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Quantitative Research in Political Science I

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Partialling Out[1]

Another way to write the k'th element of the vector of OLS estimates $\widehat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$ is:

$$\widehat{\beta}_k = \frac{cov\left(y_i, \widetilde{x}_{ki}\right)}{var\left(\widetilde{x}_{ki}\right)},$$

where $\widetilde{x}_{ki}$ is the residual of the regression of x_k on all the other x's in the model, i.e.,

$$\begin{aligned} x_{ki} &= \widehat{\delta}_0 + \sum_{j \neq k} \widehat{\delta}_j x_{ji} + \widetilde{x}_{ki} \\ \widetilde{x}_{ki} &= x_{ki} - \widehat{\delta}_0 - \sum_{j \neq k} \widehat{\delta}_j x_{ji} \\ \widetilde{x}_{ki} &= x_{ki} - \widehat{x}_{ki} \end{aligned}$$

This shows us that each coefficient in a multivariate regression is the bivariate slope coefficient for the corresponding regressor after **partialling out** the variation of all the other covariates with y and their covariation with x_k.

A [plain-language] proof: re-write y_i as:

$$y_i = \widehat{\beta}_0 + \widehat{\beta}_1 x_{1i} + ... + \widehat{\beta}_k x_{ki} + ... + \widehat{\beta}_K x_{Ki} + \widehat{u}_i$$

Now note that:

- because $\widetilde{x}_k$ is a linear combination of all the x's, it is by construction uncorrelated with the $\widehat{u}$;
- because $\widetilde{x}_k$ is a residual from a regression on all the other x's in the model, it is by construction uncorrelated with these x's
- and for the same reason, $cov\left(\widetilde{x}_{ki}, x_{ki}\right)$ is just $var\left(\widetilde{x}_{ki}\right)$
- thus $cov\left(y_i, \widetilde{x}_{ki}\right)$ simplifies to

$$\begin{aligned} cov\left(y_i, \widetilde{x}_{ki}\right) &= cov\left(\widehat{\beta}_k x_{ki}, \widetilde{x}_{ki}\right) \\ &= \widehat{\beta}_k var\left(\widetilde{x}_{ki}\right) \end{aligned}$$

so

$$\widehat{\beta}_k = \frac{cov\left(y_i, \widetilde{x}_{ki}\right)}{var\left(\widetilde{x}_{ki}\right)}.$$

[1] borrowing heavily from Angrist and Pischke's *Mostly Harmless Econometrics* (pp. 35-36).

Note that the magnitude of $\widehat{\beta}_k$ is larger to the extent that y varies with the variation in x_k after accounting for x_k's covariation with the other x's in the model. This nicely corresponds to our notion of $\widehat{\beta}_k$ as an estimate of the *ceteris paribus*/all things being equal relationship between y and x_k.

It can also be shown that

$$\widehat{\beta}_k = \frac{cov\left(\widetilde{y}_{ki}, \widetilde{x}_{ki}\right)}{var\left(\widetilde{x}_{ki}\right)},$$

where $\widetilde{y}_{ki}$ is the residual from the regression of y on all the x's *except* x_k in the model.

One thing that's nice about both of these formulations is that in a bivariate plot of either y on $\widetilde{x}_k$ or $\widetilde{y}_k$ on $\widetilde{x}_k$, the slope of the (bivariate) best-fit line is, in fact, $\widehat{\beta}_k$. Here's an example from my book. I'm interested in the relationship between the extent to which a political party's "owned" issues are salient in a U.S. presidential election campaign and how well the party performs in the election. The table displays a series of regression equations modeling y (election performance) as a function of x_k (the salience of the party's owned issues) and other potential confounding x's. The figures plot (first) y on x and (second) $\widetilde{y}_k$ on $\widetilde{x}_k$.

