The Lax Parametrix for the Half Wave Equation

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In this talk, we consider a motivating example that gave rise to much of the general theory of Fourier integral operators: the study of variable-coefficient wave equations. This show a precise example of how Fourier integral operators can be used as a tool to generalize the tools of harmonic analysis we normally use to analyze constant coefficient differential operators like the wave equation, and apply them to variable coefficient analogues.

1 Euclidean Half-Wave Propogators

Let's start with a quick review. Consider a solution $u(x,t)$ to the wave equation

$$(\partial_t^2 - \Delta)u = 0$$

on $\mathbb{R}^d$. We take Fourier transforms on both sides; if $\widehat{u}(\xi,t)$ denotes the Fourier transform of u in the x-variable, then we conclude that

$$(\partial_t^2 + 4\pi^2|\xi|^2) \cdot \widehat{u}(\xi,t) = 0.$$

This is an ordinary differential equation in the t variable for each fixed ξ, which we can solve, given that $u(x,0) = f(x)$, and $\partial_t u(x,0) = g(x)$, to write

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)\cos(2\pi t|\xi|) + \widehat{g}(\xi)\frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}.$$

It's often easier to see what's going on if we work with complex exponentials, so we rewrite this as

$$\left(\frac{\widehat{f}(\xi) + (2\pi|\xi|)^{-1}\widehat{g}(\xi)}{2}\right)e^{2\pi it|\xi|} + \left(\frac{\widehat{f}(\xi) - (2\pi|\xi|)^{-1}\widehat{g}(\xi)}{2i}\right)e^{-2\pi it|\xi|}.$$

If we write $u = v + w$, where

$$\widehat{v}(\xi,t) = \left(\frac{\widehat{f}(\xi) + \widehat{g}(\xi)}{2}\right)e^{2\pi it|\xi|} \quad \text{and} \quad \widehat{w}(\xi,t) = \left(\frac{\widehat{f}(\xi) - i(2\pi|\xi|)^{-1}\widehat{g}(\xi)}{2}\right)e^{-2\pi it|\xi|},$$

then we have decomposed u into the sum of solutions to the *half-wave equations*

$$\left(\partial_t - i\sqrt{-\Delta}\right)v = 0 \quad \text{and} \quad \left(\partial_t + i\sqrt{-\Delta}\right)w = 0.$$

The two operators $\partial_t - i\sqrt{-\Delta}$ and $\partial_t + i\sqrt{-\Delta}$ are identical, up to a time-reversal symmetry, so we focus on solutions to the equation $\partial_t - i\sqrt{-\Delta} = 0$. Solutions to the half-wave equation behave similarly to solutions to the wave equation, with one notable exception: the wave equation has a finite speed of propogation, whereas the half-wave equation does not.

The fact that the half-wave equation $(\partial_t - i\sqrt{-\Delta})u = 0$ is a first-order operator makes the Cauchy problem somewhat simpler to study, since we need less initial data than in the wave equation. We can therefore define operators $e^{it\sqrt{-\Delta}}$ such that

$$v(x,t) = (e^{it\sqrt{-\Delta}}v_0)(x).$$

Let's analyze the solution operator

$$Sf(x,t) = (e^{it\sqrt{-\Delta}}f)(x)$$

to the half-wave equation, from the perspective of FIO theory.

We can write

$$\begin{aligned} Sf(x,t) &= e^{2\pi it\sqrt{-\Delta}}f(x) \\ &= \int e^{2\pi i(t|\xi| + \xi\cdot(x-y))} f(y)\, dy\, d\xi \\ &= \int a(x,t,y,\xi) e^{2\pi i\phi(x,t,y,\xi)} f(y)\, dy\, d\xi, \end{aligned}$$

where

$$a(x,t,y,\xi) = 1$$

is a symbol of order zero, and the phase

$$\phi(x,t,y,\xi) = t|\xi| + \xi\cdot(x-y)$$

is non-degenerate. Thus S is a Fourier integral operator from $\mathbb{R}^d$ to $\mathbb{R}^d \times \mathbb{R}$, of order $-1/4$, with canonical relation defined by the three equations

$$\left\{ x = y + t\frac{\xi}{|\xi|} \text{ and } \xi = \eta \text{ and } \tau = |\xi| \right\}.$$

In this talk, we will construct approximate solutions (parametrices) for variable-coefficient analogues of the half-wave and wave equations using Fourier integral operators.

2 Variable-Coefficient Half-Wave Equations

A natural variable-coefficient extension of the wave equation $\partial_t^2 - \Delta = 0$ is an equation of the form $\partial_t^2 - Q = 0$, where Q is a pseudo-differential operator of order two, which is *formally positive*[1], *elliptic*[2], and classical [3]. For those not

[1] A Schwartz operator Q is formally positive if for any $f \in C_c^\infty(\mathbb{R}^d)$, $\langle Qf, f\rangle \geqslant 0$.

[2] A ΨDO $Q(x,D)$ is elliptic if it's principal symbol $q(x,\xi)$ satisfies $q(x,\xi) > 0$ for $\xi \neq 0$.

[3] A symbol $Q(x,\xi)$ of order μ is classical if we have an asymptotic expansion $Q \sim \sum q_{\mu-j}$, where q_k is a smooth, homogeneous function of order k.

too comfortable with pseudodifferential operators, any formally positive, elliptic second order differential operator Q fits these assumptions. As a non-trivial example, consider a Riemannian metric g on $\mathbb{R}^d$, introduce the Laplace-Beltrami operator

$$\Delta_g f = |g|^{-1/2} \sum_j \frac{\partial}{\partial x_j} \left\{ |g|^{1/2} \sum_k g^{jk} \partial_k f \right\},$$

and set $Q = -\Delta_g$. If we discard the first order terms of this operator, we find that the second order terms of Δ_g is

$$\sum_{j,k} g^{jk} \partial_j \partial_k.$$

Thus the principal symbol of Q is

$$q(x,\xi) = -\sum_{j,k} g^{jk} (2\pi i \xi_j)(2\pi i \xi_k) = 4\pi^2 |\xi|_g^2,$$

which is immediately verified to be elliptic.

