Math 1560: Number Theory *Lecture Notes*

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These are lecture notes for Math 1560: Number Theory taught at Brown University by Nicole Looper in the Spring of 2022.

Notes last updated February 1, 2022.

Contents

§0 January 27, 2022

§0.1 Course Logistics

- Mostly refer to syllabus for any information that you might need.
- Midterm is planned for March 17.
- Final exam schedule can be found on CAB.

§0.2 Introduction to Number Theory

Number theory can be split into two branches: analytic number theory and algebraic number theory.

What is number theory? Number theory is the study of integers and their analogues in algebraic number fields.

Prime numbers are a key focus of number theory, and the study of different properties of primes constitutes different fields of number theory:

i. The study of their distributional properties, which is analytic number theory.

ii. As building blocks for algebraic numbers, which is algebraic number theory.

§0.2.1 Examples of Analytic Number Theory

Here are some examples of analytic number theory and their statements:

- Prime Number Theorem
- Twin Prime Conjecture
- Goldbach's conjecture

Theorem 0.1 (Prime Number Theorem)
Let $\pi(x)$ be the number of primes between 1 and x, then

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\ln(x)} = 1.$$

Conjecture 0.2 (Twin Prime Conjecture)

Twin primes are pairs of primes p, q of the form $q = p + 2$. Examples include $(3,5), (11,13), \dots$. The conjecture postulates that there are infinitely many twin primes.

Conjecture 0.3 (Goldbach's conjecture)

Any positive even integer greater than 2 can be written as the sum of 2 primes.

§0.2.2 Examples of Algebraic Number Theory

Analyzing the factorization (rings of integers) of number fields is one topic of algebraic number theory.

Example 0.4

2 is prime (irreducible) in $\mathbb{Z}$.

Yet 2 is not prime in $\mathbb{Z}[i]$ (the Gaussian integers). This is because

$$2 = \underbrace{(1+i)(1-i)}_{\text{associates}}$$

we have that $(1+i) = i(1-i)$. We also note the property that the principal ideals $(2) = (1+i)^2$ are equal.

In this example, we say that 2 "ramifies" in the ring of integers.

Fermat's Last Theorem is another such example.

Recall: that a *Pythagorean triple* is a triple of the form $a, b, c \in \mathbb{Z}_+$ such that

$$a^2 + b^2 = c^2$$

Are there examples of such numbers with different exponents (say, k^{th} powers for $k \geq 3$)?

Theorem 0.5 (Fermat's Last)

There are no positive integers $a, b, c \in \mathbb{Z}_+$ satisfying

$$a^k + b^k = c^k$$

for $k \geq 3$.

The answer is no! (Proved by Andrew Wiles)

Conjecture 0.6 (*abc* Conjecture, informally)

We say *powerful numbers* are positive integers whose prime factorization contains relatively few distinct primes (appropriately weighted) with an exponent of 1.

Example

$2^{10}3^7$ is powerful, $2^{10}3^75$ is powerful, 1 is powerful.

If a, b are *very powerful* coprime numbers, then $a + b$ is predicted to be *not powerful*.

Example 0.7

Consider 2^{10} and 3^{15}. We have

$$2^{10} + 3^{15} = 14,349,931 = \underbrace{31 \cdot 462 \cdot 901}_{\text{not powerful}}$$

What about another example, like $3^{15} + 5$? The *abc* conjecture also predicts that this number is not so powerful...[1]

§1 February 1, 2022

Happy Lunar New Year!

(Thanks Qinan and Andrew for allowing me to shamelessly copy their notes.)

§1.1 Divisibility and Factorization

We start with some commonly used notation:

Definition 1.1 (Divisibility)

We use $a \mid b$ to mean "a divides b" and $a \nmid b$ to mean "a does not divide b".

Now for a series of definitions:

Definition 1.2 (Primality)

A positive integer $p \geq 2$ is said to be prime if its only positive divisors are 1 and p.

[1] After lecture Jiahua: It's a prime!?

Definition 1.3 (Positive Integers)

$\mathbb{Z}_+$ will denote the positive integers.

Definition 1.4 (Order)

For a nonzero $n \in \mathbb{Z}$ and a prime p, there is a nonnegative integer a such that $p^a \mid n$ but $p^{a+1} \nmid n$. This number a is called the order of n at p, denoted by $\operatorname{ord}_p n$.

For $n = 0$, we set $\operatorname{ord}_p 0 = \infty$. We also have $\operatorname{ord}_p n = 0 \Leftrightarrow p \nmid n$.

We prove a lemma as warm-up:

Lemma 1.5 (Existence of Factorization)

Every nonzero integer can be written as a product of primes.

We make an exception for -1. *The empty product is* 1 *so* 1 *is fine.*

Proof. Suppose for the sake of contradiction otherwise, that some nonzero integer can be written as a product of primes. Let N be the smallest integer greater than 2 that cannot be written as a product of primes.

N had better not be a prime number itself (since then it would be a product of itself). Then we can write $N = a \cdot b$ where $1 < a, b < N$.

Since we took N as the least such number that cannot be written as a product of primes, a and b which are less than N can be written as a product of primes. Then N is a product of primes since a and b individually are. This is a contradiction! Thus it had better be the case that *every* nonzero integer can be written as a product of primes. $\square$

This is the theorem we will eventually work toward proving:

Theorem 1.6 (Unique Factorization)

Every nonzero integer n yields a *unique* prime factorization

$$n = (-1)^{\varepsilon} \cdot \prod_p p^{a(p)}, a(p) \geq 0$$

where $\varepsilon = 0$ or 1, and $\varepsilon, a(p)$ are uniquely determined by n. Moreover, we note that $a(p) = \operatorname{ord}_p n$.

