Radial Multipliers

Jacob Denson

October 25, 2022

Table Of Contents

Part I

Review of Literature

Chapter 1

General Introduction

The question of the regularity of translation-invariant operators on $\mathbf{R}^d$ has proved central to the development of modern harmonic analysis. Indeed, the regularity of scuh operators underpins any subtle understanding of the Fourier transform, since with essentially any such operator T, we can associate a tempered distribution $m : \mathbf{R}^d \to \mathbf{C}$, known as the *symbol* of T, such that for any Schwartz function $f \in \mathcal{S}(\mathbf{R}^d)$,

$$Tf(x) = \int m(\xi) \widehat{f}(\xi) e^{2\pi i \xi \cdot x} \, d\xi,$$

i.e. such that $\widehat{Tf} = m \cdot \widehat{f}$. This is why such operators are also called *Fourier multipliers*. Using the spectral calculus of unbounded operators, one can also write this operator as $m(D)$, where $D = (2\pi i)^{-1} \nabla$ is a self-adjoint normalization of the gradient operator. Thus the study of the boundedness of translation invariant operators is closely connected to the study of the interactions of the projections

$$E_\xi f(x) = \widehat{f}(\xi) e^{2\pi i \xi \cdot x},$$

which act as projections onto the eigenspaces of the components of D, since we can write

$$m(D) = \int m(\xi) E_\xi.$$

Thus $m(D)$ is represented as a weighted average of the operators $\{E_\xi\}$.

The study of translation invariant operators emerges from many classical questions in analysis, like that of the convergence properties of Fourier series, or in mathematical physics, through the study of the heat, wave, and Schrödinger equation. These operators also naturally have rotational symmetry, so it is natural to restrict our attention to translation-invariant operators which are also rotation-invariant. These operators are precisely those represented by symbols $m : \mathbf{R}^d \to \mathbf{C}$ which are *radial*, i.e. such that

$$m(\xi) = h(|\xi|)$$

for some function $h : [0, \infty) \to \mathbf{C}$. This is the class of *radial Fourier multipliers*. The spectral calculus again implies one can write $m(D) = h(\sqrt{-\Delta})$, where Δ is the Laplacian on $\mathbf{R}^d$. Thus the study of radial multipliers is closely connected to interactions between the spherical projection operators

$$E_\lambda f(x) = \int_{|\xi|=1} \widehat{f}(\xi) e^{2\pi i \xi \cdot x},$$

for $0 < \lambda < \infty$, which are the projections onto the eigenspaces of $\sqrt{-\Delta}$, since we then have

$$h(\sqrt{-\Delta}) = \int h(\lambda) E_\lambda.$$

Thus studying the regularity of radial Fourier multipliers allows us to understand the behaviour of weighted averages of the operators $\{E_\lambda\}$.

Stated as above, we can extend the study of radial multipliers from $\mathbf{R}^d$ to a general *geodesically complete* Riemannian manifold X. On such a manifold we have a Laplace-Beltrami operator Δ which is an essentially self-adjoint unbounded operator on $L^2(X)$, and one can consider a spectral calculus. The operator $\sqrt{-\Delta}$ will be self-adjoint, and one can consider the study of operators of the form $h(\sqrt{-\Delta})$ for functions $h : [0, \infty) \to \mathbf{C}$. Some techniques of analyzing radial multipliers on $\mathbf{R}^d$ extend to the Riemannian case, whereas in other cases new tools are required.

This research project studies necessary and sufficient conditions to guarantee the L^p boundedness of radial multiplier operators, both in the Euclidean setting, and also in the setting of Riemannian manifolds.

1.1 Radial Multipliers on Euclidean Space

The general study of the boundedness of Fourier multipliers was initiated in the 1960s. It was quickly realized that the most fundamental estimates were those of the form

$$\|Tf\|_{L^q(\mathbf{R}^d)} \lesssim \|f\|_{L^p(\mathbf{R}^d)},$$

for $1 \leqslant p \leqslant 2$, and $q \geqslant p$. It is therefore natural to introduce the spaces $M^{p,q}(\mathbf{R}^d)$, consisting of all symbols m which induce a Fourier multiplier operator T bounded from $L^p(\mathbf{R}^d)$ to $L^q(\mathbf{R}^d)$. The space $M^{p,q}(\mathbf{R}^d)$ is then naturally a Banach space under the operator norm

$$\|m\|_{M^{p,q}(\mathbf{R}^d)} = \sup \left\{ \frac{\|Tf\|_{L^q(\mathbf{R}^d)}}{\|f\|_{L^p(\mathbf{R}^d)}} : f \in \mathcal{S}(\mathbf{R}^d) \right\}.$$

For notational convenience, $M^{p,p}(\mathbf{R}^d)$ is denoted by $M^p(\mathbf{R}^d)$.

It was very simple to characterize the spaces $M^{1,q}(\mathbf{R}^d)$, by virtue of the fact that the theory of boundedness of operators with domain $L^1(\mathbf{R}^d)$ is often relatively trivial. For any symbol m, if $k = \widehat{m}$, then

$$\|m\|_{M^{1,q}(\mathbf{R}^d)} = \begin{cases} \|k\|_{L^q(\mathbf{R}^d)} & : q > 1 \\ \|k\|_{M(\mathbf{R}^d)} & : q = 1, \end{cases}.$$

where $M(\mathbf{R}^d)$ is the space of finite signed Borel measures equipped with the total variation norm. Duality also allows us to characterize the spaces $M^{p,\infty}(\mathbf{R}^d)$, since in general one has an isometric equivalence $M^{p,q}(\mathbf{R}^d) = M^{q^*,p^*}(\mathbf{R}^d)$, and so

$$M^{p,\infty}(\mathbf{R}^d) = M^{1,p^*}(\mathbf{R}^d).$$

The fact that the Fourier transform is unitary also allowed the spaces $M^{p,2}(\mathbf{R}^d)$ to be characterized, for $1 \leqslant p \leqslant 2$, by the identity

$$\|m\|_{M^{p,2}(\mathbf{R}^d)} = \|m\|_{L^q(\mathbf{R}^d)},$$

where $q = 2p/(2-p)$. However, characterizing the remaining spaces $M^{p,q}$, i.e. for $p \in (1,2]$ and $q \in [p,\infty) - \{2\}$, proved more challenging. In the past 60 years there has been no tractable characterization of the spaces $M^{p,q}(\mathbf{R}^d)$ for any other pair of exponents p or q.

One tool that has proved useful outside of this range is *Littlewood-Paley* theory, which makes it natural to restrict to the study of Fourier multipliers compactly suported on dyadic annuli. We fix a smooth bump function

$\phi \in C_c^\infty(\mathbf{R}^d)$ supported on $\{|\xi| \sim 1\}$, and such that $1 = \sum_j \mathrm{Dil}_{2^j}\phi$. Then given a symbol m, we define

$$m_t = (\mathrm{Dil}_{1/t} m) \cdot \phi.$$

Thus m_t describes the behaviour of the multiplier m restricted to the annulus of frequencies $|\xi| \sim t$, rescaled so that this behaviour is now lying on the annulus $|\xi| \sim 1$. Littlewood-Paley theory implies that for $1 < p, q < \infty$, then

$$\|m\|_{M^{p,q}(\mathbf{R}^d)} \sim_{p,q} \sup_{t>0} t^{d(1/p-1/q)} \|m_t\|_{M^{p,q}(\mathbf{R}^d)}.$$

In the study of multipliers we have not already characterized, it is therefore natural to restrict oneself to the study of multipliers with a compactly supported symbol; in $\mathbf{R}^d$ we can rescale, so we can assume we are working with a multiplier supported on the annulus $1/2 \leqslant |\xi| \leqslant 2$. In the sequel, we will call these *unit scale multipliers*.

One common heuristic to this theory is that the regularity of the symbol m, or equivalently, the decay of the convolution kernel k away from the origin, implies some boundedness of the symbol, viewed as a multiplier. The most well known condition of this form for $1 < p < \infty$ is the Hörmander-Mikhlin multiplier theorem, which shows that for $1 < p < \infty$, and $\varepsilon > 0$, if m is a unit scale multiplier, and k is it's convolution kernel, then

$$\|m\|_{M^p(\mathbf{R}^d)} \lesssim_{p,\varepsilon} \int \frac{k(x)}{\langle x \rangle^{1+\varepsilon}}\, dx.$$

This implies the slightly weaker inequality

$$\|m\|_{M^p(\mathbf{R}^d)} \lesssim_{\varepsilon} \|m\|_{L^2_{d/2+\varepsilon}}.$$

All these results apply via a Paley-Wiener decomposition to general multipliers, i.e. with the right hand side replaced with a supremum of the quantities associated with the rescaled multipliers $\{m_t\}$.

Conversely, some control over the mass of the convolution kernel k is necessary in order to conclude that $m \in M^{p,q}(\mathbf{R}^d)$ for some exponents p and q. This is because if k is the convolution kernel corresponding to a unit scale multiplier m, and $\phi \in C_c^\infty(\mathbf{R}^d)$ has Fourier transform equal to one on the annulus $1/4 \leqslant |\xi| \leqslant 4$, then

$$\|k\|_{L^q(\mathbf{R}^d)} = \|k * \phi\|_{L^q(\mathbf{R}^d)} \lesssim \|m\|_{M^{p,q}(\mathbf{R}^d)}.$$

For a general, non compactly supported multiplier m, if k_t is the convolution kernel associated with the multiplier m_t, then one obtains the condition

$$\sup_{t>0} \left\{ t^{d(1/p-1/q)} \cdot \|k_t\|_{L^q(\mathbf{R}^d)} \right\} \lesssim \|m\|_{M^{p,q}(\mathbf{R}^d)},$$

One can phrase this bound in terms of the homogeneous Besov spaces $\dot{B}^{p,q}_s(\mathbf{R}^d)$, the space consisting of all distributions u on $\mathbf{R}^d$ such that the norm

$$\|u\|_{\dot{B}^{p,q}_s(\mathbf{R}^d)} = \left(\sum_{j=-\infty}^{\infty} \left(2^{js} \|P_j u\|_{L^p(\mathbf{R}^d)} \right)^q \right)^{1/q} = \|2^{js} P_j u\|_{l^q(\mathbf{Z}) L^p(\mathbf{R}^d)},$$

is finite, where $P_j = \phi(D/2^j)$ is the Littlewood-Paley projection operator onto a dyadic frequency band of radius 2^j. If k is the convolution kernel of a multiplier m, one can rescale the condition above to read that

$$\sup_{t>0} \left\{ t^{-d/p^*} \|P_t k\|_{L^q(\mathbf{R}^d)} \right\} \lesssim \|m\|_{M^{p,q}(\mathbf{R}^d)},$$

i.e. that

$$\|k\|_{\dot{B}^{q,\infty}_{-d/p^*}} \lesssim \|m\|_{M^{p,q}(\mathbf{R}^d)}.$$

Thus we conclude that k must satisfy some (admittedly weak) regularity assumptions to be the convolution kernel of a bounded Fourier multiplier.

Despite the lack of a complete characterization of the classes $M^{p,q}(\mathbf{R}^d)$, it is surprising that we *can* conjecture a characterization of the subspace of $M^{p,q}(\mathbf{R}^d)$ for *radial symbols* in this class, for an appropriate range of exponents. This conjecture is best phrased in terms of the result of [3], which concerned radial multipliers m whose associated operator T is bounded from the L^p norm to the L^q norm *restricted to radial functions*, i.e. such that the norm

$$\|m\|_{M^{p,q}_{\text{rad}}(\mathbf{R}^d)} = \sup \left\{ \frac{\|Tf\|_{L^q(\mathbf{R}^d)}}{\|f\|_{L^p(\mathbf{R}^d)}} : f \in \mathcal{S}(\mathbf{R}^d) \text{ and } f \text{ is radial} \right\}$$

is finite. The main result of [3] was that if $d > 1$, if $1 < p < 2d/(d+1)$, and if $p \leqslant q < 2$, then $M^{p,q}_{\text{rad}}(\mathbf{R}^d)$ is a subset of $L^1_{\text{loc}}(\mathbf{R}^d)$, and for any unit scale, integrable, radial multiplier m,

$$\|m\|_{M^{p,q}_{\text{rad}}(\mathbf{R}^d)} \sim_{p,q,d} \|k\|_{L^q(\mathbf{R}^d)}.$$

More generally, for any locally integrable radial symbol m,

$$\|m\|_{M^{p,q}_{\text{rad}}(\mathbf{R}^d)} \sim_{p,q,d} \sup_{t>0} t^{d(1/p-1/q)} \|k_t\|_{L^q(\mathbf{R}^d)} = \|k\|_{\dot{B}^{q,\infty}_{-d/p^*}}.$$

Moreover, this condition give *precisely the range* under which this characterization holds. It is natural to conjecture that the same constraint continues to hold when we remove the constraint that our inputs f are radial, i.e. that for unit scale, integrable, radial symbols m, for $d > 1$, $1 < p < 2d/(d+1)$, and for $p \leqslant q < 2$,

$$\|m\|_{M^{p,q}} \sim_{p,q,d} \|k\|_{L^q(\mathbf{R}^d)}$$

and for general locally integrable symbols m,

$$\|m\|_{M^{p,q}} \sim_{p,q,d} \|k\|_{\dot{B}^{q,\infty}_{-d/p^*}}$$

In the sequel, we call this the *radial multiplier conjecture* in $\mathbf{R}^d$. TODO: Look up counterexample which shows that these results cannot be obtained if m is not radial.

Remark. For $p = 1$, the radial multiplier conjecture is true *for compactly supported multipliers*, and one does not even need to assume that the multiplier is radial in this case. Indeed, for any (not even compactly supported) multiplier m we have

$$\|m\|_{M^{1,q}(\mathbf{R}^d)} = \|k\|_{L^q(\mathbf{R}^d)}.$$

Littlewood-Paley says this quantity is proportional to

$$\left(\sum \|P_j k\|_{L^q(\mathbf{R}^d)}^2\right)^{1/2} = \|k\|_{\dot{B}^{q,2}_0}.$$

On the other hand, for non compactly supported multipliers m the radial multiplier conjecture should say that

$$\|m\|_{M^{1,q}(\mathbf{R}^d)} \sim \|k\|_{\dot{B}^{q,\infty}_0},$$

so the result fails ‘in the second order exponents’.

Remark. Let $m(\xi) = h(|\xi|)\mathbf{I}(|\xi| \leqslant 1)$, where $h \in C_c^\infty(\mathbf{R})$ is supported on $1/4 \leqslant r \leqslant 4$, and is equal to one for $1/2 \leqslant r \leqslant 2$. Then m differs from the 'ball multiplier' by a compactly supported, smooth symbol, and thus m has all the L^p mapping properties that the ball multiplier has. In particular, it is a result of Fefferman (TODO: Cite) that the ball multiplier does not lie in $M^{p,q}(\mathbf{R}^d)$ when $d > 1$ for any values of p and q except when $p = q = 2$, so the same is true of the multiplier m given above. Now if k is the convolution kernel of m, then polar coordinates gives that

$$k(x) = \int m(\xi) e^{2\pi i \xi \cdot x} = \frac{1}{|x|^{d/2-1}} \int_{1/4}^{1} r^{d/2} h(r) J_{d/2-1}(rx).$$

For $|x| \geqslant 1$, Bessel function asymptotics gives that there is some constant a, depending on d, such that

$$k(x) = |x|^{-\frac{d-1}{2}} \int_{1/4}^{1} r^{\frac{d-1}{2}} h(r) \cos(r|x| + a)\, dr + O\left(|x|^{-\frac{d+1}{2}}\right).$$

Integration by parts then gives that $|k(x)| \lesssim |x|^{-\frac{d+1}{2}}$. Since k is bounded near the origin (i.e. by Paley-Wiener), we conclude that $k \in L^q(\mathbf{R}^d)$ for $q > 2d/(d+1)$. In particular, this multiplier gives a counterexample to the radial multiplier conjecture for $q > 2d/(d+1)$. TODO: Is there a counterexmaple for $q = 2d/(d+1)$?

Remark. Given a function h on $[0, \infty)$, we define the *d-dimensional Fourier-Bessel transform* of h as

$$\mathcal{B}_d h(r) = r^{-\frac{d-2}{2}} \int_0^\infty \rho^{d/2} h(\rho) J_{\frac{d-2}{2}}(\rho r)\, d\rho,$$

where J_α is the standard Bessel function of order α. Then if $m(\xi) = h(|\xi|)$, then we have $\{\mathcal{F} m\}(x) = \{\mathcal{B}_d h\}(|x|)$. The condition in the radial multiplier conjecture for unit scale multipliers becomes that

$$\|m\|_{M^{p,q}(\mathbf{R}^d)} \sim \left(\int_0^\infty r^{d-1} |\mathcal{B}_d h(r)|^q\, dr \right)^{1/q}.$$

We therefore that, if we let $h_t = \phi \cdot \mathrm{Dil}_{1/t} h$, then the condition in the radial multiplier conjecture becomes

$$\sup_{t>0} t^{d(1/p-1/q)} \left(\int_0^\infty r^{d-1} |\{\mathcal{B}_d h_t\}(r)|^q \right)^{1/q} < \infty.$$

One can also convert this into a statement involving the standard one-dimensional Fourier transform. One can use Bessel function asymptotics to convert this into a condition on the one-dimensional Fourier transform of h (extended to an even function on $\mathbf{R}$). Indeed, there exists a differential operator L with constant coefficients and order at most $(d-1)/2$ such that we can write

$$\begin{aligned}\mathcal{B}_d h(r) &= r^{-\frac{d-2}{2}} \int_{1/2}^{2} \rho^{\frac{d}{2}} h(\rho) \left(\cos(\rho r + a)|\rho r|^{-1/2} + O(|\rho r|^{-3/2}) \right) \, d\rho \\ &= r^{-\frac{d-1}{2}} \int_{1/2}^{2} \rho^{\frac{d-1}{2}} h(\rho) \cos(\rho r + a) \, d\rho + O\left(r^{-\frac{d+1}{2}} \|h\|_{L^1} \right) \\ &= r^{-\frac{d-1}{2}} \left(L\widehat{h}(r) + \int_{-\infty}^{\infty} \widehat{h}(r-s) s^{-\frac{d+1}{2}} \, ds \right) + O\left(r^{-\frac{d+1}{2}} \|h\|_{L^1} \right).\end{aligned}$$

TODO: Finish this calculation; Theorem 1.2 of [3] guarantees the radial multiplier condition is equivalent to

$$\sup_{t>0} t^{d(1/p-1/q)} \left(\int (1+|s|)^{(d-1)(1-q/2)} |\widehat{h}_t(s)|^q \, ds \right)^{1/q} < \infty.$$

In particular, if h is supported at a unit scale, then the condition is that

$$\left(\int_0^\infty (1+|s|)^{(d-1)(1-q/2)} |\widehat{h}(s)|^q \, ds \right)^{1/q} < \infty.$$

Thus for most values of s, we have $|\widehat{h}(s)| \lesssim |s|^{-(d-1)(1/q-1/2)}$.

