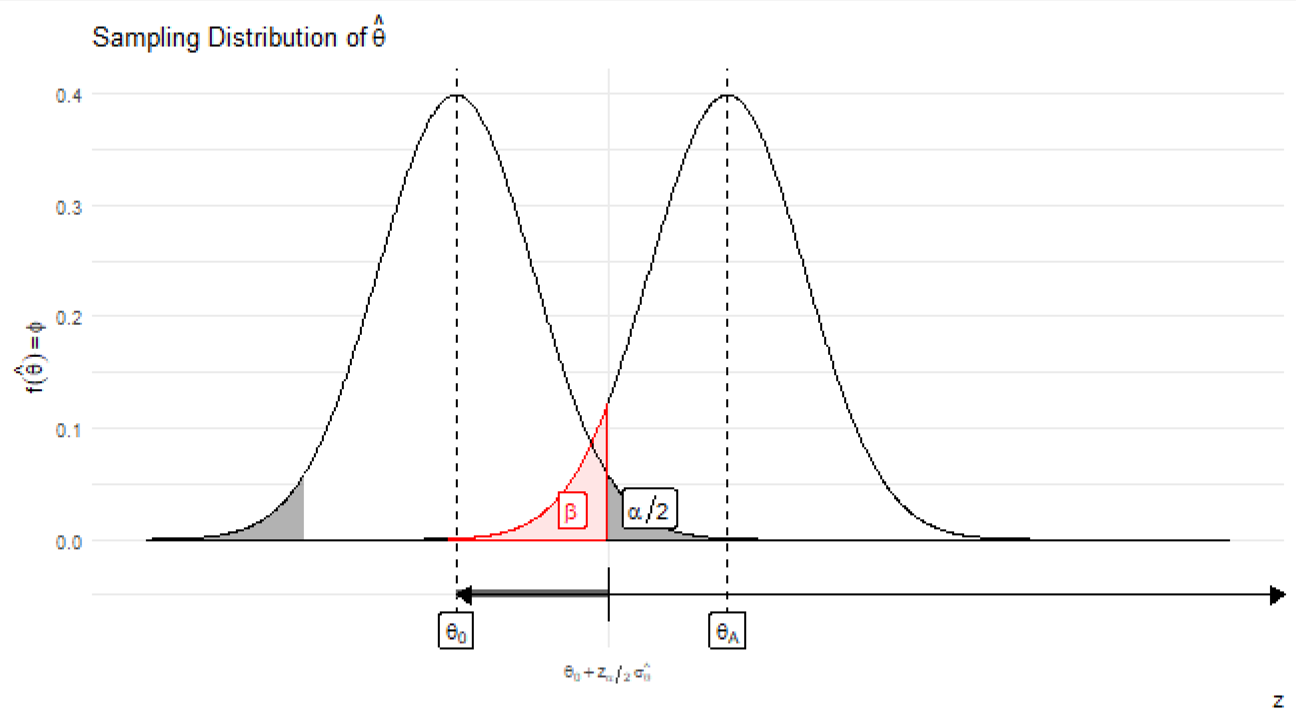
$$Power = 1 - \Phi\left(\frac{\theta_0 - \theta_A}{\frac{\sigma}{\sqrt{n}}} + z_\alpha\right)$$

- Stare at this for a second: can you figure out the following signs?

$$\frac{\partial \text{Power}}{\partial \alpha}$$
$$\frac{\partial \text{Power}}{\partial \sigma}$$
$$\frac{\partial \text{Power}}{\partial n}$$
$$\frac{\partial \text{Power}}{\partial (|\theta_0 - \theta_A|)}$$

$\frac{\partial \text{Power}}{\partial \alpha}$

- We know that $\frac{\partial z}{\partial \alpha} < 0$ so therefore $\frac{\partial \Phi\left(\frac{\theta_0-\theta_A}{\frac{\sigma}{\sqrt{n}}}+z_\alpha\right)}{\partial \alpha} < 0$ so $\frac{\partial \text{Power}}{\partial \alpha} > 0$
- Visually:



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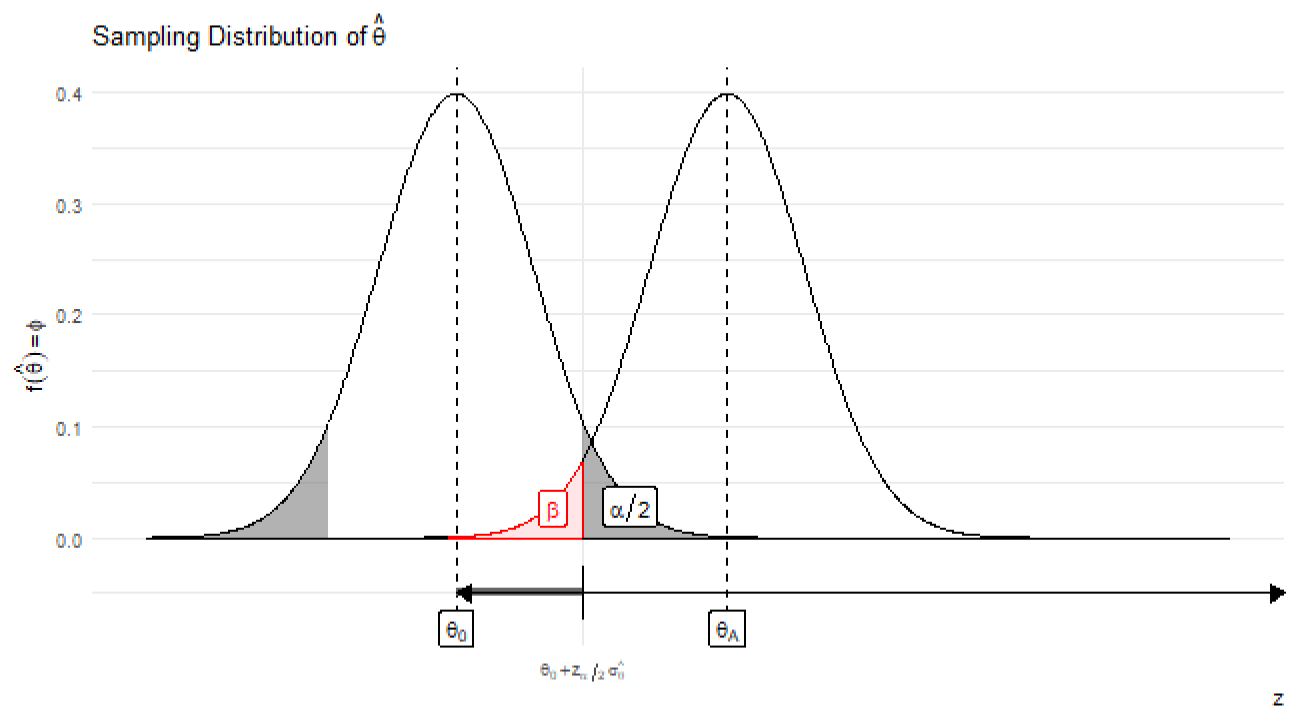

