Lecture 8 Notes

Tuesday, September 26, 2023 9:23 AM

$\hat{\theta}$: estimator

θ : true parameter

$\bar{Y} = \frac{1}{n}\sum Y_i = \mu$

$B(\hat{\theta}) = E(\hat{\theta}) - \theta$

σ^2

Proposed $S^2 = \frac{1}{n}\left(\sum (Y_i - \bar{Y})^2\right)$

$$B(S^2) = E(S^2) - \sigma^2$$
$$= E\left[\frac{1}{n}\sum (Y_i - \bar{Y})^2\right] - \sigma^2$$
$$= \frac{1}{n}E\left[\sum (Y_i - \bar{Y})^2\right] - \sigma^2$$
$$= \frac{1}{n}E\left[\sum (Y_i^2 - 2Y_i\bar{Y} + \bar{Y}^2)\right] - \sigma^2$$
$$= \frac{1}{n}E\left[\sum Y_i^2 - 2\bar{Y}\sum Y_i + \sum \bar{Y}^2\right]$$

$\bar{Y} = \frac{1}{n}\sum Y_i$

$n\bar{Y} = \sum Y_i$

$$= \frac{1}{n}E\left[\sum Y_i^2 - 2\bar{Y}n\bar{Y} + \sum \bar{Y}^2\right]$$

$\sum \bar{Y}^2 \rightarrow n\bar{Y}^2$

$$= \frac{1}{n}E\left[\sum Y_i^2 - 2n\bar{Y}^2 + n\bar{Y}^2\right]$$
$$= \frac{1}{n}\left(E\left[\sum Y_i^2 - n\bar{Y}^2\right]\right)$$
$$= \frac{1}{n}\left(\overset{\#1}{\boxed{\sum E[Y_i^2]}} - n\overset{\#2}{\boxed{E[\bar{Y}^2]}} +/- E[Y_i]^2\right) \; ; \; +/- E[\bar{Y}]^2$$

$= E[Y_i^2] + E[Y_i]^2 - E[Y_i]^2$

$= \sum \underbrace{E[Y_i^2] - E[Y_i]^2}_{VAR(Y_i)} + E[Y_i]^2$

$n\left(\underbrace{E[\bar{Y}^2] - E[\bar{Y}]^2}_{VAR(\bar{Y})} + E[\bar{Y}]^2\right)$

Combining

$$= \frac{1}{n}\left(\sum\left(\underbrace{VAR(Y_i)}_{} + \underbrace{E[Y_i]^2}_{}\right) - n\left(VAR(\bar{Y}) + E[\bar{Y}]^2\right)\right)$$
$$= \frac{1}{n}\left(\sum(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right)\right)$$
$$= \frac{1}{n}\left(n(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right)\right)$$
$$= \frac{1}{n}\left(n\sigma^2 + \cancel{n\mu^2} - \frac{n\sigma^2}{\cancel{n}} - \cancel{n\mu^2}\right)$$
$$= \frac{1}{n}(n\sigma^2 - \sigma)$$

$$= \frac{1}{n}\left(\sigma^2(n-1)\right)$$

$$= \frac{\sigma^2(n-1)}{n}$$

$$= \frac{n-1}{n}\sigma^2$$

$$\neq \sigma^2$$

$$B(S^2) = E(S^2) - \sigma^2$$

$$= \frac{n-1}{n}\sigma^2 - \sigma^2$$

$$= \left(\frac{n-1}{n} - 1\right)\sigma^2$$

$$= \left(\frac{n-1}{n} - \frac{n}{n}\right)\sigma^2$$

$$= -\frac{1}{n}\sigma^2$$

$$= -\frac{\sigma^2}{n}$$

$$S_U^2 = \frac{\sum (Y_i - \bar{Y})^2}{(n-1)}$$

Interval Estimator:

1. a rule
2. specifying how we use a sample
3. to calculate $\underline{\underline{2}}$ numbers
4. form end points of an interval
5. contains / traps / encloses a parameter θ

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Qualities

1. contain θ
2. to be narrow

Confidence Intervals (CIs)

- upper & lower bounds
- $\hat{\theta}_L$ & $\hat{\theta}_H$
-

Confidence Coefficient

$$\boxed{P(\hat{\theta}_L \leq \theta \leq \hat{\theta}_H) = 1 - \alpha}$$

- The fraction of the time
- in repeated sampling
- that the CI contains θ

$$Z = \frac{\hat{\theta} - \theta}{\sigma_{\hat{\theta}}}$$

$$P(-Z_{\alpha/2} \leq Z \leq Z_{\alpha/2}) = 1 - \alpha$$

$$\downarrow$$

$$P\left(-z_{\alpha/2} \leq \frac{\hat{\theta}-\theta}{\sigma_{\hat{\theta}}} \leq z_{\alpha/2}\right)$$

$$P\left(-z_{\alpha/2}\sigma_{\hat{\theta}} \leq \hat{\theta}-\theta \leq z_{\alpha/2}\sigma_{\hat{\theta}}\right)$$

$$P\left(-z\sigma-\hat{\theta} \leq -\theta \leq z\sigma-\hat{\theta}\right)$$

$$P\left(\hat{\theta}+z\sigma \geq \theta \geq \hat{\theta}-z\sigma\right)$$

$$P\left(\hat{\theta}-z\sigma \leq \theta \leq \hat{\theta}+z\sigma\right)$$

$$\left[\hat{\theta}-1.96\sigma,\ \hat{\theta}+1.96\sigma\right]$$

$$\sigma_{\hat{\theta}} \equiv \sqrt{\mathrm{VAR}(\hat{\theta})} = \sqrt{\frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}$$