We now know, by the results of [6] that the radial multiplier conjecture is true when $d \geqslant 4$ and when $1 < p < (2d-2)/(d+1)$. When $d = 4$, this was improved by [2], who showed that the conjecture is true here when $1 < p < 36/29$, where $36/29 \approx (2d-1.79)/(d+1)$. When $d = 3$, [2] also established a *restricted weak type* bound

$$\|Tf\|_{L^p(\mathbf{R}^n)} \lesssim \|f\|_{L^{p,1}(\mathbf{R}^n)}$$

when $d = 3$ and $1 < p < 13/12$, where $13/12 \approx (2d-1.66)/(d+1)$ But the radial multiplier conjecture has not yet been completely resolved in any dimension n, we do not have any strong type L^p bounds when $d = 3$, and we don't have any bounds whatsoever when $d = 2$. One goal of this research project is to investigate whether one can use modern research techniques to improve upon these bounds.

The full proof of the radial multiplier is likely far beyond current research techniques. Indeed, it remains a major open problem in harmonic analysis to determine the range of exponents for which *specific* radial Fourier multipliers are bounded in the range where the conjecture would apply, such as the Fourier multiplier on $\mathbf{R}^d$ with symbol $m_\lambda(\xi) = (1-|\xi|)_+^\lambda$, the family of *Bochner-Riesz multipliers*. The radial multiplier conjecture characterizes the range of the Bochner-Riesz multipliers, and thus the conjecture would also imply the Kakeya and restriction conjectures. All three of these results are major unsolved problems in harmonic analysis. On the other hand, the Bochner Riesz conjecture is completely resolved when $d = 2$, while in contrast, no results related to the radial multiplier conjecture are known in this dimension at all. And in any dimension $d > 2$, the range under which the Bochner-Riesz multiplier is known to hold [4] is strictly larger than the range under which the radial multiplier conjecture is known to hold, even for the restricted weak-type bounds obtained in [2]. Thus it still seems within hope that the techniques recently applied to improve results for Bochner-Riesz problem, such as broad-narrow analysis [1], the polynomial Wolff axioms [7], and methods of incidence geometry and polynomial partitioning [11] can be applied to give improvements to current results characterizing the boundedness of general radial Fourier multipliers.

Our hopes are further emboldened when we consult the proofs in [6] and [2], which reduce the radial multiplier conjecture to the study of upper bounds of quantities of the form

$$\left\| \sum_{(y,r)\in\mathcal{E}} F_{y,r} \right\|_{L^p(\mathbf{R}^n)},$$

where $\mathcal{E} \subset \mathbf{R}^n \times (0,\infty)$ is a finite collection of pairs, and $F_{y,r}$ is an oscillating function supported on a $O(1)$ neighborhood of a sphere of radius r centered at a point y. The L^p norm of this sum is closely related to the study of the tangential intersections of these spheres, a problem successfully studied in more combinatorial settings using incidence geometry and polynomial partitioning methods [12], which provides further estimates that these methods might yield further estimates on the radial multiplier conjecture.

When $d = 3$, the results of [2] are only able to obtain bounds on the L^p sums in the last paragraph when $\mathcal{E}$ is a Cartesian product of two subsets of $(0, \infty)$ and $\mathbf{R}^d$. This is why only restricted weak-type bounds have been obtained in this dimension. It is therefore an interesting question whether different techniques enable one to extend the L^p bounds of these sums when the set $\mathcal{E}$ is *not* a Cartesian product, which would allow us to upgrade the result of [2] in $d = 3$ to give strong L^p bounds. This question also has independent interest, because it would imply new results for the 'endpoint' local smoothing conjecture, which concerns the regularity of solutions to the wave equation in $\mathbf{R}^d$. Incidence geometry has been recently applied to yield results on the 'non-endpoint' local smoothing conjecture [5], which again suggests these techniques might be applied to yield the estimates needed to upgrade the result of [2] to give strong L^p-type bounds.

1.2 Multipliers on Riemannian Manifolds

Fix a geodesically complete Riemannian manifold X. We can then define operators $h(\sqrt{-\Delta})$, which are analogues to the radial multipliers studied in the Euclidean setting. Just like multiplier operators on $\mathbf{R}^n$ are crucial to an understanding of the interactions between the functions $e_\xi(x) = e^{2\pi i \xi \cdot x}$ on $\mathbf{R}^n$, understanding the operators $h(\sqrt{-\Delta})$ is crucial to understanding the interactions of eigenfunctions of the Laplace-Beltrami operator on X. We let $M^{p,q}(X, \sqrt{-\Delta})$ denote the family of all symbols $h : \mathbf{R} \to \mathbf{C}$ such that the operator $T_h = h(\sqrt{-\Delta})$ is bounded from $L^p(X)$ to $L^q(X)$, with the analogous operator norm, though, when there is no ambiguity, we will overload notation and write this space as $M^{p,q}(X)$.

To avoid technicalities, we will focus on a compact Riemannian manifold X, which is automatically geodesically complete. For such manifolds, there is a problem which prevents a direct generalization of the radial multiplier conjecture. For such a manifold, there exists $0 = \lambda_1 < \lambda_2 \leqslant \dots$ with $\lambda_i \to \infty$, and an orthonormal family of eigenfunctions $\{e_n\}$ in $C^\infty(X)$, forming a basis for $L^2(X)$, such that

$$\Delta f = \sum -\lambda_n^2 \langle f, e_n \rangle \cdot e_n.$$

Thus for any function $h : [0,\infty) \to \mathbf{C}$,

$$h\left(\sqrt{-\Delta}\right) f = \sum h(\lambda_n)\langle f, e_n\rangle \cdot e_n.$$

The operators $h(\sqrt{-\Delta})$ act as an analogue of radial multipliers on $\mathbf{R}^d$.

The study of multipliers on a Riemannian manifold has a certain technical problem, which the Euclidean case did not have. If h has compact support, this sum will be finite, and thus by the triangle inequality, trivially bounded from $L^p(X)$ to $L^q(X)$ for any exponents p and q. Thus $M^{p,q}(X)$ trivially contains all compactly supported radial multipliers. This trivializes the study of compactly supported radial multipliers in some sense, which is the complete opposite of the Euclidean case, where Littlewood-Paley allowed us to reduce the study of general multipliers to compactly supported radial multipliers. The key here is that Euclidean multipliers automatically have rescaling symmetries, whereas this is not present in the case of compact Riemannian manifolds. To get around this we add a rescaling into the definition of our operator norm, i.e. we study conditions that ensure we have a bound of the form

$$\sup_{t>0} t^{d(1/q-1/p)} \|\mathrm{Dil}_t h\|_{M^{p,q}(X)} < \infty.$$

We let $M^{p,q}_{\mathrm{Dil}}(X)$ denote the family of all multipliers for which the inequality above holds, and give it the norm induced by the quantity on the left hand side. A transference principle of Mitjagin [9] shows that if X is a compact Riemannian manifold, and $m : \mathbf{R}^d \to \mathbf{C}$ is radial, with $m(\xi) = h(|\xi|)$, then

$$\|m\|_{M^{p,q}(\mathbf{R}^d)} \lesssim_{X,p,q} \|h\|_{M^{p,q}_{\mathrm{Dil}}(X)}.$$

Thus, in some sense, the dilation invariant Fourier multiplier problem on a compact manifold X is at least as hard as it is on $\mathbf{R}^n$. Another goal of this research project is to try and extend the radial multiplier conjecture to the setting of dilation invariant bounds for multipliers of the Laplacian on Riemannian manifolds.

As in the case $X = \mathbf{R}^d$, the study of multipliers in $M^{2,2}_{\mathrm{Dil}}(X)$ is trivial.

Indeed, applying orthogonality, we calculate that

$$\begin{aligned}
\|h(\sqrt{-\Delta})f\|_{L^2(X)} &= \left\| \sum_\lambda h(\lambda) E_\lambda f \right\|_{L^2(X)} \\
&= \left(\sum_\lambda |h(\lambda)|^2 \|E_\lambda f\|_{L^2(X)}^2 \right)^{1/2} \\
&\leqslant \left(\sup_{\lambda \in \sigma(\sqrt{-\Delta})} |h(\lambda)| \right) \left(\sum_\lambda \|E_\lambda f\|_{L^2(X)}^2 \right)^{1/2} \\
&= \left(\sup_{\lambda \in \sigma(\sqrt{-\Delta})} |h(\lambda)| \right) \|f\|_{L^2(X)}.
\end{aligned}$$

Taking f to be an eigenfunction with eigenvalue λ which maximizes the value of $|h(\lambda)|$ shows this inequality is tight, i.e. we have

$$\|h\|_{M^{2,2}(X)} = \sup_{\lambda \in \sigma(\sqrt{-\Delta})} |h(\lambda)|.$$

Now applying an arbitrary dilation to h, we conclude that

$$\|h\|_{M^{2,2}_{\mathrm{Dil}}(X)} = \sup_{\lambda > 0} |h(\lambda)|.$$

Thus we have found a simple characterization of the space $M^{2,2}_{\mathrm{Dil}}(X)$.

The spaces $M^{1,q}_{\mathrm{Dil}}(X)$ are a little more tricky, since we do not have a precise theory of the Fourier transform in the setting of general Riemannian manifolds. To take a look at these bounds, we recall that $L^1 \to L^q$ bounds of an operator are characterized by Schur's test. If $\{e_n\}$ is a $C^\infty(X)$ basis of eigenfunctions on X, with $\Delta e_n = -\lambda_n^2 e_n$, then

$$h(\sqrt{-\Delta})f(x) = \sum h(\lambda_n) \langle f, e_n \langle e_n(x) = \int \left(\sum_n h(\lambda_n) e_n(x) \overline{e_n(y)} \right) f(y)\, dy.$$

Thus the kernel of $h(\sqrt{-\Delta})$ is precisely $K(x,y) = \sum_n h(\lambda_n) e_n(x) \overline{e_n(y)}$, and we conclude by Schur's test that

$$\|h\|_{M^{1,q}(X)} = \left\| \sum_n h(\lambda_n) e_n(x) \overline{e_n(y)} \right\|_{L^\infty_y L^q_x}.$$

In the case $X = \mathbf{R}^n$, the analogous kernel is $K(x,y) = \int_{\mathbf{R}^d} h(|\xi|) e^{2\pi i \xi \cdot x} \overline{e^{2\pi i \xi \cdot y}}$, which can be explicitly reduced to $K(x,y) = \mathcal{B}_d h(|x-y|)$, and the condition of being contained in $M^{1,q}(\mathbf{R}^d)$ then becomes that

$$\left(\int r^{d-1} |\mathcal{B}_d h(r)|^q \, dr \right)^{1/q} < \infty.$$

If h is compactly supported, then this condition is equivalent to the condition that

$$\left(\int (1+|t|)^{(d-1)(1-q/2)} |\widehat{h}(t)|^q \, dt \right)^{1/q} < \infty.$$

In the general setting we do not have quite as nice a formula, but we can still *force* the Fourier transform into the equation to see if it can be used to understand these quantities (which will be necessary for studying the radial multiplier conjecture). We thus write

$$\begin{aligned} \sum_n h(\lambda_n) e_n(x) \overline{e_n(y)} &= \sum_n \left(\int \widehat{h}(t) e^{2\pi i t \lambda_n} e_n(x) \overline{e_n(y)} \, dt \right) \\ &= \int \widehat{h}(t) \left(\sum_n e^{2\pi i t \lambda_n} e_n(x) \overline{e_n(y)} \right) \, dt \\ &= \int \widehat{h}(t) W_t(x,y) \, dt, \end{aligned}$$

where $W_t(x,y) = \sum_n e^{2\pi i t \lambda_n} e_n(x) \overline{e_n(y)}$ is the kernel of the *half-wave propogator* $e^{2\pi i t \sqrt{-\Delta}}$ on X. The connection between radial multipliers on X and the Fourier transform of their symbol is therefore closely related to the study of the half-wave equation $\partial_t = \sqrt{-\Delta}$ on X. If we are to expect an inequality of the form

$$\| \int \widehat{h}(t) W_t(x,y) \, dt \|_{L_y^\infty L_x^q} \lesssim \left(\int (1+|t|)^{(d-1)(1-q/2)} |\widehat{h}(t)|^q \, dt \right)^{1/q}$$

to hold for a general function $\widehat{h}$, then (by Hölder) we must expect that

$$\| (1+|t|)^{(d-1)(1/q-1/2)} W_t(x,y) \|_{L_y^\infty L_x^q L_t^{q^*}} < \infty.$$

TODO: In particular, we know this characterization must be true in $\mathbf{R}^d$, so if K_t is the fundamental solution of the wave equation, then we must have that for all $1 \leqslant q \leqslant \infty$,

$$\|(1+|t|)^{(d-1)(1/q-1/2)} K_t(x)\|_{L^q_x L^{q*}_t} < \infty.$$

Is this true? TODO: Ask Andreas about this.

Directly translating the assumptions of the radial multiplier conjecture to this setting yields the following statement: If $h : [0,\infty) \to \mathbf{R}$ is a function, and we define

$$A_{p,q}(h) = \sup_{t>0} t^{d(1/p-1/q)} \left(\int_{t/2 \leqslant |s| \leqslant 2t} |\widehat{h}(s)|^q (1+|s|)^{(d-1)(1-q/2)}\, ds \right)^{1/q},$$

then for what values of p and q is is true that the inequality

$$\|h\|_{M^{p,q}_{\mathrm{Dil}}(X)} \lesssim A_{p,q}(h)$$

still holds. Mitjagin's result implies that we require $1 < p < 2d/(d+1)$ and $p \leqslant q < 2$, and we conjecture that, perhaps under appropriate assumptions on X, we can achieve similar ranges of exponents as have been obtained for the Euclidean radial multiplier conjecture.

On general compact manifolds, there are difficulties arising from a generalization of the radial multiplier conjecture, connected to the fact that analogues of the Kakeya / Nikodym conjecture are false in this general setting [8]. But these problems do not arise for constant curvature manifolds, like the sphere. The sphere also has over special properties which make it especially amenable to analysis, such as the fact that solutions to the wave equation on spheres are periodic. Best of all, there are already results which achieve the analogue of [3] on the sphere. Thus it seems reasonable that current research techniques can obtain interesting results for radial multipliers on the sphere, at least in the ranges established in [6] or even [2].

1.3 Summary

In conclusion, the results of [6] and [2] indicate three lines of questioning about radial Fourier multiplier operators, which current research techniques place us in reach of resolving. The first question is whether we can extend the range of exponents upon which the conjecture of [3] is true, at least in the case $d = 2$ where Bochner-Riesz has been solved. The second is whether we can use more sophisticated arguments to prove the L^p sum bounds obtained in [2] when $d = 3$ when the sums are no longer Cartesian products, thus obtaining strong L^p characterizations in this settiong, as well as new results about the endpoint local smoothing conjecture. The third question is whether we can generalize these bounds obtained in these two papers to study radial Fourier multipliers on the sphere.

Chapter 2

Notes on Bochner-Riesz

The goal of this section is to compare and contrast approaches to understanding the Bochner-Riesz conjecture on Euclidean space and on compact Riemannian manifolds, in order to reflect on the differences in understanding multipliers on $\mathbf{R}^d$ vs on a compact manifold X before we attack the more general multiplier problem in this setting. We define the Riesz multipliers via symbols $r_\rho^\delta : [0,\infty] \to [0,\infty)$, defined for $\rho > 0$ and a real number δ by setting, for $\tau > 0$,

$$r_\rho^\delta(\tau) = (1 - \tau/\rho)_+^\delta.$$

Here $s_+ = \max(s,0)$. The resulting radial multipliers on $\mathbf{R}^n$, and on a compact Riemannian manifold X, will be denoted by

$$R_\rho^\delta = r_\rho^\delta\left(\sqrt{-\Delta}\right).$$

The goal of the Bochner-Riesz conjecture is to determine bounds on the operators $\{R_\rho^\delta\}$ invariant under dilation of the symbol.

2.1 Bochner Riesz Bounds Via Tomas-Stein

2.1.1 Euclidean Case

Let's review a reduction of Bochner-Riesz to Tomas Stein:

- First, we can *rescale the problem*. If $r^\delta = r_1^\delta$, then

$$r_\rho^\delta(\lambda) = r^\delta(\lambda/\rho).$$

Thus if $R^\delta = R_1^\delta$, then $R_\rho^\delta = R^\delta \circ \mathrm{Dil}_{1/\rho}$, and so the operators $\{R_\rho^\delta\}$ are uniformly bounded from L^p to L^p for all ρ if and only if R^δ is bounded from L^p to L^p.

- We now perform a *spatial decomposition*. Let k^δ be the convolution kernel corresponding to the operator R^δ. We break up the effects of the operator spatially into dyadic annuli, i.e. writing

$$k^\delta(x) = \sum_{j=0}^{\infty} k_j^\delta(2^j x),$$

where k_0^δ is supported on $|x| \leqslant 2$, and all of the other kernels k_j^δ are supported on the annuli $\{1/2 \leqslant |x| \leqslant 1\}$, and can be written as

$$k_j^\delta(x) = \phi \cdot \mathrm{Dil}_{1/2^j} k^\delta$$

for some $\phi \in C_c^\infty$ supported on the annulus $\{1/2 \leqslant |x| \leqslant 2\}$ and equal to one on the annulus $\{3/4 \leqslant |x| \leqslant 3/2\}$. We analyze each of the convolution kernels separately and then collect up each of the bounds we obtain by applying the triangle inequality. Thus we let R_j^δ be the operator with convolution kernel k_j^δ. Provided we can obtain a bound of the form

$$\|R_j^\delta f\|_{L^p(\mathbf{R}^d)} \lesssim 2^{-\varepsilon j} \|f\|_{L^p(\mathbf{R}^d)}$$

for some $\varepsilon > 0$, and some implicit constant uniform in j, we can sum up the bounds using the triangle inequality to bound R^δ.

- Spatial localization means that the operators $\{R_j^\delta\}$ are *local*, i.e. for any function f, the support of $R_j^\delta f$ is contained in a $O(1)$ neighborhood of the support of f. A decomposition argument, thus implies that it suffices to obtain a bound of the form

$$\|R_j^\delta f\|_{L^p(\mathbf{R}^d)} \lesssim 2^{-\varepsilon j} \|f\|_{L^p(\mathbf{R}^d)}$$

for functions f *supported on balls of radius* 1, since the general bound will follow from this.

