Optimization Techniques for Big Data Analysis

Chapter 6. Coordinate Descent Methods

Master of Science in Signal Theory and Communications
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1. Coordinate descent

2. Gradient Coordinate Descent

3. Block Coordinate Descent

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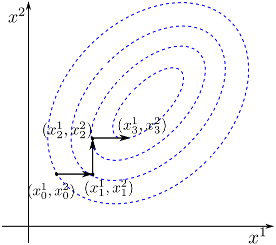

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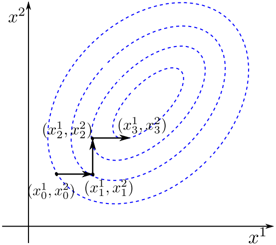

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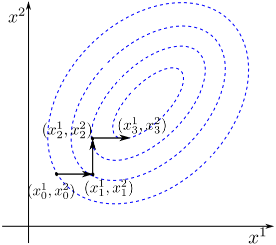

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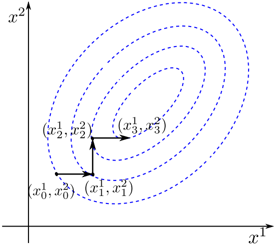

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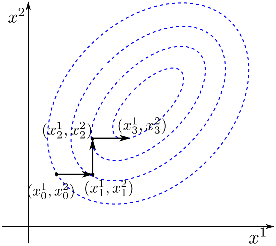

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We can also group the variables into block of dimension m_j, and optimise one block at a time; that's call **Block Coordinate Descent**.

Block Coordinate Descent

The BCD algorithm consists of solving our block-structured problem in an iterative manner. On iteration k we compute

$$\begin{aligned} x_{k+1,j} &= \arg\min_{x_j \in X_j} f\left(x_j, x_{k,-j}\right) \\ x_{k+1,l} &= x_{k,l}, \qquad \forall l \neq j \end{aligned}$$

where $x_{k,-j} \triangleq (x_{k,1}, \cdots, x_{k,j-1}, x_{k,j+1}, \cdots, x_{k,d})$. In the next iteration, a different coordinate, for instance, $j+1$, is updated.

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The method is very intuitive and simple to implement and very popular in many applications. However, it does not have guaranteed convergence for an arbitrary function f.

Example_6_1

Ridge regression:

$$\arg\min_{\mathbf{w}\in\mathbb{R}^{d+1}} f(\mathbf{w}) = \arg\min_{\mathbf{w}\in\mathbb{R}^{d+1}} \frac{1}{2}\left\|\mathbf{y}-\mathbf{X}\mathbf{w}\right\|_2^2 + \lambda\|\mathbf{w}\|_2^2$$

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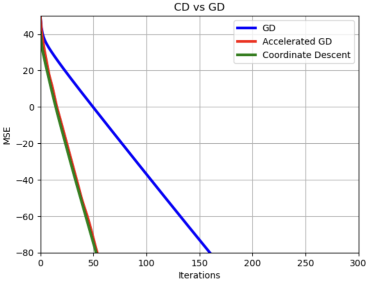

Gradient Coordinate Descent.

$$x_{k+1,j} = x_{k,j} - \eta \nabla_{x_j} f(\mathbf{x}_k)$$
$$x_{k+1,j} = x_{k,j} \quad \forall j \neq i$$

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1. If f is non-smooth, we could incorporate projected or proximal updates.
2. The SGD is also applicable, where an instantaneous estimate substitutes the gradient.
3. It could also be improved using Nesterov or Quasi-Newton principles.

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4. Randomized rule: In the randomized scheme, every block has a non-zero probability of being updated, and these probabilities are varied according to some information over the estimated errors.

case_study_6_1

Let us assume the general problem

$$\arg\min_{\mathbf{w}} \frac{1}{n} \|\mathbf{X}\mathbf{w} - \mathbf{y}\|_2^2 + \frac{\lambda}{2} \|\mathbf{w}\|_2^2$$

Instead of updating one single variable at a time, we can make blocks of variables of size $m_j \geq 1$ and solve the problem in a parallel fashion.

