

# Averaging over Curves

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Consider a smooth family of curves  $\gamma : \mathbb{R}^2 \rightarrow \mathbb{R}$ , and consider the associated averaging operator

$$Af(v, x) = \int f(x + \gamma_v(t)) \phi(v, t) dt,$$

where  $\gamma_v''(t) = 0$ , and  $\phi$  is smooth with compact support. We can write this operator as

$$Af(v, x) = (f * \mu_v)(x),$$

where  $\mu_v$  is the Borel measure such that for any bounded, measurable function  $g$ ,

$$\int g(x) d\mu_v(x) = \int g(\gamma_v(t)) \phi(t) dt.$$

We can then write

$$\hat{\mu}_v(\xi) = \int e^{-2\pi i \xi \cdot x} d\mu_v(x) = \int e^{-2\pi i \xi \cdot \gamma_v(t)} \phi(t) dt.$$

This is an oscillatory integral, which is stationary at points  $t$  where  $\xi \cdot \gamma_v'(t) = 0$ . Under the assumption that  $\gamma_v''$  is non-vanishing, these stationary points are non-degenerate, and so provided we choose  $\phi$  to have small support, for each  $\xi \in \mathbb{R}^d$ , there is at most one value of  $t$  such that  $\xi \cdot \gamma_v'(t) = 0$ . Let us write this value by  $t_0(\xi)$ . We can then find a smooth function  $\psi_v : \mathbb{R}^d \rightarrow \mathbb{R}$  such that on the domain of  $t_0$ ,

$$\psi_v(\xi) = \xi \cdot \gamma_v(t_0(\xi)).$$

Then the theory of stationary phase guarantees that

$$\hat{\mu}_v(\xi) = e^{2\pi i \psi_v(\xi)} b(\xi),$$

where  $b$  is a symbol of order  $-1/2$ , with microsupport on the domain of  $t_0$ . Using the multiplication formula for the Fourier transform, we can thus write

$$Af(v, x) = \int \hat{\mu}_v(\xi) \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi = \int b(\xi) e^{2\pi i [\psi_v(\xi) + \xi \cdot (x - y)]} f(y) d\xi dy.$$

This is a Fourier integral operator with phase

$$\phi(v, x, y, \xi) = \psi_v(\xi) + \xi \cdot (x - y).$$

Let's compute it's canonical relation.

We have  $(\nabla_\xi \phi)(v, x, y, \xi) = \nabla_\xi \psi_v(\xi) + (x - y)$ . Since  $\psi_v(\xi) = -\xi \cdot \gamma_v(t_0(\xi))$ , the chain rule implies that

$$(\nabla_\xi \psi_v)(\xi) = -\gamma_v(t_0) - (\xi \cdot \gamma'_v(t_0))(\nabla_\xi t_0) = -\gamma_v(t_0).$$

Thus the stationary points occur for values of  $\xi$  such that  $x - y = \gamma_v(t_0(\xi))$ . We then have

$$\nabla_x \phi(v, x, y, \xi) = \xi \quad \text{and} \quad \nabla_y \phi(v, x, y, \xi) = -\xi$$

and

$$\nabla_v \phi(v, x, y, \xi) = \partial_v \psi_v(\xi).$$

Thus the canonical relation of the Fourier integral operator is

$$\mathcal{C} = \left\{ (v, x, y, \nu, \xi, \eta) : \nu = \partial_v \gamma_v(t_0(\xi)) \text{ and } x = y + \gamma_v(t_0(\xi)) \text{ and } \xi = \eta \right\}.$$

The projection of  $\mathcal{C}$  onto the  $(y, \eta)$  variables give a submersion, and the projection of  $\mathcal{C}$  onto  $(v, x)$  also form a submersion. For each fixed  $z = (v, x)$ , let

$$\Gamma_z = \left\{ (\nu, \xi) : \nu = \partial_v \gamma_v(t_0(\xi)) \right\}$$

be the projection of  $\mathcal{C}$  onto the  $(\nu, \xi)$  variables at  $(v, x)$ . The cinematic curvature condition amounts to saying that  $\Gamma_z$  is a hypersurface of dimension 2 in  $\mathbb{R}^3$ , with one non-vanishing principal curvature. If we write  $\Phi(v, \xi) = \partial_v \gamma_v(t_0(\xi))$ , then this amounts to saying that the matrix  $D_\xi \Phi$  has rank one. By the chain rule, this will hold if  $\nabla_\xi t_0$  is non-zero at  $\xi$ , and  $\partial_{v,t}^2 \gamma_v$  is non-zero at  $t_0(\xi)$ . But  $\nabla_\xi t_0 \neq 0$  using the fact that  $\gamma''_v \neq 0$ , so the cinematic curvature condition holds under the assumption that  $\gamma''_v \neq 0$ , and  $\partial_{v,t}^2 \gamma \neq 0$ . Thus the cinematic curvature condition is satisfied under the assumption that each of the curve you are averaging over has non-vanishing curvature, and if  $\partial_{v,t}^2 \gamma \neq 0$ , i.e. the tangent vectors of  $\gamma$  change as we vary  $v$ .