TABLE 3.3. *Issue salience, economic conditions, and incumbency in presidential elections, 1960–2012*

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	DV: Republican Candidate Share of Two-Party Vote			
	I	II	III	IV (1980 election omitted)
Republican issue salience advantage	.22** (.06)		.07 (.05)	.10+ (.05)
Republican incumbent		13.35* (5.28)	13.18* (4.96)	7.34 (5.00)
Change in GDP, Q4 to Q2 of election year (percentage points)		−2.54*** (.50)	−2.02* (.79)	−4.40+ (2.04)
Republican incumbent x change in GDP		4.69*** (.71)	3.71** (1.06)	5.87* (1.94)
Number of consecutive terms Republican has held presidency		−5.12** (1.35)	−4.76** (1.41)	−4.59* (1.37)
Intercept	50.18*** (1.19)	44.67*** (1.91)	44.61*** (1.97)	50.37*** (3.18)
N	14	14	14	13
R^2	.47	.84	.87	.88
Adjusted R^2	.42	.77	.79	.80
SEE	4.4	2.8	2.7	2.7
p–value of coefficient on issue salience advantage term	.005		.25	.06

OLS. Estimates significantly different from zero at +p<.10; *p<.05; **p<.01; ***p<.001 (two-tailed tests, robust standard errors).

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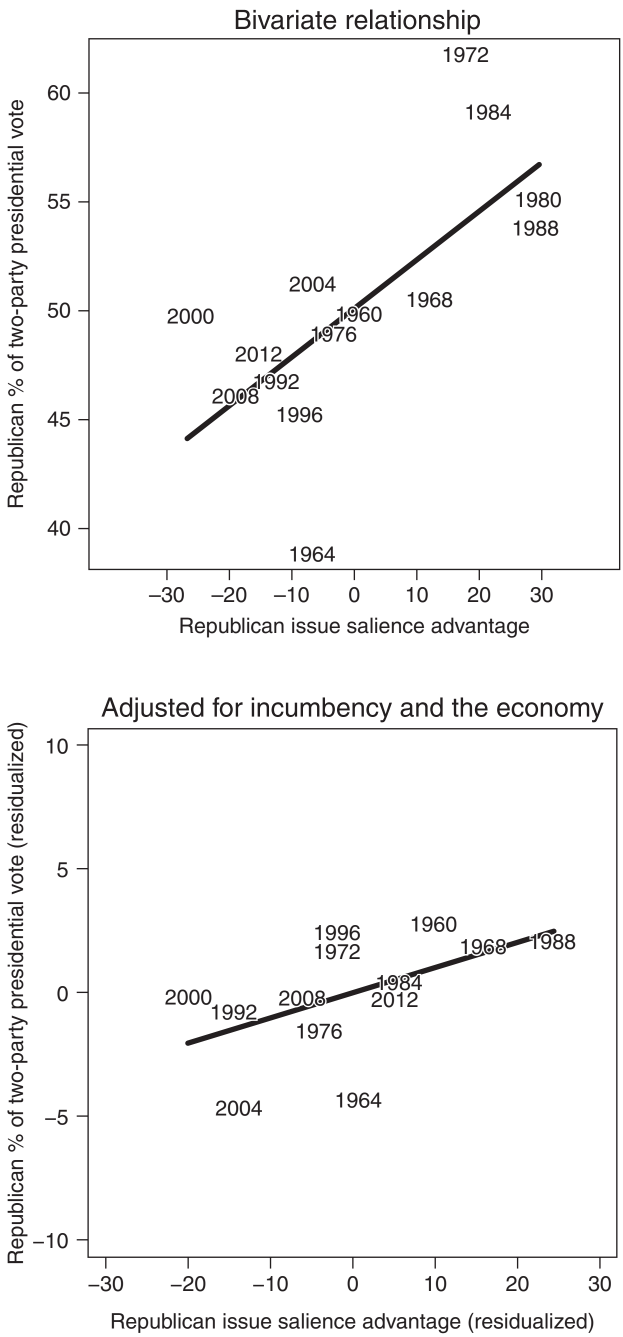

FIGURE 3.5. Issue salience, issue ownership, and vote share in U.S. presidential elections, 1960–2012.
Source: Estimates in Table 3.3.