Research Statement

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I am a harmonic analyst. My research over the past few years has focused on the study of radial Fourier multiplier operators on Euclidean space, and their analogues on compact manifolds, through an understanding of the geometry and regularity of wave propagation. In addition, I have explored problems in geometric measure theory, investigating when 'structure' occurs in fractals of large dimension. Both areas of research have raised interesting questions which I plan to pursue in my postgraduate work.

During my PhD, my work on multipliers has concentrating on relating bounds on Fourier multiplier operators to bounds for analogous operators on compact manifolds. Through this work, I was able to prove a 'transference principle' [3] for zonal multipliers operators on the sphere S^d. For $d \geq 4$ and a range of L^p spaces, this principle shows that bounds for a radial Fourier multiplier with symbol $m(|\xi|)$ imply bounds for a zonal multiplier operators on S^d induced by the same symbol m. In the process, I also completely characterized those symbols m whose dilates give a uniformly bounded family of zonal multiplier operators on $L^p(S^d)$. This is the first such characterization for any $p \neq 2$; more broadly, no comparable characterization or transference principle of this form has been established for any analogous family of multiplier operators on any other compact manifold.

My work in geometric measure theory focuses on constructing sets of large fractal dimension avoiding certain point configurations. Before starting my PhD, I had worked with Malabika Pramanik and Joshua Zahl, obtaining a method [4] for constructing sets with large Hausdorff dimension avoiding certain point configurations. During my PhD, I continued this line of research by establishing several probabilistic extensions of the methods of the previous paper to address the more difficult problem of constructing sets of large Fourier dimension avoiding configurations [2]. This method remains the only construction method for constructing sets of large Fourier dimension avoiding nonlinear configurations, and remains the best current method for constructing sets avoiding general 'linear' point configurations when $d > 1$.

In the near future, I hope to generalize the bounds obtained in [3] to the more general setting of multipliers for eigenfunction expansions of Laplace-Beltrami operators on Riemannian manifolds M with periodic geodesic flow. One obstruction to this generalization at a 'single frequency scale' is obtaining control of iterates of a pseudodifferential operator on M called the 'return operator'. Another obstruction when 'combining frequency scales' is an endpoint refinement of the local smoothing inequality for the wave equation on M. I am also interested of obtaining bounds on manifolds whose geodesic flow has well-controlled dynamical properties, such as forming an integrable system. Related to my work in geometric measure theory, I hope to apply the probabilistic methods I exploited in the construction of sets of large Fourier dimension to construct random fractals which exhibit good $l^2 L^p$ decoupling properties. And I am interested in determining the interrelation of patterns with the study of multipliers on manifolds, in particular studying Falconer distance problems on Riemannian manifolds to local smoothing bounds for the wave equation on manifolds, as well as exploring analogues of the Fourier dimension of sets on a Riemannian manifolds.

The remainder of this summary provides context and describes the results I have obtained during my PhD in further detail, finishing with a further elaboration of future work and it's feasibility given the tools I have gained from my previous work.

Multiplier Operators On $\mathbb{R}^d$ And On Manifolds

Radial Multipliers

Multipliers have long been a central object in harmonic analysis. In his pioneering work, Fourier showed solutions to the classical equations of physics are described by Fourier multipliers, operators T defined by a function $m : \mathbb{R}^d \to \mathbb{C}$, the symbol of T, such that

$$Tf(x) = \int_{\mathbb{R}^d} m(\xi)\widehat{f}(\xi)e^{2\pi i\xi\cdot x}\, dx.$$

Of particular interest are the *radial* multipliers, whose symbol is a radial function. We denote the radial multiplier with symbol $m(\xi) = a(|\xi|)$ by T_a in the sequel. Any translation-invariant operator on $\mathbb{R}^d$ is a Fourier multiplier operator, explaining their broad applicability in areas as diverse as partial differential equations, number theory, complex variables, and ergodic theory.

In harmonic analysis, it has proved incredibly profitable to study the boundedness of Fourier multipliers with respect to various L^p norms. It seems to be one of the few tractable ways of quantifying how different types of planar waves interact with one another, thus underpinning all deeper understandings of the Fourier transform. The need for an understanding of the L^p boundedness properties of a general Fourier multiplier became of central interest in the 1960s, brought on by the spur of applications the Calderon-Zygmund school and their contemporaries brought to the theory. Necessary conditions on a symbol to ensure the corresponding Fourier multiplier was bounded on L^p were found, but finding necessary and sufficient conditions which guarantee L^p boundedness proved to be an impenetrable, if not potentially impossible problem. No results were obtained in the past half century, aside from trivial cases where $p \in \{1, 2, \infty\}$.

It thus came as a surprise when recently several arguments [1,6,8,11] emerged giving necessary and sufficient conditions on a symbol a for the *radial* Fourier multiplier T_a to be bounded on $L^p(\mathbb{R}^d)$. Consider a decomposition $a(\rho) = \sum a_k(\rho/2^k)$, where $a_k(\rho) = 0$ for $\rho \notin [1, 2]$. For $1 \leq p \leq 2$, in order for T_a to be bounded on $L^p(\mathbb{R}^d)$, testing by Schwartz functions reveals it is necessary that $\sup_j C_p(a_j) < \infty$, where

$$C_p(a) = \left(\int_0^\infty \left[(1 + |t|)^{(d-1)(1/p-1/2)}\widehat{a}(t) \right]^p\, dt \right)^{1/p},$$

Duality implies the boundedness of T_a on $L^p(\mathbb{R}^d)$ is equivalent to it's boundedness on $L^{p'}(\mathbb{R}^d)$ when $1/p+1/p' = 1$, and so for $2 \leq p \leq \infty$ it is natural to define $C_p(a) = C_{p'}(a)$. Using Bochner-Riesz multipliers as endpoint examples, it is natural to conjecture the condition $\sup_j C_p(a_j) < \infty$ is not only necessary, but also *sufficient* to guarantee L^p boundedness for $|1/p - 1/2| > 1/2d$. For radial input functions this conjecture is true [6], though resolving this conjecture for general inputs is likely far beyond current research techniques, given that it implies the Bochner-Riesz conjecture, and thus also the restriction and Kakeya conjectures.

