

# Mechanics

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Classical mechanics attempts to describe the motion of objects in space, where the relative motions and masses of those objects are neither too small nor too large, so that relativity and quantum effects can be discounted. Starting with the work of physicists like Newton, Galileo, and Kepler, mechanics has evolved into a large field of techniques. In these notes, we cover the classical methods, as well as introducing the more modern, 18th century *analytical mechanics* of physicists like Lagrange and Hamilton.

# Chapter 1

## Prelude: Dimensional Analyses

In most physical theories, we take the given state of a system, and assign certain *coordinates*, definite quantities that identify a particular physical state. We can then obtain relations between these coordinates, given by various equations, that determine how a physical system evolves over time. Often, we have many different coordinate systems to choose from. This should be entirely natural from our experiences prior to learning physics, since we all measure the world from different perspectives (a chair is one metre to the left of me, whereas it may be three metres to the right of you) but we can use the same logical structure to understand how the universe behaves (except in certain exceptional cases, such as when sitting in an accelerating vehicle).

Often, the state space of a physical system can be identified with vectors in a vector space  $V$ , and the family of acceptable coordinate systems is closed under *dilations* of that vector space. These kind of symmetries give rise to the concept of a *unit* in physics. Various quantities expressed in our coordinate system will transform in a predictable way under a dilation of a coordinate system, and so these may also be ascribed units reflecting how they transform.

In classical mechanics, the state of a system can be identified with the spatial positions of all objects under consideration, the masses of those objects, and the forces exerted upon each of those objects. The fundamental units chosen are thus *time*, *mass*, and *space*, and these are ‘independent units’ since they can all be dilated independently of one another. In relativistic mechanics and electromagnetics, time and space are no longer independent units (we can measure one unit of space as the length travelled by light in one unit of time), and so only *time*, *mass*, and *charge* are fundamental units. The coordinates describing the position of a particle are purely spatial - dilating the spatial coordinates by a factor  $S$  dilates the coordinates describing the position of a particle by that same factor  $S$ , and so the position of a particle is described using ‘spatial units’. The *velocity* of a particle is defined in terms of the time and space coordinate. Dilating the spatial coordinates by a factor  $S$  and the temporal units by a factor  $T$  dilates the velocity of a particle by a factor  $S/T$ , and so the units of velocity are ‘space’ over ‘time’. Similarly, the acceleration has a dilation symmetry characterized by ‘space’ over ‘time squared’, i.e. it dilates like  $S/T^2$ . Momentum has units of ‘mass’ times ‘space’ over ‘time’. In all cases, the unit of a quantity is defined by how it

behaves under a dilation symmetry of each separate coordinate.

The advantage of keeping track of the ‘units’ of various quantities is that it allows us to perform what physicists call a ‘dimensional analysis’. Suppose that we establish an equation of the form  $A = B$  between two different physical quantities  $A$  and  $B$ , and this identity holds under arbitrary dilations of the coordinate system. Provided  $A$  and  $B$  can be associated with a certain unit then  $A$  and  $B$  *must* have the same units; if they didn’t, then a scaling symmetry would lead the equation to break. On the other hand, one may establish such identities in a *particular* coordinate system, and then establish a general equation by adding constants with units which make the equations universal. For instance, in a certain coordinate system we might establish the equation determining the motion of a particle on the end of a spring of the form  $m\ddot{x} = -x$ . The left hand side has units of the form  $S/T^2$ , and the right side has units of  $S$ . In an arbitrary coordinate system, the equation must therefore be of the form  $m\ddot{x} = -kx$ , where  $k$  is a physical quantity with units  $M/T^2$ .

Physicists prefer to work between coordinate systems in which the units are fixed, i.e. which preserve measurements of time, mass, and space – if we did not agree with what the length of a centimetre was it would be very difficult to communicate the results of experiments. Because of this, in classical mechanics the only remaining symmetries of the acceptable coordinate systems are rotation and translation symmetries in space. In relativistic mechanics the remaining symmetries are rotations in *space-time*.

## Chapter 2

### Newton's Laws

The fundamentals of classical mechanics are based on Newton's laws of motion, who in the 1600s, collecting insights from prior physicists like *Gallileo Gallilei*, *Johannes Keplar*, *Christopher Wren*, *John Wallis*, and *Christiaan Huygens*, developed the *action reaction* principle from the preexisting principle of conservation of momentum, abstracted and sophisticated the concept of a *force* beyond that of objects in direct contact with one another, and developed a theory of *gravitational attraction*, which identified the force which held Earth's objects together with the force that determined the motion of planets. The resulting success of Newton's systemification is why Classical Mechanics is often called *Newtonian Mechanics*, and its impact on physics cannot be overstated; the identification and study of the laws of the *forces* of nature became the primary study of physics ever after.

Newton's *Principia* is heavily inspired by Euclid's *Elements*, proceeding from axioms and logically deriving the physical consequences. The standards of rigour are somewhat loose from the standards of today, with definitions and initial proofs beginning in a somewhat shaky, circular fashion, before eventually finding their footing. We will attempt to interpret the principles in a more precise way. Before we do this though, we should briefly talk about how we *interpret* the mathematical system.

From a modern viewpoint, we only interpret Newton's Laws only as *sufficient approximations* to phenomena we can observe in experimentation. The axioms may fail to be true precisely, because another theory describes phenomena to a higher degree of precision (e.g. quantum mechanics or relativity). Or the axioms may fail to be true precisely because in order to derive useful consequences of the theory we are forced to ignore various minor phenomena that complicate a situation to a point where we cannot mathematically derive measurable consequences. Despite the fact that we cannot take Newton's Laws as precisely describing the real world, they are also useful because they are compact representations of the state of our knowledge and understanding of the physical mechanics of our universe.

Also, unlike the standard 20th century approach of mathematicians, who often take their fundamental objects of study as essentially undefined, physicists after the 20th century tend to agree that fundamental properties must be interpreted *operationally*, i.e. they must be

defined by how they can be measured from some particular observation<sup>1</sup>. This makes sense from an empirical point of view, how else could the accuracy of a theory be validated if it's consequences on properties of physical objects could not be directly measured? There are some subtle philosophical problems with this approach: force and mass, for example, are metaphysical concepts one cannot directly observe, and we can only measure them as *consequences of Newton's Laws*. We will take the operational definition mainly to orient how we can empirically test Newton's theory.

## 2.1 The Laws

The basic properties of physical objects in Newton's system are physical objects with *mass*, and which are impacted by *forces*, which have both magnitude and are applied in a particular direction. In the basic setup, it is natural to represent the positions of points as taking values in the Euclidean space  $\mathbb{E}^3$ , masses taking values in  $(0, \infty)$ , and forces by vectors in  $\mathbb{R}^3$ . We should remark that in the original formulation of Newton's mechanics, written often in a geometric language without the use of the tools of the Calculus, often interprets forces as acting instantaneously at a moment in time (what modern physicists would call an *impulse*), rather than acting dynamically over time.

Newton states his *First Law* as follows:

*Every body perseveres in its state of being at rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed therein.*

In modern language, the first law states that an object not acted upon by a force travels with constant velocity, and is often called the *law of inertia* (inertia is the tendency of an object to remain at rest or at a constant velocity).

