High Codimension Curves Can't Be Salem

April 17, 2023

1 New Strategies

Let $U \subset \mathbb{R}^k$ be an open set, and consider a smooth immersion $\gamma : U \to \mathbb{R}^d$. For a Borel probability measure μ supported on U, and $\xi \in \mathbb{R}^d$, we let

$$I(\mu, \xi) = \int_U e^{2\pi i \xi \cdot \gamma(x)}\, d\mu(x) = \widehat{\gamma_* \mu}(\xi).$$

Our goal is to prove the following Lemma.

TODO: By a translation argument, we may assume that $\gamma : 2Q \to \mathbb{R}^d$

Lemma 1. *Let Q be a closed, axis-oriented cube, such that $2Q \subset U$. Suppose that there exists a Borel probability measure μ supported on Q such that*

$$\sup_{\xi \in \mathbb{R}^d} |\xi|^{s/2} |\widehat{\gamma_* \mu}(\xi)| < \infty.$$

Then there exists a non-negative smooth function ϕ, supported in $2Q$, such that

$$\int_U \phi(x)\, dx = 1,$$

i.e. such that the measure $\mu_\phi = \phi\, dx$ is a probability measure, and such that

$$\sup_{\xi \in \mathbb{R}^d} |\xi|^{s/2} |\widehat{\gamma_* \mu_\phi}(\xi)| < \infty.$$

Proof. Since $x \mapsto \gamma(x)$ is an immersion, for any fixed x_0, there exists a coordinate system z, defined in a neighborhood of $\gamma(x)$, such that

$$z(\gamma(x)) = (x, 0).$$

Then $\{dz^1, \dots, dz^k, \xi_0 dx\}$ are linearly independent covector fields in a neighborhood of $\gamma(x_0)$, and thus there exists a coordinate system w, defined in a neighborhood of $\gamma(x_0)$, such that $(w^1, \dots, w^k) = (z^1, \dots, z^k)$, and $dw^{k+1} = \xi_0 \cdot dx$. Now, for each $v \in \mathbb{R}^k$ with $|v| < \delta$, we define a diffeomorphism A_v in a neighborhood of $\gamma(x_0)$ by setting

$$(w \circ A_v \circ w^{-1})(w^1, \dots, w^k) = (w^1, \dots, w^k) + (v, 0).$$

These diffeomorphisms are chosen precisely so that, for each x in a neighborhood of $\gamma(x_0)$,

$$A_v(\gamma(x)) = \gamma(x+v),$$

because $w(\gamma(x)) = (x,0)$ and $w(\gamma(x+v)) = (x+v,0)$, and so

$$w(A_v(\gamma(x))) = (x+v,0) = w(A_v(\gamma(x+v))).$$

and also, for $|v| < \delta$,

$$DA_v(y)^T(\xi_0) = \xi_0,$$

which can be verified in the language of differential forms by noting that

$$A_v^*(\xi_0 dx) = A_v^*(dw^{k+1}) = d(w^{k+1} \circ A_v) = dw^{k+1} = \xi_0 dx,$$

i.e. so that the covector field $\xi_0 dx$ is preserved by the diffeomorphisms $\{A_v\}$.

Consider a smooth, non-negative bump function ψ on $\mathbb{R}^d$, which is equal to one on a neighborhood of $\gamma(x_0)$. For small v, consider the measure $\mu_v = \mathrm{Trans}_v \mu$. We calculate using the multiplication formula that

$$\begin{aligned} \widehat{\gamma_* \mu_v}(\lambda \xi_0) &= \int_U e^{2\pi i \lambda \xi_0 \cdot \gamma(x+v)}\ d\mu(x) \\ &= \int_U e^{2\pi i \lambda \xi_0 \cdot A_v(\gamma(x))}\ d\mu(x) \\ &= \int_{\mathbb{R}^d_y} e^{2\pi i \lambda \xi_0 \cdot A_v(y)}\ d(\gamma_* \mu)(y). \end{aligned}$$