We will not analyze these wave equations directly. If we define $P = Q^{1/2}$, well defined by the calculus of pseudodifferential operators, then P is a formally positive, classical pseudodifferential operator of order one. Using similar tricks to the wave equation, we can split solutions to the wave equation $\partial_t^2 - Q$ into the sum of two solutions to the *half-wave equations* $\partial_t - iP = 0$ and $\partial_t + iP = 0$. By time symmetry, we can focus solely on studying the solutions to the operator

$$L = \partial_t - iP,$$

where P is a first order *pseudodifferential operator*. The pseudodifferential operator P has principal symbol $p(x,\xi) = q(x,\xi)^{1/2}$; in particular, the operator $P = \sqrt{-\Delta_g}$ has principal symbol $p(x,\xi) = 2\pi|\xi|_g$.

We are not interested in the study of the existence or uniqueness of solutions to this PDE, but the problem of *regularity*, e.g. the mapping properties of the solution and propogator operators in L^p norms or Sobolev spaces. We thus refer to the literature on hyperbolic equations, which states that, given the assumptions above, for any compact set $K \subset \mathbb{R}^d$, there exists $\varepsilon > 0$ such that smooth solutions

$$u : \mathbb{R}^d \times [-\varepsilon, +\varepsilon] \to \mathbb{R}$$

to the half-wave equation exist, and (via energy type arguments) are the unique such solutions in $L_t^\infty L_x^2$ to solve the wave equation with some initial condition given by a smooth, compactly supported functions on K. We are interested in finding integral expressions which give approximate expressions for u, which can be used as a tool to study the regularity of solutions to the half-wave equation.

By an integral expression 'approximating' solutions to the half-wave equation, we mean finding a *parametrix* A for the solution operator S to the Cauchy problem $Lu = 0$ which has a good integral expression (as a Fourier integral operator). A parametrix is an operator A such that $R = A - S$ is a *smoothing operator*, i.e. a Schwartz operator whose kernel is a smooth function. The mapping properties of the operator S with respect to Sobolev norms then immediately reduces to the mapping properties of the operator A, because the operator R has trivial mapping properties. For example, if u is a compactly supported distribution, then Ru is a smooth function, and moreover, R maps any compactly supported function in one Sobolev space continuously into a function locally lying in *any* other Sobolev space, i.e. mapping H^s_c continuously into $H^{s_1}_{x,\text{loc}} H^{s_2}_{t,\text{loc}}$ for any three parameters s, s_1, and s_2.

The reason parametrices arise is that in many variable coefficient problems, it is often possible to find operators A which can be expressed in a simple manner, whereas none may exist for S. This in particular arises from the perspective of harmonic analysis, since it is often the case that we can find good approximations to solutions to partial differential equations for *high frequency data*, but such that these approximations tend not to work so well for *low frequency data*. From the perspective of parametrices, this is not a problem since low frequency data is automatically smooth, and thus does not need to be approximated as well as high frequency data, which must be approximated to an extent good enough that the approximation differs from the true solution by something smooth.

Our proof technique will show that solutions to the half-wave equation induced by initial data given by a high-frequency wave packet, localized in space near a point x_0, and oscillating at a frequency ξ_0, where $|\xi_0|$ is large, will at each time t, look like a wave packet localized near a point $x(t)$ and oscillating a frequency $\xi(t)$, where x and ξ are functions solving the *Hamiltonian system*

$$\frac{dx}{dt} = -(\nabla_\xi p)(x,\xi) \quad \frac{d\xi}{dt} = (\nabla_x p)(x,\xi).$$

In particular, we will find that the *singularities* of the kernel to the solution operator to the half-wave equation are supported on the integral curves of this system. This formalizes a number of heuristics used by physicists well before they were studied by mathematicians, namely, that visible light (a high-frequency wave relative to the scale of the human eye) travels according to Fermat's principle of least time, and that high energy particles in quantum mechanics whose state is given by a wave packet behave like classical particles, moving through space according to the principles of classical mechanics. The methods we use here to construct asymptotics for the half-wave equation are thus descendents of the *WKB methods* of quantum physicists of the 1920s, and going even further back, descendents of the analytical methods of geometric optics discovered by Fresnel and Airy in the 1800s.

3 High-Frequency Asymptotic Solutions

Fix $x_0 \in K$, as well as three quantities $0 < r < R$, and $\varepsilon > 0$, to be specified later. Our goal is to find a general family of 'high-frequency asymptotic solutions' to the half-wave equation, supported on the ball $B_R(x_0) = \{x : |x - x_0| \leqslant R\}$, for $|t| \leqslant \varepsilon$, given some initial conditions supported on the smaller ball $B_r(x_0)$.

Let us describe what we mean by 'high-frequency asymptotic solutions'. Fix an expression of the form

$$u_\lambda(x,t) = e^{2\pi i \lambda \phi(x,t)} a(x,t,\lambda),$$

where a is a classical symbol of order zero in the λ variable, defined for $|t| \leqslant \varepsilon$, and with $\operatorname{supp}_x(u_\lambda) \subset B_R(x_0)$, and where ϕ is a smooth, real-valued function, such that $\nabla_x \phi(x,t) \neq 0$ on the support of a. This latter condition is necessary to interpret u_λ as a function 'oscillating at a magnitude λ'. Indeed, if the condition is true, the principle of nonstationary phase shows that the Fourier transform of u_λ rapidly decays outside the annulus of frequencies $|\xi| \sim \lambda$. As $\lambda \to \infty$, the solution u_λ thus begins to oscillate more and more rapidly.