There is an implicit assumption in the first law that we are observing motion from a *particular coordinate system*. If I sit on a roundabout in a playground, from my perspective objects will rotate without any force acting upon them. Thus Newton's Law implicitly postulates the existence of *inertial coordinate systems*, i.e. coordinate systems under which the first law applies.

Given the existence of one inertial coordinate system  $T : \mathbb{E}^3 \times \mathbb{R} \rightarrow \mathbb{R}^{3+1}$ , any other inertial system can be obtained by composing  $T$  with a *Gallilean symmetry*, i.e. an isometry  $G : \mathbb{R}^{3+1} \rightarrow \mathbb{R}^{3+1}$  of the form

$$G(x, t) = (A(x - x_0) - (t - t_0)v, t - t_0)$$

for values  $x_0 \in \mathbb{R}^3$ ,  $t_0 \in \mathbb{R}$ ,  $v_0 \in \mathbb{R}^3$ , and  $A \in O(3)$ . Physically, the new coordinate system measures time  $t_0$  units differently to the original coordinate system, is centered around a point starting at  $x_0$  and travelling at a velocity  $v$  in the original coordinate system, and measuring spatial coordinates along the axis given by the unit vectors  $Ae_1$ ,  $Ae_2$ , and  $Ae_3$ .

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<sup>1</sup>Defining physical properties operationally is a modern perspective - Newton doesn't define physical properties operationally, for instance, defining mass in the principia somewhat circularly as "a measure of the same, arising from it's density and bulk conjunctly".

Newton's Second Law states that

*The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that forces is impressed.*

In modern language, we state the law states that in an inertial system, the rate of change of the momentum of an object is directly proportional to the force acting upon it. Choosing our units to make the constant of proportionality equal to one, the law can be summarized by the famous equation  $F = ma$ , though Newton never stated the law in this way. Our interpretation of the second law does not imply our interpretation of the first law, since the first law also implies the existence of an inertial coordinate system.

Finally, we have Newton's Third Law. It is equivalent to the law of conservation of momentum already known to physicists, but stated in the novel 'action-reaction' form:

*To every action there is always opposed an equal reaction: or the mutual actions of two objects upon each other are always equal, and directed to contrary parts.*

Newtons law states that forces always come in pairs, *between two objects*: at an instant in time, if an object  $A$  exerts a force  $F$  on an object  $B$ , then Newton's Law states that object  $B$  will exert a force  $-F$  on object  $A$ .

The Third Law is quite unintuitive to the ways we normally think about interacting with the world. We talk about 'pushing a door open', ignoring the fact that the door, at the same time, emits a mysterious force that pushes us backwards. It can be perhaps made more intuitive by a reduction to the scale of atoms. All things are made of protons and electrons, and from this perspective pushing a door open becomes a symmetric situation: the negatively charged electrons in my hand and the negatively charged electrons in the door repel one another (though things are not so symmetric when we consider situations involving protons, electrons, and neutrons, and yet the third law still appears to apply in this situation).

The Third Law is equivalent to the principle of the conservation of momentum. In a system of  $N$  objects with masses  $m_1, \dots, m_N$  and position  $x_1(t), \dots, x_N(t)$ , the momentum

$$m_1 v_1 + \dots + m_N v_N$$

is invariant of  $t$ . Indeed, the derivative of this quantity in time is equal to

$$m_1 a_1 + \dots + m_N a_N = F_1 + \dots + F_N,$$

where  $F_j$  is the cumulative force on the object  $j$ . The Third Law implies that  $F_1 + \dots + F_N = 0$ . Conversely, the conservation of momentum implies the Third Law provided we assume that forces only emerge from pairwise actions, i.e.

$$F_j = \sum_{i \neq j} F_{ij},$$

and that the forces are invariant under rotational symmetries. Ron Maimon on [Physics Stack Exchange](#) gives a wide array of counterexamples without this assumption, including physical

examples under which it is reasonable to consider non-pairwise forces between two different objects (three particles forming the vertices of a soap bubble).

These are the three laws. Taken on their own, they say nothing about how objects will behave, because we do not know *what forces between objects are out there*. The three laws only constrain *what forces can do to objects*. In particular, the masses and forces on a given family of objects and an instance of time is *underdetermined* from the information we can instantaneously observe. A simple example of this underdetermined is a system of three objects at rest; this could either be the result that these objects exert no forces on one another, or forces on each object cancelling out with one another.

On the other hand, we are only directly able to observe the accelerations of these  $N$  objects. We thus only observe  $3N$  parameters of the problem, and since  $3N < N + 3\binom{N}{2}$  for  $N > 3$ , it is impossible to recover the masses and forces on a family of objects solely from what can be directly observed.

On the other hand, if we make assumptions on the relative strength and behaviour of the various forces, it *is* possible to recover the masses of the objects, and once th

The definition by itself is somewhat circular (masses are defined in terms of a force assumed to be constant for two different objects, and then all forces can be determined once the mass is known). However, once we account that this definition remains

but experience proves that the process doesn't lead to inconsistencies at the scale of physics under which we apply the principles of classical mechanics, and so in practice this circular definition doesn't prove to be a problem.

We begin with some observations that should appear mostly obvious, but are incredibly important to further developments. The first axiom tells us how space can be described – in classical mechanics, we do this in the style of Euclid.  $n$ -dimensional affine space is a set  $\mathbf{E}^n$  together with a faithful, transitive free group action from the additive group  $\mathbf{R}^n$ , denoted by  $+$ . From this, it follows that for any two points  $a$  and  $b$  in  $\mathbf{E}^n$ , there is a unique vector  $v$  in  $\mathbf{R}^n$  such that  $a + v = b$ . We denote  $v$  by  $b - a$ . From this, we may assign a metric structure on  $\mathbf{E}^n$ , defining  $d(a, b) = \|b - a\|$ . The standard metric properties on  $\mathbf{R}^n$  then follow in  $\mathbf{E}^n$ .

Now the space used in euclidean geometry is  $\mathbf{E}^3$ , since there is no notion of time in geometric studies. To enhance this, we add an additional dimension to our space, and use  $\mathbf{E}^4$  to describe space, known as Galilean spacetime. In addition to this, we need a linear transformation  $t$  from  $\mathbf{R}^4$  to  $\mathbf{R}$ , which describes the 'time displacement' in our system. The use of this transformation is to determine the time between two events (an event is just another word for a point in  $\mathbf{E}^4$ ). If  $a$  and  $b$  are events, then  $t(b - a)$  is the time between the two events. If  $t(b - a) = 0$ , then  $a$  and  $b$  are called simultaneous events. The set of all such simultaneous events to some point  $a$  is called the space of simultaneous events relative to  $a$ . Euclid's discourse on geometry arises from any specific simultaneous subspace. However, it is then implicitly assumed that the space of points simultaneous to any point  $a$  is three-dimensional – we must therefore assert that the kernel of  $t$  is three dimensional. Space has now been described in a rigorous manner. Because  $\mathbf{E}^4$  is an affine space, it has no inherent coordinate system to use (unless you ascribe to some geocentric model of the universe). The Galilean group consists of all invertible affine transformations of  $\mathbf{E}^4$  which preserve time and



space measurements between events. That is,  $T$  is an element of the Galilean group if

$$t(y - x) = t(T(y) - T(x))$$

$$\|y - x\| = \|T(y) - T(x)\|$$

We will see that galilean transformations are those that preserve Newton's laws.

