Analysis SEP Problems & Solutions

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Contents

1 Basic Analysis Notes

Let's begin by reviewing some fundamental techniques in analysis. These include techniques from inequalities, calculus, sequences and series, and integration techniques. In this section we give a brief summary of the techniques that we feel can often be applied to the basic analysis questions that occur on qualifying exams.

- The use of asymptotic notation can often simplify calculations: in these notes we use both Vinogradov and Bachmann-Landau notation. For some quantities x and y depending on some parameters, we write $x \lesssim y$, or $x = O(y)$, if there exists a constant $c > 0$ so that $x \leqslant cy$ holds. If the constant c depends on some parameter z, we write $x \lesssim_z y$ or $x = O_z(y)$. We also write $x \sim y$ if $x \lesssim y$ and $y \lesssim x$.

- Swapping Limits with it's Limit Inferior and Superior: To show a sequence $\{c_n\}$ converges to c as $n \to \infty$, it suffices to show
$$\limsup_{n\to\infty} c_n \leqslant c \quad \text{and} \quad \liminf_{n\to\infty} c_n \geqslant c.$$
This strategy is closely related to the method of "giving yourself a ε of room". To show an inequality $a \leqslant b$, it suffices to show that $a \leqslant b + \varepsilon$ is true for any $\varepsilon > 0$. And to show an identity $a = b$, it suffices to show $a \leqslant b + \varepsilon$ and $b \leqslant a + \varepsilon$ for any $\varepsilon > 0$.

- A *dyadic decomposition* is often useful to obtain rough bounds for quantities, i.e. breaking up regions of integration and regions of summation which have total width given by a power of two. The exponential increase in the size of these regions means we can often apply rather crude estimates for the behaviour of a sum or integral on these regions in order to obtain a bound on the overall sum.

- The *Stone-Weirstrass theorem* tells us that in $\mathbb{R}^n$, the family of multi-variate polynomials on $\mathbb{R}^n$ form a dense subspace of $C(K)$, for any compact set $K \subset \mathbb{R}^n$, where $C(K)$ is the Banach space given by the L^∞ norm. It follows that this class is also dense in most other function spaces encountered in analysis, e.g. for the spaces $L^p(\mathbb{R}^n)$ for $1 \leqslant p < \infty$.

- Inequalities involving the trigonometric functions often come up on the qualifying exam, and are useful to remember. For instance, the inequality $x/2 \leqslant \sin x \leqslant x$ holds for $x \in [0, \pi/2]$, and the inequality $x - x^2/2 \leqslant \sin x \leqslant x$ holds for all $x \in \mathbb{R}$. Similarily, $1 - x^2/2 \leqslant \cos x \leqslant 1 - x^2/6$ for all $x \in \mathbb{R}$.

- The *mean value theorem* implies that for $a < b$, and for $f \in C^1[a, b]$, there exists $c \in [a, b]$ such that
$$f(c) = f(a) + bf'(c).$$
More generally, in $\mathbb{R}^d$ we have *Taylor's formula*, which allows us to write a function $f \in C^{k+1}(\mathbb{R}^n)$ as
$$f(x) = \sum\nolimits_{|\alpha| \leqslant k} D^\alpha f(x_0)(x - x_0)^\alpha + \sum\nolimits_{|\alpha| = k+1} R_\alpha(x)(x - x_0)^\alpha,$$
where α ranges over multi-indices, and
$$R_\alpha(x) = \frac{|\alpha|}{\alpha!} \int_0^1 (1 - t)^{|\alpha|-1} (D^\alpha f)(x_0 + t(x - x_0)) = o(|x - x_0|^{k+1}).$$

- The Cauchy-Schwarz inequality implies that
$$\left|\sum a_n b_n\right| \leqslant \left(\sum |a_n|^2\right)^{1/2} \left(|b_n|^2\right)^{1/2}$$
and more generally, we have *Hölder's inequality*
$$|\sum a_n b_n| \leqslant \left(\sum |a_n|^p\right)^{1/p} \left(|b_n|^q\right)^{1/q}$$
where $1 \leqslant p, q \leqslant \infty$ and $1/p + 1/q = 1$. The same inequalities also hold when we swap sums with integrals.

- The *Arzela-Ascoli theorem*, which says that a set K of functions in $C[0, 1]$ is compact in the L^∞ norm if K is *uniformly bounded*, i.e. there is $M > 0$ such that $\|f\|_{L^\infty} \leqslant M$ for all $f \in K$, and *uniformly equicontinuous*, i.e. for any $\varepsilon > 0$, there is $\delta > 0$ such that if $|x - y| \leqslant \delta$ then for any $f \in K$, $|f(x) - f(y)| \leqslant \varepsilon$.

Sequences, Series, and Integrals

There are many questions on analysis qualifying exams asking to determine whether a given infinite series

$$\sum_{n=1}^{\infty} a_n$$

converges. Depending on the sequence, one of various techniques may apply:

- *Breaking a sum into layers*: If a_n is a non-negative integer for each n, then
$$\sum\nolimits_n a_n = \sum\nolimits_{k \geqslant 1} \#\{n : a_n = k\} \cdot k.$$
More generally, for any positive sequence $\{a_n\}$ we may apply a dyadic decomposition, so that the convergence of $\sum a_n$ is equivalent to the convergence of
$$\sum\nolimits_{k \in \mathbb{Z}} \#\{n : 2^k \leqslant a_n < 2^{k+1}\} \cdot 2^k.$$
Similarily, for $a : X \to [0, \infty)$ on a measure space X, the integral $\int_X a(x)\, dx$ is finite if and only if
$$\sum\nolimits_{k \in \mathbb{Z}} |\{x \in X : 2^k \leqslant a(x) < 2^{k+1}\}| \cdot 2^k < \infty.$$

- If the sequence $\{a_n\}$ is positive and non-increasing, then one can apply *Cauchy's condensation theorem*, which says that the convergence of the series is equivalent to convergence of the sequence
$$\sum_{k=1}^{\infty} 2^k \cdot a_{2^k}.$$
This is often useful if the sequence $\{a_n\}$ grows somewhat logarithmically, since logarithmic growth in $\{a_n\}$ will be turned into linear growth in the sequence $\{a_{2^k}\}$.

- One can often convert sums into integrals, and vice versa. If $a_n = a(n)$ for some function $a : [1, \infty) \to \mathbb{R}$, then it is often true that the convergence of $\sum a_n$ is equivalent to the convergence of the integral
$$\int_1^{\infty} a(x)\, dx.$$
This is true, for instance, if $\{a_n\}$ and the function a are both positive and non-increasing, or if $\sum_{k=1}^{\infty} |\Delta a(k)| < \infty$, where $\Delta a(k) = \int_0^1 |a(k+x) - a(k)|\, dx$.

- If a function f is smooth (the derivative of f is well behaved), and a function g is oscillating very fast, then one can often control an integral via an integration by parts, e.g. writing
$$\int_a^b f(x) g(x)\, dx = f(b) G(b) - \int_a^b f'(x) G(x)\, dx$$
where $G(x) = \int_a^x g(x)\, dx$ is likely small since g is oscillating fast. A similar method in the theory of series is using *summation by parts*, e.g. writing
$$\sum_{k=1}^{n} a_k b_k = a_k B_n + \sum_{k=1}^{n-1} (a_k - a_{k+1}) B_k,$$
where $B_k = \sum_{i \leqslant k} b_k$, which might help understand the sum if the discrete derivative $a_k - a_{k+1}$ is well behaved, and the sequence $\{b_k\}$ is oscillating fast.

- If a sequence $\{a_n\}$ *converges absolutely*, then any rearrangement of $\{a_n\}$ converges, and converges to the same value, i.e. for any bijection $\pi : \{1, 2, \dots\} \to \{1, 2, \dots\}$, $\sum a_n = \sum a_{\pi(n)}$.

- The *Leibnitz test* says if $\{a_n\}$ is positive, non-increasing, and $\lim_{n \to \infty} a_n = 0$ then $\sum (-1)^n a_n$ converges.

- A *power series* $\sum a_n z^n$ converges for any $z \in \mathbb{C}$ absolutely when $|z| \leqslant \limsup_{n \to \infty} |a_n|^{1/n}$.

2 Day 1: Warm Up Question

1. (Fall 2016) For $n \geqslant 2$ an integer, define

$$F(n) = \max\left\{k \in \mathbb{Z} : 2^k/k \leqslant n\right\}.$$

Does the infinite series

$$\sum_{n=2}^{\infty} 2^{-F(n)}$$

converge or diverge?

3 Day 1: Basic Analysis

2. (Fall 2017) Let $\{a_n\}$ be a sequence of complex numbers and let

$$c_n = n^{-5} \sum_{k=1}^{n} k^4 a_k.$$

   (a) Prove or Disprove: If $\lim_{n\to\infty} a_n = a$ exists, then $\lim_{n\to\infty} c_n = c$ exists.

   (b) Prove or Disprove: If $\lim_{n\to\infty} c_n = c$ exists, then $\lim_{n\to\infty} a_n = a$ exists.

3. (Fall 2018) For $c_k \in \mathbb{R}$, say that Πc_k converges if $\lim_{K\to\infty} \prod_{k=1}^{K} c_k = C$ exists with $C \neq 0, \infty$.

   (a) Prove that if $0 < a_k < 1$ for all k, or if $-1 < a_k < 0$, for all k, then $\prod(1+a_k)$ converges if and only if $\sum_k a_k$ converges.

   (b) However, prove that $\prod_{k>1}\left(1+\frac{(-1)^k}{\sqrt{k}}\right)$ diverges.

4. (Fall 2019) Let f be a continuous function on $\mathbb{R}$ satisfying

$$|f(x)| \leqslant \frac{1}{1+x^2}.$$

   Define a function F on $\mathbb{R}$ by

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n).$$

   (a) Prove that F is continuous and periodic with period 1.

   (b) Prove that if G is continuous and periodic with period one, then

$$\int_0^1 F(x)G(x) = \int_{-\infty}^{\infty} f(x)G(x)\, dx.$$

5. (Fall 2015) Let $a_1, a_2, \ldots$ be a sequence of positive real numbers and assume that

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = 1.$$

   (a) Show that $\lim_{n\to\infty} a_n n^{-1} = 0$.

   (b) If $b_n = \max(a_1, \ldots, a_n)$, show that $\lim_{n\to\infty} b_n n^{-1} = 0$.

   (c) Show that

$$\lim_{n\to\infty} \frac{a_1^\beta + \cdots + a_n^\beta}{n^\beta} = \begin{cases} 0 & : \beta > 1 \\ \infty & : \beta < 1. \end{cases}.$$

6. (Fall 2021) Let $f \in C^1[0,1]$. Show that for every $\varepsilon > 0$, there exists a polynomial p such that

$$\|f-p\|_{L^\infty[0,1]} + \|f'-p'\|_{L^\infty[0,1]} \leqslant \varepsilon.$$

4 Day 2: Warm Up Question

7. (Fall 2023) Let I be a compact interval. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n n^{-1/2} \cos(x/n)$$

converges uniformly on I.

5 Day 2: Basic Analysis

8. (Spring 2017, Spring 2011, and Spring 2007) Show that the sequence of functions

$$S_n(x) = \sum_{k=1}^{n} \frac{\sin(kx)}{k}, \qquad n = 1, 2, 3, \ldots,$$

is uniformly bounded in $\mathbb{R}$.

9. (Spring 2018 and Spring 2021) Determine if

$$\sum_{k=1}^{\infty} \frac{\cos(\sqrt{k})}{k}$$

converges.

10. (Fall 2015) Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin(x/n).$$

   (a) Show that the series converges pointwise to some function f on $\mathbb{R}$.
   (b) Is f continuous on $\mathbb{R}$? Does $f'(x)$ exist for all $x \in \mathbb{R}$?
   (c) Does the series converge uniformly on $\mathbb{R}$?

11. (Fall 2019) Show that if $K \subset \mathbb{R}^n$, and every continuous function on K is bounded, then K is compact.

12. (Spring 2015) Prove that the integral

$$f(a) = \int_0^{\infty} \frac{\sin(x^2 + ax)}{x} \, dx$$

converges for $a \geqslant 0$, and f is continuous on $[0, \infty)$.

13. (Fall 2017) Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \int_0^{n} \frac{\sin(sx)}{\sqrt{s}} \, ds.$$

   (a) Show that $\{f_n\}$ converges locally uniformly on $(0, \infty)$.
   (b) Show that $\{f_n\}$ does *not* converge uniformly on $(0, 1]$.
   (c) Does the sequence $\{f_n\}$ converge uniformly on $[1, \infty)$ as $n \to \infty$?

14. (Fall 2021) Does the improper integral

$$\int_2^{\infty} \frac{x \sin(e^x)}{x + \sin(e^x)} \, dx$$

converge?

