Sheet 8: More hyperbolic geometry

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The final mark out of 10 (which counts towards your grade) will be calculated as your mark on Q1 plus your mark on Q2 plus your best mark from Q3–5. I will also award stars: silver for a total mark of 12 or more on any four questions, gold for a total mark of 15 or more on all questions.

Question 1. (2 marks)
An ideal hyperbolic triangle is a triple of hyperbolic lines A, B, C such that each pair intersect precisely once on the boundary of hyperbolic space. For example, the semicircle $\{z \ : \ |z| = 1, \ \mathrm{Im}(z) > 0\}$ and the two vertical half-lines $\{z = -1 + ib \ : \ 0 < b \in \mathbf{R}\}$ and $\{z = 1 + ib \ : \ 0 < b \in \mathbf{R}\}$ at $x = -1$ and $x = 1$ form an ideal triangle (the three "vertices" are at $-1, 1$ and ∞). What is the area of an ideal triangle? Why are any two ideal triangles are related by an isometry of the hyperbolic plane? *[Hint: Try to use 3-transitivity.]*

Question 2. (5 marks)
Working in the disc model of hyperbolic 2-space, let $\gamma(t) = rt$ be the straight-line path starting at the origin when $t = 0$ and finishing at radius r at time 1 and let $\delta(t) = re^{2\pi i t}$ be the circular path at radius r. A hyperbolic circle of hyperbolic radius R is defined to be the set of points a fixed hyperbolic distance R away from a fixed point.

(a) Find the hyperbolic length of γ and the hyperbolic length of δ as functions of r.

(b) Deduce that a hyperbolic circle of hyperbolic radius R centred at the origin is an ordinary Euclidean circle and that the hyperbolic circumference is $2\pi \sinh(R)$.

(c) Show that the area circumscribed by a hyperbolic circle of hyperbolic radius R i $2\pi(\cosh(R) - 1)$.

(d) We have now seen that a hyperbolic circle centred at the origin looks (in the disc model) like an ordinary Euclidean circle. What if the centre is taken to be at a different point? What if we look at the circle in the upper half-plane?

Question 3. (3 marks)

(a) Working in the upper half-plane model of hyperbolic space, which elements of $PSL(2, \mathbf{R})$ send the positive imaginary half-axis to itself?

(b) Let ℓ be a straight ray in the upper half-plane starting at 0. For any $z \in \ell$, let C_z be the unique semicircle centred at 0 passing through z. Let z' denote the point where C_z intersects the positive imaginary axis. Prove that the hyperbolic length of the segment of C_z between z and z' depends only on the ray ℓ and not on the specific choice of a point $z \in \ell$. *[Hint: Use part (a).]*

Question 4. (3 marks)
Let ABC be a hyperbolic triangle with edge lengths a, b, c opposite angles α, β, γ. Starting from the hyperbolic cosine rule $\cosh(a) = \cosh(b)\cosh(c) - \cos(\alpha)\sinh(b)\sinh(c)$, prove the hyperbolic sine rule:

$$\frac{\sinh(a)}{\sin(\alpha)} = \frac{\sinh(b)}{\sin(\beta)} = \frac{\sinh(c)}{\sin(\gamma)}.$$

[Hint: You want to show that $\sinh^2(b)\sinh^2(c)\sin^2(\alpha)$ can be written in terms of a, b, c in a completely symmetric way.]

Question 5. (3 marks)
Consider the semicircle C centred at $r \in \mathbf{R}$ with radius r. What is its image under the Möbius transformation $g(z) = -1/z$? What are the images under g of the points $A = r + ri \in C$ and $B = r(1 + e^{i\pi/4}) \in C$? Hence or otherwise, find the length of the segment of C connecting A to B.