Let us consider some elementary examples of galilean transformations. Let our euclidean space be  $\mathbf{R} \times \mathbf{R}^3$ . The first galilean transformation is a motion by constant velocity. For any vector  $v$ , the map

$$f(t, x) = (t, x + vt)$$

is Galilean. The second is the change of origin; for any number  $s$  and vector  $s'$ ,

$$f(t, x) = (t + s, x + s')$$

is also Galilean. Finally, if  $G$  is in  $O(3)$ , then

$$f(t, x) = (t, Gx)$$

is Galilean, and is a rotation of the coordinate axis. It can be shown that every galilean transformation arises uniquely as the composition of a translation, rotation and motion of constant velocity, so that the dimension of the Galilean group is  $3 + 4 + 3 = 10$ .

Every euclidean space  $\mathbf{E}^4$  can be put in one-to-one correspondence with  $\mathbf{R} \times \mathbf{R}^3$ . We call any such one to one correspondence  $\varphi$  a galilean coordinate system. If  $\varphi$  and  $\varphi'$  are coordinate systems, then  $\varphi$  is Galilean with respect to  $\varphi'$  if  $\varphi \circ \varphi'^{-1}$  is a Galilean transformation.

Empty space is easy to describe. The problem of physics is to describe space when objects are introduced into it. A motion in  $\mathbf{R}^n$  is a differentiable mapping  $x$  from some interval  $I$  in  $\mathbf{R}$  to  $\mathbf{R}^n$ . The derivative  $\dot{x}$  is called the velocity, and the second derivative  $\ddot{x}$  is called the acceleration. The image of this mapping is called a trajectory, and the graph of the function is a curve in  $\mathbf{R} \times \mathbf{R}^n$ . A curve in  $\mathbf{E}^4$  is called a world line if it appears in some (and hence every) galilean coordinate system as the graph of a motion. The direct product of  $\mathbf{R}^3$   $n$  times is called the configuration space of a system of  $n$  points.  $n$  motions  $x_1, x_2, \dots, x_n$  in  $\mathbf{R}^3$  define a single motion  $\mathbf{x} : \mathbf{R} \rightarrow \mathbf{R}^{3n}$ .

If we know  $\mathbf{x}(t_0)$  and  $\dot{\mathbf{x}}(t_0)$  at a single time  $t_0$ , Newton's principle of determinacy guarantees that we can determine  $\mathbf{x}$  at all times both before and after  $t_0$ . In particular, this is actually a notion equivalent to the second law that Newton describes. That is, Newton's second law prescribes exactly that in each galilean reference frame there exists a function  $F : \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R} \rightarrow \mathbf{R}^N$  such that

$$\ddot{\mathbf{x}}(t) = F(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)$$

Provided this function  $F$  is smooth enough, which is always assumed in physics, then the theorem of uniqueness and existence of solutions to ordinary differential equations states that there is a unique motion  $\mathbf{x}$  determined by any initial conditions  $\mathbf{x}(t_0)$  and  $\dot{\mathbf{x}}(t_0)$ . The

importance of the second law is thus paramount to the description of motions in space that classical mechanics attempts to describe.

Newton's first law classically states that objects which are at rest remain at rest unless acted upon by some force. Of course, this cannot be completely true regardless of the frame of reference objects are measured in. If I sit on a roundabout, objects that are at rest relative to me begin to move as the roundabout begins to rotate, and they have no force acted on them. Obviously, the first law requires some higher sense of rigour. Nowadays, the only use of the first law should more relevantly be called the Galilean principle of relativity. This principle states that in  $\mathbf{E}^4$  there exists a class of galilean coordinate frames called inertial, having the following property. Let  $f$  is inertial, and if  $\mathbf{x}$  is a motion in  $f$  such that  $\ddot{\mathbf{x}} = F$  is satisfied. If  $f'$  is a reference frame galilean with respect to  $f$ , then not only should  $F'$  be equal to  $F$ , but also  $\mathbf{x}'(f' \circ f^{-1} \circ \mathbf{x})$  should satisfy  $\ddot{\mathbf{x}}' = F'$ .

We can deduce many interesting facts about inertial reference frames from the first law. Since the map

$$\varphi(t, x) = (t + s, x)$$

is a galilean transformation, then if  $F$  is the force in an inertial reference frame, then  $F(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) = F(\mathbf{x}, \dot{\mathbf{x}}, t + s)$  for any real number  $s$ . Thus  $F$  is really only a function of position and velocity. This expresses the knowledge that the laws of physics are irrelevant with respect to time. Therefore, if we see that a force depends on the time of a system, we know that there we are either missing some forces, or we are not measuring the force with respect to an inertial reference frame.

The second simplification results from the fact that the map

$$\varphi(t, x) = (t, x + r)$$

is galilean for any vector  $r$ , so that  $F(\mathbf{x}, \dot{\mathbf{x}}, t) = F(\mathbf{x} + r, \dot{\mathbf{x}}, t)$ . We must therefore have the force  $F$  be related only to relative positions  $x_i - x_j$ . Similarly, since a shift by constant velocity is galilean,  $F$  can also only be described by relative velocities  $\dot{x}_i - \dot{x}_j$ . This tells us that space is homogenous – it has the same properties at all points. Furthermore, it tells us that forces are intrinsically linked to the motions of particles themselves, since the forces are related only to a particles position between one another.

Our final notion is that space is isotropic – it prefers no sense of direction. If  $G$  is a rotation, since  $G$  is also a galilean transformation, then we must have  $F(\dot{x}, \ddot{x}, t) = F(G\dot{x}, G\ddot{x}, t)$ .

1. There exists an inertial reference frame in which objects at rest remain at rest unless acted upon by some force.
2. The net force on an object is equal to the change in its momentum.

Newton's first law is also known as Galileo's law of inertia. Mathematically, the second law states that, if  $r(t)$  parameterizes the movement of some body in space in relation to time, and if  $F_{net}(t)$  represents the force, then, since

$$p(t) = mr'(t)$$

described the momentum of a particle,

$$F_{net}(t) = mr''(t)$$

assuming mass remains constant throughout the objects movement.

Separated from Newton's second law, forces are constrained by some other physical law, derived by experimentation, and mainly described by a differential equation. The goal of classical mechanics, is, given a set of particles in an initial position, given the physical laws present, determine completely the motion of each particle.

Let us consider an elementary example, the motion of a single particle in one dimension due to the effects of gravity. Suppose  $r(0) = h$ , and  $r'(0) = 0$ . A simplification of Newton's laws of gravitation, which are accurate for small distances, tells us that the force due to gravity is described

$$F_G(t) = mg$$

where  $m$  is the particles mass, and  $g$  is the gravitational acceleration. Assuming this is the only force on the particle (so  $F_{net} = F_G$ ), we obtain from Newton's second law that

$$mr''(t) = F_G(t) = mg$$

Dividing by  $m$  (assuming the particle has some mass),

$$r''(t) = g$$

Functions that have the same derivative on an open interval differ by a constant on that interval. Thus, for some constant  $c_0$

$$r'(t) = gt + c_0$$

As the initial velocity of 0,  $c_0 = r'(0) = 0$ , and  $r'(t) = gt$ . Performing an antidifferentiation again, for some constant  $c_1$ ,

$$r(t) = \frac{g}{2}t^2 + c_1$$

and  $c_1 = r(0) = h$ . Hence our motion is completely defined by

$$r(t) = \frac{g}{2}t^2 + h$$

In the manner of classical mechanics, this simple problem has been 'solved'.

