Characterizations of Bounded Spectral Multipliers on Manifolds with Periodic Geodesic Flow

Jacob Denson*

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Abstract

It is a well known result of Mitjagin that on a compact Riemannian manifold M, and for a function $m : (0, \infty) \to \mathbb{C}$, if the spectral multiplier operators $m(\sqrt{-\Delta}/R)$ are uniformly bounded on $L^p(M)$ for $R > 0$, then the radial function $m(|\cdot|) : \mathbb{R}^d \to \mathbb{C}$ induces a bounded Fourier multiplier operator on $L^p(\mathbb{R}^d)$. In this paper, we prove the converse for manifolds in which the geodesic flow is periodic, of dimension $d \geqslant 4$ and for $(d-1)^{-1} \leqslant |1/p - 1/2| \leqslant 1/2$. In the process, we find an effective characterization of the functions m for which the operators $m(\sqrt{-\Delta}/R)$ are uniformly bounded in $L^p(M)$ for this range of p, which can be viewed as a variable-coefficient analogue of the results of Heo, Nasarov and Seeger.

1 Introduction

Let M be a compact Riemannian manifold of dimension d, let Δ be it's Laplace-Beltrami operator, and let $P = \sqrt{-\Delta}$. For any bounded function $m : (0, \infty) \to \mathbb{C}$, we can define a spectral multiplier operator $m(P)$. In this paper, we study the relation between the L^p boundedness of the dilated multipliers $m_\rho(P)$ for $\rho > 0$, where $m_\rho(\lambda) = m(\lambda/\rho)$, and the L^p boundedness of the radial Fourier multiplier operator T_m on $\mathbb{R}^d$ defined by setting

$$T_m f(x) = \int_{\mathbb{R}^d} m(|\xi|) \widehat{f}(\xi) e^{2\pi i \xi \cdot x} \, d\xi \qquad \text{for } f : \mathbb{R}^d \to \mathbb{C}.$$

In particular, we show that for any $p \in [1, \infty]$, if d is sufficiently large, then the L^p boundedness of T_m is equivalent to the uniform L^p boundedness of the operators $m_\rho(P)$, provided that the geodesic flow on M is periodic.

Theorem 1. *Suppose M is a compact Riemannian manifold, and the geodesic flow on M is periodic. If $d \geqslant 4$, and $1/(d-1) \leqslant |1/p - 1/2| \leqslant 1/2$, then*

$$\sup_\rho \|m_\rho(P)\| \sim \|T_m\|,$$

where $\|T_m\|$ is the operator norm on $L^p(\mathbb{R}^d)$, and $\|m_\rho(P)\|$ the operator norm on $L^p(M)$. The implicit constant depends only on M and p

*University of Madison Wisconsin, Madison, WI, jcdenson@wisc.edu

One side of this inequality is already known. A classical transplantation theorem of Mitjagin [7] states that

$$\|T_m\| \lesssim \sup_\rho \|m_\rho(P)\|. \tag{1.1}$$

Moreover, this inequality is known to hold for the larger range of exponents $1 \leqslant p \leqslant \infty$ and for general compact manifolds M (see [5] for an english translation of Mitjagin's proof). The result is intuitive if we consider the geometry in (1.1); if we let Δ_ρ be the Laplace-Beltrami operator on M associated with the dilated metric $g_\rho = \rho^{-2} g$, and define $P_\rho = \sqrt{-\Delta_\rho}$, then $m_\rho(P) = m(P_\rho)$. As $\rho \to \infty$, the metric g_ρ gives the manifold M less curvature and more volume, and as $\rho \to \infty$ we might therefore expect M equipped with Δ_ρ to behave more and more like $\mathbb{R}^d$ equipped with the usual Laplace operator $\Delta_{\mathbb{R}^d} = \sum \partial_j^2$. Indeed, Mitjagin's proof essentially shows that in coordinates, $m_\rho(P)$ converges to $m(\Delta_{\mathbb{R}^d})$ as $\rho \to \infty$, in an appropriate sense, and since $m(\Delta_{\mathbb{R}^d})$ is just the operator T_m, (1.1) follows.

The novel result in this paper is a proof of the converse inequality to (1.1) under the assumption that the geodesic flow on M is periodic, i.e. a proof that

$$\sup_\rho \|T_\rho\| \lesssim \|T_m\| \qquad \text{if} \quad \frac{1}{d-1} \leqslant \left| \frac{1}{p} - \frac{1}{2} \right| \leqslant \frac{1}{2}. \tag{1.2}$$

Equation (1.2) is more surprising than equation (1.1), since it is not geometrically intuitive why L^p bounds of a Fourier multiplier on a flat space should imply uniform L^p bounds for a family of spectral mulipliers on a curved space. No variants of (1.2) are currently known for any manifold and any value of $p \neq 2$; the only exception is in the study of multipliers on $\mathbb{T}^d$, where one has more robust tools, like the Poisson summation formula, to relate L^p bounds for multipliers on $\mathbb{T}^d$ and L^p bounds for multipliers on $\mathbb{R}^d$ (see Section 3.6.2 of [2]). Note also that $\mathbb{T}^d$ is a Riemannian manifold with no curvature.

The path to proving (1.2) without being able to relate the geometry of the two spaces is hinted at by the main result of [3], which states that, for $(d-1)^{-1} \leqslant |1/p - 1/2| \leqslant 1/2$, $\|T_m\| \sim C_p(m)$, where

$$C_p(m) = \sup_{h>0} \left(\int_0^\infty \left[\langle t \rangle^{\alpha(p)} |\widehat{m}_h(t)| \right]^p dt \right)^{1/p}.$$

Here $\alpha(p) = (d-1)|1/p - 1/2|$, and $m_h(\lambda) = m(2^h \lambda)\chi(\lambda)$ for any fixed choice of a smooth, compactly supported function χ with support on $[1/2, 2]$ for which $\sum \chi(2^h \lambda) = 1$. Since $\sup_{\rho > 0} C_p(m_\rho) \sim C_p(m)$, and duality implies that the operator norm of $m(P)$ on $L^p(M)$ is equal to the operator norm of $m(P)$ on $L^{p'}(M)$ for $1/p + 1/p' = 1$, Theorem 1 therefore follows from the following estimate.

Proposition 2. *Suppose $1 \leqslant p \leqslant 2(d-1)/(d+1)$. Then for any function $m : (0,\infty) \to \mathbb{C}$,*

$$\|m(P)\| \lesssim C_p(m),$$

where the implicit constant depends only on p and the manifold M.

The techniques that underlie the proof of Proposition 2 are also inspired by the proof methods of [3]. As in that proof, we begin by considering a decomposition $m(\cdot) = \sum_j m_j(\cdot/2^j)$, where $\mathrm{supp}(m_j) \subset [1/2, 2]$. We will see that the behaviour of the multipliers $m_j(P/2^j)$ is fairly benign for $j \lesssim 1$. Proposition 2 then follows if we can show that for each $j \gtrsim 10$,

$$\|m_j(P/2^j)\| \lesssim \|\langle t\rangle^{s_p} \widehat{m}_j(t)\|_{L^p(\mathbb{R})}, \tag{1.3}$$

and also that

$$\Big\|\sum_{j \gtrsim 1} m_j(P/2^j)\Big\| \lesssim \sup_{j \gtrsim 1} \|m_j(P/2^j)\|. \tag{1.4}$$

We prove (1.3) by obtaining some new quasi-orthogonality estimates for averages of solutions to the half-wave equation on M, which reduces our analysis to a geometric 'density-decomposition' argument involving geodesic annuli on M. We then use variants of Littlewood-Paley theory and the theory of atomic decompositions to obtain (1.4).

2 Preliminary Setup

Let us define our problem more precisely, which will also give us a chance to introduce some notation. Spectral theory guarantees the existence of a discrete set $\Lambda_M \subset (0, \infty)$, and an orthogonal decomposition $L^2(M) = \bigoplus_{\lambda \in \Lambda_M} \mathcal{V}_\lambda$, where $\mathcal{V}_\lambda$ is a finite dimensional subspace of $C^\infty(M)$ such that $Pf = \lambda f$ for all $f \in \mathcal{V}_\lambda$. If we let $\mathcal{P}_\lambda$ be the orthogonal projection operator onto $\mathcal{V}_\lambda$, then for any bounded function $m : (0, \infty) \to \mathbb{C}$, we define $m(P)$ by setting

$$m(P) = \sum_{\lambda \in \Lambda_M} m(\lambda)\, \mathcal{P}_\lambda.$$

As discussed in the last section, we consider a dyadic decomposition $m(P) = \sum_j m_j(P/2^j)$, where $m_j(P/2^j) = \sum_\lambda m_j(\lambda/2^j)\mathcal{P}_\lambda$. To exploit the integrability of the functions $\{\widehat{m}_j\}$, we must relate the Fourier transform of a function to it's associated spectral multiplier operator. A standard method in this setting is to apply the Fourier inversion formula; given a function $h : \mathbb{R} \to \mathbb{C}$, we have

$$h(P) = \int_{-\infty}^{\infty} \widehat{h}(t) e^{2\pi i t P}\, dt,$$

where

$$e^{2\pi i t P} = \sum_\lambda e^{2\pi i t \lambda} \mathcal{P}_\lambda$$

is the multiplier operator on M which, as t varies, gives solutions to the half-wave equation $\partial_t = 2\pi i P$ on M. In our situation, we have

$$m_j(P/2^j) = \int_{-\infty}^{\infty} 2^j \widehat{m}_j(2^j t) e^{2\pi i P t}\, dt,$$

and so our study of multipliers reduces to studying averages of the half-wave propogators.

The half-wave equation on M is hyperbolic, and like other hyperbolic partial differential equations, the singularities of solutions to the half-wave propogate along characteristic curves, which is this case are the geodesics of the manifold M. For high frequency initial conditions, we should expect solutions to be concentrated near these geodesics. To obtain quantitative information following this intuition, we 'localize in frequency'. Fix $\beta_0 \in C_c^\infty(\mathbb{R})$ with $\text{supp}(\beta) \subset [1/4, 4]$ and with $\beta_0(\lambda) = 1$ for $\lambda \in [1/2, 2]$. Then $\beta(t) = \beta_0(t)^2$ has the exact same properties. For $R > 0$, we define $Q_R = \beta(P/R)$. Then Q_R has range contained in the finite dimensional subspace V_R of $C^\infty(M)$ spanned by eigenfunctions of P with eigenvalues in $[R/4, 4R]$. Since P is elliptic, it is often a useful heuristic that elements of V_R have similar properties to a function on $\mathbb{R}^d$ with Fourier support on the annulus $\{\xi : R/4 \leqslant |\xi| \leqslant 4R\}$. In particular, the uncertainty principle tells us the kernel of the operator $Q_R \circ e^{2\pi i t P} \circ Q_R$ should be smooth, and locally constant at the scale $1/R$.

The symbol calculus allows us to introduce the operators $\{Q_R\}$ to our multipliers, writing

$$m_j(P/2^j) = Q_{2^j} \circ m_j(P/2^j) \circ Q_{2^j} = \int_{\mathbb{R}} 2^j \widehat{m}_j(2^j t)(Q_{2^j} \circ e^{2\pi i t P} \circ Q_{2^j})\, dt.$$

We only have explicit formulas defining solutions to the half-wave equation for small times; this is why we must exploit the fact that the manifold M has periodic geodesic flow. Normalizing the metric on M appropriately, we may assume that the geodesic flow has period one. It follows that $e^{2\pi i(t+n)P} = e^{2\pi i t P}$ for any $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. We may then write

$$m_j(P/2^j) = \int_{-1/2}^{1/2} b_j(t)(Q_{2^j} \circ e^{2\pi i t P} \circ Q_{2^j})\, dt,$$

where $b_j : [-1/2, 1/2] \to \mathbb{C}$ is the periodization

$$b_j(t) = \sum_{n \in \mathbb{Z}} 2^j \widehat{m}_j(2^j(t+n)).$$

We then split our analysis of these multipliers into two regimes. In the first regime, over times $|t| \leqslant \varepsilon_M$, we perform a further wave packet decomposition at a frequency scale 2^j, whereas we do not perform this further wave packet decomposition over times $|t| > \varepsilon_M$, since we have better estimates on b_j over these times. We summarize this decomposition in the following lemma, whose proof we relegate to the appendix.

Lemma 3. *Let $\mathcal{T}_j = \mathbb{Z}/2^j \cap [-\varepsilon_M, \varepsilon_M]$. Then we can write*

$$b_j = \sum_{t_0 \in \mathcal{T}} b^I_{j,t_0} + b^{II}_j,$$

such that the following properties hold:

- *supp*$(b^I_{j,t_0}) \subset [t_0 - 2/2^j, t_0 + 2/2^j]$ *and supp*$(b^{II}_j) \subset [-1/2, 1/2] - [-\varepsilon_M/2, \varepsilon_M/2]$.
- *For each $k \geqslant 0$,*

$$\left(\sum_{t_0 \in \mathcal{T}_j} \left[\|\partial_t^k b^I_{j,t_0}\|_{L^1(\mathbb{R})} \langle 2^j t_0 \rangle^{\alpha(p)} \right]^p \right)^{1/p} \lesssim_k 2^{j(k-1/p')} C_p(m)$$

and

$$\|b^{II}_j\|_{L^p(\mathbb{R})} \lesssim 2^{-j(1/p' + \alpha(p))} C_p(m).$$

We can thus write

$$m_j(P/2^j) = M_j^I + M_j^{II} = \left(\sum_{t_0 \in \mathcal{T}_j} M_{j,t_0}^I \right) + M_j^{II}$$

where

$$M_{j,t_0}^I = \int b_{j,t_0}^I(t)(Q_{2^j} \circ e^{2\pi itP} \circ Q_{2^j})\, dt \quad \text{and} \quad M_j^{II} = \int b_j^{II}(t)(Q_{2^j} \circ e^{2\pi itP} \circ Q_{2^j})\, dt.$$

We will obtain control over the operator M_j^I by understanding the interactions of functions of the form $f_{x_0,t_0} = M_{j,t_0}^I u_{x_0}$, where $u_{x_0} : M \to \mathbb{C}$ is a function with $\text{supp}(u_0)$ contained in $B(x_0, 2/2^j)$, the radius $2/2^j$ geodesic ball centered at some point $x_0 \in M$. We begin our analysis of these interactions in the next section.

3 Estimates For High-Frequency Wave Packets

The discussion at the end of the introduction motivated us to consider functions obtained by taking averages of the wave equation over a local set of times, with initial conditions localized to a particular frequency. In this section, we obtain pointwise bounds and orthogonality estimates for such functions, which we summarize in the following proposition.