6 Day 3: Warm Up Question

15. (Spring 2018 and Spring 2021) Let $u : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $|u'(x)| \leqslant a < 1/2$ everywhere. Show that $g(x, y) = (x + u(2y), y + u(x))$ is surjective.

7 Day 3: Basic Analysis

16. (Fall 2018) Prove that for $1 \leqslant p \leqslant 2$ and $0 < b < a$,

$$(a+b)^p + (a-b)^p \geqslant 2a^p + p(p-1)a^{p-2}b^2.$$

17. (Spring 2015) Let g be a non-constant differentiable real function on a finite interval $[a, b]$, with $g(a) = g(b) = 0$. Show that there exists $c \in (a, b)$ such that

$$|g'(c)| > \frac{4}{(b-a)^2} \int_a^b |g(t)|\, dt.$$

18. (Fall 2019) If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable on $\mathbb{R}^n - \{0\}$, continuous at 0, and

$$\lim_{x \to 0} \frac{\partial f}{\partial x^i}(x) = 0,$$

for $1 \leqslant i \leqslant n$, then f is differentiable at 0.

19. (Spring 2017) Show there exists a constant $C > 0$ such that for any pair of sequences $\{a_k\}$ and $\{b_n\}$,

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_m}{n+m} \lesssim \left(\sum_{n=1}^{\infty} a_n^2 \right)^{1/2} \left(\sum_{m=1}^{\infty} b_m^2 \right)^{1/2}.$$

8 Measure Theory Notes

It is useful to keep in mind *Littlewood's Three Principles*:

- Every finite measure set is *nearly* the union of a finite collection of disjoint sets. One instance of this principle is that if $E \subset \mathbb{R}^d$ is measurable, then for any $\varepsilon > 0$, there is a disjoint family of cubes $\{Q_i\}$ such that $|E\Delta Q_i| \leqslant \varepsilon$.
- Every measurable function is *nearly* continuous. One instance of this principle is that $C_c(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d)$. A more technical instance is *Lusin's theorem*, that for every measurable function $f : E_0 \to \mathbb{C}$, where $|E| < \infty$, and for any $\varepsilon > 0$, there is $E \subset E_0$ with $|E\Delta E_0| \leqslant \varepsilon$, such that $f|_E$ is a continuous function.
- Every almost everywhere convergent sequence of measurable functions is *nearly* uniformly convergent. One instance of this principle is *Egorov's theorem*, that if $\{f_n\}$ is a sequence of measurable functions on a finite measure set $E_0 \subset \mathbb{R}^d$ converging pointwise to some function $f : E_0 \to \mathbb{C}$, then for any $\varepsilon > 0$, we can find $E \subset E_0$ with $|E\Delta E_0| < \varepsilon$ such that $\{f_n\}$ converges uniformly to f on E.

Here are some other results which are often useful:

- The *monotone convergence theorem*: If $\{f_n\}$ is a monotone sequence of non-negative measurable functions converging to some function f pointwise almost everywhere as $n \to \infty$, then
$$\lim_n \int f_n(x)\, dx = \int f(x)\, dx.$$
- The *dominated convergence theorem*: If $\{f_n\}$ is a sequence of measurable functions converging pointwise almost everywhere to some function f, and $|f_n| \leqslant g$ for some integrable function g, then
$$\lim_n \int f_n(x)\, dx = \int f(x)\, dx.$$
- *Fatou's lemma*, which says that if $\{f_n\}$ are a family of non-negative, measurable functions, then
$$\int \left(\liminf_{n\to\infty} f_n(x)\right)\, dx \leqslant \liminf_{n\to\infty} \left(\int f_n(x)\, dx\right).$$
- The *Borel-Cantelli Lemma*: If $\{E_n\}$ is a sequence of measurable sets such that
$$\sum_{n=1}^{\infty} |E_n| < \infty,$$
then $\limsup_n E_n = \bigcap_{n=1}^{\infty} \bigcup_{m\geqslant n} E_m$ is a set of measure zero.

One should also have an aptitude for manipulating sets to calculate their measure, but it is difficult to summarize these techniques here: they come from practice in various problems.

9 Convergence of Measurable Functions Notes

Let's discuss the notions of convergence one can have for random variables, and for measurable functions. Random variables are just measurable functions on a measure space with total measure one, so for every notion of convergence we obtain for measurable functions, we will obtain an analogous definition for random variables, which is the same definition, but written in a more probabilistic language. Let's begin by listing out the main types of convergence one can have on a measure space Ω, equipped with a measure μ:

- A sequence of measurable functions $\{f_n : \Omega \to \mathbb{R}\}$ *converges in measure* to a function $f : \Omega \to \mathbb{R}$ if, for any $\varepsilon > 0$,
$$\limsup_{n\to\infty} \mu\Big(\{x \in \Omega : |f_n(x) - f(x)| \geqslant \varepsilon\}\Big) = 0.$$
Analogously, for a sequence of random variables $\{X_n : \Omega \to \mathbb{R}\}$, we say these random variables *converge in probability* to a random variable $X : \Omega \to \mathbb{R}$ if, for any $\varepsilon > 0$,
$$\limsup_{n\to\infty} \mathbb{P}\left(|X_n - X| \geqslant \varepsilon\right) = 0.$$

- A sequence of measurable functions $\{f_n : \Omega \to \mathbb{R}\}$ *converges almost everywhere* to a measurable function $f : \Omega \to \mathbb{R}$ if, there exists a measurable set $\Omega_0 \subset \Omega$ with $\mu(\Omega_0^c) = 0$, such that for any $x \in \Omega_0$, $\lim_{n\to\infty} f_n(x) = f(x)$. Equivalently,
$$\mu\Big(\{x \in \Omega : \limsup_{n\to\infty} |f_n(x) - f(x)| \neq 0\}\Big) = 0.$$
Analogously, a sequence of random variables $\{X_n : \Omega \to \mathbb{R}\}$ *converges almost surely* to a random variable $X : \Omega \to \mathbb{R}$ if there exists a measurable set $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for any $x \in \Omega_0$, $\lim_{n\to\infty} f_n(x) = f(x)$. Equivalently,
$$\mathbb{P}\left(\limsup_{n\to\infty} |X_n - X| \neq 0\right) = 0.$$

- A sequence of measurable functions $\{f_n : \Omega \to \mathbb{R}\}$ *converges in L^p* to a function $f : \Omega \to \mathbb{R}$ if
$$\limsup_{n\to\infty} \int |f_n(x) - f(x)|^p \, dx = 0.$$
In order for this definition to makes sense, one normally assumes that each function in the family $\{f_n\}$, and the limiting function f, lies in $L^p(\Omega)$, the space of all functions $g : \Omega \to \mathbb{R}$ such that
$$\int |g(x)|^p \, dx < \infty.$$
A sequence of random variables $\{X_n : \Omega \to \mathbb{R}\}$ *converges in L^p* to a random variable $X : \Omega \to \mathbb{R}$ if
$$\limsup_{n\to\infty} \mathbb{E}\left[|\right] X_n - X|^p = 0.$$
In order for this definition to make sense, one normally assumes that each random variable in the family $\{X_n\}$, and the limiting variable X, lies in $L^p(\Omega)$, the space of all random variables Y such that $\mathbb{E}\left[|\right] Y|^p < \infty$.

The notions of convergence in measure and convergence almost everywhere are very closely related, by virtue of the fact that they look at quantities associated with how fast a function is converging pointwise. Convergence in measure tells us that for any $\varepsilon > 0$ and $\delta > 0$, if n is suitably large, then all points $x \in \Omega$ outside a set of measure δ will satisfy $|f_n(x) - f(x)| \leqslant \varepsilon$. Convergence almost everywhere tells us that for any point x outside a set of measure zero, for any $\varepsilon > 0$, if n is taken suitably large, then $|f_n(x) - f(x)| \leqslant \varepsilon$.

These definitions are *almost* exactly the same, but the problem is in the order of the quantifiers, which means that, in general, neither condition implies the other condition. In other words, there exists sequences of functions converging in measure, but not converging almost surely, and there also exists sequences of functions converging almost surely, but not converging in measure. For convergence in measure, the value n selected is allowed to depend on ε and δ, but not the point x (except that we may throw away a set of 'bad points' that has measure at most δ). For convergence in measure, the value n is allowed to depend on ε and x (but we cannot throw away a set of 'bad points' with measure δ as in convergence in measure).

Convergence in Measure Implies Convergence Almost Everywhere

One way convergence in measure can imply convergence almost everywhere is if one has a more quantified estimate on the rate of convergence result and applying the *Borel-Cantelli Lemma*. For instance, suppose we can justify that, for some sequence $\{f_n\}$,

$$\mu\Big(\{x \in \Omega : |f_n(x) - f(x)| \geqslant 1/n\}\Big) \leqslant 1/2^n.$$

If we define $E_n = \{x \in \Omega : |f_n(x) - f(x)| \geqslant 1/n\}$, then we find that

$$\sum_{n=1}^{\infty} \mu(E_n) \leqslant \sum_{n=1}^{\infty} 1/2^n < \infty.$$

Thus the Borel Cantelli Lemma implies that the complement of the set

$$E^* = \bigcup_{n_0=1}^{\infty} \bigcap_{n=n_0}^{\infty} E_n = \limsup_{n \to \infty} E_n$$

is a set of measure zero. But

$$E^* = \{x \in \Omega : \limsup_{n \to \infty} n \cdot |f_n(x) - f(x)| \leqslant 1\}$$

and so if $x \in E^*$, then $f_n(x)$ converges to $f(x)$. More explicitly, if $x \in E^*$, then there exists n_0 such that $x \in E_n$ for all $n \geqslant n_0$, which means that

$$|f_n(x) - f(x)| \leqslant 1/n.$$

for all such n. This implies that $f_n(x) \to f(x)$ for all $x \in E^*$, and thus the sequence $\{f_n\}$ converges to f almost everywhere.

We now use another secret trick: *For any sequence converging qualitatively, by taking a clever subsequence we can often introduce a more quantitative convergence.* Thus if $\{f_n\}$ is a sequence converging in measure to a function f, then for any n, we take $\varepsilon = 1/n$. We can then choose an integer k_n such that

$$\mu\Big(\{x \in \Omega : |f_{k_n}(x) - f(x)| \geqslant 1/n\}\Big) \leqslant 1/2^n.$$

The Borel-Cantelli method in the last paragraph can then by applied to the subsequence $\{f_{k_n}\}$. Thus we conclude every sequence converging in measure has a *subsequence* converging almost everywhere.

On the other hand, there are sequences converging in probability but not almost everywhere. The standard example is the *typewriter sequence*, given for $f_n : [0,1] \to \{0,1\}$ defined to be the indicator function of the set

$$E_n = \{[x] : \log n \leqslant x \leqslant \log(n+1)\},$$

where $[x]$ denotes the decimal part of x. Then E_n has Lebesgue measure at most $\log(n+1) - \log(n) \lesssim 1/n$. This implies $\{f_n\}$ converges to zero in measure since for $\varepsilon < 1$,

$$\mu\Big(\{x \in [0,1] : |f_n(x)| \geqslant \varepsilon\}\Big) = \mu(E_n) \leqslant 1/n,$$

and so

$$\lim_{n\to\infty}\mu\Big(\{x\in[0,1]:|f_n(x)|\geqslant\varepsilon\}\Big)=\lim_{n\to\infty}\mu(E_n)\lesssim\lim_{n\to\infty}1/n=0.$$

On the other hand, for any $x\in[0,1]$, x is contained in infinitely many of the sets E_n and so

$$\limsup|f_n(x)|=1.$$

Thus f_n does not converge pointwise to zero for any particular value.

