Sheet 6: More Möbius maps

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The final mark out of 10 (which counts towards your grade) will be calculated as your mark on Q1 plus your mark on Q2 plus your best mark from Q3–5. I will also award stars: silver for a total mark of 12 or more on any four questions, gold for a total mark of 15 or more on all questions.

Question 1. (2 marks)
Suppose that z_1, z_2, z_3, z_4 are distinct points in $\mathbf{C} \cup \{\infty\}$. For which $w \in \mathbf{C} \cup \{\infty\}$ is it impossible that $[z_1, z_2; z_3, z_4] = w$?

Answer 1. Without loss of generality, we can take $z_1 = 0$, $z_2 = \infty$, $z_4 = 1$ which gives $w = [0, \infty; z, 1] = z$ and z can be anything except $0, 1, \infty$.

Question 2. (5 marks)
Let $S\colon S^2 \to \mathbf{C} \cup \{\infty\}$ denote the stereographic projection and let $x = (x_1, x_2, x_3)$ be a point on S^2.

(a) Show that $|S(x)| \leq 1$ if and only if $x_3 \leq 0$ (so that the southern hemisphere projects stereographically to the unit disc).

(b) What is the stereographic projection of the spherical triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$?

(c) Suppose that $z = S(x_1, x_2, x_3)$ and let θ be the angle between $(0, 0, 1)$ and (x_1, x_2, x_3). Show that $|z| = \tan(\theta/2)$. [This is the geometric origin of the famous "$tan(x/2)$ substitution".]

(d) Prove that four points in $\mathbf{C} \cup \{\infty\}$ lie on a circle if and only if their cross-ratio is real.

(e) Deduce that if A, B, C, D are points on a circle (cyclically ordered) then the internal angles in the quadrilateral $ABCD$ at B and at D add up to π and that the angle between AB and AC equals the angle between DB and DC.

[Hint: Consider the arguments of the cross-ratios $[D, B; A, C]$ and $[A, B; C, D]$.]

Answer 2. (a) We have $|S(x)|^2 = \left|\frac{x_1+ix_2}{1-x_3}\right|^2 = \frac{x_1^2+x_2^2}{(1-x_3)^2}$ and since $x_1^2 + x_2^2 = 1 - x_3^2$ we get

$$|S(x)| = \frac{1 - x_3^2}{(1 - x_3)^2} = \frac{1 + x_3}{1 - x_3}$$

which is ≤ 1 if and only if $1 + x_3 \leq 1 - x_3$ i.e. $x_3 \leq 1$.

(b) The vertices map to 1, i and ∞. Since spherical circles project to circles and straight lines, the edges map to: the segment of the unit circle connecting 1 and i, the half-lines $\{i + it \ : \ t \geq 0\}$ and $\{1 + t \ : \ t \geq 0\}$. So the whole triangle projects to

$$\{x + iy \ : \ x^2 + y^2 \geq 1, \ x \geq 0, \ y \geq 0\}.$$

(c) We have $x_3 = \cos\theta$ and $x_3 = \frac{1-|z|^2}{1+|z|^2}$ so $|z| = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$ but

$$\begin{aligned} 1 - \cos\theta &= 1 - \cos^2(\theta/2) + \sin^2(\theta/2) \\ &= 2\sin^2(\theta/2) \\ 1 + \cos\theta &= 1 + \cos^2(\theta/2) - \sin^2(\theta/2) \\ &= 2\cos^2(\theta/2) \\ \therefore \frac{1 - \cos\theta}{1 + \cos\theta} &= \frac{\sin^2(\theta/2)}{\cos^2(\theta/2)} \\ \therefore \sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} &= \tan(\theta/2). \end{aligned}$$

(d) Let T be the Möbius transformation taking z_1, z_2, z_3 to $0, 1, \infty$ and let $z = Tz_4$. Then the cross-ratio of the four points is z. If z_1, z_2, z_3, z_4 lie on a circle C then $0, 1, \infty, z$ lie on a circle/straight line TC, but $0, 1, \infty$ lie on a unique circle/straight line, namely $\mathbf{R} \cup \{\infty\}$, hence $z \in \mathbf{R}$. Conversely, if $z \in \mathbf{R}$ then $0, 1, \infty, z$ lie on a common straight line and hence z_1, z_2, z_3, z_4 lie on a common circle ($T^{-1}(\mathbf{R} \cup \{\infty\})$).