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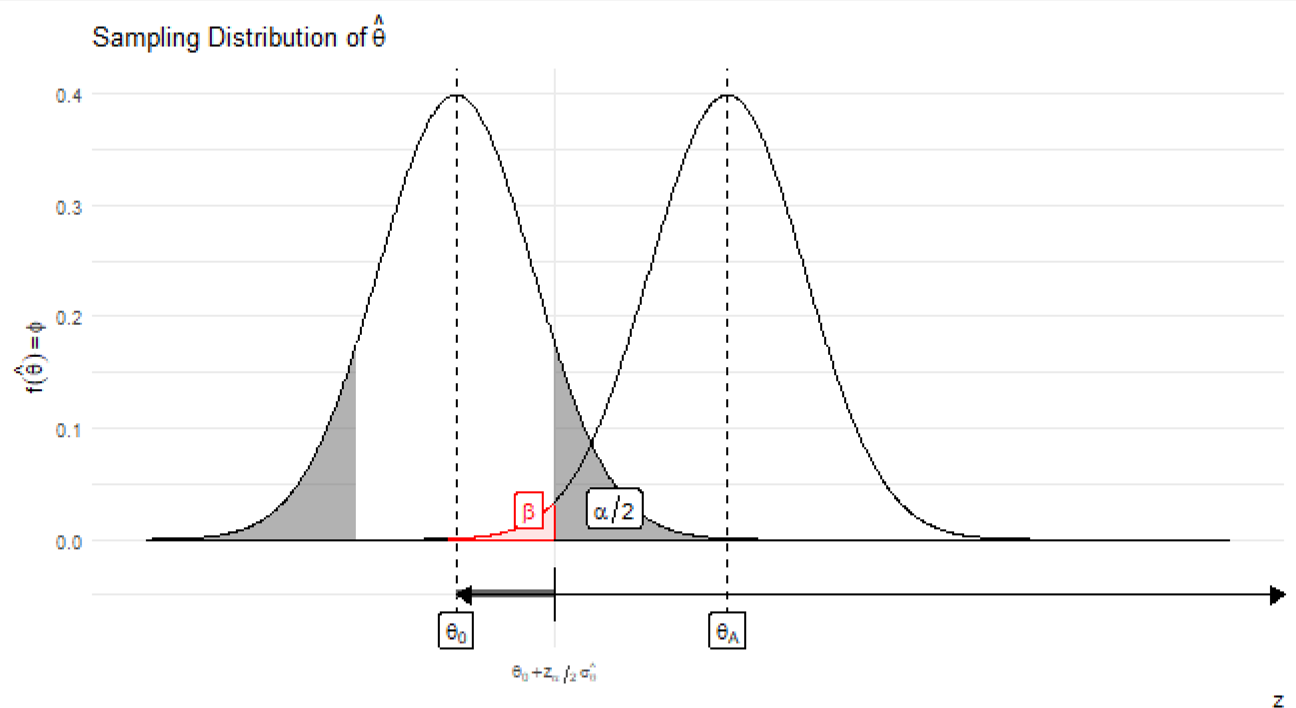
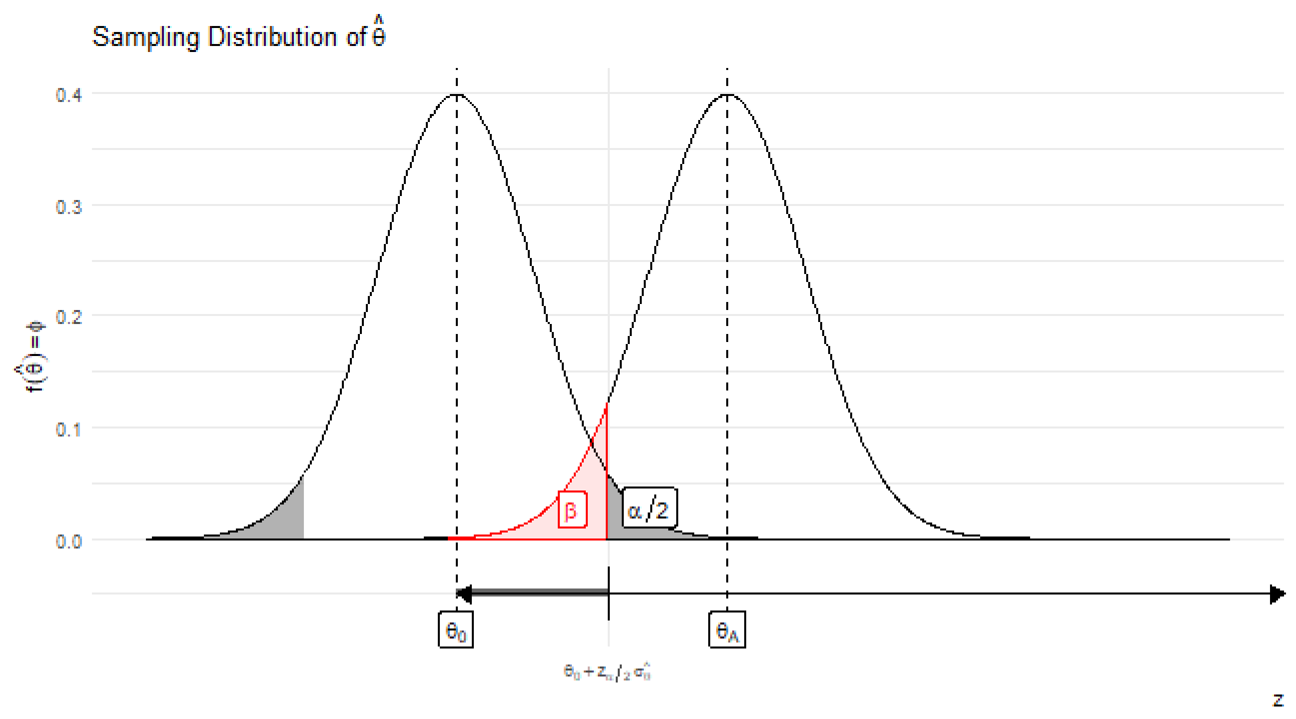


$\frac{\partial \text{Power}}{\partial \sigma}$

- We know that $\frac{\partial \Phi\left(\frac{(\theta_0-\theta_A)\sqrt{n}}{\sigma}+z_\alpha\right)}{\partial \sigma} > 0$ (since $\theta_0 - \theta_A < 0$) so $\frac{\partial \text{Power}}{\partial \sigma} < 0$
- Visually:



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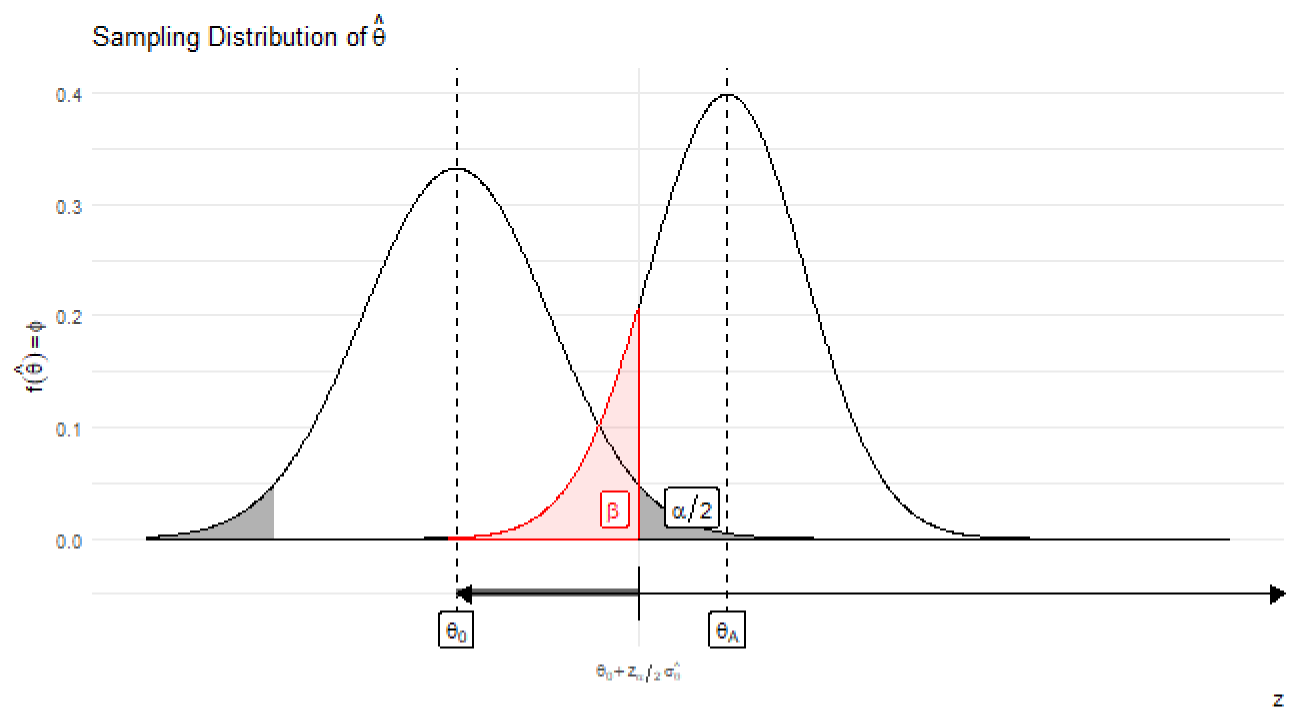

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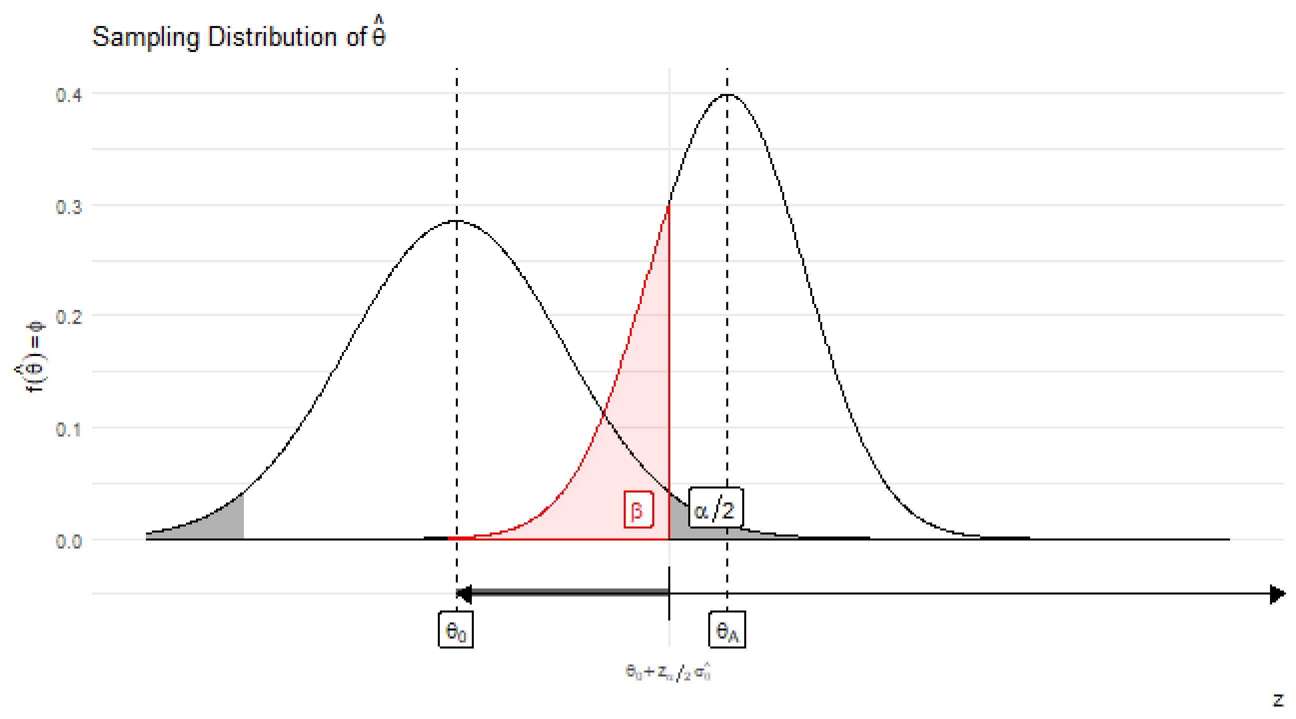

