Exercises

8.1 Using the identity

$$(\hat{\theta} - \theta) = [\hat{\theta} - E(\hat{\theta})] + [E(\hat{\theta}) - \theta] = [\hat{\theta} - E(\hat{\theta})] + B(\hat{\theta}),$$

show that

$$\text{MSE}(\hat{\theta}) = E[(\hat{\theta} - \theta)^2] = V(\hat{\theta}) + (B(\hat{\theta}))^2.$$

8.2 **a** If $\hat{\theta}$ is an unbiased estimator for θ, what is $B(\hat{\theta})$?

b If $B(\hat{\theta}) = 5$, what is $E(\hat{\theta})$?

8.3 Suppose that $\hat{\theta}$ is an estimator for a parameter θ and $E(\hat{\theta}) = a\theta + b$ for some nonzero constants a and b.

a In terms of a, b, and θ, what is $B(\hat{\theta})$?

b Find a function of $\hat{\theta}$—say, $\hat{\theta}^\star$—that is an unbiased estimator for θ.

8.4 Refer to Exercise 8.1.

a If $\hat{\theta}$ is an unbiased estimator for θ, how does $\text{MSE}(\hat{\theta})$ compare to $V(\hat{\theta})$?

b If $\hat{\theta}$ is an biased estimator for θ, how does $\text{MSE}(\hat{\theta})$ compare to $V(\hat{\theta})$?

8.5 Refer to Exercises 8.1 and consider the unbiased estimator $\hat{\theta}^\star$ that you proposed in Exercise 8.3.

a Express $\text{MSE}(\hat{\theta}^\star)$ as a function of $V(\hat{\theta})$.

b Give an example of a value of a for which $\text{MSE}(\hat{\theta}^\star) < \text{MSE}(\hat{\theta})$.

c Give an example of values for a and b for which $\text{MSE}(\hat{\theta}^\star) > \text{MSE}(\hat{\theta})$.

8.6 Suppose that $E(\hat{\theta}_1) = E(\hat{\theta}_2) = \theta$, $V(\hat{\theta}_1) = \sigma_1^2$, and $V(\hat{\theta}_2) = \sigma_2^2$. Consider the estimator $\hat{\theta}_3 = a\hat{\theta}_1 + (1 - a)\hat{\theta}_2$.

a Show that $\hat{\theta}_3$ is an unbiased estimator for θ.

b If $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent, how should the constant a be chosen in order to minimize the variance of $\hat{\theta}_3$?

8.7 Consider the situation described in Exercise 8.6. How should the constant a be chosen to minimize the variance of $\hat{\theta}_3$ if $\hat{\theta}_1$ and $\hat{\theta}_2$ are not independent but are such that $\text{Cov}(\hat{\theta}_1, \hat{\theta}_2) = c \neq 0$?

8.8 Suppose that Y_1, Y_2, Y_3 denote a random sample from an exponential distribution with density function

$$f(y) = \begin{cases} \left(\dfrac{1}{\theta}\right) e^{-y/\theta}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Consider the following five estimators of θ:

$$\hat{\theta}_1 = Y_1, \qquad \hat{\theta}_2 = \frac{Y_1 + Y_2}{2}, \qquad \hat{\theta}_3 = \frac{Y_1 + 2Y_2}{3}, \qquad \hat{\theta}_4 = \min(Y_1, Y_2, Y_3), \qquad \hat{\theta}_5 = \overline{Y}.$$

a Which of these estimators are unbiased?

b Among the unbiased estimators, which has the smallest variance?

***8.16** Suppose that $Y_1, Y_2, \dots, Y_n$ constitute a random sample from a normal distribution with parameters μ and σ^2.[1]

a Show that $S = \sqrt{S^2}$ is a biased estimator of σ. [*Hint:* Recall the distribution of $(n-1)S^2/\sigma^2$ and the result given in Exercise 4.112.]

b Adjust S to form an unbiased estimator of σ.

c Find an unbiased estimator of $\mu - z_\alpha\sigma$, the point that cuts off a lower-tail area of α under this normal curve.

8.17 If Y has a binomial distribution with parameters n and p, then $\hat{p}_1 = Y/n$ is an unbiased estimator of p. Another estimator of p is $\hat{p}_2 = (Y+1)/(n+2)$.

a Derive the bias of $\hat{p}_2$.

b Derive $\text{MSE}(\hat{p}_1)$ and $\text{MSE}(\hat{p}_2)$.

c For what values of p is $\text{MSE}(\hat{p}_1) < \text{MSE}(\hat{p}_2)$?

8.18 Let $Y_1, Y_2, \dots, Y_n$ denote a random sample of size n from a population with a uniform distribution on the interval $(0, \theta)$. Consider $Y_{(1)} = \min(Y_1, Y_2, \dots, Y_n)$, the smallest-order statistic. Use the methods of Section 6.7 to derive $E(Y_{(1)})$. Find a multiple of $Y_{(1)}$ that is an unbiased estimator for θ.

8.19 Suppose that $Y_1, Y_2, \dots, Y_n$ denote a random sample of size n from a population with an exponential distribution whose density is given by

$$f(y) = \begin{cases} (1/\theta)e^{-y/\theta}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

If $Y_{(1)} = \min(Y_1, Y_2, \dots, Y_n)$ denotes the smallest-order statistic, show that $\hat{\theta} = nY_{(1)}$ is an unbiased estimator for θ and find $\text{MSE}(\hat{\theta})$. [*Hint:* Recall the results of Exercise 6.81.]

***8.20** Suppose that $Y_1,\ Y_2,\ Y_3,\ Y_4$ denote a random sample of size 4 from a population with an exponential distribution whose density is given by

$$f(y) \begin{cases} (1/\theta)e^{-y/\theta}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

a Let $X = \sqrt{Y_1Y_2}$. Find a multiple of X that is an unbiased estimator for θ. [*Hint:* Use your knowledge of the gamma distribution and the fact that $\Gamma(1/2) = \sqrt{\pi}$ to find $E(\sqrt{Y_1})$. Recall that the variables Y_i are independent.]

b Let $W = \sqrt{Y_1Y_2Y_3Y_4}$. Find a multiple of W that is an unbiased estimator for θ^2. [Recall the hint for part (a).]

8.3 Some Common Unbiased Point Estimators

Some formal methods for deriving point estimators for target parameters are presented in Chapter 9. In this section, we focus on some estimators that merit consideration on the basis of intuition. For example, it seems natural to use the sample mean

1. Exercises preceded by an asterisk are optional.