LMI Properties and Applications in Systems, Stability, and Control Theory

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June 14, 2019

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1 Preliminaries

1.1 Introduction

The purpose of this document is to collect and organize properties, tricks, and applications related to linear matrix inequalities (LMIs) from a number of references together in a single document. Proofs of the properties presented in this document are not included when they can be found in the cited references in the interest of brevity. Illustrative examples are included whenever necessary to fully explain a certain property. Multiple equivalent forms of LMIs are often presented to give the reader a choice of which form may be best suited for a particular problem at hand. The equivalency of some of the LMIs in this document may be straightforward to more experienced readers, but the authors believe that some readers may benefit from the presentation of multiple equivalent LMIs.

The document is organized as follows. In the remaining portions of Section 1, the notation used throughout the document is presented and some fundamental LMI properties are discussed. Section 2 features a collection of LMI properties and tricks that are interesting and potentially useful. The LMI properties and tricks in this section are grouped together based on similarities when possible. Applications of LMIs in systems and stability theory is included in Section 3. Section 4 presents a number of LMI-based optimal controller synthesis methods, while Section 5 presents LMI-based optimal estimation synthesis methods.

The authors would like to thank the following individuals for alerting us of errors, and providing useful comments and suggestions for improvement: Leila Bridgeman, Jyot Buch, Robyn Fortune, Peter Seiler.

Please note that this document is a work in progress. If you notice any errors or inaccuracies, or have any suggestions of content that should be included in this document, please email either of the authors at `rcaverly@umn.edu` or `james.richard.forbes@mcgill.ca` so that changes to future versions can be made.

1.2 Notation

In this document, matrices are denoted by boldface letters (e.g., $\mathbf{A} \in \mathbb{R}^{n \times n}$), column matrices are denoted by lowercase boldface letters (e.g., $\mathbf{x} \in \mathbb{R}^n$), scalars are denoted by simple letters (e.g., $\gamma \in \mathbb{R}$), and operators are denoted by script letters (e.g., $\mathcal{G} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$). The set of n by m real matrices is denoted as $\mathbb{R}^{n \times m}$, the set of n by m complex matrices is denoted as $\mathbb{C}^{n \times m}$, and the set of n by n symmetric matrices is denoted as $\mathbb{S}^n$. The identity matrix is written as $\mathbf{1}$ and a matrix filled with zeros is written as $\mathbf{0}$. The dimensions of $\mathbf{1}$ and $\mathbf{0}$ are specified when necessary. Repeated blocks within symmetric matrices are replaced by $*$ for brevity and clarity. The conjugate transpose or Hermitian transpose of the matrix $\mathbf{V} \in \mathbb{C}^{n \times m}$ is denoted by $\mathbf{V}^{\mathsf{H}}$. The notation $\mathrm{He}\{\cdot\}$ is used as a shorthand in situations with limited space, where $\mathrm{He}\{\cdot\} = (\cdot) + (\cdot)^{\mathsf{H}}$. The real and imaginary parts of the complex number $z \in \mathbb{C}$ are denoted as $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively. The Kroenecker product of two matrices is denoted by $\otimes$.

Consider the square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$. The eigenvalues of $\mathbf{A}$ are denoted by $\lambda_i(\mathbf{A})$, $i = 1, 2, \ldots, n$. The matrix $\mathbf{A}$ is Hurwitz if all of its eigenvalues are in the open left-half complex plane (i.e., $\mathrm{Re}\,(\lambda_i(\mathbf{A})) < 0$, $i = 1, \ldots, n$). A matrix is Schur if all of its eigenvalues are strictly within a unit disk centered at the origin of the complex plane (i.e., $|\lambda_i(\mathbf{A})| < 1$, $i = 1, \ldots, n$). If $\mathbf{A} \in \mathbb{S}^n$, then the minimum eigenvalue of $\mathbf{A}$ is denoted by $\underline{\lambda}(\mathbf{A})$ and its maximum eigenvalue is denoted by $\bar{\lambda}(\mathbf{A})$.

Consider the matrix $\mathbf{B} \in \mathbb{R}^{n\times m}$. The minimum singular value of $\mathbf{B}$ is denoted by $\underline{\sigma}(\mathbf{B})$ and its maximum singular value is denoted by $\bar{\sigma}(\mathbf{B})$. The range and nullspace of $\mathbf{B}$ are denoted by $\mathcal{R}(\mathbf{B})$ and $\mathcal{N}(\mathbf{B})$, respectively. The Frobenius norm of $\mathbf{B}$ is $\|\mathbf{B}\|_F = \sqrt{\operatorname{tr}(\mathbf{B}^{\mathsf{H}}\mathbf{B})}$.

A state-space realization of the continuous-time linear time-invariant (LTI) system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t),\end{aligned}$$

is often written compactly as $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ in this document. The argument of time is often omitted in continuous-time state-space realizations, unless needed to prevent ambiguity.

A state-space realization of the discrete-time LTI system

$$\begin{aligned}\mathbf{x}_{k+1} &= \mathbf{A}_{\mathrm{d}}\mathbf{x}_k + \mathbf{B}_{\mathrm{d}}\mathbf{u}_k,\\ \mathbf{y}_k &= \mathbf{C}_{\mathrm{d}}\mathbf{x}_k + \mathbf{D}_{\mathrm{d}}\mathbf{u}_k,\end{aligned}$$

is often written compactly as $(\mathbf{A}_{\mathrm{d}}, \mathbf{B}_{\mathrm{d}}, \mathbf{C}_{\mathrm{d}}, \mathbf{D}_{\mathrm{d}})$.

The $\mathcal{H}_\infty$ norm of the LTI system $\boldsymbol{\mathcal{G}}$ is denoted by $\|\boldsymbol{\mathcal{G}}\|_\infty$ and the $\mathcal{H}_2$ norm of $\boldsymbol{\mathcal{G}}$ is denoted by $\|\boldsymbol{\mathcal{G}}\|_2$.

The inner product spaces $\mathcal{L}_2$ and $\mathcal{L}_{2e}$ for continuous-time signals are defined as follows.

$$\begin{aligned}\mathcal{L}_2 &= \left\{ \mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n \;\middle|\; \|\mathbf{x}\|_2^2 = \int_0^\infty \mathbf{x}^{\mathsf{T}}(t)\mathbf{x}(t)\mathrm{d}t < \infty \right\},\\ \mathcal{L}_{2e} &= \left\{ \mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n \;\middle|\; \|\mathbf{x}\|_{2T}^2 = \int_0^T \mathbf{x}^{\mathsf{T}}(t)\mathbf{x}(t)\mathrm{d}t < \infty,\ \forall T \in \mathbb{R}_{\geq 0} \right\}.\end{aligned}$$

The inner product sequence spaces ℓ_2 and ℓ_{2e} for discrete-time signals are defined as follows.

$$\begin{aligned}\ell_2 &= \left\{ \mathbf{x} : \mathbb{Z}_{\geq 0} \to \mathbb{R}^n \;\middle|\; \|\mathbf{x}\|_2^2 = \sum_{k=0}^{\infty} \mathbf{x}_k^{\mathsf{T}}\mathbf{x}_k < \infty \right\},\\ \ell_{2e} &= \left\{ \mathbf{x} : \mathbb{Z}_{\geq 0} \to \mathbb{R}^n \;\middle|\; \|\mathbf{x}\|_{2N}^2 = \sum_{k=0}^{N} \mathbf{x}_k^{\mathsf{T}}\mathbf{x}_k < \infty,\ \forall N \in \mathbb{Z}_{\geq 0} \right\}.\end{aligned}$$

1.3 Definitions and Fundamental LMI Properties

1.3.1 Definiteness of a Matrix

Definition 1.1. [1, pp. 429–430] Consider the symmetric matrix $\mathbf{A} \in \mathbb{S}^n$. The matrix $\mathbf{A}$ is

a) *positive definite* if $\mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} > 0$, $\forall \mathbf{x} \neq \mathbf{0} \in \mathbb{R}^n$,

b) *positive semi-definite* if $\mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} \geq 0$, $\forall \mathbf{x} \in \mathbb{R}^n$,

c) *negative definite* if $\mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} < 0$, $\forall \mathbf{x} \neq \mathbf{0} \in \mathbb{R}^n$,

d) *negative semi-definite* if $\mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} \leq 0$, $\forall \mathbf{x} \in \mathbb{R}^n$,

e) and indefinite if $\mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x}$ is neither positive nor negative.

Theorem 1.2. [1, pp. 430–431], [2, p. 703] Consider the symmetric matrix $\mathbf{A} \in \mathbb{S}^n$. The matrix $\mathbf{A}$ is

a) *positive definite* if and only if $\underline{\lambda}(\mathbf{A}) > 0$,

b) *positive semi-definite* if and only if $\underline{\lambda}(\mathbf{A}) \geq 0$,

c) *negative definite* if and only if $\bar{\lambda}(\mathbf{A}) < 0$,

d) *negative semi-definite* if and only if $\bar{\lambda}(\mathbf{A}) \leq 0$,

e) and indefinite if and only if $\underline{\lambda}(\mathbf{A}) < 0$ and $\bar{\lambda}(\mathbf{A}) > 0$.

Proof. To see why the sign of $\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x}$ is dictated by the eigenvalues of $\mathbf{A}$, let $\mathbf{A} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}$, where $\mathbf{V}^{-1} = \mathbf{V}^\mathsf{T}$ because $\mathbf{A}$ is symmetric. Notice that

$$\begin{aligned}
\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} &= \mathbf{x}^\mathsf{T}\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}\mathbf{x} \\
&= \left(\mathbf{V}^\mathsf{T}\mathbf{x}\right)^\mathsf{T}\boldsymbol{\Lambda}\mathbf{V}^\mathsf{T}\mathbf{x} \\
&= \mathbf{z}^\mathsf{T}\boldsymbol{\Lambda}\mathbf{z} \\
&= \sum_{i=1}^{n}\lambda_i(\mathbf{A})z_i^2,
\end{aligned}$$

where $\mathbf{z} = \mathbf{V}^\mathsf{T}\mathbf{x} = \begin{bmatrix} z_1 & z_2 & \cdots & z_n \end{bmatrix}^\mathsf{T}$. □

When evaluating the sign of the quadratic form $\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x}$, there is no loss of generality in restricting $\mathbf{A}$ to be symmetric. This is seen through the next two examples.

Example 1.1. Consider the skew-symmetric matrix $\mathbf{A} = -\mathbf{A}^\mathsf{T} \in \mathbb{R}^{n\times n}$. Evaluating the quadratic form $\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x}$ yields

$$\begin{aligned}
\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} &= \tfrac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} + \tfrac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} \\
&= \tfrac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} + \tfrac{1}{2}\left(\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x}\right)^\mathsf{T} \\
&= \tfrac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} + \tfrac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{A}^\mathsf{T}\mathbf{x} \\
&= \tfrac{1}{2}\mathbf{x}^\mathsf{T}\left(\mathbf{A} - \mathbf{A}\right)\mathbf{x} \\
&= 0.
\end{aligned}$$

Therefore, $\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} = 0$ for all skew-symmetic matrices.

Example 1.2. Consider the matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, which can be decomposed as

$$\begin{aligned}
\mathbf{A} &= \tfrac{1}{2}\mathbf{A} + \tfrac{1}{2}\mathbf{A} \\
&= \tfrac{1}{2}\mathbf{A} + \tfrac{1}{2}\mathbf{A} + \tfrac{1}{2}\left(\mathbf{A}^\mathsf{T} - \mathbf{A}^\mathsf{T}\right) \\
&= \underbrace{\tfrac{1}{2}\left(\mathbf{A} + \mathbf{A}^\mathsf{T}\right)}_{\mathbf{A}_{\text{sym}}} + \underbrace{\tfrac{1}{2}\left(\mathbf{A} - \mathbf{A}^\mathsf{T}\right)}_{\mathbf{A}_{\text{skew}}},
\end{aligned}$$

where $\mathbf{A}_{\text{sym}} = \mathbf{A}_{\text{sym}}^{\mathsf{T}} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^{\mathsf{T}}\right)$ is the symmetric part of $\mathbf{A}$ and $\mathbf{A}_{\text{skew}} = -\mathbf{A}_{\text{skew}}^{\mathsf{T}} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^{\mathsf{T}}\right)$ is the skew-symmetric part of $\mathbf{A}$. Evaluating the quadratic form $\mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x}$ yields

$$\begin{aligned}
\mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} &= \tfrac{1}{2}\mathbf{x}^{\mathsf{T}}\left(\mathbf{A}_{\text{sym}} + \mathbf{A}_{\text{skew}}\right)\mathbf{x} \\
&= \tfrac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{A}_{\text{sym}}\mathbf{x} + \cancelto{0}{\tfrac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{A}_{\text{skew}}\mathbf{x}} \\
&= \tfrac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{A}_{\text{sym}}\mathbf{x}.
\end{aligned}$$

This confirms that when determining the definiteness of a matrix there is no loss of generality in restricting the matrix to be symmetric.

The positive definiteness and positive semidefiniteness of a matrix are denoted by > 0 and ≥ 0, respectively (e.g., $\mathbf{A} = \mathbf{A}^{\mathsf{T}} > 0$ is positive definite and $\mathbf{B} = \mathbf{B}^{\mathsf{T}} \geq 0$ is positive semidefinite). Similarly, the negative definiteness and negative semidefiniteness of a matrix are denoted by < 0 and ≤ 0, respectively (e.g., $\mathbf{C} = \mathbf{C}^{\mathsf{T}} < 0$ is negative definite and $\mathbf{D} = \mathbf{D}^{\mathsf{T}} \leq 0$ is negative semidefinite). For brevity, the transpose component of a definiteness statement is omitted in this document, for example, $\mathbf{A} = \mathbf{A}^{\mathsf{T}} > 0$ is simply written as $\mathbf{A} > 0$.

1.3.2 Matrix Inequalities and LMIs

Definition 1.3. A matrix inequality, $\mathbf{G} : \mathbb{R}^m \to \mathbb{S}^n$, in the variable $\mathbf{x} \in \mathbb{R}^m$ is an expression of the form

$$\mathbf{G}(\mathbf{x}) = \mathbf{G}_0 + \sum_{i=1}^{p} f_i(\mathbf{x})\mathbf{G}_i \leq 0,$$

where $\mathbf{x}^{\mathsf{T}} = \begin{bmatrix} x_1 \cdots x_m \end{bmatrix}$, $\mathbf{G}_0 \in \mathbb{S}^n$, and $\mathbf{G}_i \in \mathbb{R}^{n \times n}$, $i = 1, \ldots, p$.

Definition 1.4. A bilinear matrix inequality (BMI), $\mathbf{H} : \mathbb{R}^m \to \mathbb{S}^n$, in the variable $\mathbf{x} \in \mathbb{R}^m$ is an expression of the form

$$\mathbf{H}(\mathbf{x}) = \mathbf{H}_0 + \sum_{i=1}^{m} x_i \mathbf{H}_i + \sum_{i=1}^{m}\sum_{j=1}^{m} x_i x_j \mathbf{H}_{i,j} \leq 0,$$

where $\mathbf{x}^{\mathsf{T}} = \begin{bmatrix} x_1 \cdots x_m \end{bmatrix}$, and $\mathbf{H}_i$, $\mathbf{H}_{i,j} \in \mathbb{S}^n$, $i = 0, \ldots, m$, $j = 0, \ldots, m$.

Definition 1.5. [3, p. 7], [4, pp. 15–16] An LMI, $\mathbf{F} : \mathbb{R}^m \to \mathbb{S}^n$, in the variable $\mathbf{x} \in \mathbb{R}^m$ is an expression of the form

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_0 + \sum_{i=1}^{m} x_i \mathbf{F}_i \leq 0, \tag{1.1}$$

where $\mathbf{x}^{\mathsf{T}} = \begin{bmatrix} x_1 \cdots x_m \end{bmatrix}$ and $\mathbf{F}_i \in \mathbb{S}^n$, $i = 0, \ldots, m$.

Example 1.3. [3, pp. 8–9] Consider the matrices $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{Q} > 0$. It is desired to find a symmetric matrix $\mathbf{P} \in \mathbb{S}^n$ satisfying the inequality

$$\mathbf{P}\mathbf{A} + \mathbf{A}^{\mathsf{T}}\mathbf{P} + \mathbf{Q} < 0, \tag{1.2}$$

where $\mathbf{P} > 0$. The elements of $\mathbf{P}$ are the design variables in this problem, and although (1.2) is indeed an LMI in the matrix $\mathbf{P}$, it does not look like the LMI in (1.1). For simplicity, let us consider the case of $n = 2$ so that each matrix is of dimension 2×2, and $\mathbf{x} = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}^\mathsf{T}$. Writing the matrix $\mathbf{P}$ in terms of a basis $\mathbf{E}_i \in \mathbb{S}^2$, $i = 1, 2, 3$, yields

$$\mathbf{P} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = p_1 \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{E}_1} + p_2 \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{E}_2} + p_3 \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{E}_3}.$$

Note that the matrices $\mathbf{E}_i$ are linearly independent and symmetric, thus forming a basis for the symmetric matrix $\mathbf{P}$. The matrix inequality in (1.2) can be written as

$$p_1 \left(\mathbf{E}_1\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{E}_1\right) + p_2 \left(\mathbf{E}_2\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{E}_2\right) + p_3 \left(\mathbf{E}_3\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{E}_3\right).$$

Defining $\mathbf{F}_0 = \mathbf{Q}$ and $\mathbf{F}_i = \mathbf{F}_i^\mathsf{T} = \mathbf{E}_i\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{E}_i$, $i = 1, 2, 3$, yields

$$\mathbf{F}_0 + \sum_{i=1}^{3} p_i \mathbf{F}_i < 0,$$

which now resembles the definition of an LMI in (1.1). Throughout this document, LMIs are typically written in the matrix form of (1.2), rather than the scalar form of (1.1).

1.3.3 Convexity of LMIs

Definition 1.6. [5, p. 138] A set, $\mathcal{S}$, in a real inner product space is convex if for all $\mathbf{x}, \mathbf{y} \in \mathcal{S}$ and $\alpha \in \mathbb{R}$, where $0 \leq \alpha \leq 1$, it holds that $\alpha\mathbf{x} + (1 - \alpha)\mathbf{y} \in \mathcal{S}$.

Lemma 1.1. The set of solutions to an LMI is convex. That is, the set $\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^m \mid \mathbf{F}(\mathbf{x}) \leq 0\}$ is a convex set, where $\mathbf{F} : \mathbb{R}^m \to \mathbb{S}^n$ is an LMI.

Proof. Consider $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ and $\alpha \in [0, 1]$, and suppose that $\mathbf{x}$ and $\mathbf{y}$ satisfy (1.1). The LMI $\mathbf{F} : \mathbb{R}^m \to \mathbb{S}^n$ is convex, since

$$\begin{aligned}
\mathbf{F}(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) &= \mathbf{F}_0 + \sum_{i=1}^{m} \left(\alpha x_i + (1 - \alpha) y_i\right) \mathbf{F}_i \\
&= \mathbf{F}_0 - \alpha\mathbf{F}_0 + \alpha\mathbf{F}_0 + \alpha \sum_{i=1}^{m} x_i \mathbf{F}_1 + (1 - \alpha) \sum_{i=1}^{m} y_i \mathbf{F}_i \\
&= \alpha\mathbf{F}_0 + \alpha \sum_{i=1}^{m} x_i \mathbf{F}_i + (1 - \alpha)\mathbf{F}_0 + (1 - \alpha) \sum_{i=1}^{m} y_i \mathbf{F}_i \\
&= \alpha\mathbf{F}(\mathbf{x}) + (1 - \alpha)\mathbf{F}(\mathbf{y}).
\end{aligned}$$

$\square$

From Lemma 1.1, it is known that an optimization problem with a convex objective function and LMI constraints is convex. The following is a non-exhaustive list of scalar convex objective functions involving matrix variables that can be minimized in conjunction with LMI constraints to yield a semidefinite programming (SDP) problem.

- [6, p. 71] $\mathcal{J}(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{P}\mathbf{x} + \mathbf{q}^\mathsf{T}\mathbf{x} + r$, where $\mathbf{x}$, $\mathbf{q} \in \mathbb{R}^n$, $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{P} > 0$, and $r \in \mathbb{R}$.

- Special case when $\mathbf{q} = 0$ and $r = 0$: $\mathcal{J}(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{P}\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{P} \in \mathbb{S}^n$, and $\mathbf{P} > 0$.
- Special case when $\mathbf{P} = 2 \cdot \mathbf{1}$, $\mathbf{q} = 0$, and $r = 0$: $\mathcal{J}(\mathbf{x}) = \mathbf{x}^\mathsf{T}\mathbf{x} = \|\mathbf{x}\|_2^2$, where $\mathbf{x} \in \mathbb{R}^n$.

- $\mathcal{J}(\mathbf{X}) = \operatorname{tr}\left(\mathbf{X}^\mathsf{T}\mathbf{P}\mathbf{X} + \mathbf{Q}^\mathsf{T}\mathbf{X} + \mathbf{X}^\mathsf{T}\mathbf{R} + \mathbf{S}\right)$, where $\mathbf{X}$, $\mathbf{Q}$, $\mathbf{R} \in \mathbb{R}^{n\times m}$, $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{S} \in \mathbb{R}^{n\times n}$, and $\mathbf{P} \geq 0$.
  - Special case when $\mathbf{Q} = \mathbf{R} = \mathbf{0}$ and $\mathbf{S} = \mathbf{0}$: $\mathcal{J}(\mathbf{X}) = \operatorname{tr}\left(\mathbf{X}^\mathsf{T}\mathbf{P}\mathbf{X}\right)$, where $\mathbf{X} \in \mathbb{R}^{n\times m}$, $\mathbf{P} \in \mathbb{S}^n$, and $\mathbf{P} > 0$.
  - Special case when $\mathbf{P} = \mathbf{1}$, $\mathbf{Q} = \mathbf{R} = \mathbf{0}$, and $\mathbf{S} = \mathbf{0}$: $\mathcal{J}(\mathbf{X}) = \operatorname{tr}\left(\mathbf{X}^\mathsf{T}\mathbf{X}\right) = \|\mathbf{X}\|_F^2$, where $\mathbf{X} \in \mathbb{R}^{n\times m}$.
  - [3, p. 88] Special case when $\mathbf{P} = \mathbf{0}$, $\mathbf{R} = \mathbf{0}$ and $\mathbf{S} = \mathbf{0}$: $\mathcal{J}(\mathbf{X}) = \operatorname{tr}(\mathbf{Q}^\mathsf{T}\mathbf{X})$, where $\mathbf{X}$, $\mathbf{Q} \in \mathbb{R}^{n\times m}$.
  - [2, p. 718] Special case when $\mathbf{P} = \mathbf{1}$, $\mathbf{Q} = \mathbf{R} = \mathbf{0}$, $\mathbf{S} = \mathbf{0}$, and $\mathbf{X} \in \mathbb{S}^n$: $\mathcal{J}(\mathbf{X}) = \operatorname{tr}(\mathbf{X}^2)$, where $\mathbf{X} \in \mathbb{S}^n$.

- [3, p. 14] $\mathcal{J}(\mathbf{X}) = \log\left(\det(\mathbf{X}^{-1})\right) = -\log\left(\det(\mathbf{X})\right)$, where $\mathbf{X} \in \mathbb{S}^n$ and $\mathbf{X} > 0$.

1.3.4 Relative Definiteness of a Matrix [2, pp. 703–704]

The definiteness of a matrix can be found relative to another matrix. For example, consider the matrices $\mathbf{A} \in \mathbb{S}^n$ and $\mathbf{B} \in \mathbb{S}^n$. The matrix inequality $\mathbf{A} < \mathbf{B}$ is equivalent to $\mathbf{A} - \mathbf{B} < 0$ or $\mathbf{B} - \mathbf{A} > 0$.

Knowing the relative definiteness of matrices can be useful. For example, if in the previous example we have $\mathbf{A} < \mathbf{B}$ and also know that $\mathbf{A} > 0$, then we know that $\mathbf{B} > 0$. This follows from $0 < \mathbf{A} < \mathbf{B}$. For more facts involving the relative definiteness of matrices, see [2, pp. 703–704].

1.3.5 Strict and Nonstrict Matrix Inequalities

A strict matrix inequality can be converted to a nonstrict matrix inequality. For example, $\mathbf{A} > 0$ is implied by $\mathbf{A} \geq \epsilon\mathbf{1}$, where $\epsilon \in \mathbb{R}_{>0}$. Similarly, $\mathbf{B} < 0$ is implied by $\mathbf{B} \leq -\epsilon\mathbf{1}$, where $\epsilon \in \mathbb{R}_{>0}$.

Converting a strict matrix inequality into a nonstrict matrix inequality is useful when working with LMI solvers that cannot handle strict constraints.

1.3.6 Concatenation of LMIs

A useful property of LMIs is that multiple LMIs can be concatenated together to form a single LMI. For example, satisfying the LMIs $\mathbf{A} < 0$ and $\mathbf{B} < 0$ is equivalent to satisfying the concatenated LMI

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix} < 0.$$

More generally, satisfying the LMIs $\mathbf{A}_i < 0$, $i = 1, \ldots, n$ is equivalent to satisfying the concatenated LMI $\operatorname{diag}\{\mathbf{A}_1, \ldots, \mathbf{A}_n\} < 0$.

1.4 LMI Solvers

There are many semidefinite programming software packages that accept LMI constraints. The authors have experience with `SeDuMi` [7], `SDPT3` [8], and `Mosek` [9], though other software packages are available, such as `CSDP` [10], and `LMILab` [11]. There are advantages and disadvantages to each of these solvers, and sometimes one solver may give a solution to a given problem when others do not. For this reason, it is useful to have multiple solvers available. Comparisons of various LMI solvers and benchmark problems are found in [12–14].

The solvers `SeDuMi`, `SDPT3`, and many others listed are available for free, while `Mosek` is a commercial software package. A free academic license of `Mosek` can be requested for research in academic institutions or educational purposes.

The openly-distributed toolboxes `Yalmip` [15] and `CVX` [16] can be used within `Matlab` to interface with `SeDuMi`, `SDPT3`, or `Mosek`.

2 LMI Properties and Tricks

This section presents a compilation of LMI properties and tricks from the literature. Many of these properties are used in subsequent sections to reformulate LMIs or transform matrix inequalities into LMIs.

2.1 Change of Variables [3, pp. 100–101], [17, p. 480]

A BMI can sometimes be converted into an LMI using a change of variables.

Example 2.1. [17, p. 480] Consider $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{K} \in \mathbb{R}^{m\times n}$, and $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{Q} > 0$. The matrix inequality given by

$$\mathbf{Q}\mathbf{A}^\mathsf{T} + \mathbf{A}\mathbf{Q} - \mathbf{Q}\mathbf{K}^\mathsf{T}\mathbf{B}^\mathsf{T} - \mathbf{B}\mathbf{K}\mathbf{Q} < 0,$$

is bilinear in the variables $\mathbf{Q}$ and $\mathbf{K}$. Define a change of variable as $\mathbf{F} = \mathbf{K}\mathbf{Q}$ to obtain

$$\mathbf{Q}\mathbf{A}^\mathsf{T} + \mathbf{A}\mathbf{Q} - \mathbf{F}^\mathsf{T}\mathbf{B}^\mathsf{T} - \mathbf{B}\mathbf{F} < 0,$$

which is an LMI in the variables $\mathbf{Q}$ and $\mathbf{F}$. Once this LMI is solved, the original variable can be recovered by $\mathbf{K} = \mathbf{F}\mathbf{Q}^{-1}$.

It is important that a change of variables is chosen to be a one-to-one mapping in order for the new matrix inequality to be equivalent to the original matrix inequality. In Example 2.1 the change of variable $\mathbf{F} = \mathbf{K}\mathbf{Q}$ is a one-to-one mapping since $\mathbf{Q}^{-1}$ is invertible, which gives a unique solution for the reverse change of variable $\mathbf{K} = \mathbf{F}\mathbf{Q}^{-1}$.

2.2 Congruence Transformation [3, p. 15], [17, p. 481]

Consider $\mathbf{Q} \in \mathbb{S}^n$ and $\mathbf{W} \in \mathbb{R}^{n\times n}$, where $\operatorname{rank}(\mathbf{W}) = n$. The matrix inequality $\mathbf{Q} < 0$ is satisfied if and only if $\mathbf{W}\mathbf{Q}\mathbf{W}^\mathsf{T} < 0$ or equivalently $\mathbf{W}^\mathsf{T}\mathbf{Q}\mathbf{W} < 0$.

Example 2.2. [17, p. 481] Consider $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{K} \in \mathbb{R}^{m\times p}$, $\mathbf{C}^\mathsf{T} \in \mathbb{R}^{n\times p}$, $\mathbf{P} \in \mathbb{S}^n$, and $\mathbf{V} \in \mathbb{S}^p$, where $\mathbf{P} > 0$ and $\mathbf{V} > 0$. The matrix inequality given by

$$\mathbf{Q} = \begin{bmatrix} \mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{P}\mathbf{A} & -\mathbf{P}\mathbf{B}\mathbf{K} + \mathbf{C}^\mathsf{T}\mathbf{V} \\ * & -2\mathbf{V} \end{bmatrix} < 0,$$

is linear in the variable $\mathbf{V}$ and bilinear in the variable pair $(\mathbf{P}, \mathbf{K})$. Choose the matrix $\mathbf{W} = \operatorname{diag}\{\mathbf{P}^{-1}, \mathbf{V}^{-1}\}$ to obtain an equivalent BMI given by

$$\mathbf{W}\mathbf{Q}\mathbf{W}^\mathsf{T} = \begin{bmatrix} \mathbf{P}^{-1}\mathbf{A}^\mathsf{T} + \mathbf{A}\mathbf{P}^{-1} & -\mathbf{B}\mathbf{K}\mathbf{V}^{-1} + \mathbf{P}^{-1}\mathbf{C}^\mathsf{T} \\ * & -2\mathbf{V}^{-1} \end{bmatrix} < 0. \tag{2.1}$$

Using a change of variable $\mathbf{X} = \mathbf{P}^{-1}$, $\mathbf{U} = \mathbf{V}^{-1}$, and $\mathbf{F} = \mathbf{K}\mathbf{V}^{-1}$, (2.1) becomes

$$\mathbf{W}\mathbf{Q}\mathbf{W}^\mathsf{T} = \begin{bmatrix} \mathbf{X}\mathbf{A}^\mathsf{T} + \mathbf{A}\mathbf{X} & -\mathbf{B}\mathbf{F} + \mathbf{X}\mathbf{C}^\mathsf{T} \\ * & -2\mathbf{U} \end{bmatrix} < 0, \tag{2.2}$$

which is an LMI in the variables $\mathbf{X}$, $\mathbf{U}$, and $\mathbf{F}$. Once (2.2) is solved, the original variable $\mathbf{K}$ is recovered by the reverse change of variable $\mathbf{K} = \mathbf{F}\mathbf{U}^{-1}$.

A congruence transformation preserves the definiteness of a matrix by ensuring that $\mathbf{Q} < 0$ and $\mathbf{W}^\mathsf{T}\mathbf{Q}\mathbf{W} < 0$ are equivalent. A congruence transformation is related, but not equivalent to a similarity transformation $\mathbf{T}\mathbf{Q}\mathbf{T}^{-1}$, which preserves not only the definiteness, but also the eigenvalues of a matrix. A congruence transformation is equivalent to a similarity transformation in the special case when $\mathbf{W}^\mathsf{T} = \mathbf{W}^{-1}$.

2.3 Schur Complement

2.3.1 Strict Schur Complement [3, pp. 7–8], [17, p. 481]

Consider $\mathbf{A} \in \mathbb{S}^n$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, and $\mathbf{C} \in \mathbb{S}^m$. The following statements are equivalent.

a) $\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\mathsf{T} & \mathbf{C} \end{bmatrix} < 0.$

b) $\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^\mathsf{T} < 0$, $\mathbf{C} < 0$.

c) $\mathbf{C} - \mathbf{B}^\mathsf{T}\mathbf{A}^{-1}\mathbf{B} < 0$, $\mathbf{A} < 0$.

2.3.2 Nonstrict Schur Complement [3, p. 28]

Consider $\mathbf{A} \in \mathbb{S}^n$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, and $\mathbf{C} \in \mathbb{S}^m$. The following statements are equivalent.

a) $\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\mathsf{T} & \mathbf{C} \end{bmatrix} \leq 0.$

b) $\mathbf{A} - \mathbf{B}\mathbf{C}^{+}\mathbf{B}^\mathsf{T} < 0$, $\mathbf{C} \leq 0$, $\mathbf{B}(\mathbf{1} - \mathbf{C}\mathbf{C}^{+}) = \mathbf{0}$, where $\mathbf{C}^{+}$ is the Moore-Penrose inverse of $\mathbf{C}$.

c) $\mathbf{C} - \mathbf{B}^\mathsf{T}\mathbf{A}^{+}\mathbf{B} < 0$, $\mathbf{A} \leq 0$, $\mathbf{B}^\mathsf{T}(\mathbf{1} - \mathbf{A}\mathbf{A}^{+}) = \mathbf{0}$, where $\mathbf{A}^{+}$ is the Moore-Penrose inverse of $\mathbf{A}$.

2.3.3 Schur Complement Lemma-Based Properties

1. [18, p. 100] Consider $\mathbf{P}_{11} \in \mathbb{S}^n$, $\mathbf{P}_{12} \in \mathbb{R}^{n\times m}$, $\mathbf{P}_{22}$, $\mathbf{X} \in \mathbb{S}^m$, $\mathbf{P}_{13} \in \mathbb{R}^{n\times p}$, $\mathbf{P}_{23} \in \mathbb{R}^{m\times p}$, and $\mathbf{P}_{33} \in \mathbb{S}^p$. There exists $\mathbf{X}$ such that

$$\begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} & \mathbf{P}_{13} \\ * & \mathbf{P}_{22} + \mathbf{X} & \mathbf{P}_{23} \\ * & * & \mathbf{P}_{33} \end{bmatrix} < 0, \tag{2.3}$$

   if and only if

$$\begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{13} \\ * & \mathbf{P}_{33} \end{bmatrix} < 0.$$

   Any matrix $\mathbf{X} \in \mathbb{S}^m$ satisfying

$$\mathbf{X} < -\mathbf{P}_{22} + \begin{bmatrix} \mathbf{P}_{12}^\mathsf{T} & \mathbf{P}_{23} \end{bmatrix} \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{13} \\ * & \mathbf{P}_{33} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{P}_{12} \\ \mathbf{P}_{23}^\mathsf{T} \end{bmatrix} \tag{2.4}$$

   is a solution to (2.3). That is, (2.4) $\Longrightarrow$ (2.3).

2. [18, p. 101] Consider $\mathbf{P}_{11} \in \mathbb{S}^n$, $\mathbf{P}_{12}$, $\mathbf{X} \in \mathbb{R}^{n\times m}$, $\mathbf{P}_{22} \in \mathbb{S}^m$, $\mathbf{P}_{13} \in \mathbb{R}^{n\times p}$, $\mathbf{P}_{23} \in \mathbb{R}^{m\times p}$, and $\mathbf{P}_{33} \in \mathbb{S}^p$. There exists $\mathbf{X}$ such that

$$\begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} + \mathbf{X}^\mathsf{T} & \mathbf{P}_{13} \\ * & \mathbf{P}_{22} & \mathbf{P}_{23} \\ * & * & \mathbf{P}_{33} \end{bmatrix} < 0 \tag{2.5}$$

if and only if

$$\begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{13} \\ * & \mathbf{P}_{33} \end{bmatrix} < 0, \quad \text{and} \quad \begin{bmatrix} \mathbf{P}_{22} & \mathbf{P}_{23} \\ * & \mathbf{P}_{33} \end{bmatrix} < 0. \tag{2.6}$$

If the two matrix inequalities in (2.6) hold, then a solution to (2.5) is given by

$$\mathbf{X} = \mathbf{P}_{23}\mathbf{P}_{33}^{-1}\mathbf{P}_{13}^\mathsf{T} - \mathbf{P}_{12}^\mathsf{T}.$$

Proof. Necessity ((2.5) $\Longrightarrow$ (2.6)) comes from the requirement that the submatrices corresponding to the principle minors of (2.5) are negative definite. Sufficiency ((2.6) $\Longrightarrow$ (2.5)) is shown by rewriting the matrix inequalities of (2.6) in the equivalent form

$$\mathbf{P}_{11} - \mathbf{P}_{13}^\mathsf{T}\mathbf{P}_{33}^{-1}\mathbf{P}_{13} < 0, \quad \text{and} \quad \mathbf{P}_{22} - \mathbf{P}_{23}^\mathsf{T}\mathbf{P}_{33}^{-1}\mathbf{P}_{23} < 0. \tag{2.7}$$

Concatenating the two matrix inequalities in (2.7) and choosing $\mathbf{X} = \mathbf{P}_{23}\mathbf{P}_{33}^{-1}\mathbf{P}_{13}^\mathsf{T} - \mathbf{P}_{12}^\mathsf{T}$ gives the equivalent matrix inequality

$$\begin{bmatrix} \mathbf{P}_{11} - \mathbf{P}_{13}^\mathsf{T}\mathbf{P}_{33}^{-1}\mathbf{P}_{13} & \mathbf{P}_{12} - \mathbf{P}_{13}^\mathsf{T}\mathbf{P}_{33}^{-1}\mathbf{P}_{23} + \mathbf{X}^\mathsf{T} \\ * & \mathbf{P}_{22} - \mathbf{P}_{23}^\mathsf{T}\mathbf{P}_{33}^{-1}\mathbf{P}_{23} \end{bmatrix} < 0,$$

or

$$\begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} + \mathbf{X}^\mathsf{T} \\ * & \mathbf{P}_{22} \end{bmatrix} - \begin{bmatrix} \mathbf{P}_{13}^\mathsf{T} \\ \mathbf{P}_{23}^\mathsf{T} \end{bmatrix} \mathbf{P}_{33}^{-1} \begin{bmatrix} \mathbf{P}_{13} & \mathbf{P}_{23} \end{bmatrix} < 0,$$

which is equivalent to (2.5) using the Schur complemet lemma. □

Permutation of the columns and rows of (2.5) yields the following equivalent result.

[19, pp. 41–42] Consider $\mathbf{P}_{11} \in \mathbb{S}^n$, $\mathbf{P}_{12}$, $\mathbf{X} \in \mathbb{R}^{n\times m}$, $\mathbf{P}_{22} \in \mathbb{S}^m$, $\mathbf{P}_{13} \in \mathbb{R}^{n\times p}$, $\mathbf{P}_{23} \in \mathbb{R}^{m\times p}$, and $\mathbf{P}_{33} \in \mathbb{S}^p$. There exists $\mathbf{X}$ such that

$$\begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} & \mathbf{P}_{13} \\ * & \mathbf{P}_{22} & \mathbf{P}_{23} + \mathbf{X}^\mathsf{T} \\ * & * & \mathbf{P}_{33} \end{bmatrix} < 0 \tag{2.8}$$

if and only if

$$\begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ * & \mathbf{P}_{22} \end{bmatrix} < 0, \quad \text{and} \quad \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{13} \\ * & \mathbf{P}_{33} \end{bmatrix} < 0. \tag{2.9}$$

If the matrix inequalities in (2.9) hold, then a solution to (2.8) is given by

$$\mathbf{X} = \mathbf{P}_{13}^\mathsf{T}\mathbf{P}_{11}^{-1}\mathbf{P}_{12} - \mathbf{P}_{23}^\mathsf{T}.$$

3. [19, p. 41] Consider $\mathbf{P}_{11}, \mathbf{X} \in \mathbb{S}^n$, $\mathbf{P}_{12} \in \mathbb{R}^{n\times m}$, and $\mathbf{P}_{22} \in \mathbb{S}^m$, where $\mathbf{X} > 0$. There exists $\mathbf{X}$ such that
$$\begin{bmatrix} \mathbf{P}_{11} - \mathbf{X} & \mathbf{P}_{12} & \mathbf{X} \\ * & \mathbf{P}_{22} & \mathbf{0} \\ * & * & -\mathbf{X} \end{bmatrix} < 0, \tag{2.10}$$
if and only if
$$\begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ * & \mathbf{P}_{22} \end{bmatrix} < 0. \tag{2.11}$$

Proof. The matrix inequality in (2.10) can be rewritten using the Schur complement lemma as
$$\begin{aligned} \begin{bmatrix} \mathbf{P}_{11} - \mathbf{X} & \mathbf{P}_{12} \\ * & \mathbf{P}_{22} \end{bmatrix} - \begin{bmatrix} \mathbf{X} \\ \mathbf{0} \end{bmatrix} \mathbf{X}^{-1} \begin{bmatrix} \mathbf{X} & \mathbf{0} \end{bmatrix} &< 0 \\ \begin{bmatrix} \mathbf{P}_{11} - \mathbf{X} & \mathbf{P}_{12} \\ * & \mathbf{P}_{22} \end{bmatrix} - \begin{bmatrix} \mathbf{X} & \mathbf{0} \\ * & \mathbf{0} \end{bmatrix} &< 0 \\ \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ * & \mathbf{P}_{22} \end{bmatrix} &< 0. \end{aligned}$$

□

4. [20] Consider $\mathbf{X} \in \mathbb{S}^n$, $\mathbf{H} \in \mathbb{R}^{m\times n}$, $\mathbf{G} \in \mathbb{R}^{m\times m}$, and $\mathbf{P} \in \mathbb{S}^m$, where $\mathbf{P} > 0$. The matrix inequality given by
$$\begin{bmatrix} \mathbf{X} & \mathbf{H}^\mathsf{T} \\ * & \mathbf{G} + \mathbf{G}^\mathsf{T} - \mathbf{P} \end{bmatrix} > 0, \tag{2.12}$$
implies
$$\mathbf{X} > \mathbf{H}^\mathsf{T}\mathbf{G}^{-1}\mathbf{P}\mathbf{G}^{-\mathsf{T}}\mathbf{H}. \tag{2.13}$$
For $\mathbf{G} = \mathbf{P}$, this relationship becomes the Schur complement lemma.

Proof. Using the Schur complement lemma on (2.12) gives
$$\mathbf{X} > \mathbf{H}^\mathsf{T} \left(\mathbf{G} + \mathbf{G}^\mathsf{T} - \mathbf{P}\right)^{-1} \mathbf{H}.$$
Using the property $\mathbf{G} + \mathbf{G}^\mathsf{T} - \mathbf{P} \leq \mathbf{G}^\mathsf{T}\mathbf{P}^{-1}\mathbf{G}$ (see the special case of Young's relation in Section 2.4.3), or equivalently $\left(\mathbf{G} + \mathbf{G}^\mathsf{T} - \mathbf{P}\right)^{-1} \geq \mathbf{G}^{-1}\mathbf{P}\mathbf{G}^{-\mathsf{T}}$ gives
$$\mathbf{X} > \mathbf{H}^\mathsf{T} \left(\mathbf{G} + \mathbf{G}^\mathsf{T} - \mathbf{P}\right)^{-1} \mathbf{H} \leq \mathbf{H}^\mathsf{T}\mathbf{G}^{-1}\mathbf{P}\mathbf{G}^{-\mathsf{T}}\mathbf{H},$$
thus implying (2.13). □

Variations of this property are listed as follows.

(a) [20] Consider $\mathbf{X} \in \mathbb{S}^n$, $\mathbf{H} \in \mathbb{R}^{n\times n}$, $\mathbf{G} \in \mathbb{R}^{m\times n}$, and $\mathbf{P} \in \mathbb{S}^m$, where $\mathbf{P} > 0$. The matrix inequality given by
$$\begin{bmatrix} \mathbf{H} + \mathbf{H}^\mathsf{T} - \mathbf{X} & \mathbf{G}^\mathsf{T} \\ * & \mathbf{P} \end{bmatrix} > 0, \tag{2.14}$$
implies
$$\mathbf{X} < \mathbf{H}^\mathsf{T}\mathbf{G}^{-1}\mathbf{P}\mathbf{G}^{-\mathsf{T}}\mathbf{H}.$$

(b) [21] Consider $\mathbf{A} \in \mathbb{S}^n$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{G} \in \mathbb{R}^{m\times m}$, $\mathbf{P} \in \mathbb{S}^m$, and $\beta \in \mathbb{R}$. The matrix inequality given by

$$\begin{bmatrix} \mathbf{A} & \mathbf{BG} \\ * & -\beta\left(\mathbf{G}+\mathbf{G}^\mathsf{T}\right)+\beta^2\mathbf{P} \end{bmatrix} < 0,$$

implies the matrix inequality $\mathbf{A}+\mathbf{BPB}^\mathsf{T} < 0$.

5. [22] Consider $\mathbf{P}_1 \in \mathbb{S}^n$, $\mathbf{P}_2, \mathbf{X} \in \mathbb{S}^q$, $\mathbf{Q}_1 \in \mathbb{R}^{n\times m}$, $\mathbf{Q}_2 \in \mathbb{R}^{q\times p}$, $\mathbf{R}_1 \in \mathbb{S}^m$, and $\mathbf{R}_2 \in \mathbb{S}^p$. The matrix inequalities given by

$$\begin{bmatrix} \mathbf{P}_1 - \mathbf{LXL}^\mathsf{T} & \mathbf{Q}_1 \\ * & \mathbf{R}_1 \end{bmatrix} > 0, \quad \begin{bmatrix} \mathbf{P}_2 + \mathbf{X} & \mathbf{Q}_2 \\ * & \mathbf{R}_2 \end{bmatrix} > 0, \tag{2.15}$$

are satisfied if and only if

$$\begin{bmatrix} \mathbf{P}_1 + \mathbf{LP}_2\mathbf{L}^\mathsf{T} & \mathbf{Q}_1 & \mathbf{LQ}_2 \\ * & \mathbf{R}_1 & \mathbf{0} \\ * & * & \mathbf{R}_2 \end{bmatrix} > 0. \tag{2.16}$$

Proof. The proof is found in [22] and is very similar to the proof of Property 2. □

6. [22, 23] Consider $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{R} \in \mathbb{S}^m$, $\mathbf{S} \in \mathbb{S}^p$, $\mathbf{Q} \in \mathbb{R}^{n\times m}$, $\mathbf{X} \in \mathbb{R}^{n\times p}$, $\mathbf{V} \in \mathbb{R}^{m\times p}$, and $\mathbf{E} \in \mathbb{R}^{p\times m}$. The matrix inequalities given by

$$\begin{bmatrix} \mathbf{P} & \mathbf{Q} \\ * & \mathbf{R} - \mathbf{VE} - \mathbf{E}^\mathsf{T}\mathbf{V}^\mathsf{T} + \mathbf{E}^\mathsf{T}\mathbf{SE} \end{bmatrix} > 0, \quad \begin{bmatrix} \mathbf{R} & \mathbf{V} \\ * & \mathbf{S} \end{bmatrix} > 0, \tag{2.17}$$

are satisfied if and only if

$$\begin{bmatrix} \mathbf{P} & \mathbf{Q}+\mathbf{XE} & \mathbf{X} \\ * & \mathbf{R} & \mathbf{V} \\ * & * & \mathbf{S} \end{bmatrix} > 0. \tag{2.18}$$

Proof. The proof is found in [22] and is very similar to the proof of Property 2. □

7. [24], [25, p. 229] Consider $\mathbf{P}_1, \mathbf{Q} \in \mathbb{S}^n$, $\mathbf{P}_2, \mathbf{Q}_2 \in \mathbb{R}^{n\times m}$, and $\mathbf{P}_3, \mathbf{Q}_3 \in \mathbb{S}^m$, where $\mathbf{P}_1 > 0$, $\mathbf{P}_3 > 0$, $\mathbf{Q}_1 > 0$, and $\mathbf{Q}_3 > 0$. There exist $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{Q}_2$, and $\mathbf{Q}_3$ such that

$$\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ * & \mathbf{P}_3 \end{bmatrix} > 0, \quad \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ * & \mathbf{P}_3 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ * & \mathbf{Q}_3 \end{bmatrix}, \tag{2.19}$$

if and only if

$$\begin{bmatrix} \mathbf{P}_1 & \mathbf{1} \\ * & \mathbf{Q}_1 \end{bmatrix} \geq 0, \quad \operatorname{rank}\left(\begin{bmatrix} \mathbf{P}_1 & \mathbf{1} \\ * & \mathbf{Q}_1 \end{bmatrix}\right) \leq n+m. \tag{2.20}$$

Provided $\mathbf{P}_1$ and $\mathbf{Q}_1$ satisfy (2.20), a solution to (2.19) is given by $\mathbf{P}_3 = \mathbf{1}$, $\mathbf{Q}_2 = -\mathbf{Q}_1\mathbf{P}_2$, $\mathbf{Q}_3 = \mathbf{P}_2^\mathsf{T}\mathbf{Q}_1\mathbf{P}_2 + \mathbf{1}$, and $\mathbf{P}_2$ satisfies $\mathbf{P}_2\mathbf{P}_2^\mathsf{T} = \mathbf{P}_1 - \mathbf{Q}_1^{-1}$.

2.4 Young's Relation (Completion of the Squares)

2.4.1 Young's Relation [26,27]

Consider $\mathbf{X}$, $\mathbf{Y} \in \mathbb{R}^{n\times m}$ and $\mathbf{S} \in \mathbb{S}^n$, where $\mathbf{S} > 0$. The matrix inequality given by

$$\mathbf{X}^{\mathsf{T}}\mathbf{Y} + \mathbf{Y}^{\mathsf{T}}\mathbf{X} \leq \mathbf{X}^{\mathsf{T}}\mathbf{S}^{-1}\mathbf{X} + \mathbf{Y}^{\mathsf{T}}\mathbf{S}\mathbf{Y},$$

is known as Young's relation or Young's inequality.

Young's relation can be derived from a completion of the squares as follows.

$$\begin{aligned} 0 &\leq (\mathbf{X} - \mathbf{S}\mathbf{Y})^{\mathsf{T}}\,\mathbf{S}^{-1}\,(\mathbf{X} - \mathbf{S}\mathbf{Y}) \\ 0 &\leq \mathbf{X}^{\mathsf{T}}\mathbf{S}^{-1}\mathbf{X} + \mathbf{Y}^{\mathsf{T}}\mathbf{S}\mathbf{Y} - \mathbf{X}^{\mathsf{T}}\mathbf{Y} - \mathbf{Y}^{\mathsf{T}}\mathbf{X} \\ \mathbf{X}^{\mathsf{T}}\mathbf{Y} + \mathbf{Y}^{\mathsf{T}}\mathbf{X} &\leq \mathbf{X}^{\mathsf{T}}\mathbf{S}^{-1}\mathbf{X} + \mathbf{Y}^{\mathsf{T}}\mathbf{S}\mathbf{Y}, \end{aligned}$$

which is Young's relation.

2.4.2 Reformulation of Young's Relation [26]

Consider $\mathbf{X}$, $\mathbf{Y} \in \mathbb{R}^{n\times m}$ and $\mathbf{S} \in \mathbb{S}^n$, where $\mathbf{S} > 0$. The matrix inequality given by

$$\mathbf{X}^{\mathsf{T}}\mathbf{Y} + \mathbf{Y}^{\mathsf{T}}\mathbf{X} \leq \tfrac{1}{2}\left(\mathbf{X} + \mathbf{S}\mathbf{Y}\right)^{\mathsf{T}}\mathbf{S}^{-1}\left(\mathbf{X} + \mathbf{S}\mathbf{Y}\right),$$

is a reformulation of Young's relation.

2.4.3 Special Cases of Young's Relation

1. Consider $\mathbf{X}$,$\mathbf{Y} \in \mathbb{R}^{n\times m}$. A special case of Young's relation with $\mathbf{S} = \mathbf{1}$ is given by

$$\mathbf{X}^{\mathsf{T}}\mathbf{Y} + \mathbf{Y}^{\mathsf{T}}\mathbf{X} \leq \mathbf{X}^{\mathsf{T}}\mathbf{X} + \mathbf{Y}^{\mathsf{T}}\mathbf{Y}. \tag{2.21}$$

2. [20] Consider $\mathbf{G} \in \mathbb{R}^{n\times n}$ and $\mathbf{S} \in \mathbb{S}^n$, where $\mathbf{S} > 0$. A special case of Young's relation with $\mathbf{X} = \mathbf{G}$ and $\mathbf{Y} = \mathbf{1}$ is given by

$$\mathbf{G}^{\mathsf{T}}\mathbf{S}^{-1}\mathbf{G} \geq \mathbf{G} + \mathbf{G}^{\mathsf{T}} - \mathbf{S}.$$

3. [2, p. 732] Consider $\mathbf{G} \in \mathbb{R}^{n\times n}$ and $\alpha \in \mathbb{R}_{>0}$. A special case of Young's relation with $\mathbf{X} = \mathbf{G}$, $\mathbf{Y} = \mathbf{1}$, and $\mathbf{S} = \alpha\mathbf{1}$ is given by

$$\alpha^{-1}\mathbf{G}^{\mathsf{T}}\mathbf{G} \geq \mathbf{G} + \mathbf{G}^{\mathsf{T}} - \alpha\mathbf{1}.$$

4. [2, p. 732] Consider $\mathbf{G} \in \mathbb{R}^{n\times n}$ and $\alpha \in \mathbb{R}_{>0}$. A special case of Young's relation with $\mathbf{X} = \mathbf{G}$, $\mathbf{Y} = \mathbf{G}^{\mathsf{T}}$, and $\mathbf{S} = \alpha\mathbf{1}$ is given by

$$\mathbf{G}^2 + \left(\mathbf{G}^{\mathsf{T}}\right)^2 \leq \alpha^{-1}\mathbf{G}^{\mathsf{T}}\mathbf{G} + \alpha\mathbf{G}\mathbf{G}^{\mathsf{T}}.$$

5. [28, p. 38], [29] Consider the column matrices $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$, and $\mathbf{S} \in \mathbb{S}^n$, where $\mathbf{S} > 0$. A special case of Young's relation with $\mathbf{X} = \mathbf{x}$ and $\mathbf{Y} = \mathbf{y}$ is given by

$$2\mathbf{x}^{\mathsf{T}}\mathbf{y} \leq \mathbf{x}^{\mathsf{T}}\mathbf{S}^{-1}\mathbf{x} + \mathbf{y}^{\mathsf{T}}\mathbf{S}\mathbf{y}. \tag{2.22}$$

6. Consider $\mathbf{X} \in \mathbb{R}^{n\times m}$, $\mathbf{F} \in \mathbb{R}^{n\times q}$, $\bar{\mathbf{Y}} \in \mathbb{R}^{q\times m}$, and $\mathbf{S} \in \mathbb{S}^n$, where $\mathbf{S} > 0$. A special case of Young's relation with $\mathbf{Y} = \mathbf{F}\bar{\mathbf{Y}}$ is given by

$$\mathbf{X}^\mathsf{T}\mathbf{F}\bar{\mathbf{Y}} + \bar{\mathbf{Y}}^\mathsf{T}\mathbf{F}^\mathsf{T}\mathbf{X} \leq \mathbf{X}^\mathsf{T}\mathbf{S}^{-1}\mathbf{X} + \bar{\mathbf{Y}}^\mathsf{T}\mathbf{F}^\mathsf{T}\mathbf{S}\mathbf{F}\bar{\mathbf{Y}}. \tag{2.23}$$

7. [19, pp. 29–30] Consider $\mathbf{X} \in \mathbb{R}^{n\times m}$, $\bar{\mathbf{Y}} \in \mathbb{R}^{n\times m}$, $\mathbf{F} \in \mathbb{S}^n$, and $\delta \in \mathbb{R}_{>0}$, where $\mathbf{F} > 0$. A special case of Young's relation with $\mathbf{Y} = \mathbf{F}\bar{\mathbf{Y}}$ and $\mathbf{S} = (\delta\mathbf{F})^{-1}$ is given by

$$\mathbf{X}^\mathsf{T}\mathbf{F}\bar{\mathbf{Y}} + \bar{\mathbf{Y}}^\mathsf{T}\mathbf{F}\mathbf{X} \leq \delta\mathbf{X}^\mathsf{T}\mathbf{F}\mathbf{X} + \delta^{-1}\bar{\mathbf{Y}}^\mathsf{T}\mathbf{F}\bar{\mathbf{Y}}.$$

8. [30] Consider $\mathbf{X} \in \mathbb{R}^{n\times m}$, $\mathbf{F} \in \mathbb{R}^{n\times q}$, $\bar{\mathbf{Y}} \in \mathbb{R}^{q\times m}$, and $\epsilon \in \mathbb{R}_{>0}$, where $\mathbf{F}^\mathsf{T}\mathbf{F} \leq \mathbf{1}$. A special case of the matrix inequality (2.23) with $\mathbf{S} = \epsilon\mathbf{1}$ is given by

$$\mathbf{X}^\mathsf{T}\mathbf{F}\bar{\mathbf{Y}} + \bar{\mathbf{Y}}^\mathsf{T}\mathbf{F}^\mathsf{T}\mathbf{X} \leq \epsilon^{-1}\mathbf{X}^\mathsf{T}\mathbf{X} + \epsilon\bar{\mathbf{Y}}^\mathsf{T}\bar{\mathbf{Y}}. \tag{2.24}$$

Proof. Substituting $\mathbf{S} = \epsilon\mathbf{1}$ into (2.23) yields

$$\mathbf{X}^\mathsf{T}\mathbf{F}\bar{\mathbf{Y}} + \bar{\mathbf{Y}}^\mathsf{T}\mathbf{F}^\mathsf{T}\mathbf{X} \leq \epsilon\mathbf{X}^\mathsf{T}\mathbf{X} + \epsilon^{-1}\bar{\mathbf{Y}}^\mathsf{T}\mathbf{F}^\mathsf{T}\mathbf{F}\bar{\mathbf{Y}}. \tag{2.25}$$

Premultiplying $\mathbf{F}^\mathsf{T}\mathbf{F} \leq \mathbf{1}$ by $\bar{\mathbf{Y}}^\mathsf{T}$, postmultiplying by $\bar{\mathbf{Y}}$, and multiplying both sides by ϵ^{-1} leads to

$$\epsilon^{-1}\bar{\mathbf{Y}}^\mathsf{T}\mathbf{F}^\mathsf{T}\mathbf{F}\bar{\mathbf{Y}} \leq \epsilon^{-1}\bar{\mathbf{Y}}^\mathsf{T}\bar{\mathbf{Y}}. \tag{2.26}$$

Substituting (2.26) into (2.25) yields (2.24). □

9. Consider $\mathbf{X} \in \mathbb{R}^{n\times m}$, $\mathbf{F} \in \mathbb{R}^{n\times q}$, $\mathbf{Y} \in \mathbb{R}^{q\times m}$, and $\mathbf{S} \in \mathbb{S}^n$, where $\mathbf{S} > 0$. Applying Young's relation gives the matrix inequality

$$\tfrac{1}{2}\left(\mathbf{X} + \mathbf{F}\mathbf{Y}\right)^\mathsf{T}\mathbf{S}^{-1}\left(\mathbf{X} + \mathbf{F}\mathbf{Y}\right) \leq \mathbf{X}^\mathsf{T}\mathbf{S}^{-1}\mathbf{X} + \mathbf{Y}^\mathsf{T}\mathbf{F}^\mathsf{T}\mathbf{S}^{-1}\mathbf{F}\mathbf{Y}. \tag{2.27}$$

Proof. Expanding the left-hand side of (2.27) yields

$$\tfrac{1}{2}\left(\mathbf{X} + \mathbf{F}\mathbf{Y}\right)^\mathsf{T}\mathbf{S}^{-1}\left(\mathbf{X} + \mathbf{F}\mathbf{Y}\right) = \tfrac{1}{2}\left(\mathbf{X}^\mathsf{T}\mathbf{S}^{-1}\mathbf{X} + \mathbf{X}^\mathsf{T}\mathbf{S}^{-1}\mathbf{F}\mathbf{Y} + \mathbf{Y}^\mathsf{T}\mathbf{F}^{-1}\mathbf{S}^{-1}\mathbf{X} + \mathbf{Y}^\mathsf{T}\mathbf{F}^\mathsf{T}\mathbf{S}^{-1}\mathbf{F}\mathbf{Y}\right) \tag{2.28}$$

From Young's relation it can be shown that

$$\mathbf{X}^\mathsf{T}\mathbf{S}^{-1}\mathbf{F}\mathbf{Y} + \mathbf{Y}^\mathsf{T}\mathbf{F}^{-1}\mathbf{S}^{-1}\mathbf{X} \leq \mathbf{X}^\mathsf{T}\mathbf{S}^{-1}\mathbf{X} + \mathbf{Y}^\mathsf{T}\mathbf{F}^\mathsf{T}\mathbf{S}^{-1}\mathbf{F}\mathbf{Y}. \tag{2.29}$$

Substituting (2.29) into (2.28) gives (2.27). □

10. Consider $\mathbf{X}$, $\mathbf{Y} \in \mathbb{R}^{n\times m}$, and $\mathbf{S} \in \mathbb{S}^n$, where $\mathbf{S} > 0$. A special case of (2.27) with $\mathbf{F} = \mathbf{S}$ is given by

$$\tfrac{1}{2}\left(\mathbf{X} + \mathbf{S}\mathbf{Y}\right)^\mathsf{T}\mathbf{S}^{-1}\left(\mathbf{X} + \mathbf{S}\mathbf{Y}\right) \leq \mathbf{X}^\mathsf{T}\mathbf{S}^{-1}\mathbf{X} + \mathbf{Y}^\mathsf{T}\mathbf{S}\mathbf{Y}.$$

11. [28, p. 38], [29] Consider $\mathbf{X} \in \mathbb{R}^{n\times m}$, $\mathbf{D} \in \mathbb{R}^{n\times r}$, $\mathbf{F} \in \mathbb{R}^{r\times q}$, $\mathbf{E} \in \mathbb{R}^{q\times m}$, $\mathbf{P} \in \mathbb{S}^n$, and $\epsilon \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$, $\mathbf{F}^\mathsf{T}\mathbf{F} \leq \mathbf{1}$, and $\mathbf{P} - \epsilon\mathbf{D}\mathbf{D}^\mathsf{T} > 0$. Then the matrix inequality given by

$$(\mathbf{X} + \mathbf{D}\mathbf{F}\mathbf{E})^\mathsf{T}\mathbf{P}^{-1}(\mathbf{X} + \mathbf{D}\mathbf{F}\mathbf{E}) \leq \epsilon^{-1}\mathbf{E}^\mathsf{T}\mathbf{E} + \mathbf{X}^\mathsf{T}(\mathbf{P} - \epsilon\mathbf{D}\mathbf{D}^\mathsf{T})^{-1}\mathbf{X}, \tag{2.30}$$

holds.