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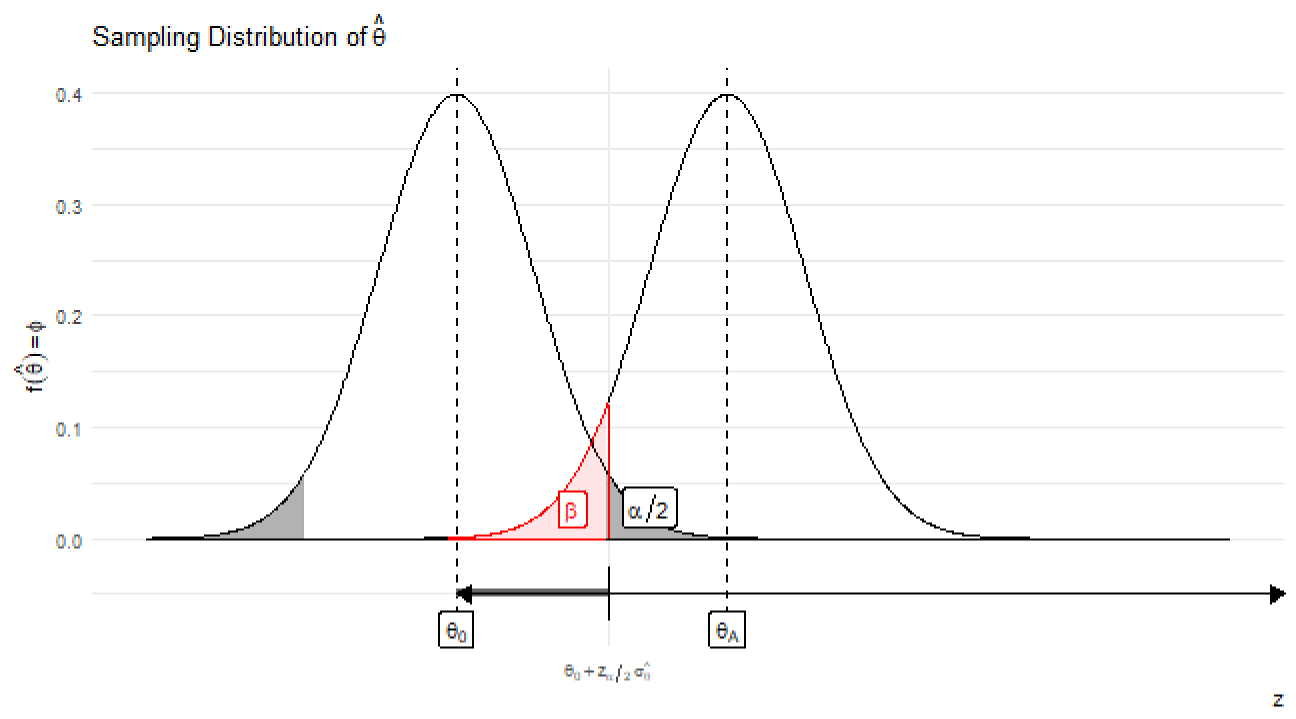

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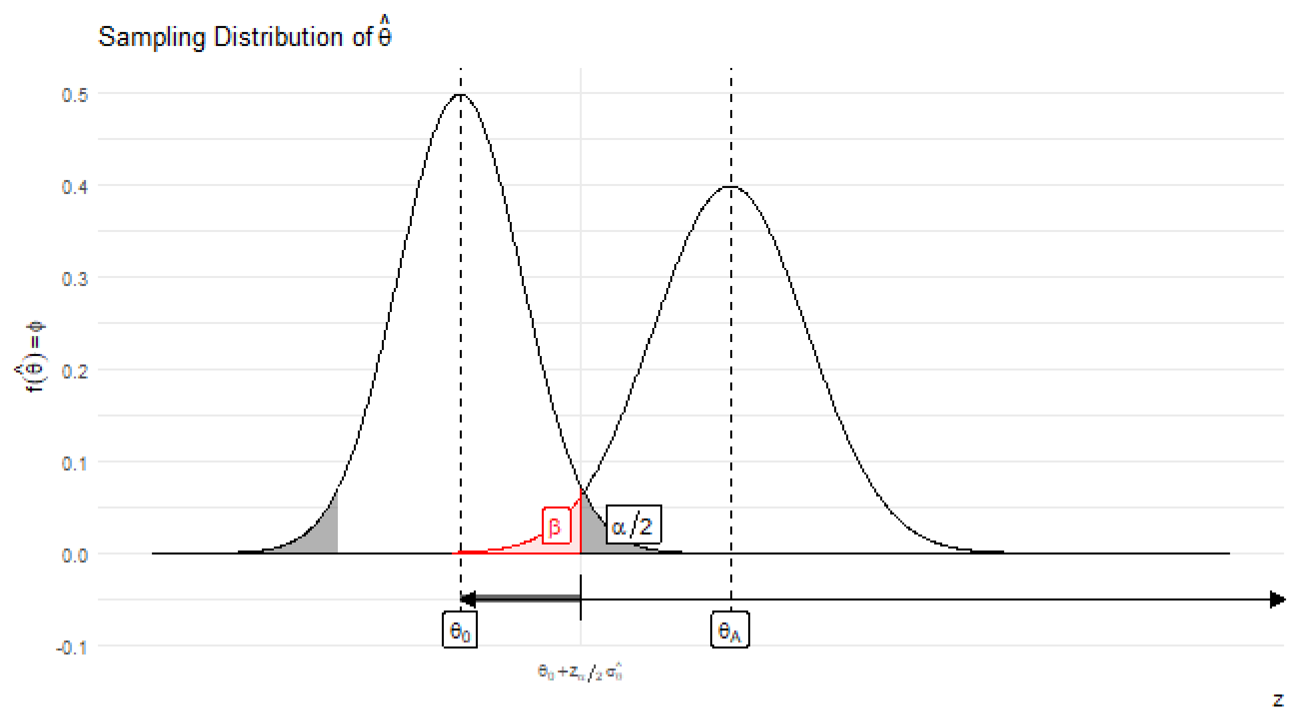

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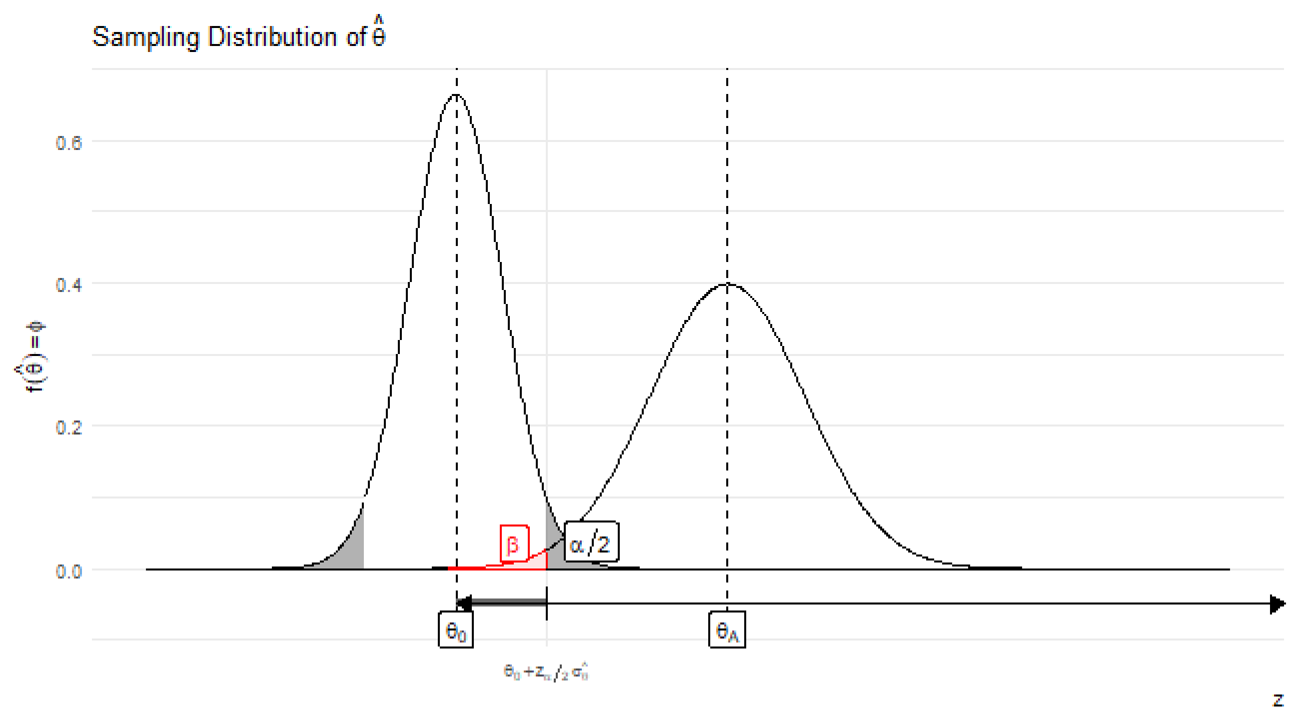