§1.2 Euclidean and Principal Ideal Domains

Before this proof, we first recall a conclusion from Math 1530:

Lemma 1.7 (Division Lemma)
If $a, b \in \mathbb{Z}$ and $b > 0$, then there exists $q, r \in \mathbb{Z}$ such that

$$a = bq + r$$

with $0 \leq r < b$.

Proof. Consider the set

$$S = \{a - xb \mid x \in \mathbb{Z}\}$$

We note that S contains *some* positive elements. Let $r = a - qb$ be the least nonnegative element of S.

We claim that $0 \leq r < b$. Suppose for the sake of contradiction otherwise, then $r = a - qb \geq b$ gives $a - qb - b \leq 0$ and $a - (q+1)b \leq 0$. Which is a contradiction since we took r to be a the least nonnegative element in S and we've found such smaller element $a - (q+1)b$.

Then it had better be that $0 \leq r < b$ for some $r, q \in \mathbb{Z}$. $\square$

Corollary 1.8
$\mathbb{Z}$ is a Euclidean domain, with a Euclidean function given by lemma 1.7.

$R[x]$ for field R is also a Euclidean domain, with $\lambda = \deg$.

Definition 1.9 (Euclidean Domain)
Let R be an integral domain. R is a <u>Euclidean domain</u> if there exists a function $\lambda : R \setminus \{0\} \to \mathbb{N}$ such that if $a, b \in R$ with $b \neq 0$, then there exists some $c, d \in R$ with the property that $a = cb + d$ with $d = 0$ or $\lambda(d) < \lambda(b)$.

Example 1.10
$\mathbb{Z}$ is a Euclidean domain with λ function given in lemma 1.7.

Proposition 1.11
If R is a Euclidean domain, then R is a principal ideal domain. That is, if $I \subseteq R$ is an ideal, then $\exists a \in R$ such that $I = Ra = \{ra \mid r \in R\}$.

Proof. Assume WLOG that I is not the trivial ideal $I \neq (0)$. Let $0 \neq a \in I$ such that $\lambda(a) \leq \lambda(b) \forall b \in I, b \neq 0$.

We claim that $I = (a) = Ra$.

We know that $Ra \subseteq I$ since I is an ideal. Let $b \in I$. Then $\exists c, d \in R$ such that $b = ca + d$ where $d = 0$ or $\lambda(d) < \lambda(a)$. Now we have $d = b - ca \in I$, so we can't have $\lambda(d) < \lambda(a)$. Thus $d = 0$, so $b = ca \in Ra$.

Hence we have $I \subseteq Ra$. Together, we conclude that $I = Ra$. □

Definition 1.12 (Principal Ideals, PIDs)
If $I = (a)$ for some $a \in I$, then I is said to be a principal ideal.

R is a principal ideal domain (PID) if every ideal of R is principal.

Here are some important properties of PIDs:

1. Nonunit irreducible elements are exactly the prime elements in R.

   Recall: $p \in R$ is irreducible if $a \mid p \Rightarrow a$ is either a unit or an associate of p.

   $p \in R$ is prime if $p \mid ab \Rightarrow p \mid a$ or $p \mid b$ and p is a nonzero, nonunit of R.

2. GCDs always exist in PIDs.

§1.3 Unique Prime Factorization

We're nearly ready to prove unique factorization, after a lemma:

Lemma 1.13
Suppose p is a prime, and $a, b \in Z$. Then $\operatorname{ord}_p(ab) = \operatorname{ord}_p a + \operatorname{ord}_p b$.

Proof. WLOG, assume $a, b \neq 0$. We let

$$\alpha = \operatorname{ord}_p a$$
$$\beta = \operatorname{ord}_p b$$

Then we have

$$a = p^\alpha \cdot c \text{ where } p \nmid c$$
$$b = p^\beta \cdot d \text{ where } p \nmid d$$

Thus, $ab = p^{\alpha+\beta} \cdot cd$. We have that $p \nmid cd$ since $p \nmid c$ and $p \nmid d$ (we rely on the fact that if p is irreducible, p is prime). Thus we have that $\operatorname{ord}_p(ab) = \alpha + \beta$. □

Proof. (of theorem 1.6, that $\mathbb{Z}$ is a UFD). Recall that for a nonzero $n \in \mathbb{Z}$, we write

$$n = (-1)^{\varepsilon} \prod_p p^a(p), \text{where } \varepsilon = 0 \text{ or } 1 \text{ and } a(p) \geq 0$$

Given a positive prime q, we take ord_q of both sides. By lemma 1.13, this yields

$$\operatorname{ord}_q n = \varepsilon \cdot \operatorname{ord}_q(-1) + \sigma_p a(p) \operatorname{ord}_q(p)$$

Since we have that $\operatorname{ord}_q(-1) = 0$ and $\operatorname{ord}_q(p) = 0, \forall p \neq q$, we've uniquely determined $a(q)$ since $\operatorname{ord}_q(n) = a(q)$. That is, $a(q)$ is *uniquely determined* for all primes q. So n has a *unique* prime factorization. □

§1.4 Greatest Common Divisors

Definition 1.14

Let R be an integral domain. Then $d \in R$ is said to be a gcd of two elements a, b if

i) $d \mid a$ and $d \mid b$,

ii) if $d' \mid a$ and $d' \mid b$, then $d' \mid d$.

> **Remark.** An aside for ring theory enthusiasts: gcd domains are a class of rings mroe general than PIDs or UFDs.

We will denote (a, b) as the gcd of a and b.

Caution, however! gcd's are only unique up to units.

Example

-5 and 5 are both gcds of -5 and 10 since -1 is a unit.

We will ake the convention that the gcd of 2 integers is the positive gcd, that is, $(-5, 10) = 5$.

An edge case is that $\gcd(0, 0) = 0$.