Sheet 5: Möbius maps

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The final mark out of 10 (which counts towards your grade) will be calculated as your mark on Q1 plus your mark on Q2 plus your best mark from Q3–5. I will also award stars: silver for a total mark of 12 or more on any four questions, gold for a total mark of 15 or more on all questions.

Question 1. (2 marks)
Suppose that $x = (x_1, x_2, x_3)$ and $-x = (-x_1, -x_2, -x_3)$ are antipodal points on the unit sphere (so $x_1^2 + x_2^2 + x_3^2 = 1$). Let $S\colon S^2 \to \mathbf{C} \cup \{\infty\}$ denote the stereographic projection. Prove that $S(-x) = -\frac{1}{\overline{S(x)}}$.

Answer 1. We have $S(x) = \frac{x_1 + ix_2}{1 - x_3}$ so

$$\begin{aligned} S(-x) &= \frac{-(x_1 + ix_2)}{1 + x_3} \\ -\frac{1}{\overline{S(x)}} &= -\frac{1 - x_3}{x_1 - ix_2} \\ &= \frac{-(x_1 + ix_2)(1 - x_3)}{x_1^2 + x_2^2} \\ &= \frac{-(x_1 + ix_2)}{1 + x_3} \frac{1 - x_3}{1 - x_3} \end{aligned}$$

where we have used $x_1^2 + x_2^2 = 1 - x_3^2 = (1 - x_3)(1 + x_3)$.

Question 2. (5 marks)

(a) Prove that the Möbius group is generated by the Möbius maps $t_b z = z + b$ ($b \in \mathbf{C}$), $h_\lambda z = \lambda z$ ($\lambda \in \mathbf{C} \setminus \{0\}$) and $z \mapsto 1/z$.

(b) Prove that the subgroup of Möbius transformations $\frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbf{R}$ and $ad - bc = 1$ is generated by the Möbius maps t_b, ($b \in \mathbf{R}$), h_λ, ($\lambda \in \mathbf{R}$, $\lambda > 0$) and $z \mapsto -1/z$.

(c) What are the fixed points ($z = Tz$) of the Möbius transformation $Tz = \frac{z\cos\theta - \sin\theta}{z\sin\theta + \cos\theta}$?

(d) Let T be the Möbius transformation from (d). Describe the rotation $\pi \circ T \circ S$ on S^2 by giving its axis and the angle of rotation.

Answer 2. (a) We have

$$t_\alpha(h_\beta(r(t_d(h_c(z))))) = \alpha + \frac{\beta}{cz+d} = \frac{\alpha cz + \beta + \alpha d}{cz+d},$$

(where $r(z) = 1/z$) so if we pick $\alpha = a/c$ and $\beta = b - ad/c$ then we get $(az+b)/(cz+d)$ as required.

(b) Note that all of the transformations in question live in the group $PSL(2, \mathbf{R})$: for example, $h_\lambda(z) = \sqrt{\lambda}z/(1/\sqrt{\lambda})$ so we can take $a = 1/d = \sqrt{\lambda}$, $b = c = 0$, which has $ad - bc = 1$. The only difference from part 1 is that we need to use $-r$ and can only rescale by positive numbers, so:

- if $c > 0$, we need to pick $\beta = -b + ad/c = 1/c > 0$.
- if $c < 0$ we need to use h_{-c}, t_{-d} and $\beta = b - ad/c$.

(c) We have $z = Tz$ if and only if $z\cos\theta - \sin\theta = z^2\sin\theta + z\cos\theta$, i.e. $z^2 = -1$. Therefore the fixed points are $\pm i$.