Note that $\nabla_y \{\xi_0 \cdot A_v(y)\} = A_v(y)^T \xi_0 = \xi_0$, so that

$$\begin{aligned} \xi_0 \cdot A_v(y) &= \xi_0 \cdot A_v(\gamma(x_0)) + \xi_0 \cdot (y - \gamma(x_0)) \\ &= \xi_0 \cdot [\gamma(x_0+v) - \gamma(x_0)] + \xi_0 \cdot y. \end{aligned}$$

Thus

$$\widehat{\gamma_* \mu_v}(\lambda \xi_0) = e^{2\pi i \lambda \xi_0 \cdot [\gamma(x_0+v) - \gamma(x_0)]} \widehat{\gamma_* \mu}(\lambda \xi_0).$$

Write $\phi = \xi_0 \cdot A_v(y) - \eta \cdot y$. Then

$$\nabla_y \phi = DA_v(y)^T \xi_0 - \eta = \xi_0 - \eta$$

is independent of y. Thus we can write

$$\phi = c(\xi_0, v, \eta) + (\xi_0 - \eta) \cdot y.$$

Then

$$|I(y,\lambda,\nu)||\widehat{\psi}(\eta - \xi_0)|$$

We can upper bound the magnitude of I using nonstationary phase, i.e. because we can write

$$I(\eta, v, \lambda) = \int_{\mathbb{R}^d_y} \psi(y) e^{2\pi i \lambda \phi(y,\eta,v)}\ dy,$$

where

$$\phi(y, \eta, v) = [\xi_0 \cdot A_v(y) - \eta \cdot y].$$

Then $\nabla_y \phi(y, \eta, v) = DA_v(y)^T \xi_0 - \eta = \xi_0 - \eta$, i.e. so that we actually have

$$\phi(y, \eta, v) = c(\xi_0, v) + (\xi_0 - \eta) \cdot y.$$

But this means that

$$I(\eta, v, \lambda) = c(\xi_0, v)\widehat{\psi}$$

where

$$I(\eta, v, \lambda) = \int_{\mathbb{R}^d_y} \psi(y) e^{2\pi i \lambda [\xi_0 \cdot A_v(y) - \eta \cdot y]}\, dy = \int_{\mathbb{R}^d_y} \psi(y) e^{2\pi i \lambda \phi(y;\eta,v)}\, dy.$$

We calculate that

$$\nabla_y \phi(y; \eta, \lambda, v) = DA_v(y)^T \xi_0 - \eta.$$

Our choice of diffeomorphisms $\{A_v\}$ implies that $DA_v(y)^T \xi_0 = \xi_0$ for all y. Thus

$$\nabla_y \phi(y; \eta, \lambda, v) = \xi_0 - \eta.$$

Thus we can apply integration by parts to conclude that

$$|I(\eta, v, \lambda)| \lesssim_N \lambda^{-N} |\xi_0 - \eta|^{-N}.$$

Thus we conclude that

$$\begin{aligned}
\lambda^d \int_{|\eta - \xi_0| \geqslant \lambda^{-\alpha}} & I(\eta, v, \lambda) \widehat{\gamma_* \mu}(\lambda \eta)\, d\eta \\
&\lesssim_N \lambda^{d-N} \int_{|\eta - \xi_0| \geqslant \lambda^{-\alpha}} |\xi_0 - \eta|^{-N} \lesssim 1 \\
&= \lambda^{d-N} \int_{\lambda^{-\alpha}}^{\infty} t^{d-1-N}\, dt \\
&\lesssim \lambda^{(1-\alpha)(d-N)}.
\end{aligned}$$