In a lemma shortly following this discussion, we will show that for any choice of a and ϕ as above, there exists a classical symbol b of order 1 such that

$$Lu_\lambda(x,t) = e^{2\pi i \lambda \phi(x,t)} b(x,t,\lambda).$$

For *some* choices of a and ϕ, it might be true that the higher order parts of b are eliminated, i.e. so that b is of order much smaller than 1. If a and ϕ are chosen in a *very* particular way, it might be true that all finite order parts of b are eliminated, so that b is a symbol of order $-\infty$. In such a situation, we say $\{u_\lambda\}$ is a '*high-frequency asymptotic solution*' to the wave equation. If this is the case, then

$$|\partial_x^\alpha \partial_t^\beta \{Lu_\lambda\}| \lesssim_{\alpha,\beta,N} \lambda^{-N} \quad \text{for all } N > 0,$$

which justifies that u_λ behaves like a solution to the half-wave equation as $\lambda \to \infty$. We will prove the following 'Cauchy' initial value problem for high-frequency asymptotic solutions to the equation, given that our phase satisfies an *eikonal equation*.

Theorem 1. *Fix (x_0, ξ_0), and suppose φ is a smooth-real valued function on $B_R(x_0)$, solving the eikonal equation*

$$p(x, \nabla_x \varphi(x)) = p(x_0, \xi_0),$$

where p is the principal symbol of P, such that $(\nabla_x \varphi)(x_0) = \xi_0$. Set

$$\phi(x,t) = \varphi(x) + t \cdot p(x_0, \xi_0).$$

Then there exists $\varepsilon > 0$ and $r > 0$ such that any classical symbol $a(x,0,\lambda)$ of order zero supported on $|x - x_0| \leqslant r$, extends to a unique classical symbol $a(x,t,\lambda)$ of order zero, supported on $|x - x_0| \leqslant R$ and defined for $|t| \leqslant \varepsilon$, such that the associated family of functions

$$u_\lambda(x,t) = e^{2\pi i \lambda \phi(x,t)} a(x,t,\lambda)$$

are high-frequency asymptotic solutions to the half-wave equation.

In order to prove this result, we need to obtain some formulas that tell us what the symbol b looks like, whose existence was postulated above, in terms of the phase ϕ, the operator P, and the symbol a. In order to prove the theorem above, we'll construct a recursively by slowly fixing the contributions of the higher order parts of a. One then studies the lower order terms separately, so it is wise to make a study of the functions

$$u_\lambda(x,t) = e^{2\pi i\lambda\phi(x,t)}a(x,t,\lambda),$$

where a is a classical symbol of some arbitrary order μ, rather than just a symbol of order zero. This is done in the following Lemma, whose proof can be found on the online version of these notes.

Lemma 2. *Let $p(x,\xi)$ be the principal symbol of P. Consider*

$$u_\lambda(x,t) = e^{2\pi i\lambda\phi(x,t)}a(x,t,\lambda),$$

where a and ϕ are as above, i.e. a is a symbol of order μ. Then

$$b(x,t,\lambda) = e^{-2\pi i\lambda\phi(x)}(Lu_\lambda)(x,t)$$

is a classical symbol of order $\mu+1$, with principal symbol

$$2\pi i\lambda\Big(\partial_t\phi - p\left(x,\nabla_x\phi\right)\Big)a_\mu,$$

and with order μ part given by

$$\begin{aligned}2\pi i\lambda\Big(\partial_t\phi - p\left(x,\nabla_x\phi\right)\Big)&a_{\mu-1}\\ &+\partial_t a_\mu - (\nabla_\xi p)(x,\nabla_x\phi)\cdot(\nabla_x a_\mu) - is\cdot a_\mu,\end{aligned}$$

for a smooth, real-valued function s depending only on ϕ and P.

Remark. The result of this lemma shows why we must choose φ to satisfy the eikonal equation in order for Theorem 1 to hold, i.e. because otherwise the order 1 part of $e^{-2\pi i\lambda\phi(x)}Lu_\lambda$ can never vanish. Under the assumption that φ satisfies the eikonal equation, we thus conclude in the Theorem above that for any symbol a of order μ, the function b is a symbol of order μ, with principal symbol

$$\partial_t a_\mu - (\nabla_\xi p)(x,\nabla_x\phi)\cdot(\nabla_x a_\mu) - is\cdot a_\mu,$$

because the eikonal equation causes two of the three lines in the symbol expansion above to vanish.

We are now ready to prove Theorem 1.

Proof of Theorem 1. Write

$$a(x,t,\lambda) \sim \sum_{k=0}^{\infty}\lambda^{-k}a_k(x,t),$$

where a_k is a smooth function. Fix $v > 0$ to be determined later. We prove the following result by induction, which yields the required claim: *For each $n \geqslant 0$, there exists a unique choice of $a_0, \dots, a_n$ such that b is a symbol of order less than $-n$, and moreover, for $0 \leqslant i \leqslant n$, a_i is supported on the cone*

$$\Sigma(x_0, r, v) = \{(x,t) : |x - x_0| \leqslant r + v|t|\}.$$

Let us begin with the case $n = 0$. Plugging a into the result of Lemma 2, we see that b can only have order less than zero provided

$$\partial_t a_0 - (\nabla_\xi p)(x, \nabla_x \phi) \cdot (\nabla_x a_0) - is \cdot a_0 = 0.$$

If we consider the *real* vector field

$$X(x,t) = \partial_t - (\nabla_\xi p)(x, \nabla_x \phi) \cdot \nabla_x,$$

defined on $\mathbb{R}^d \times \mathbb{R}$, then the equation above becomes

$$X\{a_0\} = isa_0.$$

This is a *transport equation*, which can be solved using the methods of characteristics. In particular, suppose v is chosen larger than the speed of propogation for the transport equation, restricted to $B_R(x_0)$. Then, provided we choose $R > r + v\varepsilon$, then a smooth solution to the transport equation exists with initial conditions $a_0(\cdot, 0)$, and is the unique such solution supported on

$$\Sigma(x_0, r, v) = \{(x,t) : |x - x_0| \leqslant r + v|t|\}.$$

for $|t| \leqslant \varepsilon$. The function a_0 must agree with this solution in order for b to be a symbol of order -1, so we conclude that a_0 is uniquely determined, verifying the base case of the inductive statement.

