Sheet 3: Quaternions and rotations

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The final mark out of 10 (which counts towards your grade) will be calculated as your mark on Q1 plus your mark on Q2 plus your best mark from Q3–5. I will also award stars: silver for a total mark of 12 or more on any four questions, gold for a total mark of 15 or more on all questions.

Question 1. (2 marks)
In each case, find unit quaternions q such that the map $\mathrm{Im}(\mathbf{H}) \to \mathrm{Im}(\mathbf{H})$, $x \mapsto qxq^{-1}$, is the specified orthogonal transformation:

(a) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(b) $\begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$.

Answer 1. (a) This first matrix is a $\pi/2$-rotation around the z-axis, so should be effected by the quaternion $\cos(\pi/4) + k\sin(\pi/4) = \frac{1}{\sqrt{2}}(1+k)$. We can check that this is indeed the case:

$$\begin{aligned}\frac{1}{\sqrt{2}}(1+k)(ix+jy+kz)\frac{1}{\sqrt{2}}(1-k) &= \frac{1}{2}((i+j)x + (j-i)y + (k-1)z)(1-k) \\ &= \frac{1}{2}((i+j+j-i)x + (j-i-i-j)y + (k-1+1+k)z) \\ &= -iy + jx + kz\end{aligned}$$

so $(1,0,0)$ is sent to $(0,1,0)$, $(0,1,0)$ is sent to $(-1,0,0)$ and $(0,0,1)$ is sent to $(0,0,1)$, as required.

(b) This matrix corresponds to a $-\pi/4$-rotation around the y-axis, so should be effected by the quaternion $\cos(\pi/8) - j\sin(\pi/8)$.

Question 2. (5 marks)

(a) Let $q_1 = t_1 + ix_1 + jy_1 + kz_1$ and $q_2 = t_2 + ix_2 + jy_2 + kz_2$. Show that

$$\mathrm{Re}(\bar{q}_1 q_2) = t_1 t_2 + x_1 x_2 + y_1 y_2 + z_1 z_2.$$

(b) Suppose that $t_1 = t_2 = 0$. Show that $q_1 q_2 = -q_1 \cdot q_2 + q_1 \times q_2$, i.e. the real part equals $-(x_1x_2 + y_1y_2 + z_1z_2)$ and the imaginary part equals $i(y_1z_2 - y_2z_1) + j(z_1x_2 - z_2x_2) + k(x_1y_2 - x_2y_1)$.

(c) Show that $A\colon G \times G \to \mathrm{Maps}(\mathbf{H}, \mathbf{H})$, $A(g,h)x = gxh^{-1}$ defines an action of $G \times G$ on $\mathbf{H}$.

(d) Show that $A(g)\colon x \mapsto gxh^{-1}$ is an isometry of $\mathbf{H}$ when $g, h \in G$.

(e) Parts (c) and (d) imply that we have a homomorphism $G \times G \to O(4)$. Show that the kernel of this homomorphism is $\{(1,1), (-1,-1)\}$. [Hint: Consider the effect of the isometry $x \mapsto gxh^{-1}$ on the quaternions $x = 1$, $x = i$, $x = j$]

Answer 2. (a) We have

$$(t_1 - ix_1 - jy_1 - kz_1)(t_2 + ix_2 + jy_2 + k_2) = t_1t_2 - i^2x_1x_2 - j^2y_1y_2 - k^2z_1z_2 + i(\cdots) + j(\cdots) + k(\cdots).$$

(b) We have

$$(ix_1 + jy_1 + kz_1)(ix_2 + jy_2 + kz_2) = -(x_1x_2 + y_1y_2 + z_1z_2) + ijx_1y_2 + jiy_1x_2 + \cdots$$

and $ij = -ji = k$, etc.

(c) We need to show that $A(1,1)(x) = 1x1 = x$ and that

$$\begin{aligned} A(g_1,h_1)A(g_2,h_2)x &= A(g_1,h_1)g_2xh_2^{-1}) \\ &= g_1g_2xh_2^{-1}h_1^{-1} \\ &= g_1g_2x(h_1h_2)^{-1} \\ &= A(g_1g_2, h_1h_2)x. \end{aligned}$$

(d) We have $|gxh^{-1}| = |g||x||h^{-1}| = |x|$ since $|g| = |h| = |h^{-1}| = 1$.