If $\alpha = 1 - [s/2(N-d)]$, we obtain that this integral is $O(\lambda^{-s/2})$. Taking N arbitrarily large allows us to pick α arbitrarily close to one. Then

$$\begin{aligned}
\lambda^d \int_{|\eta - \xi_0| \leqslant \lambda^{1-\varepsilon/d}} & I(\eta, v, \lambda) \widehat{\gamma_* \mu}(\lambda \eta)\, d\eta \\
&\leqslant \lambda^d \int_{|\eta - \xi_0| \leqslant \lambda^{1-\varepsilon/d}} \lambda^{-s/2} \\
&= \lambda^{d-(1-\varepsilon/d)d - s/2} = \lambda^{\varepsilon - s/2}.
\end{aligned}$$

Combining these calculations allows us to conclude that

$$|\widehat{\gamma_* \mu_v}(\lambda \xi_0)| \lesssim_\varepsilon \lambda^{\varepsilon - s/2}.$$

We start with some basic techniques from the study of differential manifolds. Write the standard coordinates of $\mathbb{R}^k$ by $(x^1, \dots, x^k)$, and the standard coordinates of $\mathbb{R}^d$ by $(y^1, \dots, y^d)$. Applying implicit function theorem type techniques (see Theorem 10 of Spivak, Vol 1, Chapter 2), for any $x_0 \in \mathbb{R}^k$, we can find a coordinate system z defined in a neighborhood of $\gamma(x_0)$ such that

$$z(\gamma(x)) = (x, 0).$$

Set $w^j(x) = z^j(x)$ for $1 \leqslant j \leqslant k$, and let $w^{k+1}(x) = x \cdot \xi_0$. Then $dw^{k+1} = \xi_0 dx$, and $\{dw^1, \dots, dw^{k+1}\}$ are linearly independent at $\gamma(x_0)$, so we can extend these functions to a coordinate system w defined in a neighborhood of $\gamma(x_0)$. Now we consider a family of diffeomorphisms $\{A_v\}$ defined in a neighborhood of $\gamma(x_0)$, and for small $v \in \mathbb{R}^k$, such that

$$(w \circ A_v \circ w^{-1})(w^1, \dots, w^d) = (w^1, \dots, w^d) + (v, 0).$$

Then $\{A_v\}$ is chosen precisely so that for x in a neighborhood of x_0,

$$A_v(\gamma(x)) = \gamma(x + v),$$

and also,

$$A_v^*(\xi_0 dx) = A_v^*(dw^{k+1}) = d(w^{k+1} \circ A_v) = dw^{k+1} = \xi_0 dx.$$

Thus the covector field $\xi_0 dx$ is preserved by the family of diffeomorphisms $\{A_v\}$.

□

if and only if there exists a smooth function $\phi : U \to \mathbb{R}$, supported on a compact subset of U, such that if $\nu = \gamma_*(\phi\, dx)$, then

$$|\widehat{\nu}(\xi)| \lesssim |\xi|^{-s/2}.$$

We do this by using stationary phase to show that 'translates' of μ continue to have good FOurier decay estimates, which allows us to show that a convolution of μ with a smooth, compactly supported

2 Old Strategy

Let $\gamma : I \to \mathbb{R}^3$ be a smooth, parametric curve defined on an interval $I \subset \mathbb{R}$, and let $\Gamma = \gamma(I)$ denote the parametric curve's trace. The Hausdorff dimension of Γ is equal to one, being the image of an interval under a diffeomorphism. We claim that the Fourier dimension of Γ is $2/3$, so that Γ is never a Salem set. Marstrand projection theorem variants for Fourier dimension imply that the Fourier dimension of any curve in $\mathbb{R}^d$ for $d \geqslant 3$ has Fourier dimension at most $2/3$, though I imagine similar techniques to those described here can prove the Fourier dimension of such a curve is equal to $2/d$.