Lets now address the case $n > 0$, which is not too different. By the inductive hypothesis, in order for b to be a symbol of order at most $-n$, the symbols $a_0, \dots, a_{n-1}$ are uniquely determined given their values at $t = 0$, and supported on $\Sigma(x_0, r, v)$. Define $b_n(x, t, \lambda)$ to be the order $-n$ part of

$$L\left\{ e^{2\pi i \lambda \phi(x,t)} \sum_{k<n} \lambda^{-k} a_k(x,t) \right\}.$$

Applying Lemma 2 again, we find that the function

$$L\left\{ e^{2\pi i \lambda \phi(x,t)} \sum_{k \geqslant n} \lambda^{-k} a_k(x,t) \right\}$$

is a symbol of order $-n$, with principal part

$$X\{a_k\} - isa_k.$$

Thus b is a symbol of order less than $-n$ if and only if the *non-homogeneous* transport equation

$$X\{a_n\} = isa_n + b_n$$

holds. Since $\{a_0, \dots, a_k\}$ all have support on $\Sigma(x_0, r, v)$, then (because P is pseudolocal), the function b_n also has support on $\Sigma(x_0, r, v)$. But then the theory of transport equations implies that if $R > r + v\varepsilon$, then a unique smooth solution to this equation exists which is smooth, compactly supported on $\Sigma(x_0, r, v)$, and agrees with the values that a_n must take at time $t = 0$. But this means that we have verified the inductive step of the argument, which completes the proof of the theorem. $\square$

Remark. In the theory of PDEs of Hamilton-Jacobi type, one constructs solutions φ to the eikonal equation above by the *method of characteristics*. For any hypersurface Σ in $\mathbb{R}^d$, and for any smooth choice of vectors $v(x) \in T_x \mathbb{R}^d - T_x\Sigma$, for all $x \in \Sigma$, there exists a unique choice of φ vanishing on Σ, and defined in a neighborhood of x_0, such that $(\nabla_x \varphi)(x) = v(x)$ for all $x \in \Sigma$. This choice has the property that if $\{\Phi_t\}$ is the phase flow along the integral curves of the Hamiltonian equations

$$\frac{dx}{dt} = -(\nabla_\xi p)(x, \xi) \quad \frac{d\xi}{dt} = (\nabla_x p)(x, \xi)$$

and if $(x, \xi) = \Phi_t(x_1, v(x_1))$ for some $x_1 \in \Sigma$, then

$$(\nabla_x \varphi)(x) = \xi.$$

In Theorem 1 above, we choose $(\nabla_x \varphi)(x_0) = \xi_0$, and thus choose u_λ to have initial conditions supported in a small neighborhood of x_0, and frequency localized near ξ_0. This means the vector field

$$X = \partial_t - (\nabla_x \phi) \cdot \nabla_x$$

will *approximately* act by moving a along the integral curve $t \mapsto \Phi_t(x_0, \xi_0)$, as expected by the physical theories we discussed in the last section.

4 Construction of the Parametrix

By finding asymptotic solutions to the half-wave equation in the generality above, we've essentially gotten the idea of constructing the parametrix to the half-wave equation – the idea now is to take a general input, break it up into the superposition of wave packets that are localized in space and frequency, and then apply the asymptotic solution constructed above for each of these wave packets, which behaves better as these wave packets are localized to higher and higher frequencies.

It's best to break down our solution into a *continuous* superposition of wave packets rather than the usual discrete decomposition that comes up in decoupling theory. Let's review a simple approach, due to Gabor, which won't quite work for our purposes, but gives us intuition for how the continuous superposition comes about. Consider the *Gabor transform*

$$Gf(x_0, \xi_0) = \int f(x)\eta(x - x_0)e^{-2\pi i x\cdot\xi_0}\, dx,$$

where η is some fixed, non-negative, function η supported on the set $\{|x| \leqslant r\}$, and with

$$\int \eta(x)^2\, dx = 1.$$

This is a unitary transformation, with adjoint given by

$$(G^*h)(x) = \iint h(x_0, \xi_0)\eta(x - x_0)e^{2\pi i x\cdot\xi_0}\, dx_0\, d\xi_0,$$

so we obtain a 'localized Fourier inversion formula'

$$f(x) = \int Gf(x_0, \xi_0)\eta(x - x_0)e^{2\pi i x\cdot\xi_0}\, dy\, d\xi,$$

which expresses f as a superposition of the wave packets

$$x \mapsto \eta(x - x_0)e^{2\pi i x\cdot\xi_0}.$$

This approach doesn't quite work for our purposes, since our choice of asymptotic solutions to the half-wave equation leads us to try and decompose a given initial condition into wave packets of the form

$$x \mapsto s(x, x_0, \xi_0)e^{2\pi i\varphi(x,x_0,\xi_0)}.$$

for some function s, where φ satisfies the eikonal equation $p(x, \nabla_x\varphi(x, x_0, \xi_0)) = p(x_0, \xi_0)$ as in the last few sections[4]. But the fact that we can always choose a solution to the eikonal equation satisfying $\varphi(x, x_0, \xi_0) \approx (x - x_0)\cdot\xi_0$ intuitively shows that this family of wave packets is enough to represent wave packets of each frequency and position, and that a similar approach should work as for the Gabor transform.

