

# Math 1560: Number Theory *Lecture Notes*

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## §0 January 27, 2022

### §0.1 Course Logistics

- Mostly refer to syllabus for any information that you might need.
- Midterm is planned for March 17.
- Final exam schedule can be found on CAB.

### §0.2 Introduction to Number Theory

Number theory can be split into two branches: analytic number theory and algebraic number theory.

*What is number theory?* Number theory is the study of integers and their analogues in algebraic number fields.

Prime numbers are a key focus of number theory, and the study of different properties of primes constitutes different fields of number theory:

- i. The study of their distributional properties, which is analytic number theory.
- ii. As building blocks for algebraic numbers, which is algebraic number theory.

#### §0.2.1 Examples of Analytic Number Theory

Here are some examples of analytic number theory and their statements:

- Prime Number Theorem
- Twin Prime Conjecture
- Goldbach's conjecture

**Theorem 0.1** (Prime Number Theorem)

Let  $\pi(x)$  be the number of primes between 1 and  $x$ , then

$$\lim_{x \rightarrow \infty} \frac{\pi(x)}{x / \ln(x)} = 1.$$

**Conjecture 0.2** (Twin Prime Conjecture)

Twin primes are pairs of primes  $p, q$  of the form  $q = p + 2$ . Examples include  $(3, 5), (11, 13), \dots$ . The conjecture postulates that there are infinitely many twin primes.

**Conjecture 0.3** (Goldbach's conjecture)

Any positive even integer greater than 2 can be written as the sum of 2 primes.

**§0.2.2 Examples of Algebraic Number Theory**

Analyzing the factorization (rings of integers) of number fields is one topic of algebraic number theory.

**Example 0.4**

2 is prime (irreducible) in  $\mathbb{Z}$ .

Yet 2 is not prime in  $\mathbb{Z}[i]$  (the Gaussian integers). This is because

$$2 = \underbrace{(1+i)(1-i)}_{\text{associates}}$$

we have that  $(1+i) = i(1-i)$ . We also note the property that the principal ideals  $(2) = (1+i)^2$  are equal.

In this example, we say that 2 “ramifies” in the ring of integers.

Fermat's Last Theorem is another such example.

*Recall:* that a *Pythagorean triple* is a triple of the form  $a, b, c \in \mathbb{Z}_+$  such that

$$a^2 + b^2 = c^2$$

Are there examples of such numbers with different exponents (say,  $k^{\text{th}}$  powers for  $k \geq 3$ )?

**Theorem 0.5** (Fermat's Last)

There are no positive integers  $a, b, c \in \mathbb{Z}_+$  satisfying

$$a^k + b^k = c^k$$

for  $k \geq 3$ .

The answer is no! (Proved by Andrew Wiles)

**Conjecture 0.6** (*abc Conjecture, informally*)

We say *powerful numbers* are positive integers whose prime factorization contains relatively few distinct primes (appropriately weighted) with an exponent of 1.

**Example**

$2^{10}3^7$  is powerful,  $2^{10}3^75$  is powerful, 1 is powerful.

If  $a, b$  are *very powerful* coprime numbers, then  $a + b$  is predicted to be *not powerful*.

**Example 0.7**

Consider  $2^{10}$  and  $3^{15}$ . We have

$$2^{10} + 3^{15} = 14,349,931 = \underbrace{31 \cdot 462 \cdot 901}_{\text{not powerful}}$$

What about another example, like  $3^{15} + 5$ ? The *abc* conjecture also predicts that this number is not so powerful...<sup>1</sup>

## §1 February 1, 2022

Happy Lunar  
New Year! 🏮

(Thanks Qinan and Andrew for allowing me to shamelessly copy their notes.)

### §1.1 Divisibility and Factorization

We start with some commonly used notation:

**Definition 1.1** (Divisibility)

We use  $a \mid b$  to mean “ $a$  divides  $b$ ” and  $a \nmid b$  to mean “ $a$  does not divide  $b$ ”.

Now for a series of definitions:

**Definition 1.2** (Primality)

A positive integer  $p \geq 2$  is said to be prime if its only positive divisors are 1 and  $p$ .

<sup>1</sup>After lecture Jiahua: It's a prime!?

**Definition 1.3 (Positive Integers)**

$\mathbb{Z}_+$  will denote the positive integers.

**Definition 1.4 (Order)**

For a nonzero  $n \in \mathbb{Z}$  and a prime  $p$ , there is a nonnegative integer  $a$  such that  $p^a \mid n$  but  $p^{a+1} \nmid n$ . This number  $a$  is called the order of  $n$  at  $p$ , denoted by  $\text{ord}_p n$ .

For  $n = 0$ , we set  $\text{ord}_p 0 = \infty$ . We also have  $\text{ord}_p n = 0 \Leftrightarrow p \nmid n$ .

We prove a lemma as warm-up:

**Lemma 1.5 (Existence of Factorization)**

Every nonzero integer can be written as a product of primes.

*We make an exception for  $-1$ . The empty product is 1 so 1 is fine.*

*Proof.* Suppose for the sake of contradiction otherwise, that some nonzero integer can be written as a product of primes. Let  $N$  be the smallest integer greater than 2 that cannot be written as a product of primes.

$N$  had better not be a prime number itself (since then it would be a product of itself). Then we can write  $N = a \cdot b$  where  $1 < a, b < N$ .

Since we took  $N$  as the least such number that cannot be written as a product of primes,  $a$  and  $b$  which are less than  $N$  can be written as a product of primes. Then  $N$  is a product of primes since  $a$  and  $b$  individually are. This is a contradiction! Thus it had better be the case that *every* nonzero integer can be written as a product of primes.  $\square$

This is the theorem we will eventually work toward proving:

**Theorem 1.6 (Unique Factorization)**

Every nonzero integer  $n$  yields a *unique* prime factorization

$$n = (-1)^\varepsilon \cdot \prod_p p^{a(p)}, a(p) \geq 0$$

where  $\varepsilon = 0$  or  $1$ , and  $\varepsilon, a(p)$  are uniquely determined by  $n$ . Moreover, we note that  $a(p) = \text{ord}_p n$ .

## §1.2 Euclidean and Principal Ideal Domains

Before this proof, we first recall a conclusion from Math 1530:

### Lemma 1.7 (Division Lemma)

If  $a, b \in \mathbb{Z}$  and  $b > 0$ , then there exists  $q, r \in \mathbb{Z}$  such that

$$a = bq + r$$

with  $0 \leq r < b$ .

*Proof.* Consider the set

$$S = \{a - xb \mid x \in \mathbb{Z}\}$$

We note that  $S$  contains *some* positive elements. Let  $r = a - qb$  be the least nonnegative element of  $S$ .

We claim that  $0 \leq r < b$ . Suppose for the sake of contradiction otherwise, then  $r = a - qb \geq b$  gives  $a - qb - b \leq 0$  and  $a - (q+1)b \leq 0$ . Which is a contradiction since we took  $r$  to be the least nonnegative element in  $S$  and we've found such smaller element  $a - (q+1)b$ .

Then it had better be that  $0 \leq r < b$  for some  $r, q \in \mathbb{Z}$ . □

### Corollary 1.8

$\mathbb{Z}$  is a Euclidean domain, with a Euclidean function given by [lemma 1.7](#).

### Definition 1.9 (Euclidean Domain)

Let  $R$  be an integral domain.  $R$  is a Euclidean domain if there exists a function  $\lambda : R \setminus \{0\} \rightarrow \mathbb{N}$  such that if  $a, b \in R$  with  $b \neq 0$ , then there exists some  $c, d \in R$  with the property that  $a = cb + d$  with  $d = 0$  or  $\lambda(d) < \lambda(b)$ .

### Example 1.10

$\mathbb{Z}$  is a Euclidean domain with  $\lambda$  function given in [lemma 1.7](#).

$R[x]$  for field  $R$  is also a Euclidean domain, with  $\lambda = \deg$ .

### Proposition 1.11

If  $R$  is a Euclidean domain, then  $R$  is a principal ideal domain. That is, if  $I \subseteq R$  is an ideal, then  $\exists a \in R$  such that  $I = Ra = \{ra \mid r \in R\}$ .

*Proof.* Assume WLOG that  $I$  is not the trivial ideal  $I \neq (0)$ . Let  $0 \neq a \in I$  such that  $\lambda(a) \leq \lambda(b) \forall b \in I, b \neq 0$ .

We claim that  $I = (a) = Ra$ .

We know that  $Ra \subseteq I$  since  $I$  is an ideal. Let  $b \in I$ . Then  $\exists c, d \in R$  such that  $b = ca + d$  where  $d = 0$  or  $\lambda(d) < \lambda(a)$ . Now we have  $d = b - ca \in I$ , so we can't have  $\lambda(d) < \lambda(a)$ . Thus  $d = 0$ , so  $b = ca \in Ra$ .

Hence we have  $I \subseteq Ra$ . Together, we conclude that  $I = Ra$ . □

### Definition 1.12 (Principal Ideals, PIDs)

If  $I = (a)$  for some  $a \in I$ , then  $I$  is said to be a principal ideal.

$R$  is a principal ideal domain (PID) if every ideal of  $R$  is principal.

Here are some important properties of PIDs:

1. Nonunit irreducible elements are exactly the prime elements in  $R$ .

*Recall:*  $p \in R$  is irreducible if  $a \mid p \Rightarrow a$  is either a unit or an associate of  $p$ .

$p \in R$  is prime if  $p \mid ab \Rightarrow p \mid a$  or  $p \mid b$  and  $p$  is a nonzero, nonunit of  $R$ .

2. GCDs always exist in PIDs.

## §1.3 Unique Prime Factorization

We're nearly ready to prove unique factorization, after a lemma:

### Lemma 1.13

Suppose  $p$  is a prime, and  $a, b \in \mathbb{Z}$ . Then  $\text{ord}_p(ab) = \text{ord}_p a + \text{ord}_p b$ .