(d) Since $\pi(\pm i) = (0, \pm 1, 0)$, the fixed points of $g = \pi \circ T \circ S$ are $(0, \pm 1, 0)$ so g is a rotation around the y-axis. We have $\pi(1) = (1, 0, 0)$ and $T(1) = \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = \frac{1 - \tan\theta}{1 + \tan\theta}$ so $\pi(T(1))$ has x_3-coordinate $\frac{1 - \left(\frac{1-\tan\theta}{1+\tan\theta}\right)^2}{1 + \left(\frac{1-\tan\theta}{1+\tan\theta}\right)^2} = \frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$, x_2-coordinate zero and x_1-coordinate $\cos 2\theta$, we see that this is a rotation by 2θ around the y-axis.

Question 3. (3 marks)
Draw an approximate map of the world under stereographic projection.

Answer 3.

Question 4. (3 marks) Let $U, V \subset \mathbf{C}$ be open sets and suppose that $f\colon U \to V$ is a *holomorphic map*, in other words for all $z \in U$ there exists a complex number $f'(z)$ such that

$$f(z+w) = f(z) + f'(z)w + \mathcal{O}(w^2)$$

(where $\mathcal{O}(w^2)$ means higher order terms in w). Suppose that $z_0 \in U$ is a point for which $f'(z_0) \neq 0$. If $\gamma(t)$ is a curve in $\mathbf{C}$, let $\dot{\gamma}(0)$ denote the vector $\frac{d}{dt}\big|_{t=0}(\gamma(t))$. Let $\gamma_1(t)$ and $\gamma_2(t)$ be curves with $\gamma_1(0) = \gamma_2(0) = z_0$ and suppose that the angle between the vectors $\dot{\gamma}_1(0)$ and $\dot{\gamma}_2(0)$ at z_0 is θ. Let $\delta_n(t) = f(\gamma_n(t))$, $n = 1, 2$. Prove that $\dot{\delta}_1(0)$ and $\dot{\delta}_2(0)$ meet at an angle θ at the point $f(z_0)$.

[This is a precise way of saying that holomorphic maps are conformal wherever their derivatives are nonvanishing. Note that Möbius maps are holomorphic.]

Answer 4. We have $f'(z_0) = re^{i\theta}$. By the chain rule, $\frac{d}{dt}\big|_{t=0} f(\gamma(t)) = f'(z_0)\dot{\gamma}(0)$, i.e. $\dot{\gamma}(0)$ is rotated by θ and rescaled by r. Similarly for $\dot{\delta}(0)$. Since both vectors are rotated by θ, the angle between them is unchanged.

Question 5. (3 marks)
Let T be a Möbius transformation $T(z) = \frac{az+b}{cz+d}$. If $c \neq 0$, show that T has either one or two fixed points in $\mathbf{C} \cup \{\infty\}$ and that it has precisely one fixed point if and only if $(a+d)^2 - 4(ad-bc) = 0$. If $c = 0$ show that T has either two fixed points (if $a \neq d$), or one fixed point (if $a = d$ and $b \neq 0$), or else T is the identity.

Find the fixed points of the following Möbius maps:

(a) $Tz = 1/z$,

(b) $Tz = e^{i\theta} z$ (for $\theta \in (0, 2\pi)$),

(c) $Tz = z + 1$.

Answer 5. A fixed point z satisfies $z = \frac{az+b}{cz+d}$ so a fixed point is a root of

$$cz^2 + dz - az - b = 0.$$

If $c \neq 0$ this is a quadratic equation with solutions $\frac{(a-d) \pm \sqrt{(a-d)^2 + 4bc}}{2c}$. When $(a-d)^2 + 4bc = 0$, i.e. $(a+d)^2 - 4(ad-bc)$, there is precisely one solution $(a-d)/2c)$ and otherwise there are two. If $c = 0$ then the fixed point equation is $az + b = dz$. This is always satisfied by $z = \infty$ and has:

- the additional solution $-b/(a-d)$ if $a \neq d$;
- no additional solutions if $a = d$ but $b \neq 0$;
- infinitely many solutions if $a = d$ and $b = 0$ (when T is the identity).

Finally

Map	Fixed point(s)
$1/z$	$1, -1$
$e^{i\theta} z$	$0, \infty$
$z + 1$	∞