Proof. Define

$$\mathbf{W} = \left(\epsilon^{-1}\mathbf{1} - \mathbf{D}^\mathsf{T}\mathbf{P}^{-1}\mathbf{D}\right)^{-1/2}\mathbf{D}^\mathsf{T}\mathbf{P}^{-1}\mathbf{X} - \left(\epsilon^{-1}\mathbf{1} - \mathbf{D}^\mathsf{T}\mathbf{P}^{-1}\mathbf{D}\right)^{1/2}\mathbf{F}\mathbf{E},$$

where $\left(\epsilon^{-1}\mathbf{1} - \mathbf{D}^\mathsf{T}\mathbf{P}^{-1}\mathbf{D}\right)^{-1/2}$ exists due to the matrix inversion lemma [2, p. 304] since $\mathbf{P} - \epsilon\mathbf{D}\mathbf{D}^\mathsf{T} > 0$. Expanding the terms in $\mathbf{W}^\mathsf{T}\mathbf{W} \geq 0$ yields

$$\begin{aligned}\mathbf{X}^\mathsf{T}\mathbf{P}^{-1}\mathbf{D}\left(\epsilon^{-1}\mathbf{1} - \mathbf{D}^\mathsf{T}\mathbf{P}^{-1}\mathbf{D}\right)^{-1}\mathbf{D}^\mathsf{T}\mathbf{P}^{-1}\mathbf{X} - \mathbf{X}^\mathsf{T}\mathbf{P}^{-1}\mathbf{D}\mathbf{F}\mathbf{E} - \mathbf{E}^\mathsf{T}\mathbf{F}^\mathsf{T}\mathbf{D}^\mathsf{T}\mathbf{P}^{-1}\mathbf{X}& \\ + \mathbf{E}^\mathsf{T}\mathbf{F}^\mathsf{T}\left(\epsilon^{-1}\mathbf{1} - \mathbf{D}^\mathsf{T}\mathbf{P}^{-1}\mathbf{D}\right)\mathbf{F}\mathbf{E} \geq 0.&\end{aligned}$$

Adding $\mathbf{X}^\mathsf{T}\mathbf{P}^{-1}\mathbf{X}$ to both sides of the inequality and rearranging gives

$$\begin{aligned}&\mathbf{X}^\mathsf{T}\mathbf{P}^{-1}\mathbf{X} + \mathbf{X}^\mathsf{T}\mathbf{P}^{-1}\mathbf{D}\mathbf{F}\mathbf{E} + \mathbf{E}^\mathsf{T}\mathbf{F}^\mathsf{T}\mathbf{D}^\mathsf{T}\mathbf{P}^{-1}\mathbf{X} + \mathbf{E}^\mathsf{T}\mathbf{F}^\mathsf{T}\mathbf{D}^\mathsf{T}\mathbf{P}^{-1}\mathbf{D}\mathbf{F}\mathbf{E} \\ &\quad\leq \epsilon^{-1}\mathbf{E}^\mathsf{T}\mathbf{F}^\mathsf{T}\mathbf{F}\mathbf{E} + \mathbf{X}^\mathsf{T}\left(\mathbf{P}^{-1}\mathbf{D}(\epsilon^{-1}\mathbf{1} - \mathbf{D}^\mathsf{T}\mathbf{P}^{-1}\mathbf{D})^{-1}\mathbf{D}^\mathsf{T}\mathbf{P}^{-1} + \mathbf{P}^{-1}\right)\mathbf{X}. \end{aligned}\tag{2.31}$$

Using the matrix inversion lemma [2, p. 304], it is known that

$$(\mathbf{P} - \epsilon\mathbf{D}\mathbf{D}^\mathsf{T})^{-1} = \mathbf{P}^{-1}\mathbf{D}(\epsilon^{-1}\mathbf{1} - \mathbf{D}^\mathsf{T}\mathbf{P}^{-1}\mathbf{D})^{-1}\mathbf{D}^\mathsf{T}\mathbf{P}^{-1} + \mathbf{P}^{-1}. \tag{2.32}$$

Substituting (2.32) into (2.31), factoring the left side of the inequality, and knowing $\mathbf{F}^\mathsf{T}\mathbf{F} \leq \mathbf{1}$ gives (2.30). □

12. [29,31] Consider $\mathbf{X} \in \mathbb{R}^{n\times m}$, $\mathbf{D} \in \mathbb{R}^{n\times r}$, $\mathbf{F} \in \mathbb{R}^{r\times q}$, $\mathbf{E} \in \mathbb{R}^{q\times m}$, $\mathbf{P} \in \mathbb{S}^n$, and $\epsilon \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$, $\mathbf{F}^\mathsf{T}\mathbf{F} \leq \mathbf{1}$, and $\epsilon\mathbf{1} - \mathbf{D}^\mathsf{T}\mathbf{P}\mathbf{D} > 0$. Then the matrix inequality given by

$$(\mathbf{X} + \mathbf{D}\mathbf{F}\mathbf{E})^\mathsf{T}\mathbf{P}(\mathbf{X} + \mathbf{D}\mathbf{F}\mathbf{E}) \leq \epsilon\mathbf{E}^\mathsf{T}\mathbf{E} + \mathbf{X}^\mathsf{T}\mathbf{P}\mathbf{D}(\epsilon\mathbf{1} - \mathbf{D}^\mathsf{T}\mathbf{P}\mathbf{D})^{-1}\mathbf{D}^\mathsf{T}\mathbf{P}\mathbf{X} + \mathbf{X}^\mathsf{T}\mathbf{P}\mathbf{X}, \tag{2.33}$$

holds.

Proof. Define

$$\mathbf{W} = \left(\epsilon\mathbf{1} - \mathbf{D}^\mathsf{T}\mathbf{P}\mathbf{D}\right)^{-1/2}\mathbf{D}^\mathsf{T}\mathbf{P}\mathbf{X} - \left(\epsilon\mathbf{1} - \mathbf{D}^\mathsf{T}\mathbf{P}\mathbf{D}\right)^{1/2}\mathbf{F}\mathbf{E},$$

where $\left(\epsilon\mathbf{1} - \mathbf{D}^\mathsf{T}\mathbf{P}\mathbf{D}\right)^{-1/2}$ exists since $\epsilon\mathbf{1} - \mathbf{D}^\mathsf{T}\mathbf{P}\mathbf{D} > 0$. Expanding the terms in $\mathbf{W}^\mathsf{T}\mathbf{W} \geq 0$ yields

$$\mathbf{X}^\mathsf{T}\mathbf{P}\mathbf{D}\left(\epsilon\mathbf{1} - \mathbf{D}^\mathsf{T}\mathbf{P}\mathbf{D}\right)^{-1}\mathbf{D}^\mathsf{T}\mathbf{P}\mathbf{X} - \mathbf{X}^\mathsf{T}\mathbf{P}\mathbf{D}\mathbf{F}\mathbf{E} - \mathbf{E}^\mathsf{T}\mathbf{F}^\mathsf{T}\mathbf{D}^\mathsf{T}\mathbf{P}\mathbf{X} + \mathbf{E}^\mathsf{T}\mathbf{F}^\mathsf{T}\left(\epsilon\mathbf{1} - \mathbf{D}^\mathsf{T}\mathbf{P}\mathbf{D}\right)\mathbf{F}\mathbf{E} \geq 0.$$

Adding $\mathbf{X}^\mathsf{T}\mathbf{P}\mathbf{X}$ to both sides of the inequality and rearranging gives

$$\begin{aligned}&\mathbf{X}^\mathsf{T}\mathbf{P}\mathbf{X} + \mathbf{X}^\mathsf{T}\mathbf{P}\mathbf{D}\mathbf{F}\mathbf{E} + \mathbf{E}^\mathsf{T}\mathbf{F}^\mathsf{T}\mathbf{D}^\mathsf{T}\mathbf{P}\mathbf{X} + \mathbf{E}^\mathsf{T}\mathbf{F}^\mathsf{T}\mathbf{D}^\mathsf{T}\mathbf{P}\mathbf{D}\mathbf{F}\mathbf{E} \\ &\quad\leq \epsilon\mathbf{E}^\mathsf{T}\mathbf{F}^\mathsf{T}\mathbf{F}\mathbf{E} + \mathbf{X}^\mathsf{T}\mathbf{P}\mathbf{D}(\epsilon\mathbf{1} - \mathbf{D}^\mathsf{T}\mathbf{P}\mathbf{D})^{-1}\mathbf{D}^\mathsf{T}\mathbf{P}\mathbf{X} + \mathbf{X}^\mathsf{T}\mathbf{P}\mathbf{X}.\end{aligned}$$

Factoring the left side of the inequality and knowing $\mathbf{F}^\mathsf{T}\mathbf{F} \geq \mathbf{1}$ gives (2.33). □

13. [32, p. 11] Consider $\mathbf{N} \in \mathbb{R}^{n\times n}$, $\mathbf{E} \in \mathbb{R}^{n\times m}$, $\mathbf{H} \in \mathbb{R}^{m\times p}$, $\mathbf{F} \in \mathbb{R}^{p\times n}$, $\mathbf{J} \in \mathbb{S}^n$, and $\epsilon \in \mathbb{R}_{>0}$, where $\mathbf{J} > 0$ and $\mathbf{F}^\mathsf{T}\mathbf{F} \leq \mathbf{1}$. With some manipulation, a special case of (2.23) with $\mathbf{X} = \mathbf{H}^\mathsf{T}\mathbf{E}^\mathsf{T}\mathbf{N}^\mathsf{T}$ and $\bar{\mathbf{Y}} = \mathbf{1}$ is given by

$$-\mathbf{N}\left(\mathbf{1} - \mathbf{E}\mathbf{H}\mathbf{F}\right)\mathbf{J}^{-1}\left(\mathbf{1} - \mathbf{E}\mathbf{H}\mathbf{F}\right)^\mathsf{T}\mathbf{N}^\mathsf{T} \leq \mathbf{J} - \mathbf{N} - \mathbf{N}^\mathsf{T} + \epsilon^{-1}\mathbf{N}\mathbf{E}\mathbf{H}\mathbf{H}^\mathsf{T}\mathbf{E}^\mathsf{T}\mathbf{N}^\mathsf{T} + \epsilon\mathbf{1}.$$

14. [32, p. 11] Consider $\mathbf{N} \in \mathbb{R}^{n\times n}$, $\mathbf{F} \in \mathbb{R}^{n\times m}$, $\mathbf{E} \in \mathbb{R}^{m\times p}$, $\mathbf{H} \in \mathbb{R}^{p\times n}$, $\mathbf{J} \in \mathbb{S}^n$, and $\epsilon \in \mathbb{R}_{>0}$, where $\mathbf{J} > 0$ and $\mathbf{F}^\mathsf{T}\mathbf{F} \leq \mathbf{1}$. With some manipulation, a special case of (2.23) with $\mathbf{X} = \mathbf{N}\mathbf{H}\mathbf{E}$ and $\bar{\mathbf{Y}} = \mathbf{1}$ is given by

$$-\mathbf{N}^\mathsf{T}\left(\mathbf{1} - \mathbf{F}\mathbf{E}\mathbf{H}\right)^\mathsf{T}\mathbf{J}^{-1}\left(\mathbf{1} - \mathbf{F}\mathbf{E}\mathbf{H}\right)\mathbf{N} \leq \mathbf{J} - \mathbf{N} - \mathbf{N}^\mathsf{T} + \epsilon^{-1}\mathbf{N}^\mathsf{T}\mathbf{H}^\mathsf{T}\mathbf{E}^\mathsf{T}\mathbf{E}\mathbf{H}\mathbf{N} + \epsilon\mathbf{1}.$$

2.4.4 Young's Relation-Based Properties

1. [33] Consider $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{n\times m}$ and $\mathbf{Z} \in \mathbb{S}^m$. The matrix inequality given by

$$\mathbf{Z} + \mathbf{X}^\mathsf{T}\mathbf{Y} + \mathbf{Y}^\mathsf{T}\mathbf{X} > 0,$$

is satisfied if and only if there exist $\mathbf{Q} \in \mathbb{S}^m$, $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{G}_1 \in \mathbb{R}^{n\times n}$, $\mathbf{G}_2 \in \mathbb{R}^{n\times m}$, $\mathbf{F} \in \mathbb{R}^{m\times n}$, and $\mathbf{H} \in \mathbb{R}^{m\times m}$, where $\mathbf{Q} > 0$ and $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{Y} \\ * & \mathbf{Q} \end{bmatrix} > 0 \quad \text{and} \quad \begin{bmatrix} \mathbf{Z} + \mathbf{Q} + \mathbf{X}^\mathsf{T}\mathbf{P}\mathbf{X} & \mathbf{F} - \mathbf{X}^\mathsf{T}\mathbf{G}_1 & \mathbf{H} - \mathbf{X}^\mathsf{T}\mathbf{G}_2 \\ * & \mathbf{G}_1 + \mathbf{G}_1^\mathsf{T} - \mathbf{P} & \mathbf{F}^\mathsf{T} + \mathbf{G}_2 - \mathbf{Y} \\ * & * & \mathbf{H}^\mathsf{T} + \mathbf{H} - \mathbf{Q} \end{bmatrix} > 0.$$

2. [33] Consider $\mathbf{X} \in \mathbb{R}^{n\times n}$ and $\mathbf{W} \in \mathbb{S}^n$, where $\mathbf{X}$ is full rank and $\mathbf{W} > 0$. The matrix inequality given by

$$\mathbf{X}^\mathsf{T}\mathbf{X} - \mathbf{W} > 0,$$

is satisfied if there exists $\lambda \in \mathbb{R}_{>0}$ such that

$$\begin{bmatrix} \lambda\mathbf{1} & \lambda\mathbf{1} & \mathbf{0} \\ * & \mathbf{X} + \mathbf{X}^\mathsf{T} & \mathbf{W}^{\frac{1}{2}} \\ * & * & \lambda\mathbf{1} \end{bmatrix} > 0.$$

2.4.5 Iterative Convex Overbounding [34, 35]

Iterative convex overbounding is a technique based on Young's relation that is useful when solving an optimization problem with a BMI constraint.

Consider the matrices $\mathbf{Q} = \mathbf{Q}^\mathsf{T} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{R} \in \mathbb{R}^{m\times p}$, $\mathbf{D} \in \mathbb{R}^{p\times q}$, $\mathbf{S} \in \mathbb{R}^{q\times r}$, and $\mathbf{C} \in \mathbb{R}^{r\times n}$, where $\mathbf{S}$ and $\mathbf{R}$ are design variables in the BMI given by

$$\mathbf{Q} + \mathbf{BRDSC} + \mathbf{C}^\mathsf{T}\mathbf{S}^\mathsf{T}\mathbf{D}^\mathsf{T}\mathbf{R}^\mathsf{T}\mathbf{B}^\mathsf{T} < 0. \tag{2.34}$$

Suppose that $\mathbf{S}_0$ and $\mathbf{R}_0$ are known to satisfy (2.34). The BMI of (2.34) is implied by the LMI

$$\begin{bmatrix} \mathbf{Q} + \phi(\mathbf{R},\mathbf{S}) + \phi^\mathsf{T}(\mathbf{R},\mathbf{S}) & \mathbf{B}\left(\mathbf{R} - \mathbf{R}_0\right)\mathbf{U} & \mathbf{C}^\mathsf{T}\left(\mathbf{S} - \mathbf{S}_0\right)^\mathsf{T}\mathbf{V}^\mathsf{T} \\ * & \mathbf{W}^{-1} & \mathbf{0} \\ * & * & -\mathbf{W} \end{bmatrix} < 0, \tag{2.35}$$

where $\phi(\mathbf{R},\mathbf{S}) = \mathbf{B}\left(\mathbf{RDS}_0 + \mathbf{R}_0\mathbf{DS} - \mathbf{R}_0\mathbf{DS}_0\right)\mathbf{C}$, $\mathbf{W} > 0$ is an arbitrary matrix, $\mathbf{D} = \mathbf{UV}$, and the matrices $\mathbf{U}$ and $\mathbf{V}^\mathsf{T}$ have full column rank. The LMI of (2.35) is equivalent to the BMI of (2.34) when $\mathbf{R} = \mathbf{R}_0$ and $\mathbf{S} = \mathbf{S}_0$, and is therefore non-conservative for values of $\mathbf{R}$ and $\mathbf{S}$ and are close to the previously known solutions $\mathbf{R}_0$ and $\mathbf{S}_0$.

Alternatively, the BMI of (2.34) is implied by the LMI

$$\begin{bmatrix} \mathbf{Q} + \phi(\mathbf{R},\mathbf{S}) + \phi^\mathsf{T}(\mathbf{R},\mathbf{S}) & \mathbf{Z}^\mathsf{T}\mathbf{U}^\mathsf{T}\left(\mathbf{R} - \mathbf{R}_0\right)^\mathsf{T}\mathbf{B}^\mathsf{T} + \mathbf{V}\left(\mathbf{S} - \mathbf{S}_0\right)\mathbf{C} \\ * & -\mathbf{Z} \end{bmatrix} < 0, \tag{2.36}$$

where $\mathbf{Z} > 0$ is an arbitrary matrix, $\mathbf{D} = \mathbf{UV}$, and the matrices $\mathbf{U}$ and $\mathbf{V}^\mathsf{T}$ have full column rank. Again, the LMI of (2.36) is equivalent to the BMI of (2.34) when $\mathbf{R} = \mathbf{R}_0$ and $\mathbf{S} = \mathbf{S}_0$, and is therefore non-conservative for values of $\mathbf{R}$ and $\mathbf{S}$ and are close to the previously known solutions $\mathbf{R}_0$ and $\mathbf{S}_0$.

A benefit of convex overbounding compared to a linearization approach, is that in addition to ensuring conservatism or error is reduced in the neighborhood of $\mathbf{R} = \mathbf{R}_0$ and $\mathbf{S} = \mathbf{S}_0$, the LMIs of (2.35) and (2.36) imply (2.34).

Iterative convex overbounding is particularly useful when used to solve an optimization problem with BMI constraints. For example, choose $\mathbf{R}_0$ and $\mathbf{S}_0$ that are initial feasible solutions to (2.34). Then solve for $\mathbf{R}$ and $\mathbf{S}$ that minimize a specified objective function and satisfy (2.35) or (2.36), which imply (2.34) without conservatism when $\mathbf{R} = \mathbf{R}_0$ and $\mathbf{S} = \mathbf{S}_0$. Set $\mathbf{R}_0 = \mathbf{R}$ and $\mathbf{S}_0 = \mathbf{S}$, and repeat until the objective function meets a specified stopping criteria. The benefits of this procedure are that its individual steps are convex optimization problems with very little conservatism in the neighborhood of the solution from the previous iteration, and that it tends to converge quickly to a solution. However, there is no guarantee that the method will converge to even a local solution.

Example 2.3. Consider a special case of (2.34) given by

$$\mathbf{Q} + \mathbf{R}\mathbf{S} + \mathbf{S}^{\mathsf{T}}\mathbf{R}^{\mathsf{T}} < 0, \tag{2.37}$$

where $\mathbf{Q} \in \mathbb{S}^n$, $\mathbf{R} \in \mathbb{R}^{n\times m}$, and $\mathbf{S} \in \mathbb{R}^{m\times n}$. The BMI of (2.37) is implied by the LMI

$$\begin{bmatrix} \mathbf{Q} + \mathbf{R}\mathbf{S}_0 + \mathbf{S}_0^{\mathsf{T}}\mathbf{R}^{\mathsf{T}} + \mathbf{R}_0\mathbf{S} + \mathbf{S}^{\mathsf{T}}\mathbf{R}_0^{\mathsf{T}} - \mathbf{R}_0\mathbf{S}_0 - \mathbf{S}_0^{\mathsf{T}}\mathbf{R}_0^{\mathsf{T}} & \mathbf{R} - \mathbf{R}_0 & \mathbf{S}^{\mathsf{T}} - \mathbf{S}_0^{\mathsf{T}} \\ * & -\mathbf{W}^{-1} & \mathbf{0} \\ * & * & -\mathbf{W} \end{bmatrix} < 0,$$

where $\mathbf{W} > 0$ is an arbitrary matrix. Alternatively, the BMI of (2.37) is implied by the LMI

$$\begin{bmatrix} \mathbf{Q} + \mathbf{R}\mathbf{S}_0 + \mathbf{S}_0^{\mathsf{T}}\mathbf{R}^{\mathsf{T}} + \mathbf{R}_0\mathbf{S} + \mathbf{S}^{\mathsf{T}}\mathbf{R}_0^{\mathsf{T}} - \mathbf{R}_0\mathbf{S}_0 - \mathbf{S}_0^{\mathsf{T}}\mathbf{R}_0^{\mathsf{T}} & \mathbf{Z}\left(\mathbf{R} - \mathbf{R}_0\right)^{\mathsf{T}} + \mathbf{S} - \mathbf{S}_0 \\ * & -\mathbf{Z} \end{bmatrix} < 0,$$

where $\mathbf{Z} > 0$ is an arbitrary matrix.

2.5 Projection Lemma (Matrix Elimination Lemma)

2.5.1 Strict Projection Lemma [24], [3, pp. 22–23], [17, pp. 483–484]

Consider $\mathbf{\Psi} \in \mathbb{S}^n$, $\mathbf{G} \in \mathbb{R}^{n\times m}$, $\mathbf{\Lambda} \in \mathbb{R}^{m\times p}$, and $\mathbf{H} \in \mathbb{R}^{n\times p}$. There exists $\mathbf{\Lambda}$ such that

$$\mathbf{\Psi} + \mathbf{G}\mathbf{\Lambda}\mathbf{H}^{\mathsf{T}} + \mathbf{H}\mathbf{\Lambda}^{\mathsf{T}}\mathbf{G}^{\mathsf{T}} < 0, \tag{2.38}$$

if and only if

$$\mathbf{N}_G^{\mathsf{T}}\mathbf{\Psi}\mathbf{N}_G < 0,$$
$$\mathbf{N}_H^{\mathsf{T}}\mathbf{\Psi}\mathbf{N}_H < 0,$$

where $\mathcal{R}(\mathbf{N}_G) = \mathcal{N}(\mathbf{G}^{\mathsf{T}})$ and $\mathcal{R}(\mathbf{N}_H) = \mathcal{N}(\mathbf{H}^{\mathsf{T}})$.

2.5.2 Nonstrict Projection Lemma [36, p. 93]

Consider $\mathbf{\Psi} \in \mathbb{S}^n$, $\mathbf{G} \in \mathbb{R}^{n\times m}$, $\mathbf{\Lambda} \in \mathbb{R}^{m\times p}$, and $\mathbf{H} \in \mathbb{R}^{n\times p}$, where $\mathcal{R}(\mathbf{G})$ and $\mathcal{R}(\mathbf{H})$ linearly independent. There exists $\mathbf{\Lambda}$ such that

$$\mathbf{\Psi} + \mathbf{G}\mathbf{\Lambda}\mathbf{H}^{\mathsf{T}} + \mathbf{H}\mathbf{\Lambda}^{\mathsf{T}}\mathbf{G}^{\mathsf{T}} \leq 0,$$

if and only if

$$\mathbf{N}_G^\mathsf{T}\mathbf{\Psi}\mathbf{N}_G \le 0,$$
$$\mathbf{N}_H^\mathsf{T}\mathbf{\Psi}\mathbf{N}_H \le 0,$$

where $\mathcal{R}(\mathbf{N}_G) = \mathcal{N}(\mathbf{G}^\mathsf{T})$ and $\mathcal{R}(\mathbf{N}_H) = \mathcal{N}(\mathbf{H}^\mathsf{T})$.

2.5.3 Reciprocal Projection Lemma [37]

Consider $\mathbf{P}$, $\mathbf{\Psi} \in \mathbb{S}^n$ and $\mathbf{W}$, $\mathbf{S} \in \mathbb{R}^{n\times n}$. There exists $\mathbf{W}$ such that

$$\begin{bmatrix} \mathbf{\Psi} + \mathbf{P} - \left(\mathbf{W} + \mathbf{W}^\mathsf{T}\right) & \mathbf{S}^\mathsf{T} + \mathbf{W}^\mathsf{T} \\ * & -\mathbf{P} \end{bmatrix} < 0,$$

if and only if $\mathbf{\Psi} + \mathbf{S} + \mathbf{S}^\mathsf{T} < 0$.

2.5.4 Projection Lemma-Based Properties

1. [38] Consider $\mathbf{A} \in \mathbb{S}^n$, $\mathbf{B}$, $\mathbf{J} \in \mathbb{R}^{n\times m}$, $\mathbf{G} \in \mathbb{R}^{m\times m}$, and $\mathbf{P} \in \mathbb{S}^m$. The matrix inequality given by
$$\begin{bmatrix} \mathbf{A} + \mathbf{B}\mathbf{J}^\mathsf{T} + \mathbf{J}\mathbf{B}^\mathsf{T} & -\mathbf{J} + \mathbf{B}\mathbf{G} \\ * & -\left(\mathbf{G} + \mathbf{G}^\mathsf{T}\right) + \mathbf{P} \end{bmatrix} < 0, \tag{2.39}$$
implies the matrix inequality
$$\mathbf{A} + \mathbf{B}\mathbf{P}\mathbf{B}^\mathsf{T} < 0. \tag{2.40}$$
If the matrices $\mathbf{J}$ and $\mathbf{G}$ are free (i.e., they are design variables), then the matrix inequalities (2.39) and (2.40) are equivalent [39].

2. [40] Consider $\mathbf{T} \in \mathbb{S}^n$ and $\mathbf{A}$, $\mathbf{J}$, $\mathbf{G}$, $\mathbf{P} \in \mathbb{R}^{n\times n}$. The matrix inequality given by
$$\begin{bmatrix} \mathbf{T} + \mathbf{A}^\mathsf{T}\mathbf{J}^\mathsf{T} + \mathbf{J}\mathbf{A} & \mathbf{P} - \mathbf{J} + \mathbf{A}^\mathsf{T}\mathbf{G} \\ * & -\left(\mathbf{G} + \mathbf{G}^\mathsf{T}\right) \end{bmatrix} < 0 \tag{2.41}$$
implies the matrix inequality
$$\mathbf{T} + \mathbf{A}^\mathsf{T}\mathbf{P}^\mathsf{T} + \mathbf{P}\mathbf{A} < 0. \tag{2.42}$$
If the matrices $\mathbf{J}$ and $\mathbf{G}$ are free (i.e., they are design variables), then the matrix inequalities (2.41) and (2.42) are equivalent [39].

3. [39] Consider $\mathbf{T}_1$, $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{A}$, $\mathbf{J}_1$, $\mathbf{G} \in \mathbb{R}^{n\times n}$, $\mathbf{T}_2 \in \mathbb{R}^{n\times m}$, $\mathbf{J}_2 \in \mathbb{R}^{m\times n}$, and $\mathbf{T}_3 \in \mathbb{S}^m$, where $\mathbf{P} > 0$ and $\mathbf{T}_3 < 0$. The matrix inequality given by
$$\begin{bmatrix} \mathbf{T}_1 + \mathbf{A}^\mathsf{T}\mathbf{J}_1^\mathsf{T} + \mathbf{J}_1\mathbf{A} & \mathbf{T}_2 + \mathbf{A}^\mathsf{T}\mathbf{J}_2^\mathsf{T} & \mathbf{P} - \mathbf{J}_1 + \mathbf{A}^\mathsf{T}\mathbf{G} \\ * & \mathbf{T}_3 & -\mathbf{J}_2 \\ * & * & -\left(\mathbf{G} + \mathbf{G}^\mathsf{T}\right) \end{bmatrix} < 0 \tag{2.43}$$
implies the matrix inequality
$$\begin{bmatrix} \mathbf{T}_1 + \mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{P}\mathbf{A} & \mathbf{T}_2 \\ * & \mathbf{T}_3 \end{bmatrix} < 0. \tag{2.44}$$
If the matrices $\mathbf{J}_1$, $\mathbf{J}_2$, and $\mathbf{G}$ are free (i.e., they are design variables), then the matrix inequalities (2.43) and (2.44) are equivalent.

4. [32, p. 9] Consider $\mathbf{T} \in \mathbb{S}^n$, $\mathbf{A}, \mathbf{G}, \mathbf{P} \in \mathbb{R}^{n \times n}$, and $\beta \in \mathbb{R}$, where $\mathbf{T} < 0$. The matrix inequality given by
$$\begin{bmatrix} \mathbf{T} & \beta\mathbf{P} + \mathbf{A}^\mathsf{T}\mathbf{G} \\ * & -\beta\left(\mathbf{G} + \mathbf{G}^\mathsf{T}\right) \end{bmatrix} < 0,$$
implies the matrix inequality $\mathbf{T} + \mathbf{A}^\mathsf{T}\mathbf{P}^\mathsf{T} + \mathbf{P}\mathbf{A} < 0$.

2.6 Finsler's Lemma

2.6.1 Finsler's Lemma [41], [3, pp. 22–23], [17, pp. 483–484]

Consider $\boldsymbol{\Psi} \in \mathbb{S}^n$, $\mathbf{G} \in \mathbb{R}^{n \times m}$, $\boldsymbol{\Lambda} \in \mathbb{R}^{m \times p}$, $\mathbf{H} \in \mathbb{R}^{n \times p}$, and $\sigma \in \mathbb{R}$. There exists $\boldsymbol{\Lambda}$ such that

$$\boldsymbol{\Psi} + \mathbf{G}\boldsymbol{\Lambda}\mathbf{H}^\mathsf{T} + \mathbf{H}\boldsymbol{\Lambda}^\mathsf{T}\mathbf{G}^\mathsf{T} < 0,$$

if and only if there exists σ such that

$$\boldsymbol{\Psi} - \sigma\mathbf{G}\mathbf{G}^\mathsf{T} < 0,$$
$$\boldsymbol{\Psi} - \sigma\mathbf{H}\mathbf{H}^\mathsf{T} < 0.$$

2.6.2 Alternative Form of Finsler's Lemma [30]

Consider $\boldsymbol{\Psi} \in \mathbb{S}^n$, $\mathbf{Z} \in \mathbb{R}^{p \times n}$, $\mathbf{x} \in \mathbb{R}^n$, and $\sigma \in \mathbb{R}_{>0}$. If there exists $\mathbf{Z}$ such that

$$\mathbf{x}^\mathsf{T}\boldsymbol{\Psi}\mathbf{x} < 0,$$

holds for all $\mathbf{x} \neq \mathbf{0}$ satisfying $\mathbf{Z}\mathbf{x} = \mathbf{0}$, then there exists σ such that

$$\boldsymbol{\Psi} - \sigma\mathbf{Z}^\mathsf{T}\mathbf{Z} < 0.$$

2.6.3 Modified Finsler's Lemma [28, p. 37], [42, 43]

Consider $\boldsymbol{\Psi} \in \mathbb{S}^n$, $\mathbf{G} \in \mathbb{R}^{n \times m}$, $\boldsymbol{\Lambda} \in \mathbb{R}^{m \times p}$, $\mathbf{H} \in \mathbb{R}^{n \times p}$, and $\epsilon \in \mathbb{R}_{>0}$, where $\boldsymbol{\Lambda}^\mathsf{T}\boldsymbol{\Lambda} \leq \mathbf{R}$ and $\mathbf{R} > 0$. There exists $\boldsymbol{\Lambda}$ such that

$$\boldsymbol{\Psi} + \mathbf{G}\boldsymbol{\Lambda}\mathbf{H}^\mathsf{T} + \mathbf{H}\boldsymbol{\Lambda}^\mathsf{T}\mathbf{G}^\mathsf{T} < 0,$$

if and only if there exists ϵ such that

$$\boldsymbol{\Psi} + \epsilon^{-1}\mathbf{G}\mathbf{G}^\mathsf{T} + \epsilon\mathbf{H}\mathbf{R}\mathbf{H}^\mathsf{T} < 0.$$

Proof. To be proven. □

2.7 Discussion on the Schur Complement, Young's Relation, Convex Overbounding, and the Projection Lemma

The Schur complement, Young's relation, and the projection lemma are three of the most common tools used to transform a BMI into an LMI. The sign of the BMI determines which one is suitable to transform the BMI into an LMI. For example, consider the case of a BMI in the variable $\mathbf{X} \in \mathbb{R}^{m \times n}$ of the form

$$\mathbf{P} + \mathbf{X}^\mathsf{T}\mathbf{S}\mathbf{X} < 0, \tag{2.45}$$

where $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{S} \in \mathbb{S}^m$, and $\mathbf{S} > 0$. The Schur complement is used to obtain an equivalent LMI given by

$$\begin{bmatrix} \mathbf{P} & \mathbf{X}^{\mathsf{T}} \\ * & -\mathbf{S}^{-1} \end{bmatrix} < 0.$$

This LMI can also be written as

$$\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ * & -\mathbf{S}^{-1} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \mathbf{X} \begin{bmatrix} \mathbf{1} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} \mathbf{X}^{\mathsf{T}} \begin{bmatrix} \mathbf{0} & \mathbf{1} \end{bmatrix} < 0. \tag{2.46}$$

Applying the Projection Lemma, it is known that there exists $\mathbf{X}$ satisfying (2.46) if and only if $\mathbf{P} < 0$ and $\mathbf{S}^{-1} > 0$, since $\mathcal{N}\left(\begin{bmatrix} \mathbf{1} & \mathbf{0} \end{bmatrix}\right) = \mathcal{R}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right)$, $\mathcal{N}\left(\begin{bmatrix} \mathbf{0} & \mathbf{1} \end{bmatrix}\right) = \mathcal{R}\left(\begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}\right)$, and

$$\mathbf{P} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ * & -\mathbf{S}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}, \quad -\mathbf{S}^{-1} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ * & -\mathbf{S}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}.$$

Notice that the Projection Lemma gives two matrix inequalities that do not depend on the variable $\mathbf{X}$. This is why the Projection Lemma is also known as the Matrix Elimination Lemma.

Alternatively, consider the BMI

$$\mathbf{P} - \mathbf{X}^{\mathsf{T}}\mathbf{S}\mathbf{X} < 0, \tag{2.47}$$

where $\mathbf{X} \in \mathbb{R}^{m \times n}$, $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{S} \in \mathbb{S}^m$, and $\mathbf{S} > 0$. Young's relation is used to obtain an LMI in $\mathbf{X}$ given by

$$\mathbf{P} - \mathbf{X}^{\mathsf{T}}\mathbf{Y} - \mathbf{Y}^{\mathsf{T}}\mathbf{X} + \mathbf{Y}^{\mathsf{T}}\mathbf{S}^{-1}\mathbf{Y} < 0, \tag{2.48}$$

which implies the BMI of (2.47). Notice that (2.48) involves a new variable $\mathbf{Y} \in \mathbb{R}^{m \times n}$. Using the Schur complement on (2.48) yields

$$\begin{bmatrix} \mathbf{P} - \mathbf{X}^{\mathsf{T}}\mathbf{Y} - \mathbf{Y}^{\mathsf{T}}\mathbf{X} & \mathbf{Y}^{\mathsf{T}} \\ * & \mathbf{S}^{-1} \end{bmatrix} < 0,$$

which is an LMI in $\mathbf{X}$ for a fixed $\mathbf{Y}$.

It is desirable to use the Schur complement of the Projection Lemma over Young's relation whenever possible, as they provides an LMI or LMIs that are equivalent to the original BMI. When using Young's relation, the resulting LMI implies the original BMI, but is not equivalent. This introduces conservatism into an optimization problem.

If a previously-known solution $\mathbf{X}_0$ to (2.47) is available, then convex overbounding can be used to reduce conservatism in the neighborhood of $\mathbf{X}_0$. The BMI of (2.47) is equivalent to the BMI

$$\mathbf{P} - (\mathbf{X} - \mathbf{X}_0)^{\mathsf{T}} \mathbf{S} (\mathbf{X} - \mathbf{X}_0) - \mathbf{X}^{\mathsf{T}}\mathbf{S}\mathbf{X}_0 - \mathbf{X}_0^{\mathsf{T}}\mathbf{S}\mathbf{X} + \mathbf{X}_0^{\mathsf{T}}\mathbf{S}\mathbf{X}_0 < 0. \tag{2.49}$$

Since the term $(\mathbf{X} - \mathbf{X}_0)^{\mathsf{T}} \mathbf{S} (\mathbf{X} - \mathbf{X}_0)$ is positive definite, (2.49) is implied by the LMI

$$\mathbf{P} - \mathbf{X}^{\mathsf{T}}\mathbf{S}\mathbf{X}_0 - \mathbf{X}_0^{\mathsf{T}}\mathbf{S}\mathbf{X} + \mathbf{X}_0^{\mathsf{T}}\mathbf{S}\mathbf{X}_0 < 0. \tag{2.50}$$

The LMI of (2.50) is in general conservative, but this conservatism disappears when $\mathbf{X} = \mathbf{X}_0$ and is reduced when $\mathbf{X}$ is close to $\mathbf{X}_0$.

2.8 Dilation

Matrix inequalities can be dilated to obtain a larger matrix inequality, often with additional design variables. This can be a useful technique to separate design variables in a BMI.

A common technique to dilate an LMI involves the use the projection lemma in reverse or the reciprocal projection lemma. For instance, consider the following example taken from [37] and inspired by the dilated bounded real lemma matrix inequality in [19, pp. 153–155] involving the matrices $\mathbf{P} \in \mathbb{S}^n$ and $\mathbf{A} \in \mathbb{R}^{n\times n}$, where $\mathbf{P} > 0$. The matrix inequality

$$\begin{bmatrix} \mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} - \mathbf{P} & \mathbf{P} \\ * & -\mathbf{P} \end{bmatrix} < 0, \tag{2.51}$$

can be rewritten as

$$\begin{bmatrix} \mathbf{A}^\mathsf{T} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{P} & \mathbf{0} \\ * & -\mathbf{P} & \mathbf{0} \\ * & * & -\mathbf{P} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} < 0. \tag{2.52}$$

Since $\mathbf{P} > 0$, it is also known that

$$\begin{bmatrix} -\mathbf{P} & \mathbf{0} \\ * & -\mathbf{P} \end{bmatrix} < 0,$$

which can be rewritten as

$$\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{P} & \mathbf{0} \\ * & -\mathbf{P} & \mathbf{0} \\ * & * & -\mathbf{P} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} < 0. \tag{2.53}$$

The matrix inequalities in (2.52) and (2.53) are in the form of the strict projection lemma. Specifically, (2.52) is in the form of $\mathbf{N}_G^\mathsf{T}(\mathbf{A})\mathbf{\Phi}(\mathbf{P})\mathbf{N}_G(\mathbf{A}) < 0$, where

$$\mathbf{\Phi}(\mathbf{P}) = \begin{bmatrix} \mathbf{0} & \mathbf{P} & \mathbf{0} \\ * & -\mathbf{P} & \mathbf{0} \\ * & * & -\mathbf{P} \end{bmatrix}, \quad \mathbf{N}_G(\mathbf{A}) = \begin{bmatrix} \mathbf{A} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}.$$

The matrix inequality of (2.53) is in the form of $\mathbf{N}_H^\mathsf{T}\mathbf{\Phi}(\mathbf{P})\mathbf{N}_H < 0$, where

$$\mathbf{N}_H = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}.$$

The projection lemma states that (2.52) and (2.53) are equivalent to

$$\mathbf{\Phi}(\mathbf{P}) + \mathbf{G}(\mathbf{A})\mathbf{V}\mathbf{H}^\mathsf{T} + \mathbf{H}\mathbf{V}^\mathsf{T}\mathbf{G}^\mathsf{T}(\mathbf{A}), \tag{2.54}$$

where $\mathcal{N}(\mathbf{G}^\mathsf{T}(\mathbf{A})) = \mathcal{R}(\mathbf{N}_G(\mathbf{A}))$, $\mathcal{N}(\mathbf{H}^\mathsf{T}) = \mathcal{R}(\mathbf{N}_H)$, and $\mathbf{V} \in \mathbb{R}^{n\times n}$. Choosing

$$\mathbf{G}(\mathbf{A}) = \begin{bmatrix} -\mathbf{1} \\ \mathbf{A}^\mathsf{T} \\ \mathbf{1} \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

the matrix inequality of (2.54) can be rewritten as

$$\begin{bmatrix} \mathbf{0} & \mathbf{P} & \mathbf{0} \\ * & -\mathbf{P} & \mathbf{0} \\ * & * & -\mathbf{P} \end{bmatrix} + \begin{bmatrix} -\mathbf{1} \\ \mathbf{A}^\mathsf{T} \\ \mathbf{1} \end{bmatrix} \mathbf{V} \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{V}^\mathsf{T} \begin{bmatrix} -\mathbf{1} & \mathbf{A} & \mathbf{1} \end{bmatrix} < 0,$$

or equivalently

$$\begin{bmatrix} -\left(\mathbf{V}+\mathbf{V}^{\mathsf{T}}\right) & \mathbf{V}^{\mathsf{T}}\mathbf{A}+\mathbf{P} & \mathbf{V}^{\mathsf{T}} \\ * & -\mathbf{P} & \mathbf{0} \\ * & * & -\mathbf{P} \end{bmatrix} < 0. \tag{2.55}$$

Therefore, the matrix inequality of (2.52) with $\mathbf{P} > 0$ is equivalent to the dilated matrix inequality of (2.55).

2.8.1 Examples of Dilated Matrix Inequalities

Examples of some useful dilated matrix inequalities are presented here, while dilated forms of a number of important matrix inequalities are included as equivalent matrix inequalities in their respective sections.

1. [44] Consider the matrices $\mathbf{A}$, $\mathbf{G} \in \mathbb{R}^{n\times n}$, $\mathbf{\Delta} \in \mathbb{R}^{m\times n}$, $\mathbf{P} \in \mathbb{S}^n$, δ_1, δ_2, a, $b \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$ and $b = a^{-1}$. The matrix inequality

$$\mathbf{AP}+\mathbf{PA}^{\mathsf{T}}+\delta_1\mathbf{P}+\delta_2\mathbf{APA}^{\mathsf{T}}+\mathbf{P}\mathbf{\Delta}^{\mathsf{T}}\mathbf{\Delta}\mathbf{P} < 0 \tag{2.56}$$

is equivalent to the matrix inequality

$$\begin{bmatrix} \mathbf{0} & -\mathbf{P} & \mathbf{P} & \mathbf{0} & \mathbf{P}\mathbf{\Delta}^{\mathsf{T}} \\ * & \mathbf{0} & \mathbf{0} & -\mathbf{P} & \mathbf{0} \\ * & * & -\delta_1^{-1}\mathbf{P} & \mathbf{0} & \mathbf{0} \\ * & * & * & -\delta_2^{-1}\mathbf{P} & \mathbf{0} \\ * & * & * & * & -\mathbf{1} \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathbf{A} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{G} \begin{bmatrix} \mathbf{1} & -b\mathbf{1} & b\mathbf{1} & \mathbf{1} & b\mathbf{\Delta}^{\mathsf{T}} \end{bmatrix} \right\} < 0. \tag{2.57}$$

Moreover, for every solution $\mathbf{P} > 0$ of (2.56), $\mathbf{P}$ and $\mathbf{G} = -a\left(\mathbf{A} - a\mathbf{1}\right)^{-1}\mathbf{P}$ will be solutions of (2.57).

2. [32, pp. 7–8] Consider the matrices $\mathbf{A}$, $\mathbf{V} \in \mathbb{R}^{n\times n}$, $\mathbf{P}$, $\mathbf{X} \in \mathbb{S}^n$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, $\mathbf{D} \in \mathbb{R}^{p\times m}$, $\mathbf{R} \in \mathbb{S}^m$, and $\mathbf{S} \in \mathbb{S}^p$, where $\mathbf{P} > 0$, $\mathbf{R} > 0$, $\mathbf{S} > 0$, and $\mathbf{X} > 0$. The matrix inequality given by

$$\begin{bmatrix} -\mathbf{V}-\mathbf{V}^{\mathsf{T}} & \mathbf{VA}+\mathbf{P} & \mathbf{VB} & \mathbf{0} & \mathbf{V} \\ * & -2\mathbf{P}+\mathbf{X} & \mathbf{0} & \mathbf{C}^{\mathsf{T}} & \mathbf{0} \\ * & * & -\mathbf{R} & \mathbf{D}^{\mathsf{T}} & \mathbf{0} \\ * & * & * & -\mathbf{S} & \mathbf{0} \\ * & * & * & * & -\mathbf{X} \end{bmatrix} < 0,$$

implies the matrix inequality

$$\begin{bmatrix} \mathbf{PA}+\mathbf{A}^{\mathsf{T}}\mathbf{P} & \mathbf{PB} & \mathbf{C}^{\mathsf{T}} \\ * & -\mathbf{R} & \mathbf{D}^{\mathsf{T}} \\ * & * & -\mathbf{S} \end{bmatrix} < 0.$$

3. [32, p. 9] Consider the matrices $\mathbf{A}$, $\mathbf{V} \in \mathbb{R}^{n\times n}$, $\mathbf{Q}$, $\mathbf{X} \in \mathbb{S}^n$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, $\mathbf{D} \in \mathbb{R}^{p\times m}$, $\mathbf{R} \in \mathbb{S}^m$, and $\mathbf{S} \in \mathbb{S}^p$, where $\mathbf{Q} > 0$, $\mathbf{R} > 0$, $\mathbf{S} > 0$, and $\mathbf{X} > 0$. The matrix inequality given by

$$\begin{bmatrix} -\mathbf{V}-\mathbf{V}^{\mathsf{T}} & \mathbf{V}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}+\mathbf{Q} & \mathbf{0} & \mathbf{V}^{\mathsf{T}}\mathbf{C} & \mathbf{V}^{\mathsf{T}} \\ * & -2\mathbf{Q}+\mathbf{X} & \mathbf{B} & \mathbf{0} & \mathbf{0} \\ * & * & -\mathbf{R} & \mathbf{D}^{\mathsf{T}} & \mathbf{0} \\ * & * & * & -\mathbf{S} & \mathbf{0} \\ * & * & * & * & -\mathbf{X} \end{bmatrix} < 0$$

implies the matrix inequality

$$\begin{bmatrix} \mathbf{AQ} + \mathbf{QA}^\mathsf{T} & \mathbf{B} & \mathbf{QC}^\mathsf{T} \\ * & -\mathbf{R} & \mathbf{D}^\mathsf{T} \\ * & * & -\mathbf{S} \end{bmatrix} < 0.$$

2.9 The S-Procedure [3, pp. 23–24], [17, pp. 482–483]

Consider $\mathbf{x} \in \mathbb{R}^n$, $\tau_i \in \mathbb{R}_{\geq 0}$, and the quadratic functions $F_0(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$, $F_i(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$, where $i = 1, \ldots, m$. The inequality $F_0(\mathbf{x}) \leq 0$ is satisfied when $F_i(\mathbf{x}) \geq 0$, $i = 1, \ldots, m$, if

$$F_0(\mathbf{x}) + \sum_{i=1}^{m} \tau_i F_i(\mathbf{x}) \leq 0.$$

Example 2.4. [3, p. 24], [17, p. 483] Consider $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{z} \in \mathbb{R}^m$, and $\tau \in \mathbb{R}_{\geq 0}$. The problem of solving for $\mathbf{P} > 0$ such that

$$\begin{bmatrix} \mathbf{x}^\mathsf{T} & \mathbf{z}^\mathsf{T} \end{bmatrix} \begin{bmatrix} \mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{PA} & \mathbf{PB} \\ * & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} < 0$$

whenever $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{z}$ satisfy the constraint $\mathbf{z}^\mathsf{T}\mathbf{z} \leq \mathbf{x}^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{C}\mathbf{x}$ is equivalent to finding $\mathbf{P} > 0$ and τ such that

$$\begin{bmatrix} \mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{PA} + \tau\mathbf{C}^\mathsf{T}\mathbf{C} & \mathbf{PB} \\ * & -\tau\mathbf{1} \end{bmatrix} < 0.$$

2.10 Singular Values

2.10.1 Maximum Singular Value [3, p. 8], [45]

Consider $\mathbf{A} \in \mathbb{R}^{n \times m}$ and $\gamma \in \mathbb{R}_{>0}$. The maximum singular value of $\mathbf{A}$ is strictly less than γ (i.e., $\bar{\sigma}(\mathbf{A}) < \gamma$) if and only if $\mathbf{AA}^\mathsf{T} < \gamma^2\mathbf{1}$. Using the Schur complement, $\mathbf{AA}^\mathsf{T} < \gamma^2\mathbf{1}$ is equivalent to

$$\begin{bmatrix} \gamma\mathbf{1} & \mathbf{A} \\ * & \gamma\mathbf{1} \end{bmatrix} > 0.$$

Equivalently, $\bar{\sigma}(\mathbf{A}) < \gamma$ if and only if $\mathbf{A}^\mathsf{T}\mathbf{A} < \gamma^2\mathbf{1}$ or

$$\begin{bmatrix} \gamma\mathbf{1} & \mathbf{A}^\mathsf{T} \\ * & \gamma\mathbf{1} \end{bmatrix} > 0.$$

2.10.2 Maximum Singular Value of a Complex Matrix [46]

Consider $\mathbf{A} \in \mathbb{C}^{n \times m}$ and $\gamma \in \mathbb{R}_{>0}$. The maximum singular value of $\mathbf{A}$ is strictly less than γ (i.e., $\bar{\sigma}(\mathbf{A}) < \gamma$) if and only if $\mathbf{AA}^\mathsf{H} < \gamma^2\mathbf{1}$. Using the Schur complement, $\mathbf{AA}^\mathsf{H} < \gamma^2\mathbf{1}$ is equivalent to

$$\begin{bmatrix} \gamma\mathbf{1} & \mathbf{A} \\ \mathbf{A}^\mathsf{H} & \gamma\mathbf{1} \end{bmatrix} > 0.$$

Equivalently, $\bar{\sigma}(\mathbf{A}) < \gamma$ if and only if $\mathbf{A}^\mathsf{H}\mathbf{A} < \gamma^2\mathbf{1}$ or

$$\begin{bmatrix} \gamma\mathbf{1} & \mathbf{A}^\mathsf{H} \\ \mathbf{A} & \gamma\mathbf{1} \end{bmatrix} > 0.$$

2.10.3 Minimum Singular Value

Consider $\mathbf{A} \in \mathbb{R}^{n \times m}$ and $\nu \in \mathbb{R}_{\geq 0}$. If $n \leq m$, the minimum singular value of $\mathbf{A}$ is strictly greater than ν (i.e., $\underline{\sigma}(\mathbf{A}) > \nu$) if and only if $\mathbf{A}\mathbf{A}^{\mathsf{T}} > \nu^2 \mathbf{1}$. If $m \leq n$, $\underline{\sigma}(\mathbf{A}) > \nu$ if and only if $\mathbf{A}^{\mathsf{T}}\mathbf{A} > \nu^2 \mathbf{1}$.

2.10.4 Minimum Singular Value of a Complex Matrix

Consider $\mathbf{A} \in \mathbb{C}^{n \times m}$ and $\nu \in \mathbb{R}_{\geq 0}$. If $n \leq m$, the minimum singular value of $\mathbf{A}$ is strictly greater than ν (i.e., $\underline{\sigma}(\mathbf{A}) > \nu$) if and only if $\mathbf{A}\mathbf{A}^{\mathsf{H}} > \nu^2 \mathbf{1}$. If $m \leq n$, $\underline{\sigma}(\mathbf{A}) > \nu$ if and only if $\mathbf{A}^{\mathsf{H}}\mathbf{A} > \nu^2 \mathbf{1}$.