Suppose we add an additional drag force to resist the downward velocity,

$$F_d = -\beta r'$$

so that

$$F_{net} = F_d + F_g = -\beta r' + mg$$

By Newton's 2nd,

$$mr''(t) = mg - \beta r'(t)$$

$$r''(t) = g - \frac{\beta}{m} r'(t)$$

If  $g - \frac{\beta}{m} r'(t) \neq 0$  for any  $t$  in the interval considered,

$$\frac{r''(t)}{g - \frac{\beta}{m} r'(t)} = 1$$

Hence

$$-\frac{m}{\beta} \log |g - \frac{\beta}{m} r'(t)| = t + c_0$$

$$\log |g - \frac{\beta}{m} r'(t)| = -\frac{\beta}{m} (t + c_0)$$

$$|g - \frac{\beta}{m} r'(t)| = e^{-\frac{\beta}{m}(t+c_0)}$$

When  $g - \frac{\beta}{m} r'(t) > 0$  for all  $t$ ,

$$r'(t) = -\frac{m}{\beta} (e^{-\frac{\beta}{m}(t+c_0)} - g)$$

Via some antidifferentiation,

$$r(t) = \frac{gm}{\beta} t + \left(\frac{m}{\beta}\right)^2 e^{-\frac{\beta}{m}(t+c_0)} + c_1$$

Since  $0 = r'(0) = -\frac{m}{\beta} (e^{-\frac{c_0\beta}{m}} - g)$ ,  $e^{-\frac{c_0\beta}{m}} = g$ , hence

$$c_0 = -\frac{m}{\beta} \log(g)$$

This allows us to simplify the motion above. Since

$$e^{-\frac{\beta}{m}(t - \frac{m}{\beta} \log(g))} = e^{-\frac{\beta}{m}t} e^{\log(g)} = g e^{-\frac{\beta}{m}t}$$

$$r(t) = \frac{gm}{\beta} t + \left(\frac{m}{\beta}\right)^2 g e^{-\frac{\beta}{m}t} + c_1$$

and then  $r(0) = h = c_1 + \left(\frac{m}{\beta}\right)^2 g$ , hence  $c_1 = h - \left(\frac{m}{\beta}\right)^2 g$ , and the motion is completely described by

$$r(t) = \frac{gm}{\beta} t + \left(\frac{m}{\beta}\right)^2 g e^{-\frac{\beta}{m}t} + h - \left(\frac{m}{\beta}\right)^2 g$$

Experimental tests tells us this equation is more accurate than the last. We also gain something more interesting than just the equation.

$$\lim_{t \rightarrow \infty} r'(t) = \lim_{t \rightarrow \infty} \left( \frac{gm}{\beta} - \frac{mg}{\beta} e^{-\frac{\beta}{m}t} \right) = \frac{gm}{\beta}$$

Which we call the terminal velocity of an object. The constant  $\beta$  is proportional to the square of the length of a body. In many objects, the mass is proportional to the cube of the length of a body. Hence the terminal velocity is proportional to the length.

We have two equations for motion:

$$r(t) = \frac{gt^2}{2} + h$$

$$r(t) = \frac{gm}{\beta}t + \left(\frac{m}{\beta}\right)^2 g e^{-\frac{\beta}{m}t} + h - \left(\frac{m}{\beta}\right)^2 g$$

The first function somehow represents the second when  $\beta = 0$ , in some sense. Of course, we cannot simply set  $\beta = 0$ , since this would mean we would have to divide by zero. We remember that for any complex number  $z$ ,

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

Hence

$$e^{-\frac{\beta}{m}t} = \sum_{k=0}^{\infty} \frac{\left(-\frac{\beta}{m}t\right)^k}{k!}$$

and

$$\begin{aligned} r(t) &= \frac{mg}{\beta}t + \left(\frac{m}{\beta}\right)^2 g \sum_{k=0}^{\infty} \frac{\left(-\frac{\beta}{m}t\right)^k}{k!} + h - \left(\frac{m}{\beta}\right)^2 g \\ &= \frac{mg}{\beta}t + \left(\frac{m}{\beta}\right)^2 g - \frac{mg}{\beta}t + \frac{gt^2}{2} + \left(\frac{m}{\beta}\right)^2 g \sum_{k=3}^{\infty} \frac{\left(-\frac{\beta}{m}t\right)^k}{k!} + h - \left(\frac{m}{\beta}\right)^2 g \\ &= h + \frac{gt^2}{2} + g \sum_{k=3}^{\infty} \frac{\left(\frac{\beta}{m}\right)^{k-2} (-t)^k}{k!} \end{aligned}$$

And now  $\beta$  is only in the numerator, hence when  $\beta$  is set to zero, we obtain the first formula.

# Chapter 3

## The Motion of the Planets

### 3.1 Medieval Astronomy Before Kepler

Medieval astronomy, like many of the medieval sciences, is based on the insights of the Greeks, in particular of Aristotle. Aristotle was a believer in a geocentric model of the solar system, with the seven classical planets (named after the Greek Gods in order of brightness, Helios, Selene, Ares, Hermes, Zeus, Aphrodite, and Cronus) orbiting the earth on orbits at a uniform speed, of different radii. There were likely other theories proposed at the time, but this is hard to tell. Archimedes, in his book *The Sand Reckoner*, claimed that Aristarchus of Samos had a theory that the earth rotated daily, while revolving around the sun. But Aristarchus' works are lost.

However, it was observed that the outer planets exhibit *retrograde motion*, i.e. they appear to slow down, stop, and move backwards, before continuing their orbit. Planets also appear to increase and decrease in size over time. Apollonius of Perga in the 3rd century BCE proposed that planets travel on a larger circle, known as the *deferent*, while also travelling on an *epicycle*, a smaller circle. This theory was further analyzed by Apollonius of Perga and Hipparchus of Rhodes. The angular velocity on each circle is uniform.

In the 2nd century, Ptolemy proposed his *equant* model. Planets continue to orbit on epicycles around a central point, but this point is *not earth*. Earth instead deviates from this central point (the orbit is 'eccentric'), and the planets have uniform angular velocity with respect to the point opposite the earth from the central point of orbit.

Between the 9th and 13th century, Ptolemy's theory was critiqued and recalculated by Islamic scholars. Ibn al-Haytham wrote a critique of the theory in the 10th century, and the Maragha school found errors and refined Ptolemy's theory. The school's devices were used by Copernicus in the development of his heliocentric model.

In the early 16th century, likely between the years 1508 and 1514, Nicolaus Copernicus began advocating for a heliocentric model of the planets orbits. He proposed that the earth was only the centre of orbit for the moon, and that the apparent orbit of the other planets around the earth was primarily a result of the earth's motion, revolution about its centre, and revolution about its inclination. On the other hand, he still assumed the motion of the

planets was uniform, on circles about a fixed pole. In order to account for observations, he thus had to introduce several corrections to his theory, such as that the poles of each planet are not quite the same.