Heo, Nazarov, and Seeger [8] have proved the conjecture for $d \geq 4$ and $|1/p-1/2| > (d-1)^{-1}$. Cladek [1] improved the range of the conjecture for compactly supported a. She proved the result when $d = 4$ and $|1/p - 1/2| > 11/36$ and when $d = 3$ and $|1/p - 1/2| > 11/26$. Also of note is the work of Kim [11] also extended the bounds of [8] to *quasi-radial multipliers*, Fourier multipliers with a symbol $q(\xi)$ which is smooth, non-negative, homogeneous of order one, and whose level sets are hypersurfaces of non-vanishing Gauss curvature. Nonetheless, the full conjecture remains unsolved for all $d \geq 2$.

Often bounds on multipliers are obtained by assuming smoothness properties of the symbol a. The bound $\sup_j C_p(a_j) < \infty$ can be viewed in some sense as such a condition, but is not equivalent to the boundedness of any Sobolev, Besov, or Triebel-Lizorkin norm. However, the bound is implied if the functions $\{a_j\}$ uniformly lie in the Besov space $B_p(\mathbb{R}) := B^{d(1/p-1/2)}_{2,p}(\mathbb{R})$, which roughly speaking says that the functions $\{a_j\}$ have $d(1/p-1/2)$ derivatives in L^2. One could conjecture that for $|1/p-1/2| > 1/2d$, the operator T_a is bounded on $L^p(\mathbb{R}^d)$ if $\sup_j \|a_j\|_{B_p(\mathbb{R})} < \infty$.

This conjecture is weaker than the last, and only gives necessary, not sufficient, conditions for boundedness. Nonetheless, this weaker conjecture has been verified by Lee, Rogers, and Seeger [14] to be true for all $d \geq 2$ in the Stein-Tomas range $|1/p - 1/2| > (d+1)^{-1}$, but the full range still remains open for all $d \geq 2$.

We remark that various high powered techniques have recently been developed towards an understanding of the Bochner-Riesz conjecture, such as broad-narrow analysis, decoupling, and the polynomial method. However, these methods are difficult to apply when studying the two conjectures introduced above, since they are *endpoint results*. More precisely, in arguments related to the Bochner-Riesz conjecture, one allows for inequalities to have a multiplicative loss of factors of the form R^{ε} or $\log R$, where R is the frequency scale of the analysis. This is negligible to the analysis, since the Bochner-Riesz multipliers are conjectured to be bounded on L^p for an *open* interval of exponents, and so methods involving interpolation between L^p spaces allow us to remove these multiplicative factors when making conclusions. But an arbitrary multiplier bounded on $L^p(\mathbb{R}^d)$ may not be bounded on $L^{p'}(\mathbb{R}^d)$ for any $p' < p$, and so such methods are unavailable to us.

Zonal Multipliers and More General Operators on Manifolds

A theory of multiplier operators analogous to Fourier multipliers can be developed on the sphere S^d. Roughly speaking, Fourier multipliers are operators on $\mathbb{R}^d$ with $e^{2\pi i \xi \cdot x}$ as eigenfunctions. Zonal multipliers on S^d are those operators with the *spherical harmonics* as eigenfunctions, i.e. the restrictions to S^d of homogeneous harmonic polynomials on $\mathbb{R}^{d+1}$. Every function $f \in L^2(S^d)$ can be uniquely expanded as $\sum_{k=0}^{\infty} H_k f$, where $H_k f$ is a degree k spherical harmonics. A *zonal multiplier* is then an operator on S^d defined in terms of a function $a : \mathbb{N} \to \mathbb{C}$ by setting

$$Z_a f = \sum_{k=0}^{\infty} a(k) H_k f.$$

Every rotation invariant operator on S^d is a zonal multiplier, and thus such operators arise in diverse applications, including celestial mechanics, physics, and computer graphics.

Classic methods for studying zonal multipliers involve the analysis of special functions and orthogonal polynomials. But in the 1960s Lars Hörmander introduced the powerful theory of Fourier integral operators, a much more robust method which allows one to apply more modern techniques of harmonic analysis the theory. This theory is more robust in other senses, and in particular allows for the study of the much more general setup of multiplier operators of a general first order self-adjoint pseudodifferential operator P on a manifold M. For any such operator P, if Λ is the set of eigenvalues for P, then every function $f \in L^2(M)$ has an orthogonal decomposition $f = \sum_{\lambda \in \Lambda} f_\lambda$ where $P f_\lambda = \lambda f_\lambda$. For any symbol $a : \Lambda \to \mathbb{C}$, we define a multiplier operator $a(P) f = \sum_{\lambda \in \Lambda} a(\lambda) f_\lambda$. We note that if Δ is the Laplace-Beltrami operator on S^d, then $Pf = k(k+d-1) f$ for any spherical harmonic f of degree k. Thus if $P = \sqrt{\alpha^2 - \Delta}$, where $\alpha = (d-1)/2$, then $Pf = kf$ for a degree k harmonic f, and so any zonal multiplier Z_a can also be written as $a(P)$. In this general setup, Hörmander's idea was to use the Fourier inversion formula to write

$$a(P) = \int \widehat{a}(t) e^{2\pi i t P} \, dt,$$

The multiplier operators $e^{2\pi i t P}$, as t varies, give solutions to the *half-wave equation* $\partial_t = iP$ on the manifold M, one part of the full wave equation $\partial_t^2 - P^2 = 0$. Thus the study of the boundedness of the operators $a(P)$ is connected to the regularity of the wave equation on M. In particular, zonal multipliers are related to the wave equation $\partial_t^2 - \Delta = \alpha^2$ on S^d. We note that this method also connects the study of zonal harmonics to radial Fourier multipliers, since we can also write T_a as $a(\sqrt{-\Delta})$, where Δ is the usual Laplacian on $\mathbb{R}^d$.