- We *reduce to L^2 bounds*: Now that f is suported on a ball of radius 1, R_j^δ is supported on a ball of radius $O(1)$, and so for $p \leqslant 2$ we have

$$\|R_j^\delta f\|_{L^p(\mathbf{R}^d)} \lesssim \|R_j^\delta f\|_{L^2(\mathbf{R}^d)}.$$

  Thus it suffices to obtain a bound of the form $\|R_j^\delta f\|_{L^2(\mathbf{R}^d)} \lesssim \|f\|_{L^p(\mathbf{R}^d)}$. Switching from the L^p norm to the L^2 norm is the most inefficient part of the proof, but it enables us to apply more powerful tools which we only have in $L^2(\mathbf{R}^d)$. Getting around this reduction is key to improving the currently known Bochner-Riesz bounds.

- We reduce the problem to Tomas-Stein. Since we are now in $L^2(\mathbf{R}^d)$, we can apply Plancherel. If ψ_j^δ is the Fourier transform of k_j^δ, then we obtain that

$$\|R_j^\delta f\|_{L^2(\mathbf{R}^d)} = \|\psi_j^\delta \cdot \widehat{f}\|_{L^2(\mathbf{R}^d)}.$$

  A stationary phase calculation shows that ψ_j^δ has the majority of its mass on an annulus of radius 2^j and width $O(1)$, and has magnitude $O(2^{-j\delta})$ there, i.e.

$$|\psi_j^\delta(\xi)| \lesssim_N 2^{-\delta j} \langle 2^j - |\xi| \rangle^{-N}.$$

  Thus by Tomas-Stein, if R_S denotes the restriction operator to the unit sphere S, we find that

$$\begin{aligned}
\|\psi_j^\delta \cdot \widehat{f}\|_{L^2(\mathbf{R}^d)} &\lesssim_N 2^{-\delta j} \left(\int_0^\infty \langle 2^j |1-r| \rangle^{-2N} \int_{|\xi|=1} |\widehat{f}(r\xi)|^2 \, d\sigma \, r^{d-1} dr \right)^{1/2} \\
&\lesssim 2^{-\delta j} \left(\int_0^\infty \langle 2^j |1-r| \rangle^{-2N} \|R_S \circ \mathrm{Dil}_r f\|_{L^2(S^{n-1})}^2 \frac{dr}{r} \right)^{1/2} \\
&\lesssim 2^{-\delta j} \|f\|_{L^p(\mathbf{R}^d)} \left(\int_0^\infty \langle 2^j |1-r| \rangle^{-2N} r^{2d/p-1} dr \right)^{1/2} \\
&\lesssim 2^{-\delta j} \|f\|_{L^p(\mathbf{R}^d)}.
\end{aligned}$$

  This bound is summable in j, which yields the required result.

Let us end our discussion of the Euclidean case by expanding on the computation of the inequality

$$|\psi_j^\delta(\xi)| \lesssim_N 2^{-j\delta} \langle 2^j - |\xi| \rangle^{-N}.$$

Before this, let's see why the result is *intuitive*. The function ψ_j^δ is obtained by localizing the frequency multiplier m^δ on the spatial side and then rescaling. Thus our result is intuitively saying that the phase-portrait of the multiplier is concentrated on a neighborhood of the set

$$\{(x,\xi) : |\xi| \leqslant 1 \text{ and } ||\xi| - 1| = 1/|x|\}.$$

This makes sense, since the 'high frequency' components of m^δ should be distributed near the boundary of the unit ball, since this is where the symbol becomes singular; that the spatial part should be inversely proportional to the distance to the boundary can be detected by taking derivatives of m in the frequency variable, i.e. noting that if $||\xi| - 1| \sim 1/2^j$, then

$$|\nabla^N m^\delta(\xi)| \lesssim_{N,\delta} (1 - |\xi|)^{\delta - N} \sim 2^{-j\delta} 2^{jN}.$$

And we see the derivative grows in N as a power of 2^j, which is inversely propoertional to $||\xi| - 1|$. Working more precisely, we have

$$\psi_j^\delta = 2^{-jd} \left[\widehat{\phi} * \mathrm{Dil}_{2^j} m^\delta \right].$$

The function is the average of $\widehat{\phi}$ over a ball of radius $O(2^j)$ so we immediately obtain a bound by using the rapid decay of $\widehat{\phi}$, thus obtaining that

$$|\psi_j^\delta(\xi)| \lesssim_N \langle 2^j - |\xi| \rangle^{-N}.$$

Thus we see that ψ_j^δ has the majority of it's support on the ball of radius 2^j. But we can do much better than this using the fact that $\widehat{\phi}$ is *oscillatory*, since ϕ is supported away from the origin, and m^δ is *mostly* smooth. More precisely, $\widehat{\phi}$ oscillates at frequencies ~ 1, so we should expect integration by parts to yield useful decay on a quantity $\widehat{\phi} * \mathrm{Dil}_{2^j} f$ if we had a bound $|\nabla^N f| \ll 2^{Nj}$ for large $N > 0$. This is true of m^δ away from a thickness $O(2^{-j})$ annulus containing the unit ball. Thus we are motivated to define $m^\delta = a_j^\delta + b_j^\delta$, where

$$a_j^\delta(\xi) = m^\delta(\xi)\eta(2^j(1 - |\xi|)) \quad \text{and} \quad b_j^\delta(\xi) = m_j^\delta(\xi)(1 - \eta(2^j(1 - |\xi|)))$$

where $\eta(t)$ is supported on $|t| \leqslant 1$ and equal to one for $|t| \leqslant 1/2$. The function b_j^δ is therefore supported on $|\xi| \leqslant 1 - 1/2^{j+1}$. For $N > 0$, we have

$$|\nabla^N m_j^\delta(\xi)| \lesssim_{N,\delta} (1 - |\xi|)^{\delta - N}.$$

By the product rule, $\nabla^N b_j^\delta$ is a sum of derivatives of m_j^δ and of derivatives of $1-\eta(2^j(|\xi|-1))$. The support of any derivative of the latter is supported on $|\xi| \geqslant 1 - 1/2^j$. Thus we have

$$|\nabla^N b_j^\delta(\xi)| \lesssim_N (1-|\xi|)^{\delta - N} \mathbf{I}(|\xi| \leqslant 1 - 1/2^{j+1}) + 2^{j(N-\delta)} \mathbf{I}(1 - 1/2^j \leqslant |\xi| \leqslant 1).$$

Since ϕ is supported away from the origin, we may antidifferentiate $\widehat{\phi}$ arbitrarily many times without any singular behaviour emerging. But now averaging the Nth antiderivative of $\widehat{\phi}$, which is rapidly decaying, with the Nth derivative of $\mathrm{Dil}_{2^j} b_j^\delta$, which is rapidly decaying outside of an annulus of width 1 and radius 2^j, we find that

$$|(\widehat{\phi} * \mathrm{Dil}_{2^j} b_j^\delta)(\xi)| \lesssim 2^{j(d-\delta)} \langle 2^j - \xi \rangle^{-N}.$$

The multiplier a_j^δ is not so smooth, but it is supported on a very thin annulus of radius 1 and thickness $O(2^{-j})$, and m^δ has magnitude at most $2^{-\delta j}$ on this annulus, which gives that

$$|(\widehat{\phi} * \mathrm{Dil}_{2^j} a_j^\delta)(\xi)| \lesssim 2^{-j\delta} \int_{||\eta| - 2^j| \leqslant 1} |\widehat{\phi}(\xi - \eta)|\, d\eta \lesssim_N 2^{j(d-\delta)} \langle 2^j - \xi \rangle^{-N}.$$

Putting these results together gives the required bound.

2.1.2 Manifold Case

The analogue of the Tomas Stein theorem on a compact Riemannian manifold X is a result due to Sogge, so let's see if we can obtain a result for compact manifolds using similar techniques:

- The first problem is that on a compact Riemannian manifold we do not have a rescaling symmetry which we can use to reduce the study of the Bochner-Riesz multipliers R_ρ^δ to the case $\rho = 1$. Thus we must analyze a general multiplier of the form R_ρ^δ for all $\rho > 0$. The case of small ρ is easily dealt with using the triangle inequality, so we may assume that $\rho \gtrsim 1$ in what follows.

- Now we try and reduce to Sogge's spectral cluster bounds, which are analogous to the Tomas-Stein bounds in $\mathbf{R}^d$. If we are able to justify that $K^\delta_{\rho,j}$ behaves like a spectral band projection operator, as in the Euclidean setting, we'd be able to apply this bound. Plancherel does not quite have an analogy to the L^2 setting on a manifold. But we can instead use the wave operator and it's parametrices, i.e. that

$$\begin{aligned} R^\delta_\rho &= \sum_\lambda r^\delta(\lambda/\rho) E_\lambda \\ &= \rho \int_0^\infty \widehat{r^\delta}(\rho t) e^{2\pi i t\sqrt{-\Delta}}\, dt \\ &= c_\delta \cdot \rho^{-\delta} \int_0^\infty e^{2\pi i \rho t}(t + i0)^{-\delta-1} e^{2\pi i t\sqrt{-\Delta}}\, dt. \end{aligned}$$

The singularity in the definition of this integral occurs at $t = 0$, so the operator should, for large t, be relatively well behaved.

- Since we expect the function is well behaved for large t, let's bound these terms so we may reduce to controlling the integral over $t \lesssim 1$. Fix $\alpha \in C_c^\infty(\mathbf{R})$ equal to one in a neighborhood of zero, and consider the behaviour of R^δ_ρ for large t, i.e. the operator

$$R^\delta_\rho = c_\delta \cdot \rho^{-\delta} \int_0^\infty (1 - \alpha(t)) e^{-2\pi i \rho t} t^{-\delta-1} e^{2\pi i t\sqrt{-\Delta}}\, dt.$$

If ψ is the inverse Fourier transform of $c_\delta t^{-\delta-1}(1 - \alpha(t))$, then ψ is bounded and rapidly decreasing because all of the derivatives of it's Fourier transform are smooth and integrable. We thus can revert back to the multiplier setting and write

$$R^\delta_\rho = \rho^{-\delta} \sum_\lambda \psi(\lambda - \rho) E_\lambda.$$

The rapid decay here means we can be fairly lazy in controlling this operator, for instance, employing the Sobolev embedding bound

$$\|E_\lambda f\|_{L^2(X)} \lesssim \langle \lambda \rangle^{d(1/p-1/2)-1/2} \|f\|_{L^p(X)}$$

and the triangle inequality, using the rapid decay to obtain that

$$\|R^\delta_\rho f\|_{L^p(X)} \lesssim \langle \rho \rangle^{-[\delta - d(1/p-1/2)+1/2]} \|f\|_{L^p(X)},$$

which is better than what we need. Thus we now need only bound the operator

$$\tilde{R}^\delta_\rho = c_\delta \cdot \rho^{-\delta} \int_0^\infty \alpha(t) e^{2\pi i \rho t} (t + i0)^{-\delta-1} e^{2\pi i t \sqrt{-\Delta}}\, dt.$$

The advantage of doing this is because we only have understanding of the wave operator through Fourier integral operators (through the Lax parametrix) for times $t \lesssim 1$.

- We now 'spatially localize' as in the Euclidean case, though things look different here since we are dealing with the wave equation. We choose β such that

$$1 = \eta + \sum_{j=1}^\infty \mathrm{Dil}_{2^j} \beta.$$

We then write

$$R^\delta_\rho = \sum_{j=0}^{O(\log \rho)} R^\delta_{\rho,j}$$

where for $j > 0$

$$R^\delta_{\rho,j} = c_\delta \cdot \rho^{-\delta} \int_0^\infty \alpha(t) (\mathrm{Dil}_{2^j} \beta)(\rho t) e^{-2\pi i \rho t} t^{-\delta-1} e^{2\pi i \sqrt{-\Delta}}\, dt,$$

and

$$\begin{aligned} R^\delta_{\rho,0} &= c_\delta \cdot \rho^{-\delta} \int_0^\infty \alpha(t) \eta(\rho t) e^{-2\pi i \rho t} t^{-\delta-1} e^{2\pi i \sqrt{-\Delta}}\, dt \\ &= c_\delta \cdot \rho^{-\delta} \int_0^\infty \eta(\rho t) e^{-2\pi i \rho t} t^{-\delta-1} e^{2\pi i \sqrt{-\Delta}}\, dt, \end{aligned}$$

where the last identity follows because the result the support of the integral is on $t \lesssim 1/\rho$, and we are assuming ρ is large so that α may be assumed equal to one on the support of the integral. Thus R^δ_ρ is an integral over $t \sim 2^j/\rho$. This is analogous to the spatial decomposition we performed in the Euclidean setting, except now we have the wave equation involved, and the 'pseudolocal' finite speed of propogation for the wave equation now must substitute for the explicit spatial localization we obtained in the Euclidean decomposition.

- Despite the singularity that occurs at the origin, the case $j = 0$ is simplest to deal with. If we define

$$m(\lambda) = c_\delta(\widehat{\eta} * r_\delta)$$

then $R^\delta_{\rho,0}$ is a multiplier operator with symbol

$$m_\rho(\lambda) = \rho^{-\delta}\mathrm{Dil}_\rho m.$$

We have estimates of the form

$$|\nabla^N m(\lambda)| \lesssim_N \langle\lambda\rangle^{-M}.$$

Thus

$$|\nabla^N m_\rho(\lambda)| \lesssim_N \rho^{-\delta-N}\langle\lambda/\rho\rangle^{-M}.$$

In particular, taking $M = N$ and $M = 0$ yields that

$$|\nabla^N m_\rho(\lambda)| \lesssim_N \rho^{-\delta}\langle\lambda\rangle^{-N}.$$

Thus $\{\rho^\delta m_\rho\}$ are a uniformly bounded family of symbols of order zero. Thus (TODO: Review estimates for multipliers given by a symbol) we can obtain that

$$\|m_\rho(\sqrt{-\Delta})f\|_{L^p(X)} \lesssim \rho^{-\delta}\|f\|_{L^p(X)} \lesssim \|f\|_{L^p(X)}.$$

TODO: Check there isn't an error here since the $\rho^{-\delta}$ terms helps us out, but shouldn't our bounds be scale invariant?

- Now we deal with the $j > 0$ terms, and we must use the pseudolocal finite speed of propogation of the wave equation as a substitute for explicit localization. Since we have localized to times $t \lesssim 1$. We deal with this by using the Lax parametrix for the wave equation, but first we must ensure the remainder terms from employing the parametrix are well behaved. For $t \lesssim 1$, we can write $e^{2\pi i t\sqrt{-\Delta}} = Q(t) + R(t)$, where $Q(t)$ is a Fourier integral operator supported on a $O(1)$ neighborhood of the diagonal $\Delta = \{(x,x) : x \in X\}$, and with kernel given in coordinates by

$$(x,y) \mapsto \int e^{2\pi i[\phi(x,y,\xi)+t|\xi|]}q(t,x,y,\xi)\, d\xi$$

where q is a symbol of order zero, and ϕ is homogeneous of order one in ξ, with $\phi(x,y,\xi) \approx (x-y)\cdot\xi$, in the sense that

$$|\nabla_\xi^N [\phi(x,y,\xi) - (x-y)\cdot\xi]| \lesssim_N |x-y|^2 |\xi|^{1-N}$$

for all $N > 0$. The operators $\{R(t)\}$ are smoothing, i.e. with a joint kernel A uniformly in $C^\infty([-1,1]\times X\times X)$. Thus we write

$$\begin{aligned} R^\delta_{\rho,j} &= c_\delta \cdot \rho^{-\delta} \int_0^\infty \alpha(t)(\mathrm{Dil}_{2^j}\beta)(\rho t) e^{-2\pi i \rho t} t^{-\delta-1}(Q(t)+R(t))\, dt \\ &= R^\delta_{\rho,j,Q} + R^\delta_{\rho,j,R}. \end{aligned}$$

Let's control the $R(t)$ term. Computing the integral of the kernel defining $R^\delta_{\rho,j,R}$ leads to a term of the form

$$c_\delta 2^j \rho^{-1-\delta} (\widehat{\alpha A} * \mathrm{Dil}_{\rho/2^j} \widehat{\beta} * r^\delta)(\rho).$$

The function αA is smooth and compactly supported in the t variable, so it's Fourier transform is rapidly decaying. The same is true of $\widehat{\beta}$, except it is rescaled so we can imagine the majority of it's mass occurs on $|\lambda| \lesssim \rho/2^j$. Finally, r^δ is concentrated on $|\lambda| \lesssim 1$. Thus the kernel is pointwise bound from above by a constant times

$$2^j \rho^{-1-\delta} \int_{\rho - O(1)}^{\rho + O(1)} (\widehat{\alpha A} * \mathrm{Dil}_{\rho/2^j} \widehat{\beta})(\lambda)\, d\lambda.$$

Taking advantage of the oscillation of $\widehat{\beta}$, and the smoothness of $\widehat{\alpha A}$, i.e. integrating by parts, one can shows that for $|\lambda - \rho| \lesssim 1$

$$|(\widehat{\alpha A} * \mathrm{Dil}_{\rho/2^j}\widehat{\beta})(\lambda)| \lesssim_{N,M} (\rho/2^j)^N \cdot \rho^{-M} \cdot (\rho/2^j) = \rho^{1+N-M} 2^{-(N+1)j},$$

Taking $N = M$ gives that the kernel is bounded above by

$$2^j \rho^{-1-\delta} \left(\rho 2^{-(N+1)j} \right) = 2^{-Nj}.$$

But now trivial estimates, e.g. using Schur's lemma implies that

$$\|R^\delta_{\rho,j,R} f\|_{L^p(X)} \lesssim_N \rho^{-\delta} 2^{-Nj} \|f\|_{L^p(X)},$$

a bound that can be summed in j by taking, e.g. $N = 1$. Thus we are now reduced to the study of the oscillatory integral operators $R^\delta_{\rho,j,Q}$.

- Now let's localize. First off, the condition that $K_{\rho,j}$ is supported on the diagonal, and the compactness of X, means we need only prove the result restricted to a single coordinate chart. Let $K_{\rho,j,t}$ be the kernel of the operator $R^\delta_{\rho,j,Q}$. Intuitively, the wave equation travels at unit speed, so, since $R^\delta_{\rho,j,Q}$ involves the wave equation localized to times $t \sim 2^j/\rho$, we should expect this kernel to be localized to $|x-y| \lesssim 2^j/\rho$. In fact, we will show that the restricted kernel

  $$K'_{\rho,j,t}(x,y) = K_{\rho,j,t}(x,y) \cdot \mathbf{I}(|x-y| \geqslant 2^{j(1+\varepsilon)}/\rho)$$

  has $L^\infty_y L^1_x$ and $L^\infty_x L^1_y$ bounds of the form $O_{\varepsilon,N}(2^{-jN})$, so that Schur's lemma implies that if we write $(R^\delta_{\rho,j,Q})'$ as the operator with kernel $K'_{\rho,j,t}$, then

  $$\|(R^\delta_{\rho,j,Q})' f\|_{L^p(X)} \lesssim_N 2^{-jN} \|f\|_{L^p(X)}.$$

  This reduces us to proving localized estimates of the following form: for some $\varepsilon > 0$, and for any function f supported on a ball of radius $2^j/\rho$, we have a bound

  $$\|R^\delta_{\rho,j,Q} f\|_{L^p(O(2^j/\rho))} \lesssim 2^{-j\varepsilon} \|f\|_{L^p(X)}.$$

  Notice the localization we get here is slightly weaker than in the Euclidean setting (the operators are localized to balls of radius $O(2^{j(1+\varepsilon)}/\rho)$ for any $\varepsilon > 0$ rather than localized to balls of radius $O(2^j/\rho)$) which means our bounds here need the slightly greater decay in j (the $O(2^{-j\varepsilon})$ bound above) rather than a bound independent of j.