Most encouraged his work, though a few (such as Martin Luther and Tycho Brahe) were not willing to accept the system, since it seemed to disagree with the bible. Brahe instead advocated a 'geoheliostatic system', in which the inner planets revolve around the sun, and the outer planets revolve around the earth. Pope Clement reacted favorable to Copernicus' theories. There is no indication of how Pope Paul III reacted, to whom Copernicus' book *On The Revolutions* is dedicated. But by 1600 there was no official position from the Catholic church on the theory, and it was certainly not a heresy to advocate for the heliocentric model. It is highly unlikely that Giordano Bruno, who advocated for an extension of the Copernican model, and was burned at the stake in 1600, was burned for his cosmological views, but instead for his denial of core catholic tenets, such as the virginity of Mary and the divinity of Jesus.

In the 16th century, Tycho Brahe collected an unprecedented and comprehensive set of observations on the motion of planets, over the course of 21 years on an observatory of the island of Hven in Denmark (now Sweden). The observatory housed instruments to accurately measure the motions of the planets, but notably, no telescopes were used; Brahe observed the motion of the planets with naked eye. Johannes Kepler was an assistant of Brahe for a year, and Kepler used Brahe's observations as the basis for his laws. In this chapter, we will see how they are a consequence of Newton's Laws, which was one of the primary sources of evidence to Newton of the verisimilitude of his theory.

## 3.2 Kepler's Laws

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## Chapter 4

# Oscillations: Disguised Linear Differential Equations

We now come to a chapter that not only has applications to mechanics, but also to almost all other branches of physics. That is because most physics have motion that is inherently symmetric – it oscillates back and forth in a smooth motion. Whether this is a pendulum, spring, atomic vibration or planetary orbitation, the oscillations can all be described by the same mathematics: Linear differential equations. The best way to see how these relate is to consider an example.

The Force on a spring is described as proportional to the displacement of a particle from the equilibrium in the direction opposite to the displacement. Placing the equilibrium at the origin, we obtain the equation

$$F = -kx(t)$$

This is known as Hooke's law. Of course, any system with a stable equilibrium can be approximated by this equation which results from the Taylor polynomial, so this is an obvious approximation that applies to a wide variety of situations. If we let  $\omega$  be a number such that  $\omega^2 = k/m$ , then we are solving the differential equation

$$x'' + \omega^2 x = 0$$

This is a linear differential equation, and can be solved in terms of exponential functions. That is, we are looking for a constant  $\alpha$  such that

$$\alpha^2 e^{\alpha t} + \omega^2 e^{\alpha t} = 0$$

Hence  $\alpha^2 = -\omega^2$ , and  $\alpha = \pm \sqrt{-\omega^2} = \pm i\omega$ , and our general solution to the differential equation is

$$x(t) = Ae^{i\omega t} + Be^{-i\omega t}$$

for some constants  $A$  and  $B$ . A rule in complex analysis tells us that

$$e^{i\theta} = \cos \theta + i \sin \theta$$



Hence we can write our solution as

$$\begin{aligned}x(t) &= [A \cos(\omega t) + B \cos(-\omega t)] + i[A \sin(\omega t) + B \sin(-\omega t)] \\&= (A + B) \cos(\omega t) + i(A - B) \sin(\omega t)\end{aligned}$$

We require that our coefficients are real numbers. That is, both  $A + B$  and  $i(A - B)$  are real. Hence if

$$A = m + ni \qquad B = p + qi$$

Then we require that

$$A + B = (m + p) + (n + q)i \qquad (A - B)i = (m - p)i - (n - q)$$

are real, so that  $n = -q$ , and  $m = p$ , so  $A + B = 2m$ , and  $(A - B)i = 2n$ , and we obtain the solution (since  $m$  and  $n$  are arbitrary real numbers),

$$x(t) = A \cos(\omega t) + B \sin(\omega t)$$

If the initial velocity is zero,

$$x'(0) = B \cos(0) - A \sin(0) = 0$$

we must have  $B = 0$ , so that

$$x(t) = A \cos(\omega t)$$

# Chapter 5

## Energy

Suppose that the force on an object moving in a single axis depends only on the position of that object (denoted by  $s$ ). Then, since

$$F \circ s = m\ddot{s}$$

we may multiply  $\dot{s}$  on both sides to conclude

$$(F \circ s)\dot{s} = m\dot{s}\ddot{s}$$

Integrating both sides, for some times  $t_0$  (where the object is at position  $x_0$ ) and  $t$ ,

$$\int_{t_0}^t (F \circ s)\dot{s} = m \int_{t_0}^t \dot{s}\ddot{s}$$

By a change of variables,

$$m \int_{t_0}^t \dot{s}\ddot{s} = m \int_{\dot{s}(t_0)}^{\dot{s}(t)} \mathbf{1} = m \left[ \frac{\dot{s}(t)^2}{2} - \frac{\dot{s}(t_0)^2}{2} \right]$$
$$\int_{t_0}^t (F \circ s)\dot{s} = \int_{x_0}^{s(t)} F$$

Therefore, if we define the potential energy  $U$  at a point  $x$  to be

$$U(x) = - \int_{x_0}^{s(t)} F = \int_{s(t)}^{x_0}$$

And if we define the kinetic energy to be

$$K(t) = \frac{m\dot{s}(t)^2}{2}$$

Then our above discussion amounts to the fact that, at any time  $t$ ,

$$(U \circ s)(t) + K(t) = \frac{m\dot{s}(t_0)^2}{2} = E$$

Which states that in this system, the conservation of energy holds.

# Chapter 6

## Mechanics in Differential Geometry

Let us try and consider a coordinate independent, *geometric* approach to mechanics. Thus we fix Euclidean space  $\mathbf{E}$ , viewed as a differential manifold, and we try and model the laws of mechanics in a coordinate independent way, with the hope that this will bring out the symmetries in the equation. We will find there are two approaches to defining momentum and force in a coordinate invariant way. First, we can consider them as *vectors* on  $\mathbf{E}$ , i.e. so that a force and momentum is given in terms of a direction and magnitude on the manifold. Alternately, we can view them as *covectors* on  $\mathbf{E}$ , so that force measures the work required to move in a particular direction on the manifold.

Let us begin by setting us the equations of motion if momentum  $p$  is viewed as elements of  $T\mathbf{E}$ . Newton's law tells us that we should view  $F$  as an element of  $T(T\mathbf{E})$ . Then Newton's law says that

$$F = \frac{dp}{dt}$$

makes sense, and  $F$  depends solely on the time, and position and velocity of the particle. To expand upon this geometrically, we consider two maps  $\pi$  and  $\Pi$  from  $T(T\mathbf{E})$  to  $T\mathbf{E}$ , where  $\pi$  is just the basepoint map, and  $\Pi$  is obtained from differentiating the projection map  $T\mathbf{E} \rightarrow \mathbf{E}$ . We let  $A$  be the *equalizer* fiber subbundle of  $T(T\mathbf{E})$  consisting of all  $\mathbf{v} \in T(T\mathbf{E})$  such that  $\pi(\mathbf{v}) = \Pi(\mathbf{v})$  (the *semi spray condition*). Then we can consider  $F$  as a kind of time dependent section  $F : I \times T\mathbf{E} \rightarrow A$ . We can model the momentum as  $p : I \rightarrow T\mathbf{E}$ , and then Newton's law becomes