Let us make the simplifying assumption that γ', γ'', and γ''' are all nonvanishing on I, and moreover, are linearly independent[1]. There exists a unique, smooth family of unit vectors $\{\xi_0(t) : t \in I\}$ in $\mathbb{R}^d$ such that

$$\xi_0(t) \cdot \gamma'(t) = \xi_0(t) \cdot \gamma''(t) = 0 \quad \text{for all } t \in I,$$

[1] We can probably use Sard's Theorem, or something similar, to reduce the study of any curve to one satisfying this assumption, but let's not get ahead of ourselves.

and with

$$\xi_0(t) \cdot \gamma'''(t) > 0 \quad \text{for all } t \in I.$$

It follows by taking a Taylor series in the t variable that we can guarantee that there exists $\varepsilon > 0$ such that for $0 < |t - s| < \varepsilon$, we have

$$\frac{\xi_0(t) \cdot \gamma'(s)}{(s-t)^{d-1}} > 0.$$

If we break up I into a finite union of almost disjoint union of intervals $\{I_j\}$, each with length less than $\varepsilon/3$, and set $\Gamma_j = \gamma(I_j)$, then it follows from (Ekström, Persson, Schmeling, 2015) that

$$\dim_{\mathbb{F}}(\Gamma) = \max_j \dim_{\mathbb{F}}(\Gamma_j).$$

We can therefore choose some j such that $\dim_{\mathbb{F}}(\Gamma_j) = 1$. Swapping out I for I_j, and Γ for Γ_j, we will assume in what follows that for all distinct $t, s \in I$, the smooth function ν agreeing with

$$\frac{\xi_0(t) \cdot \gamma'(s)}{(s-t)^{d-1}}$$

for distinct $t, s \in I$ is positive. Taking a Taylor series in the s variable, and then letting $s \to 0$ allows us to conclude that $\nu(t,t) = \xi_0(t) \cdot \gamma'''(t)$. We also consider the smooth, positive function $a(t) = (\xi_0(t) \cdot \gamma'''(t))^{1/3}$.

For a measure μ on I, a function $\gamma : I \to \mathbb{R}^3$, and $\xi \in \mathbb{R}^3$, let

$$I_\gamma(\mu, \xi) = \int e^{i\xi \cdot \gamma(t)} d\mu(t).$$

Our goal is to show that for any probability measure μ on I, and any $\varepsilon > 0$,

$$\limsup_{\xi \to \infty} |\xi|^{1/3+\varepsilon} I_\gamma(\mu, \xi) = \infty,$$

which is equivalent to proving that $\dim_{\mathbb{F}}(\Gamma) \leqslant 2/3$.

The following stationary phase result will be useful.

Lemma 2. *There exists a constant* Γ *such that if* f *is a* C^1 *function supported on* $[-10, +10]$*, then for* $t \in I$*, and* $\lambda > 0$*,*

$$I_\gamma(f, \lambda\xi_0(t)) = C\, a(t) f(t) e^{i\lambda\xi_0(t)\cdot\gamma(t)} \lambda^{-1/d} + O(\lambda^{-2/d}),$$

where the implicit constant is upper bounded by a constant multiple of $\|f\|_{L^\infty} + \|f'\|_{L^\infty}$*.*

Proof. This follows from one-dimensional stationary phase methods (see Erdelyi, in the discussion of Equation (4) of Section 2.9), because we have made the assumption that the function ν above is positive. $\square$

Conversely, we can also apply the principle of nonstationary phase.

Lemma 3. *Suppose that if f is a C^1 function supported on an interval of length L, ξ is a unit vector in $\mathbb{R}^d$, and and $|\xi \cdot \gamma'(t)| \geqslant \varepsilon$ for all $t \in I$. Then*

$$I_\gamma(f, \lambda\xi) \lesssim_\gamma \frac{L}{\lambda}\left(\frac{\|f'\|_{L^\infty}}{\varepsilon} + \frac{\|f\|_{L^\infty}}{\varepsilon^2}\right).$$