Proposition 4. *For any compact Riemannian manifold M, there exists a small geometric constant $\varepsilon_M > 0$ such that for $R \geqslant 1/\varepsilon_M$, the following estimates hold:*

- *(Pointwise Estimates) Fix any $|t_0| \leqslant \varepsilon_M$ and $x_0 \in M$. Consider any two functions $c : \mathbb{R} \to \mathbb{C}$ and $u : M \to \mathbb{C}$ with*

  $$\mathit{supp}(c) \subset [t_0 - 2/R, t_0 + 2/R] \quad \mathit{and} \quad \mathit{supp}(u) \subset B(x_0, 2/R).$$

  If we define $f : M \to \mathbb{C}$ by setting

  $$f = \int c(t)(Q_R \circ e^{2\pi itP} \circ Q_R)\{u\}\, dt.$$

  Then for any $K \geqslant 0$, and $x \in M$,

  $$|f(x)| \lesssim_M \|c\|_{L^1(\mathbb{R})} \|u\|_{L^1(M)} \frac{R^d}{(Rd_g(x,x_0))^{\frac{d-1}{2}}} \left\langle R \big| t_0 - d_g(x,x_0) \big| \right\rangle^{-K}.$$

- *(Quasi-Orthogonality Estimates) Fix $t_0, t_1 \in \mathbb{R}$ with $|t_0 - t_1| \leqslant \varepsilon_M$, and $x_0, x_1 \in M$. Consider any two pairs of functions $c_0, c_1 : \mathbb{R} \to \mathbb{C}$ and $u_0, u_1 : M \to \mathbb{C}$ with*

  $$\mathit{supp}(c_j) \subset [t_j - 2/R, t_j + 2/R] \quad \mathit{and} \quad \mathit{supp}(u_j) \subset B(x_j, 2/R).$$

  Define two functions

  $$f_j = \int c_j(t)(Q_R \circ e^{2\pi itP} \circ Q_R)\{u_j\}\, dt.$$

  Then for any $K \geqslant 0$,

  $$|\langle f_0, f_1 \rangle| \lesssim_K \frac{R^d}{(Rd_g(x_0,x_1))^{\frac{d-1}{2}}} \left\langle R \big| (t_0 - t_1) - d_g(x_1,x_0) \big| \right\rangle^{-K}.$$

Suppose c, c_0, c_1, u, u_0, and u_1 are all L^1 normalized. The pointwise estimate tell us that the function f is concentrated on a geodesic annulus of radius t_0 centered at x_0, with thickness $1/R$, and on this annulus it has height at most $R^{\frac{d+1}{2}}|t_0|^{-\frac{d-1}{2}}$. The quasi-orthogonality estimate tells us that the two functions f_0 and f_1 are only significantly correlated with one another if the annuli on which f_0 and f_1 are externally or internally tangent to one another, and then the inner product $\langle f_0, f_1 \rangle$ has magnitude at most $R^{\frac{d+1}{2}}|t_0 - t_1|^{\frac{d-1}{2}}$. This estimate is then an analogue of Lemma 3.3 of [3], though with different exponents because here we are using the half wave equation to define our functions f_j, whereas in [4] the functions are simply defined by taking a smooth functions adapted to the respective annuli.

The remainder of this section is devoted to a proof of Proposition 4. Since R is fixed, we will write Q_R as Q in the sequel. For both estimates, we want to consider the operators in coordinates, so we can use the *Lax-Hörmander Parametrix* to understand the wave propogators in terms of various oscillatory integrals. Start by covering M by a finite family of open sets $\{V_\alpha\}$, chosen such that for each α, there is a coordinate chart U_α such that the neighborhood $N(V_\alpha, 0.5)$ is contained in U_α. Let $\{\eta_\alpha\}$ be a partition of unity subordinate to $\{V_\alpha\}$. It will be convenient to define $V_\alpha^* = N(V_\alpha, 0.1)$. The next Lemma allows us to approximate the operator Q, and the propogators $e^{2\pi i t P}$ with operators which have more explicit representations in the coordinate system $\{U_\alpha\}$, by an error term which is negligible to the results of Proposition 4.

Lemma 5. *For each α, and $|t| \leqslant 1/100$, there exists Schwartz operators Q_α and $W_\alpha(t)$, each with kernel supported on $U_\alpha \times V_\alpha^*$, such that the following properties hold:*

- *For $f \in L^1(M)$ with supp$(f) \subset V_\alpha^*$,*

$$\textit{supp}(Q_\alpha f) \subset N(\textit{supp}(f), 0.1) \quad \textit{and} \quad \textit{supp}(W_\alpha(t) f) \subset N(\textit{supp}(f), 0.1).$$

  Moreover, for all $N \geqslant 0$,

$$\|(Q - Q_\alpha) f\|_{L^\infty(M)} \lesssim_N R^{-N} \|f\|_{L^1(M)}$$

  and

$$\left\| \Big(Q_\alpha \circ \Big(e^{2\pi i t P} - W_\alpha(t) \Big) \circ Q_\alpha \Big) \{f\} \right\|_{L^\infty(M)} \lesssim_N R^{-N} \|f\|_{L^1(M)}.$$

- *In the coordinate system of U_α, Q_α is a pseudodifferential operator of order zero given by a symbol $\sigma(x, \xi)$, where*

$$\textit{supp}(\sigma) \subset \{\xi \in \mathbb{R}^d : R/2 \leqslant |\xi| \leqslant 2R\},$$

  and σ satisfies derivative estimates of the form

$$|\partial_x^\beta \partial_\xi^\kappa \sigma(x, \xi)| \lesssim_{\beta,\kappa} R^{-|\kappa|}.$$

- *In the coordinate system U_α, the operator $W_\alpha(t)$ has a kernel $W_\alpha(t, x, y)$ with an oscillatory integral representation*

$$W_\alpha(t, x, y) = \int s(t, x, y, \xi) e^{2\pi i [\phi(x, y, \xi) + t|\xi|_y]} \, d\xi,$$

where s has compact support in it's x and y coordinates, with

$$supp_\xi(s) \subset \{\xi \in \mathbb{R}^d : R/2 \leqslant |\xi| \leqslant 2R\},$$

where s satisfies derivative estimates of the form

$$|\partial^\beta_{t,x,y} \partial^\kappa_\xi s| \lesssim_{\beta,\kappa} R^{-|\kappa|},$$

and where $|\cdot|_y$ denotes the norm on $\mathbb{R}^n_\xi$ induced by the Riemannian metric on S^d on the contangent space $T^*_y S^d$.

We relegate the proof of Lemma 4 to the appendix, the proof being a fairly technical calculation involving the calculus of Fourier integral operators. Let us now proceed with the proof of the pointwise bounds in Proposition 4 using this lemma. Given $u : M \to \mathbb{C}$, write $u = \sum u_\alpha$, where $u_\alpha = \eta_\alpha u$. Lemma 13 implies that if we define

$$f_\alpha = \int c(t)(Q_\alpha \circ W_\alpha(t) \circ Q_\alpha)\{u_\alpha\}\; dt,$$

then

$$\left\| f - \sum\nolimits_\alpha f_\alpha \right\|_{L^\infty(M)} \lesssim_N R^{-N} \|f\|_{L^1(M)}.$$

This error is negligible to the bounds we want to obtain in Proposition 4. We will bound each of the functions $\{f_\alpha\}$ separately from one another, applying the triangle inequality to get the main pointwise bounds.

To obtain the pointwise bounds, it suffices to expand out the implicit integrals in the definition of f_α, writing, in the coordinate system U_α,

$$\begin{aligned} f_\alpha(x) = \int c(t)\sigma(x,\eta) & e^{2\pi i \eta\cdot(x-y)} \\ & s(t,y,z,\xi) e^{2\pi i [\phi(y,z,\xi) + tp(z,\xi)]} \\ & \quad \sigma(z,\theta) e^{2\pi i \theta\cdot(z-w)} (\eta_\alpha u)(w) \\ & \qquad dt\; dy\; dz\; dw\; d\theta\; d\xi\; d\eta. \end{aligned}$$

The integral looks highly complicated, but can be simplified considerably by noticing that most variables are highly localized. To begin with, we note that since s is smooth and compactly supported in all it's variables, so s should roughly behave like a linear combination of tensor products of it's variables. Using Fourier series, we can write

$$s(t,y,z,\xi) = \sum_{n \in \mathbb{Z}^d} s_{n,1}(y) s_{n,2}(t,z,\xi),$$

where $s_{n,1}(y) = e^{2\pi i n\cdot y}$, and where

$$|\partial^\alpha_{t,z} \partial^\kappa_\xi \{s_{n,2}\}| \lesssim_{\alpha,k,N} |n|^{-N} R^{-|\kappa|}.$$

If we write $a_n(x,\xi) = a_{n,1}(x, R\xi) a_{n,2}(R\xi)$, where

$$a_{n,1}(x,\xi) = \int \sigma(x,\eta) s_{n,1}(y) e^{2\pi i [\eta\cdot(x-y) - \phi(x,x_0,\xi)]}\; dy\; d\eta$$

and

$$a_{n,2}(\xi) = \int c(t) s_{n,2}(t,z,\xi)\sigma(z,\theta)(\eta_\alpha u)(w) e^{2\pi i[\phi(y,z,\xi)+tp(z,\xi)+\theta\cdot(z-w)-t_0|\xi|_{x_0}]}\, dt\, dz\, dw\, d\theta$$

then

$$f_\alpha(x) = R^d \sum_{n\in\mathbb{Z}^d} \int a_n(x,\xi) e^{2\pi i R[\phi(x,x_0,\xi)+t_0|\xi|_{x_0}]}\, d\xi.$$

We have $\operatorname{supp}(a_n) \subset \{|\xi| \sim 1\}$ and

$$|(\nabla_\xi^\kappa a_n)(x,\xi)| \lesssim_{\kappa,N} |n|^{-N} \|c\|_{L^1(\mathbb{R})}.$$

To obtain an efficient upper bound on this oscillatory integral, it will be convenient to change coordinate systems in a way better respecting the Riemannian metric at x_0, i.e. finding a smooth family of diffeomorphisms $\{F_{x_0} : S^{d-1} \to S^{d-1}\}$ such that $|F_{x_0}|_{x_0} = 1$. We can choose this function such that $F_{x_0}(-x) = -F_{x_0}(x)$. Then if $a_n'(x,\rho,\eta) = a_n(x,\rho F_{x_0}(\eta)) J F_{x_0}(\eta)$, then a change of variables gives that

$$R^d \int a_n(x,\xi) e^{2\pi i R[\phi(x,x_0,\xi)+t_0|\xi|_{x_0}]} = R^d \int_0^\infty \rho^{d-1} \int_{|\eta|=1} a_n'(x,\rho,\eta) e^{2\pi i R\rho[\phi(x,x_0,F_{x_0}(\eta))+t_0]}\, d\eta\, d\rho.$$

For each fixed ρ, we claim that the phase has exactly two stationary points in the η variable, at the values $\pm\eta_0$, where x_1 lies on the geodesic passing through x_0 tangent to the vector $\eta_0^\sharp$ (here we are using the musical isomorphism to map the cotangent vector η_0 to a tangent vector). Moreover, at these values,

$$\phi(x_1, x_0, F_{x_0}(\pm\eta_0)) = \pm d_g(x_1,x_0),$$

and the Hessian at $\pm\eta_0$ is (positive / negative) definite, with each eigenvalue having magnitude exceeding a constant multiple of $d_g(x_1,x_0)$. It follows from the principle of stationary phase, that the integral above can be written as

$$\frac{R^d}{[Rd_g(x_1,x_0)]^{\frac{d-1}{2}}} \sum_\pm \int_0^\infty \rho^{\frac{d-1}{2}} a_{n,\pm}''(x,\rho) e^{2\pi i R\rho[t_0 \pm d_g(x_1,x_0)]}\, d\rho,$$

where $a_{n,\pm}''$ is supported on $|\rho| \sim 1$, and

$$|\partial_\rho^m a_{n,\pm}''| \lesssim_K |n|^{-K} \|c\|_{L^1(\mathbb{R})}.$$

Integrating by parts in the ρ variable if $\pm d_g(x_1,x_0)+t_0$ is large, and then taking in absolute values, we conclude that

$$\left| \int a_n(x,\xi) e^{2\pi i R[\phi(x_1,x_0,\xi)+t_0|\xi|_{x_0}]} \right| \lesssim_{K_1,K_2} |n|^{-K_1} \frac{\|c\|_{L^1(\mathbb{R})}}{(Rd_g(x_1,x_0))^{\frac{d-1}{2}}} \sum_\pm \langle R|t_0 \pm d_g(x_1,x_0)|\rangle^{-K_2}.$$

Taking $K_1 \geqslant d+1$ and $K_2 = K$, and then summing in the n variable, we conclude that

$$\begin{aligned} |f_\alpha(x)| &= \left| R^d \sum_n \int a_n(x,\xi) e^{2\pi i R[\phi(x_1,x_0,\xi)+t'|\xi|_{x_0}]} \right| \\ &\lesssim_K \|c\|_{L^1(\mathbb{R})} \frac{R^d}{(Rd_g(x_1,x_0))^{\frac{d-1}{2}}} \sum_\pm \langle R|t_0 \pm d_g(x_1,x_0)|\rangle^{-K}. \end{aligned}$$

Thus we have proved the bounds required.

The quasi-orthogonality arguments are obtained by a largely analogous method. One major difference is that we can use the self-adjointness of the operators Q, and the unitary group structure of $\{e^{2\pi i t P}\}$, to write

$$\begin{aligned}
\langle f_0, f_1 \rangle &= \int c_0(t) c_1(s) \Big\langle (Q \circ e^{2\pi i t P} \circ Q)\{u_0\}, (Q \circ e^{2\pi i s P} \circ Q)\{u_1\} \Big\rangle \\
&= \int c_0(t) c_1(s) \Big\langle (Q^2 \circ e^{2\pi i (t-s) P} \circ Q^2)\{u_0\}, u_1 \Big\rangle \\
&= \int c(t) \Big\langle (Q^2 \circ e^{2\pi i t P} \circ Q^2)\{u_0\}, u_1 \Big\rangle,
\end{aligned}$$

where $c(t) = \int c_0(u) c_1(u-t)\, du$ is essentially the convolution of the functions, satisfying

$$\|c\|_{L^1(\mathbb{R})} \lesssim \|c_0\|_{L^1(\mathbb{R})} \|c_1\|_{L^1(\mathbb{R})} \quad \text{and} \quad \operatorname{supp}(c) \subset [(t_0 - t_1) - 4/R, (t_0 - t_1) + 4/R].$$

After this, one proceeds exactly as in the proof of the pointwise estimate. We write the inner product as

$$\sum_\alpha \int c(t) \Big\langle (Q^2 \circ e^{2\pi i t P} \circ Q^2)\{\eta_\alpha u_0\}, u_1 \Big\rangle.$$

Then we use Lemma 13 too replace $Q^2 \circ e^{2\pi i t P} \circ Q^2$ with $Q_\alpha^2 \circ W_\alpha(t) \circ Q_\alpha^2$ using Lemma 13, modulo a negligible error. The integral

$$\sum_\alpha \int c(t) \Big\langle (Q_\alpha^2 \circ W_\alpha(t) \circ Q_\alpha^2)\{\eta_\alpha u_0\}, u_1 \Big\rangle$$

is then only non-zero if both the supports of u_0 and u_1 are compactly contained in U_α. Thus we can switch to the coordinate system of U_α, in which we can express the inner product by oscillatory integrals of the exact same kind as those occuring in the pointwise estimate. Integrating away any the highly localized variables, and then applying stationary phase, we obtain the required estimate.