  To prove the bounds for the restricted kernel $K'_{\rho,j,t}$ above, we just apply the principle of nonstationary phase to the integral representation, which says that for $|x-y| \gtrsim 2^{j(1+\varepsilon)}/\rho$ we have, taking the Fourier inversion formula in the t variable,

  $$\begin{aligned} K'_{\rho,j,t} &= c_\delta \rho^{-\delta} \int_0^\infty \int \alpha(t)(\mathrm{Dil}_{2^j}\beta)(\rho t)(t+i0)^{-\delta-1} q(t,x,y,\xi) e^{2\pi i[\phi(x,y,\xi)+t|\xi|-\rho t]}\, d\xi\, dt \\ &= \int a^\delta_{\rho,j}(x,y,\xi,|\xi|-\rho) e^{2\pi i \phi(x,y,\xi)}\, d\xi, \end{aligned}$$

  where

  $$a^\delta_{\rho,j}(x,y,\xi,\cdot) = c_\delta 2^j \rho^{-1-\delta} (\widehat{\alpha q(\cdot,x,y},\xi) * \mathrm{Dil}_{\rho/2^j}\beta * r^\delta * q_\cdot(x,y,\xi))$$

and therefore TODO satisfies estimates of the form

$$|\nabla_t^n \nabla_\xi^m a_{\rho,j}^\delta| \lesssim_{n,m,N} 2^{-j\delta} (2^j/\rho)^n \langle 2^j \tau/\rho \rangle^{-N} \langle \xi \rangle^{-m}$$

Nonstationary phase TODO thus gives the required bounds.

- It now suffices to show that for some $\varepsilon > 0$, and for any function f supported on a ball B of radius $2^j/\rho$, we have a bound

  $$\|R_{\rho,j,Q}^\delta f\|_{L^p(O(1)\cdot B)} \lesssim 2^{-j\varepsilon} \|f\|_{L^p(X)}.$$

  Since we are localized, we can now, like in the Euclidean case, reduce to an L^2 bound, i.e. writing

  $$\|R_{\rho,j,Q}^\delta f\|_{L^p(O(1)\cdot B)} \lesssim (2^j/\rho)^{d(1/p-1/2)} \|R_{\rho,j,Q}^\delta f\|_{L^2(O(1)\cdot B)}.$$

  It now suffices to note TODO that $R_{\rho,j,Q}^\delta$ is a Fourier multiplier operator with symbol which is pointwise bouned by $O_N(2^{-j\delta}\langle 2^j \tau/\rho\rangle^{-N})$, so we can now TODO apply Sogge's version of Tomas Stein on manifolds summed over geometric intervals to yield the required bounds.

2.2 Proof Involving Carleson-Sjölin

On $\mathbf{R}^d$, we can take Fourier transforms, applying stationary phase to determine that if $K_{\rho,\delta}$ is the convolution kernel corresponding to R_ρ^δ, then

$$\begin{aligned} K_\delta(x) &= \int_0^1 \lambda^{n-1} (1-\lambda)^\delta e^{2\pi i \xi \cdot x}\, d\lambda \\ &= a_1(x) \frac{e^{2\pi i |x|}}{\langle x \rangle^{\frac{n+1}{2}+\delta}} + a_2(x) \frac{e^{-2\pi i |x|}}{\langle x \rangle^{\frac{n+1}{2}+\delta}} + O\left(\frac{1}{\langle x \rangle^{n+1}} \right), \end{aligned}$$

where a_1 and a_2 are symbols of order zero, with $|a_1(x)|, |a_2(x)| \gtrsim 1$ for $|x| \gtrsim 1$. TODO: Necessity of the $\delta(p)$ values.

Lemma 2.1. *If $a \in C_c^\infty(\mathbf{R}^n \times \mathbf{R}^n)$, and $a(x,y) = 0$ unless $1/2 \leqslant |x-y| \leqslant 2$, then*

$$\left\| \int e^{2\pi i \rho |x-y|} a(x,y) f(y) \right\|_{L^q(\mathbf{R}^n)} \lesssim \rho^{n/q} \|f\|_{L^p(\mathbf{R}^n)}$$

if $q = [(n+1)/(n-1)]p^*$ and $1 \leqslant p \leqslant 2$.

Proof. This is a non homogeneous oscillatory integral operator with wavefront set

$$\left\{ \left(x, y; \frac{x}{|x-y|}, \frac{y}{|x-y|} \right) \right\}$$

which, because of the assumption of the support of a, satisfies the Carleson Sjölin conditions, and thus the result follows. □

The required bounds now follows by applying a dyadic spatial decomposition, rescaling, and applying the result above, hich can be applied because of our explicit computation of the kernel K_δ above. TODO: Go over this argument in more detail to make sure it actually works.

Chapter 3

Heo, Nazarov, and Seeger

In this chapter we give a description of the techniques of Heo, Nazarov, and Seeger's 2011 paper *Radial Fourier Multipliers in High Dimensions* [10]. One of the main goals of this paper is to verify the radial multiplier conjecture in $\mathbf{R}^d$ for $d \geqslant 4$, and $1 < p < 2(d-1)/(d+1)$, i.e. that if $m \in L^\infty(\mathbf{Z})$ is a radial function, $d \geqslant 4$, and $\eta \in \mathcal{S}(\mathbf{R}^d)$ is nonzero, then

$$\|m\|_{M^p(\mathbf{R}^d)} \sim \sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)} \quad \text{for } p \in \left(1, \frac{2(d-1)}{d+1}\right),$$

where the implicit constant depends on p and η. We have

$$\sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)} \sim \sup_{t>0} \frac{\|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)}}{\|\mathrm{Dil}_t \eta\|_{L^p(\mathbf{R}^d)}}.$$

Thus we find that the boundedness of T_m on $\mathcal{S}(\mathbf{R}^d)$ is equivalent to it's boundedness on the family of inputs $\{\mathrm{Dil}_t \eta\}$. If we make the assumption that m is compactly supported, then the assumption is equivalent to the fact that the convolution kernel k associated with m is in $L^p(\mathbf{R}^n)$.

Another consequence of the techniques of this paper is that an 'endpoint' result for local smoothing. Namely, the techniques of the paper imply that if $d \geqslant 4$, and $q > 2 + 4/(d-3)$, then

$$\frac{1}{2L} \int_{-L}^{L} \|e^{it\sqrt{-\Delta}} f\|_{L^q(\mathbf{R}^d)}^q \, dt \lesssim_{q,d} \|(I - L^2 \Delta)^{\alpha/2} f\|_{L^q(\mathbf{R}^d)}^q,$$

where $\alpha = d(1/2 - 1/q) - 1/2$. TODO: Why is this an 'endpoint result', i.e. is it because it works for an arbitrarily L, rather than a unit time interval like local smoothing normally deals with?

3.1 Discretized Reduction

It is obvious that

$$\|m\|_{M^p(\mathbf{R}^d)} \gtrsim_\eta \sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)},$$

so it suffices to show that

$$\|m\|_{M^p(\mathbf{R}^d)} \lesssim_\eta \sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)},$$

We will show this via a discrete convolution inequality, which can also be used to prove local smoothing results for the wave equation.

Let σ_r be the surface measure for the sphere of radius r centered at the origin in $\mathbf{R}^d$. Also fix a nonzero, radial, compactly supported function $\psi \in \mathcal{S}(\mathbf{R}^d)$ whose Fourier transform is non-negative, and vanishes to high order at the origin. Given $x \in \mathbf{R}^d$ and $r \geqslant 1$, define $\chi_{xr} = \mathrm{Trans}_x(\sigma_r * \psi)$, which we view as a smooth function oscillation on a thickness ≈ 1 annulus of radius r centered at x. Our goal is to prove the following inequality.

Lemma 3.1. *For any $a : \mathbf{R}^d \times [1,\infty) \to \mathbf{C}$, and $1 \leqslant p < 2(d-1)/(d+1)$,*

$$\left\| \int_{\mathbf{R}^d} \int_1^\infty a(x,r) \chi_{x,r} \, dx \, dr \right\|_{L^p(\mathbf{R}^d)} \lesssim \left(\int_{\mathbf{R}^d} \int_1^\infty |a(x,r)|^p r^{d-1} dr dx \right)^{1/p}.$$

The implicit constant here depends on p, d, and ψ.

How does Lemma 3.1 prove the required result? Suppose $m : \mathbf{R}^d \to \mathbf{C}$ is a radial multiplier, so we can consider it's convolution kernel $k : \mathbf{R}^d \to \mathbf{C}$, which is also radial. Let $k(x) = b(|x|)$ for some function $b : [0,\infty) \to \mathbf{C}$. If we set $a(x,r) = g(x)b(r)$ for any function $g : \mathbf{R}^d \to \mathbf{C}$, then the function

$$F(x) = \int_{\mathbf{R}^d} \int_1^\infty a(x',r) \chi_{x',r} \, dx' \, dr,$$

is equal to $k * \psi * g$. In this setting, Lemma 3.1 says that

$$\|k * \psi * g\|_{L^p(\mathbf{R}^d)} \lesssim \|k\|_{L^p(\mathbf{R}^d)} \|g\|_{L^p(\mathbf{R}^d)}.$$

The left hand side is a Fourier multiplier operator applied to g, with symbol equal to $\widehat{\psi} \cdot m$, which is clearly related to the bounds we want to show.

In particular, if m is compactly supported away from the origin, let's say, on the annulus $1/2 \leqslant |\xi| \leqslant 2$. If we chose ψ so that $\widehat{\psi}$ is nonvanishing on the annulus $1/4 \leqslant |\xi| \leqslant 2$, then the multiplier $1/\widehat{\psi}$ is smooth on the support of m, and so satisfies $L^p \to L^p$ bounds for all $1 < p < \infty$ restricted to functions with Fourier support on m. In particular, we conclude that m is bounded from L^p to L^p if it's Fourier transform lies in $L^p(\mathbf{R}^d)$. We can then use other tools (Hardy space technology and the like) to study more general multipliers that aren't compactly supported.

To prove Lemma 3.1, it suffices to prove the following discretized estimate where we replace integrals with sums.

Theorem 3.2. *Fix a finite family of pairs $\mathcal{E} \subset \mathbf{R}^d \times [1,\infty)$, which is* discretized *in the sense that $|(x_1,r_1)-(x_2,r_2)| \geqslant 1$ for each distinct pair (x_1,r_1) and (x_2,r_2) in $\mathcal{E}$. Then for any $a : \mathcal{E} \to \mathbf{C}$ and $1 \leqslant p < 2(d-1)/(d+1)$,*

$$\left\| \sum_{(x,r)\in\mathcal{E}} a_r(x)\chi_{x,r} \right\|_{L^p(\mathbf{R}^d)} \lesssim \left(\sum_{(x,r)\in\mathcal{E}} |a_r(x)|^p r^{p-1} \right)^{1/p},$$

where the implicit constant depends on p, d, and ψ, but most importantly, is independant of $\mathcal{E}$.

Proof of Lemma 3.1 from Lemma 3.2. For any $a : \mathbf{R}^d \times [1,\infty) \to \mathbf{C}$,

$$\int_{\mathbf{R}^d} \int_1^\infty a(x,r)\chi_{x,r}\, dr\, dx = \int_{[0,1)^d} \int_0^1 \sum\nolimits_{n\in\mathbf{Z}^d} \sum\nolimits_{m>0} \mathrm{Trans}_{n,m}(a\chi_{x,r})\, dr\, dx$$

Minkowski's inequality thus implies that

$$\begin{aligned}
\left\| \int_{\mathbf{R}^d} \int_1^\infty a(x,r)\chi_{x,r} \right\|_{L^p(\mathbf{R}^d)} &\leqslant \int_{[0,1)^d} \int_0^1 \left\| \sum\nolimits_{n\in\mathbf{Z}^d} \sum\nolimits_{m>0} \mathrm{Trans}_{n,m}(a\chi_{x,r}) \right\|_{L^p(\mathbf{R}^d)} dr\, dx \\
&\lesssim \int_{[0,1)^d} \int_0^1 \left(\sum\nolimits_{n\in\mathbf{Z}^d} \sum\nolimits_{m>0} |a(x-n,r+m)|^p r^{d-1} \right)^{1/p} dr\, dx \\
&\leqslant \left(\int_{[0,1)^d} \int_0^1 \sum\nolimits_{n\in\mathbf{Z}^d} \sum\nolimits_{m>0} |a(x-n,r+m)|^p r^{d-1}\, dr\, dx \right)^{1/p} \\
&= \left(\int_{\mathbf{R}^d} \int_1^\infty |a(x,r)|^p r^{d-1} dr dx \right)^{1/p}. \qquad \square
\end{aligned}$$

Lemma 3.2 is further reduced by considering it as a bound on the operator $a \mapsto \sum_{(x,r)\in\mathcal{E}} a(x,r)\chi_{x,r}$. In particular, applying real interpolation, it suffices for us to prove a restricted strong type bound. Given any discretized set $\mathcal{E}$, let $\mathcal{E}_k$ be the set of $(x,r) \in \mathcal{E}$ with $2^k \leqslant r < 2^{k+1}$. Then Lemma 3.2 is implied by the following Lemma.

Lemma 3.3. *For any* $1 \leqslant p < 2(d-1)/(d+1)$ *and* $k \geqslant 1$,

$$\left\| \sum_{(x,r)\in\mathcal{E}_k} \chi_{x,r} \right\|_{L^p(\mathbf{R}^d)} \lesssim 2^{k(d-1)}\#(\mathcal{E}_k)^{1/p} = 2^k \cdot (2^{k(d-p-1)}\#(\mathcal{E}_k))^{1/p}.$$

Proof of Lemma 3.2 from Lemma 3.3. Applying a dyadic interpolation result (Lemma 2.2 of the paper), Lemma 3.3 implies that

$$\left\| \sum_{(x,r)\in\mathcal{E}} \chi_{x,r} \right\|_{L^p(\mathbf{R}^d)} \lesssim \left(\sum 2^{kp} 2^{k(d-p-1)}\#(\mathcal{E}_k) \right)^{1/p} = \left(\sum 2^{k(d-1)}\#(\mathcal{E}_k) \right)^{1/p}$$

This is a restricted strong type bound for Lemma 3.2, which we can then interpolate. □

If ψ is compactly supported, and r is sufficiently large depending on the size of this support, then $\chi_{x,r}$ is supported on an annulus with centre x, radius r, and thickness $O(1)$. Thus $\|\chi_{x,r}\|_{L^p(\mathbf{R}^d)} \sim r^{(d-1)/p}$, which implies that

$$\left\| \sum_{(x,r)\in\mathcal{E}_k} \chi_{x,r} \right\|_{L^p(\mathbf{R}^d)} \gtrsim 2^{k(d-1)/p}\#(\mathcal{E}_k)^{1/p}.$$

Thus this bound can only be true if $p \geqslant 1$, and becomes tight when $p = 1$, where we actually have

$$\left\| \sum_{(x,r)\in\mathcal{E}_k} \chi_{x,r} \right\|_{L^1(\mathbf{R}^d)} \sim 2^{k(d-1)}\#(\mathcal{E}_k)$$

because there can be no constructive interference in the L^1 norm. Understanding the sum in Lemma 3.3 for $1 < p < 2(d-1)/(d+1)$ will require an understanding of the interference patterns of annuli with comparable radius. We will use almost orthogonality principles to understand these interference patterns.

Lemma 3.4. *For any $N > 0$, $x_1, x_2 \in \mathbf{R}^d$ and $r_1, r_2 \geqslant 1$,*

$$|\langle \chi_{x_1,r_1}, \chi_{x_2,r_2} \rangle| \lesssim_N (r_1 r_2)^{(d-1)/2} (1 + |r_1 - r_2| + |x_1 - x_2|)^{-(d-1)/2} \sum_{\pm,\pm} (1 + ||x_1 - x_2| \pm r_1 \pm r_2|)^{-N}.$$

In particular,

$$|\langle \chi_{x_1,r_1}, \chi_{x_2,r_2} \rangle| \lesssim \left(\frac{r_1 r_2}{|(x_1, r_1) - (x_2, r_2)|} \right)^{(d-1)/2}$$

Remark. Suppose $r_1 \leqslant r_2$. Then Lemma 3.4 implies that χ_{x_1,r_1} and χ_{x_2,r_2} are roughly uncorrelated, except when $|x_1 - x_2|$ and $|r_1 - r_2|$ is small, and in addition, one of the following two properties hold:

- $r_1 + r_2 \approx |x_1 - x_2|$, which holds when the two annuli are 'approximately' externally tangent to one another.
- $r_2 - r_1 \approx |x_1 - x_2|$, which holds when the two annuli are 'approximately' internally tangent to one another.

Heo, Nazarov, and Seeger do not exploit the tangency information, though utilizing the tangencies seems important to improve the results they obtain. Laura Cladek's paper exploits this tangency information, to some extent, to obtain the improved result in her paper.