Using this reduction, Hörmander [9] was able to prove L^p boundedness of the analogues of the Bochner-Riesz multipliers in this setting. Sogge [20,21] improved these bounds, introducing the approach, which works within the Stein-Tomas range, of reducing the problem to certain $L^2(M) \to L^p(M)$ bounds for spectral projection operators on M. Recently, Kim [12] adapted Sogge's approach to analyze multipliers of an operator P satisfying the following assumption:

Assumption A: If $p_{\text{prin}} : T^*M \to [0, \infty)$ is the principal symbol of P, then for each $x \in M$ the 'cosphere' $S_x^* = \{\xi \in T_x^*M : p_{\text{prin}}(x, \xi) = 1\}$ has non-vanishing Gaussian curvature.

Kim proved that under Assumption A, in the Stein-Tomas range $|1/p - 1/2| > (d+1)^{-1}$, if $\sup_j \|a_j\|_{B_p(\mathbb{R})} < \infty$, then the operator $a(P)$ is bounded on $L^p(M)$, thus obtaining an analogue of the result of Lee, Rogers and Seeger. In particular, Assumption A is satisfied for the $P = \sqrt{\alpha^2 - \Delta}$ on S^d, since the cospheres of P are ellipses. Thus Kim's result applies to zonal multipliers, but also more generally, arbitrary perturbations of Laplace-Beltrami operator on Riemannian manifolds. Note, however that there are no results in the literature for any exponent p, and any manifold M and operator P, which show that an operator $a(P)$ is bounded on $L^p(M)$ if $\sup_j C_p(a_j) < \infty$. *The main goal of my research project was to remedy this.*

My Contributions To The Study of Multipliers

As mentioned above, the main goal of my PhD research into multipliers was to see if we could obtain analogues of the arguments of [1,8,11] for zonal multipliers, i.e. proving that for some range of p and all functions a, if $\sup_j C_p(a_j) < \infty$, then the zonal multiplier Z_a is bounded on $L^p(S^d)$. I was able to able to obtain such analogues. Moreover, our argument is somewhat robust, applying to multipliers for a range of different operators P. Namely, we assume P satisfies assumption A, and in addition, satisfies the following, much more strict assumption:

Assumption B: The eigenvalues of P are contained in an arithmetic progression.

When $P = \sqrt{\alpha^2 - \Delta}$ on S^d, all eigenvalues are positive integers, so assumption B is satisfied for zonal multipliers. The assumption also holds for multipliers on the *rank one symmetric spaces* $\mathbb{R}\mathbb{P}^d$, $\mathbb{C}\mathbb{P}^d$, $\mathbb{H}\mathbb{P}^d$, and $\mathbb{O}\mathbb{P}^2$, i.e. operators diagonalized by analogous functions to the spherical harmonics on these spaces. Nonetheless, Assumption B is less natural than Assumption A, and I hope to obtain bounds under weaker assumptions, which we discuss in the future work section. Under Assumption A and Assumption B, in [3] I proved a 'single scale' version of this bound, i.e. proving that if a is supported on $[1,2]$, and $|1/p - 1/2| > 1/d$ then, uniformly in j,

$$\|a(P/2^j)f\|_{L^p(M)} \lesssim C_p(a)\|f\|_{L^p(M)}.$$

In a paper to be submitted for publication shortly, I provide further arguments that for an arbitrary function a, the operator $a(P)$ is bounded on $L^p(M)$ if $\sup_j C_p(a_j) < \infty$, thus obtaining a complete analogue of the argument of [8] for zonal multipliers.

This result has several important corollaries. Firstly, it implies a *transference principle* between Fourier multipliers and zonal multipliers. Since the condition $\sup_j C_p(a_j)$ is necessary for T_a to be bounded on $L^p(\mathbb{R}^d)$, we conclude that for $|1/p - 1/2| > 1/d$, if T_a is bounded on $L^p(\mathbb{R}^d)$, then the multipliers $a(P)$ is bounded on $L^p(M)$. Aside from the study of Fourier multipliers on $\mathbb{R}^d$, this is the first transference principle of this kind. There are no results in the literature for any $p \neq 2$, any other compact manifold M, and any operator P which guarantee that $a(P)$ is bounded on $L^p(M)$ if T_a is bounded on $L^p(\mathbb{R}^d)$.

Another powerful corollary follows from a result of Mitjagin [17], which implies that if a family of multipliers of the form $\{a(P/2^j) : j > 0\}$ are uniformly bounded on $L^p(S^d)$, and P has principal symbol p_{prin}, then the Fourier multiplier T with symbol $a \circ p_{\text{prin}}$ is bounded on $L^p(\mathbb{R}^d)$. It follows that the operators $a(P/2^j)$ cannot be uniformly bounded on $L^p(M)$ unless $\sup_j C_p(a_j)$

to be finite. But since the quantity $\sup_j C_p(a_j)$ is *scale invariant* (it is unchanged if we dilate a by a factor of 2^j), the bounds discussed above allow us to conclude that the operators $a(P/2^j)$ are uniformly bounded on $L^p(M)$ if and only if $\sup_j C_p(a_j) < \infty$ for $|1/p - 1/2| > (d-1)^{-1}$. Thus we have proved necessary and sufficient conditions for the operators $\{a(P/2^j) : j > 0\}$ to be uniformly bounded on $L^p(M)$. By the uniform boundedness principle, this result also classifies all functions a such that $a(P/2^j)\{f\}$ converges in L^p to f as $j \to \infty$ for all $f \in L^p(M)$. As before, these results are the first for any $p \neq 2$ and any other compact manifold M.