$$\frac{\partial \text{Power}}{\partial(|\theta_0 - \theta_A|)}$$

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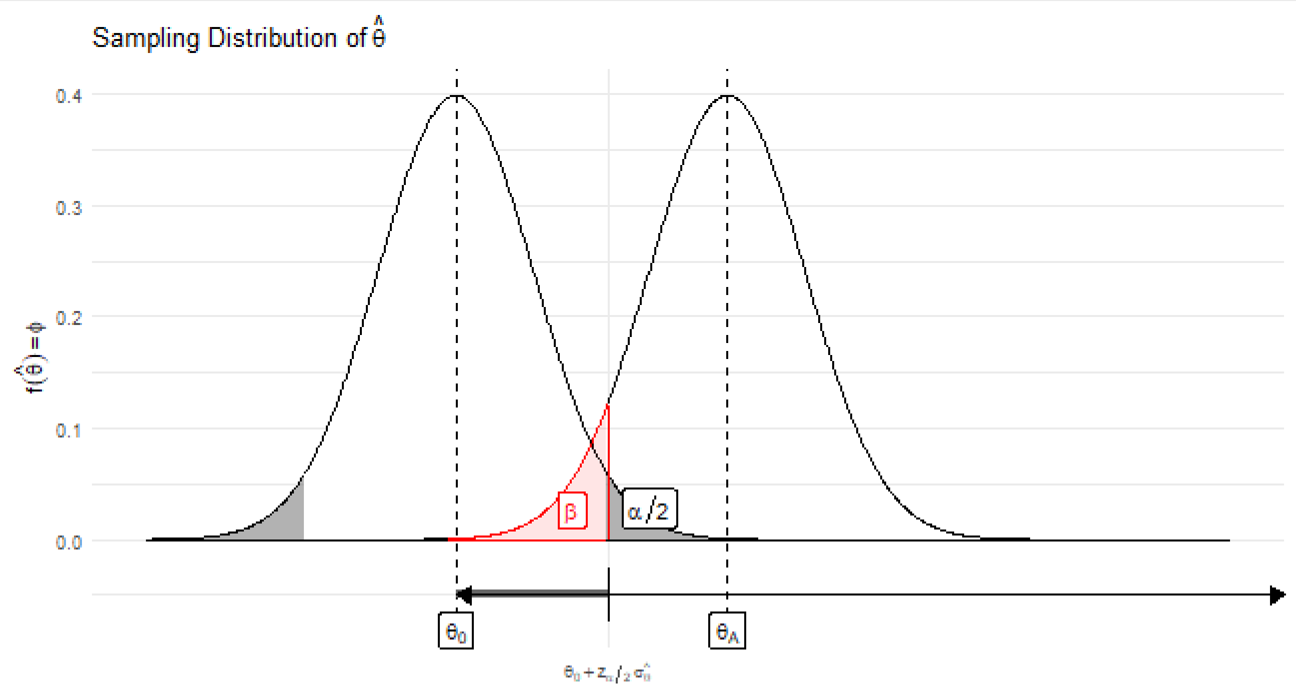

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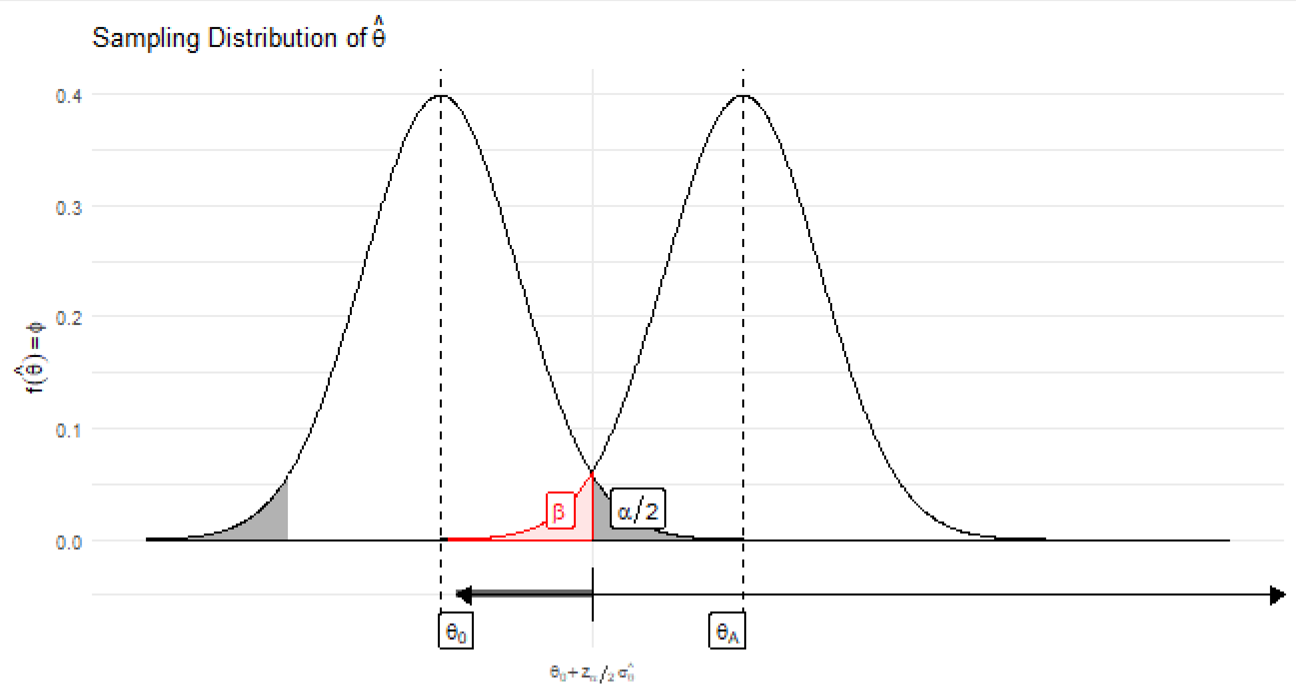