*Proof.* WLOG, assume  $a, b \neq 0$ . We let

$$\alpha = \text{ord}_p a$$

$$\beta = \text{ord}_p b$$

Then we have

$$a = p^\alpha \cdot c \text{ where } p \nmid c$$

$$b = p^\beta \cdot d \text{ where } p \nmid d$$



Thus,  $ab = p^{\alpha+\beta} \cdot cd$ . We have that  $p \nmid cd$  since  $p \nmid c$  and  $p \nmid d$  (we rely on the fact that if  $p$  is irreducible,  $p$  is prime). Thus we have that  $\text{ord}_p(ab) = \alpha + \beta$ .  $\square$

*Proof.* (of [theorem 1.6](#), that  $\mathbb{Z}$  is a UFD). Recall that for a nonzero  $n \in \mathbb{Z}$ , we write

$$n = (-1)^\varepsilon \prod_p p^{a(p)}, \text{ where } \varepsilon = 0 \text{ or } 1 \text{ and } a(p) \geq 0$$

Given a positive prime  $q$ , we take  $\text{ord}_q$  of both sides. By [lemma 1.13](#), this yields

$$\text{ord}_q n = \varepsilon \cdot \text{ord}_q(-1) + \sigma_p a(p) \text{ord}_q(p)$$

Since we have that  $\text{ord}_q(-1) = 0$  and  $\text{ord}_q(p) = 0, \forall p \neq q$ , we've uniquely determined  $a(q)$  since  $\text{ord}_q(n) = a(q)$ . That is,  $a(q)$  is *uniquely determined* for all primes  $q$ . So  $n$  has a *unique* prime factorization.  $\square$

## §1.4 Greatest Common Divisors

### Definition 1.14

Let  $R$  be an integral domain. Then  $d \in R$  is said to be a gcd of two elements  $a, b$  if

- i)  $d \mid a$  and  $d \mid b$ ,
- ii) if  $d' \mid a$  and  $d' \mid b$ , then  $d' \mid d$ .

**Remark.** An aside for ring theory enthusiasts: gcd domains are a class of rings more general than PIDs or UFDs.

We will denote  $(a, b)$  as the gcd of  $a$  and  $b$ .

**Caution, however!** gcd's are only unique up to units.

### Example

$-5$  and  $5$  are both gcds of  $-5$  and  $10$  since  $-1$  is a unit.

We will make the convention that the gcd of 2 integers is the positive gcd, that is,  $(-5, 10) = 5$ .

An edge case is that  $\text{gcd}(0, 0) = 0$ .

## §2 February 3, 2022

### §2.1 Arithmetic Functions

We look at arithmetic functions and how they act on prime numbers:

#### Definition 2.1 (Arithmetic Function)

An arithmetic function is a function  $f : \mathbb{Z}_+ \rightarrow \mathbb{C}$ .

(Typically, these are integer valued.)

#### Example 2.2

We have some examples of arithmetic functions:

- Euler  $\phi$  function.
- $\tau(n)$ , the counting function. It takes a positive integer and counts the number of positive divisors of  $n$ .

$$\tau(n) = \sum_{d|n} 1$$

- $\sigma(n)$ , the sum of divisors function. It is the sum over all the positive divisors of  $n$ .

$$\sigma(n) = \sum_{d|n} d$$

We have some properties of these functions, like *multiplicative*, *completely multiplicative*, *additive*, *completely additive*.

#### Definition 2.3 (Multiplicativity)

An arithmetic function  $f$  is multiplicative if

$$f(mn) = f(m)f(n) \quad \text{whenever } (m, n) = 1$$

$f$  is said to be totally or completely multiplicative if

$$f(mn) = f(m)f(n) \quad \forall m, n \in \mathbb{Z}_+$$

regardless of coprimality.

If  $f$  is multiplicative and  $n_1, \dots, n_k$  are positive pairwise coprime integers, then

$$f(n_1 \dots n_k) = f(n_1)f(n_2) \dots f(n_k).$$

A particular case that is useful is when we write

$$n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$$

so that assuming multiplicativity, we have that

$$f(n) = f(p_1^{e_1})f(p_2^{e_2}) \dots f(p_k^{e_k})$$

A common type of arithmetic function is a summatory function, namely a function  $f$  of the form

$$f(n) = \sum_{d|n} g(d), \quad \text{where } g \text{ is some arithmetic function.}$$

*Food for thought:* how special are summatory functions within the set of all arithmetic functions?

A special property of summatory functions is that they “inherit multiplicativity”.

**Lemma 2.4**

If  $g$  is a multiplicative function, and

$$f(n) = \sum_{d|n} g(d) \quad \forall n,$$

then  $f$  itself is multiplicative.

*Proof.* Suppose  $m, n \in \mathbb{Z}_+$  are coprime positive integers.

The divisors  $d$  of  $mn$  are the products  $a \cdot b$  where  $a \mid m$  and  $b \mid n$ . Each such pair  $a, b$  yields a uniquely determined product  $d = a \cdot b$ . Conversely, since  $(m, n) = 1$ , each divisor  $d$  of  $mn$  determines a unique divisor  $a = \gcd(d, m)$  and  $b = \gcd(d, n)$  so that  $d = a \cdot b$ .

Thus there is a *bijection* between divisors of  $mn$  and  $m, n$  separately

$$d \mid mn \longleftrightarrow (a \mid m, b \mid n)$$

Thus we have

$$\begin{aligned} f(m \cdot n) &= \sum_{d|mn} g(d) \\ &= \sum_{a|m} \sum_{b|n} g(ab) \\ &= \sum_{a|m} \sum_{b|n} g(a)g(b) \\ &= \left( \sum_{a|m} g(a) \right) \left( \sum_{b|n} g(b) \right) = f(m) \cdot f(n) \end{aligned}$$

Thus completes the proof that  $f$  is multiplicative.  $\square$

*Recall:* The functions introduced earlier

$$\tau(n) = \sum_{d|n} 1 \quad \sigma(n) = \sum_{d|n} d$$

So  $\tau$  is the summatory function of the constant 1 functions, and  $\sigma$  is the summatory function of the identity function. We know that the constant 1 function and the identity function are both completely multiplicative, so  $\sigma$  and  $\tau$  are multiplicative functions.

The implication of which is that it suffices to apply  $\tau$  and  $\sigma$  on prime powers and multiply.

Let  $p$  be a prime. Then

$$\tau(p^e) = e + 1 \quad (\text{from } p^0 \text{ to } p^e).$$

We also have

$$\sigma(p^e) = 1 + p + p^2 + \cdots + p^e = \frac{p^{e+1} - 1}{p - 1}$$

Therefore, if  $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ , then

$$\begin{aligned} \tau(n) &= \prod_{i=1}^k (e_i + 1) \\ \sigma(n) &= \prod_{i=1}^k \left( \frac{p_i^{e_i+1} - 1}{p_i - 1} \right). \end{aligned}$$

**Remark 2.5.** There are higher order divisor functions

$$\sigma_k(n) = \sum_{d|n} d^k$$

so  $\sigma_0 = \tau, \sigma_1 = \sigma, \dots$

## §2.2 Review of $\mathbb{Z}/n\mathbb{Z}$ and its units

### Definition 2.6 (Modular Congruence)

If  $a, b, m \in \mathbb{Z}$ ,  $m \neq 0$ , we say that  $a$  is congruent to  $b$  modulo  $m$  if  $m \mid b - a$ . We write

$$a \equiv b \pmod{m}, \text{ or more simply } a \equiv b \pmod{m}$$

Congruence mod  $m$  is an equivalence relation on  $\mathbb{Z}$ . If  $a \in \mathbb{Z}$ ,  $\bar{a}$  denotes the set of integers congruent to  $a \pmod{m}$ , i.e.  $\bar{a} = \{a + km \mid k \in \mathbb{Z}\}$ .

**Definition 2.7** ( $\mathbb{Z}/m\mathbb{Z}$ , Residues mod  $m$ )

The set of congruence classes mod  $m$  is denoted  $\mathbb{Z}/m\mathbb{Z}$ . This is a quotient ring of the ring of integers  $\mathbb{Z}$ .

If  $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_m$  form a complete set of congruence classes mod  $m$ , then the set of integers  $\{a_1, a_2, \dots, a_m\}$  is called a complete set of residues mod  $m$ .

$\mathbb{Z}/m\mathbb{Z}$  can be endowed with the structure of a commutative ring by setting

$$\begin{aligned}\bar{a} + \bar{b} &= \overline{a + b} \\ \text{and } \bar{a} \cdot \bar{b} &= \overline{ab},\end{aligned}$$

and proving that this is well-defined as ring operations.

**Proposition 2.8**

The set of units in  $\mathbb{Z}/m\mathbb{Z}$  is exactly

$$\{\bar{a} \mid (a, m) = 1\}$$

*Proof.* Let  $\bar{a} \in \mathbb{Z}/m\mathbb{Z}$ , then

$$\begin{aligned}\exists \bar{b} \in \mathbb{Z}/m\mathbb{Z} \text{ s.t. } \bar{b} \cdot \bar{a} &\equiv 1 \pmod{m} \\ \iff \exists b, n \in \mathbb{Z} \text{ s.t. } ba - mn &= 1\end{aligned}$$

Then by Bézout's identity...

$$\iff (a, m) = 1$$

□

**§2.3 The Euler  $\phi$  Function**

For  $n \in \mathbb{Z}_+$ ,  $\phi(n)$  is defined to be the number of integers  $1 \leq m \leq n$  coprime to  $n$ .

**Example 2.9**

We have some examples of the Euler  $\phi$  functions:

$$\begin{aligned}\phi(1) &= 1 \\ \phi(p) &= p - 1 \text{ for any prime } p\end{aligned}$$

Let  $e \geq 1$ ,

$$\phi(p^e) = p^e - p^{e-1} \text{ for prime powers, we exclude multiples of } p$$

Wouldn't be great if  $\phi$  were multiplicative? It is!

**Theorem 2.10**

If  $(m, n) = 1$ , then  $\phi(mn) = \phi(m)\phi(n)$ .

*Proof.* By the Chinese Remainder Theorem<sup>2</sup>,  $\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$  if  $(m, n) = 1$ .

Taking the unit groups on both sides, we have

$$(\mathbb{Z}/mn\mathbb{Z})^\times \cong (\mathbb{Z}/m\mathbb{Z})^\times \times (\mathbb{Z}/n\mathbb{Z})^\times$$

and the Euler  $\phi$  function is simply measuring the order of said unit groups ( $\phi(n) = |(\mathbb{Z}/n\mathbb{Z})^\times|$ ).  $\square$

Here is an important fact about the Euler  $\phi$  function:

**Proposition 2.11**

We have

$$\sum_{d|n} \phi(d) = n.$$

*Proof.* (1: a cute, snazzy proof) Consider the  $n$  rational numbers

$$\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{n}{n} = 1$$

and reduce all to lowest terms so that the numerator and denominator are coprime.