$$F(t, p) = \frac{dp}{dt}.$$

Working in coordinates, if  $(x, y, z, v_x, v_y, v_z)$  are the coordinates on  $T\mathbf{E}$  induced by the coordinates  $(x, y, z)$  on  $\mathbf{E}$ , then, we can write

$$F = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} + F_x \frac{\partial}{\partial v_x} + F_y \frac{\partial}{\partial v_y} + F_z \frac{\partial}{\partial v_z},$$

for component functions  $F_x, F_y, F_z : T\mathbf{E} \rightarrow \mathbf{R}$  (in any coordinates, the assumption that  $F$  lies in  $A$  amounts to the 'first order' components of the vector field being determined). Let

us also write the momentum in coordinates, i.e. as

$$p = p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} + p_z \frac{\partial}{\partial z},$$

where  $p_x, p_y, p_z : I \rightarrow \mathbf{R}$  are the components of momentum. Then Newton's First Law reads precisely that, in Euclidean coordinates

$$F_x = \frac{dp_x}{dt} \quad \text{and} \quad F_y = \frac{dp_y}{dt} \quad \text{and} \quad F_z = \frac{dp_z}{dt}.$$

These equations are elegant, but *do not* remain elegant in more general coordinate systems. Indeed, let us work with cylindrical coordinates  $(r, \theta, z)$ , where  $x = r \cos \theta$  and  $y = r \sin \theta$ . Then we can write

$$p = p_r \frac{\partial}{\partial r} + p_\theta \frac{\partial}{\partial \theta} + p_z \frac{\partial}{\partial z}$$

for some new component functions  $p_r, p_\theta : I \rightarrow \mathbf{R}$ . If we consider the variables  $(r, \theta, z, v_r, v_\theta, v_z)$  then we have

$$F = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} + F_r \frac{\partial}{\partial v_r} + F_\theta \frac{\partial}{\partial v_\theta} + F_z \frac{\partial}{\partial v_z},$$

for new functions  $F_r, F_\theta : I \times T\mathbf{E} \rightarrow \mathbf{R}$ . Then Newton's laws combine with a change of variables, telling us that

$$\begin{aligned} F_r &= (\cos \theta) F_x + (\sin \theta) F_y \\ &= (\cos \theta) \frac{dp_x}{dt} + (\sin \theta) \frac{dp_y}{dt} \\ &= \cos \theta \frac{d}{dt} (\cos \theta p_r - r \sin \theta p_\theta) + \sin \theta \frac{d}{dt} (\sin \theta p_r + r \cos \theta p_\theta) \\ &= \frac{dp_r}{dt} - r p_\theta \frac{d\theta}{dt} \\ &= \frac{dp_r}{dt} - \frac{r p_\theta^2}{m}, \end{aligned}$$

and

$$\begin{aligned} F_\theta &= -\frac{\sin \theta}{r} F_x + \frac{\cos \theta}{r} F_y \\ &= -\frac{\sin \theta}{r} \frac{dp_x}{dt} + \frac{\cos \theta}{r} \frac{dp_y}{dt} \\ &= -\frac{\sin \theta}{r} \frac{d}{dt} (\cos \theta p_r - r \sin \theta p_\theta) + \frac{\cos \theta}{r} \frac{d}{dt} (\sin \theta p_r + r \cos \theta p_\theta) \\ &= \frac{dp_\theta}{dt} + \frac{p_r}{r} \frac{d\theta}{dt} + \frac{dp_\theta}{dt} + \frac{p_\theta}{r} \frac{dr}{dt} \\ &= \frac{dp_\theta}{dt} + \frac{2p_r p_\theta}{rm}. \end{aligned}$$

We get more elegant equations by switching to *covector fields*.

We can also view  $p$  as a *covector*, i.e. as an element of  $T^*\mathbf{E}$ . Thus we write

$$p = p_x dx + p_y dy + p_z dz,$$

where  $p_x, p_y$ , and  $p_z$  are above. This is a coordinate dependent definition, but can be made coordinate independent. Indeed, if we define the metric  $g^2 = m dx^2 + m dy^2 + m dz^2$  on  $\mathbf{E}$ , then we get a map  $T\mathbf{E} \rightarrow T^*\mathbf{E}$  induced by the musical isomorphism with respect to this metric. A trajectory on  $\mathbf{E}$  induces a velocity vector in a coordinate independent way, and thus we get a momentum vector by applying the isomorphism. If we define force  $F$  to be a time dependent section  $F : I \times T\mathbf{E} \rightarrow A$ , where  $A$  is a fiber subbundle of  $T(T^*\mathbf{E})$  defined analogously to the bundle in the last paragraph. The musical isomorphism gives each position trajectory  $x : I \rightarrow \mathbf{E}$  a momentum trajectory  $p : I \rightarrow T^*\mathbf{E}$ . Then Newton's law now becomes

$$F \left( t, \frac{dx}{dt} \right) = \frac{dp}{dt}.$$

This approach tends to be more advantageous when considering energy approaches to Newtonian physics, i.e. Lagrangian and Hamiltonian approaches to the theory. Suppose that  $F$  is a conservative force, which in the vector case, means that there exists a function  $V : \mathbf{E} \rightarrow \mathbf{R}$  such that, in Euclidean coordinates,  $(F_x, F_y, F_z) = -\nabla V$ . This equation does not transform nicely with respect to general coordinate systems. But our formulation of momentum as a covector allows us to obtain an equivalent form that is coordinate independent. First, we begin by discussing a natural isomorphism

$$j : T(T^*\mathbf{E}) \rightarrow T^*(T\mathbf{E}).$$

This can be done by analyzing the coordinate systems induced on an appropriate coordinate patch. Given a coordinate system  $x$  on  $\mathbf{E}$ , we obtain a coordinate system  $(x, \xi)$  and  $(x, v)$  on  $T^*\mathbf{E}$  and  $T\mathbf{E}$ . Then we get coordinate systems  $(x, \xi, \alpha, \beta)$  and  $(x, v, a, b)$  on  $T(T^*\mathbf{E})$  and  $T^*(T\mathbf{E})$ . As we vary the coordinate system  $x$ , we note that  $\alpha$  and  $v$  transform covariantly,  $\xi$  and  $b$  transform contravariantly, and  $\beta$  and  $a$  transform 'weirdly', but in a way which is compatible with one another. Thus the map given in coordinates by  $j(x, \xi, \alpha, \beta) = (x, \alpha, \beta, \xi)$ , with inverse  $j^{-1}(x, v, a, b) = (x, b, v, a)$ , is well defined globally, and gives an isomorphism. Now given a potential  $V$  and a kinetic energy, we can define the *Lagrangian*  $L : T\mathbf{E} \rightarrow \mathbf{R}$  equal to the difference between potential and kinetic energy, i.e. such that for  $v \in T_x\mathbf{E}$ ,

$$L(v) = V(x) - (m/2)\langle v, v \rangle_g.$$

Then in Cartesian coordinates the differential map  $dL : T\mathbf{E} \rightarrow T^*(T\mathbf{E})$  has

$$dL(v) = -(F_x dx + F_y dy + F_z dz + p_x dv_x + p_y dv_y + p_z dv_z),$$

and so it is simple to compute that  $(j \circ dL)(v)$  is the tangent vector to the corresponding moment  $p \in T_x^*\mathbf{E}$  with coordinates

$$(j \circ dL)(v) = - \left( v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} + F_x \frac{\partial}{\partial \xi} + F_y \frac{\partial}{\partial \eta} + F_z \frac{\partial}{\partial \zeta} \right) = -F(v).$$

The equation  $j \circ dL = -F$  is now *coordinate independent*, and can be applied in an arbitrary coordinate system. In practice, we do not need to go through all this formality, since the relevant acceleration vectors are given by the components of  $j \circ dL$  corresponding to the tangent vectors pointing in the tangential directions of  $T^*\mathbf{E}$ .

