Sheet 2: Euclidean isometries

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I will mark all questions and get a total out of 16. Stars will be awarded: silver for marks of 12 or more, gold for marks of 15 or more. The final mark (which counts towards your grade) will be calculated as Q1 plus Q2 plus your best solution from Q3–5.

Question 1. (2 marks)
Let $Sx = Ax + b$ and let $Tx = x + c$. Find S^{-1} in the form $A'x + b'$. Prove that $S \circ T \circ S^{-1}$ is a translation.

Answer 1. We have $S^{-1}(S(x)) = A'(Ax + b) + b' = A'Ax + A'b + b' = x$ so $A' = A^{-1}$ and $A'b = -b'$, i.e. $b' = -A^{-1}b$. Now

$$\begin{aligned} S(T(S^{-1}(x))) &= S(T(A^{-1}x - A^{-1}b)) \\ &= S(A^{-1}x - A^{-1}b + c) \\ &= A(A^{-1}x - A^{-1}b + c) + b \\ &= x + Ac \end{aligned}$$

Question 2. (5 marks)

(a) Show that the map $\mathrm{Isom}(\mathbf{R}^n) \to O(n)$ given by $T \mapsto A$ (where $Tx = Ax + b$) is a homomorphism. What is its kernel?

(b) Is it true that $\mathrm{Isom}(\mathbf{R}^n)$ is isomorphic to $O(n) \times \mathbf{R}^n$ where $O(n)$ is the subgroup of orthogonal transformations and $\mathbf{R}^n$ is the subgroup of translations? [Hint: Consider a commutator.]

(c) Consider the map $\sigma\colon O(n) \to O(n)$, $\sigma(A) = A\det(A)$. Show that σ is a homomorphism. When n is odd, show that the image is $SO(n)$. Using this, prove that if n is odd, $O(n) \cong SO(n) \times \{\pm\mathbf{1}\}$, where $\mathbf{1}$ denotes the identity matrix.

(d) Deduce that if P is a 3-dimensional convex polytope invariant under the map $\tau\colon \mathbf{R}^3 \to \mathbf{R}^3$, $\tau(x) = -x$ then $\mathrm{Sym}(P) = \mathrm{Sym}^+(P) \times \{\pm 1\}$.

(e) Give an example of a P which is τ-invariant, and an example of a P which shows that τ-invariant is a necessary hypothesis in part (d).

Answer 2. (a) If $Sx = Ax+b$ and $Tx = Cx+d$ then $STx = A(Cx+d)+b = ACx+Ad+b$ so $ST \mapsto AC$, as required. The kernel is the group of translations.

(b) We saw in Question 1 that if $Sx = Ax + b$ and $Tx = x + c$ then $STS^{-1}x = x + Ac$. Therefore $T^{-1}STS^{-1}x = x + Ac - c$ which is not the identity unless c is fixed by A. Therefore there are elements of $O(n)$ and of the translation subgroup which do not commute (as they would if the group were a direct product).

(c) We have $AB\det(AB) = AB\det(A)\det(B) = A\det(A)B\det(B)$ as required. $A\det A$ has determinant $\det(A)^{n+1}$ so this is always equal to 1 when n is odd (since $\det(A) = \pm 1$) and the image of σ is contained in $SO(n)$; moreover, $\sigma(A) = A$ when $A \in SO(n)$, so the image equals $SO(n)$. Therefore the map $A \mapsto (A\det A, \det A)$ is a homomorphism $O(n) \to SO(n) \times \{\pm 1\}$. It is clearly bijective because $A = A\det A/\det A$.

(d) If P is invariant under τ then $-\mathbf{1} \in O(n)$ is in $\mathrm{Sym}(P)$ and hence $A\det A \in \mathrm{Sym}^+(P)$ for any $A \in \mathrm{Sym}(P)$. So, as in (c), $A \mapsto (A\det A, \det A)$ gives the desired isomorphism.

(e) The cube is τ-invariant. The tetrahedron is not, and $\mathrm{Sym}^+(T) = A_4$, $\mathrm{Sym}(T) = S_4$ and S_4 is not a direct product of A_4 with $\{\pm 1\}$.

Question 3. (3 marks)
Suppose that G is a finite group acting on a set X. For each $g \in G$, let $X^g = \{x \in X \ : \ gx = x\}$ denote the set of fixed points of g. By considering the set $\{(g,x) \in G \times X \ : \ gx = x\}$, show that $\sum_{g \in G} |X^g| = \sum_{x \in X} |\operatorname{Stab}(x)|$. Deduce that there are $\frac{1}{|G|} \sum_{g \in G} |X^g|$ different orbits in total.