Q: Given a positive divisor  $d$  of  $n$ , how many fractions have  $d$  as the denominator?

A: We have exactly  $\phi(d)$  of them.

Conversely, every denominator  $d$  is certainly a divisor of  $n$ . So we conclude that  $n = \sum_{d|n} \phi(d)$ .  $\square$

*Proof.* (2: using what we've learnt) We use the fact that  $\phi$  is multiplicative, and that this function is a summatory function of  $\phi$ , so this function itself is multiplicative. We can decompose this into prime powers. So it suffices to show this for prime powers.

<sup>2</sup>This is an easy way to prove this assuming Math 1530 (Abstract Algebra). There is another way to prove this with one hand tied behind the back, it just takes more mental muscle to do.

Let  $n = p^k$ . Let

$$f(n) = \sum_{d|n} d \phi(d).$$

Then we have

$$f(p^k) = \sum_{d|p^k} \phi(d) = 1 + (p-1) + (p^2-p) + \cdots + (p^k - p^{k-1})$$

which is a telescoping sum which leaves

$$= p^k$$

which is as intended. □

## §3 February 8, 2022

### §3.1 Dirichlet Convolutions

#### Definition 3.1 (Dirichlet Convolution)

Let  $f, g$  be arithmetic functions. Then the Dirichlet convolution/product of  $f$  and  $g$  is

$$\begin{aligned} (f * g)(n) &:= \sum_{d_1 d_2 = n} f(d_1) g(d_2) \\ &= \sum_{d|n} f(d) g(n/d) \end{aligned}$$

We do check that this has properties that we want it to have, like associativity:

$$\begin{aligned} ((f * g) * h)(n) &= (f * (g * h))(n) \\ &= \sum_{d_1 d_2 d_3 = n} f(d_1) g(d_2) h(d_3) \end{aligned}$$

It is also clearly commutative.

We also have that this product has a multiplicative identity.

#### Definition 3.2

Let  $I : \mathbb{Z}_+ \rightarrow \{0, 1\}$  be given by

$$I(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Then  $I$  is an identity for  $*$ , in the sense that  $f * I = I * f = f$ .

**Lemma 3.3**

If  $f$  is an arithmetic function such that  $f(1) \neq 0$ , then there exists an arithmetic function  $g$  such that  $f * g = I$ .

It is given recursively by

$$g(1) = \frac{1}{f(1)}$$

$$g(n) = -\frac{1}{f(1)} \cdot \sum_{d|n, d < n} g(d)f(n/d)$$

*Proof.* We want to show that given  $g$  and  $g$  defined as above, we have that  $f * g = I$ .

$n = 1$ :

$$g(1) \cdot f(1) = \frac{1}{f(1)} f(1) = 1$$

$n > 1$ :

$$\begin{aligned} \sum_{d|n} g(d)f(n/d) &= g(n) \cdot f(1) + \sum_{d|n, d < n} g(d)f(n/d) \\ &= -\frac{1}{f(1)} \cdot \sum_{d|n, d < n} g(d)f(n/d) \cdot f(1) + \sum_{d|n, d < n} g(d)f(n/d) \\ &= 0 \end{aligned}$$

So  $g$  is indeed an inverse of  $f$  since they produce the identity function  $I$ . □

**§3.2 Möbius Inversion**

The motivation of this is: given a summatory function of multiplicative functions, can we recover the multiplicative function?

**Definition 3.4 (Möbius  $\mu$  Function)**

We define  $\mu : \mathbb{Z}_+ \rightarrow \{-1, 0, 1\}$  given by

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 p_2 \dots p_k \text{ if } p_i \text{ are pairwise distinct primes} \\ 0 & \text{otherwise} \end{cases}$$

(We note that  $\mu(1) = 1$ .)



**Lemma 3.5**

$\mu$  is a multiplicative function.

*Proof.* Let  $m, n \in \mathbb{Z}_+$  such that  $(m, n) = 1$ . We write

$$\begin{aligned} m &= p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} \\ n &= q_1^{f_1} q_2^{f_2} \cdots q_l^{f_l} \end{aligned}$$

**Case 1** Some exponent  $e_i$  or  $f_i \geq 2$ . Then we have that

$$\mu(mn) = \mu(m)\mu(n) = 0$$

**Case 2** We have that

$$\begin{aligned} m &= p_1 p_2 \cdots p_k \\ n &= q_1 q_2 \cdots q_l \end{aligned}$$

where  $p_i$  and  $q_i$  are all pairwise distinct. Then  $\mu(m) = (-1)^k$  and  $\mu(n) = (-1)^l$ , so  $\mu(mn) = \mu(m)\mu(n) = (-1)^{k+l}$ .

Since these are coprime  $(m, n) = 1$ , then we have that  $\mu(mn) = (-1)^{k+l}$ .

Which is as intended, giving that  $\mu$  is a multiplicative function. □

**Lemma 3.6**

We have the property:

$$\sum_{d|n} \mu(d) = 0 \quad \forall n \geq 2.$$

Which tells us that the summatory function of  $\mu$  is  $I$ .

*Proof.* We define

$$f(n) := \sum_{d|n} \mu(d) \quad \text{is multiplicative}$$

We check this on prime powers, for prime  $p$  and  $e \geq 1$ :

$$\begin{aligned} f(p^e) &= \mu(1) + \mu(p) + \mu(p^2) + \cdots + \mu(p^e) \\ &= 1 - 1 + 0 + \cdots + 0 = 0 \end{aligned}$$

so we're done since  $f$  is multiplicative and is 0 for all power of primes. □

**Lemma 3.7**

Let  $i : \mathbb{Z}_+ \rightarrow \{1\}$  be the constant 1 function.

$$i * \mu = \mu * i = I$$

*Proof.* In the case of  $n = 1$ , we have  $i(1)\mu(1) = 1$ .

For  $n > 1$ , we have  $(i * \mu)(n) = \sum_{d|n} \mu(d) = 0$  from above. □

We see here that summatory functions can be seen as Dirichlet products: the summatory function  $F$  of  $f$  is  $F = f * i$ . What we said about summatory functions being multiplicative boils down to Dirichlet convolutions preserving multiplicativity.

*Recall:* that summatory functions inherit multiplicativity. In fact, this holds for Dirichlet products as well. If  $f, g$  are multiplicative, then so is  $f * g$ .

The proof is parallel to the proof for summatory functions, for [lemma 2.4](#).

**Theorem 3.8 (Möbius Inversion)**

Let

$$F(n) = \sum_{d|n} f(d)$$

Then we have

$$f(n) = \sum_{d|n} \mu(d) \cdot F(n/d) = \mu * F.$$

*Proof.*  $F = f * i$ , then

$$F * \mu = (f * i) * \mu = f * (i * \mu) = f * I = f.$$

which was simpler than I expected... □

**Corollary 3.9**

If  $F$  is the summatory function of  $f$ , and  $F$  is multiplicative, then  $f$  is also multiplicative, as  $f = \mu * F$  and  $\mu$  is multiplicative and convolutions with multiplicative functions are multiplicative.

**Corollary 3.10**

Corollary 3.9 gives another proof that  $\phi$  is multiplicative, as

$$\sum_{d|n} \phi(d) = \phi * i = \text{id}.$$

**§3.3 Applications of Möbius Inversion****§3.3.1 Cyclotomic Polynomials**

*Recall:* the  $n^{\text{th}}$  cyclotomic polynomial  $\Phi_n(x)$  is the unique irreducible polynomial in  $\mathbb{Z}[x]$  dividing  $x^n - 1$  but no  $x^k - 1$  for  $k < n$ .

Thus

$$\Phi_n(x) = \prod_{\substack{1 \leq k < n \\ (k,n)=1}} (x - e^{2\pi i k/n})$$

as the roots of this polynomial are exactly the primitive  $n^{\text{th}}$  roots of unity. We have that

$$\prod_{d|n} \Phi_d(x) = x^n - 1.$$

By Möbius inversion, if

$$G(n) = \prod_{d|n} g(d),$$

then we have that

$$g(n) = \prod_{d|n} G(d)^{\mu(n/d)}$$

In particular, taking  $G(n) = x^n - 1$  (with particular  $x \in \mathbb{C}$ ) as an arithmetic function, we have

$$\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(n/d)} \quad (x \in \mathbb{C})$$

Applying this identity for enough  $x \in \mathbb{C}$  yields this as an identity of polynomials.

**§3.3.2 Dynatomic Polynomials**

The roots of cyclotomic polynomials are roots of unity. Dynatomic polynomials have as roots the periodic points (of certain periods) of a polynomial.

**Definition 3.11**

Let  $K$  be a field, and let  $f \in K[x]$  of degree  $d \geq 2$ . Let

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$$

then  $P \in \overline{K}^a$  is said to be periodic under  $f$  if

$$f^n(P) = P \quad \text{for some } n \geq 1$$

<sup>a</sup>Algebraic numbers in field  $K$ , the field you get by adjoining all roots of polynomials in  $K[x]$ .

**Example 3.12**

Let  $f(x) = x^2 - 1$ . 0 is a period point under  $f$ :

$$0 \mapsto -1 \mapsto 0$$

and its period is 2.

**Remark.** If  $n$  is the smallest positive integer such that  $f^n(p) = p$  ( $p$  periodic), then we call  $n$  the exact period of  $p$  under  $f$ .

**Definition 3.13**

The  $n^{\text{th}}$  dynatomic polynomial of  $f$  is

$$\Phi_{f,n}(x) := \prod_{d|n} \left( f^d(x) - x \right)^{\mu(n/d)}$$

We hope that  $\Phi_{f,n}(x)$  has as its roots the points of exact period  $n$ ... This hope is dashed...

**Example 3.14**

$$f(x) = x^2 - \frac{3}{4}.$$

$$\begin{aligned} f^2(x) - x &= \left( x - \frac{3}{2} \right) \left( 1 - \frac{1}{2} \right)^3 \\ f(x) - x &= \left( x - \frac{3}{2} \right) \left( x + \frac{1}{2} \right) \end{aligned}$$

Thus

$$\frac{f^2(x) - x}{f(x) - x} = \left( x + \frac{1}{2} \right)^2$$

But  $x = -\frac{1}{2}$  is fixed under  $f$ .