4 Regime I: Density Arguments For Dyadic Pieces

In this section, we begin obtaining estimates for the operator M_j^I. Given a general input $u : M \to \mathbb{C}$, we consider a maximal $1/2^j$ separated subset $\mathcal{X}_j$ of M, then consider a decomposition $u = \sum_{x_0 \in \mathcal{X}_j} u_{x_0}$, where u_{x_0} is supported on $B(x_0, 2/2^j)$, such that for all $r \in [1, \infty]$,

$$\|u\|_{L^r(M)} \sim \left(\sum_{x_0 \in \mathcal{X}_j} \|u_{x_0}\|_{L^r(M)}^r \right)^{1/r}.$$

If we set $f_{x_0,t_0} = M_{j,t_0}^I \{u_{x_0}\}$, then

$$\|M_j^I u\|_{L^p(M)} = \Big\| \sum f_{x_0,t_0} \Big\|_{L^p(M)}.$$

In this section, we use the quasi-orthogonality estimates of the last section to obtain L^2 estimates on partial sums of the functions $\{f_{x_0,t_0}\}$, under a density assumption on the set of indices we are summing over. To obtain bounds on $\| \sum f_{x_0,t_0} \|_{L^p(M)}$, we will later perform a *density decomposition* to break up $\mathcal{X}_j \times \mathcal{T}_j$ into a low and high density piece, and the methods of this section, appropriately interpolated, will be used to control the L^p norm of the low density piece.

Proposition 6. *Fix $u \geqslant 1$. Consider a set $\mathcal{E} \subset \mathcal{X} \times \mathcal{T}$. Write*

$$\mathcal{E} = \bigcup_{k=0}^{\infty} \mathcal{E}_k,$$

where $\mathcal{E}_k = \{(x,t) \in \mathcal{E} : |t| \sim 2^{k-j}\}$. Suppose that each of the sets $\mathcal{E}_k$ has density type $(2^j u, 2^{k-j})$, i.e. so that for any set $B \subset \mathcal{X} \times \mathcal{T}$ with diam$(B) \leqslant 2^{k-j}$,

$$\#(\mathcal{E}_k \cap B) \leqslant 2^j u \; diam(B).$$

Then
$$\Big\| \sum_k \sum_{(x_0,t_0)\in\mathcal{E}_k} 2^{k\frac{d-1}{2}} f_{x_0,t_0} \Big\|_{L^2(S^d)} \lesssim R^d \log_2(u) u^{\frac{2}{d-1}} \sum_k 2^{k(d-1)} \#\mathcal{E}_k.$$

Write $F = \sum F_k$, where

$$F_k = 2^{k\frac{d-1}{2}} \sum_{(x_0,t_0)\in\mathcal{E}} f_{x_0,t_0}.$$

Applying Cauchy-Schwartz, we have

$$\|F\|_{L^2(S^d)}^2 \lesssim \log_2(u) \left(\sum_{k \lesssim \log_2(u)} \|F_k\|_{L^2(S^d)}^2 + \| \sum_{k \gtrsim \log_2(u)} F_k \|_{L^2(S^d)}^2 \right).$$

Without loss of generality, increasing the implicit constant, we can assume that $\{k : \mathcal{E}_k \neq \varnothing\}$ is 10-separated, and that all values of t with $(x,t) \in \mathcal{E}$ are positive (the case where all values of t being negative being treated analogously, and then combined with the positive values trivially using the triangle inequality). Thus if F_k and $F_{k'}$ are both nonzero, then $k = k'$ or $|k - k'| \geqslant 10$. For $k \geqslant k' + 10$, let us estimate $\langle F_k, F_{k'} \rangle$. We can decompose this inner product into a sum of quantities of the form $2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} \langle S_{x_0,t_0}, S_{x_1,t_1} \rangle$, where $t_0 \sim 2^k/R$ and $t_1 \sim 2^{k'}/R$. Now consider the two sets

$$\mathcal{G}_{x_0,t_0,\text{low}} = \{(x_1,t_1) \in \mathcal{E}_{k'} : |d_g(x_0,x_1) - (t_0 - t_1)| \lesssim 2^{k'+10}/R\}$$

and for $l \geqslant k' + 10$, consider the set

$$\mathcal{G}_{x_0,t_0,l} = \{(x_1,t_1) \in \mathcal{E}_{k'} : |d_g(x_0,x_1) - (t_0 - t_1)| \sim 2^l/R\}.$$

Let us use the density properties of $\mathcal{E}$ to control the size of these index sets. First, note that for any $(x_0,t_0) \in \mathcal{E}_k$ and $(x_1,t_1) \in \mathcal{E}_{k'}$, $t_0 - t_1$ lies in a radius $O(2^{k'}/R)$ interval centered at t_0:

- Let us first estimate interactions between the functions S_{x_0,t_0} and S_{x_1,t_1} with $(x_1,t_1) \in \mathcal{G}_{x_0,t_0,\mathrm{low}}$. If $(x_1,t_1) \in \mathcal{G}_{x_0,t_0,\mathrm{low}}$, then x_1 must lie in a width $O(2^{k'}/R)$ and radius $O(2^k/R)$ annulus centered at x_0. Thus $\mathcal{G}_{x_0,t_0,\mathrm{low}}$ is covered by $O(2^{(k-k')(d-1)})$ balls of radius $2^{k'}/R$. The density properties of $\mathcal{E}_{k'}$ implies that

$$\#\mathcal{G}_{x_0,t_0,l} \lesssim Ru\, 2^{(k-k')(d-1)}(2^{k'}/R) = u2^{(k-k')(d-1)+k'}.$$

Together with Lemma 12, we conclude that

$$2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}} \sum_{(x_1,t_1)\in\mathcal{G}_{x_0,t_0,\mathrm{low}}} |\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim_M R^{d-2}2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}}\left(u2^{(k-k')(d-1)+k'}\right)\left(2^{-k\frac{d-1}{2}}\right).$$

We can now sum over $\log_2(u) \lesssim k' \leqslant k-10$ and $(x_0,t_0) \in \mathcal{E}_k$ to find

$$2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}} \sum_{(x_0,t_0)\in\mathcal{E}_k}\sum_{k'\leqslant k-10}\sum_{(x_1,t_1)\in\mathcal{G}_{x_0,t_0,\mathrm{low}}} |\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim R^{d-2}2^{k(d-1)}\#\mathcal{E}_k.$$

- Next, let's estimate interactions between the functions S_{x_0,t_0} and S_{x_1,t_1} with $(x_1,t_1) \in \mathcal{G}_{x_0,t_0,l}$ with $k'+10 \leqslant l \leqslant k-5$. If $(x_1,t_1) \in \mathcal{G}_{x_0,t_0,l}$, then x_1 must lie in one of two geodesic annuli centered at x_0, each width $O(2^l/R)$ and radii $O(2^k/R)$. Thus $\mathcal{G}_{x_0,t_0,l}$ is covered by $O(2^{(l-k')}2^{(k-k')(d-1)})$ balls of radius $2^{k'}/R$, and the density of $\mathcal{E}_{k'}$ implies that

$$\#\mathcal{G}_{x_0,t_0,l} \lesssim Ru\, 2^{(l-k')}2^{(k-k')(d-1)}2^{k'}/R = u2^l2^{(k-k')(d-1)}.$$

Together with Lemma 12, we conclude that

$$2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}} \sum_{(x_1,t_1)\in\mathcal{G}_{x_0,t_0,l}} |\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim_M R^{d-2}2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}}\left(u2^l2^{(k-k')(d-1)}\right)\left(2^{-k\frac{d-1}{2}}2^{-lM}\right).$$

Picking $M > 1$, we can sum over $k'+10 \leqslant l \leqslant k-5$, $\log_2(u) \lesssim k' \leqslant k-10$, and $(x_0,t_0) \in \mathcal{E}_k$ to find

$$\sum_{(x_0,t_0)\in\mathcal{E}_k}\sum_{k'\leqslant k-10}\sum_{k'+10\leqslant l\leqslant k-5}\sum_{(x_1,t_1)\in\mathcal{G}_{x_0,t_0,l}} 2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}}|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim R^{d-2}2^{k(d-1)}\#\mathcal{E}_k.$$

- Now let's estimate the interactions between the functions S_{x_0,t_0} and S_{x_1,t_1} with $(x_1,t_1) \in \mathcal{G}_{x_0,t_0,l}$, for $k+10 \leqslant l \leqslant \log_2 R$, then x_1 must lie in a geodesic ball of radius $O(2^l/R)$ centered at x_0. Such a ball is covered by $O(2^{(l-k')d})$ balls of radius $2^{k'}/R$, and the density of $\mathcal{E}_{k'}$ implies that

$$\#\mathcal{G}_{x_0,t_0,l} \lesssim Ru\, 2^{(l-k')d}(2^{k'}/R) = u2^{(l-k')d}2^{k'}.$$

Together with Lemma 12, we conclude that

$$2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}} \sum_{(x_1,t_1)\in\mathcal{G}_{x_0,t_0,l}} |\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim_M R^{d-2}2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}}\left(u2^{(l-k')d}2^{k'}\right)\left(2^{-lM}\right).$$

Picking $M > d$, we can sum over $k-5 \leqslant l \lesssim \log R$, $\log_2(u) \lesssim k' \leqslant k-10$, and $(x_0,t_0) \in \mathcal{E}_k$ to conclude that

$$2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}} \sum_{(x_0,t_0)\in\mathcal{E}_k}\sum_{k'\leqslant k-10}\sum_{k-5\leqslant l\lesssim \log R}\sum_{(x_1,t_1)\in\mathcal{G}_{x_0,t_0,l}} R^{d-2}|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim R^{d-2}.$$

Putting these three bounds together, we conclude that

$$\sum_{\log_2(u)\lesssim k'<k} |\langle F_k, F_{k'}\rangle| \lesssim R^{d-2}\sum_k 2^{k(d-1)}\#\mathcal{E}_k.$$

In particular, we have

$$\|F\|_{L^2(S^d)}^2 \lesssim \log_2(u)\left(\sum_k \|F_k\|_{L^2(S^d)}^2 + R^{d-2}\sum_k 2^{k(d-1)}\#\mathcal{E}_k\right).$$

Next, let us fix some parameter a, and decompose $[2^k/R, 2^{k+1}/R]$ into the disjoint union of length u^a intervals

$$I_{k,\mu} = [2^k/R + (\mu-1)u^a/R, 2^k/R + \mu u^a/R] \quad \text{for } 1 \leqslant \mu \leqslant 2^k/u^a,$$

and thus considering a further decomposition $\mathcal{E}_k = \bigcup \mathcal{E}_{k,\mu}$ and $F_k = \sum F_{k,\mu}$. As before, increasing the implicit constant in the Lemma, we may assume without loss of generality that the set $\{\mu : \mathcal{E}_{k,\mu} \neq \varnothing\}$ is 10-separated. We now estimate

$$\sum_{\mu\geqslant\mu'+10} |\langle F_{k,\mu}, F_{k,\mu'}\rangle|.$$

For $(x_0,t_0) \in \mathcal{E}_{k,\mu}$ and $l \geqslant 1$, define

$$\mathcal{H}_{x_0,t_0,l} = \left\{(x_1,t_1) \in \mathcal{E}_{k,\mu'} : \max(d_g(x_0,x_1), t_0-t_1) \sim 2^l u^a/R\right\}.$$

Then $\bigcup_{l\geqslant 1}\mathcal{H}_{x_0,t_0,l}$ covers $\bigcup_{\mu\geqslant\mu'+10}\mathcal{E}_{k,\mu'}$. The density properties of $\mathcal{E}_{k,\mu'}$ imply that provided that $l \leqslant k - a\log_2 u + 10$ (so that $2^l u^a/R \leqslant 2^k/R$),

$$\#\mathcal{H}_{x_0,t_0,l} \lesssim (Ru)(2^l u^a/R) = u^{a+1}2^l$$

For $(x_1,t_1) \in \mathcal{H}_{x_0,t_0,l}$, we claim that

$$2^{k(d-1)}|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim R^{d-2}2^{k(d-1)}(2^l u^a)^{-\frac{d-1}{2}}.$$

Indeed, for such tuples we have

$$d_g(x_0,x_1) \gtrsim 2^l u^a/R \quad \text{or} \quad |d_g(x_0,x_1) - (t_0-t_1)| \gtrsim 2^l u^a/R,$$

and the estimate follows from Lemma 12 in either case. Since $d \geqslant 4$,

$$\begin{aligned}\sum_{1\leqslant l\leqslant k-a\log_2 u+10}\ \sum_{(x_1,t_1)\in\mathcal{H}_{x_0,t_0,l}} 2^{k(d-1)}|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| &\lesssim R^{d-2}\sum_{1\leqslant l\leqslant k-a\log_2 u+10}(2^{k(d-1)})(2^l u^a)^{-\frac{d-1}{2}}(u^{a+1}2^l)\\ &\lesssim R^{d-2}\sum_{1\leqslant l\leqslant k-a\log_2 u+10} 2^{k(d-1)}2^{-l\frac{d-3}{2}}u^{1-a\left(\frac{d-3}{2}\right)}\\ &\lesssim R^{d-2}2^{k(d-1)}u^{1-a\left(\frac{d-3}{2}\right)}.\end{aligned}$$

