Large Salem Sets Avoiding Polynomial Patterns

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Abstract

TODO

Adapting the discrete strategy of (TODO) to the continuous setting, and together with the translation dimension boosting argument of Schmerkin, we prove the existence of a Salem set $E \subset [0,1]$ such that for $x_1, x_2, x_3, x_4 \in E$ with $x_1 \neq x_2$ and $x_3 \neq x_4$, $x_1 - x_2 \neq (x_3 - x_4)^2$.

We construct E as follows. Fix an integer $k \geqslant 20$, and consider a family of subsets $R_n \subset \{0, \dots, k-1\}$ for each $n \geqslant 0$. Define

$$E = \left\{ \sum_{n=1}^{\infty} a_n k^{-n} : a_n \in R_n \text{ for all } n \geqslant 1 \right\}.$$

We claim that E avoids patterns if $\{R_n\}$ are chosen suitably well. Let us begin by making an apriori assumption that for each n, $1 \notin R_n - R_n$. Let us suppose that there exists $x_1, x_2, x_3, x_4 \in E$ such that $x_1 - x_2 = (x_3 - x_4)^2$. Write

$$x_1 = \sum_{n=1}^{\infty} a_n k^{-n}, \quad x_2 = \sum_{n=1}^{\infty} b_n k^{-n}, \quad x_3 = \sum_{n=1}^{\infty} c_n k^{-n}, \quad \text{and} \quad x_4 = \sum_{n=1}^{\infty} d_n k^{-n}.$$

Let $\delta_n = a_n - b_n$, and $\varepsilon_n = c_n - d_n$. Then

$$\sum_{n=1}^{\infty} \delta_n k^{-n} = \left(\sum_{n=1}^{\infty} \varepsilon_n k^{-n} \right)^2.$$

Let i be the first index such that $\delta_i \neq 0$, and j the first index where $\varepsilon_j \neq 0$. Then

$$k^{-i} < \sum_{n=1}^{\infty} \delta_n k^{-n} < k^{1-i} \quad \text{and} \quad k^{-2j} < \left(\sum_{n=1}^{\infty} \varepsilon_n k^{-n} \right)^2 < k^{2-2j}.$$

Equality is thus only possible if $k^{-2j} < k^{1-i}$ (so $i < 2j+1$) and $k^{-i} < k^{2-2j}$ (so $i > 2j-2$). Thus $i = 2j-1$ or $i = 2j$.

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Assume first that i is even, so that $2j = i$. Write $(U,V,W) = (\delta_i, \delta_{i+1}, \delta_{i+2})$ and $(A,B,C) = (\varepsilon_j, \varepsilon_{j+1}, \varepsilon_{j+2})$. Then

$$\left|\sum_{n=1}^{\infty} \delta_n k^{-n} - (Uk^{-i} + Vk^{-i-1} + Wk^{-i-2})\right| < k^{-i-2}$$

and

$$\left|\left(\sum_{n=1}^{\infty} \varepsilon_n k^{-n}\right)^2 - \Big(Ak^{-j} + Bk^{-j-1} + Ck^{-j-2}\Big)^2\right| < \Big(2A + 2Bk^{-1} + (2C+1)k^{-2}\Big)k^{-i-2},$$

and so

$$\left|\Big(U + Vk^{-1} + Wk^{-2}\Big) - \Big(A + Bk^{-1} + Ck^{-2}\Big)^2\right| < \Big(1 + 2A + 2Bk^{-1} + (2C+1)k^{-2}\Big)k^{-2}$$

We have

$$\left|\Big(U + Vk^{-1} + Wk^{-2}\Big) - \Big(A + Bk^{-1} + Ck^{-2}\Big)^2\right| > (A-1)^2 - k,$$

and so we obtain that $(A-1)^2 < 1.006k$, and thus $A < 1.23k^{1/2}$, which means

$$\left|\Big(U + Vk^{-1} + Wk^{-2}\Big) - \Big(A + Bk^{-1} + Ck^{-2}\Big)^2\right| < 3.16k^{-3/2}.$$