The trick here is to consider an inversion formula of the form

$$f(x) = \int s(x, x_0, \xi_0)e^{2\pi i\varphi(x,x_0,\xi_0)}f(x_0)\, dx_0\, d\xi_0,$$

which, if held, would imply we could decompose an arbitrary function f into wave packets. We claim that we can use the *equivalence of phase theorem* to find a symbol a of order zero such that this inversion formula holds. Indeed, suppose we can choose φ to solve the eikonal equation, subject to the constraint that $\varphi(x, x_0, \xi_0) \approx (x - x_0)\cdot\xi_0$, in the sense that for $|x - x_0| \leqslant R$,

$$(\nabla_{\xi_0}\varphi)(x, x_0, \xi_0) = 0 \quad \text{if and only if} \quad x = x_0,$$

[4]Though the Gabor transform *would* work if we considering the usual half-wave equation $\partial_t - i\sqrt{-\Delta}$, in which case the eikonal equation becomes $p(x, \nabla_x\varphi) = |\xi_0|$, which has solution $\varphi(x, x_0, \xi_0) = (x - x_0)\cdot\xi_0$

$$(\nabla_x \varphi)(x_0, x_0, \xi_0) = \xi_0 \quad \text{and} \quad (\nabla_{x_0} \varphi)(x_0, x_0, \xi_0) = -\xi_0.$$

These three assumptions imply precisely that the phase on the right hand side is non-degenerate, and is associated with the Lagrangian manifold

$$\Delta_{T^* \mathbb{R}^d} = \Big\{ (x, x_0; \xi, \xi_0) : x = x_0 \text{ and } \xi = \xi_0 \Big\}.$$

This is *also* the Lagrangian manifold associated with the phase function $(x, x_0, \xi_0) \mapsto (x - x_0) \cdot \xi_0$, and we can write

$$f(x) = \int e^{2\pi i (x - x_0) \cdot \xi_0} f(x_0) \; dx_0 \; d\xi_0.$$

The equivalence of phase function theorem thus implies that there exists a symbol s of order zero[5] such that

$$f(x) = \int e^{2\pi i (x - x_0) \cdot \xi_0} f(x_0) \; dx_0 \; d\xi_0 = \int s(x, x_0, \xi_0) e^{2\pi i \varphi(x, x_0, \xi_0)} f(x_0) \; dx_0 \; d\xi_0.$$

The equivalence of phase function theorem does not guarantee that

$$\text{supp}_{(x, x_0)}(s) \subset \{ (x, x_0) : |x - x_0| \lesssim r \},$$

but we can always replace s by the function $(x, x_0, \xi_0) \mapsto \eta(x - x_0) s(x, x_0, \xi_0)$, where η is equal to one in a neighborhood of the origin, and is supported on $|x| \leqslant r$, because our assumptions guarantee that φ is non-stationary on the support of $(1 - \eta(x - x_0)) s(x, x_0, \xi_0)$. The cost of doing this, however, is that the equation

$$f(x) = \int s(x, x_0, \xi_0) e^{2\pi i \varphi(x, x_0, \xi_0)} f(x_0) \; dx_0 \; d\xi_0$$

will now only hold *modulo a smoothing operator*, which (since we're constructing a parametrix), causes us no issues.

The rest of the construction is rather easy. If we set

$$a(x, x_0, \xi_0, \lambda) = s(x, x_0, \lambda \xi_0),$$

then a is a symbol of order zero in the λ variable. Theorem 1 allows us to find asymptotic solutions

$$a(x, t, x_0, \xi_0, \lambda)$$

to the wave equation as $\lambda \to \infty$. Our parametrix A is now defined by setting

$$Af(x, t) = \int a \left(x, t, x_0, \frac{\xi_0}{|\xi_0|}, |\xi_0| \right) e^{2\pi i \phi(x, t, x_0, \xi_0)} f(x_0) \; dx_0 \; d\xi.$$

The inversion formula we constructed in the previous paragraph implies that $A_0 - I$ is a smoothing operator, where $A_0 f(x) = Af(x, 0)$. And the fact that

[5] If one looks more carefully at the proof of the equivalence of phase theorem, we see that the principal symbol of s is the constant function $(x, x_0, \xi_0) \mapsto 1$.

a gives high-frequency asymptotic solutions to the half-wave equation for each fixed x_0 and ξ_0 implies that $L \circ A$ is equal to

$$\int b\left(x, t, x_0, \frac{\xi_0}{|\xi_0|}, |\xi_0|\right) e^{2\pi i \phi(x,t,x_0,\xi_0)}\, d\xi_0,$$

where b is a symbol of order $-\infty$ in $|\xi_0|$. But this is sufficient to conclude that $L \circ A$ is a smoothing operator.

These facts immediately justify that A is a parametrix for the solution operator S to the half-wave equation. Indeed, if we let $L \circ A$ have smooth kernel K, and let $A_0 - I$ have smooth kernel K'. Then Duhamel's principle implies that the unique solution to $L \circ A = K$ with initial conditions $A_0 = I + K'$, namely, the kernel A, can be written as

$$A(x, t, y) = S(x, t, y) + S \circ K' + \int_0^t (e^{isP} K)(x, s, y)\, ds.$$

But we can simply conclude from this that

$$A - S = S \circ K' + \int_0^t e^{isP} K$$

is a smoothing operator. Thus we have constructed a parametrix A for the half-wave equation. Fantastic!

5 Hamilton-Jacobi Theory

We have constructed a parametrix of the form

$$Af(x, t) = \int a(x, t, x_0, \xi_0) e^{2\pi i \phi(x,t,x_0,\xi_0)} f(x_0)\, d\xi_0.$$

where a is a symbol of order zero, and ϕ is a non-degenerate phase function. Thus A is a Fourier integral operator of order $-1/4$. But what is it's canonical relation? To answer this, we must return back to one of our assumptions at the beginning of these notes, namely, that we can choose a function φ which satifies the eikonal equation

$$p(x_0, \xi_0) = p(x, \nabla_x \varphi(x, x_0, \xi_0)),$$

subject to the constraint that $\varphi(x_0, x_0, \xi_0) = 0$, that $\nabla_x \varphi(x_0, x_0, \xi_0) = \xi_0$, and that for x on the hyperplane

$$\Sigma(x_0, \xi_0) = \{x : (x - x_0) \cdot \xi_0\},$$

$\nabla_x \varphi(x, x_0, \xi_0)$ is normal to $\Sigma(x_0, \xi_0)$, i.e. so that φ vanishes on $\Sigma(x_0, \xi_0)$. The Hamilton-Jacobi theory used to prove the existence and uniqueness of solutions to this equation will give us more information about the behaviour of the phase