For $l > k - a\log_2 u + 10$, a tuple (x_1, t_1) lies in $\mathcal{H}_{x_0,t_0,l}$ if and only if $d_g(x_0, x_1) \sim 2^l u^a/R$, since we always have

$$|t_0 - t_1| \lesssim 2^k/R \ll 2^l u^a/R.$$

We conclude from Lemma 12 that

$$2^{k(d-1)}|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim_M R^{d-2} 2^{k(d-1)} (2^l u^a)^{-M}.$$

Now $\mathcal{H}_{x_0,t_0,l}$ is covered by $O((2^{l-k}u^a)^d)$ balls of radius $2^k/R$, and the density properties of $\mathcal{E}_k$ imply that

$$\#\mathcal{H}_{x_0,t_0,l} \lesssim (Ru)(2^{l-k}u^a)^d(2^k/R) \lesssim u^{1+ad}2^{ld}2^{-k(d-1)}.$$

Thus, picking $M > \max(d, 1+ad)$, we conclude that

$$\begin{aligned}\sum_{l \geqslant k - a\log_2 u + 10} \sum_{(x_1,t_1)\in\mathcal{H}_{x_0,t_0,l}} 2^{k(d-1)}|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| &\lesssim R^{d-2} \sum_{l \geqslant k - a\log_2 u + 10} (2^{k(d-1)})(2^l u^a)^{-M} u^{1+ad}2^{ld}2^{-k(d-1)} \\ &\lesssim R^{d-2}.\end{aligned}$$

Putting these two bounds together, and then summing over the tuples (x_0, t_0), we conclude that

$$\sum_{\mu \geqslant \mu' + 10} |\langle F_{k,\mu}, F_{k,\mu'}\rangle| \lesssim R^{d-2}\left(1 + 2^{k(d-1)}u^{1-a\left(\frac{d-3}{2}\right)}\right)\#\mathcal{E}_{k,\mu}.$$

Now summing in μ, we conclude that

$$\|F_k\|_{L^2(S^d)}^2 \lesssim \sum_\mu \|F_{k,\mu}\|_{L^2(S^d)}^2 + R^{d-2}\left(1 + 2^{k(d-1)}u^{1-a\left(\frac{d-3}{2}\right)}\right)\#\mathcal{E}_k.$$

The functions in the sum defining $F_{k,\mu}$ are highly coupled, and it is difficult to use anything except Cauchy-Schwartz to break them apart. Since $\#(\mathcal{T} \cap I_{k,\mu}) \sim u^a$, if we set $F_{k,\mu} = \sum_{t\in\mathcal{T}\cap I_{k,\mu}} F_{k,\mu,t}$, then we find

$$\|F_{k,\mu}\|_{L^2(S^d)}^2 \lesssim u^a \sum_{t\in\mathcal{T}\cap I_{k,\mu}} \|F_{k,\mu,t}\|_{L^2(S^d)}^2.$$

Fortunately, since $\mathcal{X}$ is 1-separated, the functions in $F_{k,\mu,t}$ are quite orthogonal to one another, and so

$$\|F_{k,\mu,t}\|_{L^2(S^d)}^2 \lesssim R^{d-2}2^{k(d-1)}\#(\mathcal{E}_k \cap (S^d \times \{t\})).$$

But this means that

$$u^a \sum_t \|F_{k,\mu,t}\|_{L^2(S^d)}^2 \lesssim R^{d-2}2^{k(d-1)}u^a\#\mathcal{E}_{k,\mu}.$$

and so

$$\begin{aligned}\|F_k\|_{L^2(S^d)}^2 &\lesssim \sum_\mu \|F_{k,\mu}\|_{L^2(S^d)}^2 + R^{d-2}\left(1 + 2^{k(d-1)}u^{1-a\left(\frac{d-3}{2}\right)}\right)\#\mathcal{E}_k \\ &\lesssim R^{d-2}\left(2^{k(d-1)}u^a + (1 + 2^{k(d-1)}u^{1-a\left(\frac{d-3}{2}\right)}\right)\#\mathcal{E}_k.\end{aligned}$$

Picking $a = 2/(d-1)$, we conclude that

$$\|F_k\|_{L^2(S^d)}^2 \lesssim R^{d-2} 2^{k(d-1)} u^{\frac{2}{d-1}} \# \mathcal{E}_k.$$

Thus, returning to our bound for F, we conclude that

$$\|F\|_{L^2(S^d)}^2 \lesssim R^{d-2} \log_2(u) u^{\frac{2}{d-1}} \sum_k 2^{k(d-1)} \# \mathcal{E}_k.$$

This completes the proof of the L^2 density bound.

5 Regime I: Density Decompositions

It finally remains to analyze the operator $Q_R \circ II_R \circ Q_R$, where

$$II_R = \int \chi_{II}(t) R \widehat{h}(Rt) e^{2\pi i t P}\, dt$$

is obtained by integrating the wave propogators over times $100/R \leqslant |t| \leqslant 0.01$ respectively. To prevent notation from growing too cumbersome later on, let us eschew uses of the subscript R in our operators in this section, e.g. writing II_R as

$$II = \int b(t) e^{2\pi i t P}\, dt,$$

where $b(t) = \chi_{II}(t) R \widehat{h}(Rt)$. We then have

$$\|b(t) \langle t \rangle^{s_p}\|_{L^p(\mathbb{R})} \lesssim R^{1-1/p-s_p} C_p(h).$$

Bounding II requires a more subtle analysis of the geometric behaviour of the wave-propogator operators, and we will begin by converting our problem in coordinates on S^d, where the kernels have more explicit representations in oscillatory integrals.

We will employ some restricted weak type bounds, together with interpolation, to obtain L^p estimates on the operators $Q \circ II \circ Q$. We thus introduce a set $E \subset S^d$ and try to obtain $L^{p,\infty}$ bounds on the function $S = (Q \circ II_W \circ Q)\{E\}$. Given that Q already acts, heuristically, by localizing the behaviour of it's inputs to the frequency R, despite the choice of the set E, the uncertainty principle implies $Q\{E\}$ should be locally constant at a scale $1/R$, and so it is natural to discretize at this scale. Consider a maximal $1/2R$ separated subset $\mathcal{X}$ of S^d. Then break E down into a disjoint union of sets $\{E_{x_0} : x_0 \in \mathcal{X}\}$, where for $x_0 \in \mathcal{X}$, the set E_{x_0} is supported on the geodesic ball of radius $1/R$ about x_0. Similarily, let $\mathcal{T}$ be all points in the lattice $\mathbb{Z}/10R$ lying in the set $\{100/R \leqslant |t| \leqslant 1\}$, and write

$$b = \sum_{t \in \mathcal{T}} u(t) b_t,$$

where for each $t \in \mathcal{T}$, $u(t) = \|b\|_{L^\infty[t-10/R, t+10/R]}$, and b_t is a smooth function, compactly supported on the sidelength $1/R$ interval centered at t, satisfying

$$|\partial^\alpha b_t| \lesssim_\alpha R^{|\alpha|},$$

with implicit constants uniform in b and t. By the Plancherel-Polya theorem,

$$\|u(t)\langle t\rangle^{s_p}\|_{l^p(\mathcal{T})} \lesssim R^{1-s_p}.$$

We can then write

$$S = \sum |E_{x_0}| S_{x_0,t_0} \quad \text{where} \quad S_{x_0,t_0} = \int |E_{x_0}|^{-1} b_{t_0}(t)(Q \circ e^{2\pi i t P} \circ Q)\{E_{x_0}\}\, dt.$$

Our computation would be complete if we could show that for any coefficients $\{c(x_0,t_0) : x_0 \in \mathcal{X}, t_0 \in \mathcal{T}\}$,

$$\left\| \sum_{x_0,t_0} c(x_0,t_0) t_0^{\frac{d-1}{2}} S_{x_0,t_0} \right\|_{L^p(S^d)} \lesssim R^{s_p - 1 + d(1-1/p)} \left(\sum_{x_0,t_0} |c(x_0,t_0)|^p t_0^{d-1} \right)^{1/p}.$$

Indeed, we set $c(x_0,t_0) = |E_{x_0}| u(t_0) t_0^{-\frac{d-1}{2}}$ and apply Hölder's inequality, then the inequality above gives exactly that

$$\|S\|_{L^p(S^d)} \lesssim C_p(h)|E|^{1/p},$$

For $p = 1$, this follows from applying the triangle inequality, and using the pointwise estimates

$$|S_{x_0,t_0}(x)| \lesssim_M \frac{R^{d-1}}{(R d_g(x,x_0))^{\frac{d-1}{2}}} \left\langle R \big| t_0 - d_g(x,x_0) \big| \right\rangle^{-M}.$$

Applying interpolation, for $p > 1$ we need only prove a restricted weak type version of this inequality. In other words, we can restrict c to be the indicator function of a set $\mathcal{E}$, and take $L^{p,\infty}$ norms on the left hand side. If we write $\mathcal{E} = \bigcup_k \mathcal{E}_k$, where $\mathcal{E}_k$ is the set of $(x,t) \in \mathcal{E}$ with $|t| \sim 2^k/R$, then the inequality reads that

$$\left\| \sum_{k=1}^{\infty} 2^{k\frac{d-1}{2}} \sum_{(x_0,t_0) \in \mathcal{E}_k} S_{x_0,t_0} \right\|_{L^{p,\infty}(S^d)}^p \lesssim R^{(d-1)p-d} \left(\sum_{k=1}^{\infty} 2^{k(d-1)} \# \mathcal{E}_k \right).$$

This is equivalent to showing that for any $\lambda > 0$,

$$\left| \left\{ x : \left| \sum_k 2^{k\frac{d-1}{2}} S_{x_0,t_0}(x) \right| \geqslant \lambda \right\} \right| \lesssim \lambda^{-p} R^{(d-1)p-d} \sum_k 2^{k(d-1)} \# \mathcal{E}_k.$$

The case $\lambda \lesssim R^{d-1}$ follows from the L^1 boundedness we've already proved, so we may assume $\lambda \gtrsim R^{d-1}$ in the sequel.

To obtain this bound, we employ the method of density decompositions, introduced in [3]. Let

$$A = \left(\frac{\lambda}{R^{d-1}} \right)^{(d-1)(1-p/2)} \log \left(\frac{\lambda}{R^{d-1}} \right)^{O(1)}.$$

Then for each k, consider the collection $\mathcal{B}_k(\lambda)$ of all balls B with radius at most $2^k/R$ such that $\#\mathcal{E}_k \cap B \geqslant RA\mathrm{rad}(B)$. Applying the Vitali covering lemma, we can find a disjoint family of balls $\{B_1, \dots, B_N\}$ in $\mathcal{B}_k$ such that the balls $\{B_1^*, \dots, B_N^*\}$ obtained by dilating the balls by 5 cover $\bigcup \mathcal{B}_k(\lambda)$. Then

$$\sum \operatorname{rad}(B_j) \leqslant R^{-1} A^{-1} \# \mathcal{E}_k,$$

and the set $\widehat{\mathcal{E}}_k = \mathcal{E}_k - \bigcup \mathcal{B}_k(\lambda)$ has density type $(RA, 2^k/R)$. Then we conclude that, using the quasi-orthogonality estimates below,

$$\left\| \sum_k \sum_{(x_0,t_0) \in \widehat{\mathcal{E}}_k} 2^{k\frac{d-1}{2}} S_{x_0,t_0} \right\|_{L^2(S^d)}^2 \lesssim_p R^{d-2} \log(A) A^{\frac{2}{d-1}} \sum_k 2^{k(d-1)} \# \mathcal{E}_k.$$

Appling Chebyshev's inequality, and utilizing the choice of A above, we conclude that

$$\begin{aligned} \left| \left\{ x : \left| \sum_k \sum_{(x_0,t_0) \in \widehat{\mathcal{E}}_k} 2^{k\frac{d-1}{2}} S_{x_0,t_0}(x) \right| \geqslant \lambda/2 \right\} \right| &\lesssim R^{d-2} \log(A) A^{\frac{2}{d-1}} \sum_k 2^{k(d-1)} \# \mathcal{E}_k \\ &\lesssim \lambda^{-p} R^{(d-1)p-d} \sum_k 2^{k(d-1)} \# \mathcal{E}_k. \end{aligned}$$

Conversely, we exploit the clustering of the sets $\mathcal{E}_k - \widehat{\mathcal{E}}_k$ to bound

$$\left| \left\{ x : \left| \sum_k \sum_{(x_0,t_0) \in \mathcal{E}_k - \widehat{\mathcal{E}}_k} 2^{k\frac{d-1}{2}} S_{x_0,t_0}(x) \right| \geqslant \lambda/2 \right\} \right|$$

That is, we have found balls $B_1^* < \dots, B_N^*$, each with radius $O(2^k/R)$, such that

$$\sum \operatorname{rad}(B_j) \leqslant R^{-1} A^{-1} \# \mathcal{E}_k.$$

Let (x_j, t_j) denote the center of the ball B_j. Then the function

$$\sum_{(x_0,t_0) \in B_j} S_{x_0,t_0}$$

has mass concentrated on the geodesic annulus $\mathrm{Ann}_j \subset S^d$ with radius t_j and thickness $O(\operatorname{rad}(B_j))$, a set with measure $(2^k/R)^{d-1} \operatorname{rad}(B_j)$. For $(x_0, t_0) \in B_j$, we calculate using the pointwise bounds that

$$\begin{aligned} \int_{\mathrm{Ann}_j^c} |S_{x_0,t_0}(x)| \, dx &\lesssim R^{d-1} \int_{\operatorname{rad}(B_j) \lesssim |t_j - d_g(x,x_0)| \leqslant 1} \langle R|t_0 - d_g(x,x_0)| \rangle^{-M} \\ &\lesssim R^{d-1} \int_{\operatorname{rad}(B_j) \leqslant |t_j - s| \leqslant 1} s^{d-1} \langle R|t_0 - s| \rangle^{-M} \, ds \\ &\lesssim 2^{k(d-1)} R^{d-1} (R \operatorname{rad}(B_j))^{-M}. \end{aligned}$$

Because the set of points in $\mathcal{E}_k$ is $1/R$ separated, there can only be at most $O(R\text{rad}(B_j))^{d+1}$ values of (x_0, t_0), and so applying the triangle inequality gives that the sum of the L^1 norm outside of Ann_j is

$$\lesssim 2^{k(d-1)} R^{d-1} (R\,\text{rad}(B_j))^{d+1-M}$$

Note that since $\#\mathcal{E}_k \cap B_j \geqslant RA\text{rad}(B_j)$, and because $\mathcal{E}_k$ is $1/R$ discretized,

$$\text{rad}(B_j) \geqslant (A/R)^{\frac{1}{d-1}},$$

and this, together with Markov's inequality, is enough to justify the required bound. Conversely, since $1 < p < 2(d-1)/(d+1)$, we have

$$\begin{aligned}\sum |\text{Ann}_j| \lesssim (2^k/R)^{d-1} \sum_j \text{rad}(B_j) \\ &\lesssim (2^k/R)^{d-1} R^{-1} (L/R^{d-1})^{-(d-1)(1-p/2)} log(L/R^{d-1})^{O(1)} \\ &\lesssim \lambda^{-p} R^{(d-1)p-d} 2^{k(d-1)} \#\mathcal{E}_k,\end{aligned}$$

Summing over k completes the analysis.