# Chapter 7

## Keplar's Laws

Keplar's Laws, discovered empirically by Keplar in 1609 discuss the motion of planets about the sun, in the reference frame of the sun:

- *Keplar's First Law* states the planets move in ellipses, with the sun at one focus of the ellipse.
- *Keplar's Second Law* states that the area swept out radially by a planet, relative to the sun, is the same at all times.
- *Keplar's Third Law* states that if  $a$  is the major axis of the ellipse that a planet orbits, and  $T$  is the period of orbit of the planet, then  $a^3/T^2$  is the same for all planets orbiting the sun.

One of the major achievements of the invention of the calculus by Newton, and his resulting physical system, was his ability to show that Keplar's laws followed from the assumption that the planets are attracted to the sun by a gravitational force proportional to the mass of the planet, and following an inverse square law. Let us now see the connection.

To analyze the situation, we consider a single point particle with coordinates  $x$ , representing the planet, travelling with respect to a force field  $F$  depending only on the position of the planet. We will see how Keplar's Laws follow from properties of the force field  $F$ . It will be convenient to work in polar coordinates, so we write the motion of a planet via a parameterization  $x = re^{i\theta}$ . Recall that a force on the particle is central if, whenever the particle lies at a point  $x$ , it always lies on the line passing through  $x$  and the origin.

**Theorem 7.1.** *Keplar's Second Law is true if and only if the force on the particle is central.*

*Proof.* In a small time interval  $[t, t + \delta]$ , the particle traces out an area approximated by the triangle with vertices at the origin, at  $x(t)$ , and at  $x(t + \delta)$ . The area of this triangle is precisely

$$\frac{\det(x(t), x(t + \delta))}{2} = \frac{\det(x(t), \dot{x}(t))}{2} + O(\delta).$$

Taking Riemann sums, we conclude that the area traced out by the particle over a time interval  $[t_0, t_1]$  is precisely

$$\int_{t_0}^{t_1} \frac{\det(x(t), \dot{x}(t))}{2} dt,$$

Let

$$A(t) = \int_0^t \frac{\det(x(s), \dot{x}(s))}{2} ds.$$

Then our theorem amounts to saying that  $A(t_1 + t) - A(t_0 + t)$  is independent of  $t$ , which is only true if  $A'$  is a constant function, i.e. which holds if  $A'' = 0$ . But

$$A' = \frac{\det(x, \dot{x})}{2},$$

and thus

$$A'' = \frac{\det(x, \ddot{x})}{2}.$$

Newton's law tells us that  $\ddot{x}$  is proportional to the force  $F$ . But the fact that  $F$  is central immediately implies that  $\det(x, \ddot{x}) = 0$ , and thus  $A'' = 0$ . Thus Kepler's Second Law holds. Conversely, it could only be the case that  $A''$  was equal to zero at a particular time if  $\ddot{x}$  was always a scalar multiple of  $x$ , i.e. if and only if the force is always central on the trajectory of the particle.  $\square$

*Remark.* This proof should be compared to Newton's original proof, which purely used geometry, treating motion over two discrete instants, in which all velocities are linear (see Spivak's Calculus, Chapter 17). Newton first proves that, under the influence of no force at all; over three equally spaced instants the particle moves between the three points  $\{x, x + v, x + 2v\}$ , where  $v$  is a velocity vector, and we see the result holds in these instants, i.e. because

$$\det(x + v, x + 2v) = 2 \det(x, v) + \det(v, x) = \det(x, v).$$

Next, Newton analyzes what happens if a force is experienced instantaneously when the particle lies at the point  $x + v$ , shifting the velocity vector  $v$  to some other velocity vector  $w$ ; the fact that the force on the object is central then takes the assumption that  $w - v$  is a constant multiple of  $x + v$ . But this means that

$$\begin{aligned} \det(x + v, x + v + w) &= \det(x + v, w) \\ &= \det(x + v, v) + \det(x + v, w - v) \\ &= \det(x, v) + \lambda \det(x + v, x + v) \\ &= \det(x, x + v). \end{aligned}$$

But this means the areas of the triangles are the same. The result thus holds infinitesimally.

Next, let us analyze Kepler's First Law.



**Theorem 7.2.** *If the force on the particle is central, and satisfies an inverse square law, i.e.*

$$F = -\frac{H}{r^2}e^{i\theta},$$

*then the path of any body will be a conic section with the origin as one focus.*

*Proof.* We have seen that the area traced out by the particle is the same at all times, which amounts to saying that there exists some quantity  $L$  such that

$$\det(x, \dot{x}) = L.$$

We calculate that

$$\dot{x} = \dot{r}e^{i\theta} + ir\dot{\theta}e^{i\theta}.$$

In particular, we conclude that

$$\det(x, \dot{x}) = r^2\dot{\theta}.$$

Thus  $r^2\dot{\theta} = L$ . Now Newton's Law implies that

$$\ddot{x} = -\frac{H}{r^2}e^{i\theta} = -\frac{H}{L}\dot{\theta}e^{i\theta}.$$

The right hand side has as antiderivative  $i(H/L)e^{i\theta}$ . Thus we conclude that there exists a velocity vector  $v_0$  such that

$$\dot{x} = v_0 + i\frac{H}{L}e^{i\theta}.$$

Recalling that  $x = re^{i\theta}$ , we get that

$$\begin{aligned} L &= \det(x, \dot{x}) \\ &= r \det(e^{i\theta}, v_0 + i\frac{H}{L}e^{i\theta}) \\ &= r \left[ \det(e^{i\theta}, v_0) + \frac{H}{L} \right]. \end{aligned}$$

If  $v_0 = 0$ , then the equation above states that  $r$  is constant, and then  $r^2\dot{\theta} = L$  gives us that  $\theta$  is linear in  $t$ , and so the particle orbits on a circle about the origin, so the required claim is true. Alternatively, if  $v_0 \neq 0$ , then by applying a change of coordinate system by rotating and dilating the coordinate axis, we may assume that  $v_0$  points in the  $y$  axis and has magnitude one, i.e.  $v_0 = (0, 1)$ . Then the equation above becomes that

$$L = r \cos \theta + \frac{H}{L}.$$

Our equation is thus of the form

$$\Lambda = r[\varepsilon \cos \theta + 1],$$

where  $\Lambda = L^2/H$ , and  $\varepsilon = L/H$ . But this is *precisely* the equation describing a conic section in polar coordinates.  $\square$

Now let's prove Kepler's Third Law.