2.11 Eigenvalues of Symmetric Matrices

2.11.1 Maximum Eigenvalue [3, p. 10]

Consider $\mathbf{A} \in \mathbb{S}^{n \times n}$ and $\gamma \in \mathbb{R}$. The maximum eigenvalue of $\mathbf{A}$ is strictly less than γ (i.e., $\bar{\lambda}(\mathbf{A}) < \gamma$) if and only if $\mathbf{A} < \gamma \mathbf{1}$.

2.11.2 Minimum Eigenvalue

Consider $\mathbf{A} \in \mathbb{S}^{n \times n}$ and $\gamma \in \mathbb{R}$. The minimum eigenvalue of $\mathbf{A}$ is strictly greater than γ (i.e., $\underline{\lambda}(\mathbf{A}) > \gamma$) if and only if $\mathbf{A} > \gamma \mathbf{1}$.

2.12 Matrix Condition Number

2.12.1 Condition Number of a Matrix [3, pp. 37–38]

Consider $\mathbf{A} \in \mathbb{R}^{n \times m}$ and γ, $\mu \in \mathbb{R}_{>0}$, where the condition number of $\mathbf{A}$ is $\kappa(\mathbf{A})$. If $m \leq n$, the inequality $\kappa(\mathbf{A}) \leq \gamma$ holds if there exists μ such that

$$\mu \mathbf{1} \leq \mathbf{A}^{\mathsf{T}}\mathbf{A} \leq \gamma^2 \mu \mathbf{1}.$$

If $n \leq m$, the inequality $\kappa(\mathbf{A}) \leq \gamma$ holds if there exists μ such that

$$\mu \mathbf{1} \leq \mathbf{A}\mathbf{A}^{\mathsf{T}} \leq \gamma^2 \mu \mathbf{1}.$$

2.12.2 Condition Number of a Positive Definite Matrix [3, p. 38]

Consider $\mathbf{A} \in \mathbb{S}^{n}$ and γ, $\mu \in \mathbb{R}_{>0}$, where the condition number of $\mathbf{A}$ is $\kappa(\mathbf{A})$. The inequality $\kappa(\mathbf{A}) \leq \gamma$ holds if there exists μ such that

$$\mu \mathbf{1} \leq \mathbf{A} \leq \gamma \mu \mathbf{1}.$$

2.13 Trace of a Symmetric Matrix [19, p. 46–47]

Consider $\mathbf{P}$, $\mathbf{Q} \in \mathbb{S}^{n}$. The property $\mathrm{tr}(\mathbf{P}) < \mathrm{tr}(\mathbf{Q})$ holds if the matrix inequality $\mathbf{P} < \mathbf{Q}$ is satisfied.

2.14 Submatrix Determinants [46]

Consider $\mathbf{A} \in \mathbb{S}^n$. Let $\mathbf{A}_k \in \mathbb{S}^k$ be a submatrix of $\mathbf{A}$ consisting of its first k rows and columns, where $k \leq n$. The matrix inequality $\mathbf{A} > 0$ is satisfied if and only if

$$\det(\mathbf{A}_k) > 0, \;\; k = 1, \ldots, n.$$

2.15 Imaginary and Real Parts [17, p. 475]

Consider $\mathbf{Q}_R \in \mathbb{S}^n$, $\mathbf{Q}_I \in \mathbb{R}^{n \times n}$, and $\mathbf{Q} = \mathbf{Q}^\mathsf{H} = \mathbf{Q}_R + j\mathbf{Q}_I \in \mathbb{C}^{n \times n}$. The matrix inequality $\mathbf{Q} > 0$ is equivalent to the matrix inequality given by

$$\begin{bmatrix} \mathbf{Q}_R & \mathbf{Q}_I \\ -\mathbf{Q}_I & \mathbf{Q}_R \end{bmatrix} > 0.$$

2.16 Quadratic Inequalities

2.16.1 Weighted Norm [45]

Consider $\mathbf{W} \in \mathbb{S}^n$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and $\gamma \in \mathbb{R}_{\geq 0}$, where $\mathbf{W} > 0$. The inequality $(\mathbf{x} - \mathbf{y})^\mathsf{T}\mathbf{W}(\mathbf{x} - \mathbf{y}) \leq \gamma$ is equivalent to the matrix inequality given by

$$\begin{bmatrix} \gamma & (\mathbf{x} - \mathbf{y})^\mathsf{T} \\ * & \mathbf{W}^{-1} \end{bmatrix} \geq 0.$$

2.16.2 Quadratic Inequality

Consider $\mathbf{W} \in \mathbb{S}^n$, $\mathbf{A} \in \mathbb{R}^{n \times m}$, $\mathbf{x}, \mathbf{c} \in \mathbb{R}^m$, $\mathbf{b} \in \mathbb{R}^n$, and $d \in \mathbb{R}$, where $\mathbf{W} > 0$. The quadratic inequality $(\mathbf{A}\mathbf{x} + \mathbf{b})^\mathsf{T}\mathbf{W}(\mathbf{A}\mathbf{x} + \mathbf{b}) - \mathbf{c}^\mathsf{T}\mathbf{x} - d \leq 0$ with $\mathbf{W} > 0$ is equivalent to the matrix inequality given by

$$\begin{bmatrix} \mathbf{W}^{-1} & \mathbf{A}\mathbf{x} + \mathbf{b} \\ * & \mathbf{c}^\mathsf{T}\mathbf{x} + d \end{bmatrix} \geq 0.$$

2.17 Miscellaneous Properties and Results

1. [47, p. 19] Consider $\mathbf{M}_{11}, \mathbf{A} \in \mathbb{S}^n$, $\mathbf{M}_{12} \in \mathbb{R}^{n \times m}$, $\mathbf{M}_{22} \in \mathbb{S}^m$, $\mathbf{E}, \mathbf{F}_1 \in \mathbb{R}^{n \times n}$, and $\mathbf{F}_2 \in \mathbb{R}^{m \times n}$, where $\mathbf{M}_{11} \geq 0$ and $\mathbf{E}$ is invertible. The matrix inequality

$$\begin{bmatrix} \mathbf{E}^{-1}\mathbf{A} \\ \mathbf{1} \end{bmatrix}^\mathsf{T} \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ * & \mathbf{M}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{E}^{-1}\mathbf{A} \\ \mathbf{1} \end{bmatrix} < 0 \tag{2.58}$$

   holds if and only if there exist $\mathbf{F}_1$ and $\mathbf{F}_2$ such that

$$\begin{bmatrix} \mathbf{M}_{11} + \mathbf{F}_1\mathbf{E} + \mathbf{E}^\mathsf{T}\mathbf{F}_1^\mathsf{T} & \mathbf{M}_{12} - \mathbf{F}_1\mathbf{A} + \mathbf{E}^\mathsf{T}\mathbf{F}_2^\mathsf{T} \\ * & \mathbf{M}_{22} - \mathbf{F}_2\mathbf{A} - \mathbf{A}^\mathsf{T}\mathbf{F}_2^\mathsf{T} \end{bmatrix} < 0, \tag{2.59}$$

   Moreover, the following statements hold.

   (a) If (2.58) holds, then (2.59) holds with $\mathbf{F}_1 = -\left(\mathbf{M}_{11} + \epsilon\mathbf{W}\right)\mathbf{E}^{-1}$ and $\mathbf{F}_2 = -\mathbf{M}_{12}^\mathsf{T}\mathbf{E}^{-1}$, where $\epsilon \in \mathbb{R}_{>0}$ is sufficiently small, $\mathbf{W} \in \mathbb{S}^n$, and $\mathbf{W} > 0$.

   (b) If (2.58) holds and $\mathbf{M}_{11} > 0$, then (2.59) holds with $\mathbf{F}_1 = \mathbf{M}_{11}\mathbf{E}^{-1}$ and $\mathbf{F}_2 = -\mathbf{M}_{12}^\mathsf{T}\mathbf{E}^{-1}$.

3 LMIs in Systems and Stability Theory

3.1 Lyapunov Inequalities

3.1.1 Lyapunov Stability [2, pp. 1201–1203], [3, pp. 20–21]

Consider the matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{Q} \geq 0$. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, satisfying the Lyapunov equation

$$\mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} = \mathbf{0},$$

if and only if there exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{P}\mathbf{A} \leq 0. \tag{3.1}$$

If (3.1) holds, then $\mathrm{Re}\{\lambda_i(\mathbf{A})\} \leq 0$, $i = 1, \ldots, n$, and the equilibrium point $\bar{\mathbf{x}} = \mathbf{0}$ of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is Lyapunov stable.

The matrix inequality of (3.1) is satisfied under any of the following equivalent conditions.

1. There exists $\mathbf{X} \in \mathbb{S}^n$, where $\mathbf{X} > 0$, such that

$$\mathbf{X}\mathbf{A}^\mathsf{T} + \mathbf{A}\mathbf{X} \leq 0.$$

2. There exist $\mathbf{X} \in \mathbb{S}^n$ and $\mathbf{V} \in \mathbb{R}^n$, where $\mathbf{X} > 0$, such that

$$\begin{bmatrix} -\left(\mathbf{V} + \mathbf{V}^\mathsf{T}\right) & \mathbf{V}^\mathsf{T}\mathbf{A} + \mathbf{X} & \mathbf{V}^\mathsf{T} \\ * & -\mathbf{X} & \mathbf{0} \\ * & * & -\mathbf{X} \end{bmatrix} \leq 0.$$

Proof. Identical to the proof of (3.3) in [37], except with the use of the Nonstrict Projection Lemma, where $\mathbf{G}^\mathsf{T} = \begin{bmatrix} -\mathbf{1} & \mathbf{A} & \mathbf{1} \end{bmatrix}$ and $\mathbf{H}^\mathsf{T} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix}$, and therefore $\mathcal{R}(\mathbf{G})$ and $\mathcal{R}(\mathbf{H})$ are linearly independent. □

3. There exist $\mathbf{X} \in \mathbb{S}^n$ and $\mathbf{V} \in \mathbb{R}^n$, where $\mathbf{X} > 0$, such that

$$\begin{bmatrix} -\left(\mathbf{V} + \mathbf{V}^\mathsf{T}\right) & \mathbf{V}^\mathsf{T}\mathbf{A}^\mathsf{T} + \mathbf{X} & \mathbf{V}^\mathsf{T} \\ * & -\mathbf{X} & \mathbf{0} \\ * & * & -\mathbf{X} \end{bmatrix} \leq 0.$$

Proof. Identical to the proof of (3.4) in [37], except with the use of the Nonstrict Projection Lemma, where $\mathbf{G}^\mathsf{T} = \begin{bmatrix} -\mathbf{1} & \mathbf{A}^\mathsf{T} & \mathbf{1} \end{bmatrix}$ and $\mathbf{H}^\mathsf{T} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix}$, and therefore $\mathcal{R}(\mathbf{G})$ and $\mathcal{R}(\mathbf{H})$ are linearly independent. □

3.1.2 Asymptotic Stability [2, p. 1201–1203], [3, p. 2]

Consider the matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{Q} > 0$. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, satisfying the Lyapunov equation

$$\mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} = \mathbf{0},$$

if and only if there exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{P}\mathbf{A} < 0. \tag{3.2}$$

If (3.2) holds, then $\mathrm{Re}\{\lambda_i(\mathbf{A})\} < 0$, $i = 1, \ldots, n$, the matrix $\mathbf{A}$ is Hurwitz, and the equilibrium point $\bar{\mathbf{x}} = \mathbf{0}$ of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is asymptotically stable.

The matrix inequality of (3.2) is satisfied and the matrix $\mathbf{A}$ is Hurwitz under any of the following equivalent conditions.

1. There exists $\mathbf{X} \in \mathbb{S}^n$, where $\mathbf{X} > 0$, such that

$$\mathbf{X}\mathbf{A}^\mathsf{T} + \mathbf{A}\mathbf{X} < 0.$$

2. (*The $\mathcal{S}$-Variable Approach* [47, pp. 2–3], [48]) There exist $\mathbf{P} \in \mathbb{S}^n$ and $\mathbf{F}_1, \mathbf{F}_2 \in \mathbb{R}^{n\times n}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{F}_1\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{F}_1^\mathsf{T} & \mathbf{P} - \mathbf{F}_1 + \mathbf{A}^\mathsf{T}\mathbf{F}_2^\mathsf{T} \\ * & -(\mathbf{F}_2 + \mathbf{F}_2^\mathsf{T}) \end{bmatrix} < 0.$$

3. [37] There exist $\mathbf{Y} \in \mathbb{S}^n$ and $\mathbf{W} \in \mathbb{R}^{n\times n}$, where $\mathbf{Y} > 0$, such that

$$\begin{bmatrix} \mathbf{Y} - \left(\mathbf{W} + \mathbf{W}^\mathsf{T}\right) & \mathbf{A}\mathbf{Y} + \mathbf{W}^\mathsf{T} \\ * & -\mathbf{Y} \end{bmatrix} < 0.$$

4. [37] There exist $\mathbf{X} \in \mathbb{S}^n$ and $\mathbf{V} \in \mathbb{R}^{n\times n}$, where $\mathbf{X} > 0$, such that

$$\begin{bmatrix} -\left(\mathbf{V} + \mathbf{V}^\mathsf{T}\right) & \mathbf{V}^\mathsf{T}\mathbf{A} + \mathbf{X} & \mathbf{V}^\mathsf{T} \\ * & -\mathbf{X} & \mathbf{0} \\ * & * & -\mathbf{X} \end{bmatrix} < 0. \tag{3.3}$$

5. [37] There exist $\mathbf{X} \in \mathbb{S}^n$ and $\mathbf{V} \in \mathbb{R}^{n\times n}$, where $\mathbf{X} > 0$, such that

$$\begin{bmatrix} -\left(\mathbf{V} + \mathbf{V}^\mathsf{T}\right) & \mathbf{V}^\mathsf{T}\mathbf{A}^\mathsf{T} + \mathbf{X} & \mathbf{V}^\mathsf{T} \\ * & -\mathbf{X} & \mathbf{0} \\ * & * & -\mathbf{X} \end{bmatrix} < 0. \tag{3.4}$$

3.1.3 Discrete-Time Lyapunov Stability [2, pp. 1203–1204]

Consider the matrices $\mathbf{A}_\mathrm{d} \in \mathbb{R}^{n\times n}$ and $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{Q} \geq 0$. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, satisfying the discrete-time Lyapunov equation

$$\mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{A}_\mathrm{d} - \mathbf{P} + \mathbf{Q} = \mathbf{0}.$$

if and only if there exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{A}_\mathrm{d} - \mathbf{P} \leq 0. \tag{3.5}$$

If (3.5) holds, then $|\lambda_i(\mathbf{A}_\mathrm{d})| \leq 1$, $i = 1, \ldots, n$, and the equilibrium point $\bar{\mathbf{x}} = \mathbf{0}$ of the system $\mathbf{x}_{k+1} = \mathbf{A}_\mathrm{d}\mathbf{x}_k$ is Lyapunov stable.

The matrix inequality of (3.5) is satisfied and the eigenvalues of $\mathbf{A}_\mathrm{d}$ satisfy $|\lambda_i(\mathbf{A}_\mathrm{d})| \leq 1$, $i = 1, \ldots, n$ under any of the following equivalent conditions.

1. There exists $\mathbf{X} \in \mathbb{S}^n$, where $\mathbf{X} > 0$, such that

$$\mathbf{A}_{\mathrm{d}}\mathbf{P}\mathbf{A}_{\mathrm{d}}^{\mathsf{T}} - \mathbf{P} \leq 0.$$

2. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{A}_{\mathrm{d}}\mathbf{P} \\ * & \mathbf{P} \end{bmatrix} \geq 0.$$

3. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{A}_{\mathrm{d}}^{\mathsf{T}}\mathbf{P} \\ * & \mathbf{P} \end{bmatrix} \geq 0.$$

4. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{P}\mathbf{A}_{\mathrm{d}}^{\mathsf{T}} \\ * & \mathbf{P} \end{bmatrix} \geq 0.$$

5. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{P}\mathbf{A}_{\mathrm{d}} \\ * & \mathbf{P} \end{bmatrix} \geq 0.$$

6. [49] There exist $\mathbf{P} \in \mathbb{S}^n$ and $\mathbf{G} \in \mathbb{R}^{n \times n}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{A}_{\mathrm{d}}^{\mathsf{T}}\mathbf{G} \\ * & \mathbf{G} + \mathbf{G}^{\mathsf{T}} - \mathbf{P} \end{bmatrix} \geq 0.$$

3.1.4 Discrete-Time Asymptotic Stability [2, pp. 1203–1204], [19, pp. 97–98]

Consider the matrices $\mathbf{A}_{\mathrm{d}} \in \mathbb{R}^{n \times n}$ and $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{Q} > 0$. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, satisfying the discrete-time Lyapunov equation

$$\mathbf{A}_{\mathrm{d}}^{\mathsf{T}}\mathbf{P}\mathbf{A}_{\mathrm{d}} - \mathbf{P} + \mathbf{Q} = \mathbf{0}.$$

if and only if there exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\mathbf{A}_{\mathrm{d}}^{\mathsf{T}}\mathbf{P}\mathbf{A}_{\mathrm{d}} - \mathbf{P} < 0. \tag{3.6}$$

If (3.6) holds, then $|\lambda_i(\mathbf{A}_{\mathrm{d}})| < 1$, $i = 1, \ldots, n$, the matrix $\mathbf{A}_{\mathrm{d}}$ is Schur, and the equilibrium point $\bar{\mathbf{x}} = \mathbf{0}$ of the system $\mathbf{x}_{k+1} = \mathbf{A}_{\mathrm{d}}\mathbf{x}_k$ is asymptotically stable.

The matrix inequality of (3.6) is satisfied and the eigenvalues of $\mathbf{A}_{\mathrm{d}}$ satisfy $|\lambda_i(\mathbf{A}_{\mathrm{d}})| < 1$, $i = 1, \ldots, n$ under any of the following equivalent conditions.

1. There exists $\mathbf{X} \in \mathbb{S}^n$, where $\mathbf{X} > 0$, such that

$$\mathbf{A}_{\mathrm{d}}\mathbf{P}\mathbf{A}_{\mathrm{d}}^{\mathsf{T}} - \mathbf{P} < 0.$$

2. [19, p. 97] There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{A}_{\mathrm{d}}\mathbf{P} \\ * & \mathbf{P} \end{bmatrix} > 0.$$

3. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P} \\ * & \mathbf{P} \end{bmatrix} > 0.$$

4. [19, p. 97] There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{P}\mathbf{A}_\mathrm{d}^\mathsf{T} \\ * & \mathbf{P} \end{bmatrix} > 0.$$

5. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{P}\mathbf{A}_\mathrm{d} \\ * & \mathbf{P} \end{bmatrix} > 0.$$

6. (*The S-Variable Approach* [47, p. 3], [50]) There exist $\mathbf{P} \in \mathbb{S}^n$ and $\mathbf{F}_1, \mathbf{F}_2 \in \mathbb{R}^{n \times n}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{F}_1\mathbf{A}_\mathrm{d} + \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{F}_1^\mathsf{T} - \mathbf{P} & -\mathbf{F}_1 + \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{F}_2^\mathsf{T} \\ * & \mathbf{P} - (\mathbf{F}_2 + \mathbf{F}_2^\mathsf{T}) \end{bmatrix} < 0.$$

7. [49] There exist $\mathbf{P} \in \mathbb{S}^n$ and $\mathbf{G} \in \mathbb{R}^{n \times n}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{G} \\ * & \mathbf{G} + \mathbf{G}^\mathsf{T} - \mathbf{P} \end{bmatrix} > 0.$$

3.2 Bounded Real Lemma and the $\mathcal{H}_\infty$ Norm

3.2.1 Continuous-Time Bounded Real Lemma [24], [51, pp. 85–86]

Consider a continuous-time LTI system, $\boldsymbol{\mathcal{G}} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, with state-space realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{C} \in \mathbb{R}^{p \times n}$, and $\mathbf{D} \in \mathbb{R}^{p \times m}$. The $\mathcal{H}_\infty$ norm of $\boldsymbol{\mathcal{G}}$ is

$$\|\boldsymbol{\mathcal{G}}\|_\infty = \sup_{\mathbf{u} \in \mathcal{L}_2, \mathbf{u} \neq \mathbf{0}} \frac{\|\boldsymbol{\mathcal{G}}\mathbf{u}\|_2}{\|\mathbf{u}\|_2}.$$

The inequality $\|\boldsymbol{\mathcal{G}}\|_\infty < \gamma$ holds holds under any of the following necessary and sufficient conditions.

1. There exist $\mathbf{P} \in \mathbb{S}^n$ and $\gamma \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{P} & \mathbf{P}\mathbf{B} & \mathbf{C}^\mathsf{T} \\ * & -\gamma\mathbf{1} & \mathbf{D}^\mathsf{T} \\ * & * & -\gamma\mathbf{1} \end{bmatrix} < 0.$$

2. There exist $\mathbf{Q} \in \mathbb{S}^n$ and $\gamma \in \mathbb{R}_{>0}$, where $\mathbf{Q} > 0$, such that

$$\begin{bmatrix} \mathbf{A}\mathbf{Q} + \mathbf{Q}\mathbf{A}^\mathsf{T} & \mathbf{B} & \mathbf{Q}\mathbf{C}^\mathsf{T} \\ * & -\gamma\mathbf{1} & \mathbf{D}^\mathsf{T} \\ * & * & -\gamma\mathbf{1} \end{bmatrix} < 0.$$

3. There exist $\mathbf{P} \in \mathbb{S}^n$ and $\gamma \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{C}^\mathsf{T}\mathbf{C} & \mathbf{PB} + \mathbf{C}^\mathsf{T}\mathbf{D} \\ * & -\gamma^2\mathbf{1} + \mathbf{D}^\mathsf{T}\mathbf{D} \end{bmatrix} < 0.$$

4. There exist $\mathbf{Q} \in \mathbb{S}^n$ and $\gamma \in \mathbb{R}_{>0}$, where $\mathbf{Q} > 0$, such that

$$\begin{bmatrix} \mathbf{AQ} + \mathbf{QA}^\mathsf{T} + \mathbf{BB}^\mathsf{T} & \mathbf{QC}^\mathsf{T} + \mathbf{BD}^\mathsf{T} \\ * & -\gamma^2\mathbf{1} + \mathbf{DD}^\mathsf{T} \end{bmatrix} < 0.$$

5. There exist $\mathbf{Q} \in \mathbb{S}^n$, $\mathbf{V}_{11} \in \mathbb{R}^{n\times n}$, $\mathbf{V}_{12} \in \mathbb{R}^{n\times m}$, $\mathbf{V}_{21} \in \mathbb{R}^{m\times n}$, $\mathbf{V}_{22} \in \mathbb{R}^{m\times m}$, and $\gamma \in \mathbb{R}_{>0}$, where $\mathbf{Q} > 0$, such that

$$\begin{bmatrix} -(\mathbf{V}_{11} + \mathbf{V}_{11}^\mathsf{T}) & \mathbf{V}_{11}^\mathsf{T}\mathbf{A}^\mathsf{T} + \mathbf{V}_{21}^\mathsf{T}\mathbf{B}^\mathsf{T} + \mathbf{Q} & \mathbf{V}_{11}^\mathsf{T}\mathbf{C}^\mathsf{T} + \mathbf{V}_{21}^\mathsf{T}\mathbf{D}^\mathsf{T} & \mathbf{V}_{11}^\mathsf{T} & -\mathbf{V}_{12} - \mathbf{V}_{21}^\mathsf{T} \\ * & -\mathbf{Q} & \mathbf{0} & \mathbf{0} & \mathbf{AV}_{12} + \mathbf{BV}_{22} \\ * & * & -\gamma^2\mathbf{1} & \mathbf{0} & \mathbf{CV}_{12} + \mathbf{DV}_{22} \\ * & * & * & -\mathbf{Q} & \mathbf{V}_{12} \\ * & * & * & * & -\mathbf{1} - (\mathbf{V}_{22} + \mathbf{V}_{22}^\mathsf{T}) \end{bmatrix} < 0.$$

   Proof. Identical to the proof of (3.7) in [19, p. 156], except with $\mathbf{\Omega} = \begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{bmatrix}$. □

6. There exist $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{W}_{11} \in \mathbb{R}^{n\times n}$, $\mathbf{W}_{12} \in \mathbb{R}^{n\times p}$, $\mathbf{V}_{21} \in \mathbb{R}^{p\times n}$, $\mathbf{V}_{22} \in \mathbb{R}^{p\times p}$, and $\gamma \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} -(\mathbf{W}_{11} + \mathbf{W}_{11}^\mathsf{T}) & \mathbf{W}_{11}^\mathsf{T}\mathbf{A} + \mathbf{W}_{21}^\mathsf{T}\mathbf{C} + \mathbf{P} & \mathbf{W}_{11}^\mathsf{T}\mathbf{B} + \mathbf{W}_{21}^\mathsf{T}\mathbf{D} & \mathbf{W}_{11}^\mathsf{T} & -(\mathbf{W}_{12} + \mathbf{W}_{21}^\mathsf{T}) \\ * & -\mathbf{P} & \mathbf{0} & \mathbf{0} & \mathbf{A}^\mathsf{T}\mathbf{W}_{12} + \mathbf{C}^\mathsf{T}\mathbf{W}_{22} \\ * & * & -\gamma^2\mathbf{1} & \mathbf{0} & \mathbf{B}^\mathsf{T}\mathbf{W}_{12} + \mathbf{D}^\mathsf{T}\mathbf{W}_{22} \\ * & * & * & -\mathbf{P} & \mathbf{W}_{12} \\ * & * & * & * & -(\mathbf{1} + \mathbf{W}_{22} + \mathbf{W}_{22}^\mathsf{T}) \end{bmatrix} < 0.$$

   Proof. Identical to the proof of (3.8), except with $\mathbf{\Omega} = \begin{bmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} \\ \mathbf{W}_{21} & \mathbf{W}_{22} \end{bmatrix}$. □

   The $\mathcal{H}_\infty$ norm of $\boldsymbol{\mathcal{G}}$ is the minimum value of $\gamma \in \mathbb{R}_{>0}$ that satisfies any of the above conditions. If $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is a minimal realization, then the matrix inequalities can be nonstrict [3, pp. 26–27], [52, pp. 308–311], [53].

The inequality $\|\boldsymbol{\mathcal{G}}\|_\infty < \gamma$ also holds under any of the following equivalent sufficient conditions.

1. [19, p. 156] There exist $\mathbf{Q} \in \mathbb{S}^n$, $\mathbf{V} \in \mathbb{R}^{n\times n}$, and $\gamma \in \mathbb{R}_{>0}$, where $\mathbf{Q} > 0$, such that

$$\begin{bmatrix} -(\mathbf{V} + \mathbf{V}^\mathsf{T}) & \mathbf{V}^\mathsf{T}\mathbf{A}^\mathsf{T} + \mathbf{Q} & \mathbf{V}^\mathsf{T}\mathbf{C}^\mathsf{T} & \mathbf{V}^\mathsf{T} & \mathbf{0} \\ * & -\mathbf{Q} & \mathbf{0} & \mathbf{0} & \mathbf{B} \\ * & * & -\gamma\mathbf{1} & \mathbf{0} & \mathbf{D} \\ * & * & * & -\mathbf{Q} & \mathbf{0} \\ * & * & * & * & -\gamma\mathbf{1} \end{bmatrix} < 0. \tag{3.7}$$

2. There exist $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{W} \in \mathbb{R}^{n\times n}$, and $\gamma \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} -(\mathbf{W}+\mathbf{W}^\mathsf{T}) & \mathbf{W}^\mathsf{T}\mathbf{A}+\mathbf{P} & \mathbf{W}^\mathsf{T}\mathbf{B} & \mathbf{W}^\mathsf{T} & \mathbf{0} \\ * & -\mathbf{P} & \mathbf{0} & \mathbf{0} & \mathbf{C}^\mathsf{T} \\ * & * & -\gamma\mathbf{1} & \mathbf{0} & \mathbf{D}^\mathsf{T} \\ * & * & * & -\mathbf{P} & \mathbf{0} \\ * & * & * & * & -\gamma\mathbf{1} \end{bmatrix} < 0. \tag{3.8}$$

Proof. Identical to the proof of (3.7) in [19, p. 156], except starting with the Bounded Real Lemma in the form

$$\begin{bmatrix} \mathbf{AQ}+\mathbf{QA}^\mathsf{T}+\frac{1}{\gamma}\mathbf{QC}^\mathsf{T}\mathbf{CQ} & \mathbf{B}+\frac{1}{\gamma}\mathbf{QC}^\mathsf{T}\mathbf{D} \\ * & -\gamma\mathbf{1}+\frac{1}{\gamma}\mathbf{D}^\mathsf{T}\mathbf{D} \end{bmatrix},$$

which requires $\mathbf{\Phi} = \begin{bmatrix} -\mathbf{1} & \mathbf{A} & \mathbf{B} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} & \mathbf{D} & \mathbf{0} & -\gamma\mathbf{1} \end{bmatrix}$. □

3.2.2 Discrete-Time Bounded Real Lemma [24]

Consider a discrete-time LTI system, $\boldsymbol{\mathcal{G}} : \ell_{2e} \to \ell_{2e}$, with state-space realization $(\mathbf{A}_\mathrm{d}, \mathbf{B}_\mathrm{d}, \mathbf{C}_\mathrm{d}, \mathbf{D}_\mathrm{d})$, where $\mathbf{A}_\mathrm{d} \in \mathbb{R}^{n\times n}$, $\mathbf{B}_\mathrm{d} \in \mathbb{R}^{n\times m}$, $\mathbf{C}_\mathrm{d} \in \mathbb{R}^{p\times n}$, and $\mathbf{D}_\mathrm{d} \in \mathbb{R}^{p\times m}$. The $\mathcal{H}_\infty$ norm of $\boldsymbol{\mathcal{G}}$ is

$$\|\boldsymbol{\mathcal{G}}\|_\infty = \sup_{\mathbf{u}\in\ell_2,\mathbf{u}\neq\mathbf{0}} \frac{\|\boldsymbol{\mathcal{G}}\mathbf{u}\|_2}{\|\mathbf{u}\|_2}.$$

The inequality $\|\boldsymbol{\mathcal{G}}\|_\infty < \gamma$ holds under any of the following necessary and sufficient conditions.

1. There exist $\mathbf{P} \in \mathbb{S}^n$ and $\gamma \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{A}_\mathrm{d}-\mathbf{P} & \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d} & \mathbf{C}_\mathrm{d}^\mathsf{T} \\ * & \mathbf{B}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d}-\gamma\mathbf{1} & \mathbf{D}_\mathrm{d}^\mathsf{T} \\ * & * & -\gamma\mathbf{1} \end{bmatrix} < 0.$$

2. There exist $\mathbf{P} \in \mathbb{S}^n$ and $\gamma \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{A}_\mathrm{d}\mathbf{P}\mathbf{A}_\mathrm{d}^\mathsf{T}-\mathbf{P} & \mathbf{B}_\mathrm{d} & \mathbf{A}_\mathrm{d}\mathbf{P}\mathbf{C}_\mathrm{d}^\mathsf{T} \\ * & -\gamma\mathbf{1} & \mathbf{D}_\mathrm{d}^\mathsf{T} \\ * & * & \mathbf{C}_\mathrm{d}\mathbf{P}\mathbf{C}_\mathrm{d}^\mathsf{T}-\gamma\mathbf{1} \end{bmatrix} < 0.$$

3. [54] There exist $\mathbf{P} \in \mathbb{S}^n$ and $\gamma \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{A}_\mathrm{d}\mathbf{P} & \mathbf{B}_\mathrm{d} & \mathbf{0} \\ * & \mathbf{P} & \mathbf{0} & \mathbf{P}\mathbf{C}_\mathrm{d}^\mathsf{T} \\ * & * & \gamma\mathbf{1} & \mathbf{D}_\mathrm{d}^\mathsf{T} \\ * & * & * & \gamma\mathbf{1} \end{bmatrix} > 0.$$

4. There exist $\mathbf{Q} \in \mathbb{S}^n$ and $\gamma \in \mathbb{R}_{>0}$, where $\mathbf{Q} > 0$, such that

$$\begin{bmatrix} \mathbf{Q} & \mathbf{Q}\mathbf{A}_\mathrm{d} & \mathbf{Q}\mathbf{B}_\mathrm{d} & \mathbf{0} \\ * & \mathbf{Q} & \mathbf{0} & \mathbf{C}_\mathrm{d}^\mathsf{T} \\ * & * & \gamma\mathbf{1} & \mathbf{D}_\mathrm{d}^\mathsf{T} \\ * & * & * & \gamma\mathbf{1} \end{bmatrix} > 0.$$

5. [24] There exist $\mathbf{P} \in \mathbb{S}^n$ and $\gamma \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P}^{-1} & \mathbf{A}_\mathrm{d} & \mathbf{B}_\mathrm{d} & \mathbf{0} \\ * & \mathbf{P} & \mathbf{0} & \mathbf{C}_\mathrm{d}^\mathsf{T} \\ * & * & \gamma\mathbf{1} & \mathbf{D}_\mathrm{d}^\mathsf{T} \\ * & * & * & \gamma\mathbf{1} \end{bmatrix} > 0. \tag{3.9}$$

6. [54] There exist $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{X} \in \mathbb{R}^{n\times n}$, and $\gamma \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$ and $\mathbf{X}$ has full rank, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{A}_\mathrm{d}\mathbf{X} & \mathbf{B}_\mathrm{d} & \mathbf{0} \\ * & \mathbf{X}^\mathsf{T}\mathbf{P}^{-1}\mathbf{X} & \mathbf{0} & \mathbf{X}\mathbf{C}_\mathrm{d}^\mathsf{T} \\ * & * & \mathbf{1} & \mathbf{D}_\mathrm{d}^\mathsf{T} \\ * & * & * & \gamma^2\mathbf{1} \end{bmatrix} > 0.$$

7. There exist $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{X} \in \mathbb{R}^{n\times n}$, and $\gamma \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$ and $\mathbf{X}$ has full rank, such that

$$\begin{bmatrix} \mathbf{X}^\mathsf{T}\mathbf{P}^{-1}\mathbf{X} & \mathbf{X}\mathbf{A}_\mathrm{d} & \mathbf{X}\mathbf{B}_\mathrm{d} & \mathbf{0} \\ * & \mathbf{P} & \mathbf{0} & \mathbf{C}_\mathrm{d}^\mathsf{T} \\ * & * & \mathbf{1} & \mathbf{D}_\mathrm{d}^\mathsf{T} \\ * & * & * & \gamma^2\mathbf{1} \end{bmatrix} > 0. \tag{3.10}$$

   Proof. Apply the congruence transformation $\mathbf{W} = \mathrm{diag}\{\mathbf{X}^\mathsf{T}, \mathbf{1}, \mathbf{1}, \mathbf{1}\}$ to (3.9), where $\mathbf{W}$ has full rank since $\mathbf{X}$ has full rank. □

8. [54] There exist $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{X} \in \mathbb{R}^{n\times n}$, and $\gamma \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{A}_\mathrm{d}\mathbf{X} & \mathbf{B}_\mathrm{d} & \mathbf{0} \\ * & \mathbf{X} + \mathbf{X}^\mathsf{T} - \mathbf{P} & \mathbf{0} & \mathbf{X}\mathbf{C}_\mathrm{d}^\mathsf{T} \\ * & * & \mathbf{1} & \mathbf{D}_\mathrm{d}^\mathsf{T} \\ * & * & * & \gamma^2\mathbf{1} \end{bmatrix} > 0. \tag{3.11}$$

9. There exist $\mathbf{Q} \in \mathbb{S}^n$, $\mathbf{X} \in \mathbb{R}^{n\times n}$, and $\gamma \in \mathbb{R}_{>0}$, where $\mathbf{Q} > 0$, such that

$$\begin{bmatrix} \mathbf{X} + \mathbf{X}^\mathsf{T} - \mathbf{Q} & \mathbf{X}\mathbf{A}_\mathrm{d} & \mathbf{X}\mathbf{B}_\mathrm{d} & \mathbf{0} \\ * & \mathbf{Q} & \mathbf{0} & \mathbf{C}_\mathrm{d}^\mathsf{T} \\ * & * & \mathbf{1} & \mathbf{D}_\mathrm{d}^\mathsf{T} \\ * & * & * & \gamma^2\mathbf{1} \end{bmatrix} > 0. \tag{3.12}$$

   Proof. Same as the proof of (3.11) in [54], by which it is shown that (3.12) is equivalent to (3.10). □

The $\mathcal{H}_\infty$ norm of $\boldsymbol{\mathcal{G}}$ is the minimum value of $\gamma \in \mathbb{R}_{>0}$ that satisfies any of the above conditions. If $(\mathbf{A}_\mathrm{d}, \mathbf{B}_\mathrm{d}, \mathbf{C}_\mathrm{d}, \mathbf{D}_\mathrm{d})$ is a minimal realization, then the matrix inequalities can be nonstrict [53], [55].

3.3 $\mathcal{H}_2$ Norm

3.3.1 Continuous-Time $\mathcal{H}_2$ Norm [4, pp. 71–72]

Consider a continuous-time LTI system, $\boldsymbol{\mathcal{G}} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, with state-space realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{0})$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, and $\mathbf{A}$ is Hurwitz. The $\mathcal{H}_2$ norm of $\boldsymbol{\mathcal{G}}$ is

$$\|\boldsymbol{\mathcal{G}}\|_2 = \sqrt{\mathrm{tr}(\mathbf{C}\mathbf{W}\mathbf{C}^\mathsf{T})} = \sqrt{\mathrm{tr}(\mathbf{B}^\mathsf{T}\mathbf{M}\mathbf{B})},$$

where $\mathbf{W}, \mathbf{M} \in \mathbb{S}^n$, $\mathbf{W} > 0$, $\mathbf{M} > 0$, and

$$\mathbf{A}\mathbf{W} + \mathbf{W}\mathbf{A}^\mathsf{T} + \mathbf{B}\mathbf{B}^\mathsf{T} = \mathbf{0}, \quad \mathbf{M}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{M} + \mathbf{C}^\mathsf{T}\mathbf{C} = \mathbf{0}.$$

The inequality $\|\boldsymbol{\mathcal{G}}\|_2 < \mu$ holds under any of the following necessary and sufficient conditions.

1. [4, pp. 71–72] There exist $\mathbf{X} \in \mathbb{S}^n$ and $\mu \in \mathbb{R}_{>0}$, where $\mathbf{X} > 0$, such that

$$\begin{aligned} \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A}^\mathsf{T} + \mathbf{B}\mathbf{B}^\mathsf{T} &< 0, \\ \mathrm{tr}\left(\mathbf{C}\mathbf{X}\mathbf{C}^\mathsf{T}\right) &< \mu^2. \end{aligned}$$

2. [4, pp. 71–72] There exist $\mathbf{Y} \in \mathbb{S}^n$ and $\mu \in \mathbb{R}_{>0}$, where $\mathbf{Y} > 0$, such that

$$\begin{aligned} \mathbf{A}^\mathsf{T}\mathbf{Y} + \mathbf{Y}\mathbf{A} + \mathbf{C}^\mathsf{T}\mathbf{C} &< 0, \\ \mathrm{tr}\left(\mathbf{B}^\mathsf{T}\mathbf{Y}\mathbf{B}\right) &< \mu^2. \end{aligned}$$

3. [4, pp. 71–72] There exist $\mathbf{X} \in \mathbb{S}^n$, $\mathbf{Z} \in \mathbb{S}^m$, and $\mu \in \mathbb{R}_{>0}$, where $\mathbf{X} > 0$ and $\mathbf{Z} > 0$, such that

$$\begin{aligned} \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A}^\mathsf{T} + \mathbf{X}\mathbf{C}^\mathsf{T}\mathbf{C}\mathbf{X} &< 0, \\ \begin{bmatrix} \mathbf{Z} & \mathbf{B}^\mathsf{T} \\ * & \mathbf{X} \end{bmatrix} &> 0, \\ \mathrm{tr}\mathbf{Z} &< \mu^2. \end{aligned}$$

4. [4, pp. 71–72] There exist $\mathbf{Y} \in \mathbb{S}^n$, $\mathbf{Z} \in \mathbb{S}^m$, and $\mu \in \mathbb{R}_{>0}$, where $\mathbf{Y} > 0$ and $\mathbf{Z} > 0$, such that

$$\begin{aligned} \mathbf{A}^\mathsf{T}\mathbf{Y} + \mathbf{Y}\mathbf{A} + \mathbf{Y}\mathbf{C}^\mathsf{T}\mathbf{C}\mathbf{Y} &< 0, \\ \begin{bmatrix} \mathbf{Z} & \mathbf{C} \\ * & \mathbf{Y} \end{bmatrix} &> 0, \\ \mathrm{tr}\mathbf{Z} &< \mu^2. \end{aligned}$$

5. [37], [4, pp. 71–72] There exist $\mathbf{Y} \in \mathbb{S}^n$, $\mathbf{Z} \in \mathbb{S}^p$, and $\mu \in \mathbb{R}_{>0}$, where $\mathbf{Y} > 0$ and $\mathbf{Z} > 0$, such that

$$\begin{aligned} \begin{bmatrix} \mathbf{A}^\mathsf{T}\mathbf{Y} + \mathbf{Y}\mathbf{A} & \mathbf{Y}\mathbf{B} \\ * & -\mu\mathbf{1} \end{bmatrix} &< 0, \\ \begin{bmatrix} \mathbf{Y} & \mathbf{C}^\mathsf{T} \\ * & \mathbf{Z} \end{bmatrix} &> 0, \\ \mathrm{tr}\mathbf{Z} &< \mu. \end{aligned}$$

6. [4, pp. 71–72] There exist $\mathbf{X} \in \mathbb{S}^n$, $\mathbf{Z} \in \mathbb{S}^p$, and $\mu \in \mathbb{R}_{>0}$, where $\mathbf{X} > 0$ and $\mathbf{Z} > 0$, such that

$$\begin{bmatrix} \mathbf{X}\mathbf{A}^\mathsf{T} + \mathbf{A}\mathbf{X} & \mathbf{X}\mathbf{C}^\mathsf{T} \\ * & -\mu\mathbf{1} \end{bmatrix} < 0,$$

$$\begin{bmatrix} \mathbf{X} & \mathbf{B} \\ * & \mathbf{Z} \end{bmatrix} > 0,$$

$$\mathrm{tr}\mathbf{Z} < \mu.$$

7. [37] There exist $\mathbf{X} \in \mathbb{S}^n$, $\mathbf{Z} \in \mathbb{S}^p$, $\mathbf{V} \in \mathbb{R}^{n\times n}$, and $\mu \in \mathbb{R}_{>0}$, where $\mathbf{X} > 0$ and $\mathbf{Z} > 0$, such that

$$\begin{bmatrix} -\left(\mathbf{V} + \mathbf{V}^\mathsf{T}\right) & \mathbf{V}^\mathsf{T}\mathbf{A} + \mathbf{X} & \mathbf{V}^\mathsf{T}\mathbf{B} & \mathbf{V}^\mathsf{T} \\ * & -\mathbf{X} & \mathbf{0} & \mathbf{0} \\ * & * & -\mu^2\mathbf{1} & \mathbf{0} \\ * & * & * & -\mathbf{X} \end{bmatrix} < 0, \tag{3.13}$$

$$\begin{bmatrix} \mathbf{X} & \mathbf{C}^\mathsf{T} \\ * & \mathbf{Z} \end{bmatrix} > 0,$$

$$\mathrm{tr}\mathbf{Z} < 1.$$

8. [37] There exist $\mathbf{X} \in \mathbb{S}^n$, $\mathbf{Z} \in \mathbb{S}^m$, $\mathbf{V} \in \mathbb{R}^{n\times n}$, and $\mu \in \mathbb{R}_{>0}$, where $\mathbf{X} > 0$ and $\mathbf{Z} > 0$, such that

$$\begin{bmatrix} -\left(\mathbf{V} + \mathbf{V}^\mathsf{T}\right) & \mathbf{V}^\mathsf{T}\mathbf{A}^\mathsf{T} + \mathbf{X} & \mathbf{V}^\mathsf{T}\mathbf{C}^\mathsf{T} & \mathbf{V}^\mathsf{T} \\ * & -\mathbf{X} & \mathbf{0} & \mathbf{0} \\ * & * & -\mu^2\mathbf{1} & \mathbf{0} \\ * & * & * & -\mathbf{X} \end{bmatrix} < 0, \tag{3.14}$$

$$\begin{bmatrix} \mathbf{X} & \mathbf{B} \\ * & \mathbf{Z} \end{bmatrix} > 0,$$

$$\mathrm{tr}\mathbf{Z} < 1.$$

9. [44] There exist $\mathbf{X} \in \mathbb{S}^n$, $\mathbf{Z} \in \mathbb{S}^m$, $\mathbf{\Gamma} \in \mathbb{R}^{n\times n}$, and $\mu \in \mathbb{R}_{>0}$, where $\mathbf{X} > 0$ and $\mathbf{Z} > 0$, such that

$$\begin{bmatrix} \mathbf{0} & -\mathbf{X} & \mathbf{0} \\ * & \mathbf{0} & \mathbf{0} \\ * & * & -\mathbf{1} \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathbf{A} \\ \mathbf{1} \\ \mathbf{C} \end{bmatrix} \mathbf{\Gamma} \begin{bmatrix} \mathbf{1} & -\epsilon\mathbf{1} & \mathbf{0} \end{bmatrix} \right\} < 0,$$

$$\begin{bmatrix} \mathbf{Z} & \mathbf{B}^\mathsf{T} \\ * & \mathbf{X} \end{bmatrix} > 0,$$

$$\mathrm{tr}\mathbf{Z} < \mu^2.$$

The $\mathcal{H}_2$ norm of $\boldsymbol{\mathcal{G}}$ is the minimum value of $\mu \in \mathbb{R}_{>0}$ that satisfies any of the above conditions.

3.3.2 Discrete-Time $\mathcal{H}_2$ Norm [54]

Consider a discrete-time LTI system, $\boldsymbol{\mathcal{G}} : \ell_{2e} \to \ell_{2e}$, with state-space realization $(\mathbf{A}_\mathrm{d}, \mathbf{B}_\mathrm{d}, \mathbf{C}_\mathrm{d}, \mathbf{0})$, where $\mathbf{A}_\mathrm{d} \in \mathbb{R}^{n\times n}$, $\mathbf{B}_\mathrm{d} \in \mathbb{R}^{n\times m}$, $\mathbf{C}_\mathrm{d} \in \mathbb{R}^{p\times n}$, and $\mathbf{A}_\mathrm{d}$ is Schur. The $\mathcal{H}_2$ norm of $\boldsymbol{\mathcal{G}}$ is

$$\|\boldsymbol{\mathcal{G}}\|_2 = \sqrt{\mathrm{tr}(\mathbf{C}_\mathrm{d}\mathbf{W}\mathbf{C}_\mathrm{d}^\mathsf{T})} = \sqrt{\mathrm{tr}(\mathbf{B}_\mathrm{d}^\mathsf{T}\mathbf{M}\mathbf{B}_\mathrm{d})},$$

where $\mathbf{W}, \mathbf{M} \in \mathbb{S}^n$, $\mathbf{W} > 0$, $\mathbf{M} > 0$, and

$$\mathbf{A}_\mathrm{d}\mathbf{W}\mathbf{A}^\mathsf{T} - \mathbf{W} + \mathbf{B}_\mathrm{d}\mathbf{B}_\mathrm{d}^\mathsf{T} = \mathbf{0}, \quad \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{M}\mathbf{A}_\mathrm{d} - \mathbf{M} + \mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{C}_\mathrm{d} = \mathbf{0}.$$

The inequality $\|\mathcal{G}\|_2 < \mu$ holds under any of following necessary and sufficient conditions.

1. There exist $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{Z} \in \mathbb{S}^p$, and $\mu \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$ and $\mathbf{Z} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{A}_\mathrm{d}\mathbf{P} & \mathbf{B}_\mathrm{d} \\ * & \mathbf{P} & \mathbf{0} \\ * & * & \mathbf{1} \end{bmatrix} > 0,$$
$$\begin{bmatrix} \mathbf{Z} & \mathbf{C}_\mathrm{d}\mathbf{P} \\ * & \mathbf{P} \end{bmatrix} > 0,$$
$$\mathrm{tr}\mathbf{Z} < \mu^2.$$

2. There exist $\mathbf{Q} \in \mathbb{S}^n$, $\mathbf{Z} \in \mathbb{S}^p$, and $\mu \in \mathbb{R}_{>0}$, where $\mathbf{Q} > 0$ and $\mathbf{Z} > 0$, such that

$$\begin{bmatrix} \mathbf{Q} & \mathbf{Q}\mathbf{A}_\mathrm{d} & \mathbf{Q}\mathbf{B}_\mathrm{d} \\ * & \mathbf{Q} & \mathbf{0} \\ * & * & \mathbf{1} \end{bmatrix} > 0, \tag{3.15}$$
$$\begin{bmatrix} \mathbf{Z} & \mathbf{C}_\mathrm{d} \\ * & \mathbf{Q} \end{bmatrix} > 0,$$
$$\mathrm{tr}\mathbf{Z} < \mu^2.$$

3. [54] There exist $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{Z} \in \mathbb{S}^p$, $\mathbf{X} \in \mathbb{R}^{n\times n}$, and $\mu \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$, $\mathbf{Z} > 0$, and $\mathbf{X}$ has full rank, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{A}_\mathrm{d}\mathbf{X} & \mathbf{B}_\mathrm{d} \\ * & \mathbf{X}^\mathsf{T}\mathbf{P}^{-1}\mathbf{X} & \mathbf{0} \\ * & * & \mathbf{1} \end{bmatrix} > 0,$$
$$\begin{bmatrix} \mathbf{Z} & \mathbf{C}_\mathrm{d}\mathbf{X} \\ * & \mathbf{X}^\mathsf{T}\mathbf{P}^{-1}\mathbf{X} \end{bmatrix} > 0,$$
$$\mathrm{tr}\mathbf{Z} < \mu^2.$$

4. There exist $\mathbf{Q} \in \mathbb{S}^n$, $\mathbf{Z} \in \mathbb{S}^p$, $\mathbf{X} \in \mathbb{R}^{n\times n}$, and $\mu \in \mathbb{R}_{>0}$, where $\mathbf{Q} > 0$, $\mathbf{Z} > 0$, and $\mathbf{X}$ has full rank, such that

$$\begin{bmatrix} \mathbf{X}^\mathsf{T}\mathbf{Q}^{-1}\mathbf{X} & \mathbf{X}^\mathsf{T}\mathbf{A}_\mathrm{d} & \mathbf{X}^\mathsf{T}\mathbf{B}_\mathrm{d} \\ * & \mathbf{Q} & \mathbf{0} \\ * & * & \mathbf{1} \end{bmatrix} > 0, \tag{3.16}$$
$$\begin{bmatrix} \mathbf{Z} & \mathbf{C}_\mathrm{d} \\ * & \mathbf{Q} \end{bmatrix} > 0,$$
$$\mathrm{tr}\mathbf{Z} < \mu^2.$$

Proof. Apply the congruence transformation $\mathbf{W} = \mathrm{diag}\{\mathbf{X}^\mathsf{T}\mathbf{Q}^{-1}, \mathbf{1}, \mathbf{1}\}$ to (3.15), where $\mathbf{W}$ has full rank since $\mathbf{X}$ has full rank. $\square$

5. [54] There exist $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{Z} \in \mathbb{S}^p$, $\mathbf{X} \in \mathbb{R}^{n\times n}$, and $\mu \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$ and $\mathbf{Z} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{A}_\mathrm{d}\mathbf{X} & \mathbf{B}_\mathrm{d} \\ * & \mathbf{X} + \mathbf{X}^\mathsf{T} - \mathbf{P} & \mathbf{0} \\ * & * & \mathbf{1} \end{bmatrix} > 0, \tag{3.17}$$

$$\begin{bmatrix} \mathbf{Z} & \mathbf{C}_\mathrm{d}\mathbf{X} \\ * & \mathbf{X} + \mathbf{X}^\mathsf{T} - \mathbf{P} \end{bmatrix} > 0, \tag{3.18}$$

$$\mathrm{tr}\mathbf{Z} < \mu^2. \tag{3.19}$$

6. There exist $\mathbf{Q} \in \mathbb{S}^n$, $\mathbf{Z} \in \mathbb{S}^p$, $\mathbf{X} \in \mathbb{R}^{n\times n}$, and $\mu \in \mathbb{R}_{>0}$, where $\mathbf{Q} > 0$ and $\mathbf{Z} > 0$, such that

$$\begin{bmatrix} \mathbf{X} + \mathbf{X}^\mathsf{T} - \mathbf{Q} & \mathbf{X}^\mathsf{T}\mathbf{A}_\mathrm{d} & \mathbf{X}^\mathsf{T}\mathbf{B}_\mathrm{d} \\ * & \mathbf{Q} & \mathbf{0} \\ * & * & \mathbf{1} \end{bmatrix} > 0, \tag{3.20}$$

$$\begin{bmatrix} \mathbf{Z} & \mathbf{C}_\mathrm{d} \\ * & \mathbf{Q} \end{bmatrix} > 0,$$

$$\mathrm{tr}\mathbf{Z} < \mu^2.$$

Proof. Same as the proof of (3.17), (3.18), (3.19) in [54], by which it is shown that (3.20) is equivalent to (3.16). □

The $\mathcal{H}_2$ norm of $\mathcal{G}$ is the minimum value of $\mu \in \mathbb{R}_{>0}$ that satisfies any of the above conditions.

3.4 Generalized $\mathcal{H}_2$ Norm [4, p. 73]

Consider a continuous-time LTI system, $\mathcal{G} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, with state-space realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{0})$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, and $\mathbf{A}$ is Hurwitz. The generalized $\mathcal{H}_2$ norm of $\mathcal{G}$ is

$$\|\mathcal{G}\|_{2,\infty} = \sup_{\mathbf{u}\in\mathcal{L}_2, \mathbf{u}\neq\mathbf{0}} \frac{\|\mathcal{G}\mathbf{u}\|_\infty}{\|\mathbf{u}\|_2}.$$

The inequality $\|\mathcal{G}\|_{2,\infty} < \mu$ holds under any of following necessary and sufficient conditions.

1. There exist $\mathbf{P} \in \mathbb{S}^n$ and $\mu \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{P}\mathbf{A} & \mathbf{P}\mathbf{B} \\ * & -\mu\mathbf{1} \end{bmatrix} < 0,$$

$$\begin{bmatrix} \mathbf{P} & \mathbf{C}^\mathsf{T} \\ * & \mu\mathbf{1} \end{bmatrix} > 0.$$

2. There exist $\mathbf{Q} \in \mathbb{S}^n$ and $\mu \in \mathbb{R}_{>0}$, where $\mathbf{Q} > 0$, such that

$$\begin{bmatrix} \mathbf{Q}\mathbf{A}^\mathsf{T} + \mathbf{A}\mathbf{Q} & \mathbf{B} \\ * & -\mu\mathbf{1} \end{bmatrix} < 0,$$

$$\begin{bmatrix} \mathbf{Q} & \mathbf{Q}\mathbf{C}^\mathsf{T} \\ * & \mu\mathbf{1} \end{bmatrix} > 0.$$

3. There exist $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{V} \in \mathbb{R}^{n\times n}$, and $\mu \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} -\left(\mathbf{V}+\mathbf{V}^\mathsf{T}\right) & \mathbf{V}^\mathsf{T}\mathbf{A}+\mathbf{P} & \mathbf{V}^\mathsf{T}\mathbf{B} & \mathbf{V}^\mathsf{T} \\ * & -\mathbf{P} & \mathbf{0} & \mathbf{0} \\ * & * & -\mu\mathbf{1} & \mathbf{0} \\ * & * & * & -\mathbf{P} \end{bmatrix} < 0,$$

$$\begin{bmatrix} \mathbf{P} & \mathbf{C}^\mathsf{T} \\ * & \mu\mathbf{1} \end{bmatrix} > 0.$$

Proof. Identical to the proof in [37] used to obtain the dilated matrix inequality in (3.13). □

The generalized $\mathcal{H}_2$ norm of $\mathcal{G}$ is the minimum value of $\mu \in \mathbb{R}_{>0}$ that satisfies any of the above conditions.

3.5 Peak-to-Peak Norm [4, pp. 74–75]

Consider a continuous-time LTI system, $\mathcal{G} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, with state-space realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, $\mathbf{D} \in \mathbb{R}^{p\times m}$, and $\mathbf{A}$ is Hurwitz. The peak-to-peak norm of $\mathcal{G}$ is

$$\|\mathcal{G}\|_{\infty,\infty} = \sup_{\mathbf{u}\in\mathcal{L}_\infty,\mathbf{u}\neq\mathbf{0}} \frac{\|\mathcal{G}\mathbf{u}\|_\infty}{\|\mathbf{u}\|_\infty}.$$

The inequality $\|\mathcal{G}\|_{\infty,\infty} < \mu$ holds under any of the following equivalent sufficient conditions.

