

# Sheet 5: Möbius maps

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**The final mark out of 10 (which counts towards your grade) will be calculated as your mark on Q1 plus your mark on Q2 plus your best mark from Q3–5.** I will also award stars: silver for a total mark of 12 or more on any four questions, gold for a total mark of 15 or more on all questions.

## Question 1. (2 marks)

Suppose that  $x = (x_1, x_2, x_3)$  and  $-x = (-x_1, -x_2, -x_3)$  are antipodal points on the unit sphere (so  $x_1^2 + x_2^2 + x_3^2 = 1$ ). Let  $S: S^2 \rightarrow \mathbb{C} \cup \{\infty\}$  denote the stereographic projection. Prove that  $S(-x) = -\frac{1}{S(x)}$ .

## Question 2. (5 marks)

- (a) Prove that the Möbius group is generated by the Möbius maps  $t_b z = z + b$  ( $b \in \mathbb{C}$ ),  $h_\lambda z = \lambda z$  ( $\lambda \in \mathbb{C} \setminus \{0\}$ ) and  $z \mapsto 1/z$ .
- (b) Prove that the subgroup of Möbius transformations  $\frac{az+b}{cz+d}$  with  $a, b, c, d \in \mathbb{R}$  and  $ad - bc = 1$  is generated by the Möbius maps  $t_b$ , ( $b \in \mathbb{R}$ ),  $h_\lambda$ , ( $\lambda \in \mathbb{R}, \lambda > 0$ ) and  $z \mapsto -1/z$ .
- (c) What are the fixed points ( $z = Tz$ ) of the Möbius transformation  $Tz = \frac{z \cos \theta - \sin \theta}{z \sin \theta + \cos \theta}$ ?
- (d) Let  $T$  be the Möbius transformation from (c). Describe the rotation  $\pi \circ T \circ S$  on  $S^2$  by giving its axis and the angle of rotation.

**Question 3.** (3 marks)

Draw an approximate map of the world under stereographic projection.

**Question 4.** (3 marks) Let  $U, V \subset \mathbf{C}$  be open sets and suppose that  $f: U \rightarrow V$  is a holomorphic map, in other words for all  $z \in U$  there exists a complex number  $f'(z)$  such that

$$f(z + w) = f(z) + f'(z)w + \mathcal{O}(w^2)$$

(where  $\mathcal{O}(w^2)$  means higher order terms in  $w$ ). Suppose that  $z_0 \in U$  is a point for which  $f'(z_0) \neq 0$ . If  $\gamma(t)$  is a curve in  $\mathbf{C}$ , let  $\dot{\gamma}(0)$  denote the vector  $\frac{d}{dt}\big|_{t=0}(\gamma(t))$ . Let  $\gamma_1(t)$  and  $\gamma_2(t)$  be curves with  $\gamma_1(0) = \gamma_2(0) = z_0$  and suppose that the angle between the vectors  $\dot{\gamma}_1(0)$  and  $\dot{\gamma}_2(0)$  at  $z_0$  is  $\theta$ . Let  $\delta_n(t) = f(\gamma_n(t))$ ,  $n = 1, 2$ . Prove that  $\dot{\delta}_1(0)$  and  $\dot{\delta}_2(0)$  meet at an angle  $\theta$  at the point  $f(z_0)$ .

[This is a precise way of saying that holomorphic maps are conformal wherever their derivatives are nonvanishing. Note that Möbius maps are holomorphic.]

**Question 5.** (3 marks)

Let  $T$  be a Möbius transformation  $T(z) = \frac{az+b}{cz+d}$ . If  $c \neq 0$ , show that  $T$  has either one or two fixed points in  $\mathbf{C} \cup \{\infty\}$  and that it has precisely one fixed point if and only if  $(a+d)^2 - 4(ad-bc) = 0$ . If  $c = 0$  show that  $T$  has either two fixed points (if  $a \neq d$ ), or one fixed point (if  $a = d$  and  $b \neq 0$ ), or else  $T$  is the identity.

Find the fixed points of the following Möbius maps:

- (a)  $Tz = 1/z$ ,
- (b)  $Tz = e^{i\theta}z$  (for  $\theta \in (0, 2\pi)$ ),
- (c)  $Tz = z + 1$ .