Convergence Almost Everywhere Implies Convergence in Measure

What does convergence almost everywhere imply about convergence in L^p? Let $\{f_n\}$ be a sequence of measurable functions converging almost surely to a function f, i.e. that if

$$E^*=\{x\in\Omega:\limsup_{n\to\infty}|f_n(x)-f(x)|=0\},$$

then $\mu((E^*)^c)=0$. We claim that *in a finite measure space*, convergence almost everywhere implies convergence in measure. We must show that for any $m>0$,

$$\lim_{n\to\infty}\mu\Big(\{x\in\Omega:|f_n(x)-f(x)|\geqslant 1/m\}\Big)=0.$$

The key trick here is to introduce *monotonicity* into the problem. Let

$$E_{n,m}=\{x\in\Omega:|f_n(x)-f(x)|\geqslant 1/m\}.$$

If it was true that $E_{1,m}\supset E_{2,m}\supset\dots$, then the monotone convergence theorem would imply that (provided that we are in a *finite* measure space) that

$$\limsup_{n\to\infty}\mu(E_{n,m})=\mu\left(\bigcap_{n=1}^\infty E_{n,m}\right).$$

We have

$$\bigcap_{n=1}^\infty E_{n,m}=\left\{x\in\Omega:\sup_{n\to\infty}|f_n(x)-f(x)|\geqslant 1/m\right\}\subset(E^*)^c,$$

and so we would conclude

$$\limsup_{n\to\infty}\mu(E_{n,m})=\mu\left(\bigcap_{n=1}^\infty E_{n,m}\right)=0,$$

which would complete the argument. Unfortunately, $\{E_{n,m}\}$ are not monotone, but we can *replace* them with bigger sets that *are* monotone. Define

$$F_{n,m}=\left\{x\in\Omega:\sup_{n'\geqslant n}|f_{n'}(x)-f(x)|\geqslant 1/m\right\},$$

then $E_{n,m}\subset F_{n,m}$ for all n and m, and so

$$\limsup_{n\to\infty}\mu(E_{n,m})\leqslant\limsup_{n\to\infty}\mu(F_{n,m}).$$

It thus suffices to show the right hand side is zero for all m. But the sets $\{F_{n,m}\}$ *are* monotone decreasing, and so

$$\limsup_{n\to\infty}\mu(F_{n,m})=\mu\left(\bigcap_{n=1}^\infty F_{n,m}\right).$$

We still have

$$\bigcap_{n=1}^\infty F_{n,m}\subset(E^*)^c,$$

and so the rest of the proof from before still gives us that

$$\limsup_{n \to \infty} \mu(F_{n,m}) = \mu\left(\bigcap_{n=1}^{\infty} F_{n,m}\right) = 0.$$

Thus we have shown that in a finite measure space, convergence almost everywhere implies convergence in measure. In particular, convergence almost surely implies convergence in probability. One way to remember this is through the *weak* and *strong* law of large numbers, since the *strong* law implies the *weak* law, the weak law is about convergence in probability, and the strong law is about almost everywhere convergence (so convergence almost everywhere implies convergence in probability).

On an infinite measure space, however, convergence almost surely does not imply convergence almost everywhere. For instance, on the measure space $\Omega = \{0, 1, \dots\}$ equipped with the counting measure, we can define

$$f_n(x) = \mathbf{I}(x = n).$$

Then $f_n(x) \to 0$ for any $x \in \Omega$, but for $\varepsilon < 1$,

$$\mu\Big(\{x \in \Omega : |f_n(x)| \geqslant \varepsilon\}\Big) = \mu\Big(\{n\}\Big) = 1.$$

Thus $\{f_n\}$ does not converge to zero in measure. Similar examples exist for any infinite measure space, by taking a sequence of functions which are the indicator functions of disjoint sets of measure one.

Convergence in L^p

Note that convergence almost everywhere and convergence in measure tell us most points are close to converging when n is large, *but they tell us nothing about the points that are not close to converging, except that this set is small.* We can get more control over these non-converging points by introducing the L^p norms, which measure convergence 'on average'. For $1 \leqslant p < \infty$, convergence in L^p is equivalent to

$$\limsup_{n \to \infty} \int |f_n(x) - f(x)|^p = 0.$$

This implies *most points* x have $|f_n(x) - f(x)|$ small, i.e. Chebyshev's inequality implies that for any $\varepsilon > 0$, if n is suitably large, for all $t \geqslant 0$,

$$\mu\Big(\{x \in \Omega : |f_n(x) - f(x)| \geqslant t\}\Big) \leqslant \varepsilon/t^p.$$

Again the order of quantifiers has changed. But in this case the order of quantifiers here is stronger than convergence in measure, so that L^p convergence implies convergence in measure. But it does not imply convergence almost surely for $p < \infty$ (the typewriter sequence converges in L^p to zero for all $1 \leqslant p < \infty$). Convergence in L^∞, on the other hand, *does* imply convergence almost surely, again by looking at quantifiers.

Finally, we discuss the relation between convergence in various different L^p spaces. On a finite measure space, L^p convergence for larger p implies L^p convergence for lower L^p (by *Hölder's inequality*). The situation is reversed for 'discrete' measure spaces like the integers, convergence in L^p here for lower p implies convergence in L^p for higher L^p. For general measure spaces, L^p convergences are disjoint from one another. Examples to show this follow by taking

$$f_n(x) = H_n \mathbf{I}(x \in E_n)$$

for some measurable set E_n with $|E_n| = W_n$ for some number W_n. One can check that

$$\int |f_n(x)|^p = H_n^p W_n.$$

To show L^{p_1} convergence does not imply L^{p_2} convergence, it suffices to choose sequences $\{H_n\}$ and $\{W_n\}$ such that $H_n^{p_1} W_n \to 0$, but $H_n^{p_2} W_n$ does not converge to zero. If $p_1 < p_2$, then one will have to choose $W_n \to 0$ (which is why this argument doesn't work for 'discrete spaces' like the integers), and if $p_1 > p_2$, then one will have to choose $W_n \to \infty$ (which is why this argument doesn't work for finite measure spaces).

10 Day 4: Warm Up Problems

1. (Spring 2021 and Spring 2016) Let $E \subset \mathbb{R}$ be a Lebesgue measurable set with $|E| < \infty$. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(r) = |E \cap (E + r)|$ is continuous.

11 Day 4: Measure Theory

2. (Spring 2015) Does there exists a Borel measurable function $f : \mathbb{R} \to [0, \infty)$ such that

$$\int_a^b f(x)\, dx = \infty$$

for all real numbers $a < b$. Find an example or show that no such function exists.

3. (Fall 2018) Two parts:

   (a) Give an example, with explanation, of each of the following:

   - A sequence of functions on $\mathbb{R}$ that converges to zero in $L^1(\mathbb{R})$, but it does not converge almost anywhere on $\mathbb{R}$ to any function.
   - A sequence of functions in $L^1(\mathbb{R})$ that converges almost everywhere to zero, but it does not converge in measure to any function.

   (b) Prove that a sequence of functions on $\mathbb{R}$ that converges to zero in measure must have a subsequence that converges to zero almost everywhere. Do not quote any theorems that trivialize the problem.

4. (Spring 2017) Let $f : [0, \infty) \to \mathbb{R}$ be a continuously differentiable function for which $\|f'\|_\infty < \infty$. Define, for $x > 0$,

$$F(x) = \int_0^\infty f(x + yx)\psi(y)\, dy,$$

where ψ satisfies

$$\int_0^\infty |\psi(y)|\, dy \quad \text{and} \quad \int_0^\infty y \cdot |\psi(y)|\, dy < \infty.$$

Show that $F(x)$ is well defined for all $x \geqslant 0$, and that F is continuously differentiable.

5. (Fall 2016) Let $f : [0, 1] \to \mathbb{R}$ be continuous with $\min_{0 \leqslant x \leqslant 1} f(x) = 0$. Assume that for any $0 \leqslant a \leqslant b \leqslant 1$ we have

$$\int_a^b [f(x) - \min_{a \leqslant y \leqslant b} f(y)] dx \leqslant \frac{|b - a|}{2}.$$

Prove that for any $\lambda \geqslant 0$, we have

$$|\{x : f(x) > \lambda + 1\}| \leqslant \frac{1}{2} |\{f(x) > \lambda)\}| .$$

12 Functional Analysis Notes

Regarding the functional analysis problems, the best advice I can give is to be very familiar with applying the major results and theorems listed on the syllabus, especially in their simpler cases. Rarely do questions require knowledge of the most general statements of these theorems. Here are theorems stated in the forms most likely to be useful (based on my judgement of past qualifying problems). You may find it useful to compile a similar list for other major functional analysis results. Here we state the four main theorems of basic functional analysis:

- The *Closed Graph Theorem* says that if $T : X \to Y$ is a linear map between Banach spaces, then T is bounded / continuous if and only if the *graph*

$$G(T) = \{(x, y) \in X \times Y : Tx = y\}$$

  is closed. In practice, the closed graph theorem is most useful when attempting to show that a linear operator $T : X \to Y$ is bounded. Working directly from the definitions, to show that a linear operator T is bounded, it suffices to show that for any sequence $\{x_n\}$ in X which converges to zero, the seqeuence $\{Tx_n\}$ converges to zero. The closed graph theorem is useful because it allows us to assume the *additional constraint* on our sequence $\{x_n\}$, i.e. that $\{Tx_n\}$ converges to some $y \in Y$, and the problem is then converted into showing that $y = 0$.

- The *Open Mapping Theorem* is mostly used on the exam in the following form: If $T : X \to Y$ is a bounded linear bijection between Banach spaces, then $T^{-1} : Y \to X$ is a bounded linear operator. Quantitatively, the open mapping theorem says that if T is a bounded, linear bijection, then there exists $C > 0$ such that for any $x \in X$,

$$C^{-1}\|x\|_X \leqslant \|Tx\|_Y \leqslant C\|x\|_X,$$

  i.e. T roughly preserves the magnitude of vectors.

- The *Hahn-Banach Theorem* says that if Y is a subspace of a norm space X, and if $\phi : Y \to \mathbb{R}$ is a bounded linear functional, then there is an extension of ϕ to a map $\tilde{\phi} : X \to \mathbb{R}$ with $\|\phi\| = \|\tilde{\phi}\|$. The most common use of the Hahn-Banach theorem is that to show two vectors x_1, x_2 in a Banach space X are equal to one another, then it suffices to show that $\phi(x_1) = \phi(x_2)$ for all $\phi \in X^*$.

- The most commonly used of the four main theorems on qual problems is the *uniform boundedness principle*, which states that for a collection of linear operators $\{T_\alpha : X \to Y\}$ on a Banach space X satisfy a *uniform bound*

$$\sup_\alpha \|T_\alpha x\| \lesssim \|x\|$$

  if and only if they satisfy a *pointwise bound*

$$\sup_\alpha \|T_\alpha x\| < \infty$$

  for all $x \in X$.

Here are some useful results about finite dimensional norm spaces that are useful to keep in mind:

- If $\dim(X) < \infty$, then *all norms on X are equivalent.* In other words, for any two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X, there exists $C > 0$ such that

$$(1/C)\|x\|_1 \leqslant \|x\|_2 \leqslant C\|x\|_1.$$

- If X is a finite dimensional subspace of a norm space Y, then X is closed in Y.
- If X is a finite dimensional subspace of a norm space Y, then there exists a closed subspace X' of Y such that $Y = X \oplus X'$.

13 Day 5: Warm Up Problems

6. (Spring 2015) Let $f \in L^2[0,1]$ satisfy $\int_0^1 t^n f(t)\,dt = (n+2)^{-1}$ for $n = 0, 1, \ldots$. Must then $f(t) = t$ for almost every $t \in [0,1]$?

7. (Fall 2021) Let $\{f_n\}$ be a sequence of monotonic functions on $[0,1]$ converging to a function f in measure. Show that f coincides almost everywhere with a monotonic function f_0, and that $f_n(x) \to f_0(x)$ at every point of continuity of f_0.

14 Day 5: Functional Analysis

8. (Fall 2015) Find all $f \in L^2[0,\pi]$ such that

$$\int_0^\pi |f(x) - \sin x|^2 \, dx \leqslant \frac{4\pi}{9}$$

and

$$\int_0^\pi |f(x) - \cos x|^2 \, dx \leqslant \frac{\pi}{9}$$

9. (Spring 2017) Let $l^1(\mathbf{N})$ be the space of summable sequences, i.e.

$$l^1(\mathbf{N}) = \{x : \sum_{n=1}^{\infty} |x_n| < \infty\}.$$

Let $\{a_n\}$ be a sequence with $a_n \geqslant 0$ for all $n \in \mathbf{N}$ and consider the subset $K \subset l^1(\mathbf{N})$ defined by

$$K = \{x \in l^1(\mathbf{N}) : 0 \leqslant x_n \leqslant a_n \text{ for all } n\}.$$

Show that K is compact if and only if the sequence $\{a_n\}$ itself belongs to $l^1(\mathbf{N})$.

10. (Spring 2018, Spring 2021) Let K be a continuous function on $[0,1] \times [0,1]$. Suppose that g is a continuous function on $[0,1]$. Show that there exists a unique continuous function f on $[0,1]$ such that

$$f(x) = g(x) + \int_0^x f(y)K(x,y)dy$$

11. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support.
    (a) Prove that there exists a constant A such that

    $$\|f * \phi\|_q \leqslant A\|f\|_p$$

    for $1 \leqslant p \leqslant q \leqslant \infty$.
    (b) Show by example that such general inequality cannot hold for $p > q$.

12. (Fall 2016) Give an example of a non-empty closed subset of $L^2([0,1])$ that does not contain a vector of smallest norm. Prove your assertion.

