Math 340 Worksheet 25

The following are select solutions from the worksheet.

Exercise 1.
True or False:

- If $x \in V$ is an eigenvector for a linear operator $L : V \to V$, then so is λx for any scalar λ.
- If a 3×3 matrix A has eigenvalues -1, 2, and 5, then A is diagonalizable.
- Let A and B be 2×2 symmetric matrices. If A and B have the same trace and determinant, then A and B are similar.

Exercise 2.
Find an *orthogonal* matrix P such that $P^T AP$ is diagonal:

- $$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix}.$$
- $$A = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$
- $$A = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}.$$

Exercise 3.
If A is an orthogonal matrix, prove that 1 and -1 are the only real eigenvalues of A. Is there an orthogonal matrix which has neither 1 nor -1 as an eigenvalue?

Exercise 4.
Let V be the space of all differentiable functions on the real line. What are the eigenvalues and eigenvectors of the operator $L : V \to V$ given by *differentiation*, i.e. $Lf = f'$, taking in a function, and outputting it's *derivative*.

Exercise 5.
Construct an orthogonal basis for the subspace of $\mathbf{R}^3$ spanned by $[1, -1, 1]$, $[-2, 2, -2]$, $[2, -1, 2]$, and $[0, 0, 0]$.

Exercise 6.
Consider the orthogonal basis $[1, 0, 2]$, $[-2, 0, 1]$, and $[0, 1, 0]$ for $\mathbf{R}^3$. Write $[2, -3, 1]$ as a linear combination of these three vectors (there's an easier way that solving a system of linear equations).

Exercise 7.
Consider the orthogonal basis

$$[1, 1, 1, 1, 1, 1, 1, 1], [1, -1, 1, -1, 1, -1, 1, -1], [1, 1, -1, -1, 1, 1, -1, -1]$$

$$[1, 1, 1, 1, -1, -1, -1, -1], [1, -1, -1, 1, 1, -1, -1, 1], [1, -1, 1, -1, -1, 1, -1, 1]$$

$$[1, 1, -1, -1, -1, -1, 1, 1], [1, -1, -1, 1, -1, 1, 1, -1].$$

This is the *Haar basis* for $\mathbf{R}^8$, a very important basis in signals processing, compression, and more general theoretical computing science (the book by Ryan O'Donnell is a fun introduction for the interested reader). Write the vector $[5, 3, 2, 7, 2, 6, 5, 3]$ as a linear combination of the vectors above.