Proof. We integrate by parts, calculating that

$$\begin{aligned}\left|\int e^{i\lambda\xi\cdot\gamma(t)} f(t)\, dt\right| &= \frac{1}{\lambda}\left|\int \frac{d}{dt}\left\{e^{i\lambda\xi\cdot\gamma(t)}\right\} \frac{f(t)}{\xi\cdot\gamma'(t)}\, dt\right| \\ &= \frac{1}{\lambda}\left|\int e^{i\lambda\xi\cdot\gamma(t)}\left(\frac{f'(t)}{\xi\cdot\gamma'(t)} - \frac{f(t)}{(\xi\cdot\gamma'(t))^2}(\xi\cdot\gamma''(t))\right)\, dt\right| \\ &\lesssim_\gamma \frac{L}{\lambda}\left(\frac{\|f'\|_{L^\infty}}{\varepsilon} + \frac{\|f\|_{L^\infty}}{\varepsilon^2}\right).\end{aligned}$$

□

Lemma 4. *Let $\gamma_M(t) = (t, t^2, t^3)$ be the parameterization of the moment curve $\Gamma_M = \gamma_M(\mathbb{R})$. For any $\varepsilon \in (0, 1/100)$, if t_0 is a fixed time, ξ_0 is one of the vectors orthogonal to both $\gamma_M'(t_0)$ and $\gamma_M''(t_0)$, $\lambda \gtrsim_\varepsilon 1$, then*

$$\sup_{|\xi - \lambda\xi_0| \leqslant \varepsilon\lambda} |\xi|^{1/3} |I_{\gamma_M}(\mu, \lambda\xi)| \lesssim_\varepsilon 1.$$

Proof. Fix $\delta > 0$ and $\lambda \geqslant 1$, and suppose there was a probability measure μ compactly supported on some interval I such that

$$\sup_{|\xi - \lambda\xi_0| \leqslant \lambda\varepsilon} |\xi|^{1/3} |I_{\gamma_M}(\mu, \xi)| \leqslant \delta.$$

Define a linear transformation

$$A_h = \begin{pmatrix} 1 & 0 & 0 \\ 2h & 1 & 0 \\ 3h^2 & 3h & 1 \end{pmatrix}.$$

Then $A_h\gamma_M(t) = \gamma_M(h) + \gamma_M(t+h)$ for all $t, h \in \mathbb{R}$. If $\gamma_{M,h}(t) = \gamma_M(t+h)$, we thus have

$$\begin{aligned}I_{\gamma_{M,h}}(\mu, \xi) &= \int e^{i\xi\cdot\gamma(t+h)} d\mu(t) \\ &= e^{-i\xi\cdot\gamma(h)} \int e^{i\xi\cdot A_h\gamma(t)} d\mu(t)\, dt \\ &= e^{-i\xi\cdot\gamma(h)} \int e^{i(A_h^T\xi)\cdot\gamma(t)} d\mu(t)\, dt \\ &= e^{-i\xi\cdot\gamma(h)} I_{\gamma_M}(\mu, A_h^T\xi).\end{aligned}$$

If we consider an L^1 normalized smooth bump function $\phi : \mathbb{R} \to \mathbb{R}$ adapted to $\{|h| \leqslant \varepsilon/2\}$, and define a smooth function $f = \phi * \mu$, then

$$I_{\gamma_M}(f, \lambda\xi_0) = \int \phi(h) I_{\gamma_{M,h}}(\mu, \lambda\xi_0)\, dh = \int \phi(h) e^{-i\xi\cdot\gamma_M(h)} I(\mu, \lambda A_h^T\xi_0)\, dh.$$

Then the L^∞ norm of f and f' is $O_\varepsilon(1)$, and $f(t_0) \gtrsim_\varepsilon 1$, so we conclude that

$$I_{\gamma_M}(f, \lambda\xi_0) = C\ a(t_0)f(t_0)e^{i\lambda\xi_0}\lambda^{-1/3} + O_\varepsilon(\lambda^{-2/3}).$$