15 Day 6: Warm Up Problems

13. Let X be the set of all real-valued polynomials in a single variable. Prove that there does not exist a norm $\|\cdot\|$ on X so that $(X, \|\cdot\|)$ is a Banach space.

16 Day 6: Functional Analysis

14. (Fall 2019) Show that there is no sequence $\{a_n\}$ of positive numbers such that $\sum a_n|c_n| < \infty$ if and only if $\{c_n\}$ is a bounded sequence. Hint: Suppose that there exists a sequence and consider the map $T : l^\infty \to l^1$ given by $Tf(n) = a_n f(n)$. The set of f such that $f(n) = 0$ for all but finitely many n is dense in l^1 but not in l^∞.

15. (Fall 2020)

    Suppose that X, Y and Z are Banach spaces, and $T : X \times Y \to Z$ is a mapping such that:

    (a) For each $x \in X$, the map $y \mapsto T(x, y)$ is a bounded linear map $Y \to Z$.

    (b) For each $y \in Y$, the map $x \mapsto T(x, y)$ is a bounded linear map $X \to Z$.

    Prove there exists a constant C such that

$$\|T(x, y)\|_Z \leqslant C \|x\|_X \|y\|_Y$$

16. (Fall 2015) For $p \in (1, \infty)$, and for $f \in L^p(\mathbb{R})$ define

$$Tf(x) = \int_0^1 f(x + y)\, dy.$$

    (a) Show that $\|Tf\|_p \leqslant \|f\|_p$ and equality holds if and only if $f = 0$ almost everywhere.

    (b) (Fall 2015) Prove that the map $f \mapsto Tf - f$ does not map $L^p(\mathbb{R})$ onto $L^p(\mathbb{R})$.

17. (Fall 2014)

    (a) For any $n \geqslant 1$ an integer, there exists two positive measures μ_1^n, μ_2^n supported on $[0, 1]$ such that for any polynomial $P(x)$ with $\deg P(x) \leqslant n$ it holds:

$$P'(0) = \int_0^1 P(x) d\mu_1^n(x) - \int_0^1 P(x) d\mu_2^n(x).$$

    (b) Does there exist two finite positive measures μ_1, μ_2 supported on $[0, 1]$ such that for any polynomial $P(x)$, it holds

$$P'(0) = \int_0^1 P(x) d\mu_1(x) - \int_0^1 P(x) d\mu_2(x)?$$

17 Baire Category Notes

The *Baire Category Theorem* says that if X is a complete metric space, then for any sequence $\{U_n\}$ of open, dense subsets of X, $\bigcap_n U_n$ is dense in X. To make things less abstract, I like to think through this result in terms of logical properties:

- A logical property P of points in X is *stable* if, whenever $P(x_0)$ is true for some $x_0 \in X$, then $P(x)$ is true for x in a neighborhood of x_0.
- A logical property P of points in X is *unstable* if, whenever $P(x_0)$ is true for some $x_0 \in X$, then for any $\varepsilon > 0$, we can find $x \in X$ with $d(x, x_0) < \varepsilon$, such that $P(x)$ is false. One advantage of being quantitative here is that it suffices to show that the property is unstable for x_0 in a *dense* subspace X_0 of X.

The Baire category theorem then says that, given a countable family of properties $\{P_n\}$, such that P_n is a stable property, and the *negation* $\neg P_n$ of the property P_n is unstable, then the set of points $x \in X$ such that $P_n(x)$ is true for all n is a dense subset of X.

To see an example of this formulation of the theorem, let us go through the classic Baire category proof that there exists a continuous function on $[0, 1]$, which is differentiable nowhere. The first challenge is to find countably many properties $\{P_n\}$ of functions in $C[0, 1]$ such that a function f is differentiable nowhere if and only if $P_n(f)$ is true for all n, though we must also be careful to choose these properties to be stable. This leads to the family of properties $\{P_{N,M}\}$, where N and M are positive integers, and $P_{N,M}(f)$ is true if, for any $x_0 \in [0, 1]$, there exists $x_1 \in [0, 1]$ with $0 < |x_1 - x_0| < 1/M$ and with

$$|f(x_0) - f(x_1)| > N|x_0 - x_1|.$$

Let us now show that the properties $\{P_{N,M}\}$ are both stable, and their complement is unstable:

- Since $[0, 1]$ is compact, and for any fixed f, x_0 and x_1 such that the above properties hold, we can keep x_1 constant as we vary x_0 in a small neighborhood, we can find a finite collection of points $\{x_1(1), \dots, x_1(K)\}$ such that given any $x_0 \in [0, 1]$, there is k such that $0 < |x_0 - x_1(k)| < 1/M$ and $|f(x_0) - f(x_1)| > N|x_0 - x_1|$. Combined with the fact that we use a *strict inequality* above, this can be used to show $P_{N,M}$ is a stable property.
- Conversely, the negation of the property $P_{N,M}$ is unstable. To see this, we note that $C^\infty[0, 1]$ is dense in $C[0, 1]$. Thus to show the property is unstable, it suffices to show it is unstable at any such function $f \in C^\infty[0, 1]$. Fix $T > 0$ such that f is Lipschitz of order T for some large quantity $T \geqslant 1$. For any $R \geqslant 2M$, define a function $g_R \in C[0, 1]$ such that for $1 \leqslant k \leqslant R$, $g(k/R) = (-1)^k(10NT/R)$, and then linearly interpolating between these values. Then $\|g_R\|_{L^\infty} \leqslant 10NT/R$. Now consider the function $f_R = f + g_R$. Now if $1/R \leqslant x \leqslant 2/R$,

$$\begin{aligned} |f_R(k/R + x) - f_R(k/R)| &\geqslant |g_R(k/R + x) - g_R(k/R)| - |f(k/R + x) - f(k/R)| \\ &\geqslant 10NT/R - |f(k/R + x) - f(k/R)| \\ &\geqslant 10TN/R - 2T/R \\ &\geqslant 5TN/R > N|x|. \end{aligned}$$

  Since $R \geqslant 2M$, this justifies that $P_{N,M}(f_R)$ is true. For any $\varepsilon > 0$, if $R \geqslant 10NT/\varepsilon$, then $\|f_R - f\|_{L^\infty} \leqslant \varepsilon$. Thus we see that the property $P_{N,M}$ is unstable.

The Baire category theorem then implies that the set of $f \in C[0, 1]$ such that $P_{N,M}(f)$ is true for all N and M is dense in $C[0, 1]$, which proves the existence of continuous functions differentiable at no point.

18 Day 7: Warm Up Problems

18. (Fall 2017) Let f_n be a sequence of real functions on $\mathbb{R}$ such that each f_n' is continuous on $\mathbb{R}$. Suppose that as $n \to \infty$, f_n converges to a function $f : \mathbb{R} \to \mathbb{R}$ pointwise, and f_n' converges to a function g pointwise.

    Prove that there exists a non-empty interval (a, b) and a constant $L < \infty$ such that

    $$|f(x) - f(y)| \leqslant L|x - y|.$$

    Hint: Consider the sets $K_c = \{x : \sup_n |f_n'(x)| \leqslant c\}$.

19 Day 7: Baire Category

19. (Spring 2020) A **Hamel basis** for a vector space X is a collection $\mathcal{H} \subset X$ of vectors such that $x \in X$ can be written uniquely as a finite linear combination of elements in $\mathcal{H}$. Prove that an infinite dimensional Banach space cannot have a countable Hamel basis. (Hint: otherwise the Banach space would be first category in itself.)

20. (Fall 2016) Show that there is a continuous real valued function on $[0, 1]$ that is not monotone on any open interval $(a, b) \subset [0, 1]$.

21. (Spring 2014) Does there exist a sequence of continuous functions $f_n : [0, 1] \to \mathbb{R}$ such that $f_n \to \chi_{\mathbb{Q}}$ pointwise?

22. (Fall 2014) Let X, Y be Banach spaces and $\{T_{j,k} : j, k \in \mathbb{N}\}$ be a set of bounded linear transformations $X \to Y$. Suppose for each k, there exists $x \in X$ such that $\sup \{\|T_{j,k}x\| : j \in \mathbb{N}\} = \infty$. Then there is an $x \in X$ such that $\sup \{\|T_{j,k}x\| : j \in \mathbb{N}\} = \infty$ for all k.

20 Distribution Theory Notes

Here we detail the very basics of distribution theory. The hope is that provided one knows these basics, then without having to study much distribution theory, one can turn many problems on the exam involving distributions into more basic analysis problems, to which one can apply the tools of basic analysis, measure theory, or functional analysis.

If $U \subset \mathbb{R}^d$ is an open set, a distribution on U is a *continuous linear functional* u on the vector space $\mathcal{D}(U) := C_c^\infty(U)$ of all smooth, compactly supported functions on U. Thus with each $\phi \in C_c^\infty(U)$, the functional u associates a quantity $\langle u, \phi \rangle$, which we might also denote as

$$\int u(x)\phi(x)\ dx.$$

The set of all distributions on U is denoted $\mathcal{D}(U)^*$. There is an abstract theory of topological vector spaces that allows us to define what it means for u to be continuous, but it is not necessary to learn this theory if one remembers a more practical definition. In order for u to be continuous, one needs to show that for any compact set $K \subset U$, there exists an integer $N > 0$, possibly depending on K, such that for all $\phi \in C_c^\infty(U)$ with $\text{supp}(\phi) \subset K$,

$$|\langle u, \phi \rangle| \lesssim_K \sup_{x \in K} \sup_{|\alpha| \leqslant N} |D^\alpha \phi(x)|.$$

To check you understand this definition, prove that the linear functional u defined by setting

$$\langle u, \phi \rangle = \sum_{n=1}^\infty e^{e^n} D^{n!} \phi(n)$$

is a distribution. As other examples, for any locally integrable function $f : U \to \mathbb{C}$, or any locally finite measure μ, one can view f and μ as distributions by defining

$$\langle f, \phi \rangle = \int f(x)\phi(x)\ dx \quad \text{and} \langle \mu, \phi \rangle = \int \phi(x)\ d\mu(x).$$

Thus many mathematical objects in analysis are special cases of distributions. An important example is the Dirac delta distribution δ_x, for any $x \in U$, such that $\langle \delta_x, \phi \rangle = \phi(x)$.

Surprisingly, one can take the formal derivative of *any distribution*, by applying integration by parts. If u is a distribution on U, we define it's partial derivatives $D^i u$, which are also distributions, by the formal definition

$$\langle D^i u, \phi \rangle = -\langle u, D^i \phi \rangle.$$

As an example, we calculate that

$$\langle D^i \delta_x, \phi \rangle = -\langle \delta_x, D^i \phi \rangle = -D^i \phi(x).$$

Thus the derivative of the Dirac delta at a point x is the distribution that measures the derivatives of a given function at the point x. For more well behaved functions, the distributional derivative of a function will be equal to the derivative of the function.

A sequence of distributions $\{u_n\}$ converges *distributionally* to another distribution u if, for any $\phi \in \mathcal{D}(U)$, the quantities $\langle u_n, \phi \rangle$ converge to $\langle u, \phi \rangle$. If this is the case, it also follows that the derivatives $\{D^i u_n\}$ converge distributionally to $D^i u$.

Roughly speaking, if you start to get intuition about how to work with distributions, these facts will allow you to convert most problems about distributions to more basic problems about the convergence of numbers, or integrals, and so on, and so you can focus on trying to apply the more basic analytical tools.

21 Day 8: Intro to Distribution

23. (Fall 2020) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = |x|$. Find $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f$, where the derivative is taken in the sense of distributions.

24. (Spring 2017)
    (a) Let $f \in L^1(\mathbb{R})$ and consider the sequence of distributions $T_n(x) = \sin(nx^2)f(x)$. Show that $\lim_{n\to\infty} T_n = 0$ in the sense of distributions.
    (b) Find a distribution $T \in \mathcal{D}'(\mathbb{R})$ such that $T_n = \sin(nx^2)T$ does not converge to 0 in the sense of distributions as $n \to \infty$.

25. (Spring 2015)
    (a) If $f \in C[0, 1]$, and the distributional derivative f' of f on $(0, 1)$ is in $L^1((0, 1))$, prove that
    $$f(1) - f(0) = \int_0^1 f'(x)\, dx.$$
    (b) Let $p \in [1, \infty)$ and let $F \subset C[0, 1]$ be such that for each $f \in F$ we have $\|f\|_{L^1[0,1]} \leqslant 1$ and $\|f'\|_{L^p[0,1]} \leqslant 1$, where f' is the distributional derivative of f. Prove that F is precompact in $C[0, 1]$, or find a counter-example.