The main innovations . These techniques has been applied in other . But we were able to use Finsler geometry to obtain sharper bounds on estimates of the inner products $\langle T_\tau \rangle$, which simplifies the later analysis involving these quantities, and likely has applications in other problems.

Pattern Avoidance and Fourier Dimension

We now move onto a different topic in analysis: How large must a set $X \subset \mathbb{R}^d$ be before it must contain a certain point configuration, such as three points forming a triangle congruent to a given triangle, or four points forming a parallelogram? Discrete problems of this flavor have long been studied in combinatorics, for instance, such as when X is restricted to be a subset of the grid $\{1, \dots, N\}^d$. In the last 50 years, analysts have also begun studying analogous problems for infinite subsets $X \subset \mathbb{R}^d$, where the size of X is measured via a suitable *fractal dimension*, various different numerical statistics which measure how 'spread out' X is in space. The most common fractal dimension in use is the *Hausdorff dimension* $\dim_{\mathbb{H}}(X)$ of a set X, but we also consider the *Fourier dimension* $\dim_{\mathbb{F}}(X)$ as a refinement of Hausdorff dimension which also takes into account more subtle behavior of X related to it's correlation with the planar waves $e^{2\pi i \xi \cdot x}$ for $\xi \in \mathbb{R}^d$.

Hausdorff and Fourier dimension can both be defined in terms of finite Borel measures on X. The Hausdorff dimension of X is the least upper bound of the quantities s for which there exists a finite measure μ supported on X such that $\mu(B_r) \lesssim r^s$ for all $r > 0$ and all radius r balls $B_r \subset \mathbb{R}^d$, and the Fourier dimension is the least upper bound of s such that there is a measure μ supported on X with $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-s/2}$. Intuitively, if the condition $\mu(B_r) \lesssim r^s$ holds for a large s, then μ must have mass 'spread out' over a larger set, and this is only possible while being supported on X if X itself is spread out. Similarly, if $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-s/2}$, then μ cannot be correlated with high frequency waves, which also requires the measure μ be spread out.

Remarkably, the minimum dimension required to ensure a configuration exists can depend on the choice of dimension used. One has $\dim_{\mathbb{H}}(X) \geq \dim_{\mathbb{F}}(X)$, but for many explicit sets X this inequality is strict, with the Fourier dimension somehow measuring slightly more information about how correlated X is with planar waves. This subtle difference can emerge in the study of configurations. For instance, for each $k > 0$, a construction of Keleti [10] shows there exists a set $X \subset [0,1]$ with $\dim_{\mathbb{H}}(X) = 1$ such that for all distinct $x_1, \dots, x_k \in X$, and any integers $a_0, \dots, a_k \in \mathbb{Z}$, one has $a_1 x_1 + \dots + a_n x_n \neq a_0$. On the other hand, any set $X \subset [0,1]$ with $\dim_{\mathbb{F}}(X) > 2/k$ must necessarily contain distinct points $x_1, \dots, x_k \in X$ such that for some integers $a_0, \dots, a_k \in \mathbb{Z}$, one has $a_1 x_1 + \dots + a_n x_n = a_0$.

Unlike many other problems in harmonic analysis, we often do not have good expected *lower* bounds for the threshold dimension s_* corresponding to a given configuration, such that sets with dimension exceeding s_* must contain a given configuration, and such that sets with dimension less than s_* need not contain a given configuration. For instance, we do not know for $d > 2$ how large the Hausdorff dimension a set $X \subset \mathbb{R}^d$ must be before it contains all three vertices of an isosceles triangle, the threshold being somewhere between $d/2$ and $d-1$. Similarly, for a fixed angle $\theta \in (0, \pi)$, we do not know how large $\dim_{\mathbb{H}}(X)$ must be to guarantee X contains three distinct points A, B, and C which when connected determine an angle ABC equal to θ. If $\cos^2 \theta$ is rational, the results of Máthe [16] and Harangi, Keleti, Kiss, Maga, Máthe, Mattila, and Strenner [7] imply the threshold is somewhere between $d/4$ and $d-1$. If $\cos^2\theta$ is irrational, the threshold is somewhere between $d/8$ and $d-1$. We should not even necessarily expect the lower bounds to be the 'correct bounds' with which to make a conjecture, as we do with other problems in harmonic analysis, like the Kakeya conjecture and the Falconer distance problem; Until recently, certain results due to Łaba and Pramanik [13] seemed to imply that subsets of $[0,1]$ of Fourier dimension one must necessarily contain an arithmetic progression of length three, but Schmerkin has shown this need not be the case [19].

Given that we do not have good lower bounds with which to make definite conjectures, it is of interest to find general methods that we can use to produce counterexamples in these types of problems. That is, we wish to find methods with which to construct sets with large fractal dimension that *do not* contain certain point configurations. My research in geometric measure theory has so far focused on finding these types of methods. A good model problem to consider is, for a fixed function $f : (\mathbb{R}^d)^n \to \mathbb{R}^m$, finding a set X with large Hausdorff dimension, or a set of large Fourier dimension, such that X *avoids the zeroes* of f, in the sense that for any distinct points $x_1, \dots, x_n \in X$, $f(x_1, \dots, x_n) \neq 0$. This general model has been considered in various contexts:

(A) If $m = 1$, and f is a polynomial of degree n with rational coefficients, Máthe [16] constructs a set with Hausdorff dimension d/n avoiding the zeroes of f.

(B) If f is a C^1 submersion, Fraser and Pramanik [5] constructs a set with Hausdorff dimension $m/(n-1)$ avoiding the zeroes of f.

(C) If the zero set $f^{-1}(0)$ has Minkowski dimension at most s, I, together with my Master's thesis advisors Malabika Pramanik and Joshua Zahl [4] constructed sets of Hausdorff dimension $(dn-s)/(n-1)$ avoiding the zeroes of f.