Proof. We write

$$\begin{aligned}
\langle \chi_{x_1 r_1}, \chi_{x_2 r_2} \rangle &= \langle \widehat{\chi}_{x_1 r_1}, \widehat{\chi}_{x_2 r_2} \rangle \\
&= \int_{\mathbf{R}^d} \widehat{\sigma_{r_1} * \psi}(\xi) \cdot \overline{\widehat{\sigma_{r_2} * \psi}(\xi)} e^{2\pi i (x_2 - x_1) \cdot \xi} \, d\xi \\
&= (r_1 r_2)^{d-1} \int_{\mathbf{R}^d} \widehat{\sigma}(r_1 \xi) \overline{\widehat{\sigma}(r_2 \xi)} |\widehat{\psi}(\xi)|^2 e^{2\pi i (x_2 - x_1) \cdot \xi} \, d\xi.
\end{aligned}$$

Define functions A and B such that $B(|\xi|) = \widehat{\sigma}(\xi)$, and $A(|\xi|) = |\widehat{\psi}(\xi)|^2$. Then

$$\langle \chi_{x_1,r_1}, \chi_{x_2,r_2} \rangle = C_d (r_1 r_2)^{d-1} \int_0^\infty s^{d-1} A(s) B(r_1 s) B(r_2 s) B(|x_2 - x_1| s) \, ds.$$

Using well known asymptotics for the Fourier transform for the spherical measure, we have

$$B(s) = s^{-(d-1)/2} \sum_{n=0}^{N-1} (c_{n,+} e^{2\pi i s} + c_{n,-} e^{-2\pi i s}) s^{-n} + O_N(s^{-N}).$$

But now substituting in, assuming $A(s)$ vanishes to order $100N$ at the origin, we conclude that

$$\begin{aligned}
\langle \chi_{x_1 r_1}, \chi_{x_2 r_2} \rangle = C_d \left(\frac{r_1 r_2}{|x_1 - x_2|} \right)^{(d-1)/2} & \sum_{n,\tau} c_{n,\tau} r_1^{-n_1} r_2^{-n_2} |x_2 - x_1|^{-n_3} \\
& \left\{ \int_0^\infty A(s) s^{-(d-1)/2} s^{-n_1 - n_2 - n_3} e^{2\pi i (\tau_1 r_1 + \tau_2 r_2 + \tau_3 |x_2 - x_1|) s} \, ds \right\} \\
\lesssim_N \left(\frac{r_1 r_2}{|x_1 - x_2|} \right)^{\frac{d-1}{2}} & \left(1 + \frac{1}{|x_1 - x_2|^N} \right) \sum_{\tau} (1 + |\tau_1 r_1 + \tau_2 r_2 + \tau_3 |x_2 - x_1||)^{-5N} \\
\lesssim_N \left(\frac{r_1 r_2}{|x_1 - x_2|} \right)^{\frac{d-1}{2}} & \left(1 + \frac{1}{|x_1 - x_2|^N} \right) \sum_{\tau} (1 + |\tau_1 \tau_3 r_1 + \tau_2 \tau_3 r_2 + |x_2 - x_1||)^{-5N}.
\end{aligned}$$

This gives the result provided that $1 + |x_1 - x_2| \geqslant |r_1 - r_2|/10$ and $|x_1 - x_2| \geqslant 1$. If $1 + |x_1 - x_2| \leqslant |r_1 - r_2|/10$, then the supports of χ_{x_1, r_1} and χ_{x_2, r_2} are disjoint, so the inequality is trivial. On the other hand, if $|x_1 - x_2| \leqslant 1$, then the bound is trivial by the last sentence unless $|r_1 - r_2| \leqslant 10$, and in this case the inequality reduces to the simple inequality

$$\langle \chi_{x_1, r_1}, \chi_{x_2, r_2} \rangle \lesssim_N (r_1 r_2)^{(d-1)/2}.$$

But this follows immediately from the Cauchy-Schwartz inequality. □

The exponent $(d-1)/2$ in Lemma 3.4 is too weak to apply almost orthogonality directly to obtain L^2 bounds on $\sum_{(x,r) \in \mathcal{E}_k} \chi_{xr}$. To fix this, we apply a 'density decomposition', somewhat analogous to a Calderon Zygmund decomposition, which will enable us to obtain L^2 bounds. We say a 1-separated set $\mathcal{E}$ in $\mathbf{R}^d \times [R, 2R)$ is of *density type* (u, R) if

$$\#(B \cap \mathcal{E}) \leqslant u \cdot \operatorname{diam}(B)$$

for each ball B in $\mathbf{R}^{d+1}$ with diameter $\leqslant R$.

Theorem 3.5. *For any 1-separated set $\mathcal{E}_k \subset \mathbf{R}^d \times [2^k, 2^{k+1})$, we can consider a disjoint union $\mathcal{E}_k = \bigcup_{m=1}^{\infty} \mathcal{E}_k(2^m)$ with the following properties:*

- *For each m, $\mathcal{E}_k(2^m)$ has density type $(2^m, 2^k)$.*
- *If B is a ball of radius $r \leqslant 2^m$ containing at least $2^m \cdot r$ points of $\mathcal{E}_k$, then*
$$B \cap \mathcal{E}_k \subset \bigcup_{m' \geqslant m} \mathcal{E}_k(2^{m'}).$$
- *For each m, there are disjoint balls $\{B_i\}$, with radii $\{r_i\}$, each at most 2^k, such that*
$$\sum_i r_i \leqslant \frac{\#(\mathcal{E}_k)}{2^m}$$
such that $\bigcup B_i^*$ covers $\bigcup_{m' \geqslant m} \mathcal{E}_k(2^{m'})$, where B_i^* denotes the ball with the same center as B_i but 5 times the radius.

Proof. Vitali Covering type argument. □

Given a sum $F = \sum_{(x,r) \in \mathcal{E}_k} \chi_{x,r}$, decompose $\mathcal{E}_k$ as $\mathcal{E}_k(2^m)$, and define

$$F_m = \sum_{(x,r) \in \mathcal{E}_k(2^m)} \chi_{x,r}.$$

It follows from the covering argument above that measure of the support of F_m is $O(2^{k(d-1)-m} \#(\mathcal{E}_k))$. To Prove Lemma 3.3, it will suffice to prove the following L^2 estimate on F_m.

Lemma 3.6. *Suppose $\mathcal{E}$ is a set with density type $(2^m, 2^k)$. Then*

$$\left\| \sum_{(x,r) \in \mathcal{E}} \chi_{x,r} \right\|_{L^2(\mathbf{R}^d)} \lesssim 2^{\frac{m}{d-1} + \frac{k(d-1)}{2}} \log(2 + 2^m)^{1/2} \cdot \#(\mathcal{E}_k)^{1/2}.$$

Proof of Lemma 3.3 from Lemma 3.6. Write $F = \sum_{(x,r) \in \mathcal{E}_k} \chi_{x,r}$, and then perform a decomposition $\mathcal{E}_k = \bigcup_{m \geqslant 0} \mathcal{E}_k(2^m)$, and thus define $F = \sum_{m \geqslant 0} F_m$, where

$$F_m = \sum_{(x,r) \in \mathcal{E}(2^m)} \chi_{x,r}.$$

We have

$$\|F_m\|_{L^2(\mathbf{R}^d)} \lesssim 2^{\frac{m}{d-1}+\frac{k(d-1)}{2}} \log(2+2^m)^{1/2} \cdot \#(\mathcal{E}_k)^{1/2}.$$

If we interpolate this bound with the support bound for F_m, a kind of L^0 norm estimate, we conclude that for $0 < p \leqslant 2$,

$$\begin{aligned}
\|F_m\|_{L^p(\mathbf{R}^d)} &\leqslant |\mathrm{Supp}(F_m)|^{1/p-1/2} \|F_m\|_{L^2(\mathbf{R}^d)} \\
&\lesssim (2^{k(d-1)-m})^{1/p-1/2} 2^{\frac{m}{d-1}+\frac{k(d-1)}{2}} \log(2+2^m)^{1/2} \cdot \#(\mathcal{E}_k)^{1/p} \\
&\lesssim 2^{m(1/p_d-1/p)} \log(2+2^m)^{1/2} 2^{\frac{k(d-1)}{p}} \#(\mathcal{E}_k)^{1/p}.
\end{aligned}$$

where $p_d = 2(d-1)/(d+1)$. This bound is summable in m for $p < p_d$, which enables us to conclude that

$$\|F\|_{L^p(\mathbf{R}^d)} \lesssim 2^{\frac{k(d-1)}{p}} \#(\mathcal{E}_k)^{1/p}.$$

Thus for $1 \leqslant p < p_d$, we obtain the bound stated in Lemma 3.3. $\square$

Proving 3.6 is where the weak-orthogonality bounds from Lemma 3.4 come into play.

Proof of Lemma 3.6. Split the interval $[2^k, 2^{k+1}]$ into a family of $\lesssim 2^{(1-\alpha)k}$ intervals of length $2^{\alpha k}$, for some α to be optimized later. For integers $1 \leqslant a \leqslant 2^{(1-\alpha)k}$, let $I_a = [2^k + (a-1)2^{\alpha k}, 2^k + a2^{\alpha k}]$. Let $\mathcal{E}_a = \{(x,r) \in \mathcal{E} : r \in I_a\}$, and write $F = \sum \chi_{x,r}$, and $F_a = \sum_{(x,r)\in\mathcal{E}_a} \chi_{x,r}$. Without loss of generality, splitting up the sum appropriately, we may assume that the set of integers a for which $\mathcal{E}_a$ is non-empty is 10-separated. We calculate that

$$\|F\|_{L^2(\mathbf{R}^d)}^2 = \sum_a \|F_a\|_{L^2(\mathbf{R}^d)}^2 + 2 \sum_{a_1 < a_2} |\langle F_{a_1}, F_{a_2} \rangle|$$

Given $a_1 < a_2$, $(x_1, r_1) \in \mathcal{E}_{a_1}$, and (x_2, r_2) such that $\langle \chi_{x_1 r_1}, \chi_{x_2 r_2} \rangle \neq 0$, then $|x_1 - x_2| \leqslant 2^{k+2}$. Since $|r_1 - r_2| \leqslant 2^{k+1}$ follows because $r_1, r_2 \in [2^k, 2^{k+1}]$, it follows that $|(x_1, r_1) - (x_2, r_2)| \leqslant 3 \cdot 2^{k+1}$. For each such pair, since we may assume that $a_2 - a_1 \geqslant 10$ without loss of generality, it follows that $|r_1 - r_2| \geqslant 2^{\alpha k}$, and so applying Lemma 3.4 together with the density property, we conclude that for $d \geqslant 4$,

$$\begin{aligned}
|\langle \chi_{x_1,r_1}, F_{a_2}\rangle| &\leqslant \sum_{l=1}^{(1-\alpha)k+1} \sum_{2^l 2^{\alpha k} \leqslant |(x_1,r_1)-(x_2,r_2)| \leqslant 2^{l+1} 2^{\alpha k}} \langle \chi_{x_1,r_1}, \chi_{x_2,r_2}\rangle \\
&\lesssim \sum_{l=1}^{(1-\alpha)k+1} (2^m 2^l 2^{\alpha k}) \left(\frac{2^{2k}}{2^l 2^{\alpha k}} \right)^{(d-1)/2} \\
&\lesssim \sum_{l=1}^{(1-\alpha)k+1} 2^m (2^k)^{(d-1)-(d-3)/2\alpha} 2^{-(d-3)/2 \cdot l} \\
&\lesssim 2^m (2^k)^{(d-1)-(d-3)/2\alpha}.
\end{aligned}$$

Summing over all choices of x_1 and r_1, we conclude that

$$2 \sum_{a_1 < a_2} |\langle F_{a_1}, F_{a_2}\rangle| \lesssim 2^m (2^k)^{(d-1)-(d-3)/2\alpha} \#(\mathcal{E}).$$

Now we can study the norms $\|F_a\|_{L^2(\mathbf{R}^d)}^2$. □

Chapter 4

Cladek: Improvements to Radial Multiplier Problem Using Incidence Geometry

Chapter 5

Mockenhaupt, Seeger, and Sogge: Exploiting Wave-Equation Periodicity

The main goal of the paper *Local Smoothing of Fourier Integral Operators and Carleson-Sjölin Estimates* is to prove local regularity theorems for a class of Fourier integral operators in $I^\mu(Z, Y; \mathcal{C})$, where Y is a manifold of dimension $n \geqslant 2$, and Z is a manifold of dimension $n+1$, which naturally arise from the study of wave equations. A consequence of this result will be a local smoothing result for solutions to the wave equation, i.e. that if $2 < p < \infty$, then there is δ depending on p and n, such that if $T : Y \to Y \times \mathbf{R}$ is the solution operator to the wave equation, and Y is a compact manifold whose geodesics are periodic, then T is continuous from from $L^p_c(Y)$ to $L^p_{\alpha,\text{loc}}(Y \times \mathbf{R})$ for $\alpha \leqslant -(n-1)|1/2 - 1/p| + \delta$. Such a result is called local smoothing, since if we define $Tf(t,x) = T_t f(x)$, then the operator T_t is, for each t, a Fourier integral operator of order zero, with canonical relation

$$\mathcal{C}_t = \{(x, y; \xi, \xi) : x = y + t\widehat{\xi}\},$$

where $\widehat{\xi} = \xi/|\xi|$ is the normalization of ξ. Standard results about the regularity of hyperbolic partial differential equations show that each of the operators T_t is continuous from $L^p_c(Y)$ To $L^p_{\alpha,\text{loc}}(Y \times \mathbf{R})$ for $\alpha \leqslant -(n-1)|1/2 - 1/p|$, and that this bound is sharp. Thus T is *smoothing* in the t variable, so that for any $f \in L^p$, the functions $T_t f$ 'on average' gain a regularity of δ over the worst case regularity at each time. The local smoothing conjecture states that this result is true for any $\delta < 1/p$.

The class of Fourier integral operators studied are those satisfying the following condition: as is standard, the canonical relation $\mathcal{C}$ is a conic Lagrangian manifold of dimension $2n + 1$. The fact that $\mathcal{C}$ is Lagrangian implies $\mathcal{C}$ is locally parameterized by $(\nabla_\zeta H(\zeta, \eta), \nabla_\eta H(\zeta, \eta), \zeta, \eta)$, where H is a smooth, real homogeneous function of order one. If we assume $\mathcal{C} \to T^*Y$ is a submersion, then $D_\xi[\nabla_\eta H(\zeta, \eta)]$ has full rank, which implies $D_\eta[\nabla \xi H(\zeta, \eta)] = (D_\xi[\nabla_\eta H(\zeta, \eta)])^T$ has full rank, and thus the projection $\mathcal{C} \to T^*Z$ is an immersion. We make the further assumption that the projection $\mathcal{C} \to Z$ is a submersion, from which it follows that for each z in the image of this projection, the projection of points in $\mathcal{C}$ onto T_z^*Z is a conic hypersurface Γ_z of dimension n. The final assumption we make is that all principal curvatures of Γ_z are non-vanishing.

Remark. The projection properties of $\mathcal{C}$ imply that, in $T^*(Z \times Y)$, there exists a smooth phase ϕ defined on an open subset of $Z \times T^*Y$, homogeneous in T^*Y, such that locally we can write $\mathcal{C}$ as $(z, \nabla_z \phi(z, \eta), \nabla_\eta \phi(z, \eta), \eta)$ for $\eta \neq 0$. Then, working locally on conic sets,

$$\Gamma_z = \{(\nabla_z \phi(z, \eta))\},$$

and the curvature condition becomes that the Hessian $H_{\eta\eta}\langle \nabla_z \phi, \nu \rangle$ has constant rank $n-1$, where ν is the normal vector to Γ_z. This is a natural homogeneous analogue of the Carleson-Sjölin condition for non-homogeneous oscillatory integral operators, i.e. the Carleson-Sjölin condition is allowed to assume $H_{\eta\eta}\phi$ has rank n, which cannot be possible in our case, since ϕ is homogeneous here. An approach using the analytic interpolation method of Stein or the Strichartz / Fractional Integral approach generalizes the Carleson-Sjölin theorem to show that for any smooth, non-homogeneous phase function $\Phi : \mathbf{R}^{n+1} \times \mathbf{R}^n \to \mathbf{R}$, and any compactly supported smooth amplitude a on $\mathbf{R}^{n+1} \times \mathbf{R}^n$. Consider the operators

$$T_\lambda f(z) = \int a(z, y) e^{2\pi i \lambda \Phi(z,y)} f(y)\, dy.$$

If the associated canonical relation $\mathcal{C}$, if $\mathcal{C}$ projects submersively onto $T^*\mathbf{R}^n$, so that for each $z \in \mathbf{R}^{n+1}$ in the image of the projection map $\mathcal{C}$, the set $S_z \subset \mathbf{R}^{n+1}$ obtained from the inverse image of the projection of $\mathcal{C} \to Z$ at z is a n dimensional hypersurface with k non-vanishing curvatures. Then for $1 \leqslant p \leqslant 2$,

$$\|T_\lambda f\|_{L^q(\mathbf{R}^{n+1})} \lesssim \lambda^{-(n+1)/q} \|f\|_{L^p(\mathbf{R}^n)}.$$

where $q = p^*(1+2/k)$.

Remark. We can also see these assumptions as analogues in the framework of cinematic curvature, splitting the z coordinates into 'time-like' and 'space-like' parts. Working locally, because $\mathcal{C} \to T^*Y$ is a submersion, we can consider coordinates $z = (x,t)$ so that, with the phase ϕ introduced above, $D_x(\nabla_\eta \phi)$ has full rank n, and that $\partial_t \phi(x,t,\eta) \neq 0$. Then for each $z = (x,t)$, we can locally write $\partial_t \phi(x,t,\eta) = q(x,t,\nabla_x \phi(x,t,\eta))$, homogeneous in η, and then

$$\mathcal{C} = \{(x,t,y;\xi,\tau,\eta) : (x,\xi) = \chi_t(y,\eta), \tau = q(x,t,\xi)\},$$

where χ_t is a canonical transformation. Our curvature conditions becomes that $H_{\xi\xi} q$ has full rank $n-1$. This is the cinematic curvature condition introduced by Sogge.

Under these assumptions, the paper proves that any Fourier integral operator T in $I^{\mu-1/4}(Z,Y;\mathcal{C})$ maps $L^2_c(Y)$ to $L^q_{\text{loc}}(Z)$ if

$$2\left(\frac{n+1}{n-1}\right) \leqslant q < \infty \quad \text{and} \quad \mu \leqslant -n(1/2-1/q) + 1/q.$$

and maps $L^p_c(Y)$ to $L^p_{\text{loc}}(Z)$ if

$$p > 2 \quad \text{and} \quad \mu \leqslant -(n-1)(1/2-1/p) + \delta(p,n).$$

If we introduce time and space variables locally as in the remark above, any operator in $I^{\mu-1/4}(Z,Y;\mathcal{C})$ can be written locally as a finite sum of operators of the form

$$Tf(x) = \int_{-\infty}^{\infty} T_t f(x),$$

where

$$T_t f(x) = \int a(t,x,\eta) e^{2\pi i \phi(x,t,y,\eta)} f(y)\, dy\, d\eta.$$

is a Fourier integral operator whose canonical relation is a locally a canonical graph, then the general theory implies that each of the maps T_t maps $L^2_c(Y)$ to $L^q_{\text{loc}}(X)$ if

$$2 \leqslant q \leqslant \infty \quad \text{and} \quad \mu \leqslant -n(1/2-1/q)$$

so that here we get local smoothing of order $1/q$, and also maps $L^p_c(Y)$ to $L^p_{\mathrm{loc}}(X)$ if

$$1 < p < \infty \quad \text{and} \quad \mu \leqslant -(n-1)|1/p - 1/2|$$

so we get $\delta(p,n)$ smoothing. A consequence of the smoothing, via Sobolev embedding, is a maximal theorem result for the operator T_t, i.e. that for any finite interval I, the operator

$$Mf = \sup_{t \in I} |T_t f|$$

maps $L^p_c(Y)$ to $L^p_{\mathrm{loc}}(X)$ if $\mu < -(n-1)(1/2 - 1/p) - (1/p - \delta(p,n))$. If the local smoothing conjecture held, we would conclude that, except at the endpoint T^* has the same $L^p_c(Y)$ to $L^p_{\mathrm{loc}}(X)$ mapping properties as each of the operators T_t. We also get square function estimates, such that for any finite interval I, if we consider

$$Sf(x) = \left(\int_I |T_t f(x)|^2 \, dt \right)^{1/2},$$

then for

$$2\frac{n+1}{n-1} \leqslant q < \infty \quad \text{and} \quad \mu \leqslant -n(1/2 - 1/q) + 1/2,$$

the operator S is bounded from $L^2_c(Y)$ to $L^q_{\mathrm{loc}}(X)$.