26. (Fall 2021, Spring 2016) Prove or disprove:
    (a) There exists a distribution $u \in \mathcal{D}(\mathbb{R})$ so that the restriction to $(0, \infty)$ is given by
    $$\langle u, f\rangle = \int_0^\infty e^{1/x^2} f(x)dx$$
    for all $f \in C^\infty(\mathbb{R})$ which are compactly supported in $(0, \infty)$.
    (b) There exists a distribution $u \in \mathcal{D}'(\mathbb{R})$ so that its restriction to $(0, \infty)$ is given by
    $$\langle u, f\rangle = \int_0^\infty x^{-2} e^{i/x^2} f(x)dx$$
    for all $f \in C^\infty$ which are compactly supported in $(0, \infty)$.

27. (Fall 2017) A distribution $T \in \mathcal{S}'(\mathbb{R}^n)$ is said to be *nonnegative* if $\langle T, \phi\rangle \geqslant 0$ for every test function $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\phi(x) \geqslant 0$ for all $x \in \mathbb{R}^n$.
    (a) Suppose $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, and let T_f be the distribution defined by f. Show that $T_f \geqslant 0$ if and only if $f \geqslant 0$ for almost all $x \in \mathbb{R}^n$.
    (b) Show that if $T_n \to T$ in the sense of distributions, and if $T_n \geqslant 0$ for all n, then $T \geqslant 0$.
    (c) Suppose $\Phi : \mathbb{R} \to \mathbb{R}$ is a C^2 function with $\Phi'' \geqslant 0$ in $\mathbb{R}$, and let $f \in C^2(\mathbb{R}^n)$ have compact support. Show that $\Delta(\Phi(f(x)) \geqslant \Phi'(f(x))\Delta f(x)$.
    (d) Suppose $f \in C^2(\mathbb{R}^n)$ has compact support. Show that $\Delta|f| \geqslant \operatorname{sign}(f(x))\Delta f(x)$ holds in the sense of distributions. (Hint use (c) with $\Phi(t) = \sqrt{\varepsilon + t^2}$).

28. (Spring 2020)
    (a) Suppose Λ is a distribution on $\mathbb{R}^n$ such that $\operatorname{supp}(\Lambda) = \{0\}$. If $f \in C^\infty(\mathbb{R}^n)$ satisfies $f(0) = 0$, does it follow that $f\Lambda = 0$ as a distribution?
    (b) Suppose Λ is a distribution on $\mathbb{R}^n$ such that $\operatorname{supp}(\Lambda) \subseteq K$, where $K = \{x \in \mathbb{R}^n : |x| \leqslant 1\}$. If $f \in C^\infty(\mathbb{R}^n)$ vanishes on K, does it follow that $f\Lambda = 0$ as a distribution?

22 Fourier Analysis / Tempered Distribution Notes

With distribution theory, there are a couple simple facts that allow one to convert technical sounding problems into more basic problems. Fourier analysis consists of some more deep tools than this, so I'd recommend making a study of these tools separately. But the theory of Fourier analysis applied to tempered distributions is very similar to the theory of distributions, in the sense that if you know Fourier analysis, you can often convert problems involving tempered distributions into more basic problems about Fourier analysis / other analysis techniques.

If $f : \mathbb{R}^d \to \mathbb{C}$ is integrable, we define the Fourier transform $\widehat{f} : \mathbb{R}^d \to \mathbb{C}$ by the integral formula

$$\widehat{f}(\xi) = \int f(x) e^{-2\pi i \xi \cdot x}\, dx.$$

To avoid technical assumptions, we assume f is in the *Schwartz class* $\mathcal{S}(\mathbb{R}^d)$, which consists of smooth functions $f \in C^\infty(\mathbb{R}^d)$ which are rapidly decaying, i.e. such that for any $N > 0$ and α, $|\partial_x^\alpha f(x)| \lesssim \langle x \rangle^{-N}$. The Fourier transform is then a bijection from $\mathcal{S}(\mathbb{R}^d)$ to itself, with inverse given by

$$\check{g}(x) = \int g(\xi) e^{2\pi i \xi \cdot x}\, d\xi.$$

One can then verify that the *multiplication formula*

$$\int \widehat{f}(\xi) g(\xi)\, d\xi = \int f(x) \widehat{g}(x)\, dx$$

holds. This will enable us to use duality to extend the definition of the Fourier transform to distributions. Distributions on $\mathbb{R}^d$ were defined as continuous linear functionals on $\mathcal{D}(\mathbb{R}^d) = C_c^\infty(\mathbb{R}^d)$. Some of these distributions extend to continuous linear functionals on $\mathcal{S}(\mathbb{R}^d)$, which contains $\mathcal{D}(\mathbb{R}^d)$ as a subclass. These distributions are called *tempered*, and the class of all such tempered distributions is denoted $\mathcal{S}(\mathbb{R}^d)^*$. Again, there is an abstract theory which determines when a linear functional on $\mathcal{S}(\mathbb{R}^d)$ is continuous. But in practice, it suffices to verify the following: a tempered distribution u is a functional that associates with each $\phi \in \mathcal{S}(\mathbb{R}^d)$ a quantity $\langle u, \phi \rangle$, such that for some $N, M > 0$,

$$|\langle u, \phi \rangle| \leqslant \sup_{|\alpha| \leqslant N} \sup_{x \in \mathbb{R}^d} \frac{|D^\alpha \phi(x)|}{(1 + |x|)^M}.$$

If u is a tempered distribution, we formally a tempered distribution $\widehat{u}$ by the formula $\langle \widehat{u}, \phi \rangle = \langle u, \widehat{\phi} \rangle$, so that the multiplication formula above holds. The Fourier transform is then a bijection of $\mathcal{S}(\mathbb{R}^d)^*$, where the inverse Fourier transform is given by $\langle \check{u}, \phi \rangle = \langle u, \check{\phi} \rangle$. We now summarize several useful properties of the Fourier transform:

- Because the Fourier transform is a bijection, if $\widehat{u} = 0$, then $u = 0$.
- *The Riemann-Lebesuge Lemma*: If $f \in L^1(\mathbb{R}^d)$, then $\widehat{f} \in C_0(\mathbb{R}^d)$, and $\|\widehat{f}\|_{L^\infty(\mathbb{R}^d)} \leqslant \|f\|_{L^1(\mathbb{R}^d)}$.
- *Parseval's Formula*: The Fourier transform is an isometry of $L^2(\mathbb{R}^d)$. In particular, if u is a tempered distribution, and $\widehat{u} \in L^2(\mathbb{R}^d)$, then $u \in L^2(\mathbb{R}^d)$.
- The Fourier transform of $D^\alpha u$ is equal to $(2\pi i \xi)^\alpha \widehat{u}$.
- The Fourier transform of $(-2\pi i x)^\alpha u$ is equal to $D^\alpha \widehat{u}$.
- If we define the convolution of $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^d)$ by setting

  $$(\phi_1 * \phi_2)(x) = \int \phi_1(y) \phi_2(x - y)\, dy,$$

  then $\phi_1 * \phi_2 \in \mathcal{S}(\mathbb{R}^d)$ is Schwartz, and it's Fourier transform is $\widehat{\phi_1} \cdot \widehat{\phi_2}$. There is a way to define the convolution of a distribution u with a Schwartz function ϕ, denoted $u * \phi$, which will be a tempered distribution with Fourier transform equal to $\widehat{u}\widehat{\phi}$.

23 Day 9: Warm Up Problems

29. (Fall 2019) Let $s \in \mathbb{R}$, and let $H^s(\mathbb{R})$ be the Sobolev space on $\mathbb{R}$ with the norm

$$\|u\|_{(s)} = \left(\int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2}$$

where $\hat{u}$ is the Fourier transform of u. Let $r < s < t$ be real numbers. Prove that for every $\varepsilon > 0$ there is $C > 0$ such that

$$\|u\|_{(s)} \leqslant \varepsilon \|u\|_{(t)} + C \|u\|_{(r)}$$

for every $u \in H^t(\mathbb{R})$.

30. (Fall 2015) Let f be a tempered distribution on $\mathbb{R}$ with Fourier transform

$$\hat{f}(\xi) = 1 + \xi^{12} + \sin \xi + \operatorname{sign}(\xi).$$

Find f and f' (specify the definition of the Fourier transform you are using).

24 Day 9: Fourier Analysis + Distribution Theory

31. (Spring 2017) Let $f \in L^1(\mathbb{R}^n)$ be a function all of whose distributional derivatives $D^\alpha f$ of order $|\alpha| = m$ also belong to $L^1(\mathbb{R}^n)$. Show that if $m > n$, then $f \in C(\mathbb{R}^n)$.

32. (Fall 2019) Let $f \in L^2(\mathbb{R})$. Define

$$g(x) = \int_{-\infty}^{\infty} f(x-y)f(y)\, dy$$

Show that there exists a function $h \in L^1(\mathbb{R})$ such that

$$g(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} h(x)\, dx,$$

i.e. g is a Fourier transform of a function in $L^1(\mathbb{R})$. Hint: The following formal argument may be helpful:

$$\widehat{g}(x) = \widehat{f * f}(x) = \widehat{f}(x)^2,$$

where $*$ denotes convolution, and $\widehat{\cdot}$ denotes the Fourier transform.

33. (Fall 2015) Recall that $H^s(\mathbb{R}^n)$ is the Sobolev space consisting of all tempered distributions g on $\mathbb{R}^n$ for which the Fourier transform $\widehat{g}$ of g is locally integrable and satisfies

$$\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{g}(\xi)|^2\, d\xi < \infty.$$

Let u be a Schwartz function on $\mathbb{R}^n$ and for $a \in \mathbb{C}$, let

$$f_a(x) = |x|^a u(x).$$

Show that if $\mathrm{Re}(a) > -n/2$ and $s \in [0, \mathrm{Re}(a) + n/2)$, then $f_a \in H^s(\mathbb{R}^n)$.

25 Andreas Problems Warm Up Problem

34. (a) Let $\psi \in C_c(\mathbb{R})$. Show that

$$\lim_{N\to\infty} \frac{1}{N} \int_0^\infty \frac{\psi(x/N)}{\sqrt{1+x}} = 0.$$

(b) Let

$$J_N = \int_0^N e^{ix}\sqrt{1+x}\, dx.$$

Does $\lim_{N\to\infty} J_N$ exist?

(c) Suppose $\chi \in C_c^2(\mathbb{R})$. Prove that

$$\lim_{N\to\infty} \int_0^\infty \chi(x/N) e^{ix}\sqrt{1+x}\, dx$$

exists, and calculate the limit.

26 Andreas Problems

35. Let f be a continuous function on $\mathbb{R}$, and define
$$F_n(x) = \int_0^x (x-t)^{n-1} f(t)\, dt.$$
Prove that F_n is n times differentiable, and find a simple formula for it's nth derivative.

36. For $y > 0$, let
$$f(x,y) = \sum_{n=1}^{\infty} \frac{x}{x^2 + yn^2}.$$
   (a) Show that for each $y > 0$, the limit $g(y) = \lim_{x\to\infty} f(x,y)$ exists, and find a formula for the limit.
   (b) Determine if f converges to g uniformly on $(0,\infty)$.

37. (a) Find an explicit $\varepsilon > 0$ so that for every $x \in [0,1]$,
$$|\sqrt{x} - \sqrt{x+\varepsilon}| \leqslant \frac{1}{200}.$$
   (b) Find an explicit integer N such that there exists a polynomial P of degree at most N such that for $x \in [0,1]$,
$$|\sqrt{x} - P(x)| \leqslant 1/100.$$

38. Determine all continuous $f : [0,2] \to \mathbb{C}$ which satisfy $\int_0^2 f(x)x^n\, dx = 0$ for all $n \geqslant 0$.

39. Does the series
$$\sum_{n=1}^{\infty} (-1)^n e^{-x/n} \frac{1}{n}$$
converge uniformly on $[0,\infty)$?

40. Assume $\{a_k\}$ is a positive sequence with $\sum a_k = \infty$. Given any bounded sequence $\{b_k\}$ show that we can find an increasing sequence $\{k_n\}$ so that b_{k_n} converges as $n \to \infty$, and $\sum a_{k_n} = \infty$.

41. Let $\alpha \in \mathbb{R}$ and define $u : (1,\infty) \to \mathbb{R}$ be setting $u(x) = x^\alpha$. For which α do the differences
$$\frac{u(x+h) - u(x)}{h} \to u'(x)$$
converge uniformly on $(1,\infty)$.

42. Let f be real-valued differentiable function defined on the entire real line. Assume that
$$\frac{f(x+h) - f(x)}{h} \to f'(x)$$
uniformly as $h \to 0^+$. Show that f' is uniformly continuous. Must f itself be uniformly continuous?