6 Regime II: Local Smoothing

We now show the uniform boundedness of the operators $\{III_R\}$ on $L^p(S^d)$ in the range of p we are considering in this problem, by a reduction to an endpoint local smoothing inequality. This might seem unintuitive, since the operators III_R are obtained by averaging the wave equation over large times $|t| \gtrsim 1$, whereas local smoothing gives bounds for averages of the wave equation over times $|t| \lesssim 1$. We are able to reduce large times to small times by exploiting the *periodicity* of the half-wave equation on the sphere.

Lemma 7. *Fix $1 < p < 2d/(d+1)$, let q be the Hölder conjugate to p, and let $I = [-1/2, 1/2]$. Suppose that the sharp local smoothing inequality*

$$\left\| e^{2\pi i t P} f \right\|_{L^q(S^d) L^q_t(I)} \lesssim \|f\|_{L^q_{s_q - 1/q}(S^d)}$$

holds for all $f \in C^\infty(S^d)$. Then the operators $\{III_R\}$ satisfy a bound

$$\|(III_R \circ Q_R) f\|_{L^p(S^d)} \lesssim C_p(h) \|f\|_{L^p(S^d)},$$

with the implicit constant uniformly bounded in R. In particular,

$$\|(Q_R \circ III_R \circ Q_R) f\|_{L^p(S^d)} \lesssim C_p(h) \|f\|_{L^p(S^d)},$$

Proof. For each R, the *class* of operators of the form $\{III_R\}$ formed from a multiplier h satisfying the hypothesis of Theorem 8 is closed under taking adjoints. Indeed, if III_R is obtained from h, then III_R^* is obtained from the multiplier $\overline{h}$. Because of this self-adjointness, if we can prove that for any multiplier h satisfying the assumptions of the theorem, the operators $\{III_R\}$ are uniformly bounded in $L^q(S^d)$, where q is the Hölder conjugate to p, then it follows by duality that for any such h, it is also true that the operators $\{III_R\}$ are uniformly bounded back in the original $L^p(S^d)$ norm. In this argument we will prove such L^q estimates, because we will exploit *local smoothing* inequalities, which tend to work better with large Lebesgue exponents, precisely because Lebesgue norms with large exponents are more sensitive to functions with sharp peaks, something explicitly prevented by obtaining control over the smoothness of a function.

We begin by noting that for a pair of Hölder conjugates p and q, $s_q = s_p$. Using the periodicity of the wave equation on S^d, i.e. that

$$e^{2\pi i(t+n)P} = e^{2\pi itP} \qquad \text{for } n \in \mathbb{Z} \text{ and } t \in \mathbb{R},$$

we can write

$$III_R = \int_{-1/2}^{1/2} H_R(t) e^{2\pi itP}\, dt,$$

where

$$H_R(t) = \sum_l \chi_{III}(t) R\widehat{h}(R(t+l)) = \sum_l H_{R,l}(t).$$

Now

$$\begin{aligned}
&\left(\sum_{l \neq 0} \left(|Rl|^{s_q} \|H_{R,l}\|_{L^p[-1/2,1/2]} \right)^p \right)^{1/p} \\
&\quad \sim R \left(\int_{-1/2}^{1/2} \sum_l \left(|R(t+l)|^{s_q} |\widehat{h}(R(t+l))| \right)^p \, dt \right)^{1/p} \\
&\quad \sim R \left(\int_{|t| \geqslant 1/2} \left(|Rt|^{s_q} |\widehat{h}(Rt)| \right)^p \, dt \right)^{1/p} \\
&\quad \lesssim R^{1/q} C_p(h).
\end{aligned}$$

and

$$\begin{aligned}
\|H_{R,0}\|_{L^p[-1/2,1/2]} &= \left(\int_{-1/2}^{1/2} |\chi_{III}(t) R\widehat{h}(Rt)|^p \right)^{1/p} \\
&\leqslant \left(\int_{1/200 \leqslant |t| \leqslant 1/2} |R\widehat{h}(Rt)|^p \right)^{1/p} \\
&= R^{1/q} \left(\int_{R/3}^{R/2} |\widehat{h}(t)|^p \right)^{1/p} \\
&\lesssim R^{1/q - s_q} C_p(h).
\end{aligned}$$

Since the family of functions $\{H_{R,l}\}$ could in general be chosen arbitrarily, they can be quite correlated, and so we should expect Hölder's inequality should be efficient, in the worst case. Thus we conclude that, provided $p < 2d/(d+1)$, so that $q > 2d/(d-1)$, and thus

$$qs_q = (d-1)(q/2 - 1) > 1,$$

so we can use Hölder's inequality to conclude that

$$\begin{aligned}
\|H_R\|_{L^p[-1/2,1/2]} &\leqslant \sum_l \|H_{R,l}\|_{L^p[-1/2,1/2]} \\
&= \|H_{R,0}\|_{L^p[-1/2,1/2]} + \sum_{l\neq 0} \left(|Rl|^{s_q}\|H_{R,l}\|_{L^p[-1/2,1/2]}\right)|Rl|^{-s_q} \\
&\lesssim R^{-s_q+1/q}C_p(h) + (R^{1/q}C_p(h))\left(\sum_{l\neq 0}|Rl|^{-s_q q}\right)^{1/q} \\
&= R^{-s_q+1/q}C_p(h)\left(1+\left(\sum_{l\neq 0}|l|^{-s_q q}\right)^{1/q}\right) \\
&= R^{-s_q+1/q}C_p(h)\left(1+\left(\sum_{l\neq 0}|l|^{-s_q q}\right)^{1/q}\right) \\
&\lesssim_p R^{-s_q+1/q}C_p(h).
\end{aligned}$$

A further application of Hölder's inequality shows that

$$\begin{aligned}
|III_R| &= \left|\int_{-1/2}^{1/2} H_R(t)e^{2\pi i t P}\,dt\right| \\
&\lesssim C_p(h)R^{-s_q+1/q}\left(\int_{-1/2}^{1/2}|e^{2\pi i P}|^q\,dt\right)^{1/q}.
\end{aligned}$$

Applying the endpoint local smoothing inequality, we conclude that

$$\begin{aligned}
\|(III_R\circ Q_R)f\|_{L^q(M)} &\lesssim C_p(h)R^{-s_q+1/q}\|e^{2\pi i P}(Q_Rf)\|_{L^q_tL^q_x} \\
&\lesssim C_p(h)R^{-s_q+1/q}\|Q_Rf\|_{L^q_{s_q-1/q}(M)},
\end{aligned}$$

Applying Bernstein's inequality gives

$$\|Q_Rf\|_{L^q_{s_q-1/q}(M)} \lesssim R^{s_q-1/q}\|f\|_{L^q(M)}.$$

Thus we conclude that

$$\|(III_R\circ Q_R)f\|_{L^q(M)} \lesssim C_p(h)\|f\|_{L^q(M)}.$$

We have therefore bounded III_R uniformly in R. □

Corollary 1.2 of [6] establishes that the sharp local smoothing inequality holds for $p < 2(d-1)/(d+1)$, which covers the range of parameters studied in this paper. Thus we have obtained uniform bounds on the operators $\{III_R\}$.

7 Combining Dyadic Pieces

s

8 Appendix

9 BLAH

The idea behind the proof of the result of [3] is to reduce the analysis of radial multiplier operators to the study of the L^p norms of sums of the form

$$\left\| \sum_{(x_0,t_0)\in\mathbb{R}^d\times(0,\infty)} S_{x_0,t_0} \right\|_{L^p(\mathbb{R}^d)},$$

where for each $x_0 \in \mathbb{R}^d$ and $t_0 \in (0,\infty)$, the function S_{x_0,t_0} is supported on an annulus of radius t_0 and center x_0. We are able to translate this geometric argument to the variable coefficient setting, by replacing the condition that S_{x_0,t_0} is localized to an annulus of radius t_0 and center x_0, with the condition that S_{x_0,t_0} is obtained from the output of the *half-wave equation* $\partial_t + \Delta = 0$ on S^d at time t_0, with an input localized near the point $x_0 \in S^d$. For small times, this output can be well understood by approximating solutions to the half-wave equation by the *Lax-Hörmander Parametrix*, which describes the solutions in terms of certain oscillatory integrals.

using variable-coefficient generalizations of techniques that have been used to obtain the boundedness of radial Fourier multiplier operators of the form $h(|\cdot|)$. In particular, we prove that certain conditions on the Fourier transform of h, which are necessary for the operators $\{h(P/R)\}$ to be uniformly bounded on $L^p(S^d)$, are actually *sufficient* for a restricted range of exponents p. To state this condition, define

$$s_q = (d-1)\left|\frac{1}{q} - \frac{1}{2}\right| \quad \text{for } 1 \leqslant q \leqslant \infty,$$

and define

The condition that $C_p(h)$ is finite is necessary for the operators

$$\{h(P/R) : R > 0\}$$

to be uniformly bounded on $L^p(S^d)$. The main result of this paper is that, for a restricted range of p, this condition is also *sufficient*.

Theorem 8. *Let M be a compact Riemannian manifold of dimension d, with periodic geodesic flow. Suppose $1 \leqslant p < 2(d-1)/(d+1)$. Then for all functions $f \in L^p(S^d)$, for all functions h with $supp(h) \subset \{t : 1/2 \leqslant |t| \leqslant 2\}$, and for all $R > 0$,*

$$\|h(P/R)f\|_{L^p(S^d)} \lesssim C_p(h)\|f\|_{L^p(S^d)},$$

where the implicit constant depends only on d.

Remarks 9.

1. *Theorem 2 immediately implies by duality that for* $(d-3)/2(d-1) < p < \infty$,

$$\|h(P/R)\|_{L^p(S^d)\to L^p(S^d)} \lesssim \left(\int_0^\infty \left[\langle t\rangle^{s_{p'}}|\widehat{h}(t)|\right]^{p'}\, dt\right)^{1/p'} \quad \textit{uniformly for } R>0,$$

where $1/p + 1/p' = 1$.

2. *By a transference principle of Mitjagin [7], the uniform boundedness of the operators* $\{h(P/R) : R > 0\}$ *on* $L^p(S^d)$ *implies that the Fourier multiplier operator on* $\mathbb{R}^d$ *with symbol* $h(|\cdot|)$ *is bounded in* $L^p(\mathbb{R}^d)$. *As discussed in [3], this can only be true if* $C_p(h) < \infty$, *so the finiteness of* $C_p(h)$ *is necessary for the operators* $\{h(P/R)\}$ *to be uniformly bounded in* R. *The boundedness of* $C_p(h)$ *has already been proved sufficient provided the inputs functions* $f \in L^p(\mathbb{R}^d)$ *are restricted to* zonal *functions [1]. The main novelty of the result of this paper is that we can extend this bound to general* $f \in L^p(\mathbb{R}^d)$.

3. *If* $1 \leqslant p < 2d/(d+1)$ *and* $C_p(h) < \infty$, *then [3] implies that the operator* $h(|\cdot|)$ *is bounded as a Fourier multiplier operator on* $L^p(\mathbb{R}^d)$ *with*

$$\|h(|\cdot|)\|_{M^p(\mathbb{R}^d)} \lesssim C_p(h).$$

Interpolation and duality (see Section 2.5.5 of [2] for more details) implies that the operator is also a Fourier multiplier operator on $L^2(\mathbb{R}^d)$, *and so we conclude that*

$$\|h\|_{L^\infty(\mathbb{R})} = \|h(|\cdot|)\|_{M^2(\mathbb{R}^d)} \lesssim C_p(h).$$

4. *The projection operators* $\{\mathcal{P}_\lambda\}$ *are each individually smoothing, though not uniformly as* $\lambda \to \infty$. *They thus individually satisfy bounds of the form* $\|\mathcal{P}_\lambda f\|_{L^p(S^d)} \lesssim_\lambda \|f\|_{L^p(M)}$ *for all* $1 \leqslant p \leqslant \infty$. *It thus follows trivially from the triangle inequality, and that there are finitely many eigenvalues for* P *in* $[0, 100]$, *that for any* $R \leqslant 100$,

$$\begin{aligned}\|M_R f\|_{L^p(S^d)} &\leqslant \sum_\lambda |h(\lambda/R)|\|\mathcal{P}_\lambda f\|_{L^p(S^d)}\\ &\leqslant \|h\|_{L^\infty(0,200)} \textstyle\sum_{\lambda\in[0,200]} \|\mathcal{P}_\lambda f\|_{L^p(S^d)}\\ &\lesssim \|h\|_{L^\infty(0,\infty)}\|f\|_{L^p(S^d)}.\end{aligned}$$

Thus, in the analysis that follows, we will always assume that $R \geqslant 100$.

5. *Because* h *is a unit scale multiplier, if we fix a smooth bump function* β *supported on* $\{1/4 \leqslant |t| \leqslant 4\}$, *and equal to one on* $\{1/2 \leqslant |t| \leqslant 2\}$, *set* $\beta_R(\lambda) = \beta(\lambda/R)$, *and set* $Q_R = \beta(P/R)$, *then*

$$h(P/R) = Q_R \circ h(P/R) \circ Q_R.$$

By including the operators $\{Q_R\}$ *in our analysis, we essentially reduce our analysis to the study to inputs and outputs lying in the range of the operators* Q_R, *which is equal to the finite dimensional subspace* V_R *of* $C^\infty(S^d)$ *spanned by eigenfunctions of* P *with eigenvalue in* $R/4 \leqslant \lambda \leqslant 4R$. *Since* P *is positive-semidefinite and self-adjoint, it is* *often useful to use the heuristic that an element of V_R should behave like a function on $\mathbb{R}^d$ with Fourier support on $\{R/4 \leqslant |\xi| \leqslant 4R\}$.*

A particular application of this heuristic is an analogue of Bernstein's inequality on $\mathbb{R}^d$ (see [9], Proposition 5.1), but for functions on a Riemannian manifold lying in V_R. This analogue states that for $1 < r < \infty$, uniformly for $R \geqslant 100$ and $f \in V_R$ we have

$$\|f\|_{L^r_s(S^d)} \lesssim_{r,s} R^s \|f\|_{L^r(S^d)}. \tag{9.1}$$

See Section 3.3 of [8] for a proof.