## §4 February 10, 2022

### §4.1 Congruences *continued*

*Recall:* that for  $m \in \mathbb{Z}_+$ ,  $a, b \in \mathbb{Z}$ , the linear congruence

$$ax \equiv b \pmod{m}$$

has a solution if and only if  $(a, m) \mid b$ . Unwinding this gives Bezout's identity.

**Q: How do we actually find a solution?**

**A:** Either guess and check; or apply the following algorithm:

1) Divide all terms in the congruence by  $d = (a, m)$ .

2) If step 1 yields

$$a'x \equiv b' \pmod{m'}$$

with  $(a', m') = 1$ , then  $d := (a', b')$  is a unit mod  $m'$ , so we can divide both sides by  $d'$ .

$$a'd'^{-1}x \equiv b'd'^{-1} \pmod{m'}$$

3) Let  $a''x \equiv b'' \pmod{m}$  be the result so far. We replace  $b''$  by some  $b'' + km'^3$  such that  $(a'', b'' + km') > 1$  allows us to repeat step 2. This results in some  $a'''$  such that  $|a'''| < |a''|$ .

Given that we repeat this process, this must eventually terminate, since the absolute values of the  $a$  terms are strictly decreasing each time.

#### Example 4.1

Let  $10x \equiv 6 \pmod{14}$ .

1)  $(a, m) = (10, 14) = 2$  so we divide through by 2.

$$5x \equiv 3 \pmod{7}$$

2) Irrelevant since  $(5, 3)$  are coprime.

3) Consider integers of form  $3 + 7k$ , and see which are divisible by 5. We can take  $k = 1$ . We

<sup>3</sup>Since made  $a'$  and  $m'$  coprime, we have  $(a', m') = 1$  so we can indeed solve congruence  $a''q \equiv b'' + km'$  gives noncoprime pairs.

get

$$5x \equiv 10 \pmod{7}$$

2) Divide by  $(5, 10) = 5$  so we have  $x \equiv 2 \pmod{7}$ .

## §4.2 Simultaneous Linear Congruences

*Recall:* the CRT/Sun-tzu's theorem.

### **Theorem 4.2** (Sun-tzu's Theorem / Chinese Remainder Theorem)

Suppose that  $m = m_1 m_2 \cdots m_t$  with  $(m_i, m_j) = 1 \forall i \neq j$ .

Let  $b_1, b_2, \dots, b_t$  be integers, and consider the system of congruences

$$\begin{aligned} x_1 &\equiv b_1 \pmod{m_1} \\ x_2 &\equiv b_2 \pmod{m_2} \\ &\vdots \\ x_t &\equiv b_t \pmod{m_t} \end{aligned} \tag{*}$$

Then this system has a unique solution modulo  $m^a$ .

<sup>a</sup>That is, we have at least one solution, and we can shift it by *any* multiple of  $m$ .

*Proof.* Let  $n_i = m_1 m_2 \cdots m_t / m_i = \frac{m}{m_i}$  for each  $i$ . Since  $m_i$  is coprime to  $m_j$ ,  $\forall j \neq i$ , we have  $(n_i, m_i) = 1 \forall i$ . Then, there exists solutions  $r_i, s_i \in \mathbb{Z}$  such that

$$r_i m_i + s_i n_i = 1$$

Let  $e_i = s_i n_i$ . Then for each  $i$ ,

$$e_i \equiv 1 \pmod{m_i}$$

and  $e_i \equiv 0 \pmod{m_j}$ ,  $\forall j \neq i$ .

Our goal is to ultimately show

$$\mathbb{Z}/m\mathbb{Z} \simeq \mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z} \times \cdots \times \mathbb{Z}/m_t\mathbb{Z}$$

with each  $e_i$  generating the “ $\mathbb{Z}/m_i\mathbb{Z}$  piece”.

Set

$$x_0 = \sum_{i=1}^t b_i e_i$$

so that  $x_0 \equiv b_i \pmod{m_i} \forall i$ , so  $x_0$  is a solution to eq. (\*).

Suppose  $x_1$  is another solution. Then we have

$$x_1 - x_0 \equiv 0 \pmod{m_i} \quad \forall i, 1 \leq i \leq t$$

Since the  $m_i$  are pairwise coprime, we get  $m \mid x_1 - x_0$ . □

### §4.3 Structure of Unit Groups

*Recall:* in Math 1530, we learned Lagrange's theorem

#### **Theorem 4.3** (Lagrange's Theorem)

If  $G$  is a finite group, then for every subgroup  $H$  of  $G$ , we have  $|H| \mid |G|$ .

#### **Corollary 4.4**

If  $G$  is a finite group of order  $n$ , and  $a \in G$ , then  $a^n = e$ , where  $e$  is the identity of the group.

We've seen that  $|U(m)| = \phi(m)$ .<sup>4</sup> Applying Lagrange's theorem, we have Euler's theorem:

#### **Theorem 4.5** (Euler's Theorem)

For any  $a \in \mathbb{Z}$  with  $(a, m) = 1$ , we have  $a^{\phi(m)} \equiv 1 \pmod{m}$ .

#### **Definition 4.6**

A subset  $R$  of  $\mathbb{Z}$  is said to be a reduced set of residues mod  $m$  if  $R$  contains exactly one element from each of the  $\phi(m)$  congruence classes that are units mod  $m$ .

*Alternate proof of theorem 4.5.* Let  $R = \{r_1, r_2, \dots, r_{\phi(m)}\}$  be a reduced set of residues mod  $m$ . If  $(a, m) = 1$ , then  $aR$  is also a reduced set of residues mod  $m$ . Thus, if  $x_1, x_2, \dots, x_{\phi(m)} \in aR$  (pairwise distinct), then

$$\begin{aligned} x_1 x_2 \cdots x_{\phi(m)} &\equiv r_1 r_2 \cdots r_{\phi(m)} \pmod{m} \\ (ar_1)(ar_2) \cdots (ar_{\phi(m)}) &\equiv r_1 r_2 \cdots r_{\phi(m)} \pmod{m} \\ a^{\phi(m)} (r_1 r_2 \cdots r_{\phi(m)}) &\equiv r_1 r_2 \cdots r_{\phi(m)} \pmod{m} \end{aligned}$$

since all the  $r_i$  are units mod  $m$ , we divide through

$$a^{\phi(m)} \equiv 1 \pmod{m}$$

Which is as desired. □

---

<sup>4</sup>For notation, we use  $U(m) := (\mathbb{Z}/m\mathbb{Z})^\times$ .

We'll be studying roots of polynomials over  $\mathbb{Z}/m\mathbb{Z}$ , especially polynomials of the form  $x^d - a$ .

By Sun-tzu's theorem, the case of  $m$  being a prime power is especially important. This turns out to have a lot to do with the case that  $m = p$  is a prime itself.

First something further *afield*:

**Proposition 4.7**

If  $p$  is a prime and  $p \nmid d$  for  $d \in \mathbb{Z}_+$ , then the polynomial

$$x^d - a \in (\mathbb{Z}/p\mathbb{Z})[x], \quad a \not\equiv 0 \pmod{p}$$

has exactly  $d$  roots in some extension of  $\mathbb{F}_p$ .

Conversely, if  $p \mid d$ , then there are fewer than  $d$  roots in any extension of  $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ .

The proof uses the following proposition:

**Proposition**

A nonzero polynomial  $f \in K[x]$  is separable if and only if it is relatively prime to its derivative  $f'$ . (A separable polynomial whose roots in its algebraic closure  $\overline{K}$  whose roots are all distinct).

*Proof.*

$\Rightarrow$  **Right Direction:** Suppose  $f$  is separable and  $\alpha$  be any root of  $f$ . Then  $f(x) = (x - \alpha)h(x)$ , where  $h(\alpha) \neq 0$  since  $\alpha$  is a non-repeated root.

We have  $f'(\alpha) = h(\alpha) \neq 0$ , so  $\alpha$  is not a root of  $f'$ . Thus  $f$  and  $f'$  have no common roots, so they are coprime.

$\Leftarrow$  **Left Direction:** Prove by contrapositive. Suppose  $f$  is not separable. i.e. it has some repeated root which we call  $\alpha$ .

Then  $f(x) = (x - \alpha)^2 g(x)$ , so  $f'(x) = (x - \alpha)^2 g'(x) + 2(x - \alpha)g(x)$ . We see that  $x - \alpha$  divides both  $f$  and  $f'$  so  $(f, f') \neq 1$ .

Which concludes the bidirectional. □

*Proof of proposition 4.7.* We have  $f(x) = x^d - a$ ,  $a \not\equiv 0 \pmod{p}$  has  $d$  distinct solutions in some extension of  $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ , because

$$f'(x) = dx^{d-1} \pmod{p}$$

and with 0 as its only root but 0 is not a root of  $f$ . By above we have that  $f$  is separable.



Conversely, if  $p \mid d$ , then

$$f'(x) \equiv 0 \pmod{p},$$

so  $(f, f') \neq 1$ , meaning that  $f$  is not separable.  $\square$

**Proposition 4.8** (4.1.2 of text)

If  $p$  is a prime and if  $d \mid p - 1$ , then the polynomial

$$x^{d-1} \in (\mathbb{Z}/p\mathbb{Z})[x]$$

has exactly  $d$  roots in the base field  $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ .

*Proof.* We know this is true in the case of  $d = p - 1$  because of Fermat's Little Theorem (also Euler's Theorem).

We also note that  $(x^d - 1) \mid (x^{p-1} - 1)$ . Since  $x^{p-1} - 1$  has all roots in the base field by FLT,  $x^d - 1$  had better also retain its roots in the base field  $\mathbb{F}_p$  by contradiction.  $\square$

## §5 February 15, 2022

### §5.1 Cyclicity of Groups

#### §5.1.1 mod odd $p$

*Recall:* from last class, we had [proposition 4.8](#):

**Proposition**

If  $p$  is a prime and if  $d \mid p - 1$ , then the polynomial

$$x^{d-1} \in (\mathbb{Z}/p\mathbb{Z})[x]$$

has exactly  $d$  roots in the base field  $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ .

**Corollary 5.1**

$G := (\mathbb{Z}/p\mathbb{Z})^\times$  is cyclic.