(e) The internal angle between DA and DC is the argument of $(D - C)/(D - A)$. The internal angle between BC and BA is the argument of $(B-A)/(B-C)$. The quotient $(D - A)(B - C)/(D - C)(B - A)$ is the cross-ratio $[D, B; A, C]$ which is real because A, B, C, D lie on a line, hence the difference between the arguments of $(D-A)/(D-C)$ and $(B - A)/(B - C)$ is an integer multiple $n\pi$ (the argument of a real number is an integer multiple of π). Moreover, because $ABCD$ are cyclically ordered, if D, B, C map to $0, \infty, 1$ then A goes to a negative number, which means that n is odd. Similarly, by considering the cross-ratio $[A, B; C, D]$ we get the other identity.

Question 3. (3 marks)
Let $u, v \in \mathbf{C}$ and consider the points $\pi(u), \pi(v) \in S^2 \subset \mathbf{R}^3$ where $\pi\colon \mathbf{C} \to S^2$ is the inverse of stereographic projection. Prove that the Euclidean distance $|\pi(u) - \pi(v)|$ in $\mathbf{R}^3$ is

$$|\pi(u) - \pi(v)| = \frac{2|u-v|}{\sqrt{(1+|u|^2)(1+|v|^2)}}.$$

[*Courage* - this will probably get worse before it gets better.]

Answer 3. We have $\pi(u) = \left(\frac{2u}{1+|u|^2}, \frac{1-|u|^2}{1+|u|^2}\right)$ so

$$\begin{aligned}
|\pi(u) - \pi(v)| &= \left|\frac{2u}{1+|u|^2} - \frac{2v}{1+|v|^2}\right|^2 + \left(\frac{1-|u|^2}{1+|u|^2} - \frac{1-|v|^2}{1+|v|^2}\right)^2 \\
&= \frac{1}{(1+|u|^2)^2(1+|v|^2)^2}\Big(4\left|u(1+|v|^2) - v(1+|u|^2)\right|^2 + \\
&\quad + \left((1-|u|^2)(1+|v|^2) - (1+|u|^2)(1-|v|^2)\right)^2\Big) \\
&= \frac{4}{(1+|u|^2)^2(1+|v|^2)^2}\big(|u|^2(1+|v|^2)^2 + |v|^2(1+|u|^2) - \\
&\quad - (\bar{u}v + u\bar{v})(1+|u|^2)(1+|v|^2) + \left(|v|^2 - |u|^2\right)^2\big) \\
&= \frac{4}{(1+|u|^2)^2(1+|v|^2)^2}\big((|u|^2 + |v|^2)(1+|u|^2)(1+|v|^2) - \\
&\quad - (\bar{u}v + u\bar{v})(1+|u|^2)(1+|v|^2)\big) \\
&= \frac{4|u-v|^2}{(1+|u|^2)(1+|v|^2)}
\end{aligned}$$

and taking square roots gives the answer.

Question 4. (3 marks) Consider a Möbius map of the form $Tz = \frac{az+b}{-\bar{b}z+\bar{a}}$ with $|a|^2+|b|^2=1$. Show that if $u, v \in \mathbf{C}$ and $\pi\colon \mathbf{C}\cup\{\infty\} \to S^2$ is the inverse of stereographic projection then $|\pi(Tu) - \pi(Tv)| = |\pi(u) - \pi(v)|$. (You may assume the expression for $|\pi(u) - \pi(v)|$ from the previous question).