(e) (g,h) is in the kernel of this homomorphism if $gxh^{-1} = x$ for all x, i.e. $gx = xh$. For $x = 1$ this implies $g = h$. For $x = i$ this implies $gi = ig$. If $g = a + ib + jc + kd$ then $gi - ig = ai - b - kc + jd - ia + b - kc + jd = 2(jd - kc)$ so $gi - ig = 0$ implies $d = c = 0$. Similarly, $gj = jg$ implies $b = 0$. Therefore g is real and has length 1, so it must be ± 1. This leaves the two possibilities $(1,1)$ and $(-1,-1)$.

Question 3. (3 marks)
Suppose that y is a unit quaternion and let $H \subset \mathbf{H}$ be the plane orthogonal to y. Show that the reflection in the plane H is given in terms of quaternionic multiplication by $z \mapsto -y\bar{z}y$.

Answer 3. We know that $z \cdot y = \operatorname{Re}(\bar{z}y)$ so

$$\begin{aligned} z - 2(z \cdot y)y &= z - 2\operatorname{Re}(\bar{z}y)y \\ &= z - y(\bar{z}y + \bar{y}z)y \\ &= z - y\bar{z}y + y\bar{y}z \text{ as } y \text{ commutes with real numbers} \\ &= -y\bar{z}y \text{ as } |y| = 1. \end{aligned}$$

Question 4. (3 marks) Consider the 2-by-2 complex matrices:

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Show that $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -1$ and that $\sigma_a\sigma_b = \sigma_c = -\sigma_b\sigma_a$ where a, b, c is a cyclic permutation of $1, 2, 3$.

Consider the group of 2-by-2 complex matrices

$$SU(2) = \{A \;:\; A^\dagger = A^{-1},\ \det A = 1\} \text{ where } \dagger \text{ denotes conjugate transpose.}$$

Show that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$ then $c = -\bar{b}$, $d = \bar{a}$ and $|a|^2 + |b|^2 = 1$. If $a = t + ix$ and $b = y + iz$ then show that

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = t\mathbf{1} + x\sigma_1 + y\sigma_2 + z\sigma_3$$

and deduce that $SU(2)$ is isomorphic to the group of unit quaternions.

Answer 4. The first part is just a matrix computation. For the second part, observe that

$$A^\dagger = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

so if $A \in SU(2)$ then $c = -\bar{b}$ and $d = \bar{a}$. Moreover, $\det A = ad - bc = a\bar{a} + b\bar{b} = |a|^2 + |b|^2 = 1$. The next line is just obtained by substituting the matrices σ_m in and expanding. Finally, we use the isomorphism

$$G \to SU(2), \qquad t + ix + jy + kz \mapsto t\mathbf{1} + x\sigma_1 + y\sigma_2 + z\sigma_3.$$

This is clearly a bijection between the unit quaternions and the matrices in $SU(2)$ (the unit condition becomes the determinant one condition) so we just need to show it is a homomorphism. The multiplication

$$(t_1 + x_1\sigma_1 + y_1\sigma_2 + z_1\sigma_3)\,(t_2 + x_2\sigma_1 + y_2\sigma_2 + z_2\sigma_3)$$

is clearly bilinear in the coefficients and therefore it suffices to check that it gives the correct answer on a basis. This is what we did in the first part of the question.

Question 5. (3 marks)
We define the *exponential* of a purely imaginary quaternion q by the formula

$$e^q := \sum_{m=0}^{\infty} \frac{1}{m!} q^m.$$

If $q = u\theta$ for $|u| = 1$, prove that $e^q = \cos\theta + u\sin\theta$.

Check that

$$e^{i\pi/2} e^{j\pi/2} \neq e^{(i+j)\pi/2}$$

(i.e. the usual law of logarithms fails).

[This is just the beginning of a much more general story of exponential maps for groups. For more, see the 4th year course **Lie groups and Lie algebras**.]

Answer 5. We have

$$\begin{aligned} e^{u\theta} &= 1 + u\theta + \frac{1}{2!}u^2\theta^2 + \frac{1}{3!}u^3\theta^3 + \cdots \\ &= \left(1 - \frac{1}{2!}\theta^2 + \cdots\right) + u\left(\theta - \frac{1}{3!}\theta^3 + \cdots\right) \\ &= \cos\theta + u\sin\theta. \end{aligned}$$

since $u^2 = -1$.

For the second part we have $e^{i\pi/2} = i$, $e^{j\pi/2} = j$, $ij = k$ and $e^{(i+j)\pi/2} = e^{\frac{i+j}{\sqrt{2}}\frac{\pi}{\sqrt{2}}} = \cos\left(\frac{\pi}{\sqrt{2}}\right) + \frac{i+j}{\sqrt{2}}\sin\left(\frac{\pi}{\sqrt{2}}\right)$ so $e^{i\pi/2}e^{j\pi/2} \neq e^{(i+j)\pi/2}$.