1. There exist $\mathbf{P} \in \mathbb{S}^n$ and γ, ϵ, $\mu \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{A}^\mathsf{T}\mathbf{P}+\mathbf{P}\mathbf{A}+\lambda\mathbf{P} & \mathbf{P}\mathbf{B} \\ * & -\epsilon\mathbf{1} \end{bmatrix} < 0,$$

$$\begin{bmatrix} \lambda\mathbf{P} & \mathbf{0} & \mathbf{C}^\mathsf{T} \\ * & (\mu-\epsilon)\mathbf{1} & \mathbf{D}^\mathsf{T} \\ * & * & \mu\mathbf{1} \end{bmatrix} > 0,$$

2. There exist $\mathbf{Q} \in \mathbb{S}^n$ and γ, ϵ, $\mu \in \mathbb{R}_{>0}$, where $\mathbf{Q} > 0$, such that

$$\begin{bmatrix} \mathbf{Q}\mathbf{A}^\mathsf{T}+\mathbf{A}\mathbf{Q}+\lambda\mathbf{Q} & \mathbf{B} \\ * & -\epsilon\mathbf{1} \end{bmatrix} < 0,$$

$$\begin{bmatrix} \lambda\mathbf{Q} & \mathbf{0} & \mathbf{Q}\mathbf{C}^\mathsf{T} \\ * & (\mu-\epsilon)\mathbf{1} & \mathbf{D}^\mathsf{T} \\ * & * & \mu\mathbf{1} \end{bmatrix} > 0.$$

The peak-to-peak norm of $\mathcal{G}$ is smaller than any $\mu \in \mathbb{R}_{>0}$ that satisfies either of the above conditions.

3.6 Kalman-Yakubovich-Popov (KYP) Lemma

3.6.1 KYP Lemma for QSR Dissipative Systems [53, 56, 57]

Consider a continuous-time LTI system, $\mathcal{G} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, with minimal state-space realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, and $\mathbf{D} \in \mathbb{R}^{p\times m}$. The system $\mathcal{G}$ is QSR dissipative [58, 59] if

$$\int_0^T \big(\mathbf{y}^\mathsf{T}(t)\mathbf{Q}\mathbf{y}(t) + 2\mathbf{y}^\mathsf{T}(t)\mathbf{S}\mathbf{u}(t) + \mathbf{u}^\mathsf{T}(t)\mathbf{R}\mathbf{u}(t)\big)\,\mathrm{d}t \geq 0, \quad \forall \mathbf{u} \in \mathcal{L}_{2e}, \quad \forall T \in \mathbb{R}_{\geq 0},$$

where $\mathbf{u}(t)$ is the input to $\mathcal{G}$, $\mathbf{y}(t)$ is the output of $\mathcal{G}$, $\mathbf{Q} \in \mathbb{S}^p$, $\mathbf{S} \in \mathbb{R}^{p\times m}$, and $\mathbf{R} \in \mathbb{S}^m$. The system $\mathcal{G}$ is also QSR dissipative if and only if there exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{P} - \mathbf{C}^\mathsf{T}\mathbf{Q}\mathbf{C} & \mathbf{P}\mathbf{B} - \mathbf{C}^\mathsf{T}\mathbf{S} - \mathbf{C}^\mathsf{T}\mathbf{Q}\mathbf{D} \\ * & -\mathbf{D}^\mathsf{T}\mathbf{Q}\mathbf{D} - \big(\mathbf{D}^\mathsf{T}\mathbf{S} + \mathbf{S}^\mathsf{T}\mathbf{D}\big) - \mathbf{R} \end{bmatrix} \leq 0.$$

Note that the Bounded Real Lemma (Section 3.2.1) is a special case of the KYP Lemma for QSR dissipative systems with $\mathbf{Q} = -\mathbf{1}$, $\mathbf{S} = \mathbf{0}$, and $\mathbf{R} = \gamma^2\mathbf{1}$.

3.6.2 Discrete-Time KYP Lemma for QSR Dissipative Systems [57], [60, p. 495]

Consider a discrete-time LTI system, $\mathcal{G} : \ell_{2e} \to \ell_{2e}$, with minimal state-space realization $(\mathbf{A}_\mathrm{d}, \mathbf{B}_\mathrm{d}, \mathbf{C}_\mathrm{d}, \mathbf{D}_\mathrm{d})$, where $\mathbf{A}_\mathrm{d} \in \mathbb{R}^{n\times n}$, $\mathbf{B}_\mathrm{d} \in \mathbb{R}^{n\times m}$, $\mathbf{C}_\mathrm{d} \in \mathbb{R}^{p\times n}$, and $\mathbf{D}_\mathrm{d} \in \mathbb{R}^{p\times m}$. The system $\mathcal{G}$ is QSR dissipative [58, 59] if

$$\sum_{i=0}^{k} \big(\mathbf{y}_i^\mathsf{T}\mathbf{Q}\mathbf{y}_i + 2\mathbf{y}_i^\mathsf{T}\mathbf{S}\mathbf{u}_i + \mathbf{u}_i^\mathsf{T}\mathbf{R}\mathbf{u}_i\big) \geq 0, \quad \forall \mathbf{u} \in \ell_{2e}, \quad \forall k \in \mathbb{Z}_{\geq 0},$$

where $\mathbf{u}_k$ is the input to $\mathcal{G}$, $\mathbf{y}_k$ is the output of $\mathcal{G}$, $\mathbf{Q} \in \mathbb{S}^p$, $\mathbf{S} \in \mathbb{R}^{p\times m}$, and $\mathbf{R} \in \mathbb{S}^m$. The system $\mathcal{G}$ is also QSR dissipative if and only if there exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{A}_\mathrm{d} - \mathbf{P} - \mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{Q}\mathbf{C}_\mathrm{d} & \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d} - \mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{S} - \mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{Q}\mathbf{D}_\mathrm{d} \\ * & \mathbf{B}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d} - \mathbf{D}_\mathrm{d}^\mathsf{T}\mathbf{Q}\mathbf{D}_\mathrm{d} - \big(\mathbf{D}_\mathrm{d}^\mathsf{T}\mathbf{S} + \mathbf{S}^\mathsf{T}\mathbf{D}_\mathrm{d}\big) - \mathbf{R} \end{bmatrix} \leq 0.$$

Note that the Discrete-Time Bounded Real Lemma (Section 3.2.2) is a special case of the Discrete-Time KYP Lemma for QSR dissipative systems with $\mathbf{Q} = -\mathbf{1}$, $\mathbf{S} = \mathbf{0}$, and $\mathbf{R} = \gamma^2\mathbf{1}$.

3.6.3 KYP Lemma Without Feedthrough [61, p. 219], [62], [63, p. 14]

Consider a square, continuous-time LTI system, $\mathcal{G} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, with minimal state-space realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{0})$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, and $\mathbf{C} \in \mathbb{R}^{m\times n}$. The system $\mathcal{G}$ is positive real (PR) under either of the following equivalent necessary and sufficient conditions.

1. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\mathbf{P}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{P} \leq 0,$$
$$\mathbf{P}\mathbf{B} = \mathbf{C}^\mathsf{T}.$$

2. There exists $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{Q} > 0$, such that

$$\begin{aligned}\mathbf{AQ} + \mathbf{QA}^\mathsf{T} &\leq 0,\\ \mathbf{B} &= \mathbf{QC}^\mathsf{T}.\end{aligned}$$

This is a special case of the KYP Lemma for QSR dissipative systems with $\mathbf{Q} = \mathbf{0}$, $\mathbf{S} = \frac{1}{2} \cdot \mathbf{1}$, and $\mathbf{R} = \mathbf{0}$.

The system $\mathcal{G}$ is strictly positive real (SPR) under either of the following necessary and sufficient conditions.

1. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\begin{aligned}\mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} &< 0,\\ \mathbf{PB} &= \mathbf{C}^\mathsf{T}.\end{aligned}$$

2. There exists $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{Q} > 0$, such that

$$\begin{aligned}\mathbf{AQ} + \mathbf{QA}^\mathsf{T} &< 0,\\ \mathbf{B} &= \mathbf{QC}^\mathsf{T}.\end{aligned}$$

This is a special case of the KYP Lemma for QSR dissipative systems with $\mathbf{Q} = \epsilon \cdot \mathbf{1}$, $\mathbf{S} = \frac{1}{2} \cdot \mathbf{1}$, and $\mathbf{R} = \mathbf{0}$, where $\epsilon \in \mathbb{R}_{>0}$.

3.6.4 KYP Lemma With Feedthrough [3, p. 25], [61, p. 218], [62], [64, pp. 79–80]

Consider a square, continuous-time LTI system, $\mathcal{G} : \mathcal{L}_{2e} \rightarrow \mathcal{L}_{2e}$, with minimal state-space realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{C} \in \mathbb{R}^{m \times n}$, and $\mathbf{D} \in \mathbb{R}^{m \times m}$. The system $\mathcal{G}$ is positive real (PR) under either of the following equivalent necessary and sufficient conditions.

1. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} & \mathbf{PB} - \mathbf{C}^\mathsf{T} \\ * & -\left(\mathbf{D} + \mathbf{D}^\mathsf{T}\right) \end{bmatrix} \leq 0.$$

2. There exists $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{Q} > 0$, such that

$$\begin{bmatrix} \mathbf{AQ} + \mathbf{QA}^\mathsf{T} & \mathbf{B} - \mathbf{QC}^\mathsf{T} \\ * & -\left(\mathbf{D} + \mathbf{D}^\mathsf{T}\right) \end{bmatrix} \leq 0.$$

This is a special case of the KYP Lemma for QSR dissipative systems with $\mathbf{Q} = \mathbf{0}$, $\mathbf{S} = \frac{1}{2} \cdot \mathbf{1}$, and $\mathbf{R} = \mathbf{0}$.

The system $\mathcal{G}$ is strictly positive real (SPR) under either of the following necessary and sufficient conditions.

1. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} & \mathbf{PB} - \mathbf{C}^\mathsf{T} \\ * & -\left(\mathbf{D} + \mathbf{D}^\mathsf{T}\right) \end{bmatrix} < 0.$$

2. There exists $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{Q} > 0$, such that

$$\begin{bmatrix} \mathbf{AQ} + \mathbf{QA}^\mathsf{T} & \mathbf{B} - \mathbf{QC}^\mathsf{T} \\ * & -\left(\mathbf{D} + \mathbf{D}^\mathsf{T}\right) \end{bmatrix} < 0.$$

This is a special case of the KYP Lemma for QSR dissipative systems with $\mathbf{Q} = \epsilon\mathbf{1}$, $\mathbf{S} = \frac{1}{2} \cdot \mathbf{1}$, and $\mathbf{R} = \mathbf{0}$, where $\epsilon \in \mathbb{R}_{>0}$.

3.6.5 Discrete-Time KYP Lemma With Feedthrough [64, pp. 171–172], [65], [66]

Consider a square, discrete-time LTI system, $\boldsymbol{\mathcal{G}} : \ell_{2e} \to \ell_{2e}$, with minimal state-space realization $(\mathbf{A}_\mathrm{d}, \mathbf{B}_\mathrm{d}, \mathbf{C}_\mathrm{d}, \mathbf{D}_\mathrm{d})$, where $\mathbf{A}_\mathrm{d} \in \mathbb{R}^{n\times n}$, $\mathbf{B}_\mathrm{d} \in \mathbb{R}^{n\times m}$, $\mathbf{C}_\mathrm{d} \in \mathbb{R}^{m\times n}$, and $\mathbf{D}_\mathrm{d} \in \mathbb{R}^{m\times m}$. The system $\boldsymbol{\mathcal{G}}$ is positive real (PR) under any of the following equivalent necessary and sufficient conditions.

1. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that
$$\begin{bmatrix} \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{A}_\mathrm{d} - \mathbf{P} & \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d} - \mathbf{C}_\mathrm{d}^\mathsf{T} \\ * & \mathbf{B}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d} - \left(\mathbf{D}_\mathrm{d} + \mathbf{D}_\mathrm{d}^\mathsf{T}\right) \end{bmatrix} \leq 0.$$
2. There exists $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{Q} > 0$, such that
$$\begin{bmatrix} \mathbf{A}_\mathrm{d}\mathbf{Q}\mathbf{A}_\mathrm{d}^\mathsf{T} - \mathbf{Q} & \mathbf{A}_\mathrm{d}\mathbf{Q}\mathbf{C}_\mathrm{d}^\mathsf{T} - \mathbf{B}_\mathrm{d} \\ * & \mathbf{C}_\mathrm{d}\mathbf{Q}\mathbf{C}_\mathrm{d}^\mathsf{T} - \left(\mathbf{D}_\mathrm{d} + \mathbf{D}_\mathrm{d}^\mathsf{T}\right) \end{bmatrix} \leq 0.$$
3. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that
$$\begin{bmatrix} \mathbf{P} & \mathbf{P}\mathbf{A}_\mathrm{d} & \mathbf{P}\mathbf{B}_\mathrm{d} \\ * & \mathbf{P} & \mathbf{C}_\mathrm{d}^\mathsf{T} \\ * & * & \mathbf{D}_\mathrm{d} + \mathbf{D}_\mathrm{d}^\mathsf{T} \end{bmatrix} \geq 0.$$
4. There exists $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{Q} > 0$, such that
$$\begin{bmatrix} \mathbf{Q} & \mathbf{A}_\mathrm{d}\mathbf{Q} & \mathbf{B}_\mathrm{d} \\ * & \mathbf{Q} & \mathbf{Q}\mathbf{C}_\mathrm{d}^\mathsf{T} \\ * & * & \mathbf{D}_\mathrm{d} + \mathbf{D}_\mathrm{d}^\mathsf{T} \end{bmatrix} \geq 0.$$

This is a special case of the Discrete-Time KYP Lemma for QSR dissipative systems with $\mathbf{Q} = \mathbf{0}$, $\mathbf{S} = \frac{1}{2} \cdot \mathbf{1}$, and $\mathbf{R} = \mathbf{0}$.

The system $\boldsymbol{\mathcal{G}}$ is strictly positive real (SPR) under any of the following necessary and sufficient conditions.

1. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that
$$\begin{bmatrix} \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{A}_\mathrm{d} - \mathbf{P} & \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d} - \mathbf{C}_\mathrm{d}^\mathsf{T} \\ * & \mathbf{B}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d} - \left(\mathbf{D}_\mathrm{d} + \mathbf{D}_\mathrm{d}^\mathsf{T}\right) \end{bmatrix} < 0.$$
2. There exists $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{Q} > 0$, such that
$$\begin{bmatrix} \mathbf{A}_\mathrm{d}\mathbf{Q}\mathbf{A}_\mathrm{d}^\mathsf{T} - \mathbf{Q} & \mathbf{A}_\mathrm{d}\mathbf{Q}\mathbf{C}_\mathrm{d}^\mathsf{T} - \mathbf{B}_\mathrm{d} \\ * & \mathbf{C}_\mathrm{d}\mathbf{Q}\mathbf{C}_\mathrm{d}^\mathsf{T} - \left(\mathbf{D}_\mathrm{d} + \mathbf{D}_\mathrm{d}^\mathsf{T}\right) \end{bmatrix} < 0.$$
3. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that
$$\begin{bmatrix} \mathbf{P} & \mathbf{P}\mathbf{A}_\mathrm{d} & \mathbf{P}\mathbf{B}_\mathrm{d} \\ * & \mathbf{P} & \mathbf{C}_\mathrm{d}^\mathsf{T} \\ * & * & \mathbf{D}_\mathrm{d} + \mathbf{D}_\mathrm{d}^\mathsf{T} \end{bmatrix} > 0.$$
4. There exists $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{Q} > 0$, such that
$$\begin{bmatrix} \mathbf{Q} & \mathbf{A}_\mathrm{d}\mathbf{Q} & \mathbf{B}_\mathrm{d} \\ * & \mathbf{Q} & \mathbf{Q}\mathbf{C}_\mathrm{d}^\mathsf{T} \\ * & * & \mathbf{D}_\mathrm{d} + \mathbf{D}_\mathrm{d}^\mathsf{T} \end{bmatrix} > 0.$$

This is a special case of the Discrete-Time KYP Lemma for QSR dissipative systems with $\mathbf{Q} = \epsilon\mathbf{1}$, $\mathbf{S} = \frac{1}{2} \cdot \mathbf{1}$, and $\mathbf{R} = \mathbf{0}$, where $\epsilon \in \mathbb{R}_{>0}$.

3.6.6 KYP Lemma for Descriptor Systems [67]

Consider a square, linear time-invariant (LTI) descriptor system given by

$$\begin{aligned}\mathbf{E}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},\\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u},\end{aligned}$$

where $\mathbf{E}$, $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{m\times n}$, and $\mathbf{D} \in \mathbb{R}^{m\times m}$. The system is extended strictly positive real (ESPR) if and only if there exist $\mathbf{X} \in \mathbb{R}^{n\times n}$ and $\mathbf{W} \in \mathbb{R}^{n\times m}$ such that $\mathbf{E}^\mathsf{T}\mathbf{X} = \mathbf{X}^\mathsf{T}\mathbf{E} \geq 0$, $\mathbf{E}^\mathsf{T}\mathbf{W} = \mathbf{0}$, and

$$\begin{bmatrix}\mathbf{X}^\mathsf{T}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{X} & \mathbf{A}^\mathsf{T}\mathbf{W} + \mathbf{X}^\mathsf{T}\mathbf{B} - \mathbf{C}^\mathsf{T}\\ * & \mathbf{W}^\mathsf{T}\mathbf{B} + \mathbf{B}^\mathsf{T}\mathbf{W} - \left(\mathbf{D} + \mathbf{D}^\mathsf{T}\right)\end{bmatrix} < 0.$$

The system is also ESPR if there exists $\mathbf{X} \in \mathbb{R}^{n\times n}$ such that $\mathbf{E}^\mathsf{T}\mathbf{X} = \mathbf{X}^\mathsf{T}\mathbf{E} \geq 0$ and [68]

$$\begin{bmatrix}\mathbf{X}^\mathsf{T}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{X} & \mathbf{X}^\mathsf{T}\mathbf{B} - \mathbf{C}^\mathsf{T}\\ * & -\left(\mathbf{D} + \mathbf{D}^\mathsf{T}\right)\end{bmatrix} < 0.$$

3.7 Conic Sectors

3.7.1 Conic Sector Lemma

Consider a square, continuous-time LTI system, $\boldsymbol{\mathcal{G}} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, with minimal state-space realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{m\times n}$, and $\mathbf{D} \in \mathbb{R}^{m\times m}$.

The system $\boldsymbol{\mathcal{G}}$ is inside the cone $[a, b]$, where a, $b \in \mathbb{R}$, and $a < b$, under any of the following equivalent necessary and sufficient conditions.

1. [69] There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix}\mathbf{P}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{C}^\mathsf{T}\mathbf{C} & \mathbf{P}\mathbf{B} - \frac{a+b}{2}\mathbf{C}^\mathsf{T} + \mathbf{C}^\mathsf{T}\mathbf{D}\\ * & \mathbf{D}^\mathsf{T}\mathbf{D} - \frac{a+b}{2}\left(\mathbf{D} + \mathbf{D}^\mathsf{T}\right) + ab\mathbf{1}\end{bmatrix} \leq 0. \tag{3.21}$$

   Note that the matrix inequality of (3.22) does not allow for the case where the upper bound b is infinite.

2. [70, p. 28] There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix}\mathbf{P}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{P} + \frac{1}{b}\mathbf{C}^\mathsf{T}\mathbf{C} & \mathbf{P}\mathbf{B} - \frac{1}{2}\left(\frac{a}{b} + 1\right)\mathbf{C}^\mathsf{T} + \frac{1}{b}\mathbf{C}^\mathsf{T}\mathbf{D}\\ * & \frac{1}{b}\mathbf{D}^\mathsf{T}\mathbf{D} - \frac{1}{2}\left(\frac{a}{b} + 1\right)\left(\mathbf{D} + \mathbf{D}^\mathsf{T}\right) + a\mathbf{1}\end{bmatrix} \leq 0.$$

3. [71] There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix}\mathbf{P}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{P} & \mathbf{P}\mathbf{B} & \mathbf{C}^\mathsf{T}\\ * & -\frac{(a-b)^2}{4b}\mathbf{1} & \mathbf{D}^\mathsf{T} - \frac{a+b}{2}\mathbf{1}\\ * & * & -b\mathbf{1}\end{bmatrix} \leq 0.$$

4. There exists $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{Q} > 0$, such that

$$\begin{bmatrix}\mathbf{A}\mathbf{Q} + \mathbf{Q}\mathbf{A}^\mathsf{T} & \mathbf{B} & \mathbf{Q}\mathbf{C}^\mathsf{T}\\ * & -\frac{(a-b)^2}{4b}\mathbf{1} & \mathbf{D}^\mathsf{T} - \frac{a+b}{2}\mathbf{1}\\ * & * & -b\mathbf{1}\end{bmatrix} \leq 0.$$

The system $\mathcal{G}$ is inside the cone of radius r centered at c, where $r \in \mathbb{R}_{>0}$ and $b \in \mathbb{R}$, under any of the following equivalent necessary and sufficient conditions.

1. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{C}^\mathsf{T}\mathbf{C} & \mathbf{PB} - c\mathbf{C}^\mathsf{T} + \mathbf{C}^\mathsf{T}\mathbf{D} \\ * & \mathbf{D}^\mathsf{T}\mathbf{D} - c\left(\mathbf{D} + \mathbf{D}^\mathsf{T}\right) + \left(c^2 - r^2\right)\mathbf{1} \end{bmatrix} \leq 0. \tag{3.22}$$

   Note that the matrix inequality of (3.22) does not allow for the case where the upper bound b is infinite.

The Conic Sector Lemma is a special case of the KYP Lemma for QSR dissipative systems with $\mathbf{Q} = -\mathbf{1}$, $\mathbf{S} = \frac{a+b}{2}\mathbf{1} = c\mathbf{1}$, and $\mathbf{R} = -ab\mathbf{1} = \left(r^2 - c^2\right)\mathbf{1}$.

3.7.2 Exterior Conic Sector Lemma

Consider a square, continuous-time LTI system, $\mathcal{G} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, with state-space realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{C} \in \mathbb{R}^{m \times n}$, and $\mathbf{D} \in \mathbb{R}^{m \times m}$. The system $\mathcal{G}$ is in the exterior cone of radius r centered at c (i.e., $\mathcal{G} \in \text{excone}_r(c)$), where $r \in \mathbb{R}_{>0}$ and $c \in \mathbb{R}$, under either of the following equivalent necessary and sufficient conditions.

1. [72] There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} \geq 0$, such that

$$\begin{bmatrix} \mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} - \mathbf{C}^\mathsf{T}\mathbf{C} & \mathbf{PB} - \mathbf{C}^\mathsf{T}\left(\mathbf{D} - c\mathbf{1}\right) \\ * & r^2\mathbf{1} - \left(\mathbf{D} - c\mathbf{1}\right)^\mathsf{T}\left(\mathbf{D} - c\mathbf{1}\right) \end{bmatrix} \leq 0. \tag{3.23}$$

2. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} \geq 0$, such that

$$\begin{bmatrix} \mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} - \mathbf{C}^\mathsf{T}\mathbf{C} & \mathbf{PB} - \mathbf{C}^\mathsf{T}\left(\mathbf{D} - c\mathbf{1}\right) & \mathbf{0} \\ * & -\left(\mathbf{D} - c\mathbf{1}\right)^\mathsf{T}\left(\mathbf{D} - c\mathbf{1}\right) & r\mathbf{1} \\ * & * & -\mathbf{1} \end{bmatrix} \leq 0. \tag{3.24}$$

   Proof. Applying the Schur complement lemma to the $r^2\mathbf{1}$ term in (3.23) gives (3.24). □

3.7.3 Modified Exterior Conic Sector Lemma

Consider a square, continuous-time LTI system, $\mathcal{G} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, with state-space realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{C} \in \mathbb{R}^{m \times n}$, and $\mathbf{D} \in \mathbb{R}^{m \times m}$. The system $\mathcal{G}$ is in the exterior cone of radius r centered at c (i.e., $\mathcal{G} \in \text{excone}_r(c)$), where $r \in \mathbb{R}_{>0}$ and $c \in \mathbb{R}$, under either of the following equivalent sufficient conditions.

1. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} \geq 0$, such that

$$\begin{bmatrix} \mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} & \mathbf{PB} - \mathbf{C}^\mathsf{T}\left(\mathbf{D} - c\mathbf{1}\right) \\ * & r^2\mathbf{1} - \left(\mathbf{D} - c\mathbf{1}\right)^\mathsf{T}\left(\mathbf{D} - c\mathbf{1}\right) \end{bmatrix} \leq 0. \tag{3.25}$$

   Proof. The term $-\mathbf{C}^\mathsf{T}\mathbf{C}$ in (3.23) makes the matrix inequality "more" negative definite. Therefore,

$$\begin{bmatrix} \mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} - \mathbf{C}^\mathsf{T}\mathbf{C} & \mathbf{PB} - \mathbf{C}^\mathsf{T}\left(\mathbf{D} - c\mathbf{1}\right) \\ * & r^2\mathbf{1} - \left(\mathbf{D} - c\mathbf{1}\right)^\mathsf{T}\left(\mathbf{D} - c\mathbf{1}\right) \end{bmatrix} \leq \begin{bmatrix} \mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} & \mathbf{PB} - \mathbf{C}^\mathsf{T}\left(\mathbf{D} - c\mathbf{1}\right) \\ * & r^2\mathbf{1} - \left(\mathbf{D} - c\mathbf{1}\right)^\mathsf{T}\left(\mathbf{D} - c\mathbf{1}\right) \end{bmatrix},$$

   and (3.25) implies (3.23). □

2. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} \geq 0$, such that

$$\begin{bmatrix} \mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} & \mathbf{PB} - \mathbf{C}^\mathsf{T}(\mathbf{D} - c\mathbf{1}) & \mathbf{0} \\ * & -(\mathbf{D} - c\mathbf{1})^\mathsf{T}(\mathbf{D} - c\mathbf{1}) & r\mathbf{1} \\ * & * & -\mathbf{1} \end{bmatrix} \leq 0. \tag{3.26}$$

Proof. Applying the Schur complement lemma to the $r^2\mathbf{1}$ term in (3.25) gives (3.26). □

A system satisfying the Modified Exterior Conic Sector Lemma is Lyapunov stable if the additional restriction $\mathbf{P} > 0$ is made, which is not necessarily true for a system satisfying the Exterior Conic Sector Lemma.

The system $\boldsymbol{\mathcal{G}}$ is also in the exterior cone of radius r centered at c, where $r \in \mathbb{R}_{>0}$ and $c \in \mathbb{R}$, under either of the following equivalent sufficient conditions.

1. There exists $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{Q} > 0$, such that

$$\begin{bmatrix} \mathbf{AQ} + \mathbf{QA}^\mathsf{T} & \mathbf{B} - \mathbf{QC}^\mathsf{T}(\mathbf{D} - c\mathbf{1}) \\ * & r^2\mathbf{1} - (\mathbf{D} - c\mathbf{1})^\mathsf{T}(\mathbf{D} - c\mathbf{1}) \end{bmatrix} \leq 0.$$

2. There exists $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{Q} > 0$, such that

$$\begin{bmatrix} \mathbf{AQ} + \mathbf{QA}^\mathsf{T} & \mathbf{B} - \mathbf{QC}^\mathsf{T}(\mathbf{D} - c\mathbf{1}) & \mathbf{0} \\ * & -(\mathbf{D} - c\mathbf{1})^\mathsf{T}(\mathbf{D} - c\mathbf{1}) & r\mathbf{1} \\ * & * & -\mathbf{1} \end{bmatrix} \leq 0.$$

3.7.4 Generalized KYP (GKYP) Lemma for Conic Sectors

Consider a square, continuous-time LTI system, $\boldsymbol{\mathcal{G}} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, with state-space realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{m\times n}$, and $\mathbf{D} \in \mathbb{R}^{m\times m}$. Also consider $\mathbf{\Pi}_c(a, b) \in \mathbb{S}^m$, which is defined as

$$\mathbf{\Pi}_c(a, b) = \begin{bmatrix} -\frac{1}{b}\mathbf{1} & \frac{1}{2}\left(1 + \frac{a}{b}\right)\mathbf{1} \\ * & -a\mathbf{1} \end{bmatrix},$$

where $a \in \mathbb{R}$, $b \in \mathbb{R}_{>0}$, and $a < b$. The following generalized KYP Lemmas give conditions for $\boldsymbol{\mathcal{G}}$ to be inside the cone $[a, b]$ within finite frequency bandwidths.

1. (*Low Frequency Range* [73]) The system $\boldsymbol{\mathcal{G}}$ is inside the cone $[a, b]$ for all $\omega \in \{\omega \in \mathbb{R} \mid |\omega| < \omega_1,\ \det(j\omega\mathbf{1} - \mathbf{A}) \neq 0\}$, where $\omega_1 \in \mathbb{R}_{>0}$, $a \in \mathbb{R}$, $b \in \mathbb{R}_{>0}$, and $a < b$, if there exist $\mathbf{P}, \mathbf{Q} \in \mathbb{S}^n$ and $\bar{\omega}_1 \in \mathbb{R}_{>0}$, where $\mathbf{Q} \geq 0$, such that

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}^\mathsf{T} \begin{bmatrix} -\mathbf{Q} & \mathbf{P} \\ * & (\omega_1 - \bar{\omega}_1)^2\mathbf{Q} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}^\mathsf{T} \mathbf{\Pi}_c(a, b) \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} < 0. \tag{3.27}$$

   If $\omega_1 \to \infty$, $\mathbf{P} > 0$, and $\mathbf{Q} = \mathbf{0}$, then the traditional Conic Sector Lemma is recovered [74].

   The parameter $\bar{\omega}_1$ is included in (3.27) to effectively transform $|\omega| \leq (\omega_1 - \bar{\omega}_1)$ into the strict inequality $|\omega| < \omega_1$.

2. (*Intermediate Frequency Range* [74–76]) The system $\mathcal{G}$ is inside the cone $[a, b]$ for all $\omega \in \{\omega \in \mathbb{R} \mid \omega_1 \leq |\omega| < \omega_2,\ \det(j\omega\mathbf{1} - \mathbf{A}) \neq 0\}$, where $\omega_1, \omega_2 \in \mathbb{R}_{>0}$, $a \in \mathbb{R}$, $b \in \mathbb{R}_{>0}$, and $a < b$, if there exist $\mathbf{P}$, $\mathbf{Q} \in \mathbb{C}^n$, $\bar{\omega}_2 \in \mathbb{R}_{>0}$, and $\hat{\omega}_2 = (\omega_1 + (\omega_2 - \bar{\omega}_2))/2$, where $\mathbf{P}^{\mathsf{H}} = \mathbf{P}$, $\mathbf{Q}^{\mathsf{H}} = \mathbf{Q}$, and $\mathbf{Q} \geq 0$, such that

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} -\mathbf{Q} & \mathbf{P} + j\hat{\omega}_2\mathbf{Q} \\ \mathbf{P} - j\hat{\omega}_2\mathbf{Q} & -\omega_1(\omega_2 - \bar{\omega} - 2)\mathbf{Q} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}^{\mathsf{T}} \mathbf{\Pi}_c(a, b) \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} < 0. \tag{3.28}$$

The parameter $\bar{\omega}_2$ is included in (3.28) to effectively transform $\omega_1 \leq |\omega| \leq (\omega_2 - \bar{\omega}_2)$ into the strict inequality $\omega_1 \leq |\omega| < \omega_2$.

3. (*High Frequency Range* [75]) The system $\mathcal{G}$ is inside the cone $[a, b]$ for all $\omega \in \{\omega \in \mathbb{R} \mid \omega_2 \leq |\omega|,\ \det(j\omega\mathbf{1} - \mathbf{A}) \neq 0\}$, where $\omega_2 \in \mathbb{R}_{>0}$, $a \in \mathbb{R}$, $b \in \mathbb{R}_{>0}$, and $a < b$, if there exist $\mathbf{P}$, $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{Q} \geq 0$, such that

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \mathbf{Q} & \mathbf{P} \\ * & -\omega_2^2\mathbf{Q} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}^{\mathsf{T}} \mathbf{\Pi}_c(a, b) \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} < 0. \tag{3.29}$$

If $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is a minimal realization, then the matrix inequalities in (3.27), (3.28), and (3.29) can be nonstrict [73].

3.8 Minimum Gain

3.8.1 Minimum Gain Lemma

Consider a continuous-time LTI system, $\mathcal{G} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, with state-space realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, and $\mathbf{D} \in \mathbb{R}^{p\times m}$. The system $\mathcal{G}$ has minimum gain ν under any of the following equivalent sufficient conditions.

1. [77] There exist $\mathbf{P} \in \mathbb{S}^n$ and $\nu \in \mathbb{R}_{\geq 0}$, where $\mathbf{P} \geq 0$, such that

$$\begin{bmatrix} \mathbf{PA} + \mathbf{A}^{\mathsf{T}}\mathbf{P} - \mathbf{C}^{\mathsf{T}}\mathbf{C} & \mathbf{PB} - \mathbf{C}^{\mathsf{T}}\mathbf{D} \\ * & \nu^2\mathbf{1} - \mathbf{D}^{\mathsf{T}}\mathbf{D} \end{bmatrix} \leq 0.$$

2. [78] There exist $\mathbf{P} \in \mathbb{S}^n$ and $\nu \in \mathbb{R}_{\geq 0}$, where $\mathbf{P} \geq 0$, such that

$$\begin{bmatrix} \mathbf{PA} + \mathbf{A}^{\mathsf{T}}\mathbf{P} - \mathbf{C}^{\mathsf{T}}\mathbf{C} & \mathbf{PB} - \mathbf{C}^{\mathsf{T}}\mathbf{D} & \mathbf{0} \\ * & -\mathbf{D}^{\mathsf{T}}\mathbf{D} & \nu\mathbf{1} \\ * & * & -\mathbf{1} \end{bmatrix} \leq 0.$$

If $\mathcal{G}$ is a square system (i.e., $m = p$) or span$(\mathbf{C}) \subseteq$ span$(\mathbf{D})$, then the preceding conditions are necessary and sufficient for $\mathcal{G}$ to have minimum gain $\nu \in \mathbb{R}_{\geq 0}$ [77]. The minimum gain lemma is a special case of the exterior conic sector lemma with $a = -\nu$ and $b = \nu$.

The system $\mathcal{G}$ also has minimum gain ν under any of the following sufficient conditions.

1. There exist $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{V}_{11} \in \mathbb{R}^{n\times n}$, $\mathbf{V}_{12} \in \mathbb{R}^{n\times m}$, $\mathbf{V}_{21} \in \mathbb{R}^{p\times n}$, $\mathbf{V}_{22} \in \mathbb{R}^{p\times m}$, and $\nu \in \mathbb{R}_{\geq 0}$, where $\mathbf{P} \geq 0$, such that

$$\begin{bmatrix} -(\mathbf{V}_{11}+\mathbf{V}_{11}^\mathsf{T}) & \mathbf{V}_{11}^\mathsf{T}\mathbf{A}+\mathbf{V}_{21}^\mathsf{T}\mathbf{C}+\mathbf{P} & \mathbf{V}_{11}^\mathsf{T}\mathbf{B}+\mathbf{V}_{21}^\mathsf{T}\mathbf{D}-\mathbf{V}_{12} & \mathbf{V}_{11}^\mathsf{T} & \nu\mathbf{V}_{21}^\mathsf{T} \\ * & -\mathbf{P} & \mathbf{C}^\mathsf{T}\mathbf{V}_{22}+\mathbf{A}^\mathsf{T}\mathbf{V}_{12} & \mathbf{0} & \mathbf{0} \\ * & * & \nu\mathbf{1}+\mathbf{V}_{22}\mathbf{D}+\mathbf{D}^\mathsf{T}\mathbf{V}_{22}^\mathsf{T}+\mathbf{V}_{12}^\mathsf{T}\mathbf{B}+\mathbf{B}^\mathsf{T}\mathbf{V}_{12} & \mathbf{V}_{12}^\mathsf{T} & \nu\mathbf{V}_{22}^\mathsf{T} \\ * & * & * & -\mathbf{P} & \mathbf{0} \\ * & * & * & * & -\nu\mathbf{1} \end{bmatrix} \leq 0. \tag{3.30}$$

Proof. Applying the congruence transformation $\mathbf{W} = \mathrm{diag}\{\nu^{-1/2}\mathbf{1}, \nu^{-1/2}\mathbf{1}\}$ and defining $\bar{\mathbf{P}} = \nu^{-1}\mathbf{P}$, the matrix inequality of (2) can be rewritten as

$$\begin{bmatrix} \bar{\mathbf{P}}\mathbf{A}+\mathbf{A}^\mathsf{T}\bar{\mathbf{P}}-\nu^{-1}\mathbf{C}^\mathsf{T}\mathbf{C} & \bar{\mathbf{P}}\mathbf{B}-\nu^{-1}\mathbf{C}^\mathsf{T}\mathbf{D} \\ * & \nu\mathbf{1}-\nu^{-1}\mathbf{D}^\mathsf{T}\mathbf{D} \end{bmatrix} \leq 0. \tag{3.31}$$

Using Property 3 from Section 2.3.3 and making the assumption that $\bar{\mathbf{P}}$ is invertible, (3.31) is equivalent to

$$\begin{bmatrix} \bar{\mathbf{P}}\mathbf{A}+\mathbf{A}^\mathsf{T}\bar{\mathbf{P}}-\bar{\mathbf{P}}-\nu^{-1}\mathbf{C}^\mathsf{T}\mathbf{C} & \bar{\mathbf{P}}\mathbf{B}-\nu^{-1}\mathbf{C}^\mathsf{T}\mathbf{D} & \bar{\mathbf{P}} \\ * & \nu\mathbf{1}-\nu^{-1}\mathbf{D}^\mathsf{T}\mathbf{D} & \mathbf{0} \\ * & * & -\bar{\mathbf{P}} \end{bmatrix} \leq 0.$$

which is rewritten as

$$\begin{bmatrix} \mathbf{A}^\mathsf{T} & \mathbf{1} & \mathbf{0} & \mathbf{0} & -\nu^{-1}\mathbf{C}^\mathsf{T} \\ \mathbf{B}^\mathsf{T} & \mathbf{0} & \mathbf{1} & \mathbf{0} & -\nu^{-1}\mathbf{D}^\mathsf{T} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \bar{\mathbf{P}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & -\bar{\mathbf{P}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & \nu\mathbf{1} & \mathbf{0} & \mathbf{0} \\ * & * & * & -\bar{\mathbf{P}} & \mathbf{0} \\ * & * & * & * & -\nu\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ -\nu^{-1}\mathbf{C} & -\nu^{-1}\mathbf{D} & \mathbf{0} \end{bmatrix} \leq 0. \tag{3.32}$$

Since $\bar{\mathbf{P}} > 0$ and $\nu \in \mathbb{R}_{\geq 0}$, it is also known that

$$\begin{bmatrix} -\bar{\mathbf{P}} & \mathbf{0} & \mathbf{0} \\ * & -\bar{\mathbf{P}} & \mathbf{0} \\ * & * & -\nu\mathbf{1} \end{bmatrix} \leq 0,$$

which can be rewritten as

$$\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \bar{\mathbf{P}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & -\bar{\mathbf{P}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & \nu\mathbf{1} & \mathbf{0} & \mathbf{0} \\ * & * & * & -\bar{\mathbf{P}} & \mathbf{0} \\ * & * & * & * & -\nu\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \leq 0. \tag{3.33}$$

The matrix inequalities in (3.32) and (3.33) are in the form of the nonstrict projection lemma. Specifically, (3.32) is in the form of $\mathbf{N}_G^\mathsf{T}\mathbf{\Phi}\mathbf{N}_G \leq 0$, where

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{0} & \bar{\mathbf{P}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & -\bar{\mathbf{P}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & \nu\mathbf{1} & \mathbf{0} & \mathbf{0} \\ * & * & * & -\bar{\mathbf{P}} & \mathbf{0} \\ * & * & * & * & -\nu\mathbf{1} \end{bmatrix}, \quad \mathbf{N}_G = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ -\nu^{-1}\mathbf{C} & -\nu^{-1}\mathbf{D} & \mathbf{0} \end{bmatrix}.$$

The matrix inequality of (3.33) is in the form of $\mathbf{N}_H^\mathsf{T}\mathbf{\Phi}\mathbf{N}_H < 0$, where

$$\mathbf{N}_H = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}.$$

The nonstrict projection lemma states that (3.32) and (3.33) are equivalent to

$$\mathbf{\Phi} + \mathbf{G}\mathbf{V}\mathbf{H}^\mathsf{T} + \mathbf{H}\mathbf{V}^\mathsf{T}\mathbf{G}^\mathsf{T}, \tag{3.34}$$

where $\mathcal{N}(\mathbf{G}^\mathsf{T}) = \mathcal{R}(\mathbf{N}_G)$, $\mathcal{N}(\mathbf{H}^\mathsf{T}) = \mathcal{R}(\mathbf{N}_H)$, $\mathbf{V} \in \mathbb{R}^{n\times n}$, and $\mathcal{R}(\mathbf{G})$, $\mathcal{R}(\mathbf{H})$ are linearly independent. Choosing

$$\mathbf{G}^\mathsf{T} = \begin{bmatrix} -\mathbf{1} & \mathbf{A} & \mathbf{B} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} & \mathbf{D} & \mathbf{0} & \nu\mathbf{1} \end{bmatrix}, \quad \mathbf{H}^\mathsf{T} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{bmatrix},$$

where $\mathcal{R}(\mathbf{G})$ and $\mathcal{R}(\mathbf{H})$ are in fact linearly independent, the matrix inequality of (3.34) can be rewritten as

$$\begin{bmatrix} \mathbf{0} & \bar{\mathbf{P}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & -\bar{\mathbf{P}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & \nu\mathbf{1} & \mathbf{0} & \mathbf{0} \\ * & * & * & -\bar{\mathbf{P}} & \mathbf{0} \\ * & * & * & * & -\nu\mathbf{1} \end{bmatrix} + \begin{bmatrix} -\mathbf{1} & \mathbf{0} \\ \mathbf{A}^\mathsf{T} & \mathbf{C}^\mathsf{T} \\ \mathbf{B}^\mathsf{T} & \mathbf{D}^\mathsf{T} \\ \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \nu\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$
$$+ \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V}_{11}^\mathsf{T} & \mathbf{V}_{21}^\mathsf{T} \\ \mathbf{V}_{12}^\mathsf{T} & \mathbf{V}_{22}^\mathsf{T} \end{bmatrix} \begin{bmatrix} -\mathbf{1} & \mathbf{A} & \mathbf{B} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} & \mathbf{D} & \mathbf{0} & \nu\mathbf{1} \end{bmatrix} < 0,$$

or equivalently

$$\begin{bmatrix} -(\mathbf{V}_{11} + \mathbf{V}_{11}^\mathsf{T}) & \mathbf{V}_{11}^\mathsf{T}\mathbf{A} + \mathbf{V}_{21}^\mathsf{T}\mathbf{C} + \bar{\mathbf{P}} & \mathbf{V}_{11}^\mathsf{T}\mathbf{B} + \mathbf{V}_{21}^\mathsf{T}\mathbf{D} - \mathbf{V}_{12} & \mathbf{V}_{11}^\mathsf{T} & \nu\mathbf{V}_{21}^\mathsf{T} \\ * & -\bar{\mathbf{P}} & \mathbf{C}^\mathsf{T}\mathbf{V}_{22} + \mathbf{A}^\mathsf{T}\mathbf{V}_{12} & \mathbf{0} & \mathbf{0} \\ * & * & \nu\mathbf{1} + \mathbf{V}_{22}\mathbf{D} + \mathbf{D}^\mathsf{T}\mathbf{V}_{22}^\mathsf{T} + \mathbf{V}_{12}^\mathsf{T}\mathbf{B} + \mathbf{B}^\mathsf{T}\mathbf{V}_{12} & \mathbf{V}_{12}^\mathsf{T} & \nu\mathbf{V}_{22}^\mathsf{T} \\ * & * & * & -\bar{\mathbf{P}} & \mathbf{0} \\ * & * & * & * & -\nu\mathbf{1} \end{bmatrix} \leq 0. \tag{3.35}$$

Redefining $\mathbf{P} = \bar{\mathbf{P}}$, (3.35) is identical to (3.30). □

2. There exist $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{V}_{11} \in \mathbb{R}^{n\times n}$, and $\nu \in \mathbb{R}_{\geq 0}$, where $\mathbf{P} \geq 0$, such that

$$\begin{bmatrix} -(\mathbf{V} + \mathbf{V}^\mathsf{T}) & \mathbf{V}^\mathsf{T}\mathbf{A} + \mathbf{P} & \mathbf{V}^\mathsf{T}\mathbf{B} & \mathbf{V}^\mathsf{T} \\ * & -\mathbf{P} & -\mathbf{C}^\mathsf{T} & \mathbf{0} \\ * & * & 2\nu\mathbf{1} - (\mathbf{D} + \mathbf{D}^\mathsf{T}) & \mathbf{0} \\ * & * & * & -\mathbf{P} \end{bmatrix} < 0. \tag{3.36}$$

Proof. The matrix inequality of (3.36) is derived from (3.30) with $\mathbf{V}_{11} = \mathbf{V}$, $\mathbf{V}_{12} = \mathbf{0}$, $\mathbf{V}_{21} = \mathbf{0}$, and $\mathbf{V}_{22} = -\mathbf{1}$. The dilation in (3.30) relies on the projection lemma and becomes only a sufficient condition in this case due to the structure imposed on $\mathbf{V}_{11}$, $\mathbf{V}_{12}$, $\mathbf{V}_{21}$, and $\mathbf{V}_{22}$. □

3.8.2 Modified Minimum Gain Lemma

Consider a continuous-time LTI system, $\mathcal{G} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, with state-space realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, and $\mathbf{D} \in \mathbb{R}^{p\times m}$. The system $\mathcal{G}$ has minimum gain ν under any of the following equivalent sufficient conditions.

1. [79] There exist $\mathbf{P} \in \mathbb{S}^n$ and $\nu \in \mathbb{R}_{\geq 0}$, where $\mathbf{P} \geq 0$, such that

$$\begin{bmatrix} \mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} & \mathbf{PB} - \mathbf{C}^\mathsf{T}\mathbf{D} \\ * & \nu^2\mathbf{1} - \mathbf{D}^\mathsf{T}\mathbf{D} \end{bmatrix} \leq 0. \tag{3.37}$$

2. There exist $\mathbf{P} \in \mathbb{S}^n$ and $\nu \in \mathbb{R}_{\geq 0}$, where $\mathbf{P} \geq 0$, such that

$$\begin{bmatrix} \mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} & \mathbf{PB} - \mathbf{C}^\mathsf{T}\mathbf{D} & \mathbf{0} \\ * & -\mathbf{D}^\mathsf{T}\mathbf{D} & \nu\mathbf{1} \\ * & * & -\mathbf{1} \end{bmatrix} \leq 0. \tag{3.38}$$

Proof. Applying the Schur complement lemma to the $\nu^2\mathbf{1}$ term in (3.37) gives (3.38). □

A system satisfying the Modified Minimum Gain Lemma is Lyapunov stable if the additional restriction $\mathbf{P} > 0$ is made, which is not necessarily true for a system satisfying the Minimum Gain Lemma.

The system $\mathcal{G}$ also has minimum gain ν under any of the following equivalent sufficient conditions.

1. There exist $\mathbf{Q} \in \mathbb{S}^n$ and $\nu \in \mathbb{R}_{\geq 0}$, where $\mathbf{Q} > 0$, such that

$$\begin{bmatrix} \mathbf{AQ} + \mathbf{QA}^\mathsf{T} & \mathbf{B} - \mathbf{QC}^\mathsf{T}\mathbf{D} \\ * & \nu^2\mathbf{1} - \mathbf{D}^\mathsf{T}\mathbf{D} \end{bmatrix} \leq 0.$$

2. There exist $\mathbf{Q} \in \mathbb{S}^n$ and $\nu \in \mathbb{R}_{\geq 0}$, where $\mathbf{Q} > 0$, such that

$$\begin{bmatrix} \mathbf{AQ} + \mathbf{QA}^\mathsf{T} & \mathbf{B} - \mathbf{QC}^\mathsf{T}\mathbf{D} & \mathbf{0} \\ * & -\mathbf{D}^\mathsf{T}\mathbf{D} & \nu\mathbf{1} \\ * & * & -\mathbf{1} \end{bmatrix} \leq 0.$$

3.8.3 Discrete-Time Minimum Gain Lemma

Consider a discrete-time LTI system, $\mathcal{G} : \ell_{2e} \to \ell_{2e}$, with state-space realization $(\mathbf{A}_\mathrm{d}, \mathbf{B}_\mathrm{d}, \mathbf{C}_\mathrm{d}, \mathbf{D}_\mathrm{d})$, where $\mathbf{A}_\mathrm{d} \in \mathbb{R}^{n\times n}$, $\mathbf{B}_\mathrm{d} \in \mathbb{R}^{n\times m}$, $\mathbf{C}_\mathrm{d} \in \mathbb{R}^{p\times n}$, and $\mathbf{D}_\mathrm{d} \in \mathbb{R}^{p\times m}$. The system $\mathcal{G}$ has minimum gain ν under any of the following equivalent sufficient conditions.

1. [80, p. 30] There exist $\mathbf{P} \in \mathbb{S}^n$ and $\nu \in \mathbb{R}_{\geq 0}$, where $\mathbf{P} \geq 0$, such that

$$\begin{bmatrix} \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{A}_\mathrm{d} - \mathbf{P} - \mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{C}_\mathrm{d} & \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d} - \mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} \\ * & \mathbf{B}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d} + \nu^2\mathbf{1} - \mathbf{D}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} \end{bmatrix} \leq 0. \tag{3.39}$$

2. There exist $\mathbf{P} \in \mathbb{S}^n$ and $\nu \in \mathbb{R}_{\geq 0}$, where $\mathbf{P} \geq 0$, such that

$$\begin{bmatrix} \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{A}_\mathrm{d} - \mathbf{P} - \mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{C}_\mathrm{d} & \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d} - \mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} & \mathbf{0} \\ * & \mathbf{B}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d} - \mathbf{D}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} & \nu\mathbf{1} \\ * & * & \mathbf{1} \end{bmatrix} \leq 0. \tag{3.40}$$

Proof. Applying the Schur complement lemma to the $\nu^2\mathbf{1}$ term in (3.39) gives (3.40). $\square$

The system $\mathcal{G}$ also has minimum gain ν under any of the following equivalent sufficient conditions.

1. There exist $\mathbf{P} \in \mathbb{S}^n$ and $\nu \in \mathbb{R}_{\geq 0}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{P}\mathbf{A}_\mathrm{d} & \mathbf{P}\mathbf{B}_\mathrm{d} \\ * & \mathbf{P} + \mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{C}_\mathrm{d} & \mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} \\ * & * & \mathbf{D}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} - \nu^2\mathbf{1} \end{bmatrix} \geq 0. \tag{3.41}$$

   Proof. Under the assumption that $\mathbf{P} > 0$, the nonstrict Schur complement lemma is applied to (3.39) to yield (3.41). $\square$

2. There exist $\mathbf{P} \in \mathbb{S}^n$ and $\nu \in \mathbb{R}_{\geq 0}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{P}\mathbf{A}_\mathrm{d} & \mathbf{P}\mathbf{B}_\mathrm{d} & \mathbf{0} \\ * & \mathbf{P} + \mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{C}_\mathrm{d} & \mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} & \mathbf{0} \\ * & * & \mathbf{D}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} & \nu\mathbf{1} \\ * & * & * & \mathbf{1} \end{bmatrix} \geq 0. \tag{3.42}$$

   Proof. Applying the Schur complement lemma to the $\nu^2\mathbf{1}$ term in (3.41) gives (3.42). $\square$

3.8.4 Discrete-Time Modified Minimum Gain Lemma

Consider a discrete-time LTI system, $\mathcal{G} : \ell_{2e} \to \ell_{2e}$, with state-space realization $(\mathbf{A}_\mathrm{d}, \mathbf{B}_\mathrm{d}, \mathbf{C}_\mathrm{d}, \mathbf{D}_\mathrm{d})$, where $\mathbf{A}_\mathrm{d} \in \mathbb{R}^{n\times n}$, $\mathbf{B}_\mathrm{d} \in \mathbb{R}^{n\times m}$, $\mathbf{C}_\mathrm{d} \in \mathbb{R}^{p\times n}$, and $\mathbf{D}_\mathrm{d} \in \mathbb{R}^{p\times m}$. The system $\mathcal{G}$ has minimum gain ν under any of the following equivalent sufficient conditions.

1. There exist $\mathbf{P} \in \mathbb{S}^n$ and $\nu \in \mathbb{R}_{\geq 0}$, where $\mathbf{P} \geq 0$, such that

$$\begin{bmatrix} \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{A}_\mathrm{d} - \mathbf{P} & \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d} - \mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} \\ * & \mathbf{B}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d} + \nu^2\mathbf{1} - \mathbf{D}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} \end{bmatrix} \leq 0. \tag{3.43}$$

   Proof. The term $-\mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{C}_\mathrm{d}$ in (3.39) makes the matrix inequality "more" negative definite. Therefore,

$$\begin{bmatrix} \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{A}_\mathrm{d} - \mathbf{P} - \mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{C}_\mathrm{d} & \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d} - \mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} \\ * & \mathbf{B}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d} + \nu^2\mathbf{1} - \mathbf{D}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} \end{bmatrix} \leq \begin{bmatrix} \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{A}_\mathrm{d} - \mathbf{P} & \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d} - \mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} \\ * & \mathbf{B}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d} + \nu^2\mathbf{1} - \mathbf{D}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} \end{bmatrix},$$

   and (3.43) implies (3.39). $\square$

2. There exist $\mathbf{P} \in \mathbb{S}^n$ and $\nu \in \mathbb{R}_{\geq 0}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{A}_\mathrm{d} - \mathbf{P} & \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d} - \mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} & \mathbf{0} \\ * & \mathbf{B}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d} - \mathbf{D}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} & \nu\mathbf{1} \\ * & * & \mathbf{1} \end{bmatrix} \leq 0. \tag{3.44}$$

   Proof. Applying the Schur complement lemma to the $\nu^2\mathbf{1}$ term in (3.43) gives (3.44). $\square$

A system satisfying the Discrete-Time Modified Minimum Gain Lemma is Lyapunov stable if the additional restriction $\mathbf{P} > 0$ is made, which is not necessarily true for a system satisfying the Discrete-Time Minimum Gain Lemma.

The system $\mathcal{G}$ also has minimum gain ν under any of the following sufficient conditions.

1. There exist $\mathbf{P} \in \mathbb{S}^n$ and $\nu \in \mathbb{R}_{\geq 0}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{P}\mathbf{A}_\mathrm{d} & \mathbf{P}\mathbf{B}_\mathrm{d} \\ * & \mathbf{P} & \mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} \\ * & * & \mathbf{D}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} - \nu^2\mathbf{1} \end{bmatrix} \geq 0. \tag{3.45}$$

   Proof. Under the assumption that $\mathbf{P} > 0$, the nonstrict Schur complement lemma is applied to (3.43) to yield (3.45). □

2. There exist $\mathbf{P} \in \mathbb{S}^n$ and $\nu \in \mathbb{R}_{\geq 0}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{P}\mathbf{A}_\mathrm{d} & \mathbf{P}\mathbf{B}_\mathrm{d} & \mathbf{0} \\ * & \mathbf{P} & \mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} & \mathbf{0} \\ * & * & \mathbf{D}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} & \nu\mathbf{1} \\ * & * & * & \mathbf{1} \end{bmatrix} \geq 0. \tag{3.46}$$

   Proof. Applying the Schur complement lemma to the $\nu^2\mathbf{1}$ term in (3.45) gives (3.46). □

3. There exist $\mathbf{Q} \in \mathbb{S}^n$ and $\nu \in \mathbb{R}_{\geq 0}$, where $\mathbf{Q} > 0$, such that

$$\begin{bmatrix} \mathbf{Q} & \mathbf{A}_\mathrm{d}\mathbf{Q} & \mathbf{B}_\mathrm{d} \\ * & \mathbf{Q} & \mathbf{Q}\mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} \\ * & * & \mathbf{D}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} - \nu^2\mathbf{1} \end{bmatrix} \geq 0.$$

4. There exist $\mathbf{Q} \in \mathbb{S}^n$ and $\nu \in \mathbb{R}_{\geq 0}$, where $\mathbf{Q} > 0$, such that

$$\begin{bmatrix} \mathbf{Q} & \mathbf{A}_\mathrm{d}\mathbf{Q} & \mathbf{B}_\mathrm{d} & \mathbf{0} \\ * & \mathbf{Q} & \mathbf{Q}\mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} & \mathbf{0} \\ * & * & \mathbf{D}_\mathrm{d}^\mathsf{T}\mathbf{D}_\mathrm{d} & \nu\mathbf{1} \\ * & * & * & \mathbf{1} \end{bmatrix} \geq 0.$$

3.9 Negative Imaginary Systems

3.9.1 Negative Imaginary Lemma [81, 82]

Consider a square, continuous-time LTI system, $\mathcal{G} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, with state-space realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{m\times n}$, and $\mathbf{D} \in \mathbb{S}^m$. The system $\mathcal{G}$ is negative imaginary under either of the following equivalent necessary and sufficient conditions.

1. There exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} \geq 0$, such that

$$\begin{bmatrix} \mathbf{P}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{P} & \mathbf{P}\mathbf{B} - \mathbf{A}^\mathsf{T}\mathbf{C}^\mathsf{T} \\ * & -\left(\mathbf{C}\mathbf{B} + \mathbf{B}^\mathsf{T}\mathbf{C}^\mathsf{T}\right) \end{bmatrix} \leq 0. \tag{3.47}$$

2. There exists $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{Q} \geq 0$, such that

$$\begin{bmatrix} \mathbf{AQ} + \mathbf{QA}^\mathsf{T} & \mathbf{B} - \mathbf{QA}^\mathsf{T}\mathbf{C}^\mathsf{T} \\ * & -\left(\mathbf{CB} + \mathbf{B}^\mathsf{T}\mathbf{C}^\mathsf{T}\right) \end{bmatrix} \leq 0. \tag{3.48}$$

The system $\boldsymbol{\mathcal{G}}$ is strictly negative imaginary if $\det(\mathbf{A}) \neq 0$ and either (3.47) is satisfied with $\mathbf{P} > 0$ or (3.48) is satisfied with $\mathbf{Q} > 0$.

3.9.2 Generalized Negative Imaginary Lemma

Consider a square, continuous-time LTI system, $\boldsymbol{\mathcal{G}} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, with state-space realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{C} \in \mathbb{R}^{m \times n}$, and $\mathbf{D} \in \mathbb{S}^m$. Also consider $\mathbf{\Pi}_p \in \mathbb{S}^m$, which is defined as

$$\mathbf{\Pi}_p = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix},$$

The following generalized KYP Lemmas give conditions for $\boldsymbol{\mathcal{G}}$ to be negative imaginary within finite frequency bandwidths.

1. (*Low Frequency Range* [83]) The system $\boldsymbol{\mathcal{G}}$ is negative imaginary for all $\omega \in \{\omega \in \mathbb{R} \mid \ |\omega| < \omega_1,\ \det(j\omega\mathbf{1} - \mathbf{A}) \neq 0\}$, where $\omega_1 \in \mathbb{R}_{>0}$, if there exist $\mathbf{P}$, $\mathbf{Q} \in \mathbb{S}^n$ and $\bar{\omega}_1 \in \mathbb{R}_{>0}$, where $\mathbf{Q} \geq 0$, such that

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}^\mathsf{T} \begin{bmatrix} -\mathbf{Q} & \mathbf{P} \\ * & (\omega_1 - \bar{\omega}_1)^2\mathbf{Q} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{CA} & \mathbf{CB} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}^\mathsf{T} \mathbf{\Pi}_p \begin{bmatrix} \mathbf{CA} & \mathbf{CB} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} < 0. \tag{3.49}$$

   If $\omega_1 \to \infty$, $\mathbf{P} > 0$, and $\mathbf{Q} = \mathbf{0}$, then the traditional Negative Imaginary Lemma is recovered [83].

   The parameter $\bar{\omega}_1$ is included in (3.49) to effectively transform $|\omega| \leq (\omega_1 - \bar{\omega}_1)$ into the strict inequality $|\omega| < \omega_1$.

2. (*Intermediate Frequency Range*) The system $\boldsymbol{\mathcal{G}}$ is negative imaginary for all $\omega \in \{\omega \in \mathbb{R} \mid \omega_1 \leq |\omega| < \omega_2,\ \det(j\omega\mathbf{1} - \mathbf{A}) \neq 0\}$, where $\omega_1, \omega_2 \in \mathbb{R}_{>0}$, if there exist $\mathbf{P}$, $\mathbf{Q} \in \mathbb{C}^n$, $\bar{\omega}_2 \in \mathbb{R}_{>0}$, and $\hat{\omega}_2 = (\omega_1 + (\omega_2 - \bar{\omega}_2))/2$, where $\mathbf{P}^\mathsf{H} = \mathbf{P}$, $\mathbf{Q}^\mathsf{H} = \mathbf{Q}$, and $\mathbf{Q} \geq 0$, such that

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}^\mathsf{T} \begin{bmatrix} -\mathbf{Q} & \mathbf{P} + j\hat{\omega}_2\mathbf{Q} \\ \mathbf{P} - j\hat{\omega}_2\mathbf{Q} & -\omega_1(\omega_2 - \bar{\omega} - 2)\mathbf{Q} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{CA} & \mathbf{CB} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}^\mathsf{T} \mathbf{\Pi}_p \begin{bmatrix} \mathbf{CA} & \mathbf{CB} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} < 0. \tag{3.50}$$

   The parameter $\bar{\omega}_2$ is included in (3.50) to effectively transform $\omega_1 \leq |\omega| \leq (\omega_2 - \bar{\omega}_2)$ into the strict inequality $\omega_1 \leq |\omega| < \omega_2$.

3. (*High Frequency Range*) The system $\boldsymbol{\mathcal{G}}$ is negative imaginary for all $\omega \in \{\omega \in \mathbb{R} \mid \omega_2 \leq |\omega|,\ \det(j\omega\mathbf{1} - \mathbf{A}) \neq 0\}$, where $\omega_2 \in \mathbb{R}_{>0}$, if there exist $\mathbf{P}$, $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{Q} \geq 0$, such that

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}^\mathsf{T} \begin{bmatrix} \mathbf{Q} & \mathbf{P} \\ * & -\omega_2^2\mathbf{Q} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{CA} & \mathbf{CB} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}^\mathsf{T} \mathbf{\Pi}_p \begin{bmatrix} \mathbf{CA} & \mathbf{CB} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} < 0. \tag{3.51}$$

3.10 Algebraic Riccati Inequalities

3.10.1 Algebraic Riccati Inequality [56]

Consider $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{P}$, $\mathbf{Q} \in \mathbb{S}^n$, $\mathbf{N} \in \mathbb{R}^{n\times m}$, and $\mathbf{R} \in \mathbb{S}^m$, where $\mathbf{P} > 0$, $\mathbf{Q} \geq 0$, and $\mathbf{R} > 0$. The algebraic Riccati inequality given by

$$\mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{P}\mathbf{A} - \left(\mathbf{P}\mathbf{B} + \mathbf{N}^\mathsf{T}\right)\mathbf{R}^{-1}\left(\mathbf{B}^\mathsf{T}\mathbf{P} + \mathbf{N}\right) + \mathbf{Q} \geq 0,$$

can be rewritten using the Schur complement lemma as

$$\begin{bmatrix} \mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} & \mathbf{P}\mathbf{B} + \mathbf{N}^\mathsf{T} \\ * & \mathbf{R} \end{bmatrix} \geq 0.$$

3.10.2 Discrete-Time Algebraic Riccati Inequality [84]

Consider $\mathbf{A}_\mathrm{d} \in \mathbb{R}^{n\times n}$, $\mathbf{B}_\mathrm{d} \in \mathbb{R}^{n\times m}$, $\mathbf{P}$, $\mathbf{Q} \in \mathbb{S}^n$, and $\mathbf{R} \in \mathbb{S}^m$, where $\mathbf{P} > 0$, $\mathbf{Q} \geq 0$, and $\mathbf{R} > 0$. The discrete-time algebraic Riccati inequality given by

$$\mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{A}_\mathrm{d} - \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d}\left(\mathbf{R} + \mathbf{B}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d}\right)^{-1}\mathbf{B}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{A}_\mathrm{d} + \mathbf{Q} - \mathbf{P} \geq 0,$$

can be rewritten using the Schur complement lemma as

$$\begin{bmatrix} \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{A}_\mathrm{d} - \mathbf{P} + \mathbf{Q} & \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d} \\ * & \mathbf{R} + \mathbf{B}_\mathrm{d}^\mathsf{T}\mathbf{P}\mathbf{B}_\mathrm{d} \end{bmatrix} \geq 0.$$

Equivalently, this discrete-time algebraic Riccati inequality is satisfied under any of the following necessary and sufficient conditions.

1. There exist $\mathbf{P}$, $\mathbf{Q} \in \mathbb{S}^n$, and $\mathbf{R} \in \mathbb{S}^m$, where $\mathbf{P} > 0$, $\mathbf{Q} \geq 0$, and $\mathbf{R} > 0$, such that

$$\begin{bmatrix} \mathbf{Q} & \mathbf{0} & \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P} & \mathbf{P} \\ * & \mathbf{R} & \mathbf{B}_\mathrm{d}^\mathsf{T}\mathbf{P} & \mathbf{0} \\ * & * & \mathbf{P} & \mathbf{0} \\ * & * & * & \mathbf{P} \end{bmatrix} \geq 0.$$

2. There exist $\mathbf{X}$, $\mathbf{Q} \in \mathbb{S}^n$, and $\mathbf{R} \in \mathbb{S}^m$, where $\mathbf{X} > 0$, $\mathbf{Q} \geq 0$, and $\mathbf{R} > 0$, such that

$$\begin{bmatrix} \mathbf{Q} & \mathbf{0} & \mathbf{A}_\mathrm{d}^\mathsf{T} & \mathbf{1} \\ * & \mathbf{R} & \mathbf{B}_\mathrm{d}^\mathsf{T} & \mathbf{0} \\ * & * & \mathbf{X} & \mathbf{0} \\ * & * & * & \mathbf{X} \end{bmatrix} \geq 0.$$

3.11 Stabilizability

3.11.1 Continuous-Time Stabilizability [19, pp. 166–168]

Consider a continuous-time LTI system, $\boldsymbol{\mathcal{G}} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, with state-space realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, and $\mathbf{D} \in \mathbb{R}^{p\times m}$. The system $\boldsymbol{\mathcal{G}}$ is stabilizable if and only if there exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\mathsf{T} - \mathbf{B}\mathbf{B}^\mathsf{T} < 0.$$

The matrix $\mathbf{A} + \mathbf{B}\mathbf{K}$ is Hurwitz with $\mathbf{K} = -\frac{1}{2}\mathbf{B}^\mathsf{T}\mathbf{P}^{-1}$. Equivalently, $\boldsymbol{\mathcal{G}}$ is stabilizable if and only if there exist $\mathbf{P} \in \mathbb{S}^n$ and $\mathbf{W} \in \mathbb{R}^{m\times n}$, where $\mathbf{P} > 0$, such that

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\mathsf{T} + \mathbf{B}\mathbf{W} + \mathbf{W}^\mathsf{T}\mathbf{B}^\mathsf{T} < 0.$$

The matrix $\mathbf{A} + \mathbf{B}\mathbf{K}$ is Hurwitz with $\mathbf{K} = \mathbf{W}\mathbf{P}^{-1}$.

3.11.2 Discrete-Time Stabilizability [19, pp. 172–176]

Consider a discrete-time LTI system, $\mathcal{G} : \ell_{2e} \to \ell_{2e}$, with state-space realization $(\mathbf{A}_\mathrm{d}, \mathbf{B}_\mathrm{d}, \mathbf{C}_\mathrm{d}, \mathbf{D}_\mathrm{d})$, where $\mathbf{A}_\mathrm{d} \in \mathbb{R}^{n\times n}$, $\mathbf{B}_\mathrm{d} \in \mathbb{R}^{n\times m}$, $\mathbf{C}_\mathrm{d} \in \mathbb{R}^{p\times n}$, and $\mathbf{D}_\mathrm{d} \in \mathbb{R}^{p\times m}$. The system $\mathcal{G}$ is stabilizable if and only if there exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{P}\mathbf{A}_\mathrm{d}^\mathsf{T} \\ * & \mathbf{P} + \mathbf{B}_\mathrm{d}\mathbf{B}_\mathrm{d}^\mathsf{T} \end{bmatrix} > 0.$$

The matrix $\mathbf{A}_\mathrm{d} + \mathbf{B}_\mathrm{d}\mathbf{K}_\mathrm{d}$ is Schur with $\mathbf{K}_\mathrm{d} = -\left(2\mathbf{1} + \mathbf{B}_\mathrm{d}^\mathsf{T}\mathbf{P}^{-1}\mathbf{B}_\mathrm{d}\right)^{-1}\mathbf{B}_\mathrm{d}^\mathsf{T}\mathbf{P}^{-1}\mathbf{A}_\mathrm{d}$. Equivalently, $\mathcal{G}$ is stabilizable if and only if there exist $\mathbf{P} \in \mathbb{S}^n$ and $\mathbf{W} \in \mathbb{R}^{m\times n}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{A}_\mathrm{d}\mathbf{P} + \mathbf{B}_\mathrm{d}\mathbf{W} \\ * & \mathbf{P} \end{bmatrix} > 0.$$

The matrix $\mathbf{A}_\mathrm{d} + \mathbf{B}_\mathrm{d}\mathbf{K}_\mathrm{d}$ is Schur with $\mathbf{K}_\mathrm{d} = \mathbf{W}\mathbf{P}^{-1}$.

3.12 Detectability

3.12.1 Continuous-Time Detectability [19, pp. 170–171]

Consider a continuous-time LTI system, $\mathcal{G} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, with state-space realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, and $\mathbf{D} \in \mathbb{R}^{p\times m}$. The system $\mathcal{G}$ is detectable if and only if there exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\mathbf{P}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{P} - \mathbf{C}^\mathsf{T}\mathbf{C} < 0.$$

The matrix $\mathbf{A} + \mathbf{L}\mathbf{C}$ is Hurwitz with $\mathbf{L} = -\frac{1}{2}\mathbf{P}^{-1}\mathbf{C}^\mathsf{T}$. Equivalently, $\mathcal{G}$ is detectable if and only if there exist $\mathbf{P} \in \mathbb{S}^n$ and $\mathbf{W} \in \mathbb{R}^{p\times n}$, where $\mathbf{P} > 0$, such that

$$\mathbf{P}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{W}^\mathsf{T}\mathbf{C} + \mathbf{C}^\mathsf{T}\mathbf{W} < 0.$$

The matrix $\mathbf{A} + \mathbf{L}\mathbf{C}$ is Hurwitz with $\mathbf{L} = -\frac{1}{2}\mathbf{P}^{-1}\mathbf{W}^\mathsf{T}$.

3.12.2 Discrete-Time Detectability [19, pp. 177–178]

Consider a discrete-time LTI system, $\mathcal{G} : \ell_{2e} \to \ell_{2e}$, with state-space realization $(\mathbf{A}_\mathrm{d}, \mathbf{B}_\mathrm{d}, \mathbf{C}_\mathrm{d}, \mathbf{D}_\mathrm{d})$, where $\mathbf{A}_\mathrm{d} \in \mathbb{R}^{n\times n}$, $\mathbf{B}_\mathrm{d} \in \mathbb{R}^{n\times m}$, $\mathbf{C}_\mathrm{d} \in \mathbb{R}^{p\times n}$, and $\mathbf{D}_\mathrm{d} \in \mathbb{R}^{p\times m}$. The system $\mathcal{G}$ is detectable if and only if there exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{P}\mathbf{A}_\mathrm{d} \\ * & \mathbf{P} + \mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{C}_\mathrm{d} \end{bmatrix} > 0.$$

The matrix $\mathbf{A}_\mathrm{d} + \mathbf{L}\mathbf{C}_\mathrm{d}$ is Schur with $\mathbf{L} = -\mathbf{A}_\mathrm{d}\mathbf{P}^{-1}\mathbf{C}_\mathrm{d}^\mathsf{T}\left(2\mathbf{1} + \mathbf{C}_\mathrm{d}\mathbf{P}^{-1}\mathbf{C}_\mathrm{d}^\mathsf{T}\right)^{-1}$. Equivalently, $\mathcal{G}$ is detectable if and only if there exist $\mathbf{P} \in \mathbb{S}^n$ and $\mathbf{W} \in \mathbb{R}^{m\times n}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{P} & \mathbf{A}_\mathrm{d}^\mathsf{T}\mathbf{P} + \mathbf{C}_\mathrm{d}^\mathsf{T}\mathbf{W} \\ * & \mathbf{P} \end{bmatrix} > 0.$$

The matrix $\mathbf{A}_\mathrm{d} + \mathbf{L}\mathbf{C}_\mathrm{d}$ is Schur with $\mathbf{L} = \mathbf{P}^{-1}\mathbf{W}$.

3.13 Static Output Feedback Stabilizability

3.13.1 Continuous-Time Static Output Feedback Stabilizability [85, 86], [87, p. 120]

Consider a continuous-time LTI system, $\mathcal{G} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, with state-space realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{0})$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, and $\mathbf{C} \in \mathbb{R}^{p\times n}$. The system $\mathcal{G}$ is static output feedback stabilizable under any of the following equivalent necessary and sufficient conditions.

1. There exist $\mathbf{K} \in \mathbb{R}^{m\times p}$ and $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that
$$\begin{bmatrix} \mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{B}^\mathsf{T}\mathbf{P} & \mathbf{P}\mathbf{B} + \mathbf{C}^\mathsf{T}\mathbf{K}^\mathsf{T} \\ * & -\mathbf{1} \end{bmatrix} < 0.$$

2. There exist $\mathbf{K} \in \mathbb{R}^{m\times p}$ and $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{Q} > 0$, such that
$$\begin{bmatrix} \mathbf{Q}\mathbf{A}^\mathsf{T} + \mathbf{A}\mathbf{Q} - \mathbf{Q}\mathbf{C}^\mathsf{T}\mathbf{C}\mathbf{Q} & \mathbf{B}\mathbf{K} + \mathbf{Q}\mathbf{C}^\mathsf{T} \\ * & -\mathbf{1} \end{bmatrix} < 0.$$

3. There exist $\mathbf{K} \in \mathbb{R}^{m\times p}$ and $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{Q} > 0$, such that
$$\begin{bmatrix} \mathbf{Q}\mathbf{A}^\mathsf{T} + \mathbf{A}\mathbf{Q} - \mathbf{B}\mathbf{B}^\mathsf{T} & \mathbf{B} + \mathbf{Q}\mathbf{C}^\mathsf{T}\mathbf{K}^\mathsf{T} \\ * & -\mathbf{1} \end{bmatrix} < 0.$$

4. There exist $\mathbf{K} \in \mathbb{R}^{m\times p}$ and $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that
$$\begin{bmatrix} \mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{C}^\mathsf{T}\mathbf{C} & \mathbf{P}\mathbf{B}\mathbf{K} + \mathbf{C}^\mathsf{T} \\ * & -\mathbf{1} \end{bmatrix} < 0.$$

5. There exist $\mathbf{K} \in \mathbb{R}^{m\times p}$, $\mathbf{P}, \mathbf{X} \in \mathbb{S}^n$, where $\mathbf{P} > 0$ and $\mathbf{X} > 0$, such that
$$\begin{bmatrix} \mathbf{A}^\mathsf{T}\mathbf{X} + \mathbf{X}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{B}^\mathsf{T}\mathbf{X} - \mathbf{X}\mathbf{B}\mathbf{B}^\mathsf{T}\mathbf{P} + \mathbf{X}\mathbf{B}\mathbf{B}^\mathsf{T}\mathbf{X} & \mathbf{P}\mathbf{B} + \mathbf{C}^\mathsf{T}\mathbf{K}^\mathsf{T} \\ * & -\mathbf{1} \end{bmatrix} < 0.$$

6. There exist $\mathbf{K} \in \mathbb{R}^{m\times p}$ and $\mathbf{Q}, \mathbf{X} \in \mathbb{S}^n$, where $\mathbf{Q} > 0$ and $\mathbf{X} > 0$, such that
$$\begin{bmatrix} \mathbf{Q}\mathbf{A}^\mathsf{T} + \mathbf{A}\mathbf{Q} - \mathbf{Q}\mathbf{C}^\mathsf{T}\mathbf{C}\mathbf{X} - \mathbf{X}\mathbf{C}^\mathsf{T}\mathbf{C}\mathbf{Q} + \mathbf{X}\mathbf{C}^\mathsf{T}\mathbf{C}\mathbf{X} & \mathbf{B}\mathbf{K} + \mathbf{Q}\mathbf{C}^\mathsf{T} \\ * & -\mathbf{1} \end{bmatrix} < 0.$$

3.13.2 Discrete-Time Static Output Feedback Stabilizability

Consider a discrete-time LTI system, $\mathcal{G} : \ell_{2e} \to \ell_{2e}$, with state-space realization $(\mathbf{A}_\mathrm{d}, \mathbf{B}_\mathrm{d}, \mathbf{C}_\mathrm{d}, \mathbf{0})$, where $\mathbf{A}_\mathrm{d} \in \mathbb{R}^{n\times n}$, $\mathbf{B}_\mathrm{d} \in \mathbb{R}^{n\times m}$, and $\mathbf{C}_\mathrm{d} \in \mathbb{R}^{p\times n}$. The system $\mathcal{G}$ is static output feedback stabilizable under any of the following equivalent necessary and sufficient conditions.

1. There exist $\mathbf{K}_\mathrm{d} \in \mathbb{R}^{m\times p}$ and $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that
$$\begin{bmatrix} -\mathbf{P} & (\mathbf{A}_\mathrm{d} + \mathbf{B}_\mathrm{d}\mathbf{K}_\mathrm{d}\mathbf{C}_\mathrm{d})\,\mathbf{P} \\ * & -\mathbf{P} \end{bmatrix} < 0. \tag{3.52}$$

2. There exist $\mathbf{K}_\mathrm{d} \in \mathbb{R}^{m\times p}$ and $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that
$$\begin{bmatrix} -\mathbf{A}_\mathrm{d}\mathbf{P}\mathbf{P}\mathbf{A}_\mathrm{d}^\mathsf{T} & \mathbf{A}_\mathrm{d}\mathbf{P} + \mathbf{B}_\mathrm{d}\mathbf{K}_\mathrm{d}\mathbf{C}_\mathrm{d} & \mathbf{A}_\mathrm{d}\mathbf{P} \\ * & -\mathbf{1} & \mathbf{0} \\ * & * & -\mathbf{P} \end{bmatrix} < 0. \tag{3.53}$$

Proof. Applying the reverse Schur complement lemma to (3.52) yields

$$(\mathbf{A}_\mathrm{d} + \mathbf{B}_\mathrm{d}\mathbf{K}_\mathrm{d}\mathbf{C}_\mathrm{d})\,\mathbf{P}\,(\mathbf{A}_\mathrm{d} + \mathbf{B}_\mathrm{d}\mathbf{K}_\mathrm{d}\mathbf{C}_\mathrm{d})^\mathsf{T} - \mathbf{P} < 0.$$

Multiplying out this matrix inequality and adding $\mathbf{0} = \mathbf{A}_\mathrm{d}\mathbf{P}\mathbf{P}\mathbf{A}_\mathrm{d} - \mathbf{A}_\mathrm{d}\mathbf{P}\mathbf{P}\mathbf{A}_\mathrm{d}$ to the left-hand side gives

$$\mathbf{A}_\mathrm{d}\mathbf{P}\mathbf{A}_\mathrm{d}^\mathsf{T} - \mathbf{A}_\mathrm{d}\mathbf{P}\mathbf{P}\mathbf{A}_\mathrm{d}^\mathsf{T} + (\mathbf{A}_\mathrm{d}\mathbf{P} + \mathbf{B}_\mathrm{d}\mathbf{K}_\mathrm{d}\mathbf{C}_\mathrm{d})\,(\mathbf{A}_\mathrm{d}\mathbf{P} + \mathbf{B}_\mathrm{d}\mathbf{K}_\mathrm{d}\mathbf{C}_\mathrm{d})^\mathsf{T} < 0.$$

Applying the Schur complement lemma twice gives (3.53). □

The system $\mathcal{G}$ is also static output feedback stabilizable if there exist $\mathbf{K}_\mathrm{d} \in \mathbb{R}^{m\times p}$ and $\mathbf{P}, \mathbf{X} \in \mathbb{S}^n$, where $\mathbf{P} > 0$ and $\mathbf{X} > 0$, such that

$$\begin{bmatrix} -\mathbf{A}_\mathrm{d}\,(\mathbf{X}\mathbf{P} + \mathbf{P}\mathbf{X})\,\mathbf{A}_\mathrm{d}^\mathsf{T} & \mathbf{A}_\mathrm{d}\mathbf{P} + \mathbf{B}_\mathrm{d}\mathbf{K}_\mathrm{d}\mathbf{C}_\mathrm{d} & \mathbf{A}_\mathrm{d}\mathbf{P} & \mathbf{A}_\mathrm{d}\mathbf{X} \\ * & -\mathbf{1} & \mathbf{0} & \mathbf{0} \\ * & * & -\mathbf{P} & \mathbf{0} \\ * & * & * & -\mathbf{1} \end{bmatrix} < 0. \tag{3.54}$$

Proof. Using completion of the squares, it can be shown that

$$-\mathbf{A}_\mathrm{d}\mathbf{P}\mathbf{P}\mathbf{A}_\mathrm{d}^\mathsf{T} \le -\mathbf{A}_\mathrm{d}\,(\mathbf{X}\mathbf{P} + \mathbf{P}\mathbf{X})\,\mathbf{A}_\mathrm{d}^\mathsf{T} + \mathbf{A}_\mathrm{d}\mathbf{X}\mathbf{X}\mathbf{A}_\mathrm{d}^\mathsf{T}. \tag{3.55}$$

Substituting (3.55) into (3.53) and using the Schur complement lemma yields (3.54). The matrix inequality in (3.54) is only a sufficient condition for static output feedback stabilizability since (3.55) is an inequality. □

3.14 Strong Stabilizability

3.14.1 Continuous-Time Strong Stabilizability [88]

Consider a continuous-time LTI system, $\mathcal{G} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, with state-space realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{0})$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, and $\mathbf{C} \in \mathbb{R}^{p\times n}$, and it is assumed that $(\mathbf{A}, \mathbf{B})$ is stabilizable, $(\mathbf{A}, \mathbf{C})$ is detectable, and the transfer matrix $\mathbf{G}(s) = \mathbf{C}\,(s\mathbf{1} - \mathbf{A})^{-1}\,\mathbf{B}$ has no poles on the imaginary axis. The system $\mathcal{G}$ is strongly stabilizable if there exist $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{Z} \in \mathbb{R}^{n\times p}$, and $\gamma \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$, such that

$$\mathbf{P}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{Z}\mathbf{C} + \mathbf{C}^\mathsf{T}\mathbf{Z}^\mathsf{T} < 0,$$

$$\begin{bmatrix} \mathbf{P}\,(\mathbf{A} + \mathbf{B}\mathbf{F}) + (\mathbf{A} + \mathbf{B}\mathbf{F})^\mathsf{T}\,\mathbf{P} + \mathbf{Z}\mathbf{C} + \mathbf{C}^\mathsf{T}\mathbf{Z}^\mathsf{T} & -\mathbf{Z} & -\mathbf{X}\mathbf{B} \\ * & -\gamma\mathbf{1} & \mathbf{0} \\ * & * & -\gamma\mathbf{1} \end{bmatrix} < 0,$$

where $\mathbf{F} = -\mathbf{B}^\mathsf{T}\mathbf{X}$ and $\mathbf{X} \in \mathbb{S}_n$, $\mathbf{X} \ge 0$ is the solution to the Lyapunov equation given by

$$\mathbf{X}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{X} - \mathbf{X}\mathbf{B}\mathbf{B}^\mathsf{T}\mathbf{X} = \mathbf{0}.$$

Moreover, a controller that strongly stabilizes $\mathcal{G}$ is given by the state-space realization

$$\begin{aligned} \dot{\mathbf{x}}_c &= \left(\mathbf{A} + \mathbf{B}\mathbf{F} + \mathbf{P}^{-1}\mathbf{Z}\mathbf{C}\right)\mathbf{x} - \mathbf{P}^{-1}\mathbf{Z}\mathbf{u}, \\ \mathbf{y}_c &= -\mathbf{B}^\mathsf{T}\mathbf{X}\mathbf{x}. \end{aligned}$$

3.14.2 Discrete-Time Strong Stabilizability

Consider a discrete-time LTI system, $\mathcal{G} : \ell_{2e} \to \ell_{2e}$, with state-space realization $(\mathbf{A}_{\mathrm{d}}, \mathbf{B}_{\mathrm{d}}, \mathbf{C}_{\mathrm{d}}, \mathbf{0})$, where $\mathbf{A}_{\mathrm{d}} \in \mathbb{R}^{n\times n}$, $\mathbf{B}_{\mathrm{d}} \in \mathbb{R}^{n\times m}$, and $\mathbf{C}_{\mathrm{d}} \in \mathbb{R}^{p\times n}$, and it is assumed that $(\mathbf{A}_{\mathrm{d}}, \mathbf{B}_{\mathrm{d}})$ is stabilizable, $(\mathbf{A}_{\mathrm{d}}, \mathbf{C}_{\mathrm{d}})$ is detectable, and the transfer matrix $\mathbf{G}(z) = \mathbf{C}_{\mathrm{d}}\,(z\mathbf{1} - \mathbf{A}_{\mathrm{d}})^{-1}\,\mathbf{B}_{\mathrm{d}}$ has no poles on the unit circle. The system $\mathcal{G}$ is strongly stabilizable if there exist $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{Z} \in \mathbb{R}^{n\times p}$, and $\gamma \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$, such that

$$\begin{bmatrix} \mathbf{A}_{\mathrm{d}}^{\mathsf{T}}\mathbf{P}\mathbf{A}_{\mathrm{d}} - \mathbf{P} - \mathbf{A}_{\mathrm{d}}^{\mathsf{T}}\mathbf{Z}\mathbf{C}_{\mathrm{d}} - \mathbf{C}_{\mathrm{d}}^{\mathsf{T}}\mathbf{Z}^{\mathsf{T}}\mathbf{A}_{\mathrm{d}} & \mathbf{C}_{\mathrm{d}}^{\mathsf{T}}\mathbf{Z}^{\mathsf{T}} \\ * & -\mathbf{P} \end{bmatrix} < 0, \tag{3.56}$$

$$\begin{bmatrix} \mathbf{N}_{11} & (\mathbf{A}_{\mathrm{d}} + \mathbf{B}_{\mathrm{d}}\mathbf{F})^{\mathsf{T}}\,\mathbf{Z} & \mathbf{X}\mathbf{B}_{\mathrm{d}} & \mathbf{C}_{\mathrm{d}}^{\mathsf{T}}\mathbf{Z}^{\mathsf{T}} \\ * & -\gamma\mathbf{1} & \mathbf{0} & \mathbf{Z}^{\mathsf{T}} \\ * & * & -\gamma\mathbf{1} & \mathbf{0} \\ * & * & * & -\mathbf{P} \end{bmatrix} < 0, \tag{3.57}$$

where $\mathbf{N}_{11} = (\mathbf{A}_{\mathrm{d}} + \mathbf{B}_{\mathrm{d}}\mathbf{F})^{\mathsf{T}}\,\mathbf{P}\,(\mathbf{A}_{\mathrm{d}} + \mathbf{B}_{\mathrm{d}}\mathbf{F}) - \mathbf{P} + (\mathbf{A}_{\mathrm{d}} + \mathbf{B}_{\mathrm{d}}\mathbf{F})^{\mathsf{T}}\,\mathbf{Z}\mathbf{C}_{\mathrm{d}} + \mathbf{C}_{\mathrm{d}}^{\mathsf{T}}\mathbf{Z}^{\mathsf{T}}\,(\mathbf{A}_{\mathrm{d}} + \mathbf{B}_{\mathrm{d}}\mathbf{F})$, $\mathbf{F} = -\mathbf{B}_{\mathrm{d}}^{\mathsf{T}}\mathbf{X}$, $\mathbf{X} = \mathbf{Y}$, and $\mathbf{Y} \in \mathbb{S}_n$, $\mathbf{Y} \geq 0$ is the solution to the discrete-time Lyapunov equation given by

$$\mathbf{A}_{\mathrm{d}}\mathbf{Y}\mathbf{A}_{\mathrm{d}}^{\mathsf{T}} - \mathbf{Y} - \mathbf{B}_{\mathrm{d}}\mathbf{B}_{\mathrm{d}}^{\mathsf{T}} = \mathbf{0}.$$

Moreover, a discrete-time controller that strongly stabilizes $\mathcal{G}$ is given by the state-space realization

$$\mathbf{x}_{c,k+1} = \left(\mathbf{A}_{\mathrm{d}} + \mathbf{B}_{\mathrm{d}}\mathbf{F} + \mathbf{P}^{-1}\mathbf{Z}\mathbf{C}_{\mathrm{d}}\right)\mathbf{x}_k - \mathbf{P}^{-1}\mathbf{Z}\mathbf{u}_k, \tag{3.58}$$

$$\mathbf{y}_{c,k} = -\mathbf{B}_{\mathrm{d}}^{\mathsf{T}}\mathbf{X}\mathbf{x}_k. \tag{3.59}$$

Proof. The proof follows the same procedure as in [88] for the continuous-time case, where (3.56) ensures that the feedback controller defined by (3.58) and (3.59) renders the closed-loop system asymptotically stable and (3.57) ensures that the feedback controller defined by (3.58) and (3.59) has a finite $\mathcal{H}_\infty$ norm, and thus is asymptotically stable. □

3.15 System Zeros

3.15.1 System Zeros without Feedthrough [89]

Consider a continuous-time LTI system, $\mathcal{G} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, with minimal state-space realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{0})$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, and $\mathbf{C} \in \mathbb{R}^{p\times n}$. The transmission zeros of $\mathbf{G}(s) = \mathbf{C}\,(s\mathbf{1} - \mathbf{A})^{-1}\,\mathbf{B}$ are the eigenvalues of $\mathbf{NAM}$, where $\mathbf{N} \in \mathbb{R}^{q\times n}$, $\mathbf{M} \in \mathbb{R}^{n\times q}$, $\mathbf{CM} = \mathbf{0}$, $\mathbf{NB} = \mathbf{0}$, and $\mathbf{NM} = \mathbf{1}$. Therefore, $\mathbf{G}(s)$ is minimum phase if and only if there exists $\mathbf{P} \in \mathbb{S}^q$, where $\mathbf{P} > 0$, such that

$$\mathbf{PNAM} + \mathbf{M}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{N}^{\mathsf{T}}\mathbf{P} < 0.$$

3.15.2 System Zeros with Feedthrough

Consider a continuous-time LTI system, $\mathcal{G} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, with minimal state-space realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, $\mathbf{D} \in \mathbb{R}^{p\times m}$, $m \leq p$, and $\mathbf{D}$ is full rank. The transmission zeros of $\mathbf{G}(s) = \mathbf{C}\,(s\mathbf{1} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}$ are the eigenvalues of $\mathbf{A} - \mathbf{B}\left(\mathbf{D}^{\mathsf{T}}\mathbf{D}\right)^{-1}\mathbf{D}^{\mathsf{T}}\mathbf{C}$. Therefore, $\mathbf{G}(s)$ is minimum phase if and only if there exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\mathbf{P}\left(\mathbf{A} - \mathbf{B}\left(\mathbf{D}^{\mathsf{T}}\mathbf{D}\right)^{-1}\mathbf{D}^{\mathsf{T}}\mathbf{C}\right) + \left(\mathbf{A} - \mathbf{B}\left(\mathbf{D}^{\mathsf{T}}\mathbf{D}\right)^{-1}\mathbf{D}^{\mathsf{T}}\mathbf{C}\right)^{\mathsf{T}}\mathbf{P} < 0. \tag{3.60}$$

If the system is square ($m = p$), then $\mathbf{D}$ full rank implies $\mathbf{D}^{-1}$ exists and (3.60) simplifies to

$$\mathbf{P}\left(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right) + \left(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right)^{\mathsf{T}}\mathbf{P} < 0. \tag{3.61}$$

Proof. The system $\mathcal{G}$ can be written in state-space form as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \tag{3.62}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}. \tag{3.63}$$

Left-multiplying (3.63) by $\mathbf{D}^{\mathsf{T}}$ and rearranging yields

$$\mathbf{D}^{\mathsf{T}}\mathbf{D}\mathbf{u} = -\mathbf{D}^{\mathsf{T}}\mathbf{C}\mathbf{x} + \mathbf{D}^{\mathsf{T}}\mathbf{y}. \tag{3.64}$$

Since $\mathbf{D}$ is full rank, $\left(\mathbf{D}^{\mathsf{T}}\mathbf{D}\right)^{-1}$ exists. Therefore, left-multiplying (3.64) by $\left(\mathbf{D}^{\mathsf{T}}\mathbf{D}\right)^{-1}$ gives

$$\mathbf{u} = -\left(\mathbf{D}^{\mathsf{T}}\mathbf{D}\right)^{-1}\mathbf{D}^{\mathsf{T}}\mathbf{C}\mathbf{x} + \left(\mathbf{D}^{\mathsf{T}}\mathbf{D}\right)^{-1}\mathbf{D}^{\mathsf{T}}\mathbf{y}. \tag{3.65}$$

Substituting (3.65) into (3.62) gives the following state-space representation of the inverted transfer matrix from $\mathbf{y}$ to $\mathbf{u}$.

$$\dot{\mathbf{x}} = \left(\mathbf{A} - \mathbf{B}\left(\mathbf{D}^{\mathsf{T}}\mathbf{D}\right)^{-1}\mathbf{D}^{\mathsf{T}}\mathbf{C}\right)\mathbf{x} + \mathbf{B}\left(\mathbf{D}^{\mathsf{T}}\mathbf{D}\right)^{-1}\mathbf{D}^{\mathsf{T}}\mathbf{y}, \tag{3.66}$$

$$\mathbf{u} = -\left(\mathbf{D}^{\mathsf{T}}\mathbf{D}\right)^{-1}\mathbf{D}^{\mathsf{T}}\mathbf{C}\mathbf{x} + \left(\mathbf{D}^{\mathsf{T}}\mathbf{D}\right)^{-1}\mathbf{D}^{\mathsf{T}}\mathbf{y}. \tag{3.67}$$

The transmission zeros of $\mathbf{G}(s)$ are the poles of the inverted transfer matrix from $\mathbf{y}$ to $\mathbf{u}$, which are the eigenvalues of $\left(\mathbf{A} - \mathbf{B}\left(\mathbf{D}^{\mathsf{T}}\mathbf{D}\right)^{-1}\mathbf{D}^{\mathsf{T}}\mathbf{C}\right)$. Substituting this matrix into a Lyapunov inequality gives the desired inequality in (3.60).

If the system is square and $\mathbf{D}^{-1}$ exists, then $\left(\mathbf{D}^{\mathsf{T}}\mathbf{D}\right)^{-1}\mathbf{D}^{\mathsf{T}} = \mathbf{D}^{-1}$ and (3.60) simplifies to (3.61). □

The transfer matrix $\mathbf{G}(s)$ is also minimum phase if and only if there exist $\mathbf{P} \in \mathbb{S}^n$ and $\mathbf{Q} \in \mathbb{S}^n$, where $\mathbf{P} > 0$ and $\mathbf{Q} = \mathbf{P}^{-1}$, such that

$$\mathbf{M}^{\mathsf{T}}\left(\mathbf{P}\mathbf{A} + \mathbf{A}^{\mathsf{T}}\mathbf{P}\right)\mathbf{M} < 0, \tag{3.68}$$

$$\mathbf{N}\left(\mathbf{A}\mathbf{Q} + \mathbf{Q}\mathbf{A}^{\mathsf{T}}\right)\mathbf{N}^{\mathsf{T}} < 0, \tag{3.69}$$

where $\mathbf{N} \in \mathbb{R}^{q\times n}$, $\mathbf{M} \in \mathbb{R}^{n\times q}$, $\mathcal{R}(\mathbf{N}^{\mathsf{T}}) = \mathcal{N}(\mathbf{B}^{\mathsf{T}})$, and $\mathcal{R}(\mathbf{M}) = \mathcal{N}(\mathbf{C})$.

Proof. Applying the Strict Projection Lemma to (3.60) yields (3.68) and (3.69). □

3.16 $\mathcal{D}$-Stability [90]

Consider $\mathbf{A} \in \mathbb{R}^{n\times n}$. The matrix $\mathbf{A}$ is $\mathcal{D}$-stable if and only if there exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$[\lambda_{kl}\mathbf{P} + \phi_{kl}\mathbf{A}\mathbf{P} + \phi_{lk}\mathbf{P}\mathbf{A}^{\mathsf{T}}]_{1\leq k,l\leq m} < 0,$$

or equivalently

$$\mathbf{\Lambda} \otimes \mathbf{P} + \mathbf{\Phi} \otimes (\mathbf{A}\mathbf{P}) + \mathbf{\Phi}^{\mathsf{T}} \otimes \left(\mathbf{P}\mathbf{A}^{\mathsf{T}}\right) < 0, \tag{3.70}$$

where $\otimes$ is the Kroenecker product. The eigenvalues of a $\mathcal{D}$-stable matrix lie within the LMI region $\mathcal{D}$, which is defined as $\mathcal{D} = \{z \in \mathbb{C} : f_{\mathcal{D}}(z) < 0\}$, where

$$f_{\mathcal{D}}(z) := \mathbf{\Lambda} + z\mathbf{\Phi} + \overline{z}\mathbf{\Phi}^{\mathsf{T}} = [\lambda_{kl} + \phi_{kl}z + \phi_{lk}\overline{z}]_{1\leq k,l\leq m},$$

$\mathbf{\Lambda} \in \mathbb{S}^m$, $\mathbf{\Phi} \in \mathbb{R}^{m\times m}$, and $\overline{z}$ is the complex conjugate of z.

3.16.1 Conic Sector Region Stability via the Dilation Lemma [44]

Consider $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $k \in \mathbb{R}_{>0}$. The matrix $\mathbf{A}$ satisfies $\lambda(\mathbf{A}) \subset \mathcal{P}(k)$, where $\mathcal{P}(k) := \{\lambda \in \mathbb{C} : |\mathrm{Im}(\lambda)| < k\,|\mathrm{Re}(\lambda)|\}$, if and only if there exist $\mathbf{X} \in \mathbb{S}^n$ and $\epsilon \in \mathbb{R}_{>0}$, where $\mathbf{X} > 0$, such that

$$\begin{bmatrix} k\left(\mathbf{AX}+\mathbf{XA}^\mathsf{T}\right) & \mathbf{AX}-\mathbf{XA}^\mathsf{T} \\ * & k\left(\mathbf{AX}+\mathbf{XA}^\mathsf{T}\right) \end{bmatrix} < 0. \tag{3.71}$$

Equivalently, the matrix $\mathbf{A}$ satisfies $\lambda(\mathbf{A}) \subset \mathcal{P}(k)$ if and only if there exist $\mathbf{X} \in \mathbb{S}^n$, $\epsilon \in \mathbb{R}_{>0}$, and $\mathbf{F} \in \mathbb{R}^{n\times n}$, where $\mathbf{X} > 0$, such that

$$\begin{bmatrix} \mathbf{0} & -k\mathbf{X} & \mathbf{X} & \mathbf{0} \\ * & \mathbf{0} & \mathbf{0} & -\mathbf{X} \\ * & * & \mathbf{0} & -k\mathbf{X} \\ * & * & * & \mathbf{0} \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{F} \end{bmatrix} \begin{bmatrix} k\mathbf{1} & -\epsilon k\mathbf{1} & \epsilon\mathbf{1} & \mathbf{1} \\ -\mathbf{1} & -\epsilon\mathbf{1} & \epsilon k\mathbf{1} & k\mathbf{1} \end{bmatrix} \right\} < 0. \tag{3.72}$$

Moreover, for every $\mathbf{X}$ that satisfies (3.71), $\mathbf{X}$ and $\mathbf{F} = -\epsilon^{-1}\left(\mathbf{A} - \epsilon^{-1}\mathbf{1}\right)^{-1}\mathbf{X}$ are solutions to (3.72).

3.16.2 α-Region Stability via the Dilation Lemma [44]

Consider $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\alpha \in \mathbb{R}_{>0}$. The matrix $\mathbf{A}$ satisfies $\lambda(\mathbf{A}) \subset \mathcal{H}(\alpha)$, where $\mathcal{H}(\alpha) := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < -\alpha\}$ if and only if there exist $\mathbf{X} \in \mathbb{S}^n$ and $\epsilon \in \mathbb{R}_{>0}$, where $\mathbf{X} > 0$, such that

$$\mathbf{AX} + \mathbf{XA}^\mathsf{T} + 2\alpha\mathbf{X} < 0. \tag{3.73}$$

Equivalently, the matrix $\mathbf{A}$ satisfies $\lambda(\mathbf{A}) \subset \mathcal{H}(\alpha)$ if and only if there exist $\mathbf{X} \in \mathbb{S}^n$, $\epsilon \in \mathbb{R}_{>0}$, and $\mathbf{F} \in \mathbb{R}^{n\times n}$, where $\mathbf{X} > 0$, such that

$$\begin{bmatrix} \mathbf{0} & -\mathbf{X} & \mathbf{X} \\ * & \mathbf{0} & \mathbf{0} \\ * & * & -\frac{1}{2}\alpha^{-1}\mathbf{X} \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathbf{A} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix} \mathbf{F} \begin{bmatrix} \mathbf{1} & -\epsilon\mathbf{1} & \epsilon\mathbf{1} \end{bmatrix} \right\} < 0. \tag{3.74}$$

Moreover, for every $\mathbf{X}$ that satisfies (3.73), $\mathbf{X}$ and $\mathbf{F} = -\epsilon^{-1}\left(\mathbf{A} - \epsilon^{-1}\mathbf{1}\right)^{-1}\mathbf{X}$ are solutions to (3.74).

3.16.3 Circular Region Stability via the Dilation Lemma [44]

Consider $\mathbf{A} \in \mathbb{R}^{n\times n}$, $r \in \mathbb{R}_{>0}$, and $c \in \mathbb{R}_{<0}$, where $c < -r$. The matrix $\mathbf{A}$ satisfies $\lambda(\mathbf{A}) \subset \mathcal{G}(c, r)$, where $\mathcal{G}(c, r) := \{\lambda \in \mathbb{C} : |\lambda - c| < r\}$, if and only if there exist $\mathbf{X} \in \mathbb{S}^n$ and $\epsilon \in \mathbb{R}_{>0}$, where $\mathbf{X} > 0$, such that

$$\mathbf{AX} + \mathbf{XA}^\mathsf{T} - \frac{c^2 - r^2}{c}\mathbf{X} - \frac{1}{c}\mathbf{AXA}^\mathsf{T} < 0. \tag{3.75}$$

Equivalently, the matrix $\mathbf{A}$ satisfies $\lambda(\mathbf{A}) \subset \mathcal{G}(c, r)$ if and only if there exist $\mathbf{X} \in \mathbb{S}^n$, $\epsilon \in \mathbb{R}_{>0}$, and $\mathbf{F} \in \mathbb{R}^{n\times n}$, where $\mathbf{X} > 0$, such that

$$\begin{bmatrix} \mathbf{0} & -\mathbf{X} & \mathbf{X} & \mathbf{0} \\ * & \mathbf{0} & \mathbf{0} & -\mathbf{X} \\ * & * & \frac{c}{c^2 - r^2}\mathbf{X} & \mathbf{0} \\ * & * & * & c\mathbf{X} \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathbf{A} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{F} \begin{bmatrix} \mathbf{1} & -\epsilon\mathbf{1} & \epsilon\mathbf{1} & \mathbf{1} \end{bmatrix} \right\} < 0. \tag{3.76}$$

Moreover, for every $\mathbf{X}$ that satisfies (3.75), $\mathbf{X}$ and $\mathbf{F} = -\epsilon^{-1}\left(\mathbf{A} - \epsilon^{-1}\mathbf{1}\right)^{-1}\mathbf{X}$ are solutions to (3.76).

3.17 DC Gain of a Transfer Matrix

Consider $\gamma \in \mathbb{R}_{>0}$ and a continuous-time LTI system, $\boldsymbol{\mathcal{G}} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, with transfer matrix $\mathbf{G}(s) = \mathbf{C}\left(s\mathbf{1} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, and $\mathbf{D} \in \mathbb{R}^{p\times m}$. The DC gain of $\boldsymbol{\mathcal{G}}$ is strictly less than γ (i.e., $\bar{\sigma}(\mathbf{G}(0)) < \gamma$) if and only if

$$\begin{bmatrix} \gamma\mathbf{1} & -\mathbf{C}\mathbf{A}^{-1}\mathbf{B} + \mathbf{D} \\ * & \gamma\mathbf{1} \end{bmatrix} > 0, \tag{3.77}$$

or

$$\begin{bmatrix} \gamma\mathbf{1} & -\mathbf{B}^\mathsf{T}\mathbf{A}^{-\mathsf{T}}\mathbf{C}^\mathsf{T} + \mathbf{D}^\mathsf{T} \\ * & \gamma\mathbf{1} \end{bmatrix} > 0. \tag{3.78}$$

Proof. $\bar{\sigma}(\mathbf{G}(0)) < \gamma$ if and only if $\bar{\lambda}\left(\mathbf{G}(0)\mathbf{G}^\mathsf{T}(0)\right) < \gamma^2$, or equivalently

$$\begin{aligned} \mathbf{G}(0)\mathbf{G}^\mathsf{T}(0) - \gamma^2\mathbf{1} &< 0 \\ \mathbf{G}(0)(-\gamma^{-1}\mathbf{1})\mathbf{G}^\mathsf{T}(0) - \gamma\mathbf{1} &< 0 \\ \gamma\mathbf{1} - \mathbf{G}(0)(\gamma^{-1}\mathbf{1})\mathbf{G}^\mathsf{T}(0) &> 0 \\ \begin{bmatrix} \gamma\mathbf{1} & \mathbf{G}(0) \\ * & \gamma\mathbf{1} \end{bmatrix} &> 0. \end{aligned} \tag{3.79}$$

Substituting $\mathbf{G}(0) = -\mathbf{C}\mathbf{A}^{-1}\mathbf{B} + \mathbf{D}$ into (3.79) gives (3.77). Starting with $\bar{\sigma}(\mathbf{G}(0)) < \gamma \iff \bar{\lambda}\left(\mathbf{G}^\mathsf{T}(0)\mathbf{G}(0)\right) < \gamma^2$ in the first step of the proof and following the same steps yields (3.78). $\square$

3.18 Kharitonov-Bernstein-Haddad (KBH) Theorem [91]

Consider the set of matrices

$$\boldsymbol{\mathcal{A}} = \left\{ \mathbf{A} = \begin{bmatrix} \mathbf{0}_{(n-1)\times 1} & & \mathbf{1}_{(n-1)\times(n-1)} \\ -a_0 & \cdots & -a_{n-1} \end{bmatrix} \;\middle|\; \underline{a}_j \leq a_j \leq \bar{a}_j, \quad j = 0, 1, 2, \ldots, n-1 \right\}. \tag{3.80}$$

Every matrix in the set $\boldsymbol{\mathcal{A}}$ is Hurwitz if and only if there exist $\mathbf{P}_i \in \mathbb{S}^n$, $i = 1, 2, 3, 4$, where $\mathbf{P}_i > 0$, $i = 1, 2, 3, 4$, such that

$$\mathbf{P}_i\mathbf{A}_i + \mathbf{A}_i^\mathsf{T}\mathbf{P}_i < 0, \quad i = 1, 2, 3, 4,$$

where

$$\begin{aligned} \mathbf{A}_i &= \begin{bmatrix} \begin{bmatrix} \mathbf{0}_{(n-1)\times 1} & \mathbf{1}_{(n-1)\times(n-1)} \end{bmatrix} \\ \mathbf{a}_i \end{bmatrix}, \quad i = 1, 2, 3, 4, \\ \mathbf{a}_1 &= -\begin{bmatrix} \underline{a}_0 & \underline{a}_1 & \bar{a}_2 & \bar{a}_3 & \cdots & \underline{a}_{n-4} & \underline{a}_{n-3} & \bar{a}_{n-2} & \bar{a}_{n-1} \end{bmatrix}, \\ \mathbf{a}_2 &= -\begin{bmatrix} \underline{a}_0 & \bar{a}_1 & \bar{a}_2 & \underline{a}_3 & \cdots & \underline{a}_{n-4} & \bar{a}_{n-3} & \bar{a}_{n-2} & \underline{a}_{n-1} \end{bmatrix}, \\ \mathbf{a}_3 &= -\begin{bmatrix} \bar{a}_0 & \underline{a}_1 & \underline{a}_2 & \bar{a}_3 & \cdots & \bar{a}_{n-4} & \underline{a}_{n-3} & \underline{a}_{n-2} & \bar{a}_{n-1} \end{bmatrix}, \\ \mathbf{a}_4 &= -\begin{bmatrix} \bar{a}_0 & \bar{a}_1 & \underline{a}_2 & \underline{a}_3 & \cdots & \bar{a}_{n-4} & \bar{a}_{n-3} & \underline{a}_{n-2} & \underline{a}_{n-1} \end{bmatrix}. \end{aligned}$$

Equivalently, every matrix in the set $\boldsymbol{\mathcal{A}}$ is Hurwitz if and only if there exist $\mathbf{Q}_i \in \mathbb{S}^n$, $i = 1, 2, 3, 4$, where $\mathbf{Q}_i > 0$, $i = 1, 2, 3, 4$, such that

$$\mathbf{A}_i\mathbf{Q}_i + \mathbf{Q}_i\mathbf{A}_i^\mathsf{T} < 0, \quad i = 1, 2, 3, 4.$$

3.19 Stability of Discrete-Time System with Polytopic Uncertainty

3.19.1 Open-Loop Robust Stability [49]

Consider the set of matrices

$$\boldsymbol{\mathcal{A}} = \left\{ \mathbf{A}_{\mathrm{d}}(\alpha) \in \mathbb{R}^{n\times n} \mid \mathbf{A}_{\mathrm{d}}(\alpha) = \sum_{i=1}^{n} \alpha_i \mathbf{A}_{\mathrm{d},i},\ \mathbf{A}_{\mathrm{d},i} \in \mathbb{R}^{n\times n},\ \alpha_i \in \mathbb{R}_{\geq 0},\ \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

The discrete-time LTI system $\mathbf{x}_{k+1} = \mathbf{A}_{\mathrm{d}}(\alpha)\mathbf{x}_k$ is asymptotically stable for all $\mathbf{A}_{\mathrm{d}}(\alpha) \in \boldsymbol{\mathcal{A}}$ if there exist $\mathbf{P}_i \in \mathbb{S}^n$, $i = 1, \ldots, n$, and $\mathbf{G} \in \mathbb{R}^{n\times n}$, where $\mathbf{P}_i > 0$, $i = 1, \ldots, n$, such that

$$\begin{bmatrix} \mathbf{P}_i & \mathbf{A}_{\mathrm{d},i}^{\mathsf{T}}\mathbf{G}^{\mathsf{T}} \\ * & \mathbf{G} + \mathbf{G}^{\mathsf{T}} - \mathbf{P}_i \end{bmatrix} < 0, \quad i = 1, \ldots, n.$$

3.19.2 Closed-Loop Robust Stability [49]

Consider the set of matrices

$$\boldsymbol{\mathcal{A}} = \left\{ \mathbf{A}_{\mathrm{d}}(\alpha) \in \mathbb{R}^{n\times n} \mid \mathbf{A}_{\mathrm{d}}(\alpha) = \sum_{i=1}^{n} \alpha_i \mathbf{A}_{\mathrm{d},i},\ \mathbf{A}_{\mathrm{d},i} \in \mathbb{R}^{n\times n},\ \alpha_i \in \mathbb{R}_{\geq 0},\ \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

and

$$\boldsymbol{\mathcal{B}} = \left\{ \mathbf{B}_{\mathrm{d}}(\beta) \in \mathbb{R}^{n\times m} \mid \mathbf{B}_{\mathrm{d}}(\beta) = \sum_{i=1}^{p} \beta_i \mathbf{B}_{\mathrm{d},i}, \mathbf{B}_{\mathrm{d},i} \in \mathbb{R}^{n\times m},\ \beta_i \in \mathbb{R}_{\geq 0},\ \sum_{i=1}^{m} \beta_i = 1 \right\}.$$

The discrete-time LTI system $\mathbf{x}_{k+1} = \mathbf{A}_{\mathrm{d}}(\alpha)\mathbf{x}_k + \mathbf{B}_{\mathrm{d}}(\beta)\mathbf{u}_k$ is asymptotically stabilized by the state feedback control law $\mathbf{u}_k = -\mathbf{L}\mathbf{G}^{-1}\mathbf{u}_k$ for all $\mathbf{A}_{\mathrm{d}}(\alpha) \in \boldsymbol{\mathcal{A}}$ and $\mathbf{B}_{\mathrm{d}}(\alpha) \in \boldsymbol{\mathcal{B}}$ if there exist $\mathbf{P}_{ij} \in \mathbb{S}^n$, $i = 1, \ldots, n$, $j = 1, \ldots, p$, $\mathbf{G} \in \mathbb{R}^{n\times n}$, and $\mathbf{L} \in \mathbb{R}^{m\times n}$, where $\mathbf{P}_{ij} > 0$, $i = 1, \ldots, n$, $j = 1, \ldots, p$ and $\mathbf{G}$ is invertible, such that

$$\begin{bmatrix} \mathbf{P}_{ij} & \mathbf{A}_{\mathrm{d},i}\mathbf{G} - \mathbf{B}_{\mathrm{d},j}\mathbf{L} \\ * & \mathbf{G} + \mathbf{G}^{\mathsf{T}} - \mathbf{P}_{ij} \end{bmatrix} < 0, \quad i = 1, \ldots, n, \quad j = 1, \ldots, p.$$

3.20 Quadratic Stability

3.20.1 Continuous-Time Quadratic Stability [19, pp. 112–115]

Consider the uncertain continuous-time linear system with state-space representation

$$\dot{\mathbf{x}} = \left(\mathbf{A}_0 + \Delta\mathbf{A}(\boldsymbol{\delta}(t))\right)\mathbf{x}, \tag{3.81}$$

where $\mathbf{A}_0 \in \mathbb{R}^{n\times n}$, $\Delta\mathbf{A}(\boldsymbol{\delta}(t)) = \sum_{i=1}^{k} \delta_i(t)\mathbf{A}_i \in \mathbb{R}^{n\times n}$, $\delta_i \in \mathbb{R}$, $i = 1, \ldots, k$, $\mathbf{A}_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, k$, $\boldsymbol{\delta}^{\mathsf{T}}(t) = \begin{bmatrix} \delta_1(t) & \delta_2(t) & \cdots & \delta_k(t) \end{bmatrix} \in \boldsymbol{\Delta}$, and $\boldsymbol{\Delta}$ is the set of perturbation parameters. The uncertain system in (3.81) is quadratically stable if there exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$\left(\mathbf{A}_0 + \Delta\mathbf{A}(\boldsymbol{\delta}(t))\right)^{\mathsf{T}} \mathbf{P} + \mathbf{P}\left(\mathbf{A}_0 + \Delta\mathbf{A}(\boldsymbol{\delta}(t))\right) < 0, \quad \forall \boldsymbol{\delta}(t) \in \boldsymbol{\Delta}.$$

The following statements can be made for particular sets of perturbations.