(D) If f can be factored as $f = g \circ T$, where $T : (\mathbb{R}^d)^n \to \mathbb{R}^l$ is a full-rank, rational coefficient linear transformation, and $g : \mathbb{R}^l \to \mathbb{R}^m$ is a C^1 submersion, then I [3] have constructed a set with Hausdorff dimension m/l avoiding the zero sets of f.

(E) If $f(x_1, \dots, x_n) = a_1 x_1 + \cdots + a_n x_n$ with $\sum a_j = 0$, Pramanik and Liang [15] construct a set $X \subset [0,1]$ with Fourier dimension $\dim_{\mathbb{F}}(X) = 1$ avoiding the zeroes of f. This generalizes a construction of Schmerkin [19], who proved the result in the special case where $f(x_1, x_2, x_3) = (x_3 - x_1) - 2(x_2 - x_1)$ detects arithmetic progressions of length 3.

(F) Körner constructed subsets $X \subset [0,1]$ with Fourier dimension $(k-1)^{-1}$ such that for any integers $m_0, \dots, m_k$, and any distinct $x_1, \dots, x_k \in X$, $a_0 \neq a_1 x_1 + \cdots + a_n x_n$.

Notice that only the latter two result constructs a set with positive Fourier dimension. This is because it is much harder to construct sets with large Fourier dimension than it is to construct sets with large Hausdorff dimension. 'Random' sets often have Fourier dimension and Hausdorff dimension that agree with one another. On the other hand, most 'structured' sets have low Fourier dimension. For instance, hyper planes and the middle thirds Cantor set both have Fourier dimension zero. Constructing sets with large Fourier dimension avoiding configurations thus often requires a delicate balancing act between adding randomness and structure to the structure. Structure must be added to some degree to avoid containing a given configuration, but adding too much structure will result in your set likely having Fourier dimension zero.

As seen by Methods (E) and (F) above, the focus in the literature has mainly been on the construction of sets with large Fourier dimension avoiding patterns specified by a linear function f. This is because the Fourier transform behaves in a rather predictable manner with respect to linearity, whereas the understanding of the Fourier transform with respect to other nonlinear phenomena is poorly understood. It seems very difficult, if not impossible to adapt methods (A) and (D) above to construct sets with positive Fourier dimension, since the constructions involve constructing X at each spatial scale by choosing a good family of intervals, and then considering a large union of translates of the intervals. This is great for constructing many spatially spread out intervals, which results in a set with large Hausdorff dimension. But it is not good for ensuring Fourier decay, since a function concentrated near an arithmetic progression with separation l must have a large Fourier coefficient with frequency l^{-1}. On the other hand, methods (B) and (C) involve mostly pigeonholing arguments, so they seem the most likely to be able to be adapted to the Fourier dimension setting.

My Contributions To The Study Of Configurations

My Main Research Goal in geometric measure theory during my PhD was thus to try and adapt the methods of [5] and [2] to construct sets with large Fourier dimension avoiding patterns specified by a *nonlinear* function f, the first such result in the literature. For simplicity, I focused on the case when $m = d$ and when the function f was C^1 and full rank, as assumed in [5]. Then by the implicit function theorem, after possibly rearranging indices, we can locally write $f(x_1, \dots, x_n) = x_1 - g(x_2, \dots, x_n)$ for a function $g : (\mathbb{R}^d)^{n-1} \to \mathbb{R}^d$. In [2], under the assumption that g was a submersion in each variable $x_2, \dots, x_n$, I was able to modify the construction of [5] to construct sets with Fourier dimension $d/(n - 1/2)$ avoiding the zeroes of f. Under the further assumption that we can write $g(x_2, \dots, x_n) = ax_2 + h(x_3, \dots, x_n)$ for $a \in \mathbb{Q}$, I was able to construct sets with Fourier dimension $d/(n - 1)$ avoiding the zeroes of f, recovering the Hausdorff dimension bound of [5] in the Fourier dimension setting.

Future Lines of Research

Given the context from the previous sections, we finish this summary by describing in more detail several problems I believe may be accessible given the techniques I have used to solve previous problems.

Multipliers Associated With Periodic Geodesic Flow

It actually gives rise to a *Finsler geometry*, as I observed and used in [3] in order to obtain estimates for such multipliers.

Property I is the right version of the 'quasiradial multiplier' condition studied by Kim in [11]. If M is a Riemannian manifold, then the operator $\sqrt{-\Delta}$ satisfies Property I, since then $p(x, \xi) = |\xi|_x$ is the length of the covector ξ with respect to the metric on M, and so the cospheres S_p^* in this case are all ellipses. More generally, if M is a Finsler manifold with Finsler norm $F : TM \to [0, \infty)$, and $p(x, \xi) = F^*(x, \xi)$, where $F^*(x, \xi) = \sup_{F(x,v)=1} \xi(v)$ is the dual norm to F, then

- Analyzing the 'return time operator' to extend results on expansions of spherical harmonics to the study of the Laplace-Beltrami operator on S^d.
- Determining whether our methods extend to other manifolds whose geodesic flow is simpler to understand, such as integrable systems.
- Analyzing whether local smoothing bounds
- Constructing Random Salem Sets which satisfy a Decoupling Bound.
- Determining the relation between certain 'fractal weighted estimates' for the wave equation on $\mathbb{R}^d$ and the 'density decomposition' of multiplier bounds.

In fact, this resemblance opens up a whole new world of families of operators. Given an arbitrary elliptic self-adjoint first order classical pseudo-differential operator P

This method is highly robust and depends very little that we are working on the sphere; pretty much the only property we end up using is that the wave equation $\partial_t u = Pu$ has *periodic solutions*.