There are $3^4 = 81$ different ways of colouring in the faces of a regular tetrahedron using the colours red, white and blue. The symmetry group A_4 of the tetrahedron acts on these colourings and we say that two colourings are congruent if they are related by a rotation. The number of orbits is the number of congruence classes of colouring. Using the first part of the question, and appealing to the geometric description of the elements of A_4 on Sheet 1, prove that there are 15 congruence classes of colouring.

Answer 3. The set can be partitioned in two ways: $\bigcup_{g \in G} \{x \ : \ gx = x\}$ or $\bigcup_{x \in X} \{g \in G \ : \ gx = x\}$. Therefore it has size $\sum_{g \in G} |X^g|$ and $\sum_{x \in X} |\operatorname{Stab}(x)|$, proving the formula. Using the orbit-stabiliser theorem, we have that $\sum_{x \in X} |\operatorname{Stab}(x)| = \sum_{x \in X} |G|/|\operatorname{Orb}(x)| = |G| \sum_{x \in X} (1/|\operatorname{Orb}(x)|) = |G| \sum_{\text{orbits}} |\operatorname{Orb}(x)|/|\operatorname{Orb}(x)| = |G| \cdot N$ where N is the number of orbits.

To apply the formula we need to know the number of g-invariant colourings for each $g \in G$. For the (eight) rotations of order 3 it is easy to see that there are 3^2 possible colourings (let A be the axis of rotation; the face pierced by A can be coloured in any way and the other three faces share a common colour). For the (three) rotations of order 2 it is easy to see that there are 3^2 colourings. For the identity element, any colouring is invariant, so there are 3^4. Therefore the formula gives

$$\text{number of orbits } = \frac{1}{12}(3^4 + 3 \times 3^2 + 8 \times 3^2) = \frac{9}{12}(9 + 3 + 8) = 15.$$

Question 4. (3 marks)

(a) Let R be a rotation of the plane around 0 by an angle θ. Express R as a product of two reflections.

(b) Let ABC be a triangle with internal angles α, β, γ at A, B, C respectively and let e_A, e_B, e_C denote the edges opposite to A, B, C respectively. Let $R(p, \theta)$ denote the rotation around a point p by an angle θ anticlockwise. Using part (a) or otherwise, show that $R(C, 2\gamma)R(B, 2\beta)R(A, 2\alpha) = 1$.

Answer 4. (a) Let ℓ_1 and ℓ_2 be two lines which meet at the origin at an angle $\theta/2$ (so that ℓ_2 is at an angle $\theta/2$ anticlockwise of ℓ_1). Let r_i, $i = 1, 2$, denote the reflection in ℓ_i. Then $r_2 \circ r_1$ is a rotation by θ. To see this, note that: if $x \in \ell_1$ then $r_2(r_1(x)) = r_2(x)$ and $r_2(\ell_1)$ makes an angle θ with ℓ_1; if $x \in \ell_2$ then $r_1(x)$ makes an angle $\theta/2$ with ℓ_1 and so $r_2(r_1(\ell_2))$ makes an angle θ with ℓ_2. Since an isometry fixing 0 is determined by its action on a basis, and we have just checked that this isometry acts like a rotation on two linearly independent vectors, we deduce that this *is* a rotation.

(b) By part (a) we have $R(A, 2\alpha) = r_B r_C$, $R(B, 2\beta) = r_A r_B$ and $R(C, 2\gamma) = r_C r_A$ where r_i is the reflection in e_i. Therefore the product is $r_C r_A r_A r_B r_B r_C = 1$.

Question 5. (3 marks)
Let $H \subset \mathbf{R}^n$ be a hyperplane. Show that if T is an isometry of $\mathbf{R}^n$ such that $Tx = x$ for all $x \in H$ then T is either the identity or the reflection in H.

Answer 5. Pick a point $0 \in H$ and an orthonormal basis $e_1, ldots, e_n$ such that $e_1, \ldots, e_{n-1}$ lie on H. Then the isometry is determined by Te_n. But $\{e_1, \ldots, e_{n-1}, Te_n\}$ is an orthonormal basis, so Te_n is a unit vector on the line orthogonal to H. There are only two possibilities: $Te_n = e_n$ and $Te_n = -e_n$. These give the identity and the reflection respectively.