# Lecture 9 Notes

Thursday, September 28, 2023 9:26 AM

$$\bar{Y} \equiv \frac{1}{n} \sum Y_i$$

Bias: $E[\bar{Y}] = \mu$

$$VAR(\bar{Y}) = \frac{\sigma^2}{n}$$

$$\sigma_{\bar{Y}} = \frac{\sigma}{\sqrt{n}}$$

$$U_n \equiv Z \equiv \frac{\bar{Y} - \mu}{\sigma_{\bar{Y}}} = \frac{\bar{Y} - \mu}{\sigma / \sqrt{n}}$$

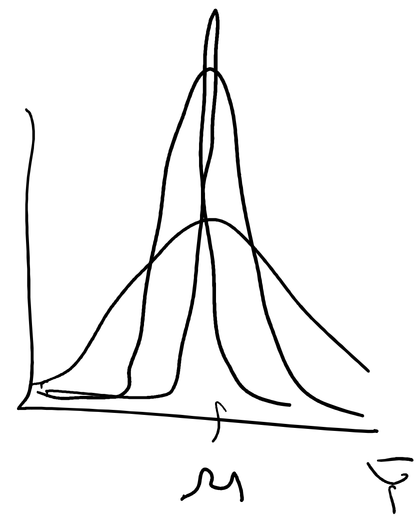
CLT: $F\left(\frac{\bar{Y} - \mu}{\sigma / \sqrt{n}}\right) \xrightarrow{P} \underline{\Phi}$

$$P\left(\bar{Y} - Z_{\alpha/2}\, \sigma_{\bar{Y}} \leq \mu \leq \bar{Y} + Z_{\alpha/2}\, \sigma_{\bar{Y}}\right) = 1 - \alpha$$

$$\sigma_{\bar{Y}} \downarrow \frac{\sigma}{\sqrt{n}} \qquad \sigma_{\bar{Y}} \downarrow \frac{\sigma}{\sqrt{n}}$$

# Consistency

$$\lim_{n\to\infty} P(|\hat{\theta}-\theta|>\varepsilon)=0$$




$$\lim_{n\to\infty} P(|\bar{Y}-\mu|>\varepsilon)=0$$

$$E[\bar{Y}]=\mu$$

$$\lim_{n\to\infty} VAR(\bar{Y})=0$$

$$=\frac{\sigma^2}{n}$$

$$\lim_{n\to\infty}\frac{\sigma^2}{n}=0$$

$$S_u^2=\frac{\sum_i (Y_i-\bar{Y})^2}{n-1}$$

$$\bar{Y}=\frac{1}{n}\sum Y_i$$

$$n\bar{Y}=\sum Y_i$$

$$=\frac{1}{n-1}\left(\sum\left(Y_i^2-2Y_i\bar{Y}+\bar{Y}^2\right)\right)$$

$$=\frac{1}{n-1}\left(\sum Y_i^2-\underline{2n\bar{Y}^2}+\underline{n\bar{Y}^2}\right)$$

$$=\frac{1}{n-1}\left(\left(\sum Y_i^2\right)-n\bar{Y}^2\right) \checkmark$$

$$=\frac{\frac{1}{n-1}\left(\left(\sum Y_i^2\right)-\cancel{n}\bar{Y}^2\right)}{n^2}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum Y_i \xrightarrow{p} \mu$$

$n^2$ $\quad -$

$$= \frac{\frac{1}{n-1}}{\frac{1}{n^2}}\left(\frac{1}{n^2}\sum\left(Y_i^2\right) - \frac{1}{n^2}n\bar{Y}^2\right)$$

$$= \frac{n^{\cancel{2}}}{n-1}\left(\frac{1}{n^{\cancel{2}}}\sum\left(Y_i^2\right) - \bar{Y}^2\right)$$

$$\bar{Y} = \frac{1}{n}\sum Y_i$$

$$= \frac{n}{n-1}\left(\frac{1}{n}\sum Y_i^2 - \bar{Y}^2\right)$$

$$\lim_{n\to\infty} \frac{1}{n}\sum Y_i^2 - \left(\frac{1}{n}\sum Y_i\right)^2$$

$$\lim_{n\to\infty} \frac{n}{n-1}\sigma^2 = 1\sigma^2$$

$$S_n^2 \xrightarrow{D} \sigma^2$$

CLT: $F\left(\frac{\bar{Y}-\mu}{\sigma/\sqrt{n}}\right) \xrightarrow{D} \Phi$ ✓

Prove: $F\left(\frac{\bar{Y}-\mu}{S_n/\sqrt{n}}\right) \xrightarrow{D} \Phi$ ?

# Slutzky's Theorem

If:

1. $F(U_n) \xrightarrow{P} \Phi$
2. $F(W_n) \xrightarrow{P} 1$

Therefore:

$$\frac{F(U_n)}{F(W_n)} \xrightarrow{P} \Phi$$

$$\frac{\bar{Y}-\mu}{S_u/\sqrt{n}} = \sqrt{n}\left(\frac{\bar{Y}-\mu}{S_u}\right)$$

$$= \sqrt{n}\left(\frac{\bar{Y}-\mu}{S_u}\right)\frac{\sigma}{\sigma}$$

$$= \sqrt{n}\left(\frac{\bar{Y}-\mu}{\sigma}\right)\frac{\sigma}{S_u}$$

$$= \sqrt{n}\left(\frac{\frac{\bar{Y}-\mu}{\sigma}}{\frac{S_u}{\sigma}}\right)$$

from CLT: $\sqrt{n}\left(\frac{\bar{Y}-\mu}{\sigma}\right) \xrightarrow{P} \Phi$

Want to prove: $\frac{S_n}{\sigma} \xrightarrow{P} 1$

$$\frac{S_n}{\sigma} = \sqrt{\frac{S_n^2}{\sigma^2}}$$

$$= \sqrt{\frac{S_n^2 \xrightarrow{P} \sigma^2}{\sigma^2 \xrightarrow{P} \sigma^2}}$$

$$= \sqrt{\frac{\sigma^2}{\sigma^2}}$$

$$= \sqrt{1}$$

$$= 1$$

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In practice: large-sample CI

$$P\left(\bar{Y} - z_{\alpha/2}\frac{S_n}{\sqrt{n}} \le \mu \le \bar{Y} + z_{\alpha/2}\frac{S_n}{\sqrt{n}}\right) = 1-\alpha$$

Example 1: