

Optimization Techniques for Big Data Analysis

Chapter 1. Introduction

Master of Science in Signal Theory and Communications

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Universidad Politécnica de Madrid

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1 Introduction

Why take this course?

Basic concepts

2 Optimization problems in Machine Learning

ML setup

Most common optimization problems in ML

Motivation

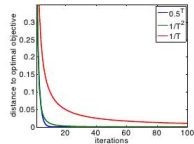
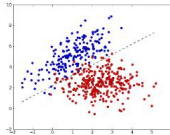
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Data

Model

Optimization



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- **Discrete optimization:**

It occurs in inference problems in structured spaces, such as Feature selection, Data subset selection, Data summarization, Architecture search etc.

Continuous optimization in ML

- **Supervised Learning:** Logistic Regression, Least Square, Support Vector Machines, Deep Models.
- **Unsupervised Learning:** k-Means Clustering, Principal Component Analysis.
- **Contextual bandits and Reinforcement learning:** Soft-Max estimators and Policy Exponential Models.
- **Recommender systems:** Matrix Completion, Non-Negative Matrix Factorization, Collaborative Filtering.

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Countless ML libraries available implement all kinds of optimization algorithms (Tensorflow, PyTorch, Scipy, Sklearn, Vowpal Wabbit, ...)

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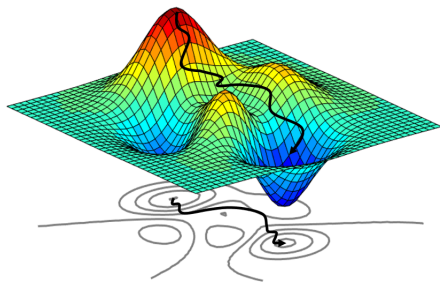
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- **Federated learning:** Run the algorithm in different nodes without sharing any data among nodes.

Nomenclature

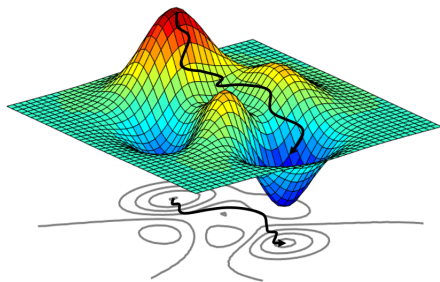
- For the most part through this course, we will use n as the number of training instances and d as the number of dimensions (features).
- Function: $f(\cdot)$
- Scalar: x ; single-input function $f(x)$
- d -dimensional vector: $\mathbf{x} \in \mathbb{R}^d$; $\mathbf{x}_i = [x_{i1}, x_{i2}, \dots, x_{id}]$
- multi-input function $f(\mathbf{x})$; vector-valued function $\mathbf{f}(\mathbf{x})$
- Vector Space: \mathcal{X}
- Vector Norm: $\|\mathbf{x}\|_L$
- Inner product: Given two vectors $\mathbf{w}, \mathbf{x} \in \mathbb{R}^d$, define the inner product $\langle \mathbf{w}, \mathbf{x} \rangle = \sum_{j=1}^d \mathbf{w}_j \mathbf{x}_j = \mathbf{w}^T \mathbf{x}$
- Random variable X ; Matrix: \mathbf{X}

Mathematical optimization



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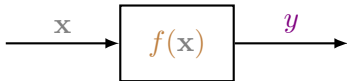
So typically we are given a problem like the following:

$$\hat{x} = \arg \max_x f(x) \text{ s.t. } g(x) < a \quad (1)$$

- $f(\cdot)$ function subject to optimization.
- $x \in \mathcal{X}$ variables/parameters that need to be adjusted.
- \mathcal{X} is the search space. \hat{x} is the optimum.
- $g(\cdot)$ restrictions. $f|_{\mathcal{G}}$, where $\mathcal{G} \subseteq \mathcal{X}$; \mathcal{G} is the feasible set.

ML set-up

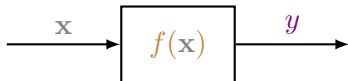
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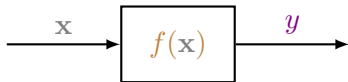


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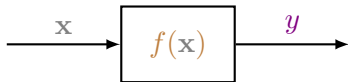
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Training $f_{\theta}(\cdot)$ corresponds to the optimization of $J(\theta)$!

Supervised Learning: Modelling

- **Data:** Given training examples $\{(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_n, y_n)\}$ where $\mathbf{x}_i \in \mathbb{R}^d$ is the feature vector and y_i is the label.
- **Model:** Denote the Model by $f_{\theta}(\mathbf{x})$ with θ being the parameters of the model. e.g. $f_{\theta}(\mathbf{x}) = \theta^T \mathbf{x}$
- **Loss Functions:** The loss function ℓ tries to measure the distance between $f_{\theta}(\mathbf{x}_i)$ and y_i .

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The generic problem denoted as the *expected loss function* can be expressed as:

$$\arg \min_{\theta} \mathcal{L}(\theta) = \mathbb{E}[\ell_{\theta}(\mathbf{x}, y)] + \lambda r(\theta) \quad (3)$$

where ℓ is the *instantaneous loss*, $r(\cdot)$ is the *regularizer* and λ a regularization factor.



Empirical loss functions

In most of this course' cases, we are dealing with a supervised learning problem, so given a dataset $\mathcal{D} = \{(\mathbf{x}_i, y_i)_{i=1}^n\}$ the loss function takes the form:

$$\mathcal{L}(\theta) = \frac{1}{n} \sum_{i=1}^n \ell_{\theta}(\mathbf{x}_i, y_i) + \lambda r(\theta) \quad (4)$$

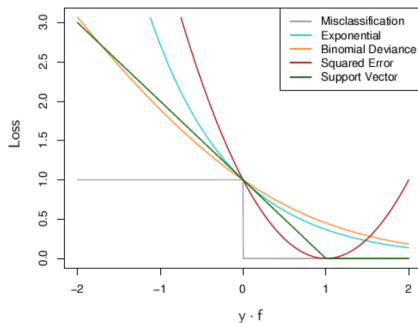
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Examples of ℓ :

- Logistic loss:
 $\log(1 + \exp(-y_i f_{\theta}(\mathbf{x}_i)))$
- Hinge Loss:
 $\max\{0, 1 - y_i f_{\theta}(\mathbf{x}_i)\}$
- Absolute Error: $|f_{\theta}(\mathbf{x}_i) - y_i|$
- Least Squares: $(f_{\theta}(\mathbf{x}_i) - y_i)^2$



Instantaneous loss function

A common simplification approximates the original function by its instantaneous value:

$$\mathcal{L}(\theta) = \ell_{\theta}(\mathbf{x}_i, y_i) + \lambda r(\theta) \quad (5)$$

- The minimization is performed by using an instantaneous version of the original gradient.

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- This simplification makes the algorithm very attractive in big data applications, both because of hardware requirements and for distributed settings.

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 - ▶ Algebraic implementations of the algorithms are crucial.
- High dimensional data: Because of the dimensionality of the parameter spaces, we can't afford second-order optimization methods. First-order or simplified second-order methods are requested.
- Distributed processing is highly desirable.

Acknowledgments

I would like to acknowledge several sources I have used to create slides

- Rishabh Iyer's course at University of Texas, DA
<https://github.com/rishabhk108/OptimizationML>
- Martin Jaggi & Nicolas Flammarion's course at EPFL
https://github.com/epfml/OptML_course

Questions?

- [1] Melvyn W Jeter. *Mathematical programming: an introduction to optimization*. Routledge, 2018.

Thank You

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