1. Consider the case where the set of perturbation parameters is defined by a regular polyhedron as

$$\boldsymbol{\Delta} = \{\boldsymbol{\delta}(t) = \begin{bmatrix}\delta_1(t) & \delta_2(t) & \cdots & \delta_k(t)\end{bmatrix} \in \mathbb{R}^k \mid \delta_i(t),\, \underline{\delta}_i,\, \bar{\delta}_i \in \mathbb{R},\ \underline{\delta}_i \le \delta_i(t) \le \bar{\delta}_i]\}.$$

The uncertain system in (3.81) is quadratically stable if and only if there exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$(\mathbf{A}_0 + \Delta\mathbf{A}(\boldsymbol{\delta}(t)))^{\mathsf{T}}\,\mathbf{P} + \mathbf{P}\,(\mathbf{A}_0 + \Delta\mathbf{A}(\boldsymbol{\delta}(t))) < 0, \quad \forall \delta_i(t) \in \{\underline{\delta}_i, \bar{\delta}_i\},\ i = 1, \ldots, k.$$

2. Consider the case where the set of perturbation parameters is defined by a polytope as

$$\boldsymbol{\Delta} = \{\boldsymbol{\delta}(t) = \begin{bmatrix}\delta_1(t) & \delta_2(t) & \cdots & \delta_k(t)\end{bmatrix} \in \mathbb{R}^k \mid \delta_i(t) \in \mathbb{R}_{\ge 0},\ \sum_{i=1}^{k} \delta_i(t) = 1\}.$$

The uncertain system in (3.81) is quadratically stable if and only if there exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$(\mathbf{A}_0 + \mathbf{A}_i)^{\mathsf{T}}\,\mathbf{P} + \mathbf{P}\,(\mathbf{A}_0 + \mathbf{A}_i) < 0, \quad i = 1, \ldots, k.$$

3.20.2 Discrete-Time Quadratic Stability [19, pp. 116–118]

Consider the uncertain discrete-time linear system with state-space representation

$$\mathbf{x}_{k+1} = (\mathbf{A}_{\mathrm{d},0} + \Delta\mathbf{A}_{\mathrm{d}}(\boldsymbol{\delta}(t)))\,\mathbf{x}_k, \tag{3.82}$$

where $\mathbf{A}_{\mathrm{d},0} \in \mathbb{R}^{n\times n}$, $\Delta\mathbf{A}_{\mathrm{d}}(\boldsymbol{\delta}(t)) = \sum_{i=1}^{k} \delta_i(t)\mathbf{A}_{\mathrm{d},i} \in \mathbb{R}^{n\times n}$, $\delta_i \in \mathbb{R}$, $i = 1, \ldots, k$, $\mathbf{A}_{\mathrm{d},i} \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, k$, $\boldsymbol{\delta}^{\mathsf{T}}(t) = \begin{bmatrix}\delta_1(t) & \delta_2(t) & \cdots & \delta_k(t)\end{bmatrix} \in \boldsymbol{\Delta}$, and $\boldsymbol{\Delta}$ is the set of perturbation parameters. The uncertain system in (3.81) is quadratically stable if there exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$(\mathbf{A}_{\mathrm{d},0} + \Delta\mathbf{A}_{\mathrm{d}}(\boldsymbol{\delta}(t)))^{\mathsf{T}}\,\mathbf{P}\,(\mathbf{A}_{\mathrm{d},0} + \Delta\mathbf{A}_{\mathrm{d}}(\boldsymbol{\delta}(t))) - \mathbf{P} < 0, \quad \forall \boldsymbol{\delta}(t) \in \boldsymbol{\Delta}.$$

The following statements can be made for particular sets of perturbations.

1. Consider the case where the set of perturbation parameters is defined by a regular polyhedron as

$$\boldsymbol{\Delta} = \{\boldsymbol{\delta}(t) = \begin{bmatrix}\delta_1(t) & \delta_2(t) & \cdots & \delta_k(t)\end{bmatrix} \in \mathbb{R}^k \mid \delta_i(t),\, \underline{\delta}_i,\, \bar{\delta}_i \in \mathbb{R},\ \underline{\delta}_i \le \delta_i(t) \le \bar{\delta}_i]\}.$$

The uncertain system in (3.81) is quadratically stable if and only if there exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$(\mathbf{A}_{\mathrm{d},0} + \Delta\mathbf{A}_{\mathrm{d}}(\boldsymbol{\delta}(t)))^{\mathsf{T}}\,\mathbf{P}\,(\mathbf{A}_{\mathrm{d},0} + \Delta\mathbf{A}_{\mathrm{d}}(\boldsymbol{\delta}(t))) - \mathbf{P} < 0, \quad \forall \delta_i(t) \in \{\underline{\delta}_i, \bar{\delta}_i\},\ i = 1, 2, \ldots, k.$$

2. Consider the case where the set of perturbation parameters is defined by a polytope as

$$\boldsymbol{\Delta} = \{\boldsymbol{\delta}(t) = \begin{bmatrix}\delta_1(t) & \delta_2(t) & \cdots & \delta_k(t)\end{bmatrix} \in \mathbb{R}^k \mid \delta_i(t) \in \mathbb{R}_{\ge 0},\ \sum_{i=1}^{k} \delta_i(t) = 1\}.$$

The uncertain system in (3.81) is quadratically stable if and only if there exists $\mathbf{P} \in \mathbb{S}^n$, where $\mathbf{P} > 0$, such that

$$(\mathbf{A}_{\mathrm{d},0} + \mathbf{A}_{\mathrm{d},i})^{\mathsf{T}}\,\mathbf{P}\,(\mathbf{A}_{\mathrm{d},0} + \mathbf{A}_{\mathrm{d},i}) - \mathbf{P} < 0, \quad i = 1, 2, \ldots, k.$$

3.21 Stability of Time-Delay Systems

Consider the continuous-time linear time-delay system with state-space representation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{A}_{\mathrm{d}}\mathbf{x}(t-d), \tag{3.83}$$

where $\mathbf{A}$, $\mathbf{A}_{\mathrm{d}} \in \mathbb{R}^{n\times n}$, d, $\bar{d} \in \mathbb{R}_{>0}$, and the initial condition is given by $\mathbf{x}(t) = \boldsymbol{\phi}(t)$, $t \in [-d, 0]$, where $\bar{d}$ is a known upper-bound on the time-delay (i.e., $0 < d \leq \bar{d}$).

3.21.1 Delay-Independent Condition [19, p. 126]

The time-delay system in (3.83) is asymptotically stable if there exist $\mathbf{P}$, $\mathbf{S} \in \mathbb{S}^n$, where $\mathbf{P} > 0$ and $\mathbf{S} > 0$, such that

$$\begin{bmatrix} \mathbf{A}^{\mathsf{T}}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{S} & \mathbf{P}\mathbf{A}_{\mathrm{d}} \\ * & -\mathbf{S} \end{bmatrix} < 0.$$

3.21.2 Delay-Dependent Condition [19, pp. 128–129]

The time-delay system in (3.83) is uniformly asymptotically stable if there exists $\mathbf{X} \in \mathbb{S}^n$ and $\beta \in \mathbb{R}_{>0}$, where $\mathbf{X} > 0$ and $\beta < 1$, such that

$$\begin{bmatrix} \mathbf{X}\left(\mathbf{A}+\mathbf{A}_{\mathrm{d}}\right)^{\mathsf{T}} + \left(\mathbf{A}+\mathbf{A}_{\mathrm{d}}\right)\mathbf{X} + \bar{d}\mathbf{A}_{\mathrm{d}}\mathbf{A}_{\mathrm{d}}^{\mathsf{T}} & \bar{d}\mathbf{X}\mathbf{A}^{\mathsf{T}} & \bar{d}\mathbf{X}\mathbf{A}_{\mathrm{d}}^{\mathsf{T}} \\ * & -\bar{d}\beta\mathbf{1} & \mathbf{0} \\ * & * & -\bar{d}(1-\beta)\mathbf{1} \end{bmatrix} < 0.$$

3.22 μ-Analysis [3, p. 38–39], [92]

Consider the matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ and the invertible matrix $\mathbf{D} \in \mathbb{R}^{n\times n}$. The inequality $\bar{\sigma}\left(\mathbf{D}\mathbf{A}\mathbf{D}^{-1}\right) < \gamma$ holds if and only if there exist $\mathbf{X} \in \mathbb{S}^n$ and $\gamma \in \mathbb{R}_{>0}$, where $\mathbf{X} > 0$, satisfying

$$\mathbf{A}^{\mathsf{T}}\mathbf{X}\mathbf{A} - \gamma^2\mathbf{X} < 0. \tag{3.84}$$

The inequality $\bar{\sigma}\left(\mathbf{D}\mathbf{A}\mathbf{D}^{-1}\right) < \gamma$ holds for $\mathbf{D} = \mathbf{X}^{\frac{1}{2}}$, where $\mathbf{X}$ satisfies (3.84).

3.23 Static Output Feedback Algebraic Loop [2, p. 1284], [80, pp. 39–40]

Consider a continuous-time LTI system, $\boldsymbol{\mathcal{G}} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, with state-space realization

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{w} + \mathbf{B}_2\mathbf{u}, \tag{3.85}$$
$$\mathbf{z} = \mathbf{C}_1\mathbf{x} + \mathbf{D}_{11}\mathbf{w} + \mathbf{D}_{12}\mathbf{u}, \tag{3.86}$$
$$\mathbf{y} = \mathbf{C}_2\mathbf{x} + \mathbf{D}_{21}\mathbf{w} + \mathbf{D}_{22}\mathbf{u},$$

where $\mathbf{x}(t) \in \mathbb{R}^{n_x}$ is the system state, $\mathbf{z}(t) \in \mathbb{R}^{n_z}$ is the performance signal, $\mathbf{y}(t) \in \mathbb{R}^{n_y}$ is the measurement signal, $\mathbf{w}(t) \in \mathbb{R}^{n_w}$ is the exogenous signal, $\mathbf{u}(t) \in \mathbb{R}^{n_u}$ is the control input signal, and the state-space matrices are real matrices with appropriate dimensions. Additionally, consider a static output feedback controller of the form $\mathbf{u} = \mathbf{K}\mathbf{y}$, where $\mathbf{K} \in \mathbb{R}^{n_u\times n_y}$ and it is assumed that the feedback interconnection is well-posed, that is, $\det(\mathbf{1} - \mathbf{K}\mathbf{D}_{22}) \neq 0$. The closed-loop system can be described by the following state-space realization.

$$\dot{\mathbf{x}} = \left(\mathbf{A} + \mathbf{B}_2\bar{\mathbf{K}}\mathbf{C}_2\right)\mathbf{x} + \left(\mathbf{B}_1 + \mathbf{B}_2\bar{\mathbf{K}}\mathbf{D}_{21}\right)\mathbf{w}, \tag{3.87}$$
$$\mathbf{z} = \left(\mathbf{C}_1 + \mathbf{D}_{12}\bar{\mathbf{K}}\mathbf{C}_2\right)\mathbf{x} + \left(\mathbf{D}_{11} + \mathbf{D}_{12}\bar{\mathbf{K}}\mathbf{D}_{21}\right)\mathbf{w}, \tag{3.88}$$

where $\bar{\mathbf{K}} = (\mathbf{1} - \mathbf{K}\mathbf{D}_{22})^{-1}\,\mathbf{K}$.

The change of variable $\bar{\mathbf{K}} = (\mathbf{1} - \mathbf{K}\mathbf{D}_{22})^{-1}\,\mathbf{K}$ allows for the simplification of matrix inequalities involving the closed-loop system.

Proof. Substituting the expression for $\mathbf{y}$ into $\mathbf{u} = \mathbf{K}\mathbf{y}$ gives

$$\mathbf{u} = \mathbf{K}\left(\mathbf{C}_2\mathbf{x} + \mathbf{D}_{21}\mathbf{w} + \mathbf{D}_{22}\mathbf{u}\right).$$

Bringing the terms with $\mathbf{u}$ to the left-hand-side of the equation, left-multiplying by $(\mathbf{1} - \mathbf{K}\mathbf{D}_{22})^{-1}$, and defining $\bar{\mathbf{K}} = (\mathbf{1} - \mathbf{K}\mathbf{D}_{22})^{-1}\,\mathbf{K}$ yields

$$\begin{aligned}
(\mathbf{1} - \mathbf{K}\mathbf{D}_{22})\,\mathbf{u} &= \mathbf{K}\mathbf{C}_2\mathbf{x} + \mathbf{K}\mathbf{D}_{21}\mathbf{w} \\
\mathbf{u} &= (\mathbf{1} - \mathbf{K}\mathbf{D}_{22})^{-1}\,\mathbf{K}\mathbf{C}_2\mathbf{x} + (\mathbf{1} - \mathbf{K}\mathbf{D}_{22})^{-1}\,\mathbf{K}\mathbf{D}_{21}\mathbf{w} \\
\mathbf{u} &= \bar{\mathbf{K}}\mathbf{C}_2\mathbf{x} + \bar{\mathbf{K}}\mathbf{D}_{21}\mathbf{w}.
\end{aligned} \tag{3.89}$$

Substituting (3.89) into (3.85) and (3.86) gives (3.87) and (3.88). □

4 LMIs in Optimal Control

This section presents controller synthesis methods using LMIs for a number of well-known optimal control problems. The derivation of the LMIs used for controller synthesis is provided in some cases, while longer derivations can be found in the cited references.

4.1 The Generalized Plant

4.1.1 The Continuous-Time Generalized Plant

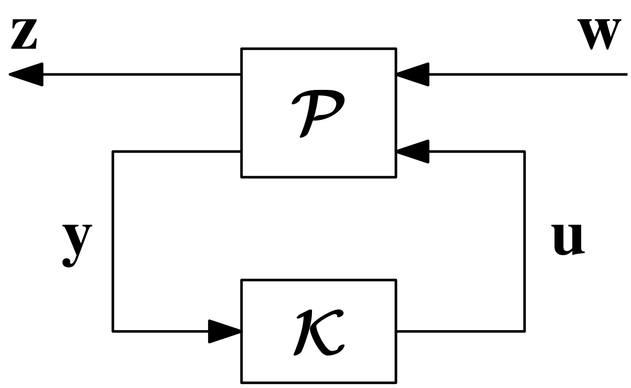

Figure 1: Block diagram of the generalized plant $\mathcal{P}$ with the controller $\mathcal{K}$.

Consider the generalized LTI plant $\mathcal{P} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, shown in Figure 1, with a minimal state-space realization [2, pp. 1291–1292], [17, pp. 104–114], [93, p. 141], [94, pp. 14–16]

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{w} + \mathbf{B}_2\mathbf{u},\\ \mathbf{z} &= \mathbf{C}_1\mathbf{x} + \mathbf{D}_{11}\mathbf{w} + \mathbf{D}_{12}\mathbf{u},\\ \mathbf{y} &= \mathbf{C}_2\mathbf{x} + \mathbf{D}_{21}\mathbf{w} + \mathbf{D}_{22}\mathbf{u},\end{aligned}$$

where $\mathbf{x}(t) \in \mathbb{R}^{n_x}$ is the system state, $\mathbf{z}(t) \in \mathbb{R}^{n_z}$ is the performance signal, $\mathbf{y}(t) \in \mathbb{R}^{n_y}$ is the measurement signal, $\mathbf{w}(t) \in \mathbb{R}^{n_w}$ is the exogenous signal, $\mathbf{u}(t) \in \mathbb{R}^{n_u}$ is the control input signal, and the state-space matrices are real matrices with appropriate dimensions. The generalized LTI plant can also be written in transfer matrix form as

$$\begin{bmatrix}\mathbf{z}(s)\\ \mathbf{y}(s)\end{bmatrix} = \mathbf{P}(s)\begin{bmatrix}\mathbf{w}(s)\\ \mathbf{u}(s)\end{bmatrix},$$

where the transfer matrix $\mathbf{P}(s) \in \mathbb{C}^{(n_z+n_y)\times(n_w+n_u)}$ is partitioned as

$$\mathbf{P}(s) = \begin{bmatrix}\mathbf{P}_{zw}(s) & \mathbf{P}_{zu}(s)\\ \mathbf{P}_{yw}(s) & \mathbf{P}_{yu}(s)\end{bmatrix} = \begin{bmatrix}\mathbf{C}_1\left(s\mathbf{1}-\mathbf{A}\right)^{-1}\mathbf{B}_1 + \mathbf{D}_{11} & \mathbf{C}_1\left(s\mathbf{1}-\mathbf{A}\right)^{-1}\mathbf{B}_2 + \mathbf{D}_{12}\\ \mathbf{C}_2\left(s\mathbf{1}-\mathbf{A}\right)^{-1}\mathbf{B}_1 + \mathbf{D}_{21} & \mathbf{C}_2\left(s\mathbf{1}-\mathbf{A}\right)^{-1}\mathbf{B}_2 + \mathbf{D}_{22}\end{bmatrix}.$$

The generalized plant, also known as the standard control problem in [2, pp. 1291–1292], [94, pp. 14–16], [95], is useful, as it is possible to represent a number of LTI systems in this form, as shown in the following example.

Example 4.1 (Basic Servo Loop Tracking [80, p. 18], [94, p. 18], [95])**.** Consider the basic servo loop shown in Figure 2 involving the LTI controller $\mathbf{K}(s) \in \mathbb{C}^{n_{yc}\times n_{uc}}$ and the plant $\mathbf{G}_p(s) \in$

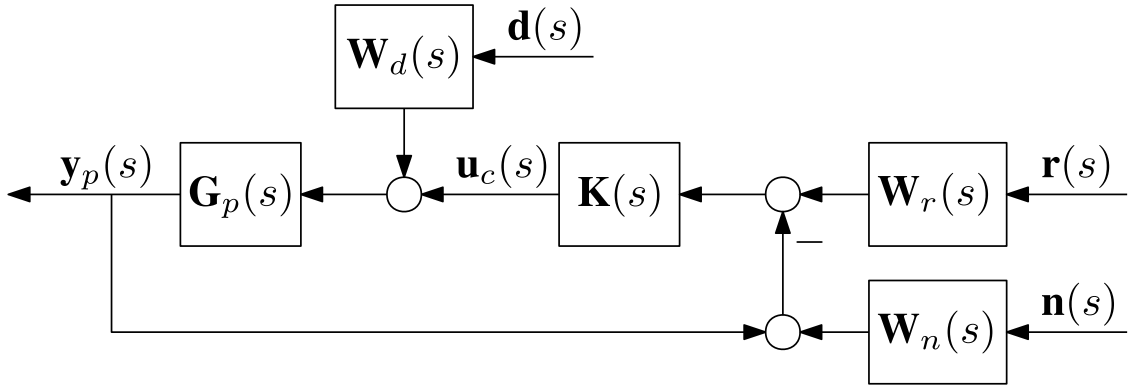

Figure 2: Block diagram of the basic servo loop with plant $\mathbf{G}_p(s)$, controller $\mathbf{K}(s)$, and weighting transfer matrices $\mathbf{W}_r(s)$, $\mathbf{W}_d(s)$, and $\mathbf{W}_n(s)$.

$\mathbb{C}^{n_{y_p} \times n_{u_p}}$, where the weighting transfer matrices are simply chosen as $\mathbf{W}_r(s) = \mathbf{1}$, $\mathbf{W}_d(s) = \mathbf{1}$, and $\mathbf{W}_n(s) = \mathbf{1}$. The plant $\mathbf{G}_p(s)$ has a minimal state-space realization $(\mathbf{A}_p, \mathbf{B}_p, \mathbf{C}_p, \mathbf{D}_p)$ and the state $\mathbf{x}_p(t)$. The performance variables are the true tracking error $\mathbf{z}_1(t) = \mathbf{e}(t) = \mathbf{r}(t) - \mathbf{y}_p(t)$ and the control effort $\mathbf{z}_2(t) = \mathbf{u}_c(t)$, where $\mathbf{z}^\mathsf{T}(t) = \begin{bmatrix}\mathbf{z}_1^\mathsf{T}(t) & \mathbf{z}_2^\mathsf{T}(t)\end{bmatrix}$. The generalized plant can be formulated with minimal state-space representation

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}_p\mathbf{x} + \begin{bmatrix}\mathbf{0} & \mathbf{B}_p & \mathbf{0}\end{bmatrix}\mathbf{w} + \mathbf{B}_p\mathbf{u}, \\
\mathbf{z} &= \begin{bmatrix}-\mathbf{C}_p \\ \mathbf{0}\end{bmatrix}\mathbf{x} + \begin{bmatrix}\mathbf{1} & -\mathbf{D}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{w} + \begin{bmatrix}-\mathbf{D}_p \\ \mathbf{1}\end{bmatrix}\mathbf{u}, \\
\mathbf{y} &= -\mathbf{C}_p\mathbf{x} + \begin{bmatrix}\mathbf{1} & -\mathbf{D}_p & -\mathbf{1}\end{bmatrix}\mathbf{w} - \mathbf{D}_p\mathbf{u},
\end{aligned}
$$

where $\mathbf{x}(t) = \mathbf{x}_p(t)$, $\mathbf{w}^\mathsf{T}(t) = \begin{bmatrix}\mathbf{r}^\mathsf{T}(t) & \mathbf{d}^\mathsf{T}(t) & \mathbf{n}^\mathsf{T}(t)\end{bmatrix}$, $\mathbf{u}(t) = \mathbf{u}_c(t)$, and $\mathbf{y}(t) = \mathbf{r}(t) - \mathbf{y}_p(t) - \mathbf{n}(t)$.

Example 4.2 (Basic Servo Loop Tracking with Weights [17, pp. 362–363], [80, p. 19], [96, pp. 169–170])**.** Consider the same basic servo loop shown in Figure 2 involving the LTI controller $\mathbf{K}(s) \in \mathbb{C}^{n_{y_c} \times n_{u_c}}$, the plant $\mathbf{G}_p(s) \in \mathbb{C}^{n_{y_p} \times n_{u_p}}$, and the weighting transfer matrices $\mathbf{W}_r(s) \in \mathbb{C}^{n_r \times n_r}$, $\mathbf{W}_d(s) \in \mathbb{C}^{n_d \times n_d}$, and $\mathbf{W}_n(s) \in \mathbb{C}^{n_n \times n_n}$. The plant $\mathbf{G}_p(s)$ has a minimal state-space realization $(\mathbf{A}_p, \mathbf{B}_p, \mathbf{C}_p, \mathbf{D}_p)$ and the weighting transfer matrices $\mathbf{W}_r(s)$, $\mathbf{W}_d(s)$, and $\mathbf{W}_n(s)$ have minimal state-space realizations $(\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{D}_r)$, $(\mathbf{A}_d, \mathbf{B}_d, \mathbf{C}_d, \mathbf{D}_d)$, and $(\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n)$, respectively. The performance variable is defined as the weighted true tracking error $\mathbf{z}_1(s) = \mathbf{W}_e(s)\mathbf{e}(s) = \mathbf{W}_e(s)\left(\mathbf{W}_r(s)\mathbf{r}(s) - \mathbf{y}_p(s)\right)$ and the weighted control effort $\mathbf{z}_2(s) = \mathbf{W}_u(s)\mathbf{u}_c(s)$, where $\mathbf{z}^\mathsf{T}(s) = \begin{bmatrix}\mathbf{z}_1^\mathsf{T}(s) & \mathbf{z}_2^\mathsf{T}(s)\end{bmatrix}$ and $\mathbf{W}_e(s) \in \mathbb{C}^{n_e \times n_e}$, $\mathbf{W}_u(s) \in \mathbb{C}^{n_u \times n_u}$ are weighting transfer matrices with minimal state-space realizations $(\mathbf{A}_e, \mathbf{B}_e, \mathbf{C}_e, \mathbf{D}_e)$ and $(\mathbf{A}_u, \mathbf{B}_u, \mathbf{C}_u, \mathbf{D}_u)$, respectively. The generalized plant can be formulated with minimal state-space representation

$$
\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix}
\mathbf{A}_p & \mathbf{0} & \mathbf{B}_p\mathbf{C}_d & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{A}_r & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{A}_d & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_n & \mathbf{0} & \mathbf{0} \\
-\mathbf{B}_e\mathbf{C}_p & \mathbf{B}_e\mathbf{C}_r & -\mathbf{B}_e\mathbf{C}_d & \mathbf{0} & \mathbf{A}_e & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_u
\end{bmatrix}\mathbf{x} + \begin{bmatrix}
\mathbf{0} & \mathbf{B}_p\mathbf{D}_d & \mathbf{0} \\
\mathbf{B}_r & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{B}_d & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{B}_n \\
\mathbf{B}_e\mathbf{D}_r & -\mathbf{B}_e\mathbf{D}_p\mathbf{D}_d & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0}
\end{bmatrix}\mathbf{w} + \begin{bmatrix}
\mathbf{B}_p \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ -\mathbf{B}_e\mathbf{D}_p \\ \mathbf{B}_u
\end{bmatrix}\mathbf{u}, \\
\mathbf{z} &= \begin{bmatrix}
-\mathbf{D}_e\mathbf{C}_p & \mathbf{D}_e\mathbf{C}_r & -\mathbf{D}_e\mathbf{D}_p\mathbf{C}_d & \mathbf{0} & \mathbf{C}_e & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C}_u
\end{bmatrix}\mathbf{x} + \begin{bmatrix}
\mathbf{D}_e\mathbf{D}_r & -\mathbf{D}_e\mathbf{D}_p\mathbf{D}_d & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0}
\end{bmatrix}\mathbf{w} + \begin{bmatrix}
-\mathbf{B}_e\mathbf{D}_p \\ \mathbf{D}_u
\end{bmatrix}\mathbf{u}, \\
\mathbf{y} &= \begin{bmatrix}-\mathbf{C}_p & \mathbf{C}_r & -\mathbf{D}_p\mathbf{C}_d & -\mathbf{C}_n & \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{x} + \begin{bmatrix}\mathbf{D}_r & -\mathbf{D}_p\mathbf{D}_d & -\mathbf{D}_n\end{bmatrix}\mathbf{w} - \mathbf{D}_p\mathbf{u},
\end{aligned}
$$

where $\mathbf{x}^\mathsf{T}(t) = \begin{bmatrix}\mathbf{x}_p^\mathsf{T}(t) & \mathbf{x}_r^\mathsf{T}(t) & \mathbf{x}_d^\mathsf{T}(t) & \mathbf{x}_n^\mathsf{T}(t) & \mathbf{x}_e^\mathsf{T}(t) & \mathbf{x}_u^\mathsf{T}(t)\end{bmatrix}$, $\mathbf{w}^\mathsf{T}(t) = \begin{bmatrix}\mathbf{r}^\mathsf{T}(t) & \mathbf{d}^\mathsf{T}(t) & \mathbf{n}^\mathsf{T}(t)\end{bmatrix}$, $\mathbf{u}(t) = \mathbf{u}_c(t)$, $\mathbf{y}(s) = \mathbf{W}_r(s)\mathbf{r}(s) - \mathbf{y}_p(s) - \mathbf{W}_n(s)\mathbf{n}(s)$, and $\mathbf{x}_r(t)$, $\mathbf{x}_d(t)$, $\mathbf{x}_n(t)$, $\mathbf{x}_e(t)$, and $\mathbf{x}_u(t)$ are the states associated with the state-space realizations of the weighting transfer matrices $\mathbf{W}_r(s)$, $\mathbf{W}_d(s)$, $\mathbf{W}_n(s)$, $\mathbf{W}_e(s)$, and $\mathbf{W}_u(s)$, respectively.

4.1.2 The Discrete-Time Generalized Plant

The discrete-time generalized LTI plant $\mathcal{P} : \ell_{2e} \to \ell_{2e}$, shown in Figure 1, is described by the state-space realization

$$\begin{aligned}\mathbf{x}_{k+1} &= \mathbf{A}_\mathrm{d}\mathbf{x}_k + \mathbf{B}_{\mathrm{d}1}\mathbf{w}_k + \mathbf{B}_{\mathrm{d}2}\mathbf{u}_k,\\ \mathbf{z}_k &= \mathbf{C}_{\mathrm{d}1}\mathbf{x}_k + \mathbf{D}_{\mathrm{d}11}\mathbf{w}_k + \mathbf{D}_{\mathrm{d}12}\mathbf{u}_k,\\ \mathbf{y}_k &= \mathbf{C}_{\mathrm{d}2}\mathbf{x}_k + \mathbf{D}_{\mathrm{d}21}\mathbf{w}_k + \mathbf{D}_{\mathrm{d}22}\mathbf{u}_k,\end{aligned}$$

where $\mathbf{x}_k \in \mathbb{R}^{n_x}$ is the system state at time step k, $\mathbf{z}_k \in \mathbb{R}^{n_z}$ is the performance signal at time step k, $\mathbf{y}_k \in \mathbb{R}^{n_y}$ is the measurement signal at time step k, $\mathbf{w}_k \in \mathbb{R}^{n_w}$ is the exogenous signal at time step k, $\mathbf{u}_k \in \mathbb{R}^{n_u}$ is the control input signal at time step k, and the state-space matrices have appropriate dimensions. The generalized LTI plant can also be written in discrete-time transfer matrix form as

$$\begin{bmatrix}\mathbf{z}(z)\\ \mathbf{y}(z)\end{bmatrix} = \mathbf{P}(z)\begin{bmatrix}\mathbf{w}(z)\\ \mathbf{u}(z)\end{bmatrix},$$

where the transfer matrix $\mathbf{P}(z) \in \mathbb{C}^{(n_z+n_y)\times(n_w+n_u)}$ is partitioned as

$$\mathbf{P}(z) = \begin{bmatrix}\mathbf{P}_{zw}(z) & \mathbf{P}_{zu}(z)\\ \mathbf{P}_{yw}(z) & \mathbf{P}_{yu}(z)\end{bmatrix} = \begin{bmatrix}\mathbf{C}_{\mathrm{d}1}\left(z\mathbf{1} - \mathbf{A}_\mathrm{d}\right)^{-1}\mathbf{B}_{\mathrm{d}1} + \mathbf{D}_{\mathrm{d}11} & \mathbf{C}_{\mathrm{d}1}\left(z\mathbf{1} - \mathbf{A}_\mathrm{d}\right)^{-1}\mathbf{B}_{\mathrm{d}2} + \mathbf{D}_{\mathrm{d}12}\\ \mathbf{C}_{\mathrm{d}2}\left(z\mathbf{1} - \mathbf{A}_\mathrm{d}\right)^{-1}\mathbf{B}_{\mathrm{d}1} + \mathbf{D}_{\mathrm{d}21} & \mathbf{C}_{\mathrm{d}2}\left(z\mathbf{1} - \mathbf{A}_\mathrm{d}\right)^{-1}\mathbf{B}_{\mathrm{d}2} + \mathbf{D}_{\mathrm{d}22}\end{bmatrix}.$$

4.2 $\mathcal{H}_2$-Optimal Control

The goal of $\mathcal{H}_2$-optimal control is to design a controller that minimizes the $\mathcal{H}_2$ norm of the closed-loop transfer matrix from **w** to **z**.

4.2.1 $\mathcal{H}_2$-Optimal Full-State Feedback Control [19, pp. 257–258]

Consider the continuous-time generalized LTI plant $\mathcal{P}$ with state-space realization

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{w} + \mathbf{B}_2\mathbf{u}, \tag{4.1}$$
$$\mathbf{z} = \mathbf{C}_1\mathbf{x} + \mathbf{D}_{12}\mathbf{u}, \tag{4.2}$$
$$\mathbf{y} = \mathbf{x},$$

where it is assumed that ($\mathbf{A}$,$\mathbf{B}_2$) is stabilizable. A full-state feedback controller $\mathcal{K} = \mathbf{K} \in \mathbb{R}^{n_u\times n_x}$ (i.e., $\mathbf{u} = \mathbf{K}\mathbf{x}$) is to be designed to minimize the $\mathcal{H}_2$ norm of the closed loop transfer matrix from the exogenous input **w** to the performance output **z**. Substituting the full-state feedback controller into (4.1) and (4.2) yields

$$\begin{aligned}\dot{\mathbf{x}} &= \left(\mathbf{A} + \mathbf{B}_2\mathbf{K}\right)\mathbf{x} + \mathbf{B}_1\mathbf{w},\\ \mathbf{z} &= \left(\mathbf{C}_1 + \mathbf{D}_{12}\mathbf{K}\right)\mathbf{x},\end{aligned}$$

and a closed-loop transfer matrix

$$\mathbf{T}(s) = (\mathbf{C}_1 + \mathbf{D}_{12}\mathbf{K})\left(s\mathbf{1} - (\mathbf{A} + \mathbf{B}_2\mathbf{K})\right)^{-1}\mathbf{B}_1.$$

Minimizing the $\mathcal{H}_2$ norm of the transfer matrix $\mathbf{T}(s)$ is equivalent to minimizing $\mathcal{J}(\mu) = \mu^2$ subject to

$$\begin{bmatrix} (\mathbf{A} + \mathbf{B}_2\mathbf{K})\mathbf{P} + \mathbf{P}(\mathbf{A} + \mathbf{B}_2\mathbf{K})^{\mathsf{T}} & \mathbf{P}(\mathbf{C}_1 + \mathbf{D}_{12}\mathbf{K})^{\mathsf{T}} \\ * & -\mathbf{1} \end{bmatrix} < 0, \tag{4.3}$$

$$\begin{bmatrix} \mathbf{Z} & \mathbf{B}_1^{\mathsf{T}} \\ * & \mathbf{P} \end{bmatrix} > 0, \tag{4.4}$$

$$\mathrm{tr}\mathbf{Z} < \mu^2, \tag{4.5}$$

where $\mathbf{P} \in \mathbb{S}^{n_x}$, $\mathbf{Z} \in \mathbb{S}^{n_w}$, $\mu \in \mathbb{R}_{>0}$, $\mathbf{P} > 0$, and $\mathbf{Z} > 0$. A change of variables is performed with $\mathbf{F} = \mathbf{KP}$ and $\nu = \mu^2$, which transforms (4.3) and (4.5) into LMIs in the variables $\mathbf{P}$, $\mathbf{F}$, $\mathbf{Z}$, and ν given by

$$\begin{bmatrix} \mathbf{AP} + \mathbf{PA}^{\mathsf{T}} + \mathbf{B}_2\mathbf{F} + \mathbf{F}^{\mathsf{T}}\mathbf{B}_2^{\mathsf{T}} & \mathbf{PC}_1^{\mathsf{T}} + \mathbf{F}^{\mathsf{T}}\mathbf{D}_{12}^{\mathsf{T}} \\ * & -\mathbf{1} \end{bmatrix} < 0, \tag{4.6}$$

$$\mathrm{tr}\mathbf{Z} < \nu. \tag{4.7}$$

Synthesis Method 4.1. The $\mathcal{H}_2$-optimal full-state feedback controller is synthesized by solving for $\mathbf{P} \in \mathbb{S}^{n_x}$, $\mathbf{Z} \in \mathbb{S}^{n_w}$, $\mathbf{F} \in \mathbb{R}^{n_u \times n_x}$, and $\nu \in \mathbb{R}_{>0}$ that minimize $\mathcal{J}(\nu) = \nu$ subject to $\mathbf{P} > 0$, $\mathbf{Z} > 0$, (4.4), (4.6), and (4.7). The $\mathcal{H}_2$-optimal full-state feedback gain is recovered by $\mathbf{K} = \mathbf{FP}^{-1}$ and the $\mathcal{H}_2$ norm of $\mathbf{T}(s)$ is $\mu = \sqrt{\nu}$.

4.2.2 Discrete-Time $\mathcal{H}_2$-Optimal Full-State Feedback Control

Consider the discrete-time generalized LTI plant $\mathcal{P}$ with state-space realization

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}_{\mathrm{d}}\mathbf{x}_k + \mathbf{B}_{\mathrm{d}1}\mathbf{w}_k + \mathbf{B}_{\mathrm{d}2}\mathbf{u}_k, \\ \mathbf{z}_k &= \mathbf{C}_{\mathrm{d}1}\mathbf{x}_k + \mathbf{D}_{\mathrm{d}12}\mathbf{u}_k, \\ \mathbf{y}_k &= \mathbf{x}_k, \end{aligned}$$

where it is assumed that ($\mathbf{A}_{\mathrm{d}}$,$\mathbf{B}_{\mathrm{d}2}$) is stabilizable. A full-state feedback controller $\mathcal{K} = \mathbf{K}_{\mathrm{d}} \in \mathbb{R}^{n_u \times n_x}$ (i.e., $\mathbf{u}_k = \mathbf{K}_{\mathrm{d}}\mathbf{x}_k$) is to be designed to minimize the $\mathcal{H}_2$ norm of the closed loop transfer matrix from the exogenous input $\mathbf{w}_k$ to the performance output $\mathbf{z}_k$, given by

$$\mathbf{T}(z) = (\mathbf{C}_{\mathrm{d}1} + \mathbf{D}_{\mathrm{d}12}\mathbf{K}_{\mathrm{d}})\left(z\mathbf{1} - (\mathbf{A}_{\mathrm{d}} + \mathbf{B}_{\mathrm{d}2}\mathbf{K}_{\mathrm{d}})\right)^{-1}\mathbf{B}_{\mathrm{d}1}.$$

Synthesis Method 4.2. The discrete-time $\mathcal{H}_2$-optimal full-state feedback controller is synthesized by solving for $\mathbf{P} \in \mathbb{S}^{n_x}$, $\mathbf{Z} \in \mathbb{S}^{n_z}$, $\mathbf{F}_{\mathrm{d}} \in \mathbb{R}^{n_u \times n_x}$, and $\nu \in \mathbb{R}_{>0}$ that minimize $\mathcal{J}(\nu) = \nu$ subject to $\mathbf{P} > 0$, $\mathbf{Z} > 0$,

$$\begin{bmatrix} \mathbf{P} & \mathbf{A}_{\mathrm{d}}\mathbf{P} + \mathbf{B}_{\mathrm{d}2}\mathbf{F}_{\mathrm{d}} & \mathbf{B}_{\mathrm{d}1} \\ * & \mathbf{P} & \mathbf{0} \\ * & * & \mathbf{1} \end{bmatrix} > 0,$$

$$\begin{bmatrix} \mathbf{Z} & \mathbf{C}_{\mathrm{d}1}\mathbf{P} + \mathbf{D}_{\mathrm{d}12}\mathbf{F}_{\mathrm{d}} \\ * & \mathbf{P} \end{bmatrix} > 0.$$

$$\mathrm{tr}\mathbf{Z} < \nu.$$

The $\mathcal{H}_2$-optimal full-state feedback gain is recovered by $\mathbf{K}_{\mathrm{d}} = \mathbf{F}_{\mathrm{d}}\mathbf{P}^{-1}$ and the $\mathcal{H}_2$ norm of $\mathbf{T}(z)$ is $\mu = \sqrt{\nu}$.

4.2.3 $\mathcal{H}_2$-Optimal Dynamic Output Feedback Control [97, 98]

Consider the continuous-time generalized LTI plant $\mathcal{P}$ with minimal state-space realization

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{w} + \mathbf{B}_2\mathbf{u},\\ \mathbf{z} &= \mathbf{C}_1\mathbf{x} + \mathbf{D}_{11}\mathbf{w} + \mathbf{D}_{12}\mathbf{u},\\ \mathbf{y} &= \mathbf{C}_2\mathbf{x} + \mathbf{D}_{21}\mathbf{w} + \mathbf{D}_{22}\mathbf{u}.\end{aligned}$$

A continuous-time dynamic output feedback LTI controller with state-space realization $(\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c, \mathbf{D}_c)$ is to be designed to minimize the $\mathcal{H}_2$ norm of the closed-loop system transfer matrix from **w** to **z**, given by

$$\mathbf{T}(s) = \mathbf{C}_{\text{CL}}\left(s\mathbf{1} - \mathbf{A}_{\text{CL}}\right)^{-1}\mathbf{B}_{\text{CL}} + \mathbf{D}_{\text{CL}},$$

where

$$\begin{aligned}\mathbf{A}_{\text{CL}} &= \begin{bmatrix}\mathbf{A} + \mathbf{B}_2\mathbf{D}_c\tilde{\mathbf{D}}^{-1}\mathbf{C}_2 & \mathbf{B}_2\left(\mathbf{1} + \mathbf{D}_c\tilde{\mathbf{D}}^{-1}\mathbf{D}_{22}\right)\mathbf{C}_c\\ \mathbf{B}_c\tilde{\mathbf{D}}^{-1}\mathbf{C}_2 & \mathbf{A}_c + \mathbf{B}_c\tilde{\mathbf{D}}^{-1}\mathbf{D}_{22}\mathbf{C}_c\end{bmatrix},\\ \mathbf{B}_{\text{CL}} &= \begin{bmatrix}\mathbf{B}_1 + \mathbf{B}_2\mathbf{D}_c\tilde{\mathbf{D}}^{-1}\mathbf{D}_{21}\\ \mathbf{B}_c\tilde{\mathbf{D}}^{-1}\mathbf{D}_{21}\end{bmatrix},\\ \mathbf{C}_{\text{CL}} &= \begin{bmatrix}\mathbf{C}_1 + \mathbf{D}_{12}\mathbf{D}_c\tilde{\mathbf{D}}^{-1}\mathbf{C}_2 & \mathbf{D}_{12}\left(\mathbf{1} + \mathbf{D}_c\tilde{\mathbf{D}}^{-1}\mathbf{D}_{22}\right)\mathbf{C}_c\end{bmatrix},\\ \mathbf{D}_{\text{CL}} &= \mathbf{D}_{11} + \mathbf{D}_{12}\mathbf{D}_c\tilde{\mathbf{D}}^{-1}\mathbf{D}_{21},\end{aligned}$$

and $\tilde{\mathbf{D}} = \mathbf{1} - \mathbf{D}_{22}\mathbf{D}_c$.

Synthesis Method 4.3. Solve for $\mathbf{A}_n \in \mathbb{R}^{n_x \times n_x}$, $\mathbf{B}_n \in \mathbb{R}^{n_x \times n_y}$, $\mathbf{C}_n \in \mathbb{R}^{n_u \times n_x}$, $\mathbf{D}_n \in \mathbb{R}^{n_u \times n_y}$, $\mathbf{X}_1, \mathbf{Y}_1 \in \mathbb{S}^{n_x}$, $\mathbf{Z} \in \mathbb{S}^{n_z}$, and $\nu \in \mathbb{R}_{>0}$ that minimize $\mathcal{J}(\nu) = \nu$ subject to $\mathbf{X}_1 > 0$, $\mathbf{Y}_1 > 0$, $\mathbf{Z} > 0$,

$$\begin{aligned}\begin{bmatrix}\mathbf{A}\mathbf{Y}_1 + \mathbf{Y}_1\mathbf{A}^\mathsf{T} + \mathbf{B}_2\mathbf{C}_n + \mathbf{C}_n^\mathsf{T}\mathbf{B}_2^\mathsf{T} & \mathbf{A} + \mathbf{A}_n^\mathsf{T} + \mathbf{B}_2\mathbf{D}_n\mathbf{C}_2 & \mathbf{B}_1 + \mathbf{B}_2\mathbf{D}_n\mathbf{D}_{21}\\ * & \mathbf{X}_1\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{X}_1 + \mathbf{B}_n\mathbf{C}_2 + \mathbf{C}_2^\mathsf{T}\mathbf{B}_n^\mathsf{T} & \mathbf{X}_1\mathbf{B}_1 + \mathbf{B}_n\mathbf{D}_{21}\\ * & * & -\mathbf{1}\end{bmatrix} &< 0,\\ \begin{bmatrix}\mathbf{X}_1 & \mathbf{1} & \mathbf{Y}_1^\mathsf{T}\mathbf{C}_1^\mathsf{T} + \mathbf{C}_n^\mathsf{T}\mathbf{D}_{12}^\mathsf{T}\\ * & \mathbf{Y}_1 & \mathbf{C}_1^\mathsf{T} + \mathbf{C}_2^\mathsf{T}\mathbf{D}_n^\mathsf{T}\mathbf{D}_{12}^\mathsf{T}\\ * & * & \mathbf{Z}\end{bmatrix} &> 0,\\ \mathbf{D}_{11} + \mathbf{D}_{12}\mathbf{D}_n\mathbf{D}_{21} &= \mathbf{0}, \quad (4.8)\\ \begin{bmatrix}\mathbf{X}_1 & \mathbf{1}\\ * & \mathbf{Y}_1\end{bmatrix} &> 0,\\ \operatorname{tr}\mathbf{Z} &< \nu.\end{aligned}$$

The controller is recovered by

$$\begin{aligned}\mathbf{A}_c &= \mathbf{A}_K - \mathbf{B}_c\left(\mathbf{1} - \mathbf{D}_{22}\mathbf{D}_c\right)^{-1}\mathbf{D}_{22}\mathbf{C}_c,\\ \mathbf{B}_c &= \mathbf{B}_K\left(\mathbf{1} - \mathbf{D}_c\mathbf{D}_{22}\right),\\ \mathbf{C}_c &= \left(\mathbf{1} - \mathbf{D}_c\mathbf{D}_{22}\right)\mathbf{C}_K,\\ \mathbf{D}_c &= \left(\mathbf{1} + \mathbf{D}_K\mathbf{D}_{22}\right)^{-1}\mathbf{D}_K,\end{aligned}$$

where

$$\begin{bmatrix} \mathbf{A}_{\scriptscriptstyle K} & \mathbf{B}_{\scriptscriptstyle K} \\ \mathbf{C}_{\scriptscriptstyle K} & \mathbf{D}_{\scriptscriptstyle K} \end{bmatrix} = \begin{bmatrix} \mathbf{X}_2 & \mathbf{X}_1\mathbf{B}_2 \\ \mathbf{0} & \mathbf{1} \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathbf{A}_n & \mathbf{B}_n \\ \mathbf{C}_n & \mathbf{D}_n \end{bmatrix} - \begin{bmatrix} \mathbf{X}_1\mathbf{A}\mathbf{Y}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{bmatrix} \mathbf{Y}_2^{\mathsf{T}} & \mathbf{0} \\ \mathbf{C}_2\mathbf{Y}_1 & \mathbf{1} \end{bmatrix}^{-1},$$

and the matrices $\mathbf{X}_2$ and $\mathbf{Y}_2$ satisfy $\mathbf{X}_2\mathbf{Y}_2^{\mathsf{T}} = \mathbf{1} - \mathbf{X}_1\mathbf{Y}_1$. If $\mathbf{D}_{22} = \mathbf{0}$, then $\mathbf{A}_c = \mathbf{A}_{\scriptscriptstyle K}$, $\mathbf{B}_c = \mathbf{B}_{\scriptscriptstyle K}$, $\mathbf{C}_c = \mathbf{C}_{\scriptscriptstyle K}$, and $\mathbf{D}_c = \mathbf{D}_{\scriptscriptstyle K}$.

Given $\mathbf{X}_1$ and $\mathbf{Y}_1$, the matrices $\mathbf{X}_2$ and $\mathbf{Y}_2$ can be found using a matrix decomposition, such as a LU decomposition or a Cholesky decomposition.

If $\mathbf{D}_{11} = \mathbf{0}$, $\mathbf{D}_{12} \neq \mathbf{0}$, and $\mathbf{D}_{21} \neq \mathbf{0}$, then it is often simplest to choose $\mathbf{D}_n = \mathbf{0}$ in order to satisfy the equality constraint of (4.8).

4.2.4 Discrete-Time $\mathcal{H}_2$-Optimal Dynamic Output Feedback Control [54]

Consider the discrete-time generalized LTI plant $\mathcal{P}$ with state-space realization

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}_{\mathrm{d}}\mathbf{x}_k + \mathbf{B}_{\mathrm{d}1}\mathbf{w}_k + \mathbf{B}_{\mathrm{d}2}\mathbf{u}_k, \\ \mathbf{z}_k &= \mathbf{C}_{\mathrm{d}1}\mathbf{x}_k + \mathbf{D}_{\mathrm{d}11}\mathbf{w}_k + \mathbf{D}_{\mathrm{d}12}\mathbf{u}_k, \\ \mathbf{y}_k &= \mathbf{C}_{\mathrm{d}2}\mathbf{x}_k + \mathbf{D}_{\mathrm{d}21}\mathbf{w}_k + \mathbf{D}_{\mathrm{d}22}\mathbf{u}_k, \end{aligned}$$

A discrete-time dynamic output feedback LTI controller with state-space realization $(\mathbf{A}_{\mathrm{d}c}, \mathbf{B}_{\mathrm{d}c}, \mathbf{C}_{\mathrm{d}c}, \mathbf{D}_{\mathrm{d}c})$ is to be designed to minimize the $\mathcal{H}_2$ norm of the closed-loop system transfer matrix from $\mathbf{w}_k$ to $\mathbf{z}_k$, given by

$$\mathbf{T}(z) = \mathbf{C}_{\mathrm{d}_{\mathrm{CL}}} \left(z\mathbf{1} - \mathbf{A}_{\mathrm{d}_{\mathrm{CL}}} \right)^{-1} \mathbf{B}_{\mathrm{d}_{\mathrm{CL}}} + \mathbf{D}_{\mathrm{d}_{\mathrm{CL}}},$$

where

$$\begin{aligned} \mathbf{A}_{\mathrm{d}_{\mathrm{CL}}} &= \begin{bmatrix} \mathbf{A}_{\mathrm{d}} + \mathbf{B}_{\mathrm{d}2}\mathbf{D}_{\mathrm{d}c}\tilde{\mathbf{D}}_{\mathrm{d}}^{-1}\mathbf{C}_{\mathrm{d}2} & \mathbf{B}_{\mathrm{d}2}\left(\mathbf{1} + \mathbf{D}_{\mathrm{d}c}\tilde{\mathbf{D}}_{\mathrm{d}}^{-1}\mathbf{D}_{\mathrm{d}22}\right)\mathbf{C}_{\mathrm{d}c} \\ \mathbf{B}_{\mathrm{d}c}\tilde{\mathbf{D}}_{\mathrm{d}}^{-1}\mathbf{C}_{\mathrm{d}2} & \mathbf{A}_{\mathrm{d}c} + \mathbf{B}_{\mathrm{d}c}\tilde{\mathbf{D}}_{\mathrm{d}}^{-1}\mathbf{D}_{\mathrm{d}22}\mathbf{C}_{\mathrm{d}c} \end{bmatrix}, \\ \mathbf{B}_{\mathrm{d}_{\mathrm{CL}}} &= \begin{bmatrix} \mathbf{B}_{\mathrm{d}1} + \mathbf{B}_{\mathrm{d}2}\mathbf{D}_{\mathrm{d}c}\tilde{\mathbf{D}}_{\mathrm{d}}^{-1}\mathbf{D}_{\mathrm{d}21} \\ \mathbf{B}_{\mathrm{d}c}\tilde{\mathbf{D}}_{\mathrm{d}}^{-1}\mathbf{D}_{\mathrm{d}21} \end{bmatrix}, \\ \mathbf{C}_{\mathrm{d}_{\mathrm{CL}}} &= \begin{bmatrix} \mathbf{C}_{\mathrm{d}1} + \mathbf{D}_{\mathrm{d}12}\mathbf{D}_{\mathrm{d}c}\tilde{\mathbf{D}}_{\mathrm{d}}^{-1}\mathbf{C}_{\mathrm{d}2} & \mathbf{D}_{\mathrm{d}12}\left(\mathbf{1} + \mathbf{D}_{\mathrm{d}c}\tilde{\mathbf{D}}_{\mathrm{d}}^{-1}\mathbf{D}_{\mathrm{d}22}\right)\mathbf{C}_{\mathrm{d}c} \end{bmatrix}, \\ \mathbf{D}_{\mathrm{d}_{\mathrm{CL}}} &= \mathbf{D}_{\mathrm{d}11} + \mathbf{D}_{\mathrm{d}12}\mathbf{D}_{\mathrm{d}c}\tilde{\mathbf{D}}_{\mathrm{d}}^{-1}\mathbf{D}_{\mathrm{d}21}, \end{aligned}$$

and $\tilde{\mathbf{D}}_{\mathrm{d}} = \mathbf{1} - \mathbf{D}_{\mathrm{d}22}\mathbf{D}_{\mathrm{d}c}$.

Synthesis Method 4.4. Solve for $\mathbf{A}_{\mathrm{d}n} \in \mathbb{R}^{n_x \times n_x}$, $\mathbf{B}_{\mathrm{d}n} \in \mathbb{R}^{n_x \times n_y}$, $\mathbf{C}_{\mathrm{d}n} \in \mathbb{R}^{n_u \times n_x}$, $\mathbf{D}_{\mathrm{d}n} \in \mathbb{R}^{n_u \times n_y}$,

$\mathbf{X}_1, \mathbf{Y}_1 \in \mathbb{S}^{n_x}$, $\mathbf{Z} \in \mathbb{S}^{n_z}$, and $\nu \in \mathbb{R}_{>0}$ that minimize $\mathcal{J}(\nu) = \nu$ subject to $\mathbf{X}_1 > 0$, $\mathbf{Y}_1 > 0$, $\mathbf{Z} > 0$,

$$\begin{aligned}
\begin{bmatrix} \mathbf{X}_1 & \mathbf{1} & \mathbf{X}_1\mathbf{A}_\mathrm{d} + \mathbf{B}_{\mathrm{d}n}\mathbf{C}_\mathrm{d2} & \mathbf{A}_{\mathrm{d}n} & \mathbf{X}_1\mathbf{B}_\mathrm{d1} + \mathbf{B}_{\mathrm{d}n}\mathbf{D}_\mathrm{d21} \\ * & \mathbf{Y}_1 & \mathbf{A}_\mathrm{d} + \mathbf{B}_\mathrm{d2}\mathbf{D}_{\mathrm{d}n}\mathbf{C}_\mathrm{d2} & \mathbf{A}_\mathrm{d}\mathbf{Y}_1 + \mathbf{B}_\mathrm{d2}\mathbf{C}_{\mathrm{d}n} & \mathbf{B}_\mathrm{d1} + \mathbf{B}_\mathrm{d2}\mathbf{D}_{\mathrm{d}n}\mathbf{D}_\mathrm{d21} \\ * & * & \mathbf{X}_1 & \mathbf{1} & \mathbf{0} \\ * & * & * & \mathbf{Y}_1 & \mathbf{0} \\ * & * & * & * & \mathbf{1} \end{bmatrix} &> 0, \\
\begin{bmatrix} \mathbf{Z} & \mathbf{C}_\mathrm{d1} + \mathbf{D}_\mathrm{d12}\mathbf{D}_{\mathrm{d}n}\mathbf{C}_\mathrm{d2} & \mathbf{C}_\mathrm{d1}\mathbf{Y}_1^\mathsf{T} + \mathbf{D}_\mathrm{d12}\mathbf{C}_{\mathrm{d}n} \\ * & \mathbf{X}_1 & \mathbf{1} \\ * & * & \mathbf{Y}_1 \end{bmatrix} &> 0, \\
\mathbf{D}_\mathrm{d11} + \mathbf{D}_\mathrm{d12}\mathbf{D}_{\mathrm{d}n}\mathbf{D}_\mathrm{d21} &= \mathbf{0}, \qquad (4.9) \\
\begin{bmatrix} \mathbf{X}_1 & \mathbf{1} \\ * & \mathbf{Y}_1 \end{bmatrix} &> 0, \\
\mathrm{tr}\mathbf{Z} &< \nu.
\end{aligned}$$

The controller is recovered by

$$\begin{aligned}
\mathbf{A}_{\mathrm{d}c} &= \mathbf{A}_{\mathrm{d}_K} - \mathbf{B}_{\mathrm{d}c}\left(\mathbf{1} - \mathbf{D}_\mathrm{d22}\mathbf{D}_{\mathrm{d}c}\right)^{-1}\mathbf{D}_\mathrm{d22}\mathbf{C}_{\mathrm{d}c}, \\
\mathbf{B}_{\mathrm{d}c} &= \mathbf{B}_{\mathrm{d}_K}\left(\mathbf{1} - \mathbf{D}_{\mathrm{d}c}\mathbf{D}_\mathrm{d22}\right), \\
\mathbf{C}_{\mathrm{d}c} &= \left(\mathbf{1} - \mathbf{D}_{\mathrm{d}c}\mathbf{D}_\mathrm{d22}\right)\mathbf{C}_{\mathrm{d}_K}, \\
\mathbf{D}_{\mathrm{d}c} &= \left(\mathbf{1} + \mathbf{D}_{\mathrm{d}_K}\mathbf{D}_\mathrm{d22}\right)^{-1}\mathbf{D}_{\mathrm{d}_K},
\end{aligned}$$

where

$$\begin{bmatrix} \mathbf{A}_{\mathrm{d}_K} & \mathbf{B}_{\mathrm{d}_K} \\ \mathbf{C}_{\mathrm{d}_K} & \mathbf{D}_{\mathrm{d}_K} \end{bmatrix} = \begin{bmatrix} \mathbf{X}_2 & \mathbf{X}_1\mathbf{B}_\mathrm{d2} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathbf{A}_{\mathrm{d}n} & \mathbf{B}_{\mathrm{d}n} \\ \mathbf{C}_{\mathrm{d}n} & \mathbf{D}_{\mathrm{d}n} \end{bmatrix} - \begin{bmatrix} \mathbf{X}_1\mathbf{A}_\mathrm{d}\mathbf{Y}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{bmatrix} \mathbf{Y}_2^\mathsf{T} & \mathbf{0} \\ \mathbf{C}_\mathrm{d2}\mathbf{Y}_1 & \mathbf{1} \end{bmatrix}^{-1},$$

and the matrices $\mathbf{X}_2$ and $\mathbf{Y}_2$ satisfy $\mathbf{X}_2\mathbf{Y}_2^\mathsf{T} = \mathbf{1} - \mathbf{X}_1\mathbf{Y}_1$. If $\mathbf{D}_\mathrm{d22} = \mathbf{0}$, then $\mathbf{A}_{\mathrm{d}c} = \mathbf{A}_{\mathrm{d}_K}$, $\mathbf{B}_{\mathrm{d}c} = \mathbf{B}_{\mathrm{d}_K}$, $\mathbf{C}_{\mathrm{d}c} = \mathbf{C}_{\mathrm{d}_K}$, and $\mathbf{D}_{\mathrm{d}c} = \mathbf{D}_{\mathrm{d}_K}$.