43. Let $u : \mathbb{R}^3 \to \mathbb{R}$ denote a smooth function and let $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ denote the Laplacian. Suppose $\Delta u = 1$ on $\mathbb{R}^3$ and that $u(x,y,z) = x^3y^3$ on the sphere of radius R centered at the origin. Compute $u(0,0,0)$.

44. Calculate the line integral
$$\int \frac{-y^3 dx + xy^2 dy}{(x^2+y^2)^2}$$
along the plane curve defined by $10x^{12} + 22y^8 = 240$, with the positive orientation.

45. Let $\mathcal{D} \subset \mathbb{R}^d$ be a compact, convex set containing the origin, with smooth boundary. For every $y \in \partial\mathcal{D}$ let $\alpha(x) \in [0, \pi)$ be the angle between the position vector x and the outer normal vector $\mathfrak{n}(x)$. Let ω_d be the surface area of the unit sphere in $\mathbb{R}^d$. Compute

$$\frac{1}{\omega_d} \int_{\partial\mathcal{D}} \frac{\cos(\alpha(x))}{|x|^{d-1}} \, d\sigma(x).$$

46. Fix $f \in C_c^1(\mathbb{R})$, and let $b > 0$. Show

$$A_b(x) = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R} - [-\varepsilon, b\varepsilon]} \frac{f(x-y)}{y} \, dy$$

exists for all $x \in \mathbb{R}$. How do A_b and $A_{b'}$ differ for $b \neq b'$.

47. Prove or disprove that

$$\lim_{\varepsilon \to 0} \int_{x^2+y^2 \geqslant \varepsilon^2} \frac{f(x,y)}{(x+iy)^3} \, dx \, dy$$

exists for every $f \in C^2(\mathbb{R}^2)$ with compact support.

48. Let

$$s_N(x) = \sum_{n=1}^{N} (-1)^n \frac{x^{3n}}{n^{2/3}}.$$

Prove that s_N converges to a limit function on $[0,1]$, and $\|s - s_N\|_{L^\infty[0,1]} \lesssim N^{-2/3}$.

49. (a) What is the volume of the region Ω in $\mathbb{R}^n$ defined by

$$\Omega = \{x \in \mathbb{R}^n : x_1, \ldots, x_n > 0, 0 < x_1 + \cdots + x_n < 1\}$$

(b) What is the area of the parallelogram spanned by the vectors $(1,1,-1,1)$ and $(2,1,2,1)$ in $\mathbb{R}^4$?

(c) What is the volume of the box spanned by the vectors $(1,1,0,0,0)$, $(0,1,1,1,0)$ and $(0,0,1,-1,1)$ in $\mathbb{R}^5$?

50. Suppose f is a positive, decreasing function on $(0, \infty)$. Let $\varepsilon > 0$ be a fixed, positive number

(a) Suppose that for $0 < x < \infty$, $f(2x) \leqslant 2^{-1-\varepsilon} f(x)$. Prove there is a constant C, depending only on ε, so that for all $a > 0$,

$$\int_a^\infty f(x) \, dx \leqslant C a f(a).$$

(b) Suppose that for all $0 < x < \infty$, $f(x) \leqslant 2^{1-\varepsilon} f(2x)$. Prove there is a constant C so that

$$\int_0^a f(x) \, dx \leqslant C a f(a).$$

(c) Suppose that for all $0 < x < \infty$, $f(2x) \geqslant f(x)/2$. Prove that $\int_1^\infty f(x) \, dx$ diverges.

51. For $a, b > 0$, let

$$F(a,b) = \int_{-\infty}^{\infty} \frac{dx}{x^4 + (x-a)^4 + (x-b)^4}.$$

For which $p > 0$ is it true that

$$\int_0^1 \int_0^1 F(a,b)^p \, da \, db < \infty?$$

Hint: First show that $F(a,b) \sim 1/b^3$ if $a \leqslant b$.

52. Let $\{f_n\}$ be a sequence of continuous functions on $[0,1]$ and assume $\sup_n |f_n(x)| < \infty$ for every $x \in [0,1]$. Show there is an interval $(a,b) \subset [0,1]$ and $M > 0$ so that $|f_n(x)| \leqslant M$ for all $x \in (a,b)$ and all n.

53. Consider a function $f : \mathbb{R} \to \mathbb{R}$. Prove that if the second derivative $f''(x_0)$ exists then

$$\lim_{h\to 0} \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} = f''(x_0).$$

54. Let f be defined on $[-2,2]$, so that

$$\frac{f(b)-f(a)}{b-a} \leqslant \frac{f(c)-f(b)}{c-b} \leqslant A$$

whenever $-2 \leqslant a < b < c \leqslant 2$ (i.e. f is a convex function), Show that there is $C \geqslant 0$ so that for $|h| \leqslant 1$¡

$$\int_{-1}^{1} |f(x+h) + f(x-h) - 2f(x)| \leqslant Ch^2.$$

55. Let K be a continuous function on the unit square $Q = [0,1]^2$ with the property that $|K(x,y)| \leqslant 1$ for all $(x,y) \in Q$. Show that there is a continuous function g on $[0,1]$ so that

$$g(x) + \int_0^1 K(x,y)g(y)\ dy = \frac{e^x}{1+x^2}$$

for $x \in [0,1]$.

56. Prove there is a unique smooth function f defined on $[0,1]$ which satisfies the integral equation

$$f(x) + \int_0^x t\cos(tx)\frac{f(t)}{1+f(t)^2} = 0$$

for all $x \in [0,1]$.

27 2025 Qual

57. Let $\{a_n\}$ be a sequence of complex numbers and suppose $\sum_{n=1}^{\infty} a_n$ converges. Prove that

$$\lim_{N\to\infty} \sum_{k=1}^{N} (1 - k/N)a_k = \sum_{n=1}^{\infty} a_n.$$

58. Let

$$f_n(x) = \int_1^n te^{it^3 x}\, dt.$$

Let $\varepsilon > 0$. Prove the sequence $\{f_n\}$ converges uniformly on (ε, ∞).

59. Let $\mathbb{R}^{n\times n}$ be the space of all real $n \times n$ matrices. Let $\|\cdot\|$ be any norm on this vector space. For $A \in \mathbb{R}^{n\times n}$, let $\mathrm{Tr}(A)$ be the trace of A. Prove that there are neighborhods U and V of the identity matrix I so that for $B \in V$ there exists a unique $A \in U$ such that

$$\frac{1}{n}\mathrm{Tr}(A)A^5 = B.$$

60. Let f be a measurable function on $[0, 1]$ with $f(x) > 0$ for all $x \in [0, 1]$, and assume $\int_0^1 f(x)\, dx < \infty$. let $0 < \gamma < 1$. Prove

$$\inf_{|E|=\gamma} \int_E f(x)\, dx > 0.$$

Here the infinum is taken over all measurable sets E with Lebesgue measure γ.

61. Assume $\{a_m\}$ and $\{b_m\}$ are sequences in l^2. Prove that

$$\sum_{k=1}^{\infty} \sum_{m=k+1}^{\infty} \frac{a_k b_m}{m} < \infty.$$

62. Let μ be a positive, finite measure on $\mathbb{R}$. Define

$$G(x) = \mu((-\infty, x])$$

and let $\beta \in \mathbb{R}$. Evaluate

$$\int_{-\infty}^{\infty} G(x + \beta) - G(x)\, dx.$$

63. Let $f \in L^1(\mathbb{T})$ (i.e. 1 periodic with $\int_0^1 |f(x)|\, dx < \infty$), and suppose f is differentiable at x_0. Prove

$$\lim_{N\to\infty} \sum_{k=-N}^{N} \hat{f}(k)e^{2\pi ikx_0} = f(x_0).$$

64. Let $\mathcal{F}$ be a set of C_c^∞ functions on $\mathbb{R}^n$ bounded in the topology of $C_c^\infty(\mathbb{R}^n)$. Prove there exists a compact set K such that for each $f \in \mathcal{F}$ is supported in K.

65. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} x^2 & : x \geqslant 0 \\ 0 & : x < 0 \end{cases}$$

Determine g', g'', and g'', and find a simple expression for $f * g'''$ if $f \in C_c^\infty(\mathbb{R})$.

28 August 2024 Qual

66. Let $K_n : [a,b]\times[a,b] \to \mathbb{R}$ be a sequence of differentiable functions such that $\|K_n\|_{L^\infty}, \|\partial_1 K_n\|_{L^\infty}, \|\partial_2 K_n\|_{L^\infty} \leqslant 1$ for all n. For $f \in C[a,b]$, define

$$A_n f(x) = \int_a^b K_n(x,t) f(t)\, dt.$$

   (a) Prove that $\{A_n f : \|f\|_{L^\infty} \leqslant 1, n \geqslant 1\}$ is a totally bounded subset of $C[a,b]$.

   (b) If we drop the assumption that $\|\partial_2 K_n\|_{L^\infty} \leqslant 1$, is the set still totally bounded?

   (c) If we drop the assumption that $\|\partial_1 K_n\|_{L^\infty \leqslant 1}$, is the set still totally bounded?

67. Show there is a unique function f in $C[0,10]$ solving the equation

$$f(x) = -15 + \cos(x) \int_0^x e^{e^{tx}} f(t)\, dt$$

   for all $x \in [0,10]$.

68. Let

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < x_1^2 \leqslant 1/2\}.$$

   Let $b \in \mathbb{R}$ and let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x) = |x|^{-b} |\log(|x|)|^{-\gamma}$$

   Determine for which b and $\gamma \in \mathbb{R}$ the function f is integrable on Ω.

69. Suppose $f_n \in L^1(\mathbb{R})$ and $\{\varepsilon_n\}$ is positive. Let $E_n = \{x : |f_n(x)| \geqslant \varepsilon_n\}$ and assume $\sum |E_n| < \infty$.

   (a) Prove $\lim_{n\to\infty} f_n = 0$ almost everywhere.

   (b) Prove that for every $\delta > 0$, there is Ω_δ with $|\Omega_\delta| < \delta$ and $\lim_n f_n = 0$ uniformly on $\mathbb{R}\setminus\Omega$.

70. Let $E \subset \mathbb{R}$. Suppose $g, f_n \in L^1(E)$, $\|f_n\|_{L^1(E)} \leqslant 1$ for each n, and $\lim_{n\to\infty} f_n = 0$ in measure. Prove that

$$\lim_{n\to\infty} \int_E \sqrt{|f_n g|} = 0.$$

71. Suppose $\{g_n\} \subset \mathcal{S}(\mathbb{R}^2)$ and $\lim_{n\to\infty} \|g_n\|_{L^2(\mathbb{R}^2)} = 0$. Show that there are $f_n \in C^2(\mathbb{R}^2)$ such that

$$\Delta f_n = f_n + g_n$$

   and $\lim_{n\to\infty} f_n(0) = 0$, and $\lim_{n\to\infty} \|\partial_{x_1}\partial_{x_2} f_n\|_{L^2(\mathbb{R}^2)} = 0$.

72. Suppose

$$\langle u, \phi \rangle = \int_0^2 \phi(0,t)\, dt$$

   and

$$\langle v, \phi \rangle = \int_0^2 \phi(t,0)\, dt.$$

   Show that $u * v$ is an absolutely continuous measure μ, and find $g \in L^1(\mathbb{R}^2)$ so that $\mu = g\, dx$.

29 More Distribution Theory Warm Up Question

73. Suppose

$$\langle u, \phi \rangle = \int_0^2 \phi(0, t)\, dt$$

and

$$\langle v, \phi \rangle = \int_0^2 \phi(t, 0)\, dt.$$

Show that $u * v$ is an absolutely continuous measure μ, and find $g \in L^1(\mathbb{R}^2)$ so that $\mu = g\, dx$.

30 More Distribution Theory Practice

74. (January 2013) Let T be a distribution on $\mathbb{R}$. Set $\tau_a\phi(x) = \phi(x-a)$, and assume that $\langle T, \tau_a\phi\rangle = \langle T, \phi\rangle$ for all $a \in \mathbb{R}$ and all test functions ϕ. Prove that T is constant (i.e. there is C so that $\langle T, \phi\rangle = C\int \phi(x)\,dx$ for all test functions ϕ).

75. (2013) Let δ be the Dirac delta function at the origin, and let δ' be it's distributional derivative. Consider

$$f_1(x) = \begin{cases} 0 & |x| > 1 \\ \sin(x^3) & |x| < 1 \end{cases}$$

and let $f_2 = \delta'$. For $j \in \{1, 2\}$, find $\sup\{s : f_j \in H^s(\mathbb{R})\}$, where

$$H^s(\mathbb{R}) = \{f : \int [|\hat{f}(\xi)|\langle\xi\rangle^s]^2\,d\xi < \infty\}.$$

76. (Spring 2014) Does there exist a distribution T such that for functions $\phi \in C_c^\infty(\mathbb{R})$ with $0 \notin \operatorname{supp}(\phi)$, $\langle T, \phi\rangle = \int \phi(x)|x|^{-d}$.

