Sheet 7: Hyperbolic geometry

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The final mark out of 10 (which counts towards your grade) will be calculated as your mark on Q1 plus your mark on Q2 plus your best mark from Q3–5. I will also award stars: silver for a total mark of 12 or more on any four questions, gold for a total mark of 15 or more on all questions.

Question 1. (2 marks)
Suppose that $a, b, c, d \in \mathbf{R}$ and $z \in \mathbf{C}$. Find the imaginary part of $\frac{az+b}{cz+d}$.

Answer 1. We have

$$\frac{az+b}{cz+d} = \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2} = \frac{ac|z|^2 + bd + bc\bar{z} + adz}{|cz+d|^2}.$$

The denominator is real and the imaginary part of the numerator $ac|z|^2 + bd + bc\bar{z} + adz$ is $(ad-bc)\,\mathrm{Im}(z)$, so the imaginary part of $(az+b)/(cz+d)$ is

$$\frac{(ad-bc)\,\mathrm{Im}(z)}{|cz+d|^2}.$$

Question 2. (5 marks)

(a) Find the subgroup of $PSL(2,\mathbf{R})$ consisting of Möbius transformations which fix the point i in the upper half-plane.

(b) Show that the subgroup you found in (a) is isomorphic to $SO(2)$, the group of rotations in 2-dimensional space.

(c) If g is a Möbius transformation with fixed point set P and h is another Möbius transformation, what is the fixed point set of hgh^{-1}?

(d) Let $t_b(z) = z + b$ and consider the group $\{t_b \in PSL(2,\mathbf{R}) \ : \ b \in \mathbf{R}\}$ be the group of translations of the upper half-plane; these all have precisely one fixed point (∞). Conjugate t_b by $h(z) = -1/z$ to get a subgroup of isometries of the hyperbolic upper half-plane which fix 0. Show that, as b varies, the orbit (under the action of this subgroup) of a point ri on the imaginary axis is a Euclidean circle centred at $ri/2$ with radius $r/2$. *[Such a circle is called a horocycle.]*

Answer 2. (a) Suppose $i = (ai + b)/(ci + d)$. Then $ai + b = -c + di$ so (since a, b, c, d are real) $a = d$ and $c = -b$. Moreover, as the original Möbius transformation is in $PSL(2,\mathbf{R})$, we can take $ad - bc = 1$, which becomes $a^2 + b^2 = 1$, so the most general element of $PSL(2,\mathbf{R})$ fixing i is the Möbius transformation $(z\cos\theta - \sin\theta)/(z\sin\theta + \cos\theta)$.

(b) There is an obvious map $F\colon SO(2) \to G$ where G is the group of Möbius transformations in (a), namely $F\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \frac{z\cos\theta - \sin\theta}{z\sin\theta + \cos\theta}$. We have seen that this is a homomorphism (it's just the restriction of the usual map $SL(2,\mathbf{R}) \to PSL(2,\mathbf{R})$ to the subgroup $SO(2)$). When is a matrix in the kernel of this map? When $\frac{z\cos\theta - \sin\theta}{z\sin\theta+\cos\theta} = z$ for all z, which means $z\cos\theta - \sin\theta = z^2\sin\theta + z\cos\theta$, or $(z^2+1)\sin\theta = 0$, which implies $\sin\theta = 0$. Therefore the kernel consists of the identity and minus the identity, so the image of the homomorphism is isomorphic to $SO(2)/\{\pm 1\}$. But this is again isomorphic to $SO(2)$ as there is a isomorphism $SO(2)/\{\pm 1\} \to SO(2)$ given by

$$G\left[\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right] = \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix}.$$

More directly, one could use the map

$$H\left[\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right] = \frac{z\cos(\theta/2) - \sin(\theta/2)}{z\sin(\theta/2) + \cos(\theta/2)}.$$

This is still a homomorphism (same proof as for F) and it's clearly surjective, but it's now injective as well: the kernel consists of those matrices $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for which $\cos(\theta/2) = \pm 1$, i.e. $\theta \in 2\pi\mathbf{Z}$, but this is just the identity matrix.

(c) The fixed point set of hgh^{-1} is $h(P)$: if $x = h(y) \in h(P)$ then $hgh^{-1}h(y) = hg(y) = h(y)$ as $y \in P$ and conversely.

(d) When we conjugate by h, we get a subgroup of Möbius transformations of the form $ht_bh^{-1}(z) = -1/(-1/z + b) = z/(1 - bz)$. If we apply this to ri then we get $ri/(1 - bri) = ri(1 + bri)/(1 + b^2r^2) = \frac{-br^2+ri}{1+b^2r^2}$. The real part is $x = -br^2/(1 + b^2r^2)$ and the imaginary part is $y = r/(1 + b^2r^2)$. Therefore $x^2 + y^2 = \frac{b^2r^2+1}{(1+b^2r^2)^2}r^2 = ry$, which gives $x^2 + (y - r/2)^2 = r^2/4$, which is the equation of a circle centred at $r/2$ with radius $r/2$.

