Structures Preventing / Enabling Gaussian Concentration

Part 2!

Theorem: For all f of degree d, $\forall \varepsilon, c, M$, there is $\tilde{f} : \mathbb{R}^N \to \mathbb{R}$ of degree d s.t.

$$\|f - \tilde{f}\|_{L^2_\gamma} \lesssim_{c,M} \varepsilon^M \|f\|_{L^2_\gamma}$$

and $\tilde{f}$ has a $(\varepsilon, \varepsilon^{-c})$ diffuse decomposition.

- $f : \mathbb{R}^N \to \mathbb{R}$
- $f = h(q_1, \dots, q_m)$
  $h : \mathbb{R}^m \to \mathbb{R}$
  $q_i : \mathbb{R}^N \to \mathbb{R}$
  $Q = (q_1, \dots, q_m)$
- (ε, C) diffuse if
  $\mathbb{P}(|Q(x) - a| \le \varepsilon) \le C\varepsilon^m$

Theorem: If f is multilinear and has a $(\tau, C, m, \varepsilon)$ decomposition then

$$|\mathbb{E}[\mathrm{sgn}(f(x))] - \mathbb{E}[\mathrm{sgn}(f/A)]| \lesssim \tau^{1/s} C + \varepsilon^{1/d}$$

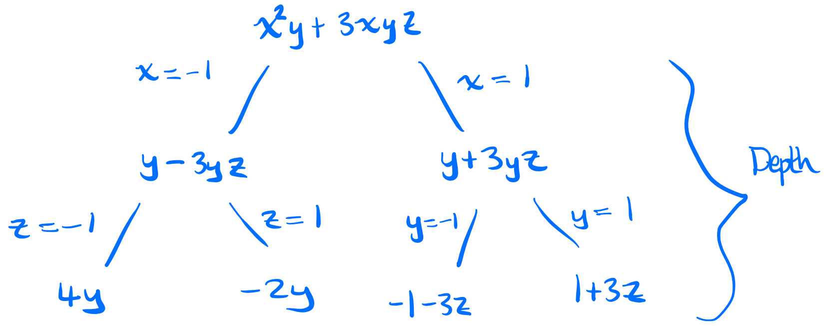
- f has a $(\tau, C, m, \varepsilon)$ decomposition if there is $\hat{f}$ s.t.

$$\|f - \tilde{f}\|_{L^2_\gamma} \le \varepsilon^2 \mathbb{V}(f(x))$$

and $\tilde{f}$ has a $(\tau^{1/s}, C)$ diffuse decomposition $(h, q_1, \dots, q_m)$ with

$$\mathrm{Inf}_i(q_j) \lesssim \tau$$

Decision Trees



Theorem If f is a degree-d multilinear polynomial, then there is a decision tree of depth $O(\tau^{-1}\log(1/\tau)^{O(d)})$ s.t. all but a fraction τ of the leaf functions f_α either have $\mathbb{V}(f_\alpha) < \tau^M \|f_\alpha\|_{2,y}^2$ or f_α has a $(\tau, \tau^{-c}, O(1), O(\tau^M))$.

So $|\mathbb{E}[\mathrm{sgn}\, f_\alpha(x)] - \mathbb{E}[\mathrm{sgn}\, f_\alpha(A)]|$

$\lesssim \tau^{1/5-c} + \tau^{M/d}$

Invariance

$$
\begin{aligned}
\mathbb{E}[\mathrm{sgn}(f(A))] &\approx \mathbb{E}[\mathrm{sgn}(\hat{f}(A))] \\
&= \mathbb{E}[\mathrm{sgn}(h(q_1(A), \dots, q_m(A)))] \\
&\approx \mathbb{E}(p(q_1(A), \dots, q_m(A))) \\
&\approx \mathbb{E}(p(q_1(x), \dots, q_m(x))) \\
&\approx \mathbb{E}(\mathrm{sgn}(h(q_1(x), \dots, q_m(x)))) \\
&= \mathbb{E}[\mathrm{sgn}(\hat{f}(x))]
\end{aligned}
$$

$\approx \mathbb{E}[\mathrm{sgn}(f(x))]$

How do we define $p : \mathbb{R}^m \to \mathbb{R}$?

Theorem: If $(h, q_1, \dots, q_m)$ is (ε, C) diffuse then there is $p : \mathbb{R}^m \to \mathbb{R}$ s.t.

