Research Statement

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My current research focuses on the study of Fourier multiplier operators, and their relation to multiplier operators on compact manifolds, as well as the study of fractal dimension and it's relation to geometric patterns in sets.

My study of patterns lead to a probabilistic construction of sets of large Fourier dimension avoiding the vertices of any isosceles triangle within any given smooth curve in $\mathbb{R}^d$. This remains the only construction in the literature of a set with significant Fourier dimension which avoids a *nonlinear* pattern.

My work on multipliers lead to a complete characterization of all operators on S^d diagonalized by the spherical harmonics which are bounded on L^p, for a certain range of p. This is the first characterization of L^p boundedness of such multipliers for any $p \neq 2$, and no other characterizations have been proved for multipliers of analogous operators on any other compact manifold.

Both projects

The work I have conducted naturally suggests several **future problems**.

- Analyzing the 'return time operator' to extend results on expansions of spherical harmonics to the study of the Laplace-Beltrami operator on S^d.
- Determining whether our methods extend to other manifolds whose geodesic flow is simpler to understand, such as integrable systems.
- Analyzing whether local smoothing bounds
- Constructing Random Salem Sets which satisfy a Decoupling Bound.
- Determining the relation between certain 'fractal weighted estimates' for the wave equation on $\mathbb{R}^d$ and the 'density decomposition' of multiplier bounds.

We provide a further elaboration of these directions later on in the summary.

Fourier Multiplier Operators

Multiplier operators are central to the study of harmonic analysis. One research project of mine considers the study of the relation between the L^p boundedness of two types of multipliers:

- *Fourier multipliers* are those operators T on $\mathbb{R}^d$ for which we can associate a function $m : \mathbb{R}^d \to \mathbb{C}$, known as the *symbol* of T, such that for any function f, the Fourier transform of Tf obeys the relation $\widehat{Tf} = m\widehat{f}$. Heuristically, this means that $Te^{2\pi i\xi\cdot x} = m(\xi)e^{2\pi i\xi\cdot x}$ for each $\xi \in \mathbb{R}^d$.

- On the sphere S^d, we have an orthogonal decomposition of $L^2(S^d)$ into the spaces $\mathcal{H}_k(S^d)$ of spherical harmonics of degree k. A *multiplier for spherical harmonics* is an operator T on S^d for which there exists $m : \mathbb{N} \to \mathbb{C}$, the *symbol* of T, such that $Tf = m(k)f$ for each $f \in \mathcal{H}_k(S^d)$.

Such operators initially arose from the study of the classical partial differential equations in physics, and continue to have applications in areas as diverse as partial differential equations, mathematical physics, number theory, and ergodic theory. Every translation invariant operator on $\mathbb{R}^d$ is a Fourier multiplier operator, and every rotation invariant operator on S^d is a multiplier of the spherical harmonic expansion on S^d.

any such operator T, we can associate a function $m : \mathbb{R}^n \to \mathbb{C}$, known as the *symbol* of T, such that for any function f, the Fourier transform of Tf obeys the relation $\widehat{Tf} = m\widehat{f}$; thus translation invariant operators are also called *Fourier multiplier operators*.

Understanding the boundedness of Fourier multiplier operators in an L^p norm for $p \neq 2$ underpins any subtle understanding of the Fourier transform. Plane waves oscillating in different directions and with different frequencies are orthogonal to one another, and thus do not interact with one another significantly in terms of the L^2 norm, as justified by Bessel's inequality. But plane waves can interact with one another in the L^p norm for $p \neq 2$, and so understanding L^p bounds for Fourier multipliers indicate when this interaction is significant or insignificant. Similarily, spherical harmonics of different degrees on S^d are orthogonal to one another, but studying the L^p bounds of multipliers of the Laplacian on the sphere is crucial to understand when the interactions of different spherical harmonics are significant or not.

The general study of the characterizations of L^p boundedness for the Fourier multipliers was initiated in the 1960s. Mathematicians quickly found simple necessary and sufficient conditions that ensure Fourier multipliers are bounded on $L^1(\mathbb{R}^d)$, $L^2(\mathbb{R}^d)$, and $L^\infty(\mathbb{R}^d)$. But the problem of finding necessary and sufficient conditions for boundedness in $L^p(\mathbb{R}^d)$ for any other exponent proved impenetrable. Indeed, many interesting problems about the boundedness of *specific* Fourier multipliers, such as the Bochner-Riesz conjecture, remain largely unsolved today.

Thus it came as a surprise in the past decade when results emerged proving necessary and sufficient conditions for *radial* Fourier multipliers to be bounded on $L^p(\mathbb{R}^d)$. First came the result of BLAH, which gave a necessary and sufficient criteria for bounds of the form $\|Tf\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$ to hold uniformly over *radial functions* f, for $|1/p - 1/2| > 1/2d$. An optimist might think this same condition causes the bound to hold uniformly over *all functions* f in the range above, a statement we call the *radial multiplier conjecture*. We now know, by the results of BLAH and BLAH, that the radial multiplier conjecture holds when $d > 4$ and $|1/p - 1/2| > 1/(d-1)$, when $d = 4$ and $|1/p - 1/2| > 11/36$, and when $d = 3$ and $|1/p - 1/2| > 11/26$. But the radial multiplier conjecture has not yet been completely solved in any dimension d, and no bounds are known at all when $d = 2$.

The natural analogue of the study of radial multipliers on $\mathbb{R}^d$ is the study of multipliers of a Laplace-Beltrami operator on a Riemannian manifold. The natural analogue of the study of quasiradial multipliers on $\mathbb{R}^d$ is the study of multipliers of an operator associated with a *Finsler geometry* on the manifold.

Pattern Avoidance

Future Lines of Research