**Theorem 7.3.** *Suppose the force on the particle is central, with a fixed constant  $H$  giving the constant in the inverse square law. Then for any ellipse with major axis  $a$  the particle can travel on with period  $T$ , we have*

$$\frac{a^3}{T^2} = \frac{H}{4\pi^2}.$$

*Thus we see, for instance, that as  $H$  increases, the particle must orbit faster and faster.*

*Proof.* Let us consider an particle  $x$  as in the last proof, travelling subject to an equation of the form

$$\Lambda = r[\varepsilon \cos \theta + 1],$$

where  $\Lambda \neq 0$ , and  $|\varepsilon| < 1$ , so the particle travels along an ellipse, with the origin as a focus. In the last proof, we calculated that

$$\ddot{x} = -\frac{L^2}{\Lambda} \frac{1}{r^2} e^{i\theta}.$$

If  $a$  is the length of the major axis of the ellipse that the particle travels through, then

$$a = \frac{\Lambda}{1 - \varepsilon^2}.$$

Similar, if  $b$  denotes the minor axis, then

$$b = \frac{\Lambda}{\sqrt{1 - \varepsilon^2}}.$$

Thus  $b^2/\Lambda = a$ . We can use this fact to calculate the *period* of the orbit of the particle. If  $A(t)$  is the area swept out by the particle from time 0 to time  $t$ , then we have seen that  $A'(t) = L/2$ , and thus  $A(t) = Lt/2$ . If the particle has period  $T$ , then  $A(T)$  is equal to the area of the entire ellipse of the particle's orbit, and thus is equal to  $\pi ab$ . But this means we conclude that

$$L = \frac{2\pi ab}{T},$$

and so

$$\frac{L^2}{\Lambda} = \frac{4\pi^2 a^2 b^2}{\Lambda T^2} = \frac{4\pi^2 a^3}{T^2}.$$

Thus

$$-\frac{H}{r^2} e^{i\theta} = -\frac{4\pi^2 a^3}{T^2} \frac{1}{r^2} e^{i\theta},$$

and so we conclude that

$$\frac{a^3}{T^2} = \frac{H}{4\pi^2},$$

and so we see that the quantity on the left hand side depends only on  $H$ . □

Finally, let's show that the consequences of Kepler's First and Third Law can only be true if the force on the particle satisfies an inverse square law.

**Theorem 7.4.** *If a force is central, then the consequences of Kepler's First and Third Law can only be true if the force satisfies an inverse square law.*

*Proof.* We first consider one consequence of Kepler's Second Law, which is satisfied by central forces. Then

$$\dot{\mathbf{x}} = \dot{r}e^{i\theta} + ir\dot{\theta}e^{i\theta}$$

and

$$\ddot{\mathbf{c}} = [\ddot{r} - \dot{\theta}^2 r] e^{i\theta} + i[2\dot{r}\dot{\theta} + r\ddot{\theta}] e^{i\theta}.$$

Since the force is central, this quantity must be a constant multiple of  $e^{i\theta(t)}$ , which implies that the latter term has to be zero, i.e.

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0,$$

and thus

$$\ddot{\mathbf{x}} = [\ddot{r} - \dot{\theta}^2 r] e^{i\theta}.$$

Now suppose an particle travels under the influence of a central force, on an ellipse with the origin as one focus. Then the particle satisfies an equation of the form

$$r[1 + \varepsilon \cos \theta] = \Lambda.$$

We claim this equation is sufficient to conclude that the force on the particle at any time satisfies the inverse square law, i.e. that

$$\ddot{r} - \dot{\theta}^2 r = \frac{-H}{r^2},$$

for some time independent constant  $H$ . Implicitly differentiating the equation constraining the motion of the particle to a conic, we have

$$\dot{r}[1 + \varepsilon \cos \theta] - \varepsilon r \dot{\theta} \sin \theta = 0,$$

and so

$$r\dot{r}[1 + \varepsilon \cos \theta] - \varepsilon r^2 \dot{\theta} \sin \theta = 0.$$

Now  $r^2 \dot{\theta} = L$  for some constant  $L$ , and thus

$$\dot{r}\Lambda - \varepsilon L \sin \theta = 0.$$

Thus we can write

$$\dot{r} = \frac{\varepsilon L}{\Lambda} \sin \theta.$$

But this means that

$$\ddot{r} = \frac{\varepsilon L}{\Lambda} \dot{\theta} \cos \theta = \frac{\varepsilon L^2}{\Lambda} \frac{\cos \theta}{r^2}.$$

Guessing from our calculations in the last proof that we should set  $H = L^2/\Lambda$ , we calculate that

$$\begin{aligned}
 \ddot{r} - \dot{\theta}^2 r &= \frac{\varepsilon L^2 \cos \theta}{\Lambda r^2} - (L/r^2)^2 r \\
 &= -\frac{L^2}{\Lambda} \frac{1}{r^2} \left( \frac{\Lambda}{r} - \varepsilon \cos \theta \right) \\
 &= -\frac{L^2}{\Lambda} \frac{1}{r^2} \left( (1 + \varepsilon \cos \theta) - \varepsilon \cos \theta \right) \\
 &= -\frac{L^2}{\Lambda} \frac{1}{r^2}.
 \end{aligned}$$

Thus we see an inverse square law emerging from our calculations. □

The planets have always been the focus behind developments in classical mechanics. Newton's theory of gravity was inspired by the elliptic motion of the planets (in fact, Newton almost discarded his theory based on the measurement of the orbit of the moon around the earth, until more accurate measurements of the radius of the earth corrected his calculations). In the 1840s, Newton's laws predicted the existence of the planet Neptune, and almost all resistance to the laws in the popular imagination ceased.

# Chapter 8

## Lagrangian Mechanics

Lagrangian Mechanics is the most basic formulation of *analytical mechanics*, as it best favors analytical arguments rather than geometric arguments. Lagrange famously said in his textbook on the subject that “*No Figures will be found in this work*”.

Lagrangian Mechanics studies mechanics via

Given some coordinate system  $x$  on an open set  $U \subset M$ , we get a coordinate system  $(x, \dot{x})$  on  $\pi^{-1}(U) \subset TM$ , where  $\pi_M : TM \rightarrow M$  is the canonical projection. Given another coordinate system  $q$  on  $U \cap N$ , when  $N$  is a submanifold of  $M$ , which induces a coordinate system  $(q, \dot{q})$  on  $\pi_N^{-1}(U \cap N) \subset TN$ , we have

$$\frac{\partial x^\alpha}{\partial \dot{q}^i} = 0,$$

simply because altering  $\dot{q}$  while keep  $q$  fixed does not change the value of  $x^\alpha$ . We also have

$$\frac{\partial \dot{x}^\alpha}{\partial \dot{q}^i} = \frac{\partial x^\alpha}{\partial q^i}.$$

To see this, we note that the chain rule equation

$$\sum \dot{q}^i \frac{\partial}{\partial q^i} = \sum \dot{q}^i \frac{\partial x^\alpha}{\partial q^i} \frac{\partial}{\partial x^\alpha}$$

implies

$$\dot{x}^\alpha = \sum \dot{q}^i \frac{\partial x^\alpha}{\partial q^i},$$

and we can then differentiate both sides with respect to  $\dot{q}^i$ .

$$\frac{\partial \dot{x}^\alpha}{\partial \dot{q}^i} = \frac{\partial x^\alpha}{\partial q^i}.$$