*Proof.* For  $d \mid (p - 1)$ , we write  $\psi(d)$  for the number of elements of  $G$  having order  $d$ .

Proposition 2 implies that<sup>5</sup>

$$\sum_{c|d} \psi(c) = d \quad (\psi * i = \text{id}, \psi = \text{id} * \mu)$$

Möbius inversion gives

$$\psi(d) = \sum_{c|d} \mu(c) \frac{d}{c}.$$

On the other hand, we have  $\text{id} = \phi * i \Rightarrow \phi = \mu * \text{id}$ . Thus  $\psi(d) = \phi(d)$  for all  $d \mid (p-1)$ . So in particular,  $\psi(p-1) = \phi(p-1) \geq 1$  for any prime  $p$ .  $\square$

### §5.1.2 mod odd power $p^e$

#### Theorem 5.2

Let  $p \in \mathbb{Z}_+$  be an odd prime, and let  $e \geq 1$ . Then  $U(p^e)$  is cyclic.

Proof overview:

1. Pick a primitive root mod  $p$ . We call it  $g$  (for generator).
2. Show that either  $g$  or  $g+p$  is a primitive root mod  $p^2$ .
3. Show that if  $h$  is any primitive root mod  $p^2$ , then  $h$  is a primitive root mod  $p^e \forall e \geq 2$ .

*Proof of theorem 5.2.*

**Step 1.** Let  $g$  be a primitive root modulo  $p$  given by [corollary 5.1](#).

**Step 2.** Let  $d$  be the order of  $g$  mod  $p^2$ . Since  $\phi(p^2) = p(p-1)$ , we have that

$$d \mid p(p-1) \quad \text{by Lagrange.}$$

By definition of  $d$ ,

$$g^d \equiv 1 \pmod{p^2}$$

so we also have

$$g^d \equiv 1 \pmod{p}$$

Thus  $(p-1) \mid d$  since  $g$  has order  $p-1$  mod  $p$ . Altogether,  $d$  is either  $p-1$  or  $p(p-1)$ . If  $d = p(p-1)$ , then we are done with step 2. So we assume the former that  $d = p-1$ .

Let  $h = g + p$ . We know that  $h$  is a primitive root mod  $p$ , so we do the same [yoga] as above and conclude that the order of  $h$  mod  $p^2$  is either  $p-1$  or  $p(p-1)$ .

---

<sup>5</sup>We throw in Lagrange's theorem, and essentially count the number of solutions to  $x^d \equiv 1$ .

By our new hypothesis,

$$g^{p-1} \equiv 1 \pmod{p^2}$$

so modulo  $p^2$ , we have

$$h^{p-1} = (g+p)^{p-1} = g^{p-1} + (p-1)g^{p-2}p + \dots + p^{p-1}$$

Modulo  $p^2$ , the only terms that survive are (expand and all  $p^2$  terms die):

$$\equiv 1 - pg^{p-2} \pmod{p^2}$$

But  $p \nmid g$ , so  $pg^{p-2} \not\equiv 0 \pmod{p}$ , and hence  $h^{p-1} \not\equiv 1 \pmod{p^2}$ . Thus the order of  $h \pmod{p^2}$  is  $p(p-1)$ , so  $h$  generates  $U(p^2)$ .

So we are done with step 2. If  $g$  is a primitive root mod  $p$ , then either  $g$  or  $g+p$  is a primitive root mod  $p^2$ .

**Step 3.** We wish to show that a primitive root mod  $p^2$  is also a primitive root mod  $p^e \forall e \geq 2$ . We induct on  $e$ .

Let  $h$  be a primitive root mod  $p^e$  for some fixed  $e \geq 2$ . Let  $d$  be the order of  $h \pmod{p^{e+1}}$ . By Lagrange, we have that  $d \mid \phi(p^{e+1}) = p^e(p-1)$ , and from step 2,

$$\phi(p^e) = p^{e-1}(p-1) \mid d$$

Hence  $d = p^e(p-1)$  or  $p^{e-1}(p-1)$ . If it's the former then we are done, so we assume latter.

We want to show that

$$h^{p^{e-1}(p-1)} \not\equiv 1 \pmod{p^{e+1}}$$

implying that  $d = p^e(p-1)$  after all.

Since  $h$  has order  $\phi(p^e) = p^{e-1}(p-1)$  in  $U(p^e)$ , we have

$$h^{p^{e-2}(p-1)} \not\equiv 1 \pmod{p^e} \tag{*}$$

However,

$$h^{p^{e-2}(p-1)} \equiv 1 \pmod{p^{e-1}} \tag{**}$$

Combining eq. (\*) and eq. (\*\*) yields

$$h^{p^{e-2}(p-1)} = 1 + kp^{e-1}$$

where  $p \nmid k$ . Therefore, we have

$$\begin{aligned} h^{p^{e-1}(p-1)} &= (1 + kp^{e-1})^p \\ &= 1 + pkp^{e-1} + \binom{p}{2} k^2 p^{2e-2} + \dots \end{aligned}$$

Subsequent terms are all divisible by  $p^{3e-3} = (p^{e-1})^3$ , and hence divisible by  $p^{e+1}$  as  $e(e-1) \geq 2+1 \forall e \geq 2$ . Thus

$$h^{p^{e-1}(p-1)} = 1 + kp^e + \frac{1}{2}k^2 p^{2e-1}(p-1) \pmod{p^{e+1}}$$

$p$  is odd, so

$$\frac{1}{2}k^2p^{2e-1}(p-1)$$

is divisible by  $p^{e+1}$ , since  $2e-1 \geq e+1$ . Thus

$$h^{p^{e-1}(p-1)} \equiv 1 + kp^e \pmod{p^e + 1}$$

Since  $p \nmid k$ , we get that  $kp^e \not\equiv 0$  so

$$h^{p^{e-1}(p-1)} \not\equiv 1 \pmod{p^e + 1}$$

This proves that  $d = p^e(p-1)$ , which is to say that  $h$  is a primitive root mod  $p^{e+1}$ .

Altogether, we have that  $U(p^e)$  is cyclic. □

### §5.1.3 mod powers of 2

#### Theorem 5.3

$U(2^e)$  is cyclic iff  $e = 1$  or  $e = 2$ .

*Proof.* Clearly  $U(2)$  and  $U(4)$  are cyclic<sup>6</sup>.

We show that  $U(2^e)$  is *not* cyclic for all  $e \geq 3$ . Notice: it suffices to show that  $U(8)$  is not cyclic, since we can find group homomorphisms down powers of 2.

$$U(8) = \{\bar{1}, \bar{3}, \bar{5}, \bar{7}\}$$

$$\text{and } \bar{1}^2 = \bar{3}^2 = \bar{5}^2 = \bar{7}^2 \pmod{8}.$$

□

## §5.2 Classification of all cyclic unit groups

#### Corollary 5.4

$U(m)$  is cyclic if and only if  $m = 1, 2, 4, p^e$  or  $2p^e$  for some odd prime  $p$ .

*Proof.* Recall that a product  $G$  of finite cyclic groups  $G_1$  and  $G_2$  is cyclic iff  $(|G_1|, |G_2|) = 1$ .<sup>7</sup> On the other hand,  $\phi(m)$  is even  $\forall m \geq 3$ . So only one of  $G_1$  and  $G_2$  needs odd power.

Combined with our structure theorems on  $U(p^e)$  for primes  $p$ , this proves the corollary since these are the only possibilities. □

<sup>6</sup>We don't have much choice since there is only one trivial group and one group of order 2, both cyclic.

<sup>7</sup>Secretly, Chinese Remainder Theorem.

## §6 February 17, 2022

### §6.1 Special Integers

#### §6.1.1 Fermat and Mersenne Primes

We make the observation that many small primes are of the form  $2^m \pm 1$  for some natural number  $m$ , for example

$$3, 5, 7, 17, 31$$

We deal with the  $+1$  and  $-1$  cases separately.

##### Lemma 6.1

If  $2^m + 1$  is prime, then  $m = 2^n$  for some  $n \geq 0$ .

*Proof.* We show the contrapositive. Suppose  $m$  is not a power of 2. We write  $m = 2^n \cdot q$  for some odd  $q > 1$ .

The polynomial

$$f(t) = t^q + 1$$

has  $t = -1$  as a root, so

$$f(t) = (t + 1)g(t) \quad \text{where } \deg f = q > 1$$

Thus

$$\begin{aligned} x^m + 1 &= f(x^{2^n}) \\ &= (x^{2^n} + 1)g(x^{2^n}) \quad \text{where } m > 2^n \end{aligned}$$

Plugging in  $x = 2$  gives

$$2^{2^n} + 1 \mid 2^m + 1, \text{ and } 2^{2^n} + 1 < 2^m + 1$$

so  $2^m + 1$  is not prime. □

##### Definition 6.2 (Fermat Numbers)

Numbers of the form  $2^{2^n} + 1$  are called Fermat numbers.

Fermat numbers that are prime are called Fermat primes.

The first few Fermat numbers happen to be prime: 3, 5, 17, 257, 65537.

**Conjecture**

Fermat conjectured that Fermat numbers are prime.

This is very false! Euler found that

$$2^{2^5} + 1 = 641 \times 6700417$$

We now turn to Mersenne numbers.

**Lemma 6.3**

If  $m > 1$  and  $a^m - 1$  is prime, then  $a = 2$  and  $m$  is prime.

*Proof.* Suppose  $m$  is composite and, so  $m = nk$ ,  $1 < k, n < m$ . Then

$$\begin{aligned} a^m - 1 &= (a^k)^n - 1 \\ &= (a^k - 1)(a^{k(n-1)} + \cdots + 1) \end{aligned}$$

This implies that  $a^m - 1$  is composite. Hence  $m$  had better be prime.

Now  $a^m - 1 = (a - 1)(a^{m-1} + \cdots + 1)$ , so we further have that  $a = 2$ . □

**Definition 6.4 (Mersenne Numbers)**

Integers of the form  $2^p - 1$  where  $p$  is a prime are called Mersenne numbers.

Mersenne numbers that are prime are called Mersenne primes.

There is a current ongoing search for more Mersenne primes on the internet. Currently, the largest known Mersenne prime (and largest known prime number) is

$$M(82,589,933)$$

That is,

$$2^{82,589,933} - 1$$

Mersenne primes are related to perfect numbers. There is a one-to-one correspondence with Mersenne primes and even perfect numbers.

