# Factorización QR

$A = QR$ , donde $A$ debe ser una matriz no singular

$Q$ = Matriz ortogonal y $Q^T Q = I$
$R$ = Matriz triangular superior

$$A = \underbrace{[\, a_1 \mid a_2 \mid \dots \mid a_m \,]}_{\text{Vectores columna}}$$

$u_1 = a_1$ y $e_1 = \dfrac{u_1}{\|u_1\|}$

$u_2 = a_2 - (a_2 e_1) e_1$ ; donde $a_2 e_1 \Rightarrow$ Producto interno de dos vectores
$e_2 = \dfrac{u_2}{\|u_2\|}$

Ejemplo $a_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ ; $e_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

$a_2 \cdot e_1 = (1 \cdot 2) + (0 \cdot 3) + (1 \cdot 1)$
$= 2 + 1 = 3$

$\vdots$

Productos internos que van siendo obtenidos.

$$A = \underbrace{[\, e_1 \mid e_2 \mid \dots \mid e_m \,]}_{Q} \underbrace{\begin{bmatrix} a_1 \cdot e_1 & a_2 \cdot e_1 & \dots & a_m \cdot e_m \\ 0 & a_2 e_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & a_m e_m \end{bmatrix}}_{R}$$

- Solo bastaría calcular los diagonales de la matriz $R$

Ejemplo $A = \begin{pmatrix} -1 & 3 \\ 1 & 5 \end{pmatrix}$

$V_1 = a_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ ya que $\|V_1\| \Rightarrow \sqrt{V_1 \cdot V_1} = \sqrt{v^T v}$

$$e_1 = \frac{V_1}{\|V_1\|} = \frac{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}{\sqrt{\sum \begin{pmatrix} -1 \\ 1 \end{pmatrix}^2}} = \frac{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}{\sqrt{-1^2 + 1^2}} \Rightarrow \frac{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}{\sqrt{2}} \Rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$V_2 = a_2 - (e_1 \cdot a_2)\, e_1$$

$$= \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \left( \left( \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right) \cdot \begin{pmatrix} 3 \\ 5 \end{pmatrix} \right) \left( \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \left( \frac{1}{\sqrt{2}} (-3 + 5) \right) \left( \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$e_2 = \frac{V_2}{\|V_2\|} = \frac{1}{4\sqrt{2}} \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Así la matriz $Q$ es

$$Q = (e_1 \mid e_2) = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$R = Q^T A$$

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 1 & 5 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1+1 & -1+3 \\ -1+1 & 3+5 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 2 \\ 0 & 8 \end{pmatrix} = \sqrt{2} \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}$$

Ejemplo 2

Descomposición QR.

1.- Calcular la descomposición QR de la siguiente matriz

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

(Ejemplo sacado del libro "Linear Algebra with applications, 3rd Edition" por Steven J. Leon)

Desarrollo: Gram-Schmidth

$A = (a_1, a_2, a_3)$ y la matriz $Q = (q_1, q_2, q_3)$ y la matriz $R$: (dimensión 3)

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$$

i) $v_1 = a_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ ; $r_{11} = \underbrace{\|v_1\|}_{\sqrt{1^2+0^2+1^2} = \sqrt{2}} = \sqrt{2}$ ; $q_1 = \dfrac{v_1}{\|v_1\|} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

ii) $v_2 = a_2 - \underbrace{(q_1 a_2)}_{r_{12}} q_1 \Rightarrow \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \left(\dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\right) \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$r_{12} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$

así $r_{22} = \|v_2\| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{\begin{pmatrix}1\\1\\-1\end{pmatrix}}{\sqrt{\sum\begin{pmatrix}1\\1\\-1\end{pmatrix}}} = \frac{\begin{pmatrix}1\\1\\-1\end{pmatrix}}{\sqrt{1^2+1^2-1^2}} = \frac{\begin{pmatrix}1\\1\\-1\end{pmatrix}}{\sqrt{3}} \Rightarrow \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\-1\end{pmatrix}$$

iii) $v_3 = a_3 - \underbrace{(q_1 a_3)}_{r_{13}} q_1 - \underbrace{(q_2 a_3)}_{r_{23}} q_2$

$r_{13} = \frac{1}{\sqrt{2}}$ y $r_{23} = 0$, así

$$v_3 = \begin{pmatrix}0\\1\\1\end{pmatrix} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} - 0 = \frac{1}{2}\begin{pmatrix}-1\\2\\1\end{pmatrix}$$

y finalmente $r_{33} = \|v_3\| = \frac{\sqrt{6}}{2}$ y $q_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\2\\1\end{pmatrix}$

Lo que finalmente nos da:

$$Q = \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6}\\ 0 & 1/\sqrt{3} & 2/\sqrt{6}\\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6}\end{pmatrix} \quad y \quad R = \begin{pmatrix}\sqrt{2} & \sqrt{2} & 1/\sqrt{2}\\ 0 & \sqrt{3} & 0\\ 0 & & \sqrt{6}/2\end{pmatrix}$$