In particular, we conclude that for $\lambda \gtrsim_\varepsilon 1$,

$$|I_{\gamma_M}(f, \lambda\xi_0)| \gtrsim C_\varepsilon \lambda^{-1/3}.$$

Now $|A_h^T\xi_0 - \xi_0| \leqslant 4h|\xi_0|$ for $|h| \leqslant 1/100$, we know by assumption that $|I(\mu, \lambda A_h^T\xi_0)| \leqslant \delta\lambda^{-1/3}$. But this means we conclude that

$$\lambda^{-1/3} \lesssim_\varepsilon \delta\lambda^{-1/3},$$

and thus that $\delta \gtrsim_\varepsilon 1$, completing the proof. □

For any measure μ on I, we fix $\delta > 0$, and consider a family of $O(\delta^{-1})$ points $\mathcal{T}$ such that the length δ intervals $\{I_t : t \in \mathcal{X}_\delta\}$ with center t cover $[0,1]$, and for each t, the middle third of the interval I_t is disjoint from $I_{t'}$ for $t \neq t'$. Consider a smooth partition of unity $\{\chi_t\}$ adapted to these intervals. For each $t \in \mathcal{T}$, define $\mu_t = \chi_t\mu$. For any $t \in \mathcal{T}$, consider the degree three polynomial curve $\gamma_t : \mathbb{R} \to \mathbb{R}^d$ given by

$$\gamma_t(s) = \gamma(t) + \gamma'(t)(s-t) + \frac{\gamma''(t)}{2}(s-t)^2 + \frac{\gamma'''(t)}{6}(s-t)^3.$$

then for any $t' \in I_t$, $|\gamma(t') - \gamma_t(t')| \lesssim \delta^4$. This means that the deviations between γ and γ_t, once localized to a δ neighborhood of t, should be undetectable for frequencies with magnitude $O(\delta^{-4})$, i.e. for $|\xi| \lesssim \delta^{-4}$, we should expect to have

$$I_\gamma(\mu, \xi) \approx \sum_t I_{\gamma_t}(\mu_t, \xi).$$

If we let B_t be the matrix with columns $\delta^{-1}\gamma'(t)$, $\delta^{-2}\gamma''(t)/2$, and $\delta^{-3}\gamma'''(t)/6$, then

$$\gamma_t(s) - \gamma(t) = B_t\gamma_M(\delta(s-t)).$$

Thus if ν_t is the dilation of $\mathrm{Trans}_{-t}\mu_t$ by a factor $1/\delta$, then

$$\begin{aligned} I_{\gamma_t}(\mu_t, \xi) &= \int e^{i\xi\cdot\gamma_t(s)} d\mu_t(s) \\ &= \int e^{i\xi\cdot[\gamma(t) + B_t\gamma_M((s-t)/\delta)]}\ d\mu_t(s) \\ &= e^{i\xi\cdot\gamma(t)} \int e^{i(B_t^T\xi)\cdot\gamma_M((s-t)/\delta)} d\mu_t(s) \\ &= e^{i\xi\cdot\gamma(t)} I_{\gamma_M}(\nu_t, B_t^T\xi). \end{aligned}$$

Thus we get

$$I_\gamma(\mu, \xi) \approx \sum_t e^{i\xi\cdot\gamma(t)} I_{\gamma_M}(\nu_t, B_t^T\xi).$$

for $|\xi| \ll \delta^{-4}$. We now consider an L^1 normalized, smooth bump function ϕ supported on a width δ interval about the origin, and define $f_t = \nu_t * \phi$. We have seen that

$$I_{\gamma_M}(f_t, B_t^T \xi) = \int \phi(h) e^{-i\xi \cdot \gamma_M(h)} I(\nu_t, A_h^T B_t^T \xi)\, dh.$$

Suppose (THIS IS THE CHEAT) we can find a matrix C_h such that $A_h^T B_t^T \xi = B_t^T C_h \xi$. Then

$$\sum_t I_{\gamma_M}(f_t, B_t^T \xi) = \sum_t \int \phi(h) e^{-i\xi \cdot \gamma_M(h)} I(\nu_t, B_t^T C_h \xi) \approx \int \phi(h) e^{-i\xi \cdot \gamma_M(h)} I_\gamma(\mu, C_h \xi)\, dh.$$

Then C_0 is the identity matrix, and so we can imagine that $|C_h \xi| \sim |\xi|$ for small h.