The natural analogue of the study of radial multipliers on $\mathbb{R}^d$ is the study of multipliers of a Laplace-Beltrami operator on a Riemannian manifold. The natural analogue of the study of quasiradial multipliers on $\mathbb{R}^d$ is the study of multipliers of an operator associated with a *Finsler geometry* on the manifold.

Relations Between Fourier Multipliers and Zonal Multipliers

One connection which explains why analogues to the bounds for radial Fourier multipliers might be found in the study of zonal multipliers is that both classes of operators are related to the Laplace operator on their respective spaces. Namely, if f is a distribution on $\mathbb{R}^d$, and $\Delta f = -\lambda^2 f$, then f has Fourier support on the sphere of radius λ centered at the origin, and so $T_a f = a(\lambda) f$. Similarly, if Δ is the Laplace-Beltrami operator on S^d, and $\Delta f = -\lambda(\lambda + d - 1)$, then f is a spherical harmonic of degree λ, and so $Z_a f = a(\lambda) f$. Using the notation of functional calculus, we can thus write $T_a = a\big(\sqrt{-\Delta}\big)$ and $Z_a = a\big(\sqrt{\alpha^2 - \Delta}\big)$, where $\alpha = (d-1)/2$, and now the resemblance is clear. In the rest of this section, we let $P = \sqrt{\alpha^2 - \Delta}$, and let $a(P/R)$ denote the zonal multiplier with symbol $a(\cdot/R)$.

On the other hand, unlike the planar waves $e^{2\pi i \xi \cdot x}$, it is difficult to understand what a general spherical harmonic might look. Dilation symmetries on $\mathbb{R}^d$ tell us high frequency planar waves are just the dilates of low frequency planar waves; on the other hand, S^d has no dilation symmetries, and high degree spherical harmonics need not look anything like low degree spherical harmonics. This could be alarming, because the transference principle we hope to prove implies a result related to dilation on S^d; namely, if the Fourier multiplier T_a is bounded on $L^p(\mathbb{R}^d)$, then the Fourier multipliers $T_{a,R}$ with symbols $a(\cdot/R)$ are all uniformly bounded on $L^p(\mathbb{R}^d)$, and thus the transference principle we hope to prove shows that the zonal multipliers $a(P/R)$ are uniformly bounded on $L^p(S^d)$. Because of the lack of dilation symmetry on S^d, the behavior of the operators $a(P/R)$ might change as R varies, which is discouraging.

Fortunately, whatever differences the operators $a(P/R)$ have as R varies are not as relevant to the study of L^p boundedness as one might at first think. This is because zonal multipliers only fail to be bounded on $L^p(S^d)$ because of 'high frequency behavior'; a zonal multiplier whose symbol is compactly supported is bounded on all of the L^p spaces. And there are various heuristics that tell us that the Laplacian on S^d begins to behave more and more similar to the Laplacian on $\mathbb{R}^d$ when restricted to high frequency eigenfunctions. For instance, on S^d, the operator P/R can be written as $\sqrt{\alpha^2/R + \Delta_R}$, where Δ_R is the Laplacian associated with the metric $g_R = R^2 g$ on S^d. As $R \to \infty$, the metric g_R gives S^d less curvature and more volume, and so we might imagine that, as $R \to \infty$, the operator P/R behaves more and more like $\sqrt{-\Delta}$ on $\mathbb{R}^d$, and thus multipliers of P/R behave more and more like radial Fourier multipliers, i.e. we might imagine that the equation $T_a = \lim_{R\to\infty} a(P/R)$ holds in a certain heuristic sense.

The last equation also leads us to believe that if the operators $a(P/R)$ are uniformly bounded on $L^p(S^d)$, then T_a is bounded on $L^p(\mathbb{R}^d)$. This is true for all $1 \leq p \leq \infty$, a classical transference result of Mitjagin [17]. On the other hand, the transference principle we prove is more unusual: the boundedness of a limit point need not necessitate uniform bounds on the limiting sequence. This might explain why the principle is more difficult to prove. Indeed, Mitjagin's argument Mitjagin's argument generalizes to show that for any compact manifold M, and any elliptic, self-adjoint pseudodifferential operator P, the uniform boundedness of the multiplier operators $a(P/R)$ on $L^p(M)$ implies the boundedness of the Fourier multiplier T on $L^p(\mathbb{R}^d)$ whose symbol is given by the principal symbol of P. Until our new transference principle, no analogous transference principle has been shown for any P and any exponent $p \neq 2$, excluding trivial cases, nor any characterization of functions a such that the operators $\{a(P/R)\}$ are uniformly bounded on $L^p(M)$ for any $p \neq 2$.

Bounding Band Limited Zonal Multipliers

To discuss the techniques I developed which lead to the aforementioned characterization, let us begin by summarizing the rough methodology by which the arguments in [1, 8, 11] are able to obtain these bounds. We begin by describing the *band limited* part of the argument:

Take a decomposition $T_a = \sum_{j \in \mathbb{Z}} T_j$, where T_j is the multiplier operator with symbol $a_j(\cdot/2^j)$. Our goal is the band limited bound $\|T_j f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$, uniformly in j We then write $T_j f = k * f$, where k is the Fourier transform of $a_j(\cdot/2^j)$. We then write $k = \sum k_\tau$ and $f = \sum f_\theta$, where the functions $\{k_\tau\}$ are supported on disjoint thickness 2^{-j} annuli, and the functions $\{f_\theta\}$ are supported on disjoint side length 2^{-k} cubes, then $T_j f = \sum_{\tau,\theta} k_\tau * f_\theta$. Using the fact that the Fourier transform is unitary, that the function f is band limited, and some Bessel function estimates, one can argue that the inner product $\langle k_\tau * f_\theta, k_{\tau'} * f_{\theta'} \rangle$ is negligible unless the annulus of radius τ centered at θ is near tangent to the annulus of radius τ' centered at θ'.