77. (Fall 2003) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x) = \max(1 - |x|^2, 0).$$

Prove that $\Delta f + 4\chi_\Omega \geqslant 0$, in the sense of distributions, i.e. that $\langle \Delta f + 4\chi_\Omega, \psi\rangle \geqslant 0$ if $\psi \geqslant 0$.

31 Day 10: Warm Up Problems

78. (Fall 2021) Let σ be a Borel probability measure on $[0, 1]$ satisfying

    1. $\sigma([1/3, 2/3]) = 0$.
    2. $\sigma([a, b]) = \sigma([1 - b, 1 - a])$ for any $0 \leqslant a < b \leqslant 1$.
    3. $\sigma([3a, 3b]) = 2\sigma([a, b])$ for any a, b such that $0 \leqslant 3a < 3b \leqslant 1$.

    σ is called the 1/3 Cantor measure on $[0, 1]$.

    (a) Find $\sigma([0, 1/8])$.

    (b) Calculate the second moment of σ, i.e. the integral

$$M = \int_0^1 x^2 d\sigma(x).$$

79. (Fall 2021) Find the spectrum of the linear operator A on $L^2(\mathbb{R})$ defined as

$$Af(x) = \int_{-\infty}^{\infty} \frac{f(y)}{1 + (x - y)^2} \, dy$$

    (The spectrum of a linear operator T is the closure of the set of all complex numbers λ such that the operator $T - \lambda$ does not have a bounded inverse). Hint: It may be useful to find the Fourier transform of $1/(1 + x^2)$.

32 Day 10: Bonus Questions

80. (Fall 2017) Let $a_1, \ldots, a_n > 0$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{1 + \sum |x_i|^{\alpha_i}}.$$

Determine for each $p > 0$ whether

$$\int |f(x)|^p \, dx < \infty.$$

81. Let $f \in L^1(\mathbb{R})$. Let

$$G(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t^2} f(t) \, dt.$$

Prove that G is a continuous function and that $\lim_{\lambda \to \infty} G(\lambda) = 0$.

82. Let (X, μ) be a σ-finite measure space. Let $\{f_n\}$ be a sequence of measurable functions and assume that $f_n \to f$ almost everywhere. Prove that there exists measurable $A_1, A_2, \dots \subset X$ such that $\mu(X - \bigcup_i A_i) = 0$, and such that $f_n|_{A_i} \to f|_{A_i}$ uniformly for each i.

83. (Spring 2017) Given for each function $f \in C^0(\mathbb{R}^2)$ we define for each $y \in \mathbb{R}$ a function $f_y \in C[0,1]$ by $f_y(x) = f(x, y)$. Assume that for each fixed y, the distributional derivative of $f_y \in \mathcal{D}'(\mathbb{R})$ defines a function $a_y \in L^p(\mathbb{R})$. Assume further that

$$\|a_y\|_p \leqslant C < \infty$$

for some constant C independent of y. Show that the distributional derivative $\partial_x f \in \mathcal{D}'(\mathbb{R}^2)$ is in $L^p_{\text{loc}}(\mathbb{R}^2)$, provided $1 < p \leqslant \infty$.

84. (Spring 2020) Let $E \subset [0,1]$ be a measurable set with positive Lebesgue measure. Moreover, it satisfies the following property: As long as x and y belong to E, we know $\frac{x+y}{2}$ belongs to E. Prove that E is an interval.

85. (a) Does $p_N = \prod_{n=2}^{N}(1 + (-1)^n/n)$ tend to a nonzero limit as $N \to \infty$.

    (b) Does $q_N = \prod_{n=2}^{N}(1 + (-1)^n/\sqrt{n})$ tend to a nonzero limit as $N \to \infty$.

33 Day 11: Warm Up Question

86. (Fall 2015) Let $\chi \in C^{\infty}(\mathbb{R})$ have a compact support and define

$$f_n(x) = n^2 \chi'(nx).$$

(a) Does f_n converge in the sense of distributions as $n \to \infty$? If so, what is the limit?

(b) Let $p \in [1, \infty)$ and $g \in L^p(\mathbb{R})$ be such that the distributional derivative of g also lies in $L^p(\mathbb{R})$. Does $f_n * g$ converge in $L^p(\mathbb{R})$ as $n \to \infty$? If so, what is the limit?

87. (Fall 2018) Prove or Disprove that in an infinite dimensional Banach space,

(a) every norm bounded set is weakly bounded,

(b) every norm closed set is weakly closed

(c) a norm bounded set has empty interior in the weak topology

34 Day 11: Bonus Questions

88. Suppose $f : [0,1]^2 \to \mathbb{R}$ is continuous, and $\partial^2 f/\partial x^2$ exists pointwise on $[0,1]$, is continuous in the x variable, and is bounded. Then $\partial f/\partial x$ is continuous.

89. Let
$$s_N(x) = \sum_{n=1}^{N} (-1)^n \frac{x^{3n}}{n^{2/3}}.$$
Prove that $s_N(x)$ converges to a limit $s(x)$ on $[0,1]$, and that there is a constant $C > 0$ so that for all $N \geqslant 1$ the inequality
$$\sup_{x\in[0,1]} |s_N(x) - s(x)| \leqslant CN^{-2/3}$$
holds.

90. (Spring 2016) Let $1 < p < \infty$, and let $\chi_{[1-\frac{1}{n},1]}$ denote the characteristic function of $[1-\frac{1}{n},1]$. For which $\alpha \in \mathbb{R}$ does the sequences $n^\alpha \chi_{[1-\frac{1}{n},1]}$ converge weakly to 0 in $L^p(\mathbb{R})$?

91. (Spring 2018) Let x_n be a sequence in a Hilbert space H. Suppose that x_n converges to x weakly. Prove that there is a subsequence x_{n_k} such that
$$\frac{1}{N}\sum_{k=1}^{N} x_{n_k}$$
converges to x (in norm) as $N \to \infty$.

92. (Fall 2015) Let $E \subset \mathbb{R}$ be a measurable set, such that $E + r = E$ for all $r \in \mathbf{Q}$. Show that $|E| = 0$ or $|E^c| = 0$.

93. (Spring 2015) Let $\{r_n\} \in [0,1]$ be an arbitrary sequence, and define the function
$$f(x) = \sum_{r_n < x} \frac{1}{2^n}$$
Show that f is Borel measurable, find all it's points of discontinuity, and find $\int_0^1 f(x)\,dx$.

35 Day 12: Warm Up Question

94. (Spring 2021) Let f be a C^1 function on $[0, \infty)$. Suppose that

$$\int_0^\infty t|f'(t)|^2\, dt < \infty$$

and

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)\, dt = L.$$

Show that $f(t) \to L$ as $t \to \infty$.

95. (Fall 2013) Let $E = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}, x_1 - x_2 \in \mathbb{Q}\}$. Is it possible to find to Lebesgue measurable sets $A_1, A_2 \subset \mathbb{R}$ such that $|A_1|, |A_2| > 0$, and $A_1 \times A_2 \subset E^c$?

96. (Spring 2021)

(a) Let H_1 and H_2 be Hilbert spaces, and let $T : H_1 \to H_2$ be a continuous linear operator. Give a precise definition of the adjoint operator T^*.

(b) Let $(a, b) \subset \mathbb{R}$ be a (possibly infinite) open interval. If $f \in L^2(a, b)$, explain what it means that the distributional derivative f' is also in $L^2(a, b)$.

(c) Let $\mathbb{R}_+$ denote the positive real axis $[0, \infty)$. Let $H^1(\mathbb{R})$ (respectively $H^1(\mathbb{R}_+)$) be the space of real-valued functions $f \in L^2(\mathbb{R})$ (respectively $f \in L^2(\mathbb{R}_+)$ such tha thte distributional derivative f' is also in $L^2(\mathbb{R})$ (respectively $L^2(\mathbb{R}_+)$). Then $H^1(\mathbb{R})$ and $H^1(\mathbb{R}_+)$ are Hilbert spaces with inner product given by

$$\langle f, g\rangle_{H^1(\mathbb{R})} = \int_{\mathbb{R}} f(x)g(x)dx + \int_{\mathbb{R}} f'(x)g'(x)dx,$$
$$\langle f, g\rangle_{H^1(\mathbb{R}_+)} = \int_{\mathbb{R}_+} f(x)g(x)dx + \int_{\mathbb{R}_+} f'(x)g'(x)dx$$

Let $T : H^1(\mathbb{R}) \to H^1(\mathbb{R}_+)$ be the mapping given by the restriction. Compute exactly the adjoint operator T^*.

36 Day 12: Bonus Questions

97. (Fall 2015) Let (X, μ) be a measure space, and let $f : X \to \mathbb{R}$ be measurable. Then if $1 \leqslant p < r < q < \infty$ and there is $C < \infty$ such that
$$\mu(\{x : |f(x)| > \lambda\}) \leqslant \frac{C}{\lambda^p + \lambda^q}$$
for every $\lambda > 0$. Then $f \in L^r(\mu)$.

98. (Spring 2020) Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Let $b_n \in \mathbb{R}$ be an increasing sequence with $\lim_{n\to\infty} b_n = \infty$. Show that
$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} b_k a_k = 0.$$

99. (Spring 2021) Let $f_n \to f$ weakly in $L^2(\mathbb{R})$ and $\|f_n\|_2 \to \|f\|_2$ as $n \to \infty$. Show that $f_n \to f$ strongly in $L^2(\mathbb{R})$.

100. (Spring 2017) Let $E \subset \mathbb{R}^n$ be a set of finite, positive measure, and let $\{t_k\}$ be a sequence with $\{t_k\} > 0$ and $\lim_k t_k = 0$. Define, for $f \in L^p(\mathbb{R}^n)$,
$$Mf(x) = \sup_k ⨍_{t_k E} |f(x - y)|\, dy.$$
Suppose furthermore that there is $C > 0$ such that
$$|\{x : Mf(x) > \lambda\}| \leqslant C\lambda^{-p}\|f\|_p^p.$$
Show that for every $f \in L^p(\mathbb{R}^n)$,
$$\lim_k ⨍_{t_k E} f(x - y)\, dy = f(x).$$
for almost every $x \in \mathbb{R}^d$.

101. (Spring 2020) Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of Lebesgue measurable functions such that f_n converges to f almost everywhere on $[0, 1]$ and such that $\|f_n\|_{L^2([0,1])} \leqslant 1$ for all n. Show that
$$\lim_{n\to\infty} \|f_n - f\|_{L^1([0,1])} = 0.$$

102. (Fall 2017, Spring 2021) Let $f : \mathbb{R} \to \mathbb{R}$ be a compactly supported function that satisfies the Hölder condition with exponent $\beta \in (0, 1)$, i.e. that there exists a constant $A < \infty$ such that for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| \leqslant A|x - y|^\beta$. Consider the function g defined by
$$g(x) = \int_{-\infty}^{\infty} \frac{f(y)}{|x - y|^\alpha}\, dy,$$
where $\alpha \in (0, \beta)$.
    (a) Prove that g is a continuous function at zero.
    (b) Prove that g is differentiable at zero. (Hint: Try the dominated convergence theorem).

103. (Fall 2018) Let $1 < p \leqslant \infty$. Let $(X, \mathcal{M}, \mu)$ be a finite measure speace. Let $\{f_n\}$ be a sequence of measurable functions converging μ-a.e. to the function f. Assume further that $\|f_n\|_p \leqslant 1$ for all n. Prove that $f_n \to f$ as $n \to \infty$ in L^r for all $1 \leqslant r < p$.

37 Day 13: Analysis Qualifying Exam 2022

104. Let $x_1, \ldots, x_{n+1}$ be pairwise distinct real numbers. Prove that there exists $C > 0$ such that if $P : \mathbb{R} \to \mathbb{R}$ is a polynomial with degree at most n, then

$$\|P\|_{L^\infty[0,1]} \leqslant C \max\left(|P(x_1)|, \ldots, |P(x_{n+1})|\right).$$

105. Given a real number x, let $\{x\}$ denote the fractional part of x. Suppose α is an irrational number and define $T : [0,1] \to [0,1]$ by

$$T(x) = \{x + \alpha\}.$$

Prove: If $A \subset [0,1]$ is measurable and $T(A) = A$ then $|A| \in \{0,1\}$.

106. Let $\{f_n\}$ be a sequence of measurable, real-valued functions on a measure space X such that $f_n \to f$ pointwise as $n \to \infty$, where $f : X \to \mathbb{R}$, and suppose that for some constant $M > 0$,

$$\int |f_n| d\mu \leqslant M \text{ for all } n \in \mathbb{N}.$$

(a) Prove that

$$\int |f| \, d\mu \leqslant M.$$

(b) Give an example to show that we may have $\int |f_n| \, d\mu = M$ for every n, but $\int |f| \, d\mu < M$.