φ, which will in turn allow us to find the canonical relation of A. Since A is a parametrix for the solution operator S, the wavefront set of S is equal to the wavefront set of A, so this will tell us where the singularities of the solutions to the half-wave equation concentrate. We will not give a detailed description of the construction, which involves a hefty amount of Lagrangian geometry. All we need know for our purposes is that φ is unique given our assumptions, and that if $\{\Phi_t\}$ is the flow on $T^* \mathbb{R}^d$ given by the Hamiltonian equations

$$\frac{dx}{dt} = -(\nabla_x p)(x,\xi) \quad \frac{d\xi}{dt} = (\nabla_\xi p)(x,\xi),$$

then

$$(\nabla_x \varphi)(x', x_0, \xi_0) = \xi' \quad \text{whenever } \nabla_x \varphi(x, x_0, \xi_0) = \xi.$$

Applying Green's theorem and Euler's homogeneous function theorem, we thus conclude that if $(x',\xi') = \Phi_t(x,\xi)$, if $(\nabla_x\varphi)(x,x_0,\xi_0) = \xi$, and if we set $(x(s),\xi(s)) = \Phi_s(x,\xi)$, then

$$\begin{aligned}
\varphi(x', x_0, \xi_0) - \varphi(x, x_0, \xi_0) &= \int_0^t (\nabla_x \varphi)(x(s), x_0, \xi_0) \cdot \partial_t x(s)\, ds \\
&= -\int_0^t \xi(s) \cdot (\nabla_\xi p)(x(s), \xi(s))\, ds \\
&= -\int_0^t p(x(s), \xi(s))\, ds \\
&= -tp(x,\xi).
\end{aligned}$$

The last equality follows from the fact that p is *constant* on the integral curves to the Hamiltonian vector field.

Theorem 3. *Let $\{\Phi_t\}$ denote the phase flow corresponding to the Hamiltonian vector field H. Then the canonical relation of the parametrix A is equal to*

$$\mathcal{C} = \Big\{ (x,\xi,t,\tau,x_0,\xi_0) : (x,\xi) = \Phi_{-t}(x_0,\xi_0) \text{ and } \tau = p(x_0,\xi_0) \Big\}.$$

Proof. The set $\mathcal{C}$ is a $2d$ dimensional submanifold of $T^*(\mathbb{R}^d_x \times \mathbb{R}^d_{x_0})$. Since the canonical relation of A is also $2d$ dimensional, it suffices to show that $\mathcal{C}$ is contained in the canonical relation A. So fix $(x,\xi,t,\tau,x_0,\xi_0) \in \mathcal{C}$. Recall that

$$\phi(x,t,x_0,\xi_0) = \varphi(x,x_0,\xi_0) + tp(x_0,\xi_0).$$

Thus we immediately see that

$$\nabla_t \phi(x,t,x_0,\xi_0) = p(x_0,\xi_0). \tag{1}$$

Because $(x,\xi) = \Phi_{-t}(x_0,\xi_0)$, and $(\nabla_x\varphi)(x_0,x_0,\xi_0) = \xi_0$, we know that

$$(\nabla_x \phi)(x,t,x_0,\xi_0) = (\nabla_x \varphi)(x,x_0,\xi_0) = \xi. \tag{2}$$

We note that for each tangent vector v to $\Sigma(x_0, \xi_0)$, we must have

$$(\nabla_x \varphi)(x_0, x_0 + hv, \xi_0) = c(h)\xi_0 \quad \text{for some function } c.$$

The gradient must be a multiple of ξ_0 by the initial conditions of φ because $x_0 \in \Sigma(x_0 + hv, \xi_0)$. Plugging the value into the eikonal equation,

$$\begin{aligned} c(h)p(x_0, \xi_0) &= p(x_0, \nabla_x \varphi(x_0, x_0 + hv, \xi_0)) \\ &= p(x_0 + hv, \xi_0), \end{aligned}$$

from which we conclude that $c(0) = 1$, and $c'(0) = p(x_0, \xi_0)^{-1}(\nabla_x p)(x_0, \xi_0) \cdot v$. The homogeneity of p implies that for all s,

$$\Phi_s(x_0, c\,\xi_0) = c\,\Phi_s(x_0, \xi_0).$$

Thus $\Phi_{-t}(x_0, c\,\xi_0) = (x, c\xi)$, and so we conclude that

$$\begin{aligned} \varphi(x, x_0 + hv, \xi_0) &= \varphi(x_0, x_0 + hv, \xi_0) + tc(h)p(x_0, \xi_0) \\ &= tc(h)p(x_0, \xi_0). \end{aligned}$$

Taking $h \to 0$, we conclude that

$$\lim_{h \to 0} \frac{\varphi(x, x_0 + hv, \xi_0)}{h} = tc'(0)p(x_0, \xi_0) = t(\nabla_x p)(x_0, \xi_0) \cdot v.$$

On the other hand, by the implicit function theorem, by the implicit function theorem, for small h, there exists unique values $t_*(h)$, $x_*(h)$, and $\xi_*(h)$ such that $x_*(h) \in \Sigma(x_0 + h\xi_0, \xi_0)$, and

$$(x_0, \xi_*(h)) = \Phi_{t_*(h)}(x_*(h), \xi_0).$$

We have $t_*(0) = 0$, $\xi_*(0) = \xi_0$, and $x_*(0) = x_0$. Expanding in h, we have

$$\begin{aligned} &(x_0, \xi_*(h)) \\ &\quad = (x_*(h), \xi_0) + t_*(h)\Big((-\nabla_x p)(x_*(h), \xi_0), (\nabla_\xi p)(x_*(h), \xi_0)\Big) + O(t_*(h)^2) \\ &\quad = (x_*(h), \xi_0) + t_*(h)\Big((-\nabla_x p)(x_*(h), \xi_0), (\nabla_\xi p)(x_*(h), \xi_0)\Big) + O(h^2). \end{aligned}$$