Given $\mathbf{X}_1$ and $\mathbf{Y}_1$, the matrices $\mathbf{X}_2$ and $\mathbf{Y}_2$ can be found using a matrix decomposition, such as a LU decomposition or a Cholesky decomposition.

If $\mathbf{D}_\mathrm{d11} = \mathbf{0}$, $\mathbf{D}_\mathrm{d12} \neq \mathbf{0}$, and $\mathbf{D}_\mathrm{d21} \neq \mathbf{0}$, then it is often simplest to choose $\mathbf{D}_{\mathrm{d}n} = \mathbf{0}$ in order to satisfy the equality constraint of (4.9).

4.3 $\mathcal{H}_\infty$-Optimal Control

The goal of $\mathcal{H}_\infty$-optimal control is to design a controller that minimizes the $\mathcal{H}_\infty$ norm of the closed-loop transfer matrix from $\mathbf{w}$ to $\mathbf{z}$.

4.3.1 $\mathcal{H}_\infty$-Optimal Full-State Feedback Control [19, pp. 251–252]

Consider the continuous-time generalized LTI plant $\mathcal{P}$ with state-space realization

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{w} + \mathbf{B}_2\mathbf{u}, && (4.10) \\
\mathbf{z} &= \mathbf{C}_1\mathbf{x} + \mathbf{D}_{11}\mathbf{w} + \mathbf{D}_{12}\mathbf{u}, && (4.11) \\
\mathbf{y} &= \mathbf{x},
\end{aligned}$$

where it is assumed that ($\mathbf{A}$,$\mathbf{B}_2$) is stabilizable. A full-state feedback controller $\mathcal{K} = \mathbf{K} \in \mathbb{R}^{n_u \times n_x}$ (i.e., $\mathbf{u} = \mathbf{K}\mathbf{x}$) is to be designed to minimize $\mathcal{H}_\infty$ norm of the closed loop transfer matrix from the exogenous input $\mathbf{w}$ to the performance output $\mathbf{z}$. Substituting the full-state feedback controller into (4.10) and (4.11) yields

$$\begin{aligned}\dot{\mathbf{x}} &= (\mathbf{A} - \mathbf{B}_2\mathbf{K})\,\mathbf{x} + \mathbf{B}_1\mathbf{w},\\ \mathbf{z} &= (\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K})\,\mathbf{x} + \mathbf{D}_{11}\mathbf{w},\end{aligned}$$

and a closed-loop transfer matrix

$$\mathbf{T}(s) = (\mathbf{C}_1 + \mathbf{D}_{12}\mathbf{K})\,(s\mathbf{1} - (\mathbf{A} + \mathbf{B}_2\mathbf{K}))^{-1}\,\mathbf{B}_1 + \mathbf{D}_{11}.$$

From the Bounded Real Lemma in Section 3.2.1, the $\mathcal{H}_\infty$ of the closed-loop system is the minimum value of $\gamma \in \mathbb{R}_{>0}$ that satisfies

$$\begin{bmatrix}\mathbf{P}\,(\mathbf{A} + \mathbf{B}_2\mathbf{K}) + (\mathbf{A} + \mathbf{B}_2\mathbf{K})^\mathsf{T}\,\mathbf{P} & \mathbf{P}\mathbf{B}_1 & (\mathbf{C}_1 + \mathbf{D}_{12}\mathbf{K})^\mathsf{T}\\ * & -\gamma\mathbf{1} & \mathbf{D}_{11}^\mathsf{T}\\ * & * & -\gamma\mathbf{1}\end{bmatrix} < 0, \tag{4.12}$$

where $\mathbf{P} \in \mathbb{S}^{n_x}$ and $\mathbf{P} > 0$. A congruence transformation is performed on (4.12) with $\mathbf{W} = \mathrm{diag}\{\mathbf{P}^{-1}, \mathbf{1}, \mathbf{1}\}$ and a change of variables is made with $\mathbf{Q} = \mathbf{P}^{-1}$ and $\mathbf{F} = \mathbf{K}\mathbf{Q}$. This yields an LMI in the design variables $\mathbf{Q}$, $\mathbf{F}$, and γ, given by

$$\begin{bmatrix}\mathbf{A}\mathbf{Q} + \mathbf{Q}\mathbf{A}^\mathsf{T} + \mathbf{B}_2\mathbf{F} + \mathbf{F}^\mathsf{T}\mathbf{B}_2^\mathsf{T} & \mathbf{B}_1 & \mathbf{Q}\mathbf{C}_1^\mathsf{T} + \mathbf{F}^\mathsf{T}\mathbf{D}_{12}^\mathsf{T}\\ * & -\gamma\mathbf{1} & \mathbf{D}_{11}^\mathsf{T}\\ * & * & -\gamma\mathbf{1}\end{bmatrix} < 0. \tag{4.13}$$

Synthesis Method 4.5. The $\mathcal{H}_\infty$-optimal full-state feedback controller is synthesized by solving for $\mathbf{Q} \in \mathbb{S}^{n_x}$ and $\mathbf{F} \in \mathbb{R}^{n_u \times n_x}$ that minimize $\mathcal{J}(\gamma) = \gamma$ subject to $\mathbf{Q} > 0$ and (4.13). The $\mathcal{H}_\infty$-optimal full-state feedback controller gain is recovered by $\mathbf{K} = \mathbf{F}\mathbf{Q}^{-1}$ and the $\mathcal{H}_\infty$ norm of $\mathbf{T}(s)$ is γ.

4.3.2 Discrete-Time $\mathcal{H}_\infty$-Optimal Full-State Feedback Control

Consider the discrete-time generalized LTI plant $\mathcal{P}$ with state-space realization

$$\begin{aligned}\mathbf{x}_{k+1} &= \mathbf{A}_\mathrm{d}\mathbf{x}_k + \mathbf{B}_{\mathrm{d}1}\mathbf{w}_k + \mathbf{B}_{\mathrm{d}2}\mathbf{u}_k,\\ \mathbf{z}_k &= \mathbf{C}_{\mathrm{d}1}\mathbf{x}_k + \mathbf{D}_{\mathrm{d}12}\mathbf{u}_k,\\ \mathbf{y}_k &= \mathbf{x}_k,\end{aligned}$$

where it is assumed that ($\mathbf{A}_\mathrm{d}$,$\mathbf{B}_{\mathrm{d}2}$) is stabilizable. A full-state feedback controller $\mathcal{K} = \mathbf{K}_\mathrm{d} \in \mathbb{R}^{n_u \times n_x}$ (i.e., $\mathbf{u}_k = \mathbf{K}_\mathrm{d}\mathbf{x}_k$) is to be designed to minimize the $\mathcal{H}_\infty$ norm of the closed loop transfer matrix from the exogenous input $\mathbf{w}_k$ to the performance output $\mathbf{z}_k$, given by

$$\mathbf{T}(z) = (\mathbf{C}_{\mathrm{d}1} + \mathbf{D}_{\mathrm{d}12}\mathbf{K}_\mathrm{d})\,(z\mathbf{1} - (\mathbf{A}_\mathrm{d} + \mathbf{B}_{\mathrm{d}2}\mathbf{K}_\mathrm{d}))^{-1}\,\mathbf{B}_{\mathrm{d}1}.$$

Synthesis Method 4.6. The discrete-time $\mathcal{H}_\infty$-optimal full-state feedback controller is synthesized by solving for $\mathbf{P} \in \mathbb{S}^{n_x}$, $\mathbf{F}_\mathrm{d} \in \mathbb{R}^{n_u \times n_x}$, and $\gamma \in \mathbb{R}_{>0}$ that minimize $\mathcal{J}(\gamma) = \gamma$ subject to $\mathbf{P} > 0$,

$$\begin{bmatrix}\mathbf{P}_\mathrm{d} & \mathbf{A}_\mathrm{d}\mathbf{P}_\mathrm{d} - \mathbf{B}_{\mathrm{d}2}\mathbf{F}_\mathrm{d} & \mathbf{B}_{\mathrm{d}1} & \mathbf{0}\\ * & \mathbf{P}_\mathrm{d} & \mathbf{0} & \mathbf{P}_\mathrm{d}\mathbf{C}_{\mathrm{d}1}^\mathsf{T} - \mathbf{F}_\mathrm{d}^\mathsf{T}\mathbf{D}_{\mathrm{d}12}^\mathsf{T}\\ * & * & \gamma\mathbf{1} & \mathbf{D}_{\mathrm{d}11}^\mathsf{T}\\ * & * & * & \gamma\mathbf{1}\end{bmatrix} > 0.$$

The $\mathcal{H}_\infty$-optimal full-state feedback gain is recovered by $\mathbf{K}_\mathrm{d} = \mathbf{F}_\mathrm{d}\mathbf{P}^{-1}$ and the $\mathcal{H}_\infty$ norm of $\mathbf{T}(z)$ is γ.

4.3.3 $\mathcal{H}_\infty$-Optimal Dynamic Output Feedback Control

Consider the continuous-time generalized LTI plant $\mathcal{P}$ with minimal state-space realization

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{w} + \mathbf{B}_2\mathbf{u},\\ \mathbf{z} &= \mathbf{C}_1\mathbf{x} + \mathbf{D}_{11}\mathbf{w} + \mathbf{D}_{12}\mathbf{u},\\ \mathbf{y} &= \mathbf{C}_2\mathbf{x} + \mathbf{D}_{21}\mathbf{w} + \mathbf{D}_{22}\mathbf{u}.\end{aligned}$$

A continuous-time dynamic output feedback LTI controller with state-space realization $(\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c, \mathbf{D}_c)$ is to be designed to minimize the $\mathcal{H}_\infty$ norm of the closed-loop system transfer matrix from **w** to **z**, given by

$$\mathbf{T}(s) = \mathbf{C}_{\text{CL}}\left(s\mathbf{1} - \mathbf{A}_{\text{CL}}\right)^{-1}\mathbf{B}_{\text{CL}} + \mathbf{D}_{\text{CL}},$$

where

$$\begin{aligned}\mathbf{A}_{\text{CL}} &= \begin{bmatrix}\mathbf{A} + \mathbf{B}_2\mathbf{D}_c\tilde{\mathbf{D}}^{-1}\mathbf{C}_2 & \mathbf{B}_2\left(\mathbf{1} + \mathbf{D}_c\tilde{\mathbf{D}}^{-1}\mathbf{D}_{22}\right)\mathbf{C}_c\\ \mathbf{B}_c\tilde{\mathbf{D}}^{-1}\mathbf{C}_2 & \mathbf{A}_c + \mathbf{B}_c\tilde{\mathbf{D}}^{-1}\mathbf{D}_{22}\mathbf{C}_c\end{bmatrix},\\ \mathbf{B}_{\text{CL}} &= \begin{bmatrix}\mathbf{B}_1 + \mathbf{B}_2\mathbf{D}_c\tilde{\mathbf{D}}^{-1}\mathbf{D}_{21}\\ \mathbf{B}_c\tilde{\mathbf{D}}^{-1}\mathbf{D}_{21}\end{bmatrix},\\ \mathbf{C}_{\text{CL}} &= \begin{bmatrix}\mathbf{C}_1 + \mathbf{D}_{12}\mathbf{D}_c\tilde{\mathbf{D}}^{-1}\mathbf{C}_2 & \mathbf{D}_{12}\left(\mathbf{1} + \mathbf{D}_c\tilde{\mathbf{D}}^{-1}\mathbf{D}_{22}\right)\mathbf{C}_c\end{bmatrix},\\ \mathbf{D}_{\text{CL}} &= \mathbf{D}_{11} + \mathbf{D}_{12}\mathbf{D}_c\tilde{\mathbf{D}}^{-1}\mathbf{D}_{21},\end{aligned}$$

and $\tilde{\mathbf{D}} = \mathbf{1} - \mathbf{D}_{22}\mathbf{D}_c$.

Two different synthesis methods for the $\mathcal{H}_\infty$-optimal dynamic output feedback control problem are presented as follows.

Synthesis Method 4.7. [97, 99, 100] Solve for $\mathbf{A}_n \in \mathbb{R}^{n_x \times n_x}$, $\mathbf{B}_n \in \mathbb{R}^{n_x \times n_y}$, $\mathbf{C}_n \in \mathbb{R}^{n_u \times n_x}$, $\mathbf{D}_n \in \mathbb{R}^{n_u \times n_y}$, $\mathbf{X}_1, \mathbf{Y}_1 \in \mathbb{S}^{n_x}$, and $\gamma \in \mathbb{R}_{>0}$ that minimize $\mathcal{J}(\gamma) = \gamma$ subject to $\mathbf{X}_1 > 0$, $\mathbf{Y}_1 > 0$,

$$\begin{bmatrix}\mathbf{N}_{11} & \mathbf{A} + \mathbf{A}_n^\mathsf{T} + \mathbf{B}_2\mathbf{D}_n\mathbf{C}_2 & \mathbf{B}_1 + \mathbf{B}_2\mathbf{D}_n\mathbf{D}_{21} & \mathbf{Y}_1^\mathsf{T}\mathbf{C}_1^\mathsf{T} + \mathbf{C}_n^\mathsf{T}\mathbf{D}_{12}^\mathsf{T}\\ * & \mathbf{X}_1\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{X}_1 + \mathbf{B}_n\mathbf{C}_2 + \mathbf{C}_2^\mathsf{T}\mathbf{B}_n^\mathsf{T} & \mathbf{X}_1\mathbf{B}_1 + \mathbf{B}_n\mathbf{D}_{21} & \mathbf{C}_1^\mathsf{T} + \mathbf{C}_2^\mathsf{T}\mathbf{D}_n^\mathsf{T}\mathbf{D}_{12}^\mathsf{T}\\ * & * & -\gamma\mathbf{1} & \mathbf{D}_{11}^\mathsf{T} + \mathbf{D}_{21}^\mathsf{T}\mathbf{D}_n^\mathsf{T}\mathbf{D}_{12}^\mathsf{T}\\ * & * & * & -\gamma\mathbf{1}\end{bmatrix} < 0,$$

$$\begin{bmatrix}\mathbf{X}_1 & \mathbf{1}\\ * & \mathbf{Y}_1\end{bmatrix} > 0,$$

where $\mathbf{N}_{11} = \mathbf{A}\mathbf{Y}_1 + \mathbf{Y}_1\mathbf{A}^\mathsf{T} + \mathbf{B}_2\mathbf{C}_n + \mathbf{C}_n^\mathsf{T}\mathbf{B}_2^\mathsf{T}$. The controller is recovered by

$$\begin{aligned}\mathbf{A}_c &= \mathbf{A}_K - \mathbf{B}_c\left(\mathbf{1} - \mathbf{D}_{22}\mathbf{D}_c\right)^{-1}\mathbf{D}_{22}\mathbf{C}_c,\\ \mathbf{B}_c &= \mathbf{B}_K\left(\mathbf{1} - \mathbf{D}_c\mathbf{D}_{22}\right),\\ \mathbf{C}_c &= \left(\mathbf{1} - \mathbf{D}_c\mathbf{D}_{22}\right)\mathbf{C}_K,\\ \mathbf{D}_c &= \left(\mathbf{1} + \mathbf{D}_K\mathbf{D}_{22}\right)^{-1}\mathbf{D}_K,\end{aligned}$$

where

$$\begin{bmatrix}\mathbf{A}_K & \mathbf{B}_K\\ \mathbf{C}_K & \mathbf{D}_K\end{bmatrix} = \begin{bmatrix}\mathbf{X}_2 & \mathbf{X}_1\mathbf{B}_2\\ \mathbf{0} & \mathbf{1}\end{bmatrix}^{-1}\left(\begin{bmatrix}\mathbf{A}_n & \mathbf{B}_n\\ \mathbf{C}_n & \mathbf{D}_n\end{bmatrix} - \begin{bmatrix}\mathbf{X}_1\mathbf{A}\mathbf{Y}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\right)\begin{bmatrix}\mathbf{Y}_2^\mathsf{T} & \mathbf{0}\\ \mathbf{C}_2\mathbf{Y}_1 & \mathbf{1}\end{bmatrix}^{-1},$$

and the matrices $\mathbf{X}_2$ and $\mathbf{Y}_2$ satisfy $\mathbf{X}_2\mathbf{Y}_2^\mathsf{T} = \mathbf{1} - \mathbf{X}_1\mathbf{Y}_1$. If $\mathbf{D}_{22} = \mathbf{0}$, then $\mathbf{A}_c = \mathbf{A}_{_K}$, $\mathbf{B}_c = \mathbf{B}_{_K}$, $\mathbf{C}_c = \mathbf{C}_{_K}$, and $\mathbf{D}_c = \mathbf{D}_{_K}$.

Given $\mathbf{X}_1$ and $\mathbf{Y}_1$, the matrices $\mathbf{X}_2$ and $\mathbf{Y}_2$ can be found using a matrix decomposition, such as a LU decomposition or a Cholesky decomposition.

Synthesis Method 4.8. [24], [25, pp. 224–232] The controller is solved for in the following two steps.

1. Solve for $\mathbf{P}$, $\mathbf{Q} \in \mathbb{S}^{n_x}$ and $\gamma \in \mathbb{R}_{>0}$, where $\mathbf{P} > 0$ and $\mathbf{Q} > 0$, that minimize $\mathcal{J}(\gamma) = \gamma$ subject to

$$\begin{bmatrix} \mathbf{N_o} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}^\mathsf{T} \begin{bmatrix} \mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} & \mathbf{PB}_1 & \mathbf{C}_1^\mathsf{T} \\ * & -\gamma\mathbf{1} & \mathbf{D}_{11}^\mathsf{T} \\ * & * & -\gamma\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{N_o} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} < 0,$$
$$\begin{bmatrix} \mathbf{N_c} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}^\mathsf{T} \begin{bmatrix} \mathbf{AQ} + \mathbf{QA}^\mathsf{T} & \mathbf{QC}_1^\mathsf{T} & \mathbf{B}_1 \\ * & -\gamma\mathbf{1} & \mathbf{D}_{11} \\ * & * & -\gamma\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{N_c} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} < 0,$$
$$\begin{bmatrix} \mathbf{P} & \mathbf{1} \\ * & \mathbf{Q} \end{bmatrix} \geq 0, \tag{4.14}$$

where $\mathcal{R}(\mathbf{N}_o) = \mathcal{N}\left(\begin{bmatrix} \mathbf{C}_2 & \mathbf{D}_{21} \end{bmatrix}\right)$ and $\mathcal{R}(\mathbf{N}_c) = \mathcal{N}\left(\begin{bmatrix} \mathbf{B}_2^\mathsf{T} & \mathbf{D}_{12}^\mathsf{T} \end{bmatrix}\right)$. Define $\mathbf{P}_{_\mathrm{CL}} = \begin{bmatrix} \mathbf{P} & \mathbf{P}_2^\mathsf{T} \\ * & \mathbf{1} \end{bmatrix}$, where $\mathbf{P}_2\mathbf{P}_2^\mathsf{T} = \mathbf{P} - \mathbf{Q}^{-1}$.

2. Fix $\mathbf{P}_{_\mathrm{CL}}$ and solve for $\mathbf{A}_n \in \mathbb{R}^{n_x \times n_x}$, $\mathbf{B}_n \in \mathbb{R}^{n_x \times n_y}$, $\mathbf{C}_n \in \mathbb{R}^{n_u \times n_x}$, $\mathbf{D}_n \in \mathbb{R}^{n_u \times n_y}$, and $\gamma \in \mathbb{R}_{>0}$ that minimize $\mathcal{J}(\gamma) = \gamma$ subject to

$$\begin{bmatrix} \mathbf{P}_{_\mathrm{CL}}\bar{\mathbf{A}} + \bar{\mathbf{A}}^\mathsf{T}\mathbf{P}_{_\mathrm{CL}} & \mathbf{P}_{_\mathrm{CL}}\bar{\mathbf{B}} & \bar{\mathbf{C}}^\mathsf{T} \\ * & -\gamma\mathbf{1} & \mathbf{D}_{11}^\mathsf{T} \\ * & * & -\gamma\mathbf{1} \end{bmatrix} + \begin{bmatrix} \mathbf{P}_{_\mathrm{CL}}\underline{\mathbf{B}} \\ \mathbf{0} \\ \underline{\mathbf{D}}_{12} \end{bmatrix} \begin{bmatrix} \mathbf{A}_n & \mathbf{B}_n \\ \mathbf{C}_n & \mathbf{D}_n \end{bmatrix} \begin{bmatrix} \underline{\mathbf{C}} & \underline{\mathbf{D}}_{21} & \mathbf{0} \end{bmatrix}$$
$$+ \begin{bmatrix} \underline{\mathbf{C}}^\mathsf{T} \\ \underline{\mathbf{D}}_{21}^\mathsf{T} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{A}_n & \mathbf{B}_n \\ \mathbf{C}_n & \mathbf{D}_n \end{bmatrix}^\mathsf{T} \begin{bmatrix} \underline{\mathbf{B}}^\mathsf{T}\mathbf{P}_{_\mathrm{CL}} & \mathbf{0} & \underline{\mathbf{D}}_{12}^\mathsf{T} \end{bmatrix} < 0,$$

where

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad \bar{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_1 - \mathbf{B}_2\bar{\mathbf{D}}_c\mathbf{D}_{21} \\ \mathbf{0} \end{bmatrix},$$
$$\bar{\mathbf{C}} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} \end{bmatrix}, \qquad \underline{\mathbf{C}} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{C}_2 & \mathbf{0} \end{bmatrix},$$
$$\underline{\mathbf{B}} = \begin{bmatrix} \mathbf{0} & -\mathbf{B}_2 \\ \mathbf{1} & \mathbf{0} \end{bmatrix}, \qquad \underline{\mathbf{D}}_{12} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_{12} \end{bmatrix},$$
$$\underline{\mathbf{D}}_{21} = \begin{bmatrix} \mathbf{0} \\ \mathbf{D}_{21} \end{bmatrix}.$$

The controller is recovered by

$$\begin{aligned}
\mathbf{A}_c &= \mathbf{A}_n - \mathbf{B}_c\left(\mathbf{1} - \mathbf{D}_{22}\mathbf{D}_c\right)^{-1}\mathbf{D}_{22}\mathbf{C}_c,\\
\mathbf{B}_c &= \mathbf{B}_n\left(\mathbf{1} - \mathbf{D}_c\mathbf{D}_{22}\right),\\
\mathbf{C}_c &= \left(\mathbf{1} - \mathbf{D}_c\mathbf{D}_{22}\right)\mathbf{C}_n,\\
\mathbf{D}_c &= \left(\mathbf{1} + \mathbf{D}_n\mathbf{D}_{22}\right)^{-1}\mathbf{D}_n.
\end{aligned}$$

If $\mathbf{D}_{22} = \mathbf{0}$, then $\mathbf{A}_c = \mathbf{A}_n$, $\mathbf{B}_c = \mathbf{B}_n$, $\mathbf{C}_c = \mathbf{C}_n$, and $\mathbf{D}_c = \mathbf{D}_n$.

Note that the purpose of the matrix inequality $\begin{bmatrix}\mathbf{P} & \mathbf{1}\\ * & \mathbf{Q}\end{bmatrix} \geq 0$ in (4.14) is to ensure that there exists $\mathbf{P}_{\text{CL}} = \begin{bmatrix}\mathbf{P} & \mathbf{P}_2^{\mathsf{T}}\\ * & \mathbf{1}\end{bmatrix} > 0$ and $\mathbf{P}_{\text{CL}}^{-1} = \begin{bmatrix}\mathbf{Q} & -\mathbf{Q}\mathbf{P}_2\\ * & \mathbf{P}_2^{\mathsf{T}}\mathbf{Q}\mathbf{P}_2 + \mathbf{1}\end{bmatrix}$. This follows from Property 7 in Section 2.3.3.

4.3.4 Discrete-Time $\mathcal{H}_\infty$-Optimal Dynamic Output Feedback Control [54]

Consider the discrete-time generalized LTI plant $\mathcal{P}$ with minimal state-space realization

$$\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{A}_{\text{d}}\mathbf{x}_k + \mathbf{B}_{\text{d}1}\mathbf{w}_k + \mathbf{B}_{\text{d}2}\mathbf{u}_k,\\
\mathbf{z}_k &= \mathbf{C}_{\text{d}1}\mathbf{x}_k + \mathbf{D}_{\text{d}11}\mathbf{w}_k + \mathbf{D}_{\text{d}12}\mathbf{u}_k,\\
\mathbf{y}_k &= \mathbf{C}_{\text{d}2}\mathbf{x}_k + \mathbf{D}_{\text{d}21}\mathbf{w}_k + \mathbf{D}_{\text{d}22}\mathbf{u}_k,
\end{aligned}$$

A discrete-time dynamic output feedback LTI controller with state-space realization $(\mathbf{A}_{\text{d}c}, \mathbf{B}_{\text{d}c}, \mathbf{C}_{\text{d}c}, \mathbf{D}_{\text{d}c})$ is to be designed to minimize the $\mathcal{H}_\infty$ norm of the closed-loop system transfer matrix from **w** to **z**, given by

$$\mathbf{T}(z) = \mathbf{C}_{\text{d}_{\text{CL}}}\left(z\mathbf{1} - \mathbf{A}_{\text{d}_{\text{CL}}}\right)^{-1}\mathbf{B}_{\text{d}_{\text{CL}}} + \mathbf{D}_{\text{d}_{\text{CL}}},$$

where

$$\begin{aligned}
\mathbf{A}_{\text{d}_{\text{CL}}} &= \begin{bmatrix}\mathbf{A}_{\text{d}} + \mathbf{B}_{\text{d}2}\mathbf{D}_{\text{d}c}\tilde{\mathbf{D}}_{\text{d}}^{-1}\mathbf{C}_{\text{d}2} & \mathbf{B}_{\text{d}2}\left(\mathbf{1} + \mathbf{D}_{\text{d}c}\tilde{\mathbf{D}}_{\text{d}}^{-1}\mathbf{D}_{\text{d}22}\right)\mathbf{C}_{\text{d}c}\\ \mathbf{B}_{\text{d}c}\tilde{\mathbf{D}}_{\text{d}}^{-1}\mathbf{C}_{\text{d}2} & \mathbf{A}_{\text{d}c} + \mathbf{B}_{\text{d}c}\tilde{\mathbf{D}}_{\text{d}}^{-1}\mathbf{D}_{\text{d}22}\mathbf{C}_{\text{d}c}\end{bmatrix},\\
\mathbf{B}_{\text{d}_{\text{CL}}} &= \begin{bmatrix}\mathbf{B}_{\text{d}1} + \mathbf{B}_{\text{d}2}\mathbf{D}_{\text{d}c}\tilde{\mathbf{D}}_{\text{d}}^{-1}\mathbf{D}_{\text{d}21}\\ \mathbf{B}_{\text{d}c}\tilde{\mathbf{D}}_{\text{d}}^{-1}\mathbf{D}_{\text{d}21}\end{bmatrix},\\
\mathbf{C}_{\text{d}_{\text{CL}}} &= \begin{bmatrix}\mathbf{C}_{\text{d}1} + \mathbf{D}_{\text{d}12}\mathbf{D}_{\text{d}c}\tilde{\mathbf{D}}_{\text{d}}^{-1}\mathbf{C}_{\text{d}2} & \mathbf{D}_{\text{d}12}\left(\mathbf{1} + \mathbf{D}_{\text{d}c}\tilde{\mathbf{D}}_{\text{d}}^{-1}\mathbf{D}_{\text{d}22}\right)\mathbf{C}_{\text{d}c}\end{bmatrix},\\
\mathbf{D}_{\text{d}_{\text{CL}}} &= \mathbf{D}_{\text{d}11} + \mathbf{D}_{\text{d}12}\mathbf{D}_{\text{d}c}\tilde{\mathbf{D}}_{\text{d}}^{-1}\mathbf{D}_{\text{d}21},
\end{aligned}$$

and $\tilde{\mathbf{D}}_{\text{d}} = \mathbf{1} - \mathbf{D}_{\text{d}22}\mathbf{D}_{\text{d}c}$.

Synthesis Method 4.9. Solve for $\mathbf{A}_{\text{d}n} \in \mathbb{R}^{n_x \times n_x}$, $\mathbf{B}_{\text{d}n} \in \mathbb{R}^{n_x \times n_y}$, $\mathbf{C}_{\text{d}n} \in \mathbb{R}^{n_u \times n_x}$, $\mathbf{D}_{\text{d}n} \in \mathbb{R}^{n_u \times n_y}$, $\mathbf{X}_1$, $\mathbf{Y}_1 \in \mathbb{S}^{n_x}$, and $\gamma \in \mathbb{R}_{>0}$ that minimize $\mathcal{J}(\gamma) = \gamma$ subject to $\mathbf{X}_1 > 0$, $\mathbf{Y}_1 > 0$,

$$\begin{bmatrix}
\mathbf{X}_1 & \mathbf{1} & \mathbf{X}_1\mathbf{A}_{\text{d}} + \mathbf{B}_{\text{d}n}\mathbf{C}_{\text{d}2} & \mathbf{A}_{\text{d}n} & \mathbf{X}_1\mathbf{B}_{\text{d}1} + \mathbf{B}_{\text{d}n}\mathbf{D}_{\text{d}21} & \mathbf{0}\\
* & \mathbf{Y}_1 & \mathbf{A}_{\text{d}} + \mathbf{B}_{\text{d}2}\mathbf{D}_{\text{d}n}\mathbf{C}_{\text{d}2} & \mathbf{A}_{\text{d}}\mathbf{Y}_1 + \mathbf{B}_{\text{d}2}\mathbf{C}_{\text{d}n} & \mathbf{B}_{\text{d}1} + \mathbf{B}_{\text{d}2}\mathbf{D}_{\text{d}n}\mathbf{D}_{\text{d}21} & \mathbf{0}\\
* & * & \mathbf{X}_1 & \mathbf{1} & \mathbf{0} & \mathbf{C}_{\text{d}1}^{\mathsf{T}} + \mathbf{C}_{\text{d}2}^{\mathsf{T}}\mathbf{D}_{\text{d}n}^{\mathsf{T}}\mathbf{D}_{\text{d}12}^{\mathsf{T}}\\
* & * & * & \mathbf{Y}_1 & \mathbf{0} & \mathbf{Y}_1\mathbf{C}_{\text{d}1}^{\mathsf{T}} + \mathbf{C}_{\text{d}n}^{\mathsf{T}}\mathbf{D}_{\text{d}12}^{\mathsf{T}}\\
* & * & * & * & -\gamma\mathbf{1} & \mathbf{D}_{\text{d}11}^{\mathsf{T}} + \mathbf{D}_{\text{d}21}^{\mathsf{T}}\mathbf{D}_{\text{d}n}^{\mathsf{T}}\mathbf{D}_{\text{d}12}^{\mathsf{T}}\\
* & * & * & * & * & -\gamma\mathbf{1}
\end{bmatrix} > 0,$$

$$\begin{bmatrix}\mathbf{X}_1 & \mathbf{1}\\ * & \mathbf{Y}_1\end{bmatrix} > 0.$$

The controller is recovered by

$$
\begin{aligned}
\mathbf{A}_{\mathrm{dc}} &= \mathbf{A}_{\mathrm{d}_K} - \mathbf{B}_{\mathrm{dc}} \left(\mathbf{1} - \mathbf{D}_{\mathrm{d22}}\mathbf{D}_{\mathrm{dc}}\right)^{-1} \mathbf{D}_{\mathrm{d22}}\mathbf{C}_{\mathrm{dc}}, \\
\mathbf{B}_{\mathrm{dc}} &= \mathbf{B}_{\mathrm{d}_K} \left(\mathbf{1} - \mathbf{D}_{\mathrm{dc}}\mathbf{D}_{\mathrm{d22}}\right), \\
\mathbf{C}_{\mathrm{dc}} &= \left(\mathbf{1} - \mathbf{D}_{\mathrm{dc}}\mathbf{D}_{\mathrm{d22}}\right) \mathbf{C}_{\mathrm{d}_K}, \\
\mathbf{D}_{\mathrm{dc}} &= \left(\mathbf{1} + \mathbf{D}_{\mathrm{d}_K}\mathbf{D}_{\mathrm{d22}}\right)^{-1} \mathbf{D}_{\mathrm{d}_K},
\end{aligned}
$$

where

$$
\begin{bmatrix} \mathbf{A}_{\mathrm{d}_K} & \mathbf{B}_{\mathrm{d}_K} \\ \mathbf{C}_{\mathrm{d}_K} & \mathbf{D}_{\mathrm{d}_K} \end{bmatrix} = \begin{bmatrix} \mathbf{X}_2 & \mathbf{X}_1\mathbf{B}_{\mathrm{d2}} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathbf{A}_{\mathrm{d}n} & \mathbf{B}_{\mathrm{d}n} \\ \mathbf{C}_{\mathrm{d}n} & \mathbf{D}_{\mathrm{d}n} \end{bmatrix} - \begin{bmatrix} \mathbf{X}_1\mathbf{A}_{\mathrm{d}}\mathbf{Y}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{bmatrix} \mathbf{Y}_2^{\mathsf{T}} & \mathbf{0} \\ \mathbf{C}_{\mathrm{d2}}\mathbf{Y}_1 & \mathbf{1} \end{bmatrix}^{-1},
$$

and the matrices $\mathbf{X}_2$ and $\mathbf{Y}_2$ satisfy $\mathbf{X}_2\mathbf{Y}_2^{\mathsf{T}} = \mathbf{1} - \mathbf{X}_1\mathbf{Y}_1$. If $\mathbf{D}_{\mathrm{d22}} = \mathbf{0}$, then $\mathbf{A}_{\mathrm{dc}} = \mathbf{A}_{\mathrm{d}_K}$, $\mathbf{B}_{\mathrm{dc}} = \mathbf{B}_{\mathrm{d}_K}$, $\mathbf{C}_{\mathrm{dc}} = \mathbf{C}_{\mathrm{d}_K}$, and $\mathbf{D}_{\mathrm{dc}} = \mathbf{D}_{\mathrm{d}_K}$.

Given $\mathbf{X}_1$ and $\mathbf{Y}_1$, the matrices $\mathbf{X}_2$ and $\mathbf{Y}_2$ can be found using a matrix decomposition, such as a LU decomposition or a Cholesky decomposition.

4.4 Mixed $\mathcal{H}_2$-$\mathcal{H}_\infty$-Optimal Control

The goal of mixed $\mathcal{H}_2$-$\mathcal{H}_\infty$-optimal control is to design a controller that minimizes the $\mathcal{H}_2$ norm of the closed-loop transfer matrix from $\mathbf{w}_1$ to $\mathbf{z}_1$, while ensuring that the $\mathcal{H}_\infty$ norm of the closed-loop transfer function from $\mathbf{w}_2$ to $\mathbf{z}_2$ is below a specified bound.

4.4.1 Mixed $\mathcal{H}_2$-$\mathcal{H}_\infty$-Optimal Full-State Feedback Control [19, pp. 329–330]

Consider the continuous-time generalized LTI plant $\mathcal{P}$ with state-space realization

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \begin{bmatrix} \mathbf{B}_{1,1} & \mathbf{B}_{1,2} \end{bmatrix} \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} + \mathbf{B}_2\mathbf{u}, \\
\begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{C}_{1,1} \\ \mathbf{C}_{1,2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{0} & \mathbf{D}_{11,12} \\ \mathbf{D}_{11,21} & \mathbf{D}_{11,22} \end{bmatrix} \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{D}_{12,1} \\ \mathbf{D}_{12,2} \end{bmatrix} \mathbf{u}, \\
\mathbf{y} &= \mathbf{x},
\end{aligned}
$$

where it is assumed that $(\mathbf{A}, \mathbf{B}_2)$ is stabilizable. A full-state feedback controller $\mathcal{K} = \mathbf{K} \in \mathbb{R}^{n_u \times n_x}$ (i.e., $\mathbf{u} = \mathbf{K}\mathbf{x}$) is to be designed to minimize the $\mathcal{H}_2$ norm of the closed-loop transfer matrix $\mathbf{T}_{11}(s)$ from the exogenous input $\mathbf{w}_1$ to the performance output $\mathbf{z}_1$ while ensuring the $\mathcal{H}_\infty$ norm of the closed-loop transfer matrix $\mathbf{T}_{22}(s)$ from the exogenous input $\mathbf{w}_2$ to the performance output $\mathbf{z}_2$ is less than γ_d, where

$$
\begin{aligned}
\mathbf{T}_{11}(s) &= \left(\mathbf{C}_{1,1} + \mathbf{D}_{12,1}\mathbf{K}\right) \left(s\mathbf{1} - (\mathbf{A} + \mathbf{B}_2\mathbf{K})\right)^{-1} \mathbf{B}_{1,1}, \\
\mathbf{T}_{22}(s) &= \left(\mathbf{C}_{1,2} + \mathbf{D}_{12,2}\mathbf{K}\right) \left(s\mathbf{1} - (\mathbf{A} + \mathbf{B}_2\mathbf{K})\right)^{-1} \mathbf{B}_{1,2} + \mathbf{D}_{11,22}.
\end{aligned}
$$

Synthesis Method 4.10. The mixed $\mathcal{H}_2$-$\mathcal{H}_\infty$-optimal full-state feedback controller is synthesized by solving for $\mathbf{P} \in \mathbb{S}^{n_x}$, $\mathbf{Z} \in \mathbb{S}^{n_w}$, $\mathbf{F} \in \mathbb{R}^{n_u \times n_x}$, and $\nu \in \mathbb{R}_{>0}$ that minimize $\mathcal{J}(\nu) = \nu$ subject to

$\mathbf{P} > 0$, $\mathbf{Z} > 0$,

$$\begin{bmatrix} \mathbf{AP} + \mathbf{PA}^{\mathsf{T}} - \mathbf{B}_2\mathbf{F} - \mathbf{F}^{\mathsf{T}}\mathbf{B}_2^{\mathsf{T}} & \mathbf{PC}_{1,1}^{\mathsf{T}} - \mathbf{F}^{\mathsf{T}}\mathbf{D}_{12,1}^{\mathsf{T}} \\ * & -\mathbf{1} \end{bmatrix} < 0,$$

$$\begin{bmatrix} \mathbf{AQ} + \mathbf{QA}^{\mathsf{T}} - \mathbf{B}_2\mathbf{F} - \mathbf{F}^{\mathsf{T}}\mathbf{B}_2^{\mathsf{T}} & \mathbf{B}_{1,2} & \mathbf{QC}_{1,2}^{\mathsf{T}} - \mathbf{F}^{\mathsf{T}}\mathbf{D}_{12,2}^{\mathsf{T}} \\ * & -\gamma_d\mathbf{1} & \mathbf{D}_{11,22}^{\mathsf{T}} \\ * & * & -\gamma_d\mathbf{1} \end{bmatrix} < 0,$$

$$\begin{bmatrix} \mathbf{Z} & \mathbf{B}_{1,1}^{\mathsf{T}} \\ * & \mathbf{P} \end{bmatrix} > 0,$$

$$\operatorname{tr}\mathbf{Z} < \nu.$$

The $\mathcal{H}_2$-optimal full-state feedback gain is recovered by $\mathbf{K} = \mathbf{FP}^{-1}$, the $\mathcal{H}_2$ norm of $\mathbf{T}_{11}(s)$ is less than $\mu = \sqrt{\nu}$, and the $\mathcal{H}_\infty$ norm of $\mathbf{T}_{22}(s)$ is less than γ_d.

4.4.2 Discrete-Time Mixed $\mathcal{H}_2$-$\mathcal{H}_\infty$-Optimal Full-State Feedback Control

Consider the discrete-time generalized LTI plant $\mathcal{P}$ with state-space realization

$$\mathbf{x}_{k+1} = \mathbf{A}_{\mathrm{d}}\mathbf{x}_k + \begin{bmatrix} \mathbf{B}_{\mathrm{d}1,1} & \mathbf{B}_{\mathrm{d}1,2} \end{bmatrix} \begin{bmatrix} \mathbf{w}_{1,k} \\ \mathbf{w}_{2,k} \end{bmatrix} + \mathbf{B}_{\mathrm{d}2}\mathbf{u}_k,$$

$$\begin{bmatrix} \mathbf{z}_{1,k} \\ \mathbf{z}_{2,k} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{\mathrm{d}1,1} \\ \mathbf{C}_{\mathrm{d}1,2} \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} \mathbf{0} & \mathbf{D}_{\mathrm{d}11,12} \\ \mathbf{D}_{\mathrm{d}11,21} & \mathbf{D}_{\mathrm{d}11,22} \end{bmatrix} \begin{bmatrix} \mathbf{w}_{1,k} \\ \mathbf{w}_{2,k} \end{bmatrix} + \begin{bmatrix} \mathbf{D}_{\mathrm{d}12,1} \\ \mathbf{D}_{\mathrm{d}12,2} \end{bmatrix} \mathbf{u}_k,$$

$$\mathbf{y}_k = \mathbf{x}_k,$$

where it is assumed that ($\mathbf{A}_{\mathrm{d}}$,$\mathbf{B}_{\mathrm{d}2}$) is stabilizable. A full-state feedback controller $\mathcal{K} = \mathbf{K}_{\mathrm{d}} \in \mathbb{R}^{n_u \times n_x}$ (i.e., $\mathbf{u}_k = \mathbf{K}_{\mathrm{d}}\mathbf{x}_k$) is to be designed to minimize the $\mathcal{H}_2$ norm of the closed loop transfer matrix $\mathbf{T}_{11}(z)$ from the exogenous input $\mathbf{w}_{1,k}$ to the performance output $\mathbf{z}_{1,k}$ while ensuring the $\mathcal{H}_\infty$ norm of the closed-loop transfer matrix $\mathbf{T}_{22}(z)$ from the exogenous input $\mathbf{w}_{2,k}$ to the performance output $\mathbf{z}_{2,k}$ is less than γ_d, where

$$\mathbf{T}_{11}(z) = (\mathbf{C}_{\mathrm{d}1,1} + \mathbf{D}_{\mathrm{d}12,1}\mathbf{K}_{\mathrm{d}})\left(z\mathbf{1} - (\mathbf{A}_{\mathrm{d}} + \mathbf{B}_{\mathrm{d}2}\mathbf{K}_{\mathrm{d}})\right)^{-1}\mathbf{B}_{\mathrm{d}1,1},$$

$$\mathbf{T}_{22}(z) = (\mathbf{C}_{\mathrm{d}1,2} + \mathbf{D}_{\mathrm{d}12,2}\mathbf{K}_{\mathrm{d}})\left(z\mathbf{1} - (\mathbf{A}_{\mathrm{d}} + \mathbf{B}_{\mathrm{d}2}\mathbf{K}_{\mathrm{d}})\right)^{-1}\mathbf{B}_{\mathrm{d}1,2} + \mathbf{D}_{\mathrm{d}11,22}.$$

Synthesis Method 4.11. The discrete-time mixed $\mathcal{H}_2$-$\mathcal{H}_\infty$-optimal full-state feedback controller is synthesized by solving for $\mathbf{P} \in \mathbb{S}^{n_x}$, $\mathbf{Z} \in \mathbb{S}^{n_w}$, $\mathbf{F}_{\mathrm{d}} \in \mathbb{R}^{n_u \times n_x}$, and $\nu \in \mathbb{R}_{>0}$ that minimize $\mathcal{J}(\nu) = \nu$ subject to $\mathbf{P} > 0$, $\mathbf{Z} > 0$,

$$\begin{bmatrix} \mathbf{P} & \mathbf{A}_{\mathrm{d}}\mathbf{P} - \mathbf{B}_{\mathrm{d}2}\mathbf{F}_{\mathrm{d}} & \mathbf{B}_{\mathrm{d}1,1} \\ * & \mathbf{P} & \mathbf{0} \\ * & * & \mathbf{1} \end{bmatrix} > 0,$$

$$\begin{bmatrix} \mathbf{P} & \mathbf{A}_{\mathrm{d}}\mathbf{P} - \mathbf{B}_{\mathrm{d}2}\mathbf{F}_{\mathrm{d}} & \mathbf{B}_{\mathrm{d}1,2} & \mathbf{0} \\ * & \mathbf{P} & \mathbf{0} & \mathbf{PC}_{\mathrm{d}1,2}^{\mathsf{T}} - \mathbf{F}_{\mathrm{d}}^{\mathsf{T}}\mathbf{D}_{\mathrm{d}12,2}^{\mathsf{T}} \\ * & * & \gamma_d\mathbf{1} & \mathbf{D}_{\mathrm{d}11,22}^{\mathsf{T}} \\ * & * & * & \gamma_d\mathbf{1} \end{bmatrix} > 0,$$

$$\begin{bmatrix} \mathbf{Z} & \mathbf{C}_{\mathrm{d}1,1}\mathbf{P} - \mathbf{D}_{\mathrm{d}12,1}\mathbf{F}_{\mathrm{d}} \\ * & \mathbf{P} \end{bmatrix} > 0.$$

$$\operatorname{tr}\mathbf{Z} < \nu.$$

The $\mathcal{H}_2$-optimal full-state feedback gain is recovered by $\mathbf{K}_{\mathrm{d}} = \mathbf{F}_{\mathrm{d}}\mathbf{P}^{-1}$, the $\mathcal{H}_2$ norm of $\mathbf{T}_{11}(z)$ is less than $\mu = \sqrt{\nu}$, and the $\mathcal{H}_\infty$ norm of $\mathbf{T}_{22}(z)$ is less than γ_d.

4.4.3 Mixed $\mathcal{H}_2$-$\mathcal{H}_\infty$-Optimal Dynamic Output Feedback Control [97,101]

Consider the continuous-time generalized LTI plant $\mathcal{P}$ with minimal state-space realization

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \begin{bmatrix}\mathbf{B}_{1,1} & \mathbf{B}_{1,2}\end{bmatrix}\begin{bmatrix}\mathbf{w}_1\\ \mathbf{w}_2\end{bmatrix} + \mathbf{B}_2\mathbf{u},\\
\begin{bmatrix}\mathbf{z}_1\\ \mathbf{z}_2\end{bmatrix} &= \begin{bmatrix}\mathbf{C}_{1,1}\\ \mathbf{C}_{1,2}\end{bmatrix}\mathbf{x} + \begin{bmatrix}\mathbf{D}_{11,11} & \mathbf{D}_{11,12}\\ \mathbf{D}_{11,21} & \mathbf{D}_{11,22}\end{bmatrix}\begin{bmatrix}\mathbf{w}_1\\ \mathbf{w}_2\end{bmatrix} + \begin{bmatrix}\mathbf{D}_{12,1}\\ \mathbf{D}_{12,2}\end{bmatrix}\mathbf{u},\\
\mathbf{y} &= \mathbf{C}_2\mathbf{x} + \begin{bmatrix}\mathbf{D}_{21,1} & \mathbf{D}_{21,2}\end{bmatrix}\begin{bmatrix}\mathbf{w}_1\\ \mathbf{w}_2\end{bmatrix} + \mathbf{D}_{22}\mathbf{u}.
\end{aligned}$$

A continuous-time dynamic output feedback LTI controller with state-space realization $(\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c, \mathbf{D}_c)$ is to be designed to minimize the $\mathcal{H}_2$ norm of the closed-loop transfer matrix $\mathbf{T}_{11}(s)$ from the exogenous input $\mathbf{w}_1$ to the performance output $\mathbf{z}_1$ while ensuring the $\mathcal{H}_\infty$ norm of the closed-loop transfer matrix $\mathbf{T}_{22}(s)$ from the exogenous input $\mathbf{w}_2$ to the performance output $\mathbf{z}_2$ is less than γ_d, where

$$\begin{aligned}
\mathbf{T}_{11}(s) &= \mathbf{C}_{\text{CL}1,1}\left(s\mathbf{1} - \mathbf{A}_{\text{CL}}\right)^{-1}\mathbf{B}_{\text{CL}1,1},\\
\mathbf{T}_{22}(s) &= \mathbf{C}_{\text{CL}1,2}\left(s\mathbf{1} - \mathbf{A}_{\text{CL}}\right)^{-1}\mathbf{B}_{\text{CL}1,2} + \mathbf{D}_{\text{CL}11,22},
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}_{\text{CL}} &= \begin{bmatrix}\mathbf{A} + \mathbf{B}_2\mathbf{D}_c\tilde{\mathbf{D}}^{-1}\mathbf{C}_2 & \mathbf{B}_2\left(\mathbf{1} + \mathbf{D}_c\tilde{\mathbf{D}}^{-1}\mathbf{D}_{22}\right)\mathbf{C}_c\\ \mathbf{B}_c\tilde{\mathbf{D}}^{-1}\mathbf{C}_2 & \mathbf{A}_c + \mathbf{B}_c\tilde{\mathbf{D}}^{-1}\mathbf{D}_{22}\mathbf{C}_c\end{bmatrix},\\
\mathbf{B}_{\text{CL}1,1} &= \begin{bmatrix}\mathbf{B}_{1,1} + \mathbf{B}_2\mathbf{D}_c\tilde{\mathbf{D}}^{-1}\mathbf{D}_{21,1}\\ \mathbf{B}_c\tilde{\mathbf{D}}^{-1}\mathbf{D}_{21,1}\end{bmatrix},\\
\mathbf{B}_{\text{CL}1,2} &= \begin{bmatrix}\mathbf{B}_{1,2} + \mathbf{B}_2\mathbf{D}_c\tilde{\mathbf{D}}^{-1}\mathbf{D}_{21,2}\\ \mathbf{B}_c\tilde{\mathbf{D}}^{-1}\mathbf{D}_{21,2}\end{bmatrix},\\
\mathbf{C}_{\text{CL}1,1} &= \begin{bmatrix}\mathbf{C}_{1,1} + \mathbf{D}_{12,1}\mathbf{D}_c\tilde{\mathbf{D}}^{-1}\mathbf{C}_{2,1} & \mathbf{D}_{12,1}\left(\mathbf{1} + \mathbf{D}_c\tilde{\mathbf{D}}^{-1}\mathbf{D}_{22}\right)\mathbf{C}_c\end{bmatrix},\\
\mathbf{C}_{\text{CL}1,2} &= \begin{bmatrix}\mathbf{C}_{1,2} + \mathbf{D}_{12,2}\mathbf{D}_c\tilde{\mathbf{D}}^{-1}\mathbf{C}_{2,2} & \mathbf{D}_{12,2}\left(\mathbf{1} + \mathbf{D}_c\tilde{\mathbf{D}}^{-1}\mathbf{D}_{22}\right)\mathbf{C}_c\end{bmatrix},\\
\mathbf{D}_{\text{CL}11,22} &= \mathbf{D}_{11,22} + \mathbf{D}_{12,2}\mathbf{D}_c\tilde{\mathbf{D}}^{-1}\mathbf{D}_{21,2},
\end{aligned}$$

and $\tilde{\mathbf{D}} = \mathbf{1} - \mathbf{D}_{22}\mathbf{D}_c$.

Synthesis Method 4.12. Solve for $\mathbf{A}_n \in \mathbb{R}^{n_x\times n_x}$, $\mathbf{B}_n \in \mathbb{R}^{n_x\times n_y}$, $\mathbf{C}_n \in \mathbb{R}^{n_u\times n_x}$, $\mathbf{D}_n \in \mathbb{R}^{n_u\times n_x}$,

$\mathbf{X}_1, \mathbf{Y}_1 \in \mathbb{S}^{n_x}$, $\mathbf{Z} \in \mathbb{S}^{n_{z_1}}$, and $\nu \in \mathbb{R}_{>0}$ that minimize $\mathcal{J}(\nu) = \nu$ subject to $\mathbf{X}_1 > 0$, $\mathbf{Y}_1 > 0$, $\mathbf{Z} > 0$,

$$\begin{bmatrix} \mathbf{N}_{11} & \mathbf{A} + \mathbf{A}_n^\mathsf{T} + \mathbf{B}_2\mathbf{D}_n\mathbf{C}_2 & \mathbf{B}_{1,1} + \mathbf{B}_2\mathbf{D}_n\mathbf{D}_{21,1} \\ * & \mathbf{X}_1\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{X}_1 + \mathbf{B}_n\mathbf{C}_2 + \mathbf{C}_2^\mathsf{T}\mathbf{B}_n^\mathsf{T} & \mathbf{X}_1\mathbf{B}_{1,1} + \mathbf{B}_n\mathbf{D}_{21,1} \\ * & * & -\mathbf{1} \end{bmatrix} < 0,$$

$$\begin{bmatrix} \mathbf{N}_{11} & \mathbf{A} + \mathbf{A}_n^\mathsf{T} + \mathbf{B}_2\mathbf{D}_n\mathbf{C}_2 & \mathbf{B}_{1,2} + \mathbf{B}_2\mathbf{D}_n\mathbf{D}_{21,2} & \mathbf{Y}_1^\mathsf{T}\mathbf{C}_{1,2}^\mathsf{T} + \mathbf{C}_n^\mathsf{T}\mathbf{D}_{12,2}^\mathsf{T} \\ * & \mathbf{X}_1\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{X}_1 + \mathbf{B}_n\mathbf{C}_2 + \mathbf{C}_2^\mathsf{T}\mathbf{B}_n^\mathsf{T} & \mathbf{X}_1\mathbf{B}_{1,2} + \mathbf{B}_n\mathbf{D}_{21,2} & \mathbf{C}_{1,2}^\mathsf{T} + \mathbf{C}_2^\mathsf{T}\mathbf{D}_n^\mathsf{T}\mathbf{D}_{12,2}^\mathsf{T} \\ * & * & -\gamma_d\mathbf{1} & \mathbf{D}_{11,22}^\mathsf{T} + \mathbf{D}_{21,2}^\mathsf{T}\mathbf{D}_n^\mathsf{T}\mathbf{D}_{12,2}^\mathsf{T} \\ * & * & * & -\gamma_d\mathbf{1} \end{bmatrix} < 0,$$

$$\begin{bmatrix} \mathbf{Y}_1 & \mathbf{1} & \mathbf{Y}_1\mathbf{C}_{1,1}^\mathsf{T} + \mathbf{C}_n^\mathsf{T}\mathbf{D}_{12,1}^\mathsf{T} \\ * & \mathbf{X}_1 & \mathbf{C}_{1,1}^\mathsf{T} + \mathbf{C}_2^\mathsf{T}\mathbf{D}_n^\mathsf{T}\mathbf{D}_{12,1}^\mathsf{T} \\ * & * & \mathbf{Z} \end{bmatrix} > 0,$$

$$\mathbf{D}_{11,11} + \mathbf{D}_{12,1}\mathbf{D}_n\mathbf{D}_{21,1} = \mathbf{0}, \tag{4.15}$$

$$\mathrm{tr}\mathbf{Z} < \nu,$$

where $\mathbf{N}_{11} = \mathbf{A}\mathbf{Y}_1 + \mathbf{Y}_1\mathbf{A}^\mathsf{T} + \mathbf{B}_2\mathbf{C}_n + \mathbf{C}_n^\mathsf{T}\mathbf{B}_2^\mathsf{T}$. The controller is recovered by

$$\begin{aligned} \mathbf{A}_c &= \mathbf{A}_{_K} - \mathbf{B}_c\left(\mathbf{1} - \mathbf{D}_{22}\mathbf{D}_c\right)^{-1}\mathbf{D}_{22}\mathbf{C}_c, \\ \mathbf{B}_c &= \mathbf{B}_{_K}\left(\mathbf{1} - \mathbf{D}_c\mathbf{D}_{22}\right), \\ \mathbf{C}_c &= \left(\mathbf{1} - \mathbf{D}_c\mathbf{D}_{22}\right)\mathbf{C}_{_K}, \\ \mathbf{D}_c &= \left(\mathbf{1} + \mathbf{D}_{_K}\mathbf{D}_{22}\right)^{-1}\mathbf{D}_{_K}, \end{aligned}$$

where

$$\begin{bmatrix} \mathbf{A}_{_K} & \mathbf{B}_{_K} \\ \mathbf{C}_{_K} & \mathbf{D}_{_K} \end{bmatrix} = \begin{bmatrix} \mathbf{X}_2 & \mathbf{X}_1\mathbf{B}_2 \\ \mathbf{0} & \mathbf{1} \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathbf{A}_n & \mathbf{B}_n \\ \mathbf{C}_n & \mathbf{D}_n \end{bmatrix} - \begin{bmatrix} \mathbf{X}_1\mathbf{A}\mathbf{Y}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{bmatrix} \mathbf{Y}_2^\mathsf{T} & \mathbf{0} \\ \mathbf{C}_2\mathbf{Y}_1 & \mathbf{1} \end{bmatrix}^{-1},$$

and the matrices $\mathbf{X}_2$ and $\mathbf{Y}_2$ satisfy $\mathbf{X}_2\mathbf{Y}_2^\mathsf{T} = \mathbf{1} - \mathbf{X}_1\mathbf{Y}_1$. If $\mathbf{D}_{22} = \mathbf{0}$, then $\mathbf{A}_c = \mathbf{A}_{_K}$, $\mathbf{B}_c = \mathbf{B}_{_K}$, $\mathbf{C}_c = \mathbf{C}_{_K}$, and $\mathbf{D}_c = \mathbf{D}_{_K}$.