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The gradient takes the same form as before, but the j represents a block of variables:

$$\nabla_j f(\mathbf{w}) = \frac{2}{n} \mathbf{X}_{:,j}^T (\mathbf{X}_{:,j}\mathbf{w}_j + \mathbf{X}_{:,-j}\mathbf{w}_{-j} - \mathbf{y}) + \lambda \mathbf{w}_j = 0$$

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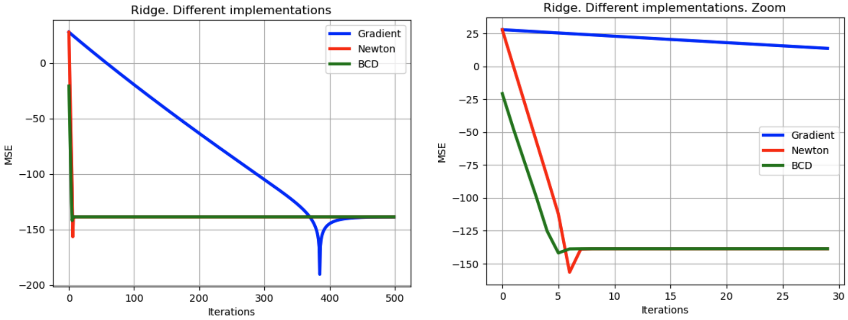
So, the closed-form solution for the iteration $k+1$ results into

$$\mathbf{w}_{k+1,j} = \left(\mathbf{X}_{:,j}^T\mathbf{X}_{:,j} + \frac{n}{2}\lambda\mathbf{I}_{m_j}\right)^{-1} \mathbf{X}_{:,j}^T \left(\mathbf{y} - \mathbf{X}_{:,-j}\mathbf{w}_{k,-j}\right)$$

Case_study_6_1

Follow the code provided in the notebook `Case_study_6_1` to obtain results as those presented in the next Figure. Pay attention to how high-speed are these algorithms.



Case_study_6_2

Let's now consider the LASSO case

$$\arg\min_{\mathbf{w}} \frac{1}{n} \left\| \mathbf{X}\mathbf{w} - \mathbf{y} \right\|_2^2 + \lambda \left\| \mathbf{w} \right\|_1$$

Case_study_6_2

Let's now consider the LASSO case

$$\arg\min_{\mathbf{w}} \frac{1}{n} \|\mathbf{X}\mathbf{w} - \mathbf{y}\|_2^2 + \lambda \|\mathbf{w}\|_1$$

By differentiating it with respect to j-th weight, we get

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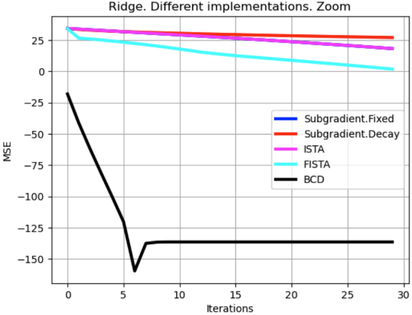
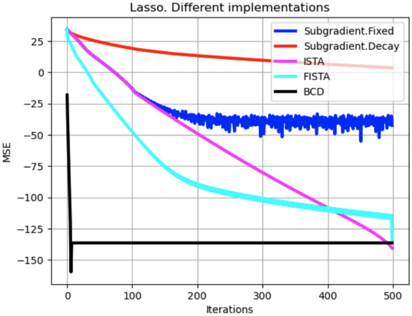
Case_study_6_2

Follow the code provided in the notebook `Case_study_6_2` to obtain the following results





Questions?

References

[1] Jorge Nocedal and J. Wright Stephen. *Numerical optimization.* Spinger, 2006.

Thank You

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