Thus

$$\begin{aligned} x_0 &= x_*(h) - t_*(h)(\nabla_x p)(x_*(h), \xi_0) + O(h^2) \\ &= x_0 + hx_*'(0) - t_*'(0)(\nabla_x p)(x_0, \xi_0) + O(h^2) \end{aligned}$$

and

$$\begin{aligned} \xi_*(h) &= \xi_0 + t_*(h)(\nabla_\xi p)(x_*(h), \xi_0) + O(h^2) \\ &= \xi_0 + ht_*'(0)(\nabla_\xi p)(x_0, \xi_0) + O(h^2). \end{aligned}$$

Thus we conclude that

$$x_*'(0) = t_*'(0)(\nabla_x p)(x_0, \xi_0) \quad \text{and} \quad \xi_*'(0) = t_*'(0)(\nabla_\xi p)(x_0, \xi_0).$$

We have $(x_*(h) - (x_0 + h\xi_0)) \cdot \xi_0 = 0$, which implies that

$$x_*(0)' \cdot \xi_0 = |\xi_0|^2.$$

Thus

$$t_*'(0) = \frac{|\xi_0|^2}{(\nabla_x p)(x_0, \xi_0) \cdot \xi_0}$$

and so

$$\xi_*'(0) = |\xi_0|^2 \frac{(\nabla_\xi p)(x_0, \xi_0)}{(\nabla_x p)(x_0, \xi_0) \cdot \xi_0}.$$

Applying the Green's theorem calculation above, we find that for any tangent vector v to $\Sigma(x_0, \xi_0)$,

$$\varphi(x, x_0 + v, \xi_0) = \varphi(x_0, x_0 + v, \xi_0) + tp(x_0, \xi_0).$$

If v is tangent to $\Sigma(x_0, \xi_0)$, $\varphi(x_0, x_0 + v, \xi_0) = 0$ since $x_0 \in \Sigma(x_0 + v, \xi_0)$, and thus we find that $\varphi(x, x_0 + v, \xi_0) = tp(x_0, \xi_0)$ is independent of v. This means that $(\nabla_{x_0}\varphi)(x, x_0, \xi_0)$ must be a scalar multiple of ξ_0. To work out which scalar multiple this is, we must calculate

$$\nabla_{x_0}\varphi(x, x_0, \xi_0) \cdot \xi_0 = \lim_{h \to 0} \frac{\varphi(x_0, x_0 + h\xi_0, \xi_0)}{h}.$$

Now if $(x(t), \xi(t)) = \Phi_t(x_0, \xi_0)$, then

$$\begin{aligned}
(x(t) - (x_0 + h\xi_0)) \cdot \xi_0 &= t[x'(0) \cdot \xi_0] - h|\xi_0|^2 + O(t^2) \\
&= -t(\nabla_\xi p)(x_0, \xi_0) \cdot \xi_0 - h|\xi_0|^2 + O(t^2) \\
&= -tp(x_0, \xi_0) - h|\xi_0|^2 + O(t^2).
\end{aligned}$$

By the implicit function theorem, for suitably small h, there exists a unique value $t^*(h)$ making the above quantity zero, and moreover,

$$t^*(h) = \frac{-|\xi_0|^2 h}{p(x_0, \xi_0)} + O(h^2).$$

If we let $x^*(h) = x(t^*(h))$, then this means that $\varphi(x^*(h), x_0 + h\xi_0, \xi_0) = 0$, so

$$\begin{aligned}
0 &= \varphi(x^*(h), x_0 + h\xi_0, \xi_0) \\
&= \varphi(x_0, x_0 + h\xi_0, \xi_0) - t^*(h)p(x_0, \xi_0).
\end{aligned}$$

Rearranging, and taking limits as $h \to 0$, we conclude that

$$\begin{aligned}
(\nabla_{x_0}\varphi)(x, x_0, \xi_0) \cdot \xi_0 &= \lim_{h \to 0} \frac{\varphi(x_0, x_0 + h\xi_0, \xi_0)}{h} \\
&= \lim_{h \to 0} \frac{t^*(h)}{h} p(x_0, \xi_0) \\
&= -|\xi_0|^2.
\end{aligned}$$

Thus we conclude that

$$(\nabla_{x_0}\varphi)(x, x_0, \xi_0) = -\xi_0. \tag{3}$$

Finally, we come to show that $\nabla_{\xi_0}\phi(x, t, x_0, \xi_0) = 0$. Set $(x(t), \xi(t)) = \Phi_t(x_0, \xi_0)$. Then $x(0) = x_0$,and

$$\nabla_{\xi_0}\phi(x_0, t, x_0, \xi_0) = \nabla_{\xi_0}\varphi(x_0, x_0, \xi_0) = 0.$$

Let

$$F(t) = \nabla_{\xi_0}\phi(x(t), t, x_0, \xi_0).$$

Then $F(0) = 0$, and the chain rule implies that

$$F_j'(t) = \sum_k \left[\frac{\partial^2\varphi}{\partial x_k \partial \xi_0^j}(x(t), t, x_0, \xi_0)\frac{dx_k(t)}{dt} \right] + \frac{\partial p}{\partial \xi_j}(x_0, \xi_0).$$

But

$$\frac{dx_k(t)}{dt} = -\frac{\partial p}{\partial \xi_k}(x(t), \xi(t)).$$

Since

$$p(x, \nabla_x \varphi(x(t), x_0, \xi_0)) = p(x_0, \xi_0),$$

taking derivatives on both sides in ξ_0 implies that for each j,

$$\sum_k \frac{\partial p}{\partial \xi_k}(x(t), x_0, \xi_0)\frac{\partial^2\varphi}{\partial \xi_0^j \partial x_k}(x(t), x_0, \xi_0) = \frac{\partial p}{\partial \xi_j}(x_0, \xi_0).$$