Using the inner product estimate, together with an argument for counting incidences, one can show that the L^2 norm of a sum $\sum_{(\tau,\theta)\in\mathcal{E}} k_\tau * f_\theta$ is well behaved if $\mathcal{E}$ is suitably 'sparse', and interpolation with a trivial L^1 estimate yields an L^p estimate on the sum. Conversely, if the set $\mathcal{E}$ is clustered, then $\sum_{(\tau,\theta)\in\mathcal{E}} k_\tau * f_\theta$ will be concentrated on only a few annuli, and so we can also get good L^p estimates. But then we can estimate $\|Tf\|_{L^p(\mathbb{R}^d)} = \|\sum k_\tau * f_\theta\|_{L^p(\mathbb{R}^d)}$ by either approach, depending on whether a sparse part of the sum dominates, or whether a clustered part of the sum dominates.

My paper [3] proves an analogue of this argument for zonal multiplier operators on S^d for $d \geq 4$ and $|1/p - 1/2| > 1/(d-1)$, in particular, obtaining the aforementioned transference principle in this range.

Giving that the argument above exploits convolution on $\mathbb{R}^d$, one might expect that we might use 'zonal convolution' in our argument, i.e. an analogue of convolution on S^d. It is likely one can use this approach to obtain L^p bounds under assumptions on the integrability of the zonal convolution kernel. However, the lack of a dilation symmetry on S^d means that the zonal convolution kernel for $a(P/R)$ is likely unrelated to the convolution kernel for $a(P)$ as R varies, so based on our previous discussion it is likely difficult to obtain a transference principle using this technique. Instead, we follow an approach due to Hörmander BLAH, successfully used in several other problems on manifolds BLAH, and use the Fourier inversion formula to write

$$Z_j f = \int_{-\infty}^{\infty} 2^j \widehat{a}_j(2^j t) e^{2\pi i t P} f \, dt,$$

where $P = \sqrt{\alpha^2 - \Delta}$ is as above, and $e^{2\pi i t P}$ are the wave propagator operators which, as t varies, give solutions to the wave equation $\partial_t^2 = \Delta - \alpha^2$ with zero velocity initial conditions, or equivalently, solutions to $\partial_t = P$, the 'half-wave equation'. Given a general input f, we perform a decomposition analogous to the method above, writing $f = \sum f_\theta$ and $T_j = \sum T_\tau$, where the functions $\{f_\theta\}$ are supported on disjoint sets of diameter 2^{-k}, and $T_\tau = \int b_\tau(t) e^{2\pi i t P} \, dt$, where $2^j \widehat{a}_j(2^j t) = \sum_\tau b_\tau(t)$ for a family of functions $\{b_\tau\}$ is supported on disjoint side length 2^{-j} intervals. We can thus write $T_j f = \sum T_\tau f_\theta$.

The behavior of the wave equation is closely tied to the behavior of geodesics on S^d. In particular, for high frequency inputs we have a near explicit representation of the wave propagator operators for $|t| < 1/2$, in a coordinate system, by an oscillatory integral

$$(e^{2\pi i t P} f)(x) \approx \int_{\mathbb{R}^d} a(t, x, y, \xi) e^{2\pi i [\phi(x,y,\xi) + t|\xi|_y]} f(y) \, d\xi \, dy,$$

where a is a symbol of order 0, $|\xi|_y = (\sum g^{jk}(y)\xi_j\xi_k)^{1/2}$ is obtained from the Riemannian metric of S^d, and ϕ solves the eikonal equation $|(\nabla_x \phi)(x,y,\xi)|_x = |\xi|_y$ subject to the constraint that $\phi(x,y,\xi) = 0$ for $(x-y)\cdot\xi = 0$.

Oscillatory integral representations of the operators $\{e^{2\pi i t P}\}$ *can* be obtained for $e^{2\pi i t P}$ simply by the fact they form a semigroup, and so for $n-1 \leq t < n$ we can write $e^{2\pi i t P} = (e^{2\pi i (t/n) P})^n$ as the composition of n oscillatory integrals. The theory of phase reduction for Fourier integral operators can help us reduce this composition, but it is difficult to control this quantity quantitatively.

simply by the fact that they are obtained by repeated compositions of the propogators with $|t| < 1/2$, and qualitative understanding of their behavior can be understand from the general theory of *Fourier integral operators*, but obtaining good control on $e^{2\pi i t P}$ for large t becomes difficult.

Using this oscillatory integral representation and the stationary phase formula, we can obtain a substitute for the Bessel estimates used in the original argument, justifying that $\langle T_\tau f_\theta, T_{\tau'} f_{\theta'} \rangle$ is negligible unless the geodesic annulus of radius τ and center θ is near tangent to the geodesic annulus of radius τ' and center θ', provided that τ is bounded away from 1. Here we use the stationary phase formula to obtain good bounds on the oscillatory integrals that emerge. However, one subtlety is showing that, restricted to values in $|\xi| = 1$, the critical points of the function $\phi(x,y,\cdot)$ are appropriately non-degenerate. Using the Hamilton-Jacobi approach to the study of the eikonal equation, one can identify the quantity $\phi(x,y,\xi)$ with the signed distance from the hyperplane $\{x' : (x'-y)\cdot\xi = 0\}$ to the point x with respect to the Riemannian metric. I came up with a geometric argument involving the second variation formula for geodesics, which justifies that the function ϕ has only two stationary points, and each is non-degenerate, with the Hessian at each point having magnitude proportional to $d_g(x,y)^{d-1}$.

After this, the argument for radial Fourier multipliers generalizes quite directly to the case of S^d, since the incidence properties required for annuli on $\mathbb{R}^d$ are roughly analogous to the properties of geodesic annuli on S^d.

