Sheet 8: More hyperbolic geometry

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The final mark out of 10 (which counts towards your grade) will be calculated as your mark on Q1 plus your mark on Q2 plus your best mark from Q3–5. I will also award stars: silver for a total mark of 12 or more on any four questions, gold for a total mark of 15 or more on all questions.

Question 1. (2 marks)
An ideal hyperbolic triangle is a triple of hyperbolic lines A, B, C such that each pair intersect precisely once on the boundary of hyperbolic space. For example, the semicircle $\{z : |z| = 1, \mathrm{Im}(z) > 0\}$ and the two vertical half-lines $\{z = -1 + ib : 0 < b \in \mathbf{R}\}$ and $\{z = 1 + ib : 0 < b \in \mathbf{R}\}$ at $x = -1$ and $x = 1$ form an ideal triangle (the three "vertices" are at $-1, 1$ and ∞). What is the area of an ideal triangle? Why are any two ideal triangles are related by an isometry of the hyperbolic plane? *[Hint: Try to use 3-transitivity.]*

Answer 1. The ideal triangle has internal angles equal to zero, so its area is $\pi - \alpha - \beta - \gamma = \pi$ (in other words, approximate it by a larger and larger sequence of actual triangles whose angles become smaller and smaller; in the limit the area tends to π). An ideal triangle is completely determined by its vertices. These vertices lie on $\mathbf{R} \cup \{\infty\}$. For any three points z_0, z_1, z_∞ in $\mathbf{C} \cup \{\infty\}$ there is a Möbius transformation taking them to $0, 1, \infty$, namely

$$\frac{z - z_0}{z - z_\infty} \frac{z_1 - z_\infty}{z_1 - z_0}.$$

. If the three points are real then this Möbius transformation is in $PGL(2, \mathbf{R})$. If necessary we can switch z_0 and z_1 to make the determinant of the corresponding $GL(2, \mathbf{R})$ matrix positive. So now we have a hyperbolic isometry taking the ideal triangle with vertices at z_0, z_1, z_∞ to the ideal triangle with vertices at $0, 1, \infty$.

Question 2. (5 marks)
Working in the disc model of hyperbolic 2-space, let $\gamma(t) = rt$ be the straight-line path starting at the origin when $t = 0$ and finishing at radius r at time 1 and let $\delta(t) = re^{2\pi i t}$ be the circular path at radius r. A hyperbolic circle of hyperbolic radius R is defined to be the set of points a fixed hyperbolic distance R away from a fixed point.

(a) Find the hyperbolic length of γ and the hyperbolic length of δ as functions of r.

(b) Deduce that a hyperbolic circle of hyperbolic radius R centred at the origin is an ordinary Euclidean circle and that the hyperbolic circumference is $2\pi \sinh(R)$.

(c) Show that the area circumscribed by a hyperbolic circle of hyperbolic radius R i $2\pi(\cosh(R) - 1)$.

(d) We have now seen that a hyperbolic circle centred at the origin looks (in the disc model) like an ordinary Euclidean circle. What if the centre is taken to be at a different point? What if we look at the circle in the upper half-plane?

Answer 2. (a) The length of δ is given by the integral $\int_0^1 \frac{2|\dot{\delta}|}{1+|\delta|^2} dt$ which is $\int_0^1 \frac{2r}{1-r^2} dt = \frac{4\pi r}{1-r^2}$. The length of γ is $\int_0^1 \frac{2dt}{1-t^2} = 2\tanh^{-1}(r)$.

(b) We see that the points at a distance R from the origin are precisely those which lie on a Euclidean circle of radius $r = \tanh(R/2)$. The hyperbolic circumference of this is just the length of δ when $r = \tanh(R/2)$, which is

$$\frac{4\pi \tanh(R/2)}{1 - \tanh^2(R/2)} = 2\pi(2\sinh(R/2)\cosh(R/2)) = 2\pi \sinh(R).$$

(c) The area is given by integrating the circumference over the radius:

$$\text{area} = \int_0^R 2\pi \sinh(x) dx = 2\pi(\cosh(R) - 1).$$

(d) If we move a hyperbolic circle centred at 0 by an isometry g then it maps to a hyperbolic circle centred at $g(0)$. Since the isometries are Möbius transformations, they take Euclidean circles to Euclidean circles, hence the hyperbolic circles centred at any point look like Euclidean circles. Similarly, since the upper half-plane and the disc are related by a Möbius transformation, hyperbolic circles in the upper half-plane also look like Euclidean circles.