We can now argue that $\dim_{\mathbb{F}}(\Gamma) \leqslant 2/3$. Suppose that instead, we could choose μ such that

$$\limsup_{\xi \to \infty} |\xi|^{2/3+\varepsilon} |I_\gamma(\mu, \xi)| < \infty.$$

Then for any $\xi \in \mathbb{R}^d$, the right hand side of the identity above satisfies estimates of the form

$$\left| \int \phi(h) e^{-i\xi \cdot \gamma_M(h)} I_\gamma(\mu, C_h \xi)\, dh \right| \lesssim |\xi|^{-1/3-\varepsilon}.$$

For $|\xi| \sim \delta^{-4}$, we get that this quantity is $\lesssim \delta^{4/3+\varepsilon}$. On the other hand, the left hand side is a sum of quantities to which we can apply stationary and nonstationary phase. If we choose $c > 0$ small enough, depending on γ, then because of the linear independence of γ', γ'', and γ''', if, for $t_0 \in \mathcal{T}$, we set $\xi = \xi_0(t_0)$, then for any $t \neq t_0$, and any $t' \in I_t$, $|\xi \cdot \gamma'(t')| \geqslant c\delta$. This implies that the principle of nonstationary phase can be applied to the quantity $I_\gamma(\nu_t, B_t^T \xi)$. For each t_0, the function f_{t_0} has L^∞ norm at most $O(\delta^{-1} \nu_{t_0}(\mathbb{R}))$, and f_{t_0}' has L^∞ norm bounded by $O(\delta^{-2} \nu_{t_0}(\mathbb{R}))$. Applying the principle of nonstationary phase, for $t \neq t_0$ we conclude that

$$|I_{\gamma_M}(f_t, B_t^T \xi)| \lesssim \delta^{-2} \nu_{t_0}(\mathbb{R}) |\xi|^{-1}.$$

Summing over $t \neq t_0$ gives that

$$\sum_{t \neq t_0} |I_{\gamma_M}(f_t, B_t^T \xi)| \lesssim \delta^{-2} |\xi|^{-1}.$$

If we take $|\xi| \sim \delta^{-4}$, this quantity is $O(\delta^2)$. On the other hand, we have $f_{t_0}(t_0) \gtrsim \delta^{-1} \nu_{t_0}(\mathbb{R})$, and so the principle of stationary phase we calculated at the beginning of our argument shows that

$$|I_{\gamma_M}(f_{t_0}, B_{t_0}^T \xi)| \gtrsim \delta^{-1} \nu_{t_0}(\mathbb{R}) |\xi|^{-1/3}$$

so for $|\xi| \sim \delta^{-4}$, we get that this quantity is $\gtrsim \delta^{1/3} \nu_{t_0}(\mathbb{R})$. Since $\sum_t \nu_t(\mathbb{R}) = \mu(\mathbb{R}) = 1$, the pigeonhole principle implies we can pick some t_0 such that $\nu_{t_0}(\mathbb{R}) \gtrsim \delta$. But then the quantity above is $\gtrsim \delta^{4/3}$. But putting these bounds together gives that

$$|\sum_t I_{\gamma_M}(f_t, B_t^T \xi)| \geqslant |I_{\gamma_M}(f_{t_0}, B_{t_0}^T \xi)| - \sum_{t \neq t_0} |I_{\gamma_M}(f_t, B_t^T \xi)| \gtrsim \delta^{4/3}.$$

But we therefore conclude that $\delta^{4/3} \lesssim \delta^{4/3+\varepsilon}$, which gives a contradiction if δ is taken appropriately small.