**Definition 6.5** (Perfect Number)

$n \in \mathbb{Z}_+$  is called perfect if

$$n = \sum_{\substack{d|n \\ d < n}} d$$

**Example 6.6**

We have

$$\begin{aligned} 6 &= 1 + 2 + 3 \\ 28 &= 1 + 2 + 4 + 7 + 14 \end{aligned}$$

**Proposition 6.7**

If  $n = 2^{p-1}(2^p - 1)$  where  $p \in \mathbb{Z}_+$ , and  $p, 2^p - 1$  are prime, then  $n$  is perfect.

*Proof.* The function  $\sigma(n) = \sum_{d|n} d$  is multiplicative. So if

$$n = 2^{p-1}(2^p - 1)$$

then

$$\sigma(n) = \sigma(2^{p-1})\sigma(2^p - 1).$$

since they are coprime. Now we also

$$\begin{aligned} \sigma(2^{p-1}) &= \frac{2^p - 1}{2 - 1} = 2^p - 1 \\ \sigma(2^p - 1) &= 1 + (2^p - 1) = 2^p \end{aligned}$$

Hence  $\sigma(n) = (2^p - 1) \cdot 2^p = 2n$ .

So  $n$  is a perfect number. □

**Proposition 6.8**

If  $n \in \mathbb{Z}_+$  is even and perfect, then  $n = 2^{p-1}(2^p - 1)$  where  $p$  and  $2^p - 1$  are both prime.

*Proof.* This is a homework exercise! □

It is currently conjectured that there are no odd perfect numbers.

### §6.1.2 Pseudoprimes and Carmichael Numbers

Homework 2 includes a problem for which a special case is Wilson's Theorem.

**Theorem 6.9** (Wilson's Theorem)

If  $p$  is a prime, then

$$(p-1)! \equiv -1 \pmod{p}$$

The converse is also true.

**Proposition 6.10**

If  $n \in \mathbb{Z}_+$  where  $n \geq 2$  is such that

$$(n-1)! \equiv 1 \pmod{n} \tag{*}$$

then  $n$  is prime.

We can think of [eq. \(\\*\)](#) as a rudimentary 'primality test'.

However, this is not a great primality test, because factorials are expensive to compute.

Recall Fermat's little theorem.

**Theorem 6.11** (Fermat's Little Theorem)

If  $p \in \mathbb{Z}_+$  is a prime and  $a \in \mathbb{Z}$ , then

$$a^p \equiv a \pmod{p}$$

Thus  $n \in \mathbb{Z}_+$  and

$$a^n \equiv a \pmod{n}$$

for some  $a \in \mathbb{Z}_+$ , then  $n$  is composite.

**Example 6.12**

If  $a = 2$ , then

$$2^n \not\equiv 2 \pmod{n} \Rightarrow n = 2 \text{ is composite}$$

**Question.** We might wonder whether a converse to this holds. Disappointingly, no.



**Example 6.13**

$2^{10} = 1024 = 1 \pmod{341}$ , so  $2^{341} = (2^{10})^{34} \cdot 2 = 2 \pmod{341}$ .

But  $341 = 11 \cdot 31$ , so 341 is composite.

**Definition 6.14 (Pseudoprime)**

We call  $n$  a pseudoprime to the base  $a$  if  $n$  is composite and happens to satisfy

$$a^n \equiv a \pmod{n}.$$

**Example 6.15**

341 is a pseudoprime to the base 2.

We might hope that if this test failed for a particular  $a$ , there exists some other  $a$  that can test whether  $n$  is composite. However, this is not the case.<sup>8</sup>

It is not true that given a composite  $n$ , there exists an  $a \in \mathbb{Z}_+$  such that  $n$  is not a pseudoprime to the base  $a$ .

**Definition 6.16 (Carmichael Numers)**

$n \in \mathbb{Z}_+$  is called a Carmichael number if  $n$  is composite and

$$a^n \equiv a \pmod{n}, \quad \forall a \in \mathbb{Z}$$

**Example 6.17**

The smallest Carmichael number is 561.

**Question.** There are variants on this question? Can you have pseudoprimes that satisfy all but one base?

**Proposition 6.18**

If a composite number  $n$  is *not* a Carmichael number, then at least half of the congruence classes  $a \in (\mathbb{Z}/n\mathbb{Z})^\times$  are such that  $n$  is *not* a pseudoprime to the base  $a$ .

*Proof.* Suppose  $n$  is a pseudoprime to the base:

$$a_1, a_2, \dots, a_r \in (\mathbb{Z}/n\mathbb{Z})^\times$$

---

<sup>8</sup>We learn in life to not be too hopeful.

and suppose we have some  $a$  such that

$$a^n \not\equiv a \pmod{n}$$

Then for all  $i$ ,

$$\begin{aligned} (a \cdot a_i)^{n-1} &= a^{n-1} a_i^{n-1} \\ &\equiv a^{n-1} \pmod{n} \\ &\not\equiv 1 \pmod{n} \end{aligned}$$

Thus  $n$  is not a pseudoprime to the bases  $a \cdot a_1, a \cdot a_2, \dots, a \cdot a_r$ . □

**Remark 6.19.** The bases for pseudoprimes form a subgroup of the group of units.

Someone  
check me on  
this.

## §7 March 1, 2022

### §7.1 Power Residues

For this section, corresponds to pages 45-46 of Ireland & Rosen are a good reference.

#### Definition 7.1 (Power Residue)

If  $m, n \in \mathbb{Z}_+$  and  $a \in \mathbb{Z}$  such that  $(a, m) = 1$ , then we say that  $a$  is an  $n^{\text{th}}$  power residue modulo  $m$  if and only if the congruence

$$x^n \equiv a \pmod{m} \tag{7.1}$$

has solutions.

Given eq. (7.1), we're interested in two questions:

- 1) Does eq. (7.1) have a solution?
- 2) If yes, then how many?

**Proposition 7.2**

If  $m \in \mathbb{Z}_+$  is such that  $U(m)$  is cyclic, and  $a \in \mathbb{Z}$  is such that  $(a, m) = 1$ , then

$$x^n \equiv a \pmod{m}$$

has solutions if and only if

$$a^{\phi(m)/d} \equiv 1 \pmod{m}$$

where  $d = (\phi(m), n)$ .

If there are solutions, then there are exactly  $d$  solutions.

*Proof.* Let  $g$  be a primitive root mod  $m$ , and let

$$a = g^b.$$

Suppose  $x = g^y$ . Then

$$\begin{aligned} x^n &\equiv a \pmod{m} \\ \iff g^{ny} &\equiv g^b \pmod{m} \\ \iff ny &\equiv b \pmod{\phi(m)} \end{aligned}$$

This is solvable if and only if  $d = (\phi(m), n) \mid b$ . If there is at least one solution, then there are exactly  $d$  solutions.

Now we show that  $d \mid b \iff a^{\phi(m)/d} \equiv 1 \pmod{m}$ .

Forward direction:

$$a^{\phi(m)/d} = g^{b \cdot \phi(m)/d} = \left(g^{\phi(m)}\right)^{b/d} = 1 \pmod{m}$$

Backward direction:

$$a^{\phi(m)/d} \equiv 1 \pmod{m} \Rightarrow g^{b \cdot \phi(m)/d} \equiv 1 \pmod{m} \Rightarrow \phi(m) \mid b \cdot \phi(m)/d \Rightarrow \frac{b}{d} \in \mathbb{Z}$$

□

We can prove this using a similar group theory theorem that we can apply directly.

**Theorem 7.3**

Let  $G$  be a cyclic group of order  $n$ , suppose  $k \in \mathbb{Z}_+$  and  $a \in G$ . Then  $a = b^k$  ( $a$  is a  $k^{\text{th}}$  power in  $G$ ) iff  $a^{n/(k,n)} = e$  iff  $x^k = a$  has  $(n, k)$  solutions in  $G$ .

The proof of this theorem uses the following lemma:

**Lemma 7.4**

Let  $G$  be a cyclic group of order  $n$  and let  $H$  be a subgroup of  $G$  of order  $d$ . Then  $x \in H$  iff  $x^d = e$  iff  $\text{ord}(x) \mid d$ .

*Proof of theorem 7.3.* Let  $H$  be a subgroup of  $k^{\text{th}}$  powers in  $G^9$ , and let  $g \in G$  be such that  $G = \langle g \rangle$ .

Then  $H = \{g^{jk} \mid j \in \mathbb{Z}\} = \langle g^k \rangle$ . Since  $\text{ord}(g^k) = \frac{n}{(k,n)}$  (*exercise*), we that  $|H| = \frac{n}{(k,n)}$ .

Consider  $\phi : G \rightarrow G$  that powers by  $k$ ,  $\phi : x \mapsto x^k$ . Then  $\text{im}(\phi) = H$ , so this implies that  $\phi$  is a  $(k,n)$ -to-1 mapping (so gives us the number of solutions to each power, and how many  $k$  powers there are).  $\square$

Knowing how to solve these modulo a group of units gives us ways using CRT/Sunzi's Theorem to solve mod composite numbers.

We write  $m = 2^e p_1^{e_1} \cdots p_r^{e_r}$  where  $p_i$  are pairwise distinct odd primes. Then

$$x^n \equiv a \pmod{m}, \quad (a, m) = 1$$

is solvable if and only if the system

$$\begin{aligned} x^n &\equiv a \pmod{2^e} \\ x^n &\equiv a \pmod{p_i^{e_i}} \\ &\vdots \\ x^n &\equiv a \pmod{p_r^{e_r}} \end{aligned}$$

is solvable.

We have that  $U(p_i^{e_i}), U(2), U(4)$  are all cyclic. Hence our prior discussion can be applied to those.

**Question.** How do we solve

$$x^n \equiv a \pmod{m}$$

where  $e \geq 3$  (for powers of 2)?

**Proposition 7.5** (4.2.2 from Text)

Let  $a \in \mathbb{Z}$  be odd,  $e \geq 3$ , and consider  $x^n \equiv a \pmod{2^e}$ .

If  $n$  is odd, then a solution exists and is unique. If  $n$  is even, a solution exists if and only if  $a \equiv 1 \pmod{4}$  and  $a^{2^{e-2}/d} \equiv 1 \pmod{2^e}$  where  $d = (n, 2^{e-2})$ . When a solution exists, there are exactly  $2d$  solutions.