Our main reason to focus on this paper is the results of the latter half of the paper applying these techniques to radial multipliers on compact manifolds with periodic geodesics. Thus we consider a compact Riemannian manifold M, such that the geodesic flow is periodic with minimal period $2\pi \cdot \Pi$. We consider $m \in L^\infty(\mathbf{R})$, such that $\sup_{s>0} \|\beta \cdot \mathrm{Dil}_s m\|_{L^2_\alpha(\mathbf{R})} = A_\alpha$ is finite for some $\alpha > 1/2$ and some $\beta \in C_c^\infty(\mathbf{R})$. We define a 'radial multiplier' operator

$$Tf = \sum_\lambda m(\lambda) E_\lambda f$$

where E_λ is the projection of f onto the space of eigenfunctions for the operator $\sqrt{-\Delta}$ on M with eigenvalue λ. We can also write this operator as $m(\sqrt{-\Delta})$. Then the wave propogation operator $e^{2\pi i t \sqrt{-\Delta}}$ is periodic of period Π. The Weyl formula tells us that the number of eigenvalues of $\sqrt{-\Delta}$ which are smaller than λ is equal to $V(M) \cdot \lambda^n + O(\lambda^{n-1})$.

Theorem 5.1. *Let $m \in L^2_\alpha(\mathbf{R})$ be supported on $(1,2)$, and assume $\alpha > 1/2$, then for $2 \leqslant p \leqslant 4$, $f \in L^p(M)$, and for any integer k,*

$$\left\| \sup_{2^k \leqslant \tau \leqslant 2^{k+1}} |Dil_\tau m(\sqrt{-\Delta}) f| \right\|_{L^p(M)} \lesssim_\alpha \|m\|_{L^2_\alpha(M)} \|f\|_{L^p(M)}.$$

Proof. To understand the radial multipliers we apply the Fourier transform, writing

$$T_\tau f = (\mathrm{Dil}_\tau m)(\sqrt{-\Delta}) f = m(\sqrt{-\Delta}/\tau) f = \int_{-\infty}^{\infty} \tau \widehat{m}(t\tau) e^{2\pi i t\sqrt{-\Delta}} f \, dt.$$

If we define $\beta \in C_c^\infty((1/2, 8))$ such that $\beta(s) = 1$ for $1 \leqslant s \leqslant 4$, and set $L_k f = \mathrm{Dil}_{2^k} \beta(\sqrt{-\Delta}) f$, then for $2^k \leqslant \tau \leqslant 2^{k+1}$

$$T_\tau f = (\mathrm{Dil}_\tau m)(\sqrt{-\Delta}) f = (\mathrm{Dil}_\tau m \cdot \mathrm{Dil}_{2^k} \beta)(\sqrt{-\Delta}) = T_\tau L_k f.$$

so Cauchy-Schwartz implies that

$$\begin{aligned} |T_\tau f(x)| &= \left| \int_{-\infty}^{\infty} \tau \widehat{m}(\tau) e^{2\pi i t \sqrt{-\Delta}} L_k f(x) \, dt \right| \\ &\leqslant \|m\|_{L^2_\alpha(M)} \left(\int_{-\infty}^{\infty} \frac{\tau}{(1 + |t\tau|^2)^\alpha} |e^{2\pi i t\sqrt{-\Delta}} L_k f(x)|^2 \right)^{1/2} \\ &\leqslant \|m\|_{L^2_\alpha(M)} \left(\int_{-\infty}^{\infty} \frac{2^k}{(1 + |2^k t|^2)^\alpha} |e^{2\pi i t\sqrt{-\Delta}} L_k f(x)|^2 \right)^{1/2} \end{aligned}$$

Because of periodicity, if we set $w_k(t) = 2^k/(1 + |2^k t|^2)^\alpha$, it suffices to prove that for $\alpha > 1/2$,

$$\left\| \left(\int_0^\Pi w_k(t) |e^{2\pi i t\sqrt{-\Delta}} L_k f(x)|^2 \, dt \right)^{1/2} \right\|_{L^p(M)} \lesssim_{\alpha,p} \|f\|_{L^p(M)}.$$

This is a weighted combination of the wave propogators, roughly speaking, assigning weight 2^k for $t \lesssim 1/2^k$, and assigning weight $1/t$ to values $t \gtrsim 1/2^k$.

For a fixed $0 < \delta$, we can split this using a partition of unity into a region where $t \gtrsim \delta$ and a region where $t \lesssim \delta$, where δ is independent of k.

For each t, the wave propogation $e^{2\pi i t\sqrt{-\Delta}}$ is a Fourier integral operator of order zero (we have an explicit formula for small t, and the composition calculus for Fourier integral operators can then be used to give a representation of the propogation operators for all times t, such that the symbols of these operators are locally uniformly bounded in S^0). Thus the square function estimate above can be applied in the region where $t \gtrsim \delta$, because the weighted square integral above has weight $O_\delta(1)$ uniformly in k.

Next, we move onto the region $t \lesssim 1/2^k$. The symbol of the operator $e^{2\pi i t\sqrt{-\Delta}}$

Finally we move onto the region $1/2^k \lesssim t \lesssim \delta$. On this region we have $w_k(t) \sim 1/t$, which hints we should try using dyadic estimates. In particular, suppose that for $\gamma \leqslant \delta$, we have a family of dyadic estimates of the form

$$\left\| \left(\int_\gamma^{2\gamma} |e^{2\pi i t\sqrt{-\Delta}} L_k f|^2 \, dt \right)^{1/2} \right\|_{L^p(M)} \lesssim \gamma^{1/2}(1+\gamma 2^k)^\varepsilon \cdot \|f\|_{L^p(M)}.$$

Summing over the $O(k)$ dyadic numbers between $1/2^k$ and δ gives

$$\left\| \left(\int_{1/2^k \lesssim t \lesssim \delta} |e^{2\pi i t\sqrt{-\Delta}} L_k f|^2 \, \frac{dt}{t} \right)^{1/2} \right\|_{L^p(M)} \lesssim 2^{\varepsilon k} \|f\|_{L^p(M)}$$

If we were able to obtain this inequality for some $\varepsilon > 0$, then we could bound

that for all $0 < \gamma < \Pi/2$

If we localize near $t \lesssim 1/2^k$ by multiplying by $\phi(2^k t)$ for some compactly supported smooth ϕ supported on $|t| \lesssim 1$, then for t on the support of $\phi(2^k t)$ we have a weight proportional to 2^k, and rescaling shows that it suffices to bound the quantities

$$\left\| \left(\int \phi(t) |e^{2\pi i (t/2^k)\sqrt{-\Delta}} L_k f(x)|^2 \, dt \right)^{1/2} \right\|$$

the family of functions

$$\left\| \left(\int |\phi(t) e^{2\pi i (t/2^k)\sqrt{-\Delta}} L_k f(x)|^2 \, Dt \right)^{1/2} \right\|_{L^p_x} \lesssim \sup \|e^{2\pi i (t/2^k)\sqrt{-\Delta}} L_k f\|_{L^p_x}$$

$a_k(t) = 2^{-k/2}\widehat{\phi}(t/2^k)\beta(\tau/2^k)$
it suffices to uniformly bound quantities of the form

$$\left\| \left(\int 2^k \phi(2^k t) |e^{2\pi i \sqrt{-\Delta}} L_k f(x)|^2 \, dt \right)^{1/2} \right\|_{L^p(M)} \lesssim_{\alpha,p} \|f\|_{L^p(M)}$$

We now apply a dyadic decomposition to deal with the smaller values of t. Let us assume for simplicity of notation that $\delta < 1$, and then consider a partition of unity $1 = \sum_{j=1}^{\infty} \phi(2^j t)$ for $0 \leqslant t \leqslant 1$, and such that ϕ is localized near $1/4 \leqslant t \leqslant 2$, then our goal is to bound the quantities

$$\left\| \left(\int_{-\infty}^{\infty} \phi(2^j t) \frac{2^k}{(1 + |2^k t|^2)^{\alpha}} |A_t L_k f(x)|^2 \, dt \right)^{1/2} \right\|_{L^p(M)},$$

which are each proportional to

s

□

Chapter 6

Lee and Seeger: Decomposition Arguments For Estimating Fourier Integral Operators

In the paper *Lebesgue Space Estimates For a Class of Fourier Integral Operators Associated With Wave Propogation*, Lee and Seeger prove a variable coefficient version of the result of Heo, Nazarov, and Seeger, i.e. generalizing their result from proving results about the boundedness of radial Fourier multipliers on $\mathbf{R}^n$ to certain Fourier integral operators satisfying the cinematic curvature condition.

We consider a localized Fourier integral operator $T : \mathcal{D}(Y) \to \mathcal{D}^*(Z)$ of order $\mu - 1/4$, where $\dim(Y) = d$ and $\dim(Z) = d + 1$, with a canonical relation $\mathcal{C}$, which must be $2d + 1$ dimensional, satisfying the following properties:

- The projection map $\pi_{T^*Y} : \mathcal{C} \to T^*Y$ is a submersion. It follows that around any point $(z_0, y_0; \zeta_0, \eta_0)$ we can choose coordinate systems y on Y and (x, t) on Z centered at z_0 and y_0 such that $\zeta_0 = dx_1$, $\eta_0 = dy_1$, and the tangent plane to $\mathcal{C}$ at this point is given by

  $$dx = dy \quad \text{and} \quad d\xi = d\eta \quad \text{and} \quad d\tau = 0.$$

  In particular, it follows that $\pi_Z : \mathcal{C} \to Z$ is a submersion, and we can locally find a function $\phi(z, \eta)$, homogeneous in η, such that, locally,

  $$\mathcal{C} = \{(z, \nabla_\eta \phi(z, \eta); \nabla_z \phi(z, \eta), \eta)\}.$$

By assumption on the tangent space of $\mathcal{C}$,

$$\nabla_\eta \phi(0, e_1) = 0 \quad \text{and} \quad \nabla_z \phi(0, e_1) = e_1.$$

The equivalence of phase theorem implies we can find a symbol $a(x, t, y, \eta)$ of order μ such that, after appropriately localizing the operator T, we have

$$Tf(x) = \int a(x, t, y, \eta) e^{2\pi i [\phi(x,t,\eta) - y \cdot \eta]} f(y) \, d\eta \, dy.$$

- The last assumption implies that for each z_0, $\Sigma_{z_0} \pi_Z^{-1}(z_0)$ is a d dimensional submanifold of $\mathcal{C}$. Moreover, our choice of coordinates makes it easy to see that the natural map $\Sigma_{z_0} \to T^*_{z_0} Z$ is an immersion, whose image is the immersed hypersurface Γ_{z_0} of $T^*_{z_0}$. Indeed, the tangent plane to Σ_{z_0} at the point above is given in coordinates by

  $$dx = dy = dt = d\tau = 0 \quad \text{and} \quad d\xi = d\eta.$$

  And this is projected injectively to the plane defined by $d\tau = 0$ in $T^*_{z_0} Z$. Our other assumption we make about $\mathcal{C}$ is an assumption on *cinematic curvature*. We assume that for each z_0, the hypersurface Σ_{z_0} is a cone with l nonvanishing principal curvatures, for some $1 \leqslant l \leqslant d - 1$. Since

  $$\Sigma_{z_0} = \{ (z_0; \nabla_z \phi(z_0, \eta_0) \}.$$

  The projection assumptions imply that the $(d+1) \times d$ matrix $D_\eta \nabla_z \phi$ has full rank, and the curvature assumptions imply that the Hessian matrix $H_\eta \{ \partial \phi / \partial t \}$ has rank at least l in a neighborhood of our initial point. TODO: Why is it cinematic curvature?

Given these assumptions, the following result is obtained.

Theorem 6.1. *If $l \geqslant 3$, and $2 + 4/(l-2) < q < \infty$, and $\mu \leqslant d/q - (d-1)/2$, then T maps $L^q(Y)$ to $L^q(Z)$.*

If we take $l = d - 1$, we get the full assumption of 'cinematic curvature' and we can use this to get results about local smoothing of the wave equation on compact Riemannian manifolds, which recovers the local smoothing result of Heo, Nazarov, and Seeger obtained in their paper on radial Fourier multipliers.

Theorem 6.2. *Let $d \geqslant 4$, $2+4/(d-3) < q < \infty$, and I a compact time interval. Then if M is a compact Riemannian manifold, and $\alpha = (d-1)/2 - d/q$, then*

$$\left(\| e^{it\sqrt{-\Delta}} f \|_{L^q(M)}^q \, dt \right)^{1/q} \lesssim_I \| f \|_{L^q_\alpha(M)}.$$

Proof. For any compact time interval I, the Lax parametrix construction allows one to, modulo a smoothing operator, find a phase function $\phi(x,y,\xi) \approx (x-y)\cdot\xi$ and a symbol a of order zero supported on a neighborhood of the diagonal and on large frequencies, such that if $\phi_0(t,x,y,\eta) = \phi(x,y,\eta) + t|\eta|_g$, then, modulo smoothing operators,

$$e^{it\sqrt{-\Delta}} f = \int a(x,t,y,\eta) e^{2\pi i \phi_0(t,x,y,\eta)} f(y) \, d\eta \, dy.$$

We define

$$Tf(x,t) = e^{it\sqrt{-\Delta}} f = \int a(x,t,y,\xi) e^{2\pi i \phi_0(t,x,y,\xi)} f(y) \, d\xi \, dy,$$

which is a Fourier integral operator with canonical relation

$$\mathcal{C} = \{ (\exp_y(t\xi/|\xi|), t, y; \eta, |\eta|_g, \eta) \}.$$

One immediately sees that the projection condition is satisfied, and if we are working on a coordinate system localized smaller than the injectivity radius of M, for each $z_0 = (x_0, t_0)$, Γ_{z_0} is a spherical cone, and thus has $d-1$ nonvanishing principal curvatures. The required result then immediately follows from the main result. $\square$

6.1 Frequency Localization and Discretization

Let us describe the idea of the proof. Write $K(z,y)$ for the kernel of T, and perform a frequency decomposition, writing

$$K(z,y) = \sum_{i=1}^{\infty} 2^{i\mu} K_i(z,y)$$

where

$$K_i(z,y) = \int \chi_i(z,y,2^{-i}\eta) e^{2\pi i [\phi(x,t,\eta) - y\cdot\eta]}$$

where $\{\chi_i\}$ are supported on a common compact subset of $Z \times Y \times \Xi$ and satisfy estimates of the form We can set

$$\chi_i(z,y,\eta) = 2^{-i\mu} a(z,y,2^i\eta)\chi(\eta),$$

since then χ_i is supported on $|\eta| \sim 1$ and we have

$$(\partial_z^\alpha \partial_y^\beta \partial_\eta^\kappa \chi_i)(z,y,\eta) \lesssim_{\alpha,\beta,\kappa} 1.$$

By performing another decompsition, we may assume Ξ is an arbitrarily small neighborhood of e_1, such that for $z \in Z$ and $\eta \in \Xi$,

$$\nabla_z \phi(z,\eta) \approx e_1 \quad \text{and} \quad D_z \nabla_\eta \phi(z,\eta) \approx \begin{pmatrix} I_d \\ 0 \end{pmatrix}$$

and $H_\eta\{\partial\phi/\partial t\}$ has rank at least l. In this section, we analyze each of these operators separately. If we write T_i for the operator with kernel K_i, then here we will prove that

$$\|T_i f\|_{L^p} \lesssim 2^{i\left(\frac{d-1}{2} - \frac{d}{q}\right)} \|f\|_{L^p}.$$

for $q > 2l/(l-2)$. In Seeger, Sogge, and Stein, it is proved that

$$\|T_i f\|_{L^\infty} \lesssim 2^{i\frac{d-1}{2}} \|f\|_{L^\infty}.$$

By interpolation, it thus suffices to prove a restricted weak type inequality of the form

$$\|T_i \chi_E\|_{L^{q_l,\infty}} \lesssim 2^{i(d/l-1/2)} |E|^{1/q_l}$$

where $q_l = 2 + 4/(l-2)$. By duality, it suffices to show that for $p_l = 2 - 4/(l+2)$,

$$\|T_i^* \chi_E\|_{L^{p_l,\infty}} \lesssim 2^{i(2d-l)/2l} |E|^{1/p_l},$$

which is equivalent to prove that for $t > 0$, the measure of the set

$$\{y : |T_i^* \chi_E(y)| \geqslant t\}$$

is bounded by $O(t^{-p_l} 2^{i(2d-l)/(l+2)}|E|)$. The operator T_i^* has kernel

$$K^*(y,z) = \overline{K(z,y)} = \int \chi_i(z,y,2^{-i}\eta) e^{2\pi i(y\cdot\eta - \phi(z,\eta))}$$

so we still have a Fourier integral operator, but with a flipped canonical relation. We will obtain these bounds by proving an analogous discretized result at a scale $1/2^i$.

We consider $Z_k = 2^{-k}\mathbf{Z}^{d+1} \cap [-\varepsilon^2, \varepsilon^2]^{d+1}$, for a small constant $\varepsilon > 0$. For each $z \in Z_k$ we consider a function a_z supported on $|\eta| \sim 2^k$ so that

$$|\partial_\eta^\alpha a_z(\eta)| \leqslant 2^{-i|\alpha|}$$

for $|\alpha| \lesssim 1$. Set

$$S_z(y) = \int a_z(\eta) e^{2\pi i (y\cdot\eta - \phi(z,\eta))}\, d\eta.$$

Our job is to understand the sums $\sum S_z$.

Lemma 6.3. *For each $\mathcal{E} \subset Z_k$, the measure of the set of y such that*

$$|\sum S_z(y)| \geqslant t$$

is $\lesssim 2^{i((d+1)l/(l+2)-1)}) t^{-p_l} \cdot \#(\mathcal{E})$.

TODO: How do we recover the continuous version of the result.

6.2 L^1 Estimates

To understand the individual pieces S_z, we consider a maximal $2^{-i/2}$ separated set Θ covering the unit sphere, and perform a further decomposition

$$a_z(\eta) = \sum_{\theta\in\Theta} a_{z,\theta}(\eta),$$

where $a_{z,\theta}$ is supported in a cone with aperture $O(2^{-i/2})$ centered at θ. Then $a_{z,\theta}$ is roughly speaking, supported on a set with length $2^{i/2}$ tangent to the radial direction, and with length 2^i in the radial direction. Thus differentiating in the radial direction no longer leads to quite as good derivative estimates, namely, if $u_1, \dots, u_M$ are unit vectors tangent to θ, and $M + N \lesssim 1$, then

$$(\theta \cdot \nabla_\eta)^N \prod_{i=1}^M (u_i \cdot \nabla_\eta)\{a_{z,\theta}\} \lesssim 2^{-kN-kM/2}.$$

The decomposition of a_z of course leads to a decomposition $S_z = \sum S_{z,\theta}$.