Substituting this into the equation for F_j', together with the value of $dx(t)/dt$, we conclude that $F_j'(t) = 0$. But this implies that $F(t) = 0$ for all t, and so in particular, for $(x, \xi, t, \tau, x_0, \xi_0) \in \mathcal{C}$,

$$\nabla_{\xi_0}\phi(x, t, x_0, \xi_0) = 0. \tag{4}$$

Combining (1), (2), (3), and (4) implies that $\mathcal{C}$ is contained in the canonical relation, as was required to be shown. $\square$

Let us consider a particular example, i.e. the Laplace-Beltrami operator

$$P = \sqrt{-\Delta_g}$$

introduced in Section 2. The principal symbol of this equation is given by $p(x, \xi) = |\xi|_g$, the length of the covector ξ with respect to the Riemannian metrix g. If we plug this principal symbol into the Hamilton-Jacobi theory above, we see that the bicharacteristics of the Hamiltonian vector field H are precisely the integral curves of the *geodesic flow* in $T^*\mathbb{R}^d$. Thus we conclude that the wavefront set of the parametrix for the half-wave operator is *precisely* the 'geodesic light cone'

$$\Big\{(x, \xi, t, \tau, x_0, \xi_0) : (x, \xi) = \exp_{x_0}(-t\xi_0) \text{ and } \tau = |\xi_0|_g\Big\},$$

where $\exp_{x_0} : T^*_{x_0}\mathbb{R}^d \to \mathbb{R}^d$ denotes the geodesic map for cotangent inputs.

6 Global Time Parametrix

For simplicity, let us now work on a compact manifold M. All the techniques we have worked on so far are local. Thus given a first-order, classical, formally positive pseudodifferential operator P on M, we can apply a compactness and partition of unity argument in coordinates to find a parametrix A for the operator $\partial_t - iP$ on M which is now *global in space*, and defined for times $|t| \lesssim \varepsilon$. We will now show that given this parametrix, for any $N > 0$, we can define a parametrix on the interval $\{|t| \leqslant N\}$.

The trick here is to use the semigroup property of the wave equation, together with the *composition calculus* for Fourier integral operators we discussed last week. For $|t| \leqslant \varepsilon$, let A_t be the operator such that

$$Af(x,t) = A_t f(x).$$

Then A_t is a parametrix for the half-wave propogator $e^{2\pi i t P}$, and moreover, each operator is a Fourier integral operator of order zero, with canonical relation

$$\mathcal{C}_t = \Big\{ (x,\xi,x_0,\xi_0) : (x,\xi) = \Phi_{-t}(x_0,\xi_0) \Big\}.$$

Since M is a compact manifold, the Hamiltionian flows $\{\Phi_t\}$ are globally defined in time, and so we can actually define $\mathcal{C}_t$ for each $t \in \mathbb{R}$. Each such set will be a Lagrangian submanifold of $T^*M \times T^*M$, and moreover, will satisfy the relation

$$\mathcal{C}_{t_1} \circ \cdots \circ \mathcal{C}_{t_N} = \mathcal{C}_{t_1 + \cdots + t_N}.$$

One can immediately check that these Lagrangian manifolds are compatible, i.e. the required transversality conditions hold, so that we are able to apply the composition calculus for Fourier integral operators.

Since M is compact, solutions to the wave equation are actually globally defined in time. The family of propogators $\{e^{2\pi i t P}\}$ satisfy the semigroup property

$$e^{2\pi i t_1 P} \circ \cdots \circ e^{2\pi i t_N P} = e^{2\pi i t P},$$

if $t = t_1 + \cdots + t_N$. Now under this assumption, the composition calculus for Fourier integral operators thus allows us to conclude that

$$A_{t_1} \circ \cdots \circ A_{t_N}$$

is a Fourier integral operator of order zero, with canonical relation $\mathcal{C}_t$. Moreover, one can do some algebraic calculations to show that

$$A_{t_1} \circ \cdots \circ A_{t_N} - e^{2\pi i t P}$$

is a smoothing operator.

We can use this trick to extend our parametrix. Given our parametrix A, defined on $\{|t| \leqslant \varepsilon\}$, let's define a parametrix A' which works for times $\{|t| \leqslant 2\varepsilon\}$, by setting

$$A'f(x,t) = \begin{cases} A_t f(x) & : |t| \leqslant \varepsilon \\ (A_{t-\varepsilon} \circ A_1) f(x) & : \varepsilon \leqslant t \leqslant 2\varepsilon \\ (A_{t+\varepsilon} \circ A_{-1}) f(x) & : -2\varepsilon \leqslant t \leqslant -\varepsilon. \end{cases}$$

It is simple to check that $A' - S$ is a smoothing operator, by using the semigroup property of the half-wave equation. So we've extended the parametrix a distance ε more than we started with!

But now you should see how the trick to get an arbitrarily large interval works. Given that we want to construct a parametrix for times $|t| \leqslant N$, we just need to iterate the argument we have just given $O(N/\varepsilon)$ times. The resulting operator will be a composition of many compatible Fourier integral operators, and will thus also be a Fourier integral operator of order zero. But the symbol of such an operator becomes very difficult to understand as $N \to \infty$; it is obtained by an $O(N/\varepsilon)$-fold iterated oscillatory integral, and these things can get hairy as $N \to \infty$. We should expect this, because, if M is a strange manifold, or P is a strange pseudodifferential operator, for large times the Hamiltonian equation can loop around in very strange ways that we might expect might make the operator fairly pathological to deal with from the Hamiltonian approximation methods we have given in these notes. But for a fixed N, one can at least conclude from the calculus that the symbol is of order zero.

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