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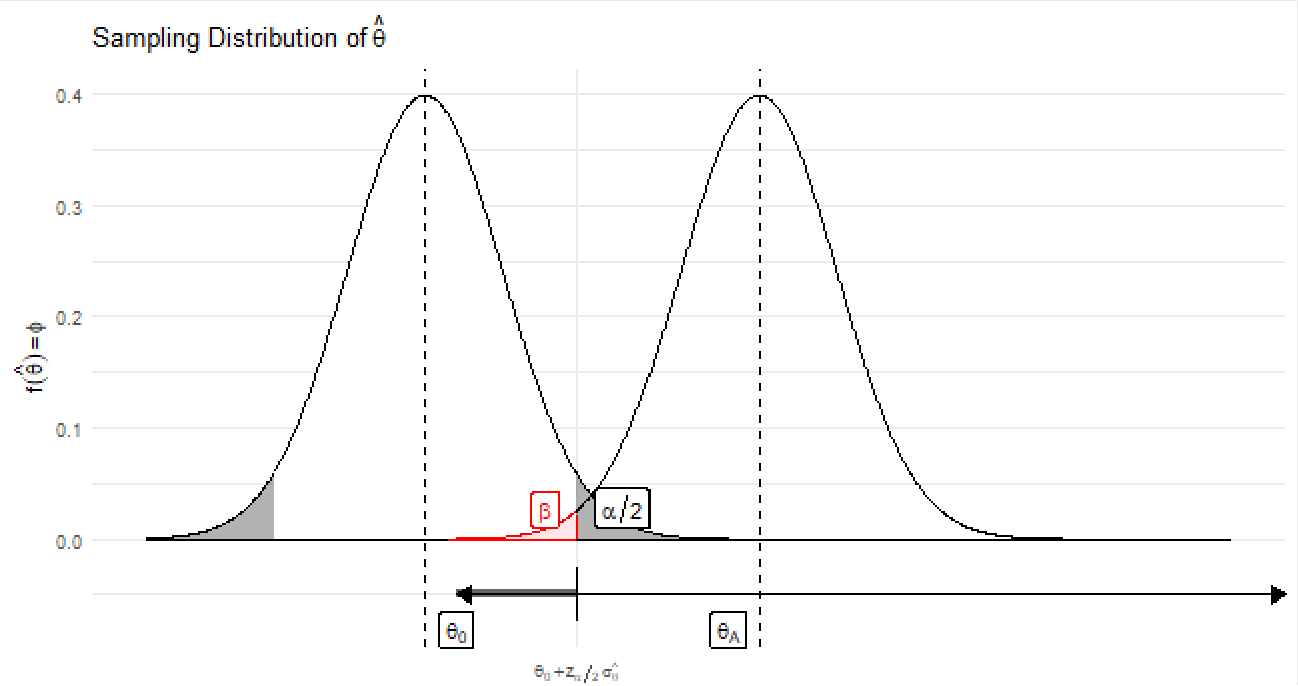

p-values

- Thus far, everything has been very black-or-white: choose α and then say **reject** or **accept**
    - Throwing away potentially useful information
- Report the p-value: **attained significance level**
    - Smallest level of significance α for which the observed data indicate that we should **reject** H_0
- The smaller the p-value, the more compelling is the evidence that the null should be rejected
    - Null should be rejected for **any value of α down to and including the p-value**

$$
\begin{aligned}
\text{Reject } H_0 &\equiv p \leq \alpha \\
\text{Accept } H_0 &\equiv \alpha \leq p
\end{aligned}
$$

Example

- Study compares reaction times of men and women. I.i.d. random samples of 50 men and 50 women drawn to produce this table.

Men	Women
$n_1 = 50$	$n_2 = 50$
$\bar{y}_1 = 3.6$	$\bar{y}_2 = 3.8$
$s_1^2 = .18$	$s_2^2 = .14$

- Do the data present sufficient evidence to suggest a difference between true mean reaction times for men and women? Use $\alpha = 0.05$. Then, report the p-value associated with this conclusion.