Now because each component of $\nabla_\eta \phi$ is homogeneous of degree 0, Euler's homogeneous function theorem says that

$$H_\eta \phi(x, \eta) \cdot \eta = 0.$$

Integration by parts (TODO: How? Also is there a typo?) yields that

$$|S_{z,\theta}(y)| \lesssim 2^{i\frac{d+1}{2}} \left(1 + 2^i |(\nabla_\xi \phi(z,\theta) - y) \cdot \theta| + 2^{k/2} |\Pi_{\theta^\perp}(\nabla_\xi \phi(z,\theta) - y)|\right)^{-O(1)}.$$

Roughly speaking, this inequality says that, roughly speaking, $S_{z,\theta}$ has magnitude $2^{i\frac{d+1}{2}}$, and is concentrated on a tube centered at $\nabla_\xi \phi(z,\theta)$, with thickness 2^{-i} in the radial direction, and thickness $2^{-i/2}$ in the tangential direction to θ. In particular, we find that

$$\|S_{z,\theta}\|_{L^1} \lesssim 1.$$

The triangle inequality (probably the best we can do in general in the L^1 setting) implies that

$$\|S_z\|_{L^1} \lesssim 2^{i(d-1)/2}.$$

This is the bound we will use in L^1.

To get more interesting bounds in other L^p spaces, we look at the orthogonality of the functions $\{S_z\}$. On the Fourier side of things, we have

$$\widehat{S_z}(\eta) = a_z(\eta) e^{-2\pi i \phi(z,\eta)}.$$

Thus by Parsevel, we have

$$\langle S_z, S_w \rangle = \langle \widehat{S_z}, \widehat{S_w} \rangle = \int a_z(\eta) a_w(\eta) e^{2\pi i [\phi(z,\eta) - \phi(w,\eta)]}.$$

TODO: Expand on rest of argument.

6.3 Adapting the Argument to Fourier Multipliers

Let $T = m(-\sqrt{\Delta})$ be a radial multiplier on $\mathbf{R}^n$, i.e. such that

$$Tf(x) = \int m(|\xi|) e^{2\pi i \xi \cdot (x-y)} f(y)\, d\xi\, dy.$$

If m is a symbol, then we can interpret T directly as a Pseudodifferential Operator. But Heo, Nazarov, and Seeger's result discuss families of multipliers m that are not even necessarily smooth, but do satisfy certain integrability conditions. To fix this, we assume a priori that we have applyied a decomposition argument, so we may assume m is compactly supported away from the origin. Then (by Paley-Wiener) $\widehat{m}$ is a smooth symbol of some finite order satisfying some integrability properties, which indicates how we might apply the theory of Fourier integral operators, i.e. by taking the Fourier transform of m, we get that

$$Tf(x) = \int \widehat{m}(\rho) e^{2\pi i [\rho|\xi| + \xi \cdot (x-y)]} f(y)\, d\rho\, d\xi\, dy.$$

This is 'almost' a Fourier integral operator, except the phase is not smooth unless $\widehat{m}$ is supported away from the origin (fixed by a decomposition argument), and the phase is non-homogeneous. To fix the non-homogeneity, we just isolate the operator in ρ, writing

$$Tf(x) = \int_{-\infty}^{\infty} \widehat{m}(\rho) T_\rho f(x)\, d\rho,$$

where

$$T_\rho f(x) = e^{2\pi i \rho \sqrt{-\Delta}} f(x) = \int e^{2\pi i [\rho|\xi| + \xi \cdot (x-y)]} f(y)\, d\xi\, dy$$

is the propogation operator for the half-wave equation $\partial_t u = \sqrt{-\Delta} \cdot u$. It has phase $\phi(x,y,\xi) = \rho|\xi| + \xi \cdot (x-y)$, and thus we have a stationary frequency value when $x = y - \rho\widehat{\xi}$, where $\widehat{\xi} = \xi/|\xi|$ is the normalization of ξ. This has canonical relation

Chapter 7

Relations to Local Smoothing

Let us now try and prove certain special cases of the radial multiplier conjecture on the sphere S^n. Thus we fix a symbol h, and study operators of the form

$$T_R = h\left(\sqrt{-\Delta}/R\right) = \sum h(\lambda/R)E_\lambda,$$

where E_λ is the projection operator onto the eigenspace corresponding to the eigenvalue λ. In particular, we wish to characterize the boundedness properties of the operators $T_{h,R}$, in terms of appropriate control of the Fourier transform of the function h. More precisely, we fix an exponent p, and assume that the quantity

$$A_p(h) = \sup_{t>0} \left(\int_{t/2 \leqslant |s| \leqslant 2t} |\widehat{h}(s)|^p (1+|s|)^{(d-1)(1-p/2)}\, dt \right)^{1/q}$$

is finite, which is a necessary condition for the multiplier $h(\sqrt{-\Delta})$ to be bounded on $L^p(\mathbf{R}^d)$ or $L^{p^*}(\mathbf{R}^d)$, and thus by the result of Mityagin, necessary for the family of operators $\{T_R : R > 0\}$ to be uniformly bounded in R on $L^p(S^n)$ or $L^{p^*}(S^n)$. For simplicity, let us assume that $\text{supp}(h)$ is contained in $1/2 \leqslant \lambda \leqslant 2$.

7.1 Attempt # 1: Exploiting Local Smoothing

Our goal is to show that, uniformly in R, we have

$$\|T_R f\|_{L^p} \lesssim \|f\|_{L^p}.$$

Since T_R is a multiplier with symbol supported on $R/2 \leqslant \lambda \leqslant 2R$, we may assume that f is also supported on this frequency range, i.e. is in the span of eigenfunctions to $\sqrt{-\Delta}$ with eigenvalue $R/2 \leqslant \lambda \leqslant 2R$. In particular, this implies that $\|f\|_{L^p_\alpha} \lesssim (1+R)^\alpha \|f\|_{L^p}$.

To exploit the fact that $A_p(h)$ is finite, we apply the Fourier transform to the sum defining T_R, writing $T_R f$ as the vector valued integral

$$T_R f = \int_{-\infty}^{\infty} R\widehat{h}(Rt) e^{2\pi i t\sqrt{-\Delta}} f.$$

where the wave propogators $\{e^{2\pi i t\sqrt{-\Delta}}\}$ give solution operators to the half wave equation

$$\frac{\partial}{\partial t} = 2\pi i \sqrt{-\Delta}.$$

We break up $T_R f = \sum_{k=0}^{\infty} T_{R,k} f$, where

$$T_{R,0} f = \int_{-\infty}^{\infty} R\widehat{h}(Rt)\rho_0(Rt) e^{2\pi i t\sqrt{-\Delta}} f$$

and for $k \geqslant 1$,

$$T_{R,k} f = \int_{-\infty}^{\infty} R\widehat{h}(Rt)\rho(Rt/2^k) e^{2\pi i t\sqrt{-\Delta}} f.$$

If m is the inverse Fourier transform of $\widehat{h} \cdot \rho_0$ then $T_{R,0} = m\left(\sqrt{-\Delta}/R\right)$. This is probably easy to bound so I'll do this one later (e.g. reducing to ΨDO theory since m is smooth and rapidly decaying). Next, Hölder's inequality, combined with the trick of first multiplying and dividing by $(1+|Rt|)^{(d-1)(1/p-1/2)}$ implies that

$$\begin{aligned}
|T_{R,k} f| &= \left| \int_{-\infty}^{\infty} R\widehat{h}(Rt)\rho(Rt/2^k) e^{2\pi i t\sqrt{-\Delta}} f \, dt \right| \\
&\leqslant R \left(\int_{|t|\sim 2^k/R} |\widehat{h}(Rt)|^p (1+|Rt|)^{(d-1)(1-p/2)} \, dt \right)^{1/p} \\
&\qquad \left(\int_{-\infty}^{\infty} |e^{2\pi i t\sqrt{-\Delta}} f|^{p^*} \rho(Rt/2^k)(1+|Rt|)^{-(d-1)(p^*/2-1)} \, dt \right)^{1/p^*} \\
&\lesssim R^{1-1/p} 2^{-k(d-1)(1/2-1/p^*)} A_p(h) \left(\int_{|t|\sim 2^k/R} |e^{2\pi i t\sqrt{-\Delta}} f|^{p^*} \, dt \right)^{1/p^*}.
\end{aligned}$$

Applying the periodicity of the wave equation, for $2^k \geqslant R$ we have

$$\left(\int_{|t| \sim 2^k/R} |e^{2\pi i t \sqrt{-\Delta}} f|^{p^*} \right)^{1/p^*} \lesssim (2^k/R)^{1/p^*} \left(\int_{|t| \lesssim 1} |e^{2\pi i t \sqrt{-\Delta}} f|^{p^*} \right)^{1/p^*}.$$

Now if h is compactly supported, then we can also replace f with the spectral projection $P_{\lambda \sim R} f$ (do this before Littlewood-Paley). If the endpoint local smoothing conjecture held, then we would have

$$\left(\int_{|t| \lesssim 1} |e^{2\pi i t \sqrt{-\Delta}} f|^{p^*} \right)^{1/p^*} \lesssim \|f\|_{L^{p^*}_{\alpha_p}},$$

where

$$\alpha_p = d(1/2 - 1/p) - 1/2.$$

Since f is a sum of well behaved eigenfunctions, Sobolev embedding should give us a bound of the form

$$\|f\|_{L^{p^*}_{\alpha_p}} \lesssim R^T \|f\|_{L^p}$$

for an appropriate power of R. If we choose T such that

$$1/p^* - \alpha_p/d = 1/p - T/d,$$

i.e. we pick

$$T = d(1/p - 1/2) - 1/2$$

TODO: Check assumptions of Sobolev embedding are true. Then we conclude that

$$\|f\|_{L^{p^*}_{\alpha_p}} \lesssim R^T \|f\|_{L^p}.$$

Putting all the bounds together, we conclude that

$$\begin{aligned} \|T_{R,k} f\|_{L^p} &\lesssim R^{1-1/p} 2^{-k(d-1)(1/2-1/p^*)} A_p(h) (2^k/R)^{1/p^*} R^T \|f\|_{L^p} \\ &\lesssim R^{d(1/p-1/2)-1/2} 2^{-k[(d-1)/2 - d/p^*]} A_p(h) \|f\|_{L^p}. \end{aligned}$$

Provided that $p < 2d/(d+1)$, this bound is summable in k. And provided that $p \geqslant 2d/(d+1)$, the bound is uniform in R. This indicates we're 'precisely' at the endpoint. In particular, local smoothing implies that for any $\varepsilon > 0$, there exists a range of p such that for each such p, there exists δ with

$$\|T_{R,k} f\|_{L^p} \lesssim 2^{-\delta k} A_{p,\varepsilon}(h) \|f\|_{L^p}.$$

where

$$A_{p,\varepsilon}(h) = \sup_{t>0} \left(\int_{t/2 \leqslant |s| \leqslant 2t} |\widehat{h}(s)|(1+|s|)^{(d-1)(1-p/2)+\varepsilon} \right).$$

Even in this case, we still need to deal with $2^k \leqslant R$, i.e. where there's no periodicity, but probably Euclidean techniques apply here since there's no overlap. But let's forget about that for now.

The small time parameterix for the half-wave operator, combined with the composition calculus of Fourier integral operators, allows us to write, for $|t| \leqslant 1$,

$$e^{2\pi i t\sqrt{-\Delta}} f = T_t f + S_t^\infty f,$$

where S_t^∞ is a *smoothing operator*, i.e. an integral operator with

$$S_t^\infty f(x) = \int K(t,x,y) f(y)\, dy,$$

where $K \in C^\infty([-1,1] \times S^n \times S^n)$, and where we can locally write

$$T_t f(x) = \int_{\mathbf{R}^n} a(t,x,y,\xi) e^{2\pi i \Phi(x,y,\xi)} f(y)\, d\xi\, dy$$

for some symbol $a \in S^0$, and some symbol $\Phi \in S^1$ satisfying

$$\Phi(x,y,t,\xi) = (x-y)\cdot\xi + t g_y(\xi,\xi) + O(|x-y|^2|\xi|).$$

To calculate the canonical relation of this operator, we look at the principal symbol of the operator $\sqrt{-\Delta}$. If g is the metric of S^n, then the principal symbol will be

$$p(x,\xi) = C\left(\sum g_{ij}(x)\xi^i\xi^j\right)^{1/2} = C|\xi|$$

for an appropriate constant C (TODO: Do this calculation more precisely). One can calculate (see Remark in Section 4.1 Sogge's book) that the canonical relation of the family of operators $\{T_t\}$ is given by

$$\mathcal{C} \subset \{(x,t,\xi,\tau,y,\eta) : (y,\eta) = \phi_t(x,\xi), \tau = p(x,\xi)\},$$

where $t \mapsto \phi_t(x,\xi)$ is the geodesic travelling at a velocity of 2π which starts at x, and travels in the direction given by ξ. We claim this canonical relation satisfies the cinematic curvature condition. Indeed, the projections

$\Pi_{y,\eta} : \mathcal{C} \to T^*S^n$ and $\Pi_{x,t} : \mathcal{C} \to T^*S^n$ are both submersions, so the Fourier integral operator is nondegenerate. For each pair (x_0, t_0), the cone

$$\mathcal{C}_{x_0,t_0} = \{(\xi, \tau) : |\xi| = C^{-1}\tau\}$$

is an n dimensional hypersurface in $T^*_{x_0,t_0}((-1,1) \times S^n)$, and it is easy to see this hypersurface is curved for all t. The endpoint local smoothing conjecture claims that if $f \in L^{p^*}(S^{n-1})$ for precisely the range of p we care about in the radial multiplier conjecture, then $Tf \in L^{p^*}_{-\alpha_p}((-1,1) \times S^{n-1})$, where $\alpha_p = n(1/2 - 1/p^*) - 1/2 = n(1/p - 1/2) - 1/2$. In particular, if we assume that Sobolev norms work on S^n the same way they work on $\mathbf{R}^n$, this means that if $f \in L^{p^*}(S^{n-1})$ has frequency supported on $|\xi| \leqslant L$, then

$$\|T_R f\|_{L^{p^*}(\mathbf{R}^d)} \lesssim \|f\|_{L^{p^*}(\mathbf{R}^d)_{\alpha_p}} \lesssim L^{\alpha_p} \|f\|_{L^{p^*}(\mathbf{R}^d)}.$$

Thus we see that local smoothing is pretty hopeless in proving the result we need to prove for general f.

The only non optimal inequality we applied here was Hölder's inequality, which would be tight if there exists a non-negative function γ such that

$$|e^{2\pi i t\sqrt{-\Delta}} f|^{p^*} (1 + |Rt|)^{-(d-1)\frac{(2-p)}{2(p-1)}} = \gamma(x)^{p^*} |\widehat{h}(Rt)|^p (1 + |Rt|)^{(d-1)(1-p/2)},$$

i.e. where

$$|e^{2\pi i t\sqrt{-\Delta}} f(x)| = \gamma(x)(1 + |Rt|)^{(d-1)\frac{(2-p)}{2}} |\widehat{h}(Rt)|^{1/p^*}.$$

TODO: Think about why local smoothing is useless. Is the theorem trivial if Hölder's inequality is applied?

7.2 Junk

Rescaling and applying Hölder's inequality, we have

$$\begin{aligned}
R\int_0^{2\pi}\int_{\mathbf{R}^d}\sum_{k=0}^{\infty} w(Rs+(2\pi k)R)a(s,x,y,\xi)e^{2\pi i\Phi(x,y,s/R,\xi)}\,d\xi\,ds \\
&= \sum_{k=0}^{\infty}\int_0^{2\pi R} w(s+(2\pi k)R)\int_{\mathbf{R}^d} a(s/R,x,y,\xi)e^{2\ pii\Phi(x,y,s/R,\xi)}\,d\xi\,ds \\
&\leqslant \sum_{k=0}^{\infty}\left(\int_0^{2\pi R}|w(s+(2\pi k)R)|^q\,ds\right)^{1/q}(s)
\end{aligned}$$

Now suppose that $\|w\|_{L^q(\mathbf{R}^d,(1+|x|)^{(d-1)(1-q/2)})} < \infty$

$$\int_0^{2\pi}\int_{\mathbf{R}^d}\sum_{k=0}^{\infty} b_t(Rs+(2\pi k)R)a(s,x,y,\xi)e^{2\pi i\Phi(x,y,\xi)}\,d\xi\,ds,$$

Let us begin with the qualitative assumption that h is compactly supported. Then, by breaking things up into a finite sum, we may assume that h is supported on $[1/2,2]$. Fix a function $\chi \in C_c^\infty(\mathbf{R})$ equal to one on $[1/2,2]$, and vanishing outside of $[1/4,4]$. Write

$$P_R = \chi\left(\sqrt{-\Delta}/R\right) = \sum \chi(\lambda/R)E_\lambda.$$

Then for any function $f \in C^\infty(S^n)$, $T_R f = T_R\{P_R f\}$. Thus when bounding the behaviour of the operator T_R, we may assume inputs are linear combinations of eigenfunctions to $\sqrt{-\Delta}$ with eigenvalues $\lambda \sim R$.

7.3 Adapting The Proof

The proof of Heo, Nazarov, and Seeger controls discrete sums of the form

$$\sum_{(y,r)\in\mathcal{E}} c(y,r)F_{y,r}$$

where $F_{y,r} = \mathrm{Trans}_y(\sigma_r * \psi)$, where ψ is radial, and it's Fourier transform is non-negative, and vanishes to high order at the origin, i.e. so it has some oscillation. A natural question to ask is whether on a compact Riemannian manifolds, there are functions analogous to the $F_{y,r}$ which we can study and control.

One option that might be comparable to the operators σ_r is the operators $e^{2\pi i r\sqrt{-\Delta}}$ which has a singularities supported near geodesic spheres, which corresponds to the multiplier $\lambda \mapsto e^{2\pi i \lambda r}$. The convolution thus corresponds to the multiplier operator

$$m_r(\xi) = e^{2\pi i \lambda r} \psi^\vee(\lambda),$$

for which

$$m_r(\sqrt{-\Delta}) = \int_{-\infty}^{\infty} \psi(t-r) e^{2\pi i t\sqrt{-\Delta}}.$$

Let K_r be the kernel of $m_r(\sqrt{-\Delta})$. Then

$$(\psi m)(\sqrt{-\Delta}) f = \int_{\infty}^{\infty} \psi(t) m(t) e^{2\pi i t\sqrt{-\Delta}} f$$

A comparable option for the operators σ_r are the operators $e^{2\pi i r\sqrt{-\Delta}}$ which have singularities suported near s.

