Averaging over Curves

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Consider a smooth family of curves $\gamma : \mathbb{R}^2 \to \mathbb{R}$, and consider the associated averaging operator

$$Af(v,x) = \int f(x + \gamma_v(t))\phi(v,t)\, dt,$$

where $\gamma_v''(t) = 0$, and ϕ is smooth with compact support. We can write this operator as

$$Af(v,x) = (f * \mu_v)(x),$$

where μ_v is the measure such that

$$\int g(x) d\mu_v(x) = \int g(\gamma_v(t))\phi(t)\, dt.$$

We can then write

$$\widehat{\mu}_v(\xi) = \int e^{-2\pi i \xi \cdot x} d\mu_v(x) = \int e^{-2\pi i \xi \cdot \gamma_v(t)} \phi(t)\, dt.$$

This is an oscillatory integral, which is stationary at points t where $\xi \cdot \gamma_v'(t) = 0$. Under the assumption that γ_v'' is non-vanishing, these stationary points are non-degenerate, and so provided we choose ϕ to have small support, for each $\xi \in \mathbb{R}^d$, there is at most one value of t such that $\xi \cdot \gamma_v'(t) = 0$. Let us write this value by $t_0(\xi)$. We can then find a smooth function $\psi_v : \mathbb{R}^d \to \mathbb{R}$ such that on the domain of t_0,

$$\psi_v(\xi) - \xi \cdot \gamma_v(t_0(\xi)).$$

Then the theory of stationary phase guarantees that

$$\widehat{\mu}_v(\xi) = e^{2\pi i \psi_v(\xi)} b(\xi),$$

where b is a symbol of order $-1/2$, with microsupport on the domain of t_0. Using the multiplication formula for the Fourier transform, we can thus write

$$Af(v,x) = \int \widehat{\mu}_v(\xi) \widehat{f}(\xi) e^{2\pi i \xi \cdot x}\, d\xi = \int b(\xi) e^{2\pi i [\psi_v(\xi) + \xi \cdot (x-y)]} f(y)\, d\xi\, dy.$$

This is a Fourier integral operator with phase

$$\phi(v,x,y,\xi) = \psi_v(\xi) + \xi \cdot (x - y).$$

Let's compute it's canonical relation.

We have $(\nabla_\xi \phi)(v,x,y,\xi) = \nabla_\xi \psi_v(\xi) + (x-y)$. Since $\psi_v(\xi) = -\xi \cdot \gamma_v(t_0(\xi))$, the chain rule implies that

$$(\nabla_\xi \psi_v)(\xi) = -\gamma_v(t_0) - (\xi \cdot \gamma_v'(t_0))(\nabla_\xi t_0) = -\gamma_v(t_0).$$

Thus the stationary points occur for values of ξ such that $x - y = \gamma_v(t_0(\xi))$. We then have

$$\nabla_x \phi(v,x,y,\xi) = \xi \quad \text{and} \quad \nabla_y \phi(v,x,y,\xi) = -\xi$$

and

$$\nabla_v \phi(v,x,y,\xi) = \partial_v \psi_v(\xi).$$

Thus the canonical relation of the Fourier integral operator is

$$\mathcal{C} = \Big\{ (v,x,y,\nu,\xi,\eta) : \nu = \partial_v \gamma_v(t_0(\xi)) \text{ and } x = y + \gamma_v(t_0(\xi)) \text{ and } \xi = \eta \Big\}.$$

The projection of $\mathcal{C}$ onto the (y,η) variables give a submersion, and the projection of $\mathcal{C}$ onto (v,x) also form a submersion. For each fixed $z = (v,x)$, let

$$\Gamma_z = \Big\{ (\nu,\xi) : \nu = \partial_v \gamma_v(t_0(\xi)) \Big\}$$

be the projection of $\mathcal{C}$ onto the (ν,ξ) variables at (v,x). The cinematic curvature condition amounts to saying that Γ_z is a hypersurface of dimension 2 in $\mathbb{R}^3$, with one non-vanishing principal curvature. If we write $\Phi(v,\xi) = \partial_v \gamma_v(t_0(\xi))$, then this amounts to saying that the matrix $D_\xi \Phi$ has rank one. By the chain rule, this will hold if $\nabla_\xi t_0$ is non-zero at ξ, and $\partial^2_{v,t} \gamma_v$ is non-zero at $t_0(\xi)$. But $\nabla_\xi t_0 \neq 0$ using the fact that $\gamma_v'' \neq 0$, so the cinematic curvature condition holds under the assumption that $\gamma_v'' \neq 0$, and $\partial^2_{v,t} \gamma \neq 0$.