Another useful inequality follows from the fact that the family of functions $\{\beta_R\}$ form a uniformly bounded subset of the Fréchet space $\mathcal{S}^0$, i.e. satisfying estimates of the form

$$|\partial_\lambda^n \{\beta_R\}(\lambda)| \lesssim_n \langle \lambda \rangle^{-n} \quad \textit{uniformly in } R > 0.$$

It follows that the operators $\{Q_R\}$ are pseudodifferential operators of order zero, uniformly bounded as operators on $L^r_s(M)$ for all $1 < r < \infty$, i.e. satisfying

$$\|Q_R f\|_{L^r_s(S^d)} \lesssim_{r,s} \|f\|_{L^r_s(S^d)}. \tag{9.2}$$

See Corollary 4.3.2 of [8] for more details.

Consider a cover

$$\{|t| < 100/R\} \cup \{50/R < |t| < 1/100\} \cup \{1/200 < |t| < \infty\}$$

of $\mathbb{R}$, and find a smooth partition of unity $\chi_{I,R}$, $\chi_{II,R}$, and $\chi_{III,R}$ adapted to these sets. Without loss of generality, we may assume all three functions are even, that $\chi_{I,R}(t) = \chi_I(Rt)$, for some smooth, compactly supported function χ_I adapted to the open set $\{|t| < 1\}$, and also assume that $\chi_{III,R} = \chi_{III}$ is independent of R. Given this partition of unity, we now write

$$h(P/R) = I_R + II_R + III_R,$$

where, for $\Pi \in \{I, II, III\}$, the operators

$$\Pi_R = \int \chi_{\Pi,R}(t) R \widehat{h}(Rt) e^{2\pi i t P}\, dt$$

isolate the study of $h(P/R)$ to the behaviour of the half-wave propogators on three different time intervals. The remainder of the argument will consist of separately obtaining L^p boundedness for each of the three operators $Q_R \circ \Pi_R \circ Q_R$, since then the triangle inequality gives the L^p boundedness of

$$(Q_R \circ I_R \circ Q_R) + (Q_R \circ II_R \circ Q_R) + (Q_R \circ II_R \circ Q_R) = h(P/R).$$

The study of the operators $\{I_R\}$ will reduce to a study of pseudodifferential operators, we will be able to apply the endpoint local smoothing inequality of [6] to control the operators $\{III_R\}$, and the study of the operators $\{II_R\}$ will be given by generalizations of the methods of [3] to a variable coefficient setting.

10 Analysis of I_R

Let us analyze

$$I_R = \int \chi_I(Rt) R\widehat{h}(Rt) e^{2\pi i t P}\, dt.$$

We are analyzing inputs to I_R coming from the composition of a general element of $C^\infty(S^d)$ with Q_R, which heuristically localizes the 'frequency support' of this function to a band of frequencies with magnitude $\sim R$. Thus, by uncertainty principle heuristics, such functions are locally constant at a scale $1/R$. The half-wave equation propogates a majority of the mass of it's input at a unit speed, and since the operators $\{I_R\}$ are obtained by averaging the half-wave equation over times $\lesssim 1/R$, we should expect that the behaviour of the operators $\{I_R\}$ to behave in a localized manner. In fact, the following analysis will show that the operators $\{I_R\}$ are pseudodifferential operators, which will allow us to conclude these operators are uniformly bounded in $L^p(S^d)$.

Lemma 10. *For all $f \in C^\infty(S^d)$,*

$$\|I_R f\|_{L^p(S^d)} \lesssim \|h\|_{L^\infty(\mathbb{R})} \|f\|_{L^p(S^d)} \lesssim C_p(h) \|f\|_{L^p(S^d)},$$

where the implicit constant is uniformly bounded in $R \geqslant 1$ and h, for $1 < p < \infty$. Thus in particular,

$$\|(Q_R \circ I_R \circ Q_R) f\|_{L^p(S^d)} \lesssim C_p(h) \|f\|_{L^p(S^d)}.$$

Proof. Let a be the inverse Fourier transform of the function $t \mapsto \chi_I(t)\widehat{h}(t)$. Then $I_R = a(P/R)$. If ψ denotes the inverse Fourier transform of χ_I, then we can write

$$a(\lambda) = \int h(\alpha)\psi(\lambda - \alpha)\, d\alpha.$$

The fact that h is a unit scale multiplier, and ψ is Schwartz, implies that

$$|\partial^\alpha a(\lambda)| \lesssim_{\alpha,N} \|h\|_{L^\infty(\mathbb{R})} \langle \lambda \rangle^{-N}.$$

If we set $a_R(\lambda) = a(\lambda/R)$, then

$$|\partial^\alpha a_R(\lambda)| \lesssim_\alpha \|h\|_{L^\infty(\mathbb{R})} \langle \lambda \rangle^{-|\alpha|},$$

with an implicit constant independent of R for $R \geqslant 1$. Thus the family of symbols $\{a_R : R \geqslant 1\}$ form a uniformly bounded subset of the Frećhet space $\mathcal{S}^0(\mathbb{R})$ of order zero symbols, and so the operators I_R are pseudodifferential operators of order zero, and uniformly bounded in the $L^p(S^d)$ norm for all $1 < p < \infty$, which yields the required claim. $\square$

11 Analysis of III_R

We now show the uniform boundedness of the operators $\{III_R\}$ on $L^p(S^d)$ in the range of p we are considering in this problem, by a reduction to an endpoint local smoothing inequality. This might seem unintuitive, since the operators III_R are obtained by averaging the wave equation over large times $|t| \gtrsim 1$, whereas local smoothing gives bounds for averages of the wave equation over times $|t| \lesssim 1$. We are able to reduce large times to small times by exploiting the *periodicity* of the half-wave equation on the sphere.

Lemma 11. *Fix $1 < p < 2d/(d+1)$, let q be the Hölder conjugate to p, and let $I = [-1/2, 1/2]$. Suppose that the sharp local smoothing inequality*

$$\left\| e^{2\pi i t P} f \right\|_{L^q(S^d) L^q_t(I)} \lesssim \|f\|_{L^q_{s_q - 1/q}(S^d)}$$

holds for all $f \in C^\infty(S^d)$. Then the operators $\{III_R\}$ satisfy a bound

$$\|(III_R \circ Q_R) f\|_{L^p(S^d)} \lesssim C_p(h) \|f\|_{L^p(S^d)},$$

with the implicit constant uniformly bounded in R. In particular,

$$\|(Q_R \circ III_R \circ Q_R) f\|_{L^p(S^d)} \lesssim C_p(h) \|f\|_{L^p(S^d)},$$

Proof. For each R, the *class* of operators of the form $\{III_R\}$ formed from a multiplier h satisfying the hypothesis of Theorem 8 is closed under taking adjoints. Indeed, if III_R is obtained from h, then III_R^* is obtained from the multiplier $\overline{h}$. Because of this self-adjointness, if we can prove that for any multiplier h satisfying the assumptions of the theorem, the operators $\{III_R\}$ are uniformly bounded in $L^q(S^d)$, where q is the Hölder conjugate to p, then it follows by duality that for any such h, it is also true that the operators $\{III_R\}$ are uniformly bounded back in the original $L^p(S^d)$ norm. In this argument we will prove such L^q estimates, because we will exploit *local smoothing* inequalities, which tend to work better with large Lebesgue exponents, precisely because Lebesgue norms with large exponents are more sensitive to functions with sharp peaks, something explicitly prevented by obtaining control over the smoothness of a function.

We begin by noting that for a pair of Hölder conjugates p and q, $s_q = s_p$. Using the periodicity of the wave equation on S^d, i.e. that

$$e^{2\pi i (t+n) P} = e^{2\pi i t P} \qquad \text{for } n \in \mathbb{Z} \text{ and } t \in \mathbb{R},$$

we can write

$$III_R = \int_{-1/2}^{1/2} H_R(t) e^{2\pi i t P} \, dt,$$

where

$$H_R(t) = \sum_l \chi_{III}(t) R \widehat{h}(R(t+l)) = \sum_l H_{R,l}(t).$$

Now

$$\begin{aligned}
&\left(\sum_{l \neq 0} \left(|Rl|^{s_q} \|H_{R,l}\|_{L^p[-1/2,1/2]} \right)^p \right)^{1/p} \\
&\quad \sim R \left(\int_{-1/2}^{1/2} \sum_l \left(|R(t+l)|^{s_q} |\widehat{h}(R(t+l))| \right)^p \, dt \right)^{1/p} \\
&\quad \sim R \left(\int_{|t| \geqslant 1/2} \left(|Rt|^{s_q} |\widehat{h}(Rt)| \right)^p \, dt \right)^{1/p} \\
&\quad \lesssim R^{1/q} C_p(h).
\end{aligned}$$

and

$$
\begin{aligned}
\|H_{R,0}\|_{L^p[-1/2,1/2]} &= \left(\int_{-1/2}^{1/2} |\chi_{III}(t) R\widehat{h}(Rt)|^p \right)^{1/p} \\
&\leqslant \left(\int_{1/200 \leqslant |t| \leqslant 1/2} |R\widehat{h}(Rt)|^p \right)^{1/p} \\
&= R^{1/q} \left(\int_{R/3}^{R/2} |\widehat{h}(t)|^p \right)^{1/p} \\
&\lesssim R^{1/q - s_q} C_p(h).
\end{aligned}
$$

Since the family of functions $\{H_{R,l}\}$ could in general be chosen arbitrarily, they can be quite correlated, and so we should expect Hölder's inequality should be efficient, in the worst case. Thus we conclude that, provided $p < 2d/(d+1)$, so that $q > 2d/(d-1)$, and thus

$$
qs_q = (d-1)(q/2 - 1) > 1,
$$

so we can use Hölder's inequality to conclude that

$$
\begin{aligned}
\|H_R\|_{L^p[-1/2,1/2]} &\leqslant \sum_l \|H_{R,l}\|_{L^p[-1/2,1/2]} \\
&= \|H_{R,0}\|_{L^p[-1/2,1/2]} + \sum_{l \neq 0} \left(|Rl|^{s_q} \|H_{R,l}\|_{L^p[-1/2,1/2]} \right) |Rl|^{-s_q} \\
&\lesssim R^{-s_q + 1/q} C_p(h) + (R^{1/q} C_p(h)) \left(\sum_{l \neq 0} |Rl|^{-s_q q} \right)^{1/q} \\
&= R^{-s_q + 1/q} C_p(h) \left(1 + \left(\sum_{l \neq 0} |l|^{-s_q q} \right)^{1/q} \right) \\
&= R^{-s_q + 1/q} C_p(h) \left(1 + \left(\sum_{l \neq 0} |l|^{-s_q q} \right)^{1/q} \right) \\
&\lesssim_p R^{-s_q + 1/q} C_p(h).
\end{aligned}
$$

A further application of Hölder's inequality shows that

$$
\begin{aligned}
|III_R| &= \left| \int_{-1/2}^{1/2} H_R(t) e^{2\pi i t P} \, dt \right| \\
&\lesssim C_p(h) R^{-s_q + 1/q} \left(\int_{-1/2}^{1/2} |e^{2\pi i P}|^q \, dt \right)^{1/q}.
\end{aligned}
$$

Applying the endpoint local smoothing inequality, we conclude that

$$
\begin{aligned}
\|(III_R \circ Q_R) f\|_{L^q(M)} &\lesssim C_p(h) R^{-s_q + 1/q} \|e^{2\pi i P}(Q_R f)\|_{L^q_t L^q_x} \\
&\lesssim C_p(h) R^{-s_q + 1/q} \|Q_R f\|_{L^q_{s_q - 1/q}(M)},
\end{aligned}
$$

Applying Bernstein's inequality gives

$$\|Q_R f\|_{L^q_{s_q-1/q}(M)} \lesssim R^{s_q-1/q}\|f\|_{L^q(M)}.$$

Thus we conclude that

$$\|(III_R \circ Q_R)f\|_{L^q(M)} \lesssim C_p(h)\|f\|_{L^q(M)}.$$

We have therefore bounded III_R uniformly in R. $\square$

Corollary 1.2 of [6] establishes that the sharp local smoothing inequality holds for $p < 2(d-1)/(d+1)$, which covers the range of parameters studied in this paper. Thus we have obtained uniform bounds on the operators $\{III_R\}$.

12 Analysis of II_R: Density Decompositions

It finally remains to analyze the operator $Q_R \circ II_R \circ Q_R$, where

$$II_R = \int \chi_{II}(t) R\widehat{h}(Rt) e^{2\pi i t P}\, dt$$

is obtained by integrating the wave propogators over times $100/R \leqslant |t| \leqslant 0.01$ respectively. To prevent notation from growing too cumbersome later on, let us eschew uses of the subscript R in our operators in this section, e.g. writing II_R as

$$II = \int b(t) e^{2\pi i t P}\, dt,$$

where $b(t) = \chi_{II}(t) R\widehat{h}(Rt)$. We then have

$$\|b(t)\langle t\rangle^{s_p}\|_{L^p(\mathbb{R})} \lesssim R^{1-1/p-s_p} C_p(h).$$

Bounding II requires a more subtle analysis of the geometric behaviour of the wave-propogator operators, and we will begin by converting our problem in coordinates on S^d, where the kernels have more explicit representations in oscillatory integrals.