*Proof. Exercise to come.*  $\square$

<sup>9</sup>This is indeed a subgroup. We use the fact that  $G$  is Abelian.

## §7.2 Quadratic Residues

Things are a lot simpler and nicer when we consider only quadratic congruences (as opposed to arbitrary residues).

### Definition 7.6 (Quadratic Residue)

Let  $a \in \mathbb{Z}$ ,  $m \in \mathbb{Z}_+$ ,  $(a, m) = 1$ . We say that  $a$  is a quadratic residue mod  $m$  if the congruence

$$x^2 \equiv a \pmod{m} \tag{7.2}$$

has a solution.

Conversely, if  $a$  is not a quadratic residue (that is, eq. (7.2) does not have a solution), we call it a nonresidue or a quadratic nonresidue.

We extract the consequences of previous propositions to get special cases of propositions 4.2.3 and 4.2.4 from text.

1) Let  $p \in \mathbb{Z}_+$  be an odd prime, and suppose  $a \in \mathbb{Z}$  with  $p \nmid a$ . Then

$$x^2 \equiv a \pmod{p}$$

is solvable iff

$$x^2 \equiv a \pmod{p^e}$$

is solvable for all  $e \geq 1$ .

2) Let  $a \in \mathbb{Z}$  be odd. Then

$$x^2 \equiv a \pmod{8}$$

is solvable iff

$$x^2 \equiv a \pmod{2^e}$$

is solvable for all  $e \geq 3$ .

**Proposition 7.7** (5.1.1 from Text)

Let

$$m = 2^e p_1^{e_1} \cdots p_r^{e_r}$$

be the prime factorization of  $m \in \mathbb{Z}_+$ , and suppose  $(a, m) = 1$ .

Then

$$x^2 \equiv a \pmod{m} \tag{7.3}$$

is solvable if and only if three conditions are satisfied:

- i. If  $e = 2$ , then  $a \equiv 1 \pmod{4}$ .
- ii. If  $e \geq 3$ , then  $a \equiv 1 \pmod{8}$ .
- iii. For each  $i$ , have

$$a^{(p_i-1)/2} \equiv 1 \pmod{p_i}$$

*Proof.* Sunzi's theorem tells us that [eq. \(7.3\)](#) is solvable iff

$$\begin{aligned} x^2 &\equiv a \pmod{2^e} \\ x^2 &\equiv a \pmod{p_1^{e_1}} \\ &\vdots \\ x^2 &\equiv a \pmod{p_r^{e_r}} \end{aligned}$$

are *all* solvable.

First consider the first equation  $x^2 \equiv a \pmod{2^e}$ . 1 is the only quadratic residue mod 4 and the same thing is true mod 8. On the other hand, [black box 2](#) gives us  $x^2 \equiv a \pmod{8}$  is solvable iff  $x^2 \equiv a \pmod{2^e}$  for  $e \geq 3$ . This gives us conditions i and ii.

Now consider  $x^2 \equiv a \pmod{p_i^{e_i}}$ . [Proposition 7.2](#) gives that  $x^2 \equiv a \pmod{p_i}$  is solvable iff  $a^{(p_i-1)/2} \equiv 1 \pmod{p_i}$ . [Black box 1](#) then tells us that

$$\begin{aligned} x^2 &\equiv a \pmod{p_i} \text{ is solvable} \\ \iff x^2 &\equiv a \pmod{p_i^{e_i}} \text{ is solvable.} \end{aligned}$$

which concludes our proof. □

**Remark.** Studying these quadratic congruences amounts to studying them modulo primes.

### §7.3 The Legendre Symbol

**Definition 7.8** (The Legendre Symbol)

Let  $p$  be an odd prime, and let  $a \in \mathbb{Z}$ .

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue mod } p \\ 0 & \text{if } p \mid a \\ -1 & \text{otherwise} \end{cases}$$

This symbol  $\left(\frac{a}{p}\right)$  is called the Legendre symbol.

**Proposition 7.9** (5.1.2 of Text)

We have the following properties of the Legendre symbol:

(a)

$$\left(\frac{a}{p}\right) = a^{(p-1)/2} \pmod{p}$$

This is called *Euler's Criterion*.

(b)

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right) \cdot \left(\frac{b}{p}\right)$$

which is to say that the Legendre symbol is totally multiplicative.

(c) If  $a \equiv b \pmod{p}$ , then

$$\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right).$$

**§8 March 3, 2022****§8.1 Quadratic Residues *continued***

*Recall:* [definition 7.8](#) and [proposition 7.9](#) from last class (right above).

We now prove the earlier proposition:

*Proof of [proposition 7.9](#).*

(c) is clear.

(a) By Fermat's Little Theorem, if  $p \nmid a$ , we have  $a^{p-1} \equiv 1 \pmod{p}$ , so

$$(a^{(p-1)/2} + 1)(a^{(p-1)/2} - 1) \equiv 0 \pmod{p}$$

Since  $\text{mod } p$  we have an integral domain, then we have that  $a^{(p-1)/2} \equiv \pm 1 \pmod{p}$ . We know that  $a^{(p-1)/2} \equiv 1 \pmod{p}$  if and only if  $a$  is a quadratic residue  $\text{mod } p$ ,

(b) This applies (a).

$$\left(\frac{ab}{p}\right) = (ab)^{(p-1)/2} = a^{(p-1)/2} \cdot b^{(p-1)/2} = \left(\frac{a}{p}\right) \left(\frac{b}{p}\right)$$

□

### Corollary 8.1

We have some corollaries:

- 1) There are exactly  $\frac{p-1}{2}$  quadratic residues and  $\frac{p-1}{2}$  quadratic non-residues  $\text{mod } p$ .
- 2) The product of two residues is a residue, the product of a residue and a non-residue is a non-residue, and the product of a non-residue and a non-residue is a residue.
- 3) If  $g$  is a primitive root modulo  $p$ , then

$$\left(\frac{g^i}{p}\right) = (-1)^i.$$

- 4) We have

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$$

which has a fancy name. This is the “*First Supplemental Law of Quadratic Reciprocity*”.

## §8.2 Gauss's Lemma

We now discuss a characterization of the Legendre symbol due to Gauss.

### Definition 8.2

For  $p \in \mathbb{Z}_+$  an odd prime,

$$S = \left\{ -\frac{p-1}{2}, -\frac{p-3}{2}, \dots, -1, 1, 2, \dots, \frac{p-1}{2} \right\}$$

is called the set of least residues mod  $p$ .



**Definition 8.3**

Let  $a \in \mathbb{Z}$  such that  $p \nmid a$ . Define  $\mu$  to be the number of negative least residues of the integers

$$a, 2a, 3a, \dots, \left(\frac{p-1}{2}\right)a$$

**Example 8.4**

If  $p = 7$  and  $a = 4$ , then  $\frac{p-1}{2} = 3$ , and  $1 \cdot 4, 2 \cdot 4, 3 \cdot 4$  are congruent to  $-3, 1, -2 \pmod{7}$ . Thus  $\mu = 2$ .

**Lemma 8.5** (Gauss's Lemma)

Let  $p \in \mathbb{Z}_+$  be an odd prime and let  $a \in \mathbb{Z}$  be such that  $p \nmid a$ . Then

$$\left(\frac{a}{p}\right) = (-1)^\mu.$$

*Proof.* It is convenient for us to partition the list  $S$  as

$$P = \{1, 2, \dots, \frac{p-1}{2}\}$$

$$N = \{-1, -2, \dots, -\frac{p-1}{2}\}$$

so that  $\mu = |aP \cap N|$ <sup>10</sup>.

A key observation is that if  $x, y \in P$  with  $x \neq y$ , then

$$ax \not\equiv \pm ay \pmod{p}$$

for otherwise,

$$a \equiv \pm y \pmod{p}$$

which is impossible since  $x$  and  $y$  are distinct elements of  $P$  (positive residue classes, so not in  $N$ ).

Thus  $aP = \{\varepsilon_i i \mid 1 \leq i \leq \frac{p-1}{2}\}$  for some  $\varepsilon_i = \pm 1$ .

Now we mimic the elementary proof of Euler's Theorem:

$$a^{(p-1)/2} \cdot \left(\frac{p-1}{2}\right)! \equiv \left(\prod_{i=1}^{(p-1)/2} \varepsilon_i\right) \cdot \left(\frac{p-1}{2}\right)!$$

$$a^{(p-1)/2} \equiv \left(\prod_{i=1}^{(p-1)/2} \varepsilon_i\right)$$

$$\equiv (-1)^\mu$$

<sup>10</sup>There's an abuse of notation here, but we conflate integers with their equivalence classes in  $S$ .

since  $\mu$  is equal to the order of the set that is contributing every  $-1$  as  $\varepsilon_i$ .

Applying [proposition 7.9](#) (Euler's criterion), this concludes the proof.  $\square$

We'll use this lemma to prove Quadratic Reciprocity later.

We now use it to prove the *Second Supplemental Law of Quadratic Reciprocity*.

**Proposition 8.6** (5.1.3, the *Second Supplemental Law of Quadratic Reciprocity*)

For  $p \in \mathbb{Z}_+$  be an odd prime.

$$\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$$

We note  $\frac{p^2-1}{8} = \frac{(p-1)(p+1)}{2 \cdot 4}$ . From this, it follows that we're really saying

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{when } p \equiv \pm 1 \pmod{8} \\ -1 & \text{when } p \equiv \pm 3 \pmod{8} \end{cases}$$

*Proof.* We apply Gauss's Lemma with  $a = 2$ .

$$2P = \{2, 4, 6, \dots, p-1\}$$

First suppose that  $p \equiv 1 \pmod{4}$ . Then  $\frac{p-1}{2}$  is even, so

$$2P = \left\{ \underbrace{2, 4, 6, \dots, \frac{p-1}{2}}_{\in P}, \underbrace{\frac{p+3}{2}, \dots, p-1}_{\in N} \right\}$$

with the first  $\frac{p-1}{4}$  elements in  $P$  and the last  $\frac{p-1}{4}$  elements in  $N$ . So  $\mu = |2P \cap N| = \frac{p-1}{4}$ , so Gauss's Lemma gives

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p-1}{4}} = \left((-1)^{\frac{p-1}{4}}\right)^{\frac{p+1}{2}} = (-1)^{\frac{p^2-1}{8}}$$

since  $\frac{p+1}{2}$  is odd.