Given $\mathbf{X}_1$ and $\mathbf{Y}_1$, the matrices $\mathbf{X}_2$ and $\mathbf{Y}_2$ can be found using a matrix decomposition, such as a LU decomposition or a Cholesky decomposition.

If $\mathbf{D}_{11,11} = \mathbf{0}$, $\mathbf{D}_{12,1} \neq \mathbf{0}$, and $\mathbf{D}_{21,1} \neq \mathbf{0}$, then it is often simplest to choose $\mathbf{D}_n = \mathbf{0}$ in order to satisfy the equality constraint of (4.15).

4.4.4 Discrete-Time Mixed $\mathcal{H}_2$-$\mathcal{H}_\infty$-Optimal Dynamic Output Feedback Control

Consider the discrete-time generalized LTI plant $\mathcal{P}$ with minimal state-space realization

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}_\mathrm{d}\mathbf{x}_k + \begin{bmatrix} \mathbf{B}_{\mathrm{d}1,1} & \mathbf{B}_{\mathrm{d}1,2} \end{bmatrix} \begin{bmatrix} \mathbf{w}_{1,k} \\ \mathbf{w}_{2,k} \end{bmatrix} + \mathbf{B}_{\mathrm{d}2}\mathbf{u}_k, \\ \begin{bmatrix} \mathbf{z}_{1,k} \\ \mathbf{z}_{2,k} \end{bmatrix} &= \begin{bmatrix} \mathbf{C}_{\mathrm{d}1,1} \\ \mathbf{C}_{\mathrm{d}1,2} \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} \mathbf{D}_{\mathrm{d}11,11} & \mathbf{D}_{\mathrm{d}11,12} \\ \mathbf{D}_{\mathrm{d}11,21} & \mathbf{D}_{\mathrm{d}11,22} \end{bmatrix} \begin{bmatrix} \mathbf{w}_{1,k} \\ \mathbf{w}_{2,k} \end{bmatrix} + \begin{bmatrix} \mathbf{D}_{\mathrm{d}12,1} \\ \mathbf{D}_{\mathrm{d}12,2} \end{bmatrix} \mathbf{u}_k, \\ \mathbf{y}_k &= \mathbf{C}_{\mathrm{d}2}\mathbf{x}_k + \begin{bmatrix} \mathbf{D}_{\mathrm{d}21,1} & \mathbf{D}_{\mathrm{d}21,2} \end{bmatrix} \begin{bmatrix} \mathbf{w}_{1,k} \\ \mathbf{w}_{2,k} \end{bmatrix} + \mathbf{D}_{\mathrm{d}22}\mathbf{u}_k. \end{aligned}$$

A discrete-time dynamic output feedback LTI controller with state-space realization $(\mathbf{A}_\mathrm{dc}, \mathbf{B}_\mathrm{dc}, \mathbf{C}_\mathrm{dc}, \mathbf{D}_\mathrm{dc})$ is to be designed to minimize the $\mathcal{H}_2$ norm of the closed loop transfer matrix $\mathbf{T}_{11}(z)$ from the exogenous input $\mathbf{w}_{1,k}$ to the performance output $\mathbf{z}_{1,k}$ while ensuring the $\mathcal{H}_\infty$ norm of the closed-loop transfer matrix $\mathbf{T}_{22}(z)$ from the exogenous input $\mathbf{w}_{2,k}$ to the performance output $\mathbf{z}_{2,k}$ is less than γ_d, where

$$\begin{aligned}
\mathbf{T}_{11}(z) &= \mathbf{C}_{\mathrm{d_{CL}}1,1}\left(z\mathbf{1} - \mathbf{A}_{\mathrm{d_{CL}}}\right)^{-1}\mathbf{B}_{\mathrm{d_{CL}}1,1}, \\
\mathbf{T}_{22}(z) &= \mathbf{C}_{\mathrm{d_{CL}}1,2}\left(z\mathbf{1} - \mathbf{A}_{\mathrm{d_{CL}}}\right)^{-1}\mathbf{B}_{\mathrm{d_{CL}}1,2} + \mathbf{D}_{\mathrm{d_{CL}}11,22},
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}_{\mathrm{d_{CL}}} &= \begin{bmatrix} \mathbf{A}_{\mathrm{d}} + \mathbf{B}_{\mathrm{d}2}\mathbf{D}_{\mathrm{d}c}\tilde{\mathbf{D}}_{\mathrm{d}}^{-1}\mathbf{C}_{\mathrm{d}2} & \mathbf{B}_{\mathrm{d}2}\left(\mathbf{1} + \mathbf{D}_{\mathrm{d}c}\tilde{\mathbf{D}}_{\mathrm{d}}^{-1}\mathbf{D}_{\mathrm{d}22}\right)\mathbf{C}_{\mathrm{d}c} \\ \mathbf{B}_{\mathrm{d}c}\tilde{\mathbf{D}}_{\mathrm{d}}^{-1}\mathbf{C}_{\mathrm{d}2} & \mathbf{A}_{\mathrm{d}c} + \mathbf{B}_{\mathrm{d}c}\tilde{\mathbf{D}}_{\mathrm{d}}^{-1}\mathbf{D}_{\mathrm{d}22}\mathbf{C}_{\mathrm{d}c} \end{bmatrix}, \\
\mathbf{B}_{\mathrm{d_{CL}}1,1} &= \begin{bmatrix} \mathbf{B}_{\mathrm{d}1,1} + \mathbf{B}_{\mathrm{d}2}\mathbf{D}_{\mathrm{d}c}\tilde{\mathbf{D}}_{\mathrm{d}}^{-1}\mathbf{D}_{\mathrm{d}21,1} \\ \mathbf{B}_{\mathrm{d}c}\tilde{\mathbf{D}}_{\mathrm{d}}^{-1}\mathbf{D}_{\mathrm{d}21,1} \end{bmatrix}, \\
\mathbf{B}_{\mathrm{d_{CL}}1,2} &= \begin{bmatrix} \mathbf{B}_{\mathrm{d}1,2} + \mathbf{B}_{\mathrm{d}2}\mathbf{D}_{\mathrm{d}c}\tilde{\mathbf{D}}_{\mathrm{d}}^{-1}\mathbf{D}_{\mathrm{d}21,2} \\ \mathbf{B}_{\mathrm{d}c}\tilde{\mathbf{D}}_{\mathrm{d}}^{-1}\mathbf{D}_{\mathrm{d}21,2} \end{bmatrix}, \\
\mathbf{C}_{\mathrm{d_{CL}}1,1} &= \begin{bmatrix} \mathbf{C}_{\mathrm{d}1,1} + \mathbf{D}_{\mathrm{d}12,1}\mathbf{D}_{\mathrm{d}c}\tilde{\mathbf{D}}_{\mathrm{d}}^{-1}\mathbf{C}_{\mathrm{d}2,1} & \mathbf{D}_{\mathrm{d}12,1}\left(\mathbf{1} + \mathbf{D}_{\mathrm{d}c}\tilde{\mathbf{D}}_{\mathrm{d}}^{-1}\mathbf{D}_{\mathrm{d}22}\right)\mathbf{C}_{\mathrm{d}c} \end{bmatrix}, \\
\mathbf{C}_{\mathrm{d_{CL}}1,2} &= \begin{bmatrix} \mathbf{C}_{\mathrm{d}1,2} + \mathbf{D}_{\mathrm{d}12,2}\mathbf{D}_{\mathrm{d}c}\tilde{\mathbf{D}}_{\mathrm{d}}^{-1}\mathbf{C}_{\mathrm{d}2,2} & \mathbf{D}_{\mathrm{d}12,2}\left(\mathbf{1} + \mathbf{D}_{\mathrm{d}c}\tilde{\mathbf{D}}_{\mathrm{d}}^{-1}\mathbf{D}_{\mathrm{d}22}\right)\mathbf{C}_{\mathrm{d}c} \end{bmatrix}, \\
\mathbf{D}_{\mathrm{d_{CL}}11,22} &= \mathbf{D}_{\mathrm{d}11,22} + \mathbf{D}_{\mathrm{d}12,2}\mathbf{D}_{\mathrm{d}c}\tilde{\mathbf{D}}_{\mathrm{d}}^{-1}\mathbf{D}_{\mathrm{d}21,2},
\end{aligned}$$

and $\tilde{\mathbf{D}}_{\mathrm{d}} = \mathbf{1} - \mathbf{D}_{\mathrm{d}22}\mathbf{D}_{\mathrm{d}c}$.

Synthesis Method 4.13. Solve for $\mathbf{A}_{\mathrm{d}n} \in \mathbb{R}^{n_x \times n_x}$, $\mathbf{B}_{\mathrm{d}n} \in \mathbb{R}^{n_x \times n_y}$, $\mathbf{C}_{\mathrm{d}n} \in \mathbb{R}^{n_u \times n_x}$, $\mathbf{D}_{\mathrm{d}n} \in \mathbb{R}^{n_u \times n_y}$, $\mathbf{X}_1$, $\mathbf{Y}_1 \in \mathbb{S}^{n_x}$, $\mathbf{Z} \in \mathbb{S}^{n_{z_1}}$, and $\nu \in \mathbb{R}_{>0}$ that minimize $\mathcal{J}(\nu) = \nu$ subject to $\mathbf{X}_1 > 0$, $\mathbf{Y}_1 > 0$, $\mathbf{Z} > 0$,

$$\begin{bmatrix} \mathbf{X}_1 & \mathbf{1} & \mathbf{X}_1\mathbf{A}_{\mathrm{d}} + \mathbf{B}_{\mathrm{d}n}\mathbf{C}_{\mathrm{d}2} & \mathbf{A}_{\mathrm{d}n} & \mathbf{X}_1\mathbf{B}_{\mathrm{d}1,1} + \mathbf{B}_{\mathrm{d}n}\mathbf{D}_{\mathrm{d}21,1} \\ * & \mathbf{Y}_1 & \mathbf{A}_{\mathrm{d}} + \mathbf{B}_{\mathrm{d}2}\mathbf{D}_{\mathrm{d}n}\mathbf{C}_{\mathrm{d}2} & \mathbf{A}_{\mathrm{d}}\mathbf{Y}_1 + \mathbf{B}_{\mathrm{d}2}\mathbf{C}_{\mathrm{d}n} & \mathbf{B}_{\mathrm{d}1,1} + \mathbf{B}_{\mathrm{d}2}\mathbf{D}_{\mathrm{d}n}\mathbf{D}_{\mathrm{d}21,1} \\ * & * & \mathbf{X}_1 & \mathbf{1} & \mathbf{0} \\ * & * & * & \mathbf{Y}_1 & \mathbf{0} \\ * & * & * & * & \mathbf{1} \end{bmatrix} > 0,$$

$$\begin{bmatrix} \mathbf{X}_1 & \mathbf{1} & \mathbf{X}_1\mathbf{A}_{\mathrm{d}} + \mathbf{B}_{\mathrm{d}n}\mathbf{C}_{\mathrm{d}2} & \mathbf{A}_{\mathrm{d}n} & \mathbf{X}_1\mathbf{B}_{\mathrm{d}1,2} + \mathbf{B}_{\mathrm{d}n}\mathbf{D}_{\mathrm{d}21,2} & \mathbf{0} \\ * & \mathbf{Y}_1 & \mathbf{A}_{\mathrm{d}} + \mathbf{B}_{\mathrm{d}2}\mathbf{D}_{\mathrm{d}n}\mathbf{C}_{\mathrm{d}2} & \mathbf{A}_{\mathrm{d}}\mathbf{Y}_1 + \mathbf{B}_{\mathrm{d}2}\mathbf{C}_{\mathrm{d}n} & \mathbf{B}_{\mathrm{d}1,2} + \mathbf{B}_{\mathrm{d}2}\mathbf{D}_{\mathrm{d}n}\mathbf{D}_{\mathrm{d}21,2} & \mathbf{0} \\ * & * & \mathbf{X}_1 & \mathbf{1} & \mathbf{0} & \mathbf{C}_{\mathrm{d}1,2}^{\mathsf{T}} + \mathbf{C}_{\mathrm{d}2}^{\mathsf{T}}\mathbf{D}_{\mathrm{d}n}^{\mathsf{T}}\mathbf{D}_{\mathrm{d}12,2}^{\mathsf{T}} \\ * & * & * & \mathbf{Y}_1 & \mathbf{0} & \mathbf{Y}_1\mathbf{C}_{\mathrm{d}1,2}^{\mathsf{T}} + \mathbf{C}_{\mathrm{d}n}^{\mathsf{T}}\mathbf{D}_{\mathrm{d}12,2}^{\mathsf{T}} \\ * & * & * & * & -\gamma_d\mathbf{1} & \mathbf{D}_{\mathrm{d}11,22}^{\mathsf{T}} + \mathbf{D}_{\mathrm{d}21,2}^{\mathsf{T}}\mathbf{D}_{\mathrm{d}n}^{\mathsf{T}}\mathbf{D}_{\mathrm{d}12,2}^{\mathsf{T}} \\ * & * & * & * & * & -\gamma_d\mathbf{1} \end{bmatrix} > 0,$$

$$\begin{bmatrix} \mathbf{Z} & \mathbf{C}_{\mathrm{d}1,1} + \mathbf{D}_{\mathrm{d}12,1}\mathbf{D}_{\mathrm{d}n}\mathbf{C}_{\mathrm{d}2} & \mathbf{C}_{\mathrm{d}1,1}\mathbf{Y}_1^{\mathsf{T}} + \mathbf{D}_{\mathrm{d}12,1}\mathbf{C}_{\mathrm{d}n} \\ * & \mathbf{X}_1 & \mathbf{1} \\ * & * & \mathbf{Y}_1 \end{bmatrix} > 0,$$

$$\mathbf{D}_{\mathrm{d}11,11} + \mathbf{D}_{\mathrm{d}12,1}\mathbf{D}_{\mathrm{d}n}\mathbf{D}_{\mathrm{d}21,1} = \mathbf{0}, \tag{4.16}$$

$$\begin{bmatrix} \mathbf{X}_1 & \mathbf{1} \\ * & \mathbf{Y}_1 \end{bmatrix} > 0,$$

$$\mathrm{tr}\mathbf{Z} < \nu.$$

The controller is recovered by

$$\begin{aligned}
\mathbf{A}_{\mathrm{d}c} &= \mathbf{A}_{\mathrm{d}_K} - \mathbf{B}_{\mathrm{d}c}\left(\mathbf{1} - \mathbf{D}_{\mathrm{d}22}\mathbf{D}_{\mathrm{d}c}\right)^{-1}\mathbf{D}_{\mathrm{d}22}\mathbf{C}_{\mathrm{d}c},\\
\mathbf{B}_{\mathrm{d}c} &= \mathbf{B}_{\mathrm{d}_K}\left(\mathbf{1} - \mathbf{D}_{\mathrm{d}c}\mathbf{D}_{\mathrm{d}22}\right),\\
\mathbf{C}_{\mathrm{d}c} &= \left(\mathbf{1} - \mathbf{D}_{\mathrm{d}c}\mathbf{D}_{\mathrm{d}22}\right)\mathbf{C}_{\mathrm{d}_K},\\
\mathbf{D}_{\mathrm{d}c} &= \left(\mathbf{1} + \mathbf{D}_{\mathrm{d}_K}\mathbf{D}_{\mathrm{d}22}\right)^{-1}\mathbf{D}_{\mathrm{d}_K},
\end{aligned}$$

where

$$\begin{bmatrix}\mathbf{A}_{\mathrm{d}_K} & \mathbf{B}_{\mathrm{d}_K}\\ \mathbf{C}_{\mathrm{d}_K} & \mathbf{D}_{\mathrm{d}_K}\end{bmatrix} = \begin{bmatrix}\mathbf{X}_2 & \mathbf{X}_1\mathbf{B}_{\mathrm{d}2}\\ \mathbf{0} & \mathbf{1}\end{bmatrix}^{-1}\left(\begin{bmatrix}\mathbf{A}_{\mathrm{d}n} & \mathbf{B}_{\mathrm{d}n}\\ \mathbf{C}_{\mathrm{d}n} & \mathbf{D}_{\mathrm{d}n}\end{bmatrix} - \begin{bmatrix}\mathbf{X}_1\mathbf{A}_{\mathrm{d}}\mathbf{Y}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\right)\begin{bmatrix}\mathbf{Y}_2^{\mathsf{T}} & \mathbf{0}\\ \mathbf{C}_{\mathrm{d}2}\mathbf{Y}_1 & \mathbf{1}\end{bmatrix}^{-1},$$

and the matrices $\mathbf{X}_2$ and $\mathbf{Y}_2$ satisfy $\mathbf{X}_2\mathbf{Y}_2^{\mathsf{T}} = \mathbf{1} - \mathbf{X}_1\mathbf{Y}_1$. If $\mathbf{D}_{\mathrm{d}22} = \mathbf{0}$, then $\mathbf{A}_{\mathrm{d}c} = \mathbf{A}_{\mathrm{d}_K}$, $\mathbf{B}_{\mathrm{d}c} = \mathbf{B}_{\mathrm{d}_K}$, $\mathbf{C}_{\mathrm{d}c} = \mathbf{C}_{\mathrm{d}_K}$, and $\mathbf{D}_{\mathrm{d}c} = \mathbf{D}_{\mathrm{d}_K}$.

Given $\mathbf{X}_1$ and $\mathbf{Y}_1$, the matrices $\mathbf{X}_2$ and $\mathbf{Y}_2$ can be found using a matrix decomposition, such as a LU decomposition or a Cholesky decomposition.

If $\mathbf{D}_{\mathrm{d}11,11} = \mathbf{0}$, $\mathbf{D}_{\mathrm{d}12,1} \neq \mathbf{0}$, and $\mathbf{D}_{\mathrm{d}21,1} \neq \mathbf{0}$, then it is often simplest to choose $\mathbf{D}_{\mathrm{d}n} = \mathbf{0}$ in order to satisfy the equality constraint of (4.16).

5 LMIs in Optimal Estimation

This section presents controller synthesis methods using LMIs for a number of well-known optimal state-estimation problems. The derivation of the LMIs used for synthesis is provided in some cases, while longer derivations can be found in the cited references.

5.1 $\mathcal{H}_2$-Optimal State Estimation

The goal of $\mathcal{H}_2$-optimal state estimation is to design an observer that minimizes the $\mathcal{H}_2$ norm of the closed-loop transfer matrix from **w** to **z**.

5.1.1 $\mathcal{H}_2$-Optimal Observer [19, p. 296]

Consider the continuous-time generalized plant $\mathcal{P}$ with state-space realization

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{w},\\ \mathbf{y} &= \mathbf{C}_2\mathbf{x} + \mathbf{D}_{21}\mathbf{w},\end{aligned}$$

where it is assumed that $(\mathbf{A},\mathbf{C}_2)$ is detectable. An observer of the form

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{L}\left(\mathbf{y} - \hat{\mathbf{y}}\right),\\ \hat{\mathbf{y}} &= \mathbf{C}_2\hat{\mathbf{x}},\end{aligned}$$

is to be designed, where $\mathbf{L} \in \mathbb{R}^{n_x \times n_y}$ is the observer gain. Defining the error state $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$, the error dynamics are found to be

$$\dot{\mathbf{e}} = \left(\mathbf{A} - \mathbf{L}\mathbf{C}_2\right)\mathbf{e} + \left(\mathbf{B}_1 - \mathbf{L}\mathbf{D}_{21}\right)\mathbf{w},$$

and the performance output is defined as

$$\mathbf{z} = \mathbf{C}_1\mathbf{e}.$$

The observer gain **L** is to be designed such that the $\mathcal{H}_2$ norm of the transfer matrix from **w** to **z**, given by

$$\mathbf{T}(s) = \mathbf{C}_1\left(s\mathbf{1} - \left(\mathbf{A} - \mathbf{L}\mathbf{C}_2\right)\right)^{-1}\left(\mathbf{B}_1 - \mathbf{L}\mathbf{D}_{21}\right),$$

is minimized. Minimizing the $\mathcal{H}_2$ norm of the transfer matrix $\mathbf{T}(s)$ is equivalent to minimizing $\mathcal{J}(\mu) = \mu^2$ subject to

$$\begin{bmatrix}\mathbf{P}\left(\mathbf{A} - \mathbf{L}\mathbf{C}_2\right) + \left(\mathbf{A} - \mathbf{L}\mathbf{C}_2\right)^{\mathsf{T}}\mathbf{P} & \mathbf{P}\left(\mathbf{B}_1 - \mathbf{L}\mathbf{D}_{21}\right)\\ * & -\mathbf{1}\end{bmatrix} < 0, \tag{5.1}$$

$$\begin{bmatrix}\mathbf{P} & \mathbf{C}_1^{\mathsf{T}}\\ * & \mathbf{Z}\end{bmatrix} > 0, \tag{5.2}$$

$$\mathrm{tr}\mathbf{Z} < \mu^2, \tag{5.3}$$

where $\mathbf{P} \in \mathbb{S}^{n_x}$, $\mathbf{Z} \in \mathbb{S}^{n_z}$, $\mu \in \mathbb{R}_{>0}$, $\mathbf{P} > 0$, and $\mathbf{Z} > 0$. A change of variables is performed with $\mathbf{G} = \mathbf{P}\mathbf{L}$ and $\nu = \mu^2$, which transforms (5.1) and (5.3) into LMIs in the variables **P**, **G**, **Z**, and ν given by

$$\begin{bmatrix}\mathbf{P}\mathbf{A} + \mathbf{A}^{\mathsf{T}}\mathbf{P} - \mathbf{G}\mathbf{C}_2 - \mathbf{C}_2^{\mathsf{T}}\mathbf{G}^{\mathsf{T}} & \mathbf{P}\mathbf{B}_1 - \mathbf{G}\mathbf{D}_{21}\\ * & -\mathbf{1}\end{bmatrix} < 0, \tag{5.4}$$

$$\mathrm{tr}\mathbf{Z} < \nu. \tag{5.5}$$

Synthesis Method 5.1. The $\mathcal{H}_2$-optimal observer gain is synthesized by solving for $\mathbf{P} \in \mathbb{S}^{n_x}$, $\mathbf{Z} \in \mathbb{S}^{n_z}$, $\mathbf{G} \in \mathbb{R}^{n_x \times n_y}$, and $\nu \in \mathbb{R}_{>0}$ that minimize $\mathcal{J}(\nu) = \nu$ subject to $\mathbf{P} > 0$, $\mathbf{Z} > 0$, (5.2), (5.4), and (5.5). The $\mathcal{H}_2$-optimal observer gain is recovered by $\mathbf{L} = \mathbf{P}^{-1}\mathbf{G}$ and the $\mathcal{H}_2$ norm of $\mathbf{T}(s)$ is $\mu = \sqrt{\nu}$.

5.1.2 Discrete-Time $\mathcal{H}_2$-Optimal Observer

Consider the discrete-time generalized LTI plant $\mathcal{P}$ with state-space realization

$$\begin{aligned}\mathbf{x}_{k+1} &= \mathbf{A}_{\mathrm{d}}\mathbf{x}_k + \mathbf{B}_{\mathrm{d}1}\mathbf{w}_k,\\ \mathbf{y}_k &= \mathbf{C}_{\mathrm{d}2}\mathbf{x}_k + \mathbf{D}_{\mathrm{d}21}\mathbf{w}_k,\end{aligned}$$

where it is assumed that ($\mathbf{A}_{\mathrm{d}}$,$\mathbf{C}_{\mathrm{d}2}$) is detectable. An observer of the form

$$\begin{aligned}\hat{\mathbf{x}}_{k+1} &= \mathbf{A}_{\mathrm{d}}\hat{\mathbf{x}}_k + \mathbf{L}_{\mathrm{d}}\left(\mathbf{y}_k - \hat{\mathbf{y}}_k\right),\\ \hat{\mathbf{y}}_k &= \mathbf{C}_{\mathrm{d}2}\hat{\mathbf{x}}_k,\end{aligned}$$

is to be designed, where $\mathbf{L}_{\mathrm{d}} \in \mathbb{R}^{n_x \times n_y}$ is the observer gain. Defining the error state $\mathbf{e}_k = \mathbf{x}_k - \hat{\mathbf{x}}_k$, the error dynamics are found to be

$$\mathbf{e}_{k+1} = \left(\mathbf{A}_{\mathrm{d}} - \mathbf{L}_{\mathrm{d}}\mathbf{C}_{\mathrm{d}2}\right)\mathbf{e}_k + \left(\mathbf{B}_{\mathrm{d}1} - \mathbf{L}_{\mathrm{d}}\mathbf{D}_{\mathrm{d}21}\right)\mathbf{w}_k,$$

and the performance output is defined as

$$\mathbf{z}_k = \mathbf{C}_{\mathrm{d}1}\mathbf{e}_k.$$

The observer gain $\mathbf{L}_{\mathrm{d}}$ is to be designed such that the $\mathcal{H}_2$ of the transfer matrix from $\mathbf{w}_k$ to $\mathbf{z}_k$, given by

$$\mathbf{T}(z) = \mathbf{C}_{\mathrm{d}1}\left(z\mathbf{1} - \left(\mathbf{A}_{\mathrm{d}} - \mathbf{L}_{\mathrm{d}}\mathbf{C}_{\mathrm{d}2}\right)\right)^{-1}\left(\mathbf{B}_{\mathrm{d}1} - \mathbf{L}_{\mathrm{d}}\mathbf{D}_{\mathrm{d}21}\right),$$

is minimized.

Synthesis Method 5.2. The discrete-time $\mathcal{H}_2$-optimal observer gain is synthesized by solving for $\mathbf{P} \in \mathbb{S}^{n_x}$, $\mathbf{Z} \in \mathbb{S}^{n_z}$, $\mathbf{G}_{\mathrm{d}} \in \mathbb{R}^{n_x \times n_y}$, and $\nu \in \mathbb{R}_{>0}$ that minimize $\mathcal{J}(\nu) = \nu$ subject to $\mathbf{P} > 0$, $\mathbf{Z} > 0$,

$$\begin{aligned}\begin{bmatrix}\mathbf{P} & \mathbf{P}\mathbf{A}_{\mathrm{d}} - \mathbf{G}_{\mathrm{d}}\mathbf{C}_{\mathrm{d}2} & \mathbf{P}\mathbf{B}_{\mathrm{d}1} - \mathbf{G}_{\mathrm{d}}\mathbf{D}_{\mathrm{d}21}\\ * & \mathbf{P} & \mathbf{0}\\ * & * & \mathbf{1}\end{bmatrix} &> 0,\\ \begin{bmatrix}\mathbf{Z} & \mathbf{P}\mathbf{C}_{\mathrm{d}1}\\ * & \mathbf{P}\end{bmatrix} &> 0.\\ \mathrm{tr}\mathbf{Z} &< \nu.\end{aligned}$$

The $\mathcal{H}_2$-optimal observer gain is recovered by $\mathbf{L}_{\mathrm{d}} = \mathbf{P}^{-1}\mathbf{G}_{\mathrm{d}}$ and the $\mathcal{H}_2$ norm of $\mathbf{T}(z)$ is $\mu = \sqrt{\nu}$.

5.2 $\mathcal{H}_\infty$-Optimal State Estimation

The goal of $\mathcal{H}_\infty$-optimal state estimation is to design an observer that minimizes the $\mathcal{H}_\infty$ norm of the closed-loop transfer matrix from $\mathbf{w}$ to $\mathbf{z}$.

5.2.1 $\mathcal{H}_\infty$-Optimal Observer [19, p. 295]

Consider the continuous-time generalized plant $\mathcal{P}$ with state-space realization

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{w},\\ \mathbf{y} &= \mathbf{C}_2\mathbf{x} + \mathbf{D}_{21}\mathbf{w},\end{aligned}$$

where it is assumed that ($\mathbf{A}$,$\mathbf{C}_2$) is detectable. An observer of the form

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{L}\left(\mathbf{y} - \hat{\mathbf{y}}\right),\\ \hat{\mathbf{y}} &= \mathbf{C}_2\hat{\mathbf{x}},\end{aligned}$$

is to be designed, where $\mathbf{L} \in \mathbb{R}^{n_x \times n_y}$ is the observer gain. Defining the error state $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$, the error dynamics are found to be

$$\dot{\mathbf{e}} = \left(\mathbf{A} - \mathbf{L}\mathbf{C}_2\right)\mathbf{e} + \left(\mathbf{B}_1 - \mathbf{L}\mathbf{D}_{21}\right)\mathbf{w},$$

and the performance output is defined as

$$\mathbf{z} = \mathbf{C}_1\mathbf{e} + \mathbf{D}_{11}\mathbf{w}.$$

The observer gain $\mathbf{L}$ is to be designed such that the $\mathcal{H}_\infty$ of the transfer matrix from $\mathbf{w}$ to $\mathbf{z}$, given by

$$\mathbf{T}(s) = \mathbf{C}_1\left(s\mathbf{1} - \left(\mathbf{A} - \mathbf{L}\mathbf{C}_2\right)\right)^{-1}\left(\mathbf{B}_1 - \mathbf{L}\mathbf{D}_{21}\right) + \mathbf{D}_{11},$$

is minimized.

Synthesis Method 5.3. The $\mathcal{H}_\infty$-optimal observer gain is synthesized by solving for $\mathbf{P} \in \mathbb{S}^{n_x}$, $\mathbf{G} \in \mathbb{R}^{n_x \times n_y}$, and $\gamma \in \mathbb{R}_{>0}$ that minimize $\mathcal{J}(\gamma) = \gamma$ subject to $\mathbf{P} > 0$ and

$$\begin{bmatrix}\mathbf{P}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{P} - \mathbf{G}\mathbf{C}_2 - \mathbf{C}_2^\mathsf{T}\mathbf{G}^\mathsf{T} & \mathbf{P}\mathbf{B}_1 - \mathbf{G}\mathbf{D}_{21} & \mathbf{C}_1\\ * & -\gamma\mathbf{1} & \mathbf{D}_{11}^\mathsf{T}\\ * & * & -\gamma\mathbf{1}\end{bmatrix} < 0.$$

The $\mathcal{H}_\infty$-optimal observer gain is recovered by $\mathbf{L} = \mathbf{P}^{-1}\mathbf{G}$ and the $\mathcal{H}_\infty$ norm of $\mathbf{T}(s)$ is γ.

5.2.2 Discrete-Time $\mathcal{H}_\infty$-Optimal Observer

Consider the discrete-time LTI plant $\mathcal{G}$ with state-space realization

$$\begin{aligned}\mathbf{x}_{k+1} &= \mathbf{A}_\mathrm{d}\mathbf{x}_k + \mathbf{B}_{\mathrm{d}1}\mathbf{w}_k,\\ \mathbf{y}_k &= \mathbf{C}_{\mathrm{d}2}\mathbf{x}_k + \mathbf{D}_{\mathrm{d}21}\mathbf{w}_k,\end{aligned}$$

where it is assumed that ($\mathbf{A}_\mathrm{d}$,$\mathbf{C}_{\mathrm{d}2}$) is detectable. An observer of the form

$$\begin{aligned}\hat{\mathbf{x}}_{k+1} &= \mathbf{A}_\mathrm{d}\hat{\mathbf{x}}_k + \mathbf{L}_\mathrm{d}\left(\mathbf{y}_k - \hat{\mathbf{y}}_k\right),\\ \hat{\mathbf{y}}_k &= \mathbf{C}_{\mathrm{d}2}\hat{\mathbf{x}}_k,\end{aligned}$$

is to be designed, where $\mathbf{L}_\mathrm{d} \in \mathbb{R}^{n_x \times n_y}$ is the observer gain. Defining the error state $\mathbf{e}_k = \mathbf{x}_k - \hat{\mathbf{x}}_k$, the error dynamics are found to be

$$\mathbf{e}_{k+1} = \left(\mathbf{A}_\mathrm{d} - \mathbf{L}_\mathrm{d}\mathbf{C}_{\mathrm{d}2}\right)\mathbf{e}_k + \left(\mathbf{B}_{\mathrm{d}1} - \mathbf{L}_\mathrm{d}\mathbf{D}_{\mathrm{d}21}\right)\mathbf{w}_k,$$

and the performance output is defined as

$$\mathbf{z}_k = \mathbf{C}_{\mathrm{d}1}\mathbf{e}_k + \mathbf{D}_{\mathrm{d}11}\mathbf{w}_k.$$

The observer gain $\mathbf{L}_{\mathrm{d}}$ is to be designed such that the $\mathcal{H}_\infty$ of the transfer matrix from $\mathbf{w}_k$ to $\mathbf{z}_k$, given by

$$\mathbf{T}(z) = \mathbf{C}_{\mathrm{d}1}\left(z\mathbf{1} - (\mathbf{A}_{\mathrm{d}} - \mathbf{L}_{\mathrm{d}}\mathbf{C}_{\mathrm{d}2})\right)^{-1}(\mathbf{B}_{\mathrm{d}1} - \mathbf{L}_{\mathrm{d}}\mathbf{D}_{\mathrm{d}21}) + \mathbf{D}_{\mathrm{d}11},$$

is minimized.

Synthesis Method 5.4. The $\mathcal{H}_\infty$-optimal observer gain is synthesized by solving for $\mathbf{P} \in \mathbb{S}^{n_x}$, $\mathbf{G}_{\mathrm{d}} \in \mathbb{R}^{n_x \times n_y}$, and $\gamma \in \mathbb{R}_{>0}$ that minimize $\mathcal{J}(\gamma) = \gamma$ subject to $\mathbf{P} > 0$ and

$$\begin{bmatrix} \mathbf{P} & \mathbf{P}\mathbf{A}_{\mathrm{d}} - \mathbf{G}_{\mathrm{d}}\mathbf{C}_{\mathrm{d}2} & \mathbf{P}\mathbf{B}_{\mathrm{d}1} - \mathbf{G}_{\mathrm{d}}\mathbf{D}_{\mathrm{d}21} & \mathbf{0} \\ * & \mathbf{P} & \mathbf{0} & \mathbf{C}_{\mathrm{d}1}^{\mathsf{T}} \\ * & * & \gamma\mathbf{1} & \mathbf{D}_{\mathrm{d}11}^{\mathsf{T}} \\ * & * & * & \gamma\mathbf{1} \end{bmatrix} > 0.$$

The $\mathcal{H}_\infty$-optimal observer gain is recovered by $\mathbf{L}_{\mathrm{d}} = \mathbf{P}^{-1}\mathbf{G}_{\mathrm{d}}$ and the $\mathcal{H}_\infty$ norm of $\mathbf{T}(z)$ is γ.

5.3 Mixed $\mathcal{H}_2$-$\mathcal{H}_\infty$-Optimal State Estimation

The goal of mixed $\mathcal{H}_2$-$\mathcal{H}_\infty$-optimal state estimation is to design an observer that minimizes the $\mathcal{H}_2$ norm of the closed-loop transfer matrix from $\mathbf{w}_1$ to $\mathbf{z}_1$, while ensuring that the $\mathcal{H}_\infty$ norm of the closed-loop transfer matrix from $\mathbf{w}_2$ to $\mathbf{z}_2$ is below a specified bound.

5.3.1 Mixed $\mathcal{H}_2$-$\mathcal{H}_\infty$-Optimal Observer

Consider the continuous-time generalized plant $\mathcal{P}$ with state-space realization

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_{1,1}\mathbf{w}_1 + \mathbf{B}_{1,2}\mathbf{w}_2, \\ \mathbf{y} &= \mathbf{C}_2\mathbf{x} + \mathbf{D}_{21,1}\mathbf{w}_1 + \mathbf{D}_{21,1}\mathbf{w}_2, \end{aligned}$$

where it is assumed that $(\mathbf{A},\mathbf{C}_2)$ is detectable. An observer of the form

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{L}\left(\mathbf{y} - \hat{\mathbf{y}}\right), \\ \hat{\mathbf{y}} &= \mathbf{C}_2\hat{\mathbf{x}}, \end{aligned}$$

is to be designed, where $\mathbf{L} \in \mathbb{R}^{n_x \times n_y}$ is the observer gain. Defining the error state $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$, the error dynamics are found to be

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{L}\mathbf{C}_2)\,\mathbf{e} + (\mathbf{B}_{1,1} - \mathbf{L}\mathbf{D}_{21,1})\,\mathbf{w}_1 + (\mathbf{B}_{1,2} - \mathbf{L}\mathbf{D}_{21,2})\,\mathbf{w}_2,$$

and the performance output is defined as

$$\begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{1,1} \\ \mathbf{C}_{1,2} \end{bmatrix}\mathbf{e} + \begin{bmatrix} \mathbf{0} & \mathbf{D}_{11,12} \\ \mathbf{D}_{11,21} & \mathbf{D}_{11,22} \end{bmatrix}\begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix}.$$

The observer gain $\mathbf{L}$ is to be designed to minimize the $\mathcal{H}_2$ norm of the closed-loop transfer matrix $\mathbf{T}_{11}(s)$ from the exogenous input $\mathbf{w}_1$ to the performance output $\mathbf{z}_1$ while ensuring the $\mathcal{H}_\infty$ norm of the closed-loop transfer matrix $\mathbf{T}_{22}(s)$ from the exogenous input $\mathbf{w}_2$ to the performance output $\mathbf{z}_2$ is less than γ_d, where

$$\begin{aligned} \mathbf{T}_{11}(s) &= \mathbf{C}_{1,1}\left(s\mathbf{1} - (\mathbf{A} - \mathbf{L}\mathbf{C}_2)\right)^{-1}(\mathbf{B}_{1,1} - \mathbf{L}\mathbf{D}_{21,1}), \\ \mathbf{T}_{22}(s) &= \mathbf{C}_{1,2}\left(s\mathbf{1} - (\mathbf{A} - \mathbf{L}\mathbf{C}_2)\right)^{-1}(\mathbf{B}_{1,2} - \mathbf{L}\mathbf{D}_{21,2}) + \mathbf{D}_{11,22}. \end{aligned}$$

Synthesis Method 5.5. The mixed $\mathcal{H}_2$-$\mathcal{H}_\infty$-optimal observer gain is synthesized by solving for $\mathbf{P} \in \mathbb{S}^{n_x}$, $\mathbf{Z} \in \mathbb{S}^{n_z}$, $\mathbf{G} \in \mathbb{R}^{n_x \times n_y}$, and $\nu \in \mathbb{R}_{>0}$ that minimize $\mathcal{J}(\nu) = \nu$ subject to $\mathbf{P} > 0$, $\mathbf{Z} > 0$,

$$\begin{bmatrix} \mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} - \mathbf{GC}_2 - \mathbf{C}_2^\mathsf{T}\mathbf{G}^\mathsf{T} & \mathbf{PB}_{1,1} - \mathbf{GD}_{21,1} \\ * & -\mathbf{1} \end{bmatrix} < 0,$$

$$\begin{bmatrix} \mathbf{PA} + \mathbf{A}^\mathsf{T}\mathbf{P} - \mathbf{GC}_2 - \mathbf{C}_2^\mathsf{T}\mathbf{G}^\mathsf{T} & \mathbf{PB}_{1,2} - \mathbf{GD}_{21,2} & \mathbf{C}_{1,2} \\ * & -\gamma_d\mathbf{1} & \mathbf{D}_{11,22}^\mathsf{T} \\ * & * & -\gamma_d\mathbf{1} \end{bmatrix} < 0,$$

$$\begin{bmatrix} \mathbf{P} & \mathbf{C}_{1,1}^\mathsf{T} \\ * & \mathbf{Z} \end{bmatrix} > 0,$$

$$\mathrm{tr}\mathbf{Z} < \nu.$$

The mixed-$\mathcal{H}_2$-$\mathcal{H}_\infty$-optimal observer gain is recovered by $\mathbf{L} = \mathbf{P}^{-1}\mathbf{G}$, the $\mathcal{H}_2$ norm of $\mathbf{T}_{11}(s)$ is less than $\mu = \sqrt{\nu}$, and the $\mathcal{H}_\infty$ norm of $\mathbf{T}_{22}(s)$ is less than γ_d.

5.3.2 Discrete-Time Mixed $\mathcal{H}_2$-$\mathcal{H}_\infty$-Optimal Observer

Consider the discrete-time generalized LTI plant $\mathcal{P}$ with state-space realization

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}_\mathrm{d}\mathbf{x}_k + \mathbf{B}_{\mathrm{d}1,1}\mathbf{w}_{1,k} + \mathbf{B}_{\mathrm{d}1,1}\mathbf{w}_{1,k}, \\ \mathbf{y}_k &= \mathbf{C}_{\mathrm{d}2}\mathbf{x}_k + \mathbf{D}_{\mathrm{d}21,1}\mathbf{w}_{1,k} + \mathbf{D}_{\mathrm{d}21,2}\mathbf{w}_{2,k}, \end{aligned}$$

where it is assumed that ($\mathbf{A}_\mathrm{d}$,$\mathbf{C}_{\mathrm{d}2}$) is detectable. An observer of the form

$$\begin{aligned} \hat{\mathbf{x}}_{k+1} &= \mathbf{A}_\mathrm{d}\hat{\mathbf{x}}_k + \mathbf{L}_\mathrm{d}\left(\mathbf{y}_k - \hat{\mathbf{y}}_k\right), \\ \hat{\mathbf{y}}_k &= \mathbf{C}_{\mathrm{d}2}\hat{\mathbf{x}}_k, \end{aligned}$$

is to be designed, where $\mathbf{L}_\mathrm{d} \in \mathbb{R}^{n_x \times n_y}$ is the observer gain. Defining the error state $\mathbf{e}_k = \mathbf{x}_k - \hat{\mathbf{x}}_k$, the error dynamics are found to be

$$\mathbf{e}_{k+1} = \left(\mathbf{A}_\mathrm{d} - \mathbf{L}_\mathrm{d}\mathbf{C}_{\mathrm{d}2}\right)\mathbf{e}_k + \left(\mathbf{B}_{\mathrm{d}1,1} - \mathbf{L}_\mathrm{d}\mathbf{D}_{\mathrm{d}21,1}\right)\mathbf{w}_{1,k} + \left(\mathbf{B}_{\mathrm{d}1,2} - \mathbf{L}_\mathrm{d}\mathbf{D}_{\mathrm{d}21,2}\right)\mathbf{w}_{2,k},$$

and the performance output is defined as

$$\begin{bmatrix} \mathbf{z}_{1,k} \\ \mathbf{z}_{2,k} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{\mathrm{d}1,1} \\ \mathbf{C}_{\mathrm{d}1,2} \end{bmatrix}\mathbf{e}_k + \begin{bmatrix} \mathbf{0} & \mathbf{D}_{\mathrm{d}11,12} \\ \mathbf{D}_{\mathrm{d}11,21} & \mathbf{D}_{\mathrm{d}11,22} \end{bmatrix}\begin{bmatrix} \mathbf{w}_{1,k} \\ \mathbf{w}_{2,k} \end{bmatrix}.$$

The observer gain $\mathbf{L}_\mathrm{d}$ is to be designed to minimize the $\mathcal{H}_2$ norm of the closed loop transfer matrix $\mathbf{T}_{11}(z)$ from the exogenous input $\mathbf{w}_{1,k}$ to the performance output $\mathbf{z}_{1,k}$ while ensuring the $\mathcal{H}_\infty$ norm of the closed-loop transfer matrix $\mathbf{T}_{22}(z)$ from the exogenous input $\mathbf{w}_{2,k}$ to the performance output $\mathbf{z}_{2,k}$ is less than γ_d, where

$$\begin{aligned} \mathbf{T}_{11}(z) &= \mathbf{C}_{\mathrm{d}1,1}\left(z\mathbf{1} - \left(\mathbf{A}_\mathrm{d} - \mathbf{L}_\mathrm{d}\mathbf{C}_{\mathrm{d}2}\right)\right)^{-1}\left(\mathbf{B}_{\mathrm{d}1,1} - \mathbf{L}_\mathrm{d}\mathbf{D}_{\mathrm{d}21,1}\right), \\ \mathbf{T}_{22}(z) &= \mathbf{C}_{\mathrm{d}1,2}\left(z\mathbf{1} - \left(\mathbf{A}_\mathrm{d} - \mathbf{L}_\mathrm{d}\mathbf{C}_{\mathrm{d}2}\right)\right)^{-1}\left(\mathbf{B}_{\mathrm{d}1,2} - \mathbf{L}_\mathrm{d}\mathbf{D}_{\mathrm{d}21,2}\right) + \mathbf{D}_{\mathrm{d}11,22}. \end{aligned}$$

Synthesis Method 5.6. The discrete-time mixed-$\mathcal{H}_2$-$\mathcal{H}_\infty$-optimal observer gain is synthesized by solving for $\mathbf{P} \in \mathbb{S}^{n_x}$, $\mathbf{Z} \in \mathbb{S}^{n_z}$, $\mathbf{G}_\mathrm{d} \in \mathbb{R}^{n_x \times n_y}$, and $\nu \in \mathbb{R}_{>0}$ that minimize $\mathcal{J}(\nu) = \nu$ subject to

$\mathbf{P} > 0, \mathbf{Z} > 0,$

$$\begin{bmatrix} \mathbf{P} & \mathbf{P}\mathbf{A}_{\mathrm{d}} - \mathbf{G}_{\mathrm{d}}\mathbf{C}_{\mathrm{d}2} & \mathbf{P}\mathbf{B}_{\mathrm{d}1,1} - \mathbf{G}_{\mathrm{d}}\mathbf{D}_{\mathrm{d}21,1} \\ * & \mathbf{P} & \mathbf{0} \\ * & * & \mathbf{1} \end{bmatrix} > 0,$$

$$\begin{bmatrix} \mathbf{P} & \mathbf{P}\mathbf{A}_{\mathrm{d}} - \mathbf{G}_{\mathrm{d}}\mathbf{C}_{\mathrm{d}2} & \mathbf{P}\mathbf{B}_{\mathrm{d}1,2} - \mathbf{G}_{\mathrm{d}}\mathbf{D}_{\mathrm{d}21,2} & \mathbf{0} \\ * & \mathbf{P} & \mathbf{0} & \mathbf{C}_{\mathrm{d}1,2}^{\mathsf{T}} \\ * & * & \gamma_d \mathbf{1} & \mathbf{D}_{\mathrm{d}11,22}^{\mathsf{T}} \\ * & * & * & \gamma_d \mathbf{1} \end{bmatrix} > 0,$$

$$\begin{bmatrix} \mathbf{Z} & \mathbf{P}\mathbf{C}_{\mathrm{d}1,1} \\ * & \mathbf{P} \end{bmatrix} > 0.$$

$$\mathrm{tr}\mathbf{Z} < \nu.$$

The mixed-$\mathcal{H}_2$-$\mathcal{H}_\infty$-optimal observer gain is recovered by $\mathbf{L}_{\mathrm{d}} = \mathbf{P}^{-1}\mathbf{G}_{\mathrm{d}}$, the $\mathcal{H}_2$ norm of $\mathbf{T}_{11}(z)$ is less than $\mu = \sqrt{\nu}$, and the $\mathcal{H}_\infty$ norm of $\mathbf{T}_{22}(z)$ is less than γ_d.

5.4 Optimal Filtering

The goal of optimal filtering is to design a filter that acts on the output $\mathbf{z}$ of the generalized plant and optimizes the transfer matrix from $\mathbf{w}$ to the filtered output. Consider the continuous-time generalized LTI plant with minimal states-space realization

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{w}, \\ \mathbf{z} &= \mathbf{C}_1\mathbf{x} + \mathbf{D}_{11}\mathbf{w}, \\ \mathbf{y} &= \mathbf{C}_2\mathbf{x} + \mathbf{D}_{21}\mathbf{w}, \end{aligned}$$

where it is assumed that $\mathbf{A}$ is Hurwitz. A continuous-time dynamic LTI filter with state-space realization

$$\begin{aligned} \dot{\mathbf{x}}_f &= \mathbf{A}_f\mathbf{x}_f + \mathbf{B}_f\mathbf{y}, \\ \hat{\mathbf{z}} &= \mathbf{C}_f\mathbf{x}_f + \mathbf{D}_f\mathbf{y}, \end{aligned}$$

is to be designed to optimize the transfer function from $\mathbf{w}$ to $\tilde{\mathbf{z}} = \mathbf{z} - \hat{\mathbf{z}}$, given by

$$\tilde{\mathbf{P}}(s) = \tilde{\mathbf{C}}_1 \left(s\mathbf{1} - \tilde{\mathbf{A}}\right)^{-1} \tilde{\mathbf{B}}_1 + \tilde{\mathbf{D}}_{11}, \tag{5.6}$$

where

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B}_f\mathbf{C}_2 & \mathbf{A}_f \end{bmatrix}, \qquad \tilde{\mathbf{B}}_1 = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_f\mathbf{D}_{21} \end{bmatrix}, \qquad \tilde{\mathbf{C}}_1 = \begin{bmatrix} \mathbf{C}_1 - \mathbf{D}_f\mathbf{C}_2 & -\mathbf{C}_f \end{bmatrix}, \qquad \tilde{\mathbf{D}}_{11} = \mathbf{D}_{11} - \mathbf{D}_f\mathbf{D}_{21}.$$

This can alternatively be formulated as a special case of synthesizing a dynamic output "feedback" controller for the generalized plant given by

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{w}, \\ \mathbf{z} &= \mathbf{C}_1\mathbf{x} + \mathbf{D}_{11}\mathbf{w} - \mathbf{u}, \\ \mathbf{y} &= \mathbf{C}_2\mathbf{x} + \mathbf{D}_{21}\mathbf{w}. \end{aligned}$$

The controller in this case is not truly a feedback controller, as it only appears as a feedthrough term in the performance channel. The synthesis methods presented in this section take advantage of this fact, resulting in a simpler formulation than applying the controller synthesis methods in Section 4.

5.4.1 $\mathcal{H}_2$-Optimal Filter [19, pp. 309–310]

An $\mathcal{H}_2$-optimal filter is designed to minimize the $\mathcal{H}_2$ norm of $\tilde{\mathbf{P}}(s)$ in (5.6). To ensure that $\tilde{\mathbf{P}}(s)$ has a finite $\mathcal{H}_2$ norm, it is required that $\mathbf{D}_f = \mathbf{D}_{11}$, which results in $\tilde{\mathbf{D}}_{11} = \mathbf{D}_{11} - \mathbf{D}_f = \mathbf{0}$.

Synthesis Method 5.7. Solve for $\mathbf{A}_n \in \mathbb{R}^{n_x \times n_x}$, $\mathbf{B}_n \in \mathbb{R}^{n_x \times n_y}$, $\mathbf{C}_f \in \mathbb{R}^{n_z \times n_x}$, $\mathbf{X}$, $\mathbf{Y} \in \mathbb{S}^{n_x}$, $\mathbf{Z} \in \mathbb{S}^{n_z}$, and $\nu \in \mathbb{R}_{>0}$ that minimize $\mathcal{J}(\nu) = \nu$ subject to $\mathbf{X} > 0$, $\mathbf{Y} > 0$, $\mathbf{Z} > 0$,

$$\begin{aligned}
\begin{bmatrix} \mathbf{YA} + \mathbf{A}^\mathsf{T}\mathbf{Y} + \mathbf{B}_n\mathbf{C}_2 + \mathbf{C}_2^\mathsf{T}\mathbf{B}_n^\mathsf{T} & \mathbf{A}_n + \mathbf{C}_2^\mathsf{T}\mathbf{B}_n^\mathsf{T} + \mathbf{A}^\mathsf{T}\mathbf{X} & \mathbf{YB}_1 + \mathbf{B}_n\mathbf{D}_{21} \\ * & \mathbf{A}_n + \mathbf{A}_n^\mathsf{T} & \mathbf{XB}_1 + \mathbf{B}_n\mathbf{D}_{21} \\ * & * & -\mathbf{1} \end{bmatrix} &< 0, \\
\begin{bmatrix} -\mathbf{Z} & \mathbf{C}_1 - \mathbf{D}_f\mathbf{C}_2 & -\mathbf{C}_f \\ * & -\mathbf{Y} & -\mathbf{X} \\ * & * & -\mathbf{X} \end{bmatrix} &< 0, \\
\mathbf{Y} - \mathbf{X} &> 0, \\
\mathrm{tr}\mathbf{Z} &< \nu.
\end{aligned}$$

The filter is recovered by $\mathbf{A}_f = \mathbf{X}^{-1}\mathbf{A}_n$, and $\mathbf{B}_f = \mathbf{X}^{-1}\mathbf{B}_n$.

5.4.2 $\mathcal{H}_\infty$-Optimal Filter [19, pp. 303–304]

An $\mathcal{H}_\infty$-optimal filter is designed to minimize the $\mathcal{H}_\infty$ norm of $\tilde{\mathbf{P}}(s)$ in (5.6).

Synthesis Method 5.8. Solve for $\mathbf{A}_n \in \mathbb{R}^{n_x \times n_x}$, $\mathbf{B}_n \in \mathbb{R}^{n_x \times n_y}$, $\mathbf{C}_f \in \mathbb{R}^{n_z \times n_x}$, $\mathbf{D}_f \in \mathbb{R}^{n_z \times n_y}$, $\mathbf{X}$, $\mathbf{Y} \in \mathbb{S}^{n_x}$, and $\gamma \in \mathbb{R}_{>0}$ that minimize $\mathcal{J}(\gamma) = \gamma$ subject to $\mathbf{X} > 0$, $\mathbf{Y} > 0$,

$$\begin{aligned}
\begin{bmatrix} \mathbf{YA} + \mathbf{A}^\mathsf{T}\mathbf{Y} + \mathbf{B}_n\mathbf{C}_2 + \mathbf{C}_2^\mathsf{T}\mathbf{B}_n^\mathsf{T} & \mathbf{A}_n + \mathbf{C}_2^\mathsf{T}\mathbf{B}_n^\mathsf{T} + \mathbf{A}^\mathsf{T}\mathbf{X} & \mathbf{YB}_1 + \mathbf{B}_n\mathbf{D}_{21} & \mathbf{C}_1^\mathsf{T} - \mathbf{C}_2^\mathsf{T}\mathbf{D}_f^\mathsf{T} \\ * & \mathbf{A}_n + \mathbf{A}_n^\mathsf{T} & \mathbf{XB}_1 + \mathbf{B}_n\mathbf{D}_{21} & -\mathbf{C}_f^\mathsf{T} \\ * & * & -\gamma\mathbf{1} & \mathbf{D}_{11}^\mathsf{T} - \mathbf{D}_{21}^\mathsf{T}\mathbf{D}_f^\mathsf{T} \\ * & * & * & -\gamma\mathbf{1} \end{bmatrix} &< 0, \\
\mathbf{Y} - \mathbf{X} &> 0.
\end{aligned}$$

The filter is recovered by $\mathbf{A}_f = \mathbf{X}^{-1}\mathbf{A}_n$ and $\mathbf{B}_f = \mathbf{X}^{-1}\mathbf{B}_n$.

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