Question 3. (3 marks)
Consider the upper half-plane model of hyperbolic 2-space. Let C_1, C_2 and C_3 be three hyperbolic lines. We say that C_3 is a common perpendicular if C_3 intersects C_1 and C_2 orthogonally at points Q and R.

(a) Assume there exists a common perpendicular for C_1 and C_2. By considering the hyperbolic triangle PQR that would be formed, prove that C_1 and C_2 cannot intersect at a third point P in the upper half-plane.

(b) When C_1 and C_2 are ultraparallel show that there is always a common perpendicular and that this is unique.

Answer 3. (a) The triangle PQR would have internal angles $\pi/2, \pi/2, \epsilon$ for some $\epsilon > 0$ and these sum to $\pi + \epsilon > \pi$ which contradicts the Gauss-Bonnet theorem.

(b) WLOG take C_1 to be the imaginary axis. Then C_3 must be a semicircle centred at 0. There is a 1-parameter family of such semicircles, an interval of which intersect C_2. At one end of this interval, the angle of intersection between C_3 and C_2 is zero (when they are first tangent), then it starts to increase, until it reaches π (when they are last tangent). Since this angle depends continuously on the radius of C_3, it takes on every value between 0 and π (by the intermediate value theorem). Therefore there exists such a C_3. If there were two, they (along with C_1 and C_2) would form a hyperbolic quadrilateral with all internal angles equal to $\pi/2$, which would contradict Gauss-Bonnet.

Question 4. (3 marks)
Consider the upper half-plane model of hyperbolic space. Let I denote the imaginary axis, so that $r_I(z) = -\bar{z}$ defines the reflection in I. Let C denote the upper unit semicircle $C = \{z \in \mathbf{C} : |z| = 1, \operatorname{Im}(z) > 0\}$. By conjugating r_I by an appropriate Möbius transformation, find the formula for the reflection r_C in the hyperbolic line C. Now let D be the upper semicircle centred at $a \in \mathbf{R}$ with radius r. By conjugating r_C by an appropriate Möbius transformation, deduce that the reflection r_D in the hyperbolic line D is given by $r_D(z) = a + \frac{r^2}{\bar{z}-a}$.

Answer 4. The Möbius transformation $g(z) = (z-1)/(z+1)$ sends 0 to -1, ∞ to 1 and i to i, so it sends I to C. Therefore $r_C = g \circ r_I \circ g^{-1}$ (because this is the unique Möbius transformation other than the identity which fixes C pointwise, i.e. the unique nontrivial element in the stabiliser of C under the action of isometries on hyperbolic lines). We have $g^{-1}(z) = (z+1)(1-z)$, so computing, we get

$$r_C(z) = g(r_I(g^{-1}(z))) = \frac{-\frac{\bar{z}-1}{\bar{z}+1} + 1}{\frac{\bar{z}-1}{\bar{z}+1} + 1} = \frac{1}{\bar{z}}.$$

Now the Möbius transformation $h(z) = \frac{z-a}{r}$ sends a to 0 and D to C, so $r_D = h^{-1} \circ r_C \circ h$. We have $h^{-1}(z) = rz + a$, so

$$r_D(z) = h^{-1}(r_C(h(z))) = a + r^2/(\bar{z} - a)$$

as required.

Question 5. (3 marks)
Find the area of a convex hyperbolic n-gon with interior angles $\alpha_1, \ldots, \alpha_n$. Prove that for every $0 < \alpha < \left(1 - \frac{2}{n}\right)\pi$ there is a regular convex hyperbolic n-gon with interior angles equal to α. Sketch this n-gon in both the disc and upper-half plane models. (The disc model is probably easier!)

Answer 5. By subdividing the polygon into $n-2$ triangles, we apply the Gauss-Bonnet theorem to each triangle and deduce that

$$\operatorname{area}(P) = (n-2)\pi - \sum \alpha_i.$$

Let μ be an nth root of unity and consider the points $r\mu^k$, $k = 0, 1, \ldots, n-1$. When $r < 1$, these form the vertices of a regular convex hyperbolic n-gon $P(r)$ in the disc model. As $r \to 1$, the interior angles go to zero and the area goes to $(n-2)\pi$. As $r \to 0$ the area goes to zero, so $\alpha = (\operatorname{area}(P(r)) - (n-2)\pi)/n \to \left(1 - \frac{2}{n}\right)\pi$. As the area and the angles change continuously in r, α takes every possible value in between 0 and $\left(1 - \frac{2}{n}\right)\pi$ (by the intermediate value theorem).