Now

$$\begin{aligned}\Bigg|\Big[\Big(U + Vk^{-1} + Wk^{-2}\Big) - \Big(A + Bk^{-1} + Ck^{-2}\Big)^2\Big]& \\ - \Big[(U - A^2 - 2AB/k) + (V/k - B^2/k^2 - 2AC/k^2)\Big]\Bigg|& \\ < 3.05k^{-1}&\end{aligned}$$

and so

$$|(U - A^2 - 2AB/k) + (V/k - B^2/k^2 - 2AC/k^2)| < 3.76k^{-1}$$

This only occurs if $|U - A^2 - 2AB/k| \leqslant 2 + 3.31k^{-1/2}$.

How about if i is odd? A similar reduction as above shows that

$$\begin{aligned}|(U + Vk^{-1} + Wk^{-2}) - k^{-1}(A + Bk^{-1} + Ck^{-2})^2|& \\ < (1 + 2Ak^{-1} + 2Bk^{-2} + (2C+1)k^{-3})k^{-2}& \\ < 3.1k^{-2}.&\end{aligned}$$

Thus

$$|(U - A^2k^{-1}) + (Vk^{-1} - 2ABk^{-2} + Wk^{-2} - 2ACk^{-3} - B^2k^{-3})| < 5.1k^{-2},$$

which implies $|U - A^2k^{-1}| \leqslant 1 + 1.1k^{-1/2}$. Thus we have reduced the problem to choosing $\{R_j, R_{2j-1}\}$ appropriately.

We find such choices computationally. First, let's suppose $\{R_j\}$ is constant for all j, equal to some common set $R \subset \{0, \dots, k-1\}$. Our goal is thus to choose R such that the difference set $R - R$ does not contain X, Y, Z, with $X, Y \neq 0$, such that

$$|X - Y^2 - 2YZ/k| \leqslant 2 + 2.1k^{-1/2} \quad \text{or} \quad |X - Y^2 k^{-1}| \leqslant 1 + 1.1k^{-1/2} \quad \text{or} \quad |X| \leqslant 1.$$

For any choice of R, the resulting set have covering number $|R|^n$ at a length scale k^{-n}, and one can show the resulting set is Ahlfors-regular, with dimension $\log_k |R|$.

$$|(a_i - b_i)k^{-i} - (c_j - d_j)^2 k^{-2j}| \leqslant k^{-i} + (2k-1)k^{-2j}.$$

If $2j \geqslant i$, then

$$|(a_i - b_i)k^{2j-i} - (c_j - d_j)^2| \leqslant k^{2j-i} + (2k-1).$$

If $i \geqslant 2j$, then we have

$$|(a_i - b_i) - (c_j - d_j)^2 k^{i-2j}| \leqslant 1 + (2k-1)k^{i-2j}.$$

If $i > 2j$, then $|(a_i - b_i) - (c_j - d_j)^2 k^{i-2j}|$

If $2j > i + 2$, then we conclude that

$$|(a_i - b_i)k^{-i} - (c_j - d_j)^2 k^{-2j}| \leqslant (1 + 2/k - 1/k^2)k^{-i}.$$

Since $a_i - b_i$ and $(c_j - d_j)^2$ are both even

TODO: Argue that we reach a contradiction unless $2j = i$, and $(a_i - b_i) = (c_j - d_j)^2$.

Provided we choose R_i so that $R_i - R_i$ is disjoint from $(R_j - R_j)^2$, except at the origin, we reach a contradiction, which allows us to conclude that the resulting set E is squarefree. We have $N(k^{-n}) \sim \prod_{j \leqslant n} |R_j|$, and so the Minkowski dimension of E is equal to

$$\frac{1}{\log k} \lim_{n \to \infty} \frac{1}{n} \sum_{j \leqslant n} \log(|R_j|).$$

We can argue that E has the same Hausdorff dimension similarily, i.e. by taking a limiting probality measure and proving a Frostman condition.