But how do we add the oscillating term

$$e^{2\pi i t\sqrt{-\Delta}} f(x) = \int e^{2\pi i (x\cdot\xi + |\xi| t)} \widehat{f}(\xi)\, d\xi.$$

Part II

Papers I Don't Understand Yet

Chapter 8

Seeger: Singular Convolution Operators in L^p Spaces

Let $m : \mathbf{R}^d \to \mathbf{C}$ be the symbol for a Fourier multiplier operator $m(D)$. If the resulting operator $m(D)$ was bounded from $L^p(\mathbf{R}^d)$ to $L^p(\mathbf{R}^d)$ with operator norm A, then the operator would also be bounded 'at all scales'. That is, if we consider a littlewood Paley decomposition, i.e. taking

$$f = \sum_{i=0}^{\infty} f_i$$

where $\widehat{f_i} = \eta_i \widehat{f}$ is supported on $2^i \leqslant |\xi| \leqslant 2^{i+1}$ for $i \geqslant 1$, and $|\xi| \leqslant 2$ for $i = 0$, then we would have estimates of the form

$$\|m(D)f_i\|_{L^p(\mathbf{R}^d)} \lesssim \|f_i\|_{L^p(\mathbf{R}^d)} \lesssim \|f\|_{L^p(\mathbf{R}^d)}, \tag{8.1}$$

where the implicit constant is uniform in i. The main focus of the paper in question is to determine whether a uniform bound of the form (8.1) implies $m(D)$ is bounded. More precisely, is it true that

$$\|m\|_{M^p(\mathbf{R}^d)} \lesssim_p \sup_{i \geqslant 0} \|m_i\|_{M^p(\mathbf{R}^d)}, \tag{8.2}$$

where $m_i = \eta_i m$.

The Hilbert transform H is a Fourier multiplier with symbol $m(\xi) = \text{sgn}(\xi)$. For each $i > 0$, $m_i(\xi) = \eta_i \text{sgn}(\xi)$, so that

$$K_i(x) = \widehat{\eta_i \text{sgn}(\xi)} = 2^i H\eta(2^i x).$$

Thus

$$\|K_i\|_{L^1(\mathbf{R})} = \|H\eta\|_{L^1(\mathbf{R})}.$$

TODO

It is clear that (8.2) is true for $p = 2$, since in this case the bound is equivalent to an inequality of the form

$$\|m\|_{L^\infty(\mathbf{R}^d)} \lesssim \sup_{i \geqslant 0} \|m_i\|_{L^\infty(\mathbf{R}^d)},$$

which is true because the supports of the symbols $\{m_i\}$ are almost all pairwise disjoint. On the other hand, (8.2) does not hold when $p = 1$ or $p = \infty$, which makes sense, since Littlewood-Paley runs into all kinds of problems for these values of p. Arguing more precisely, the condition would be equivalent to showing that for any $K : \mathbf{R}^d \to \mathbf{C}$,

$$\|K\|_{L^1(\mathbf{R}^d)} \lesssim \sup_{i \geqslant 0} \|K * \widehat{\eta}_i\|_{L^1(\mathbf{R}^d)}.$$

If

$$K_N(x) = \int_{|\xi| \leqslant 2^N} e^{2\pi i \xi \cdot x}\, d\xi$$

is the Dirichlet kernel, then $\|K_N\|_{L^1(\mathbf{R})} \sim N$. On the other hand, for $i \leqslant N - 1$, we have $K_N * \widehat{\eta}_i = \widehat{\eta}_i$, so that

$$\|K_N * \widehat{\eta}_i\|_{L^1(\mathbf{R})} = \|\widehat{\eta}_i\|_{L^1(\mathbf{R})} \lesssim 1.$$

For $i \geqslant N + 1$, we have $K_N * \widehat{\eta}_i = 0$, so that

$$\|K_N * \widehat{\eta}_i\|_{L^1(\mathbf{R})} = 0 \lesssim 1.$$

For $i = N$, we have

$$(K_N * \widehat{\eta_N})(x) = 2^N \int_0^1 \eta(\xi) e^{2\pi i 2^N (\xi \cdot x)} + \int_1^2 \eta(-\xi) e^{-2\pi i 2^N (\xi \cdot x)}\, d\xi$$

$$\int |K_N * \widehat{\eta}_i|$$

whereas one

$$K_N * \widehat{\eta}_i = \begin{cases} \widehat{\eta}_i & : i \lesssim N \\ 0 & : i \gtrsim N \end{cases},$$

and so $\|K_N * \widehat{\eta}_i\|_{L^1(\mathbf{R})} \lesssim 1$ uniformly in N and i. We can then use Baire category techniques to find a kernel K not in $L^1(\mathbf{R})$, but such that $\|K * \eta_i\|_{L^1(\mathbf{R})} \lesssim 1$, uniformly in i.

The result actually fails for $2 < p < \infty$, due to an examples of Triebel. For simplicity, let's work in $\mathbf{R}$. If we fix a bump function $\phi \in C_c^\infty(\mathbf{R})$ supported in $[-1,1]$, and set

$$m_N(\xi) = \sum_{k=N}^{2N} e^{2\pi i (2^k \xi)} \phi(\xi - 2^k),$$

then $m_N(\xi)\eta_i(\xi) = m_{N,i}(\xi)$, where $m_{N,i}(\xi) = e^{2\pi i (2^k \xi)} \phi(\xi - 2^k)$, and so $K_{N,i}(x) = \widehat{m_{N,i}}(x) = e^{2\pi i 2^k (x - 2^k)} \widehat{\phi}(x - 2^k)$, hence

$$\|m_{N,i}(D) f\|_{L^p(\mathbf{R}^d)} = \|K_{N,i} * f\|_{L^p(\mathbf{R}^d)} \leqslant \|\widehat{\phi}\|_{L^1(\mathbf{R})} \|f\|_{L^p(\mathbf{R})} \lesssim \|f\|_{L^p(\mathbf{R})}.$$

On the other hand, the operator norm of $m_N(D)$ from $L^p(\mathbf{R})$ to $L^p(\mathbf{R})$ is actually $\gtrsim_p N^{|1/p - 1/2|}$, and thus not bounded uniformly in N, so Baire category shows things don't work so well here.

This paper shows that one *can* get uniform bounds assuming an additional, very weak smoothness condition, which rules out the example m_N above. Under the most simple assumptions, if (8.1) holds, and $\|m_i\|_{\Lambda^\varepsilon} \lesssim 2^{-ik}$, where Λ^ε is the ε-Lipschitz norm, then $\|m(D)f\|_{L^r(\mathbf{R}^d)} \lesssim \|f\|_{L^r(\mathbf{R}^d)}$ whenever $|1/r - 1/2| < |1/p - 1/2|$. Under slightly stronger smoothness assumptions, we can actually conclude $\|m(D)f\|_{L^p(\mathbf{R}^d)} \lesssim \|f\|_{L^p(\mathbf{R}^d)}$.

To prove the result, we rely on Littlewood-Paley theory and the Fefferman-Stein sharp maximal function. Without loss of generality we may assume that $2 < p < \infty$. We will actually show that if for all i and $\omega \geqslant 0$,

$$\int_{|x| \geqslant \omega} |K_i(x)| \, dx \leqslant B(1 + 2^i \omega)^{-\varepsilon},$$

consistent with the fact that, if m_i was smooth, the uncertainty principle would say that K_i would live on a ball of radius $1/2^i$. We will then prove that $\|m(D)f\|_{L^p(\mathbf{R}^d)} \leqslant A\widetilde{\log}(B/A)^{|1/2 - 1/p|}$. Our goal is to show that if

$$S^\# f(x) = \sup_{x \in Q} \fint_Q \left(\sum_{i=0}^\infty \left| m_i(D) f(y) - \fint_Q m_i(D) f(z) \, dz \right|^2 \right)^{1/2} dy,$$

then $\|S^\# f\|_{L^p(\mathbf{R}^d)} \lesssim A\widetilde{\log}(B/A)^{1/2-1/p}\|f\|_{L^p(\mathbf{R}^d)}$. It then follows by Littlewood-Paley theory implies

$$\begin{aligned}
\|m(D)f\|_{L^p(\mathbf{R}^d)} &\lesssim_p \left\| \left(\sum_{k=0}^\infty |m_i(D)f|^2 \right)^{1/2} \right\|_{L^p(\mathbf{R}^d)} \\
&\leqslant \left\| M \left[\left(\sum_{k=0}^\infty |m_i(D)f|^2 \right)^{1/2} \right] \right\|_{L^p(\mathbf{R}^d)} \\
&\lesssim \left\| S^\# \left(\sum_{k=0}^\infty |m_i(D)f|^2 \right)^{1/2} \right\|_{L^p(\mathbf{R}^d)} \\
&\lesssim A\widetilde{\log}(B/A)^{1/2-1/p}.
\end{aligned}$$

To bound $S^\#$, we linearize using duality, picking Q_x for each x, and a family of functions $\chi_i(x,y)$ such that $\left(\sum |\chi_i(x,y)|^2\right)^{1/2} \leqslant 1$, such that

$$S^\# f(x) \approx ⨍_{Q_x} \sum_{i=0}^\infty \left(m_i(D)f(y) - ⨍_{Q_x} m_i(D)f(z)\,dz \right) \chi_i(x,y)\,dy.$$

Thus $S^\# f = S_1 f + S_2 f$, where if Q_x has sidelength $2^{l(x)}$,

$$S_1 f(x) = ⨍_{Q_x} \sum_{|i+l(x)| \leqslant \tilde{\log}(B/A)} \left(m_i(D)f(y) - ⨍_{Q_x} m_i(D)f(z)\,dz \right) \chi_i(x,y)\,dy$$

and

$$S_2 f(x) = ⨍_{Q_x} \sum_{|i+l(x)| \geqslant \tilde{\log}(B/A)} \left(m_i(D)f(y) - ⨍_{Q_x} m_i(D)f(z)\,dz \right) \chi_i(x,y)\,dy.$$

If $|i + l(x)| \lesssim 1$, then the uncertainty principle tells us that $m_i(D)f$ is roughly constant on squares on radius Q_x, up to some small error, so that we should expect

$$\left| m_i(D)f(y) - ⨍_{Q_x} m_i(D)f(z)\,dz \right| \lesssim \left| ⨍_{Q_x} m_i(D)f(z)\,dz \right|.$$

Thus it is natural to use the bound, $|S_1 f(x)| \lesssim M(\sum_{i=0}^\infty |m_i(D)f|^2)^{1/2}$, which implies

$$\begin{aligned}
\|S_1 f\|_{L^2(\mathbf{R}^d)} &\lesssim \|M(\sum_{i=0}^\infty |m_i(D)f|^2)^{1/2}\|_{L^2(\mathbf{R}^d)} \\
&\lesssim \left\| (\sum_{i=0}^\infty |m_i(D)f|^2)^{1/2} \right\|_{L^2(\mathbf{R}^d)} \\
&= \left(\sum_{i=0}^\infty \|m_i(D)f\|_{L^2(\mathbf{R}^d)}^2 \right)^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
\|S_1 f\|_{L^\infty(\mathbf{R}^d)} &\leqslant \|M(\sum_{|i+l(x)|\leqslant \tilde{\log}(B/A)}^\infty |m_i(D)f|^2)^{1/2}\|_{L^\infty(\mathbf{R}^d)} \\
&\leqslant \left\| \left(\sum_{|i+l(x)|\leqslant \tilde{\log}(B/A)} |m_i(D)f|^2 \right)^{1/2} \right\|_{L^\infty(\mathbf{R}^d)} \\
&\lesssim \tilde{\log}(B/A)^{1/2} \sup_i \|m_i(D)f\|_{L^\infty(\mathbf{R}^d)}
\end{aligned}$$

Interpolation gives $\|S_1 f\|_{L^p(\mathbf{R}^d)} \lesssim \tilde{\log}(B/A)^{1/2-1/p}\|m_i(D)f\|_{L^p_x(l^p_i)}$. But now Littlewood-Paley theory shows that

$$\|m_i(D)f\|_{L^p_x(l^p_i)} \leqslant A\left(\sum_{i=0}^\infty \|P_i f\|_{L^p(\mathbf{R}^d)} \right)^{1/p} \leqslant A\left(\sum_{i=0}^\infty \|P_i f\|_{L^p(\mathbf{R}^d)}^2 \right)^{1/2} \lesssim A\|f\|_{L^p}.$$

Thus $\|S_1 f\|_{L^p(\mathbf{R}^d)} \lesssim A\tilde{\log}(B/A)^{1/2-1/p}\|f\|_{L^p(\mathbf{R}^d)}$.

On the other hand, if i is much smaller than $l(x)$, we should expect the error between $m_i(D)f(y)$ and $⨍_{Q_x} m_i(D)f(z)\, dz$ to be even smaller, and if i is much bigger, then $m_i(D)f$ is no longer constant at this scale, and so the averages should be small, so $m_i(D)f(x)$ should dominate $⨍_{Q_x} m_i(D)f(z)$. Now since our assumption implues that $\|m(D)f\|_{L^2(\mathbf{R}^d)} \lesssim \|f\|_{L^2(\mathbf{R}^d)}$, it is not so difficult to prove that

$$\|S_2 f\|_{L^2(\mathbf{R}^d)} \lesssim A\|f\|_{L^2(\mathbf{R}^d)} \sim A\left\| \left(\sum |P_i f|^2 \right)^{1/2} \right\|_{L^2(\mathbf{R}^d)}.$$

The difficulty is proving $\|S_2 f\|_{L^\infty(\mathbf{R}^d)} \lesssim A\| \left(\sum |P_i f|^{1/2}\right) \|_{L^\infty(\mathbf{R}^d)}$, which we can interpolate into an inequality like above where we can apply Littlewood-Paley theory. To do this we perform another decomposition, writing

$$S_2 f = If + IIf$$

where

$$If(x) = \fint_{Q_x} \sum_{|i+l(x)| \geqslant \widetilde{\log}(B/A)} \left(m_i(D)(\mathbf{I}_{2Q_x} f)(y) - \fint_{Q_x} m_i(D)(\mathbf{I}_{2Q_x} f)(z)\, dz \right) \chi_i(x,y)\, dy.$$

and

$$If(x) = \fint_{Q_x} \sum_{|i+l(x)| \geqslant \widetilde{\log}(B/A)} \left(m_i(D)(\mathbf{I}_{(2Q_x)^c} f)(y) - \fint_{Q_x} m_i(D)(\mathbf{I}_{(2Q_x)^c} f)(z)\, dz \right) \chi_i(x,y)\, dy.$$

Now

$$\|If\|_{L^\infty} \leqslant \sup_x \fint_{Q_x} \left(\sum |m_i(D)(\mathbf{I}_{2Q_x} f)|^2 \right)^{1/2} dy \leqslant \sup_x |Q_x|^{-1/2} \left(\sum \|m_i(D)(\mathbf{I}_{2Q_x} f)\|_{L^2(\mathbf{R}^d)}^2 \right)^{1/2} \lesssim A|Q_x$$

Part III

Stuff to Read in More Detail

- Sogge, L^p Estimates For the Wave Equation and Applications (1993).

  A survey of results on regularity results for the wave equation. In particular, reviews (without proof) the ideas of Mockenhaipt, Seeger, and Sogge which give local smoothing for Fourier integral operators satisfying the cone condition, as well as mixed norm estimates for non-homogeneous results on wave equations.

- In Sogge's Book, he mentions the main developments in harmonic / microlocal analysis he couldn't discuss in the book were the following:

  - Bennett, Carbery, Tao, On the Multilinear Restriction and Kakeya Conjecture (2006).

    Introduction to multilinear methods in harmonic analysis.

  - Bourgain, Guth, Bounds on Oscillatory Integral Operators Based on Multilinear Estimates (2010).

    Application of multilinear methods to bounding oscillatory integrals.

  - Bourgain, Demeter, The Proof of the l2 Decoupling Conjecture (2014).

    Introduction to Decoupling.

  - Peetre, New Thoughts on Besov-Spaces.

    Characterizes boundedness of Fourier multipliers on homogeneous Besov spaces.

  - Johnson, Maximal Subspaces of Besov-Spaces Invariant Under Multiplication By Characters.

    Shows a Fourier multiplier operator is bounded in the L^p norm if and only if it's translates are all localizably bounded as in Seeger.

- For more background reading in microlocal analysis:

  - Hörmander, The Analysis of Linear Partial Differential Operators, Volumes I-IV.
  - Treves, Introduction to Pseudodifferential and Fourier Integral Operators, Volumes I-II.

– Taylor.

- Hormander, The Spectral Function of an Elliptic Operator - Avakumovic, Uber die Eigenfunktionen auf Geschlossenen Riemannschen Mannigfaltigkeiten - Levitan, On the Asymptotic Behaviour of the Spectral Function of a Self-Adjoint Differential Equation of Second Order.

Bibliography

[1] Jean Bourgain and Larry Guth. Bounds on oscillatory integral operators based on multilinear estimates. *Geom. Funct. Anal.*, 21(6):1239–1295, 2011.

[2] Laura Cladek. Radial Fourier multipliers in $\mathbb{R}^3$ and $\mathbb{R}^4$. *Anal. PDE*, 11(2):467–498, 2018.

[3] Gustavo Garrigós and Andreas Seeger. Characterizations of Hankel multipliers. *Math. Ann.*, 342(1):31–68, 2008.

[4] Shaoming Guo, Changkeun Oh, Wu Shukun Wang, Hong, and Ruixiang Zhang. The Bochner-Riesz problem: An old approach revisited. 2021.

[5] Larry Guth, Hong Wang, and Ruixiang Zhang. A sharp square function estimate for the cone in $\mathbb{R}^3$. *Ann. of Math.*, 192(2):551–581, 2020.

[6] Yaryong Heo, Fdor Nazarov, and Andreas Seeger. Radial Fourier multipliers in high dimensions. *Acta Math.*, 206(1):55–92, 2011.

[7] Nets Hawk Katz and Keith M. Rogers. On the polynomial Wolff axioms. *Geom. Funct. Anal.*, 28(6):1706–1716, 2018.

[8] William P. Minicozzi and Christopher D. Sogge. Negative results for Nikodym maximal functions and related oscillatory integrals in curved spaces. *Math. Res. Lett.*, 4(2):221–237, 2000.

[9] Boris S. Mitjagin. Divergenz von spektralentwicklungen in L_p-räumen. *Internat. Ser. Numer. Math.*, 25:521–530, 1974.

[10] Andreas Seeger Yaryong Heo, Fëdor Nazrov. Radial Fourier multipliers in high dimensions. 2011.

[11] Joshua Zahl. On the Wolff circular maximal function. *Illinois J. Math.*, 56(4):1281–1295, 2012.

[12] Joshua Zahl. Sphere tangencies, line incidences, and Lie's line-sphere correspondence. *Math. Proc. Cambridge Philos. Soc.*, pages 1–21, 2021.