We will employ some restricted weak type bounds, together with interpolation, to obtain L^p estimates on the operators $Q \circ II \circ Q$. We thus introduce a set $E \subset S^d$ and try to obtain $L^{p,\infty}$ bounds on the function $S = (Q \circ II_W \circ Q)\{E\}$. Given that Q already acts, heuristically, by localizing the behaviour of it's inputs to the frequency R, despite the choice of the set E, the uncertainty principle implies $Q\{E\}$ should be locally constant at a scale $1/R$, and so it is natural to discretize at this scale. Consider a maximal $1/2R$ separated subset $\mathcal{X}$ of S^d. Then break E down into a disjoint union of sets $\{E_{x_0} : x_0 \in \mathcal{X}\}$, where for $x_0 \in \mathcal{X}$, the set E_{x_0} is supported on the geodesic ball of radius $1/R$ about x_0. Similarily, let $\mathcal{T}$ be all points in the lattice $\mathbb{Z}/10R$ lying in the set $\{100/R \leqslant |t| \leqslant 1\}$, and write

$$b = \sum_{t \in \mathcal{T}} u(t) b_t,$$

where for each $t \in \mathcal{T}$, $u(t) = \|b\|_{L^\infty[t-10/R,t+10/R]}$, and b_t is a smooth function, compactly supported on the sidelength $1/R$ interval centered at t, satisfying

$$|\partial^\alpha b_t| \lesssim_\alpha R^{|\alpha|},$$

with implicit constants uniform in b and t. By the Plancherel-Polya theorem,

$$\|u(t)\langle t\rangle^{s_p}\|_{l^p(\mathcal{T})} \lesssim R^{1-s_p}.$$

We can then write

$$S = \sum |E_{x_0}| S_{x_0,t_0} \quad \text{where} \quad S_{x_0,t_0} = \int |E_{x_0}|^{-1} b_{t_0}(t)(Q \circ e^{2\pi i t P} \circ Q)\{E_{x_0}\}\, dt.$$

Our computation would be complete if we could show that for any coefficients $\{c(x_0,t_0) : x_0 \in \mathcal{X}, t_0 \in \mathcal{T}\}$,

$$\left\| \sum_{x_0,t_0} c(x_0,t_0) t_0^{\frac{d-1}{2}} S_{x_0,t_0} \right\|_{L^p(S^d)} \lesssim R^{s_p - 1 + d(1-1/p)} \left(\sum_{x_0,t_0} |c(x_0,t_0)|^p t_0^{d-1} \right)^{1/p}.$$

Indeed, we set $c(x_0,t_0) = |E_{x_0}| u(t_0) t_0^{-\frac{d-1}{2}}$ and apply Hölder's inequality, then the inequality above gives exactly that

$$\|S\|_{L^p(S^d)} \lesssim C_p(h)|E|^{1/p},$$

For $p = 1$, this follows from applying the triangle inequality, and using the pointwise estimates

$$|S_{x_0,t_0}(x)| \lesssim_M \frac{R^{d-1}}{(R d_g(x,x_0))^{\frac{d-1}{2}}} \Big\langle R\big|t_0 - d_g(x,x_0)\big| \Big\rangle^{-M}.$$

Applying interpolation, for $p > 1$ we need only prove a restricted weak type version of this inequality. In other words, we can restrict c to be the indicator function of a set $\mathcal{E}$, and take $L^{p,\infty}$ norms on the left hand side. If we write $\mathcal{E} = \bigcup_k \mathcal{E}_k$, where $\mathcal{E}_k$ is the set of $(x,t) \in \mathcal{E}$ with $|t| \sim 2^k/R$, then the inequality reads that

$$\left\| \sum_{k=1}^\infty 2^{k\frac{d-1}{2}} \sum_{(x_0,t_0)\in\mathcal{E}_k} S_{x_0,t_0} \right\|_{L^{p,\infty}(S^d)}^p \lesssim R^{(d-1)p-d} \left(\sum_{k=1}^\infty 2^{k(d-1)} \#\mathcal{E}_k \right).$$

This is equivalent to showing that for any $\lambda > 0$,

$$\left| \left\{ x : \left| \sum_k 2^{k\frac{d-1}{2}} S_{x_0,t_0}(x) \right| \geqslant \lambda \right\} \right| \lesssim \lambda^{-p} R^{(d-1)p-d} \sum_k 2^{k(d-1)} \#\mathcal{E}_k.$$

The case $\lambda \lesssim R^{d-1}$ follows from the L^1 boundedness we've already proved, so we may assume $\lambda \gtrsim R^{d-1}$ in the sequel.

To obtain this bound, we employ the method of density decompositions, introduced in [3]. Let

$$A = \left(\frac{\lambda}{R^{d-1}}\right)^{(d-1)(1-p/2)} \log\left(\frac{\lambda}{R^{d-1}}\right)^{O(1)}.$$

Then for each k, consider the collection $\mathcal{B}_k(\lambda)$ of all balls B with radius at most $2^k/R$ such that $\#\mathcal{E}_k \cap B \geqslant RA\text{rad}(B)$. Applying the Vitali covering lemma, we can find a disjoint family of balls $\{B_1, \dots, B_N\}$ in $\mathcal{B}_k$ such that the balls $\{B_1^*, \dots, B_N^*\}$ obtained by dilating the balls by 5 cover $\bigcup \mathcal{B}_k(\lambda)$. Then

$$\sum \text{rad}(B_j) \leqslant R^{-1} A^{-1} \#\mathcal{E}_k,$$

and the set $\widehat{\mathcal{E}}_k = \mathcal{E}_k - \bigcup \mathcal{B}_k(\lambda)$ has density type $(RA, 2^k/R)$. Then we conclude that, using the quasi-orthogonality estimates below,

$$\left\| \sum_k \sum_{(x_0,t_0) \in \widehat{\mathcal{E}}_k} 2^{k\frac{d-1}{2}} S_{x_0,t_0} \right\|_{L^2(S^d)}^2 \lesssim_p R^{d-2} \log(A) A^{\frac{2}{d-1}} \sum_k 2^{k(d-1)} \#\mathcal{E}_k.$$

Appling Chebyshev's inequality, and utilizing the choice of A above, we conclude that

$$\begin{aligned} \left| \left\{ x : \left| \sum_k \sum_{(x_0,t_0) \in \widehat{\mathcal{E}}_k} 2^{k\frac{d-1}{2}} S_{x_0,t_0}(x) \right| \geqslant \lambda/2 \right\} \right| &\lesssim R^{d-2} \log(A) A^{\frac{2}{d-1}} \sum_k 2^{k(d-1)} \#\mathcal{E}_k \\ &\lesssim \lambda^{-p} R^{(d-1)p-d} \sum_k 2^{k(d-1)} \#\mathcal{E}_k. \end{aligned}$$

Conversely, we exploit the clustering of the sets $\mathcal{E}_k - \widehat{\mathcal{E}}_k$ to bound

$$\left| \left\{ x : \left| \sum_k \sum_{(x_0,t_0) \in \mathcal{E}_k - \widehat{\mathcal{E}}_k} 2^{k\frac{d-1}{2}} S_{x_0,t_0}(x) \right| \geqslant \lambda/2 \right\} \right|$$

That is, we have found balls $B_1^* < \dots, B_N^*$, each with radius $O(2^k/R)$, such that

$$\sum \text{rad}(B_j) \leqslant R^{-1} A^{-1} \#\mathcal{E}_k.$$

Let (x_j, t_j) denote the center of the ball B_j. Then the function

$$\sum_{(x_0,t_0) \in B_j} S_{x_0,t_0}$$

has mass concentrated on the geodesic annulus $\text{Ann}_j \subset S^d$ with radius t_j and thickness $O(\text{rad}(B_j))$, a set with measure $(2^k/R)^{d-1}\text{rad}(B_j)$. For $(x_0, t_0) \in B_j$, we calculate using the pointwise bounds that

$$\begin{aligned} \int_{\text{Ann}_j^c} |S_{x_0,t_0}(x)| \, dx &\lesssim R^{d-1} \int_{\text{rad}(B_j) \lesssim |t_j - d_g(x,x_0)| \leqslant 1} \langle R|t_0 - d_g(x, x_0)| \rangle^{-M} \\ &\lesssim R^{d-1} \int_{\text{rad}(B_j) \leqslant |t_j - s| \leqslant 1} s^{d-1} \langle R|t_0 - s| \rangle^{-M} \, ds \\ &\lesssim 2^{k(d-1)} R^{d-1} (R\text{rad}(B_j))^{-M}. \end{aligned}$$

Because the set of points in $\mathcal{E}_k$ is $1/R$ separated, there can only be at most $O(R\text{rad}(B_j))^{d+1}$ values of (x_0, t_0), and so applying the triangle inequality gives that the sum of the L^1 norm outside of Ann_j is

$$\lesssim 2^{k(d-1)} R^{d-1} (R\,\text{rad}(B_j))^{d+1-M}$$

Note that since $\#\mathcal{E}_k \cap B_j \geqslant RA\text{rad}(B_j)$, and because $\mathcal{E}_k$ is $1/R$ discretized,

$$\text{rad}(B_j) \geqslant (A/R)^{\frac{1}{d-1}},$$

and this, together with Markov's inequality, is enough to justify the required bound. Conversely, since $1 < p < 2(d-1)/(d+1)$, we have

$$\begin{aligned}\sum |\text{Ann}_j| \lesssim (2^k/R)^{d-1} \sum_j \text{rad}(B_j) & \\ &\lesssim (2^k/R)^{d-1} R^{-1} (L/R^{d-1})^{-(d-1)(1-p/2)} log(L/R^{d-1})^{O(1)} \\ &\lesssim \lambda^{-p} R^{(d-1)p-d} 2^{k(d-1)} \#\mathcal{E}_k,\end{aligned}$$

Summing over k completes the analysis.

13 Analysis of II_R: Quasi-Orthogonality

Our first goal will be to understand how orthogonal the functions $\{S_{x_0,t_0}\}$ are to one another, which will give L^2 estimates for S, that can be interpolated with L^1 estimates to obtain the required L^p estimates. The rest of this section will be devoted to proving the following inner product estimate, which, together with a density decomposition argument, introduced in [3], can be used to obtain L^2 estimates, which we can then interpolate to obtain L^p estimates for the function S.

Lemma 12.

$$|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim_M \frac{R^{d-2}}{(Rd_g(x_0,x_1))^{\frac{d-1}{2}}} \sum_{\pm} \Big\langle R\big|(t_0-t_1) \pm d_g(x_1,x_0)\big| \Big\rangle^{-M}.$$

Let us proceed with the proof. To begin with, we can use the self-adjointness of the operators Q, the semigroup structure of $\{e^{2\pi i tP}\}$, and the fact that multipliers commute, to write

$$\begin{aligned}\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle &= \int \frac{b_{t_0}(t)\overline{b_{t_1}(s)}}{|E_{x_0}||E_{x_1}|} \Big\langle (Q\circ e^{2\pi i tP}\circ Q)\{E_{x_0}\}, (Q\circ e^{2\pi i sP}\circ Q)\{E_{x_1}\}\Big\rangle\, dt\, ds \\ &= \int \frac{b_{t_0}(t)\overline{b_{t_1}(s)}}{|E_{x_0}||E_{x_1}|} \Big\langle (Q^2\circ e^{2\pi i(t-s)P}\circ Q^2)\{E_{x_0}\}, E_{x_1}\Big\rangle \\ &= \int \frac{c_{t_0,t_1}(t)}{|E_{x_0}||E_{x_1}|}\Big\langle (Q^2\circ e^{2\pi i tP}\circ Q^2)\{E_{x_0}\}, E_{x_1}\Big\rangle,\end{aligned}$$

where

$$c_{t_0,t_1}(t) = \int b_{t_0}(u)\overline{b_{t_1}}(u-t)\, dt,$$

is the convolution of b_{t_0} with the reflection of $\overline{b_{t_1}}$ about the y-axis. Thus c_{t_0,t_1} is supported on the length $2/R$ interval centered at $t_0 - t_1$, and has L^1 norm $O(1/R^2)$ by Young's convolution inequality.

We next perform a decomposition of the inner product into various coordinate systems. Cover S^d by a finite family of sets $\{V_\alpha\}$, chosen such that for each V_α, there is a coordinate chart U_α such that the neighbourhood $N(V_\alpha, 0.5)$ is contained in U_α. Let $\{\eta_\alpha\}$ be a partition of unity subordinate to $\{V_\alpha\}$. It will also be convenient to define $V_\alpha^* = N(V_\alpha, 0.1)$. We can then write

$$\langle S_{t_0,x_0}, S_{t_1,x_1} \rangle = \sum_\alpha \int \frac{c_{t_0,t_1}(t)}{|E_{x_0}||E_{x_1}|} \langle (Q^2 \circ e^{2\pi itP} \circ Q^2)\{\eta_\alpha E_{x_0}\}, E_{x_0} \rangle \; dt.$$

We will bound each of the terms on the right separately from one another, by working with each inner product in the coordinate systems $\{U_\alpha\}$.

The next Lemma allows us to approximate the operator Q, and the propogators $e^{2\pi itP}$, with operators which have more explicit representations in the coordinate system U_α, by an error term which is negligible to our analysis. It utilizes the *Lax-Hörmander parametric* for the half-wave equation over small times, which expresses $e^{2\pi itP}$ in coordinates as a Fourier integral operator.

Lemma 13. *For each α, and $|t| \leqslant 1/100$, there exists Schwartz operators Q_α and $W_\alpha(t)$, each with kernel supported on $U_\alpha \times V_\alpha^*$, such that the following properties hold:*

- *For $f \in C^\infty(S^d)$ with supp$(f) \subset V_\alpha^*$,*

  $$\mathit{supp}(Q_\alpha f) \subset N(\mathit{supp}(f), 0.1) \quad \mathit{and} \quad \mathit{supp}(W_\alpha(t) f) \subset N(\mathit{supp}(f), 0.1).$$

  Moreover,

  $$\|(Q - Q_\alpha) f\|_{L^\infty(M)} \lesssim_N R^{-N} \|f\|_{L^1(M)}$$

  and

  $$\left\| \left(Q_\alpha \circ \left(e^{2\pi itP} - W_\alpha(t) \right) \circ Q_\alpha \right) f \right\|_{L^\infty(M)} \lesssim_N R^{-N} \|f\|_{L^1(M)}.$$

- *In the coordinate system of U_α, Q_α is a pseudodifferential operator of order zero given by a symbol $\sigma(x,\xi)$, where*

  $$\mathit{supp}(\sigma) \subset \{|\xi| \sim R\},$$

  and σ satisfies derivative estimates of the form

  $$|\partial_x^\beta \partial_\xi^\kappa \sigma(x,\xi)| \lesssim_{\beta,\kappa} R^{-|\kappa|}.$$

- *In the coordinate system U_α, the operator $W_\alpha(t)$ has a kernel $W_\alpha(t,x,y)$ with an oscillatory integral representation*

  $$W_\alpha(t,x,y) = \int s(t,x,y,\xi) e^{2\pi i[\phi(x,y,\xi) + t|\xi|_y]} \; d\xi,$$

where s has compact support in it's x and y coordinates, with

$$supp(s) \subset \{|\xi| \sim R\},$$

where s satisfies derivative estimates of the form

$$|\partial_{t,x,y}^{\beta}\partial_{\xi}^{\kappa} s| \lesssim_{\beta,\kappa} R^{-|\kappa|},$$

and where $|\cdot|_y$ denotes the norm on $\mathbb{R}^n_\xi$ induced by the Riemannian metric on S^d on the contangent space $T_y^* S^d$.