Question 3. (3 marks)

(a) Working in the upper half-plane model of hyperbolic space, which elements of $PSL(2, \mathbf{R})$ send the positive imaginary half-axis to itself?

(b) Let ℓ be a straight ray in the upper half-plane starting at 0. For any $z \in \ell$, let C_z be the unique semicircle centred at 0 passing through z. Let z' denote the point where C_z intersects the positive imaginary axis. Prove that the hyperbolic length of the segment of C_z between z and z' depends only on the ray ℓ and not on the specific choice of a point $z \in \ell$. *[Hint: Use part (a).]*

Answer 3. Such Möbius transformations must fix or switch 0 and ∞, hence they are of the form $z \mapsto az$ or $z \mapsto -a/z$, $a > 0$ real. Different choices of z on the ray ℓ are related by multiplication by a positive real number. Let z and kz be two such choices. Then $C_{kz} = kC_z$ and the positive imaginary axis is preserved, so $(kz)' = kz'$ and since $PSL(2, \mathbf{C})$ acts by hyperbolic isometries, the hyperbolic length of the segments of C_z between z, z' and kC_z between kz, kz' agree.

Question 4. (3 marks)
Let ABC be a hyperbolic triangle with edge lengths a, b, c opposite angles α, β, γ. Starting from the hyperbolic cosine rule $\cosh(a) = \cosh(b)\cosh(c) - \cos(\alpha)\sinh(b)\sinh(c)$, prove the hyperbolic sine rule:

$$\frac{\sinh(a)}{\sin(\alpha)} = \frac{\sinh(b)}{\sin(\beta)} = \frac{\sinh(c)}{\sin(\gamma)}.$$

[Hint: You want to show that $\sinh^2(b)\sinh^2(c)\sin^2(\alpha)$ *can be written in terms of* a, b, c *in a completely symmetric way.]*

Answer 4. Using the cosine rule,

$$\sinh(b)\sinh(c)\cos(\alpha) = \cosh(b)\cosh(c) - \cosh(a)$$

we get

$$\begin{aligned} \sinh^2(b)\sinh^2(c)\sin^2(\alpha) &= \sinh^2(b)\sinh^2(c)(1-\cos^2(\alpha)) \\ &= \sinh^2(b)\sinh^2(c) - (\cosh(b)\cosh(c) - \cosh(a))^2 \end{aligned}$$

Multiplying this out and using $\cosh^2 - \sinh^2 = 1$ we get

$$\sinh^2(b)\sinh^2(c)\sin^2(\alpha) = 2\cosh(a)\cosh(b)\cosh(c) - \sinh^2(a) - \sinh^2(b) - \sinh^2(c) - 2.$$

This is symmetric in a, b, c so the same argument gives

$$\sinh^2(b)\sinh^2(c)\sin^2(\alpha) = \sinh^2(c)\sinh^2(a)\sin^2(\beta) = \sinh^2(a)\sinh^2(b)\sin^2(\gamma)$$

which yields the sine rule on dividing by $\sinh^2(a)\sinh^2(b)\sinh^2(c)$.

Question 5. (3 marks)
Consider the semicircle C centred at $r \in \mathbf{R}$ with radius r. What is its image under the Möbius transformation $g(z) = -1/z$? What are the images under g of the points $A = r + ri \in C$ and $B = r(1 + e^{i\pi/4}) \in C$? Hence or otherwise, find the length of the segment of C connecting A to B.

Answer 5. The Möbius transformation sends 0 to ∞, $2r$ to $-1/2r$ and therefore it sends the semicircle C to a hyperbolic line connecting $-1/2r$ to ∞, that is a vertical half-line at $x = -1/2r$. The point A maps to $-\frac{1}{r+ri} = -\frac{r-ri}{2r^2} = \frac{-1}{2r} + \frac{i}{2r}$. The point $B = r\left(1 - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$ maps to

$$
\begin{aligned}
g(B) &= -\frac{1}{r}\left(\frac{1 - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}}{\left(1 - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}}\right) \\
&= -\frac{1}{r}\left(\frac{1 - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}}{2 - \sqrt{2}}\right) \\
&= \frac{-1}{2r} + \frac{i}{2r(\sqrt{2} - 1)}
\end{aligned}
$$

Therefore the hyperbolic length along C between A and B equals the hyperbolic length along the vertical line between $g(A)$ and $g(B)$, which is the integral

$$
\int_{1/2r}^{1/2r(\sqrt{2}-1)} \frac{dy}{y} = \ln(1/(2r(\sqrt{2} - 1))) - \ln(1/2r) = -\ln(\sqrt{2} - 1) \approx 0.88...
$$