Now suppose  $p \equiv 3 \pmod{4}$ . Then

$$2P = \left\{ \underbrace{2, 4, 6, \dots, \frac{p-3}{2}}_{\in P}, \underbrace{\frac{p+1}{2}, \dots, p-1}_{\in N} \right\}$$

The first  $\frac{p-3}{4}$  elements are in  $P$ , and the last  $\frac{p+1}{4}$  elements are in  $N$ . Then  $\mu = \frac{p+1}{4}$ , so

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p+1}{4}} = \left((-1)^{\frac{p+1}{4}}\right)^{\frac{p-1}{2}} = (-1)^{\frac{p^2-1}{8}}$$

$\square$

### §8.3 Quadratic Reciprocity

**Theorem 8.7** (Law of Quadratic Reciprocity)

Let  $p, q \in \mathbb{Z}_+$  be distinct odd positive primes. Then

$$\left(\frac{p}{q}\right) \left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2} \frac{q-1}{2}}$$

In other words,

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)$$

if and only if at least one of  $p, q$  is congruent to 1 mod 4.

*Proof. coming soon!*

□

Here's the motivation behind this: we used Euler's Criterion to easily calculate the quadratic character of some  $a \bmod p$ . With Quadratic Reciprocity, we can solve the question "If we fix odd prime  $q$ , for which  $p$  is  $q$  a quadratic residue?"

**Example 8.8**

Which odd primes  $p \in \mathbb{Z}_+$  have 3 as a quadratic residue?

Suppose  $p \equiv 1 \pmod{4}$ . Then

$$\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{3} \\ -1 & \text{if } p \equiv -1 \pmod{3} \end{cases}$$

Then  $p \equiv 1 \pmod{4}$  and  $p \equiv 1 \pmod{3}$  so  $p \equiv 1 \pmod{12}$  gives us  $\left(\frac{3}{p}\right) = 1$  and  $p \equiv 5 \pmod{12}$  gives  $\left(\frac{3}{p}\right) = -1$ . \*\*\**(Format)*

Now suppose  $p \equiv 3 \pmod{4}$ . Then

$$\left(\frac{3}{p}\right) = -\left(\frac{p}{3}\right) = \begin{cases} 1 & \text{if } p \equiv -1 \pmod{3} \\ -1 & \text{if } p \equiv 1 \pmod{3} \end{cases}$$

Thus,

$$\begin{aligned} p \equiv 11 \pmod{12} & \text{ gives } \left(\frac{3}{p}\right) = 1, \text{ and} \\ p \equiv 7 \pmod{12} & \text{ gives } \left(\frac{3}{p}\right) = -1 \end{aligned}$$

So we conclude that  $\left(\frac{3}{p}\right) = 1$  iff  $p \equiv \pm 1 \pmod{12}$ .

## §9 March 8, 2022

### §9.1 Legendre Symbol *continued*

#### Example 9.1

Determine if 219 is a quadratic residue mod 383 (we note that 383 is a prime).

$$\left(\frac{219}{383}\right) = \left(\frac{3}{383}\right) \cdot \left(\frac{73}{383}\right)$$

We now flip the Legendre Symbols using quadratic reciprocity:

$$\begin{aligned} &= -\left(\frac{383}{3}\right) \cdot \left(\frac{383}{73}\right) \\ &= -\left(\frac{2}{3}\right) \cdot \left(\frac{18}{73}\right) \\ &= 1 \cdot \left(\frac{18}{73}\right) \\ &= \left(\frac{2}{73}\right) \cdot \left(\frac{9}{73}\right) \\ &= \left(\frac{2}{73}\right) = \boxed{1} \end{aligned}$$

**Remark.** We must factor the top argument before beginning to flip using quadratic reciprocity.

### §9.2 Proof of Quadratic Reciprocity

Recall [theorem 8.7](#):

**Theorem** (Law of Quadratic Reciprocity)

Let  $p, q \in \mathbb{Z}_+$  be distinct odd positive primes. Then

$$\left(\frac{p}{q}\right) \left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2} \frac{q-1}{2}}$$

In other words,

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)$$

if and only if at least one of  $p, q$  is congruent to 1 mod 4.

*Proof of Quadratic Reciprocity (theorem 8.7), using Gauss's Lemma (lemma 8.5).* WLOG, let

$$P = \left\{1, 2, \dots, \frac{p-1}{2}\right\}, \quad N = -P,$$

$$Q = \left\{1, 2, \dots, \frac{q-1}{2}\right\}$$

We write  $\tilde{P}, \tilde{N}$  for  $P \pmod{p}$  and  $N \pmod{p}$  respectively, so that Gauss's lemma gives

$$\left(\frac{q}{p}\right) = (-1)^\mu, \quad \text{where } \mu = |q\tilde{P} \cap \tilde{N}|$$

In other words,  $\mu$  is exactly the number of  $x \in P$  such that  $qx \equiv n \pmod{p}$  for some  $n \in N$ , and hence the number of  $x \in P$  such that for  $y \in \mathbb{Z}$ ,

$$-\frac{p}{2} < qx - py < 0.$$

We now specify more precisely which  $y$  can possibly satisfy this condition. Solving these inequalities for  $y$  gives

$$\frac{qx}{p} < y < \frac{qx}{p} + \frac{1}{2}.$$

OTOH, since  $x \leq \frac{p-1}{2} \forall x \in P$ , this gives

$$y < \frac{qx}{p} + \frac{1}{2} \leq \frac{q(p-1)}{2p} + \frac{1}{2}$$

$$< \frac{q+1}{2}.$$

Thus  $0 < y < \frac{q+1}{2}$ , which means that

$$y \in Q = \left\{1, 2, \dots, \frac{q-1}{2}\right\}.$$

We've shown that  $\mu$  is the number of points  $(x, y) \in P \times Q$  such that

$$\frac{p}{2} < qx - py < 0.$$

Switching  $p$  and  $q$ , we also have

$$\left(\frac{p}{q}\right) = (-1)^\eta$$

where  $\eta$  is the number of pairs

$$(y, x) \in Q \times P$$

such that

$$-\frac{q}{2} < py - qx < 0$$

which is exactly the number of pairs

$$(x, y) \in P \times Q$$

satisfying

$$0 < qx - py < \frac{q}{2}$$

(reflecting the inequality over 0).

We note that

$$\left(\frac{p}{q}\right) \left(\frac{q}{p}\right) = (-1)^\mu (-1)^\eta = (-1)^{\mu+\eta}$$

so all that remains is counting  $\mu$  and  $\eta$ . And we have that  $\mu + \eta$  is the number of ordered pairs  $(x, y) \in P \times Q$  such that either

$$-\frac{p}{2} < qx - py < 0 \text{ or } 0 < qx - py < \frac{q}{2}$$

Noting that  $qx - py \neq 0$  since  $x$  and  $y$  are from  $P$  and  $Q$  respectively, hence we can reduce this to

$$-\frac{p}{2} < qx - py < \frac{q}{2}.$$

Graphically, we are looking at: \*\*\*(Picture!)

where  $\mu + \eta$  is the number of lattice points in the shaded region.

If  $\alpha$  is the number of lattice points in  $A$  and  $\beta$  the number of lattice points in  $B$ . Then

$$\mu + \eta = \frac{p-1}{2} \frac{q-1}{2} - (\alpha + \beta)$$

We show that  $\alpha = \beta$  so that  $\alpha + \beta \equiv 0 \pmod{2}$ .

Let  $\rho$  be the rotation given by rotating the rectangle about its center leaves it invariant.

$$\rho(x, y) = \left( \frac{p+1}{2} - x, \frac{q+1}{2} - y \right)$$

Quick check that

$$qx - py < \frac{-p}{2} \Leftrightarrow qx' - py' > \frac{q}{2}$$

Since  $\rho$  maps lattice points to lattice points, then  $\alpha = \beta$  which concludes the proof with a little extra handiwork.  $\square$

### §9.3 Jacobi Symbol

The Jacobi symbol generalizes the Legendre symbol.

#### Definition 9.2

Let  $b$  be an odd positive integer and let  $a \in \mathbb{Z}$ . Write  $b = p_1 p_2 \cdots p_m$ , where  $p_i$  are (not necessarily distinct) primes. Then we write

$$\left(\frac{a}{b}\right) = \left(\frac{a}{p_1}\right) \left(\frac{a}{p_2}\right) \cdots \left(\frac{a}{p_m}\right)$$

is called the Jacobi symbol.

We note some basic properties that the Jacobi symbol is totally multiplicative (on top and bottom!):

$$\begin{aligned} \left(\frac{a_1 a_2}{b}\right) &= \left(\frac{a_1}{b}\right) \left(\frac{a_2}{b}\right) \\ \left(\frac{a}{b_1 b_2}\right) &= \left(\frac{a}{b_1}\right) \left(\frac{a}{b_2}\right) \end{aligned}$$

**Remark.** Note that they're multiplicative *fixing* either top or bottom. That is, they don't multiply like fractions.

**Warning!**  $\left(\frac{a}{b}\right) = 1$  does not imply that  $a$  is a quadratic residue modulo  $b$  (since we could have  $-1$ 's from the factorization cancel out).

However,  $\left(\frac{a}{b}\right) = -1$  *does* imply that  $a$  is a non-residue modulo  $b$ . (it is a non-residue mod at least one of prime factors of  $b$ ).

#### Example 9.3

$$\left(\frac{2}{15}\right) = \left(\frac{2}{3}\right) \left(\frac{2}{5}\right) = (-1)(-1) = 1$$

but 2 is not a quadratic residue modulo 15.

**Proposition 9.4** (5.2.2 of Text)

We have the following properties about the Jacobi symbol:

(a)

$$\left(\frac{-1}{b}\right) = (-1)^{\frac{b-1}{2}}$$

(b)

$$\left(\frac{2}{b}\right) = (-1)^{\frac{b^2-1}{8}}$$

(c) If  $a, b \in \mathbb{Z}_+$ , then

$$\left(\frac{a}{b}\right) \left(\frac{b}{a}\right) = (-1)^{\frac{a-1}{2} \frac{b-1}{2}}$$