Thus, ignoring errors negligible to our analysis, we need only analyze

$$\left| \int \frac{c_{t_0,t_1}(t)}{|E_{x_0}||E_{x_1}|} \left\langle (Q_\alpha \circ W_\alpha(t) \circ Q_\alpha)\{\phi_\alpha E_{x_0}\}, E_{x_1} \right\rangle du \right|.$$

The behaviour of all operations in this expression are now completely localized to U_α for inputs supported on V_α^*; in particular, this expression is equal to zero unless E_{x_0} and E_{x_1} are both compactly contained in U_α, So we can now naturally work with the kernels of the operators in coordinates to upper bound the inner product, which will complete the required estimate of the inner product.

Lemma 14. *Let c be an integrable function supported on the length $10/R$ interval centered at a value t^* with $|t| \leqslant 1/100$. Then*

$$\begin{aligned} &\left| \int \frac{c(t)}{|E_{x_0}||E_{x_1}|} \left\langle (Q_\alpha \circ W_\alpha(t) \circ Q_\alpha)\{\phi_\alpha E_{x_0}\}, E_{x_1} \right\rangle dt \right| \\ &\qquad \lesssim_M R^d \frac{\|c\|_{L^1(\mathbb{R})}}{(R d_g(x_1,x_0))^{\frac{d-1}{2}}} \sum_{\pm} \left\langle R \left| t^* \pm d_g(x_1,x_0) \right| \right\rangle^{-M} \end{aligned}$$

Proof. We write the integral as

$$\begin{aligned} \int \frac{c(t)}{|E_{x_0}||E_{x_1}|} &(\eta_\alpha E_{x_1})(w)\sigma(w,\theta)e^{2\pi i\theta\cdot(w-x)} \\ &s(t,x,y,\xi)e^{2\pi i[\phi(x,y,\xi)+t|\xi|_y]}\sigma(y,\eta)e^{2\pi i\eta\cdot(y-z)}E_{x_0}(z) \\ &\quad dt\; dx\; dy\; dz\; dw\; d\theta\; d\xi\; d\eta. \end{aligned}$$

The integral looks complicated, but can be simplified considerably by noticing that all the spatial variables are highly localized. To begin with, we use the fact that s is smooth and compactly supported in all it's variables, so s should roughly behave like a linear combination of tensor products; using Fourier series, we can write

$$s(t,x,y,\xi) = \sum_{n \in \mathbb{Z}^d} s_{n,1}(x) s_{n,2}(t,y,\xi),$$

where $s_{n,1}(x) = e^{2\pi i n\cdot x}$, and where

$$|\partial_{t,y}^{\alpha}\partial_{\xi}^{\kappa}\{s_{n,2}\}| \lesssim_{\alpha,\kappa,N} |n|^{-N} R^{-|\kappa|}$$

If we define $a_n(\xi) = a_{n,1}(R\xi)a_{n,2}(R\xi)$, where

$$a_{n,1}(\xi) = |E_{x_1}|^{-1} \int (\eta_\alpha E_{x_1})(w)\sigma(w,\theta)s_{n,1}(x)e^{2\pi i[\theta\cdot(w-x)+(x-x_1)\cdot\xi]}\; d\theta\; dw\; dx$$

and

$$a_{n,2}(\xi) = |E_{x_0}|^{-1} \int c(t)s_{n,2}(t,y,\xi)\sigma(y,\zeta)E_{x_0}(z)e^{2\pi i[(\phi(t^*,x_0,\xi)-\phi(t,y,\xi))+\zeta\cdot(y-z)]}\; d\zeta\; dt\; dy\; dz,$$

then, rescaling, we can write the required integral as

$$R^d \sum_{n\in\mathbb{Z}^d} \int a_n(\xi)e^{2\pi iR[\phi(x_1,x_0,\xi)+t^*|\xi|_{x_0}]}\; d\xi.$$

Notice that $\operatorname{supp}(a_n) \subset \{|\xi| \sim 1\}$, and

$$|(\nabla_\xi^\kappa a_n)(\xi)| \lesssim_{\kappa,N} |n|^{-N}\|c\|_{L^1(\mathbb{R})}.$$

To obtain an efficient upper bound on this oscillatory integral, it will be convenient to change coordinate systems in a way better respecting the Riemannian metric at x_0, i.e. finding a smooth family of diffeomorphisms $\{F_{x_0} : S^{d-1} \to S^{d-1}\}$ such that $|F_{x_0}|_{x_0} = 1$. We can choose this function such that $F_{x_0}(-x) = -F_{x_0}(x)$. Then if $\tilde{a}_n(\rho,\eta) = a_n(\rho F_{x_0}(\eta))JF_{x_0}(\eta)$, then a change of variables gives that

$$\int a_n(\xi)e^{2\pi iR[\phi(x_1,x_0,\xi)+t^*|\xi|_{x_0}]} = \int_0^\infty \rho^{d-1}\int_{|\eta|=1} \tilde{a}_n(\rho,\eta)e^{2\pi iR\rho[\phi(x_1,x_0,F_{x_0}(\eta))+t^*]}\; d\eta\; d\rho.$$

For each fixed ρ, we claim that the phase has exactly two stationary points in the η variable, at the values $\pm\eta_0$, where x_1 lies on the geodesic passing through x_0 tangent to the vector $\eta_0^\sharp$ ($\eta_0^\sharp$ denotes the tangent vector corresponding to the cotangent vector η_0 using the musical isomorphism on the Riemannian manifold M). Moreover, at these values,

$$\phi(x_1,x_0,F_{x_0}(\pm\eta_0)) = \pm d_g(x_1,x_0),$$

and the Hessian at $\pm\eta_0$ is (positive / negative) definite, with each eigenvalue having magnitude exceeding a constant multiple of $d_g(x_1,x_0)$. It follows from the principle of stationary phase, that the integral above can be written as

$$\frac{1}{(Rd_g(x_1,x_0))^{\frac{d-1}{2}}} \sum_\pm \int_0^\infty \rho^{\frac{d-1}{2}} f_{n,\pm}(\rho)e^{2\pi iR\rho[t^*\pm d_g(x_1,x_0)]}\; d\rho,$$

where $f_{n,\pm}$ is supported on $|\rho| \sim 1$, and

$$|\partial_\rho^m f_{n,\pm}| \lesssim_{m,N} |n|^{-N}\|c\|_{L^1(\mathbb{R})}.$$

Integrating by parts in the ρ variable if $\pm d_g(x_1,x_0) + t^*$ is large, and then taking in absolute values, we conclude that

$$\left|\int a_n(\xi)e^{2\pi iR[\phi(x_1,x_0,\xi)+t^*|\xi|_{x_0}]}\right| \lesssim_{N,M} |n|^{-N}\frac{\|c\|_{L^1(\mathbb{R})}}{(Rd_g(x_1,x_0))^{\frac{d-1}{2}}} \sum_\pm \langle R|t^* \pm d_g(x_1,x_0)|\rangle^{-M}.$$

Taking $N \geqslant d+1$, and summing in the n variable, we conclude that

$$\left|\sum_n \int a_n(\xi) e^{2\pi i R[\phi(x_1,x_0,\xi)+t'|\xi|_{x_0}]}\right| \lesssim_M \frac{\|c\|_{L^1(\mathbb{R})}}{(R d_g(x_1,x_0))^{\frac{d-1}{2}}} \sum_{\pm} \langle R|t^* \pm d_g(x_1,x_0)|\rangle^{-M}.$$

But this is precisely an estimate for the quantity we wished to estimate. □

Now applying this Lemma with $c = c_{t_0,t_1}$, and then summing in α, we complete the proof of Lemma 12.

14 Analysis of II_R: L^2 Estimates

Lemma 12 of the last section implies two functions S_{x_0,t_0} and S_{x_1,t_1} can only be correlated in L^2 if $d_g(x_0,x_1) \approx |t_0 - t_1|$. We now exploit this geometry to obtain some L^2 estimates for sums of the functions S_{x_0,t_0}.

Lemma 15. *Fix $u \geqslant 1$. Consider a set $\mathcal{E} \subset \mathcal{X} \times \mathcal{T}$. Write*

$$\mathcal{E} = \bigcup_{k=0}^{\infty} \mathcal{E}_k,$$

where $\mathcal{E}_k = \{(x,t) \in \mathcal{E} : |t| \sim 2^k/R\}$. Suppose that each of the sets $\mathcal{E}_k$ has density type $(Ru, 2^k/R)$, i.e. for any set $B \subset \mathcal{X} \times \mathcal{T}$ with diam$(B) \leqslant 2^k/R$,

$$\#(\mathcal{E}_k \cap B) \leqslant Ru \; diam(B).$$

Then

$$\left\| \sum_k \sum_{(x_0,t_0)\in\mathcal{E}_k} 2^{k\frac{d-1}{2}} S_{x_0,t_0} \right\|_{L^2(S^d)}^2 \lesssim R^{d-2} \log_2(u) u^{\frac{2}{d-1}} \sum_k 2^{k(d-1)} \#\mathcal{E}_k.$$

15 Combining Dyadic Pieces

Consider a general function $h : (0,\infty) \to \mathbb{C}$, and suppose that we can write

$$h(\lambda) = \sum_{k\in\mathbb{Z}} h_k(\lambda/2^k),$$

where h_k has support on $\{|\lambda| \sim 1\}$, and for each k,

$$\left(\int_0^\infty \left[\langle t\rangle^{s_p} |\widehat{h}_k(t)| \right]^p \, dt \right)^{1/p} \leqslant C_p(h),$$

Then

$$\widehat{h}(t) = \sum (2^k R) \widehat{h}_k(2^k R t),$$

and we have can write

$$h(P/R) = \sum_k \int (2^k R)\widehat{h}_k(2^k Rt)e^{2\pi i t P}\, dt.$$

If $b_{k,0}(t) = (2^k R)\widehat{h}_k(2^k Rt)\chi(t)$,

$$\left\|\langle 2^k Rt\rangle^{s_p} b_{k,0}\right\|_{L^p[-1/2,1/2]} = (2^k R)^{1-1/p}\|\langle t\rangle^{s_p}\widehat{h}_k(t)\| = (2^k R)^{1-1/p}C_p(h).$$

If $b_{k,n}(t) = (2^k R)\widehat{h}_k(2^k R(t+n))\chi(t)$ then

$$\begin{aligned}
\left\|\langle 2^k Rt\rangle^{s_p}\sum_{n\neq 0} b_{k,n}\right\|_{L^p} &= \left\|\langle 2^k Rt\rangle^{s_p}\left(\sum_{n\neq 0}|2^k Rn|^{-p's_p}\right)^{1/p'}\left(\sum_{n\neq 0}|2^k Rn|^{ps_p}|b_{k,n}|^p\right)^{1/p}\right\|_{L^p} \\
&= \left\|\langle 2^k Rt\rangle^{s_p}(2^k R)^{-s_p}\left(\sum_{n\neq 0}|2^k Rn|^{ps_p}|b_{k,n}|^p\right)^{1/p}\right\|_{L^p} \\
&= (2^k R)^{1-1/p}C_p(h).
\end{aligned}$$

Thus we should expect just have to deal with small times?

$$\begin{aligned}
&\left(\int_{-1/2}^{1/2}\left|\langle 2^k Rt\rangle^{s_p}\sum_n (2^k R)\widehat{h}_k(2^k R(t+n))\right|^p\, dt\right)^{1/p} \\
&= (2^k R)^{1-1/p}\left(\int_{-2^k R/2}^{2^k R/2}\left|\langle t\rangle^{s_p}\sum_n \widehat{h}_k(t+2^k Rn)\right|^p\, dt\right)^{1/p} \\
&\lesssim (2^k R)^{1-1/p}\left(\int\left|\langle t\rangle^{s_p}|\widehat{h}_k(t)|\right|^p + \left|\langle t\rangle^{s_p}\left(\sum_{n\neq 0}(2^k Rn)^{-s_p p'}\right)^{1/p'}\left(\sum_{n\neq 0}(2^k Rn)^{s_p p}|\widehat{h}_k(t+2^k Rn)|^p\right)^{1/p}\right|^p\right)^{1/p} \\
&\lesssim (2^k R)^{1-1/p}\int\left(\sum_n \langle t+2^k Rn\rangle^{ps_p}|\widehat{h}_k(t+2^k Rn)|^p\right)^{1/p} \\
&\lesssim C_p(h)(2^k R)^{1-1/p}
\end{aligned}$$

Thus

$$T = \sum T_s$$

where

$$T_s = \int_{-1/2}^{1/2}(2^s R)\sum_n \widehat{h}_s(2^s R(t+n))e^{2\pi i t P}.$$

so that, by the results of the previous sections,

$$\|h_k(P/R)f\|_{L^p(M)} \lesssim C_p(h)\|f\|_{L^p(M)} \quad \text{uniformly in } R>0,$$

where the implicit constants are also independent of the functions $\{h_k\}$. Now

$$H_{k,R}(t) = \sum_l \chi_{III}(t)(2^k R)\widehat{h}_k(2^k R(t+l)) = \sum_l H_{k,R,l}(t).$$

Then we conclude that

$$\left(\sum_{l \neq 0} \left(|2^k Rl|^{s_q} \|H_{R,k,l}\|_{L^p[-1/2,1/2]}^p \right) \right)^{1/p} \lesssim (2^k R)^{1/q} C_p(h)$$

and

$$\|H_{R,k,0}\|_{L^p[-1/2,1/2]}^p \lesssim (2^k R)^{1/q - s_q} C_p(h),$$

so that by Hölder

$$\|H_{k,R}\|_{L^p[-1/2,1/2]} \lesssim (2^k R)^{-s_q + 1/q} C_p(h).$$

Thus by the local smoothing inequality,

$$\| \int H_{k,R}(t) e^{2\pi i t P} \{ \chi(P/R2^k) f \} \|_{L^p(M)} \lesssim C_p(h) \| \chi(P/R2^k) f \|_{L^p(M)}$$

Now

$$\| \sum_k H_{k,R} \chi(P/R2^k) f \|_{L^p(M)} \|$$

Let us write

$$h_s = h_{s,\mathrm{Low}} + h_{s,\mathrm{High}},$$

where $\widehat{h}_{s,\mathrm{High}}$ is supported on $|t| \geqslant T$. Then

$$h_{s,\mathrm{High}}(P/R)$$

$$H_{s,R}(t) = \sum \chi_{III}(t) R \widehat{h}_s(Rt)$$

$$h_s(\cdot/R2^s)$$

$$k_s(t) = (R2^s)\widehat{h}_s(2^s \cdot)$$

$$H_{s,R}(t) = \sum \chi_{III}(t)(R2^s)\widehat{h}_s(2^s t)$$

Then

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