Combining Dyadic Pieces With Atomic Decompositions

- Next, we consider a decomposition $f = \sum f_k$, where the Fourier transform of f_k is supported on $|\xi| \sim 2^k$, then we can write $Tf = \sum T_k f_k$. Bounds on T_k have been controlled by the previous argument, and we are now left with the job of 'recombining scales'. To obtain this bound, we consider a decomposition of each of the functions f_k into 'L^∞ atoms'. Morally speaking, we are able to write $f_k = \sum A_{k,\theta}$, where θ runs over a family of dyadic boxes, each with side length exceeding 2^{-k}, which morally we should think of as disjoint, and where $A_{k,\theta}$ is a 'molecule', which decomposes as $A_{k,\theta} = \sum a_{k,\theta,j}$, where $a_{k,\theta,j}$ is an 'L^∞ atom' on θ, in the sense that $\|a_{k,j,\theta}\|_{L^\infty(\mathbb{R}^d)} \lesssim |\theta|^{-1/p}\|a_{k,j,\theta}\|_{L^p(\mathbb{R}^d)}$ and satisfy a square function estimate, which reduces proving the bound $\|Tf\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$ to proving a bound of the form

$$\left\|\sum u_{k,j,\theta}\right\| \lesssim \left(\sum_j \left\|\left(\sum_{k,\theta} |a_{k,j,\theta}|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^d)}^p\right)^{1/p},$$

  where $u_{k,j,\theta} = T_k a_{k,j,\theta}$. Unlike the previous argument, we thus have to deal with the interaction between different frequency scales, but here we do have an additional square root cancellation to help obtain the bound.

The argument I obtained also follows this pattern, but we must introduce several new techniques when adapting the method the method to the study of spherical harmonics.

The uniformity in k actually follows immediately if we can prove the bound for $k=0$ because of the dilation symmetry on $\mathbb{R}^d$, and the fact that we have uniform control over the functions $\{a_k\}$. In the analysis of the Fourier multiplier T_0, the support of the symbol implies

The bound is then obtained by a geometric argument involving incidences of annuli. Once this is obtained, a square function bound implies control over $\sum T_k$.

First off, their assumptions are about uniform control over dyadic pieces of the symbol a. More precisely, if a is dyadically decomposed as a sum $a(\rho) = \sum_{k \in \mathbb{Z}} a_k(\rho/2^k)$, where a_k has support on $[1,2]$, then the necessary and sufficient condition for boundedness is that the quantities $C_p(a_k)$ are uniformly bounded in k, where

First, a general symbol a is dyadically decomposed as a sum $a(\rho) = \sum_{k \in \mathbb{Z}} b_k(\rho/2^k)$, where each of the functions b_k is supported on the interval $[1,2]$. If we set $a_k(\rho) = a_k(2^k \rho)$, then $T_a = \sum T_{a_k}$.n

The proofs begin by establishing bounds of the form $\|T_{a_k} f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$, uniformly in k. By applying a dilation symmetry, it suffices without loss of generality to look at a_0.

Now how

Suppose $a : [0,\infty) \to \mathbb{C}$ is compactly supported on the interval $[1/2, 2]$. Then the radial function $k(x) = \int a(|\xi|) e^{2\pi i \xi \cdot x}\, d\xi$ is the a convolution kernel for the radial multiplier operator T_a, i.e. $T_a f = k * f$ for all inputs f. The bounds on radial multipliers obtained in BLAH are then of the form $\|T_a f\|_{L^p(\mathbb{R}^d)} \lesssim \|k\|_{L^p(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}$. Dilation symmetry then immediately implies BLAH DYADIC RESULT, and then some methods of atomic decompositions and Littlewood Paley theory can be combined to obtain a tight result. On a sphere, we *can* define the zonal convolution kernel k corresponding to a zonal multiplier Z_a, such that

$$(Z_a f)(x) = \int_{S^d} k(y \cdot x) f(y)\, dy.$$

If a is supported on $[1/2, 2]$, then weighted L^p bounds on k can be used to imply bounds on Z_a, but these bounds will not scale, and it is difficult to determine how the bounds scale under dilations since there is no relation between the zonal convolution kernel k corresponding to a, and the convolution kernel corresponding to the dilations of a.

A fix is obtained by writing $Z_a f$ using the *cosine transform* of a, i.e. in terms of

$$\widehat{a}(t) = \int_0^\infty a(\rho) e^{2\pi i \rho t}\, d\rho.$$

The cosine transform *does* scale under dilations, and functional calculus and the Fourier inversion formula allows us to write Z_a in terms of $\widehat{a}$, i.e. by setting

$$Z_a f = a(P) f = \int_{-\infty}^\infty \widehat{a}(t) e^{2\pi i t P}\, dt, \tag{1}$$

where $P = \sqrt{\alpha^2 - \Delta}$, and $e^{2\pi i t P} f = e^{2\pi i t k} f$ for a spherical harmonic f of degree k. The operators $u(t) = e^{2\pi i t P} f$ give solutions to the half-wave equation $\partial_t u = Pu$. We can understand the geometric behaviour of the half-wave equation by using the theory of *Fourier integral operators.*

. Indeed, if f is a spherical harmonic of degree k, then $Z_a f = a(k) f$ and by the Fourier inversion formula $\int \widehat{a}(t) e^{2\pi i t P} f\, dt = \int \widehat{a}(t) e^{2\pi i t k} f\, dt = a(k) f(t)$. For any function f, the functions $u(t) = e^{2\pi i t P} f$ give a solution to the wave equation $\partial_t^2 u(t) = Pu(t)$ with $u(0) = f$ and $\partial_t u(0) = 0$.

The bounds on radial multipliers obtained in BLAH depend on bounds on the convolution kernel corresponding to the multiplier.

The main goal of my research project on multipliers is to understand

deconstructive interference between a family of planar waves, or spherical harmonics of different degrees. Necessary and sufficient conditions for a Fourier multiplier operator to be bounded on $L^1(\mathbb{R}^d)$ or $L^\infty(\mathbb{R}^d)$ were quickly realized.

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