Answer 4. We have

$$|\pi(u) - \pi(v)| = \frac{2|u-v|}{\sqrt{(1+|u|^2)(1+|v|^2)}}.$$

so

$$\begin{aligned}
|\pi(Tu) - \pi(Tv)| &= \frac{2|Tu - Tv|}{\sqrt{(1+|Tu|^2)(1+|Tv|^2)}} \\
&= \frac{2\left|\frac{au+b}{-\bar{b}u+\bar{a}} - \frac{av+b}{-\bar{b}v+\bar{a}}\right|}{\sqrt{\left(1+\left|\frac{au+b}{-\bar{b}u+\bar{a}}\right|^2\right)\left(1+\left|\frac{av+b}{-\bar{b}v+\bar{a}}\right|^2\right)}} \\
&= \frac{2|(au+b)(-\bar{b}v+\bar{a}) - (av+b)(-\bar{b}u+\bar{a})|}{\sqrt{(|-\bar{b}u+\bar{a}|^2 + |au+b|^2)(|-\bar{b}v+\bar{a}|^2+|av+b|^2)}} \\
&= \frac{2||a|^2u - |b|^2v - a\bar{b}uv + \bar{a}b - |a|^2v + |b|^2u - b\bar{a} + a\bar{b}uv|}{\sqrt{(|\bar{b}u|^2 + |\bar{a}|^2 + |au|^2 + |b|^2)(|\bar{b}v|^2 + |\bar{a}|^2 + |av|^2 + |b|^2)}} \\
&= \frac{2|u-v|}{\sqrt{(1+|u|^2)(1+|v|^2)}}
\end{aligned}$$

where we have used $|a|^2+|b|^2=1$ in the final step.

Question 5. (3 marks)
By considering the Möbius transformation $z \mapsto 1 - z$ (and its action on $0, 1, \infty$), show that, if A, B, C, D are four points in $\mathbf{C}$, then $[D, B; A, C] = 1 - [C, B; A, D]$. Deduce that if A, B, C, D lie ordered cyclically on a circle then

$$|A - B||C - D| + |D - A||B - C| = |D - B||A - C|.$$

Deduce that, in a regular pentagon $ABCDE$, the ratio of the length of a diagonal to the length of a side is $\frac{1+\sqrt{5}}{2}$ (i.e. the golden ratio).

[Hint: You may use the results in Q.2(e) to prove, for example, that $\frac{(D-A)(B-C)}{(D-C)(B-A)} = -\frac{|D-A||B-C|}{|D-C||B-A|}$ or $\frac{(D-B)(A-C)}{(D-C)(A-B)} = \frac{|D-B||A-C|}{|D-C||A-B|}$.]

Answer 5. Let f be the Möbius transformation sending D, B, C, A to $0, \infty, 1, z$ (where $z = [D, B; A, C]$) and let $g(z) = 1 - z$. Then $g \circ f$ sends D, B, C, A to $1, \infty, 0, 1 - z$, hence $[C, B; A, C] = 1 - z$, proving the first part. Now we see that

$$\frac{(D - A)(B - C)}{(D - C)(B - A)} = 1 - \frac{(D - B)(A - C)}{(D - C)(A - B)}$$

Q.2(e) gives us that we may replace all differences with their magnitudes (at the cost of introducing a sign for the term on the left-hand side), from which we deduce the required equality.

Take A, B, C, D, E to be the vertices of a regular pentagon. Then $x = |D - A| = |D - B| = |A - C| = |A - D|$ is the length of a diagonal and $y = |B - A| = |C - B| = |D - C|$ is the length of a side. The equation now gives $y^2 + xy = x^2$ so x/y satisfies $\tau^2 - \tau - 1 = 0$, which is solved by $(1 \pm \sqrt{5})/2$. Only one of these solutions is positive, namely $\frac{1+\sqrt{5}}{2}$.