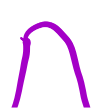
- $p(q_1, \dots, q_m) \geq \mathrm{sgn}(h(q_1, \dots, q_m))$
- $\mathbb{E}[p(q_1(x), \dots, q_m(x))] - \mathbb{E}[\mathrm{sgn}(h(q_1, \dots, q_m))]$
  $$\leq C\varepsilon \log(1/\varepsilon)^{dm/2+1}$$
- $\|D^k p\|_{L^\infty} \lesssim \varepsilon^{-k}$

Construction Let

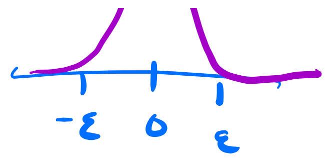
$$g(x) = \begin{cases} 1 & : \exists y : |x - y| < \varepsilon \text{ and } h(y) \geq 0 \\ -1 & : \text{otherwise} \end{cases}$$

Then $g \geq \mathrm{sgn}(h(q_1, \dots, q_m))$ even after mollifying in an ε neighborhood

Let $p = g * \eta_\varepsilon$



η



To show

$$\mathbb{E}\left[p(q_1(x),\dots,q_m(x))\right] - \mathbb{E}\left[\operatorname{sgn}\left(h(q_1(x),\dots,q_m(x))\right)\right] \lesssim C\varepsilon \log(1/\varepsilon)^{dm/2+1}$$

$$p(q_1,\dots,q_m) = \operatorname{sgn}(h(q_1,\dots,q_m))$$

except if $\underbrace{d(Q(x), h^{-1}(0)) \le 2\varepsilon}_{\text{Bound this by covering argument + diffuseness}}$

Algorithm for Thm 1

Theorem: For all f of degree d, $\forall \varepsilon, c, M$, there is $\tilde{f}: \mathbb{R}^n \to \mathbb{R}$ of degree d s.t.

$$\|f - \tilde{f}\|_{L^2_y} \lesssim_{c,M} \varepsilon^M \|f\|_{L^2_y}$$

and $\tilde{f}$ has a $(\varepsilon, \varepsilon^{-c})$ diffuse decomposition.

Associate with each $(h, q_1, \dots, q_m)$ a d-tuple $K = (K_d, \dots, K_1) \in \mathbb{N}^d$

$$K_i = \sum_{\deg(q_j) = i} 3^{m-j} \times \text{'complexity of } q_j\text{'}$$

Define ordering $K \succ K'$ if $K_i = K'_i$ $i > i_0$ and $K_{i_0} > K'_{i_0}$.

Then $\mathbb{N}^d$ has no infinite decreasing subsequence

Goal: If $(h, q_1, \dots, q_m)$ is not diffuse, can write $h(q_1, \dots, q_m) = h'(q'_1, \dots, q'_{m'})$

s.t. if $K, K' \in \mathbb{N}^d$, then $K \succ K'$

Eventually we end up with something diffuse, or where q_i are all linear + simple (and thus diffuse).

Goal: Reduce Degrees of q_i

Technique: Strong Anticoncentration

$$|f(x) - a| \gtrsim |\nabla f(x)|$$

with high probability

Simplest Case (Step 1)

$h(x) = x \qquad g(x) = f(x)$

Theorem: $f : \mathbb{R}^n \to \mathbb{R}$ has degree d,

$$\mathbb{P}\left(|f(x) - a| < \varepsilon |\nabla f(x)| \right) \lesssim \varepsilon \, 2^{O(d)} \log 1/\varepsilon$$

Thus either:

1) $|\nabla f(x)|$ is small with non-negligible probability

2) $f(x)$ is already diffuse.

If 2) we're done.

If 1) following result applies

Theorem: If $f : \mathbb{R}^n \to \mathbb{R}$ has degree d, $\|f\|_{L^2_y} = 1$, and $\mathbb{P}(|\nabla f(x)| < \varepsilon) > \varepsilon^M$

Then there are a_i, b_i with $0 < \deg a_i, \deg b_i < d$ s.t.

$\lesssim \varepsilon^{-2^{2^{2^{O(m)}}}}$ } fixed length

$$\Big\| \Big(f(x) - \sum_{i=1}^{\ell} a_i(x)\, b_i(x) \Big)^{[d]} \Big\|_{L^2_x} \lesssim \varepsilon^{1-c}$$

So replace f with $\underbrace{h(q_1, \dots, q_{m-1})}_{\deg < d} + \underbrace{q_m}_{\substack{\text{degree } d \\ \text{but} \\ O(\varepsilon^{1-c})}}$

$\{q_1, \dots, q_{m-1}\} = \{a_i\} \cup \{b_i\}$

General Step

Given $(h, q_1, \dots, q_m)$

Assume $\deg q_1 \geq \dots \geq \deg q_m$

Strong anticoncentration

$$\mathbb{P}\Big(\prod_{i=1}^{m} |q_i(x) - a_i| < \varepsilon \,\Big| \bigwedge_{i=1}^{m} \nabla q_i(x) \Big| \Big)$$

$$\lesssim \varepsilon \, 2^{O(\sum \deg q_i)} O(\sqrt{n})^{m+1} \log(1/\varepsilon)^m$$

So either diffuse, or

$$\Big| \bigwedge \nabla q_i(x) \Big| \lesssim \varepsilon^{c/2}$$

with non-negligible probability.

If true, can choose i s.t.

$$\Big| \nabla q_i - \sum_{j>i} a_j \nabla q_j \Big| \lesssim \varepsilon^{c/3^i}$$

Now replace q_i with $q_i - \sum a_j q_j$

Then $|\nabla q_i| \lesssim \varepsilon^{c/3^i}$ so can

decompose $\| q_i - \sum a_i b_i \|_{L^2_y} \lesssim \varepsilon^{1-c}$