(c) Prove that

$$\lim_{n\to\infty} \int ||f_n| - |f| - |f_n - f|| = 0.$$

107. Consider the following equation for an unknown function $f : [0,1] \to \mathbb{R}$:

$$f(x) = g(x) + \lambda \int_0^1 (x-y)^2 f(y) \, dy + \frac{1}{2} \sin(f(x)).$$

Prove that there exists a number $\lambda_0 > 0$ such that for all $\lambda \in [0, \lambda_0)$ and all continuous functions g on $[0,1]$, the equation has a unique continuous solution.

108. Given $\alpha \geqslant 0$, the α-dimensional Hausdorff measure of a set $X \subset \mathbb{R}^n$ is

$$H^\alpha(X) = \liminf_{r\to 0} \left\{ \sum_{i=1}^\infty r_i^\alpha : X \subset \bigcup_{i=1}^\infty B(x_i, r_i), r_i < r \text{ for all } i \right\}$$

(where $B(x,r)$ is the Euclidean ball with center x and radius r) and the Hausdorff dimension is

$$\dim_{\mathbf{H}}(X) = \inf\{\alpha \geqslant 0 : H^\alpha(X) = 0.\}$$

Prove:

(a) If $X \subset \mathbb{R}^n$ and μ is a finite Borel measure on X such that $\mu(X) > 0$ and $\mu(B(x,r)) \leqslant r^\alpha$ for all open balls $B(x,r)$, then $\dim_{\mathbf{H}}(X) \geqslant \alpha$.

(b) If $\mathbf{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, then $\dim_{\mathbf{H}}(\mathbf{S}^1) = 1$.

109. Let $X = [0,1]$ with Lebesgue measure and $Y = [0,1]$ with counting measure. Give an example of an integrable function $f : X \times Y \to [0, \infty)$ for which Fubini's theorem does not apply.

110. For $s > 1/2$ let $H^s(\mathbb{R}^n)$ denote the Sobolev space

$$H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi < \infty\}.$$

Use the Fourier transform to prove that if $u \in H^s(\mathbb{R}^n)$ for $s > n/2$ then $u \in L^\infty(\mathbb{R}^n)$ with the boundary

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leqslant C\|u\|_{H^s(\mathbb{R}^n)}.$$

for a constant C depending only on s and n.

111. Assume that X is a compact metric space and $T : X \to X$ is a continuous map. Let $M_1(T)$ denote the set of Borel probability measures on X such that $T_* \mu = \mu$. Prove
   (a) $M_1(T) \neq \varnothing$.
   (b) If $M_1(T) = \{\mu\}$ consits of a single measure μ, then

$$\int_X f d\mu = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n(x)$$

   for every continuous function $f : X \to \mathbb{R}$ and $x \in X$.

112. Find the Fourier transform of the following function f in $\mathbb{R}^2$:

$$f(x) = e^{ix\xi_0}|x - x_0|^{-1}.$$

38 Interchanging Limits Notes

It is common that one wishes to the question of when one can justify the interchange of integral and limit, i.e. as in the case where $f_n \to f$ and one wishes to show of

$$\lim_{n\to\infty} \int_E f_n = \int_E \lim_{n\to\infty} f_n.$$

often arises. There are several tools for justifying such an *interchange of limits*:

- Monotone Convergence Theorem
- Uniform Convergence
- Lebesgue Dominated Convergence Theorem
- General Lebesgue Dominated Convergence Theorem
- Vitali Convergence Theorem

Of these theorems, the Vitali Convergence Theorem and Uniform convergence require the domain of integration E to be a set of finite measure. However, in that case, the Vitali Convergence Theorem, which requires the f_n's to be *uniformly integrable*, provides a somewhat more general condition than the existence of a dominating function (and is often useful when a dominating function is difficult to find). One condition which implies uniform integrability is

$$\|f_n\|_p \leqslant C < \infty$$

for some $p > 1$ and all n, (though if $p = 1$ this condition is not sufficient). The proof of the Vitali Convergence theorem is essentially an application of Egorov's theorem (and was a qual question in Fall 2010).

If one needs only prove that a limit function $f = \lim_{n\to\infty} f_n$ is integrable, the above tools are sometimes overkill and one may be able to simply apply Fatou's lemma:

$$\int_E |f| \leqslant \liminf_{n\to\infty} \int_E |f_n|$$

provided that one knows the right-hand side is finite.

39 Bonus Day: Interchanging Limits

113. (Rice Qualifying Exam, Winter 2011) Let $\{f_n\}$ be a sequence of Lebesgue measurable functions defined on $[0,1]$ such that $|f_n(x)| \leqslant 1$ for all $n \geqslant 1$ and all $0 < x \leqslant 1$, and

$$\lim_{n\to\infty} f_n(x) = f(x)$$

exists for each $0 \leqslant x \leqslant 1$. Prove that

$$\lim_{n\to\infty} \int_0^1 \frac{f_n(x)}{\sqrt{|x-1/n|}} = \int_0^1 \frac{f(x)}{\sqrt{x}}\,dx$$

114. (Rice, Spring 2005) Compute

(a) $\lim_{n\to\infty} \int_0^\infty \frac{x^{n-2}}{1+x^n}$.

(b) $\lim_{n\to\infty} n\int_0^\infty \frac{\sin y}{y(1+n^2y)}\,dy$ (Hint: Substitute $x = ny$).

115. (Spring 2017) Let $f : [0,\infty) \to \mathbb{R}$ be a continuously differentiable function for which $\|f'\|_\infty < \infty$. Define, for $x > 0$,

$$F(x) = \int_0^\infty f(x + yx)\psi(y)\,dy,$$

where ψ satisfies

$$\int_0^\infty |\psi(y)|\,dy \quad \text{and} \quad \int_0^\infty y \cdot |\psi(y)|\,dy < \infty.$$

Show that $F(x)$ is well defined for all $x \geqslant 0$, and that F is continuously differentiable.

40 Arzela-Ascoli Notes

The key part of the Arzela-Ascoli theorem to know for the qual is the following:

> If $\{f_n\} \subset C[0,1]$ is a sequence which is uniformly bounded and equicontinuous, then $\{f_n\}$ has a uniformly convergent subsequence.

(Note that we can replace $[0,1]$ by any compact subset of $\mathbb{R}^d$. Also, there is a converse to the theorem, but I haven't seen it used in any qual problems. For a more general statement and discussion of this thorem, see the appendix to Rudin's *Functional Analysis*.)

- By *uniformly bounded*, we mean that $|f_n(x)| \leqslant C$ for all $x \in [0,1]$, $n \in \mathbb{N}$.
- By *equicontinuous*, we mean that for all $\epsilon > 0$, there exists δ such that $|f_n(x) - f_n(y)| < \epsilon$ whenever $|x - y| < \delta$ for all $n \in \mathbb{N}$.

A useful condition for demonstrating equicontinuity of a collection of functions is having some sort of bound on their derivatives (e.g. as in several of the problems below).

Recall that a subset K of a metric space X is *sequentially compact* if every sequence in K has a convergent subsequence whose limit belongs to K. For subsets of metric spaces, sequential compactness is equivalent to compactness. Similarly, K is *precompact* if and only if every sequence in K has a convergent subsequence (but whose limit need not belong to K).

Let X, Y be normed linear spaces. A linear operator $A : X \to Y$ is said to be *compact* if it maps bounded sets to precompact sets.

When showing that a linear operator is compact, the following condition is often useful:

A linear operator $A : X \to Y$ is compact if $(Ax_n)_{n=1}^{\infty}$ has a cauchy subsequence whenever $(x_n)_{n=1}^{\infty}$ is bounded in X.

41 Bonus Day: Arzela-Ascoli

116. (From a UBC Math 321 Midterm) Let $\{f_n\}$ be a sequence of functions in $C[a,b]$ with no uniformly convergent subsequence. Define

$$F_n(x) = \int_a^x \sin(f_n(t))\, dt.$$

Does $\{F_n\}$ has a uniformly convergent subsequence.

117. (From a UBC Math 321 Midterm) Let $\{f_n\}$ be a sequence of functions in $C[a,b]$ with no uniformly convergent subsequence. Define

$$F_n(x) = \int_a^x \sin(f_n(t))\, dt.$$

Does $\{F_n\}$ has a uniformly convergent subsequence.

118. (Fall 2004) Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of continuous functions whose derivatives f_n' in the sense of distributions belong to $L^2(0,1)$. The functions also satisfy $f_n(0) = 0$.

(a) Assume that

$$\lim_{n\to\infty} \int_0^1 f_n'(x) g(x) dx$$

exists for all $g \in L^2(0,1)$. Show that the f_n converge uniformly as $n \to \infty$.

(b) Assume that

$$\lim_{n\to\infty} \int_0^1 f_n'(x) g(x) dx$$

exists for all $g \in C([0,1])$. Do we still have the f_n converge uniformly?

119. (Spring 2014) Consider the following operator

$$Af(x) = \frac{1}{x\sqrt{1+|\log x|}} \int_0^x f(t) dt.$$

Is A bounded as an operator from $L^2[0,1]$ to $L^2[0,1]$? Is it compact?

120. (Problem 36 from the 2017 SEP) Consider the Hilbert space $L^2([0,1])$ with inner product $(f,g) := \int_0^1 f(t)\bar{g}(t)dt$. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal system of functions in $L^2([0,1])$.

(a) Suppose that $e_n' \in L^2([0,1])$ for all $n \in \mathbb{N}$. Show that

$$\sup_n \max_{x\in[0,1]} |e_n'(x)| = \infty.$$

(b) Suppose that e_n is complete, which means $(g, e_n) = 0$ for all n implies $g = 0$ almost everywhere. Prove

$$\sum_{n=1}^\infty |e_n(x)|^2 = \infty, \quad \text{almost everywhere.}$$

42 Bonus Day: Misc. Topics

121. (Rice, Winter 2008) Is it possible to construct a measurable set $E \subset \mathbb{R}$ of positive measure such that for any pair $a < b$, $|E \cap [a,b]| \leqslant 0.5(b-a)$?

122. (Spring 2010) For $\lambda > 0$, set

$$F(\lambda) = \int_0^1 e^{-10\lambda x^4 + \lambda x^6} dx$$

Prove there exists constants A and $C > 0$, such that $F(\lambda) = \frac{A}{\lambda^{\frac{1}{4}}} + E(\lambda)$ where $|E(\lambda)| \leqslant \frac{C}{\lambda^{\frac{1}{2}}}$.

123. (Fall 2010) Let $I = [0,1]$ and define for $f \in L^2(I)$ the Fourier coefficients as $\hat{f}(k) = \int_0^1 f(t)e^{-2\pi ikt}dt$ for any $k \in \mathbb{Z}$.

   (a) Let $\mathcal{G}$ be the set of all $L^2(I)$ functions with the property that $|\hat{f}(0)| \leqslant 1$ and $|\hat{f}(k)| \leqslant |k|^{-3/5}$ for any $k \in \mathbb{Z}$, $k \neq 0$. Prove that $\mathcal{G}$ is a compact subset of $L^2(I)$.

   (b) Let $\mathcal{E}$ be the set of all $L^2(I)$ functions with the property that $\sum_k |\hat{f}(k)|^{5/3} \leqslant 2016^{-2016}$. Is $\mathcal{E}$ a compact subset of $L^2(I)$?

124. (Fall 2011) Let $\ell^2(\mathbb{N})$ denote the Hilbert space of square summable sequences with inner product $(x,y) = \sum_{n=1}^{\infty} x_n y_n$, where $x = (x_1, x_2, \cdots)$ and $y = (y_1, y_2, \cdots)$.

   (a) What are the necessary and sufficient conditions on $\lambda_n > 0$ for the set

   $$S = \{(x_1, x_2 \cdots) \in \ell^2(\mathbb{N}) : |x_n| \leqslant \lambda_n, \forall n\}$$

   to be compact in $\ell^2(\mathbb{N})$?

   (b) What are the necessary and sufficient conditions on $\mu_n > 0$ for the set

   $$\left\{(x_1, x_2 \cdots) \in \ell^2(\mathbb{N}) : \sum_n \frac{|x_n|^2}{\mu_n^2} \leqslant 1\right\}$$

   to be compact in $\ell^2(\mathbb{N})$?

125. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function, let $E = \{x \in \mathbb{R} : f \text{ is not differentiable at } x\}$. Show that E is at most countable.

126. (Fall 2015) Identify all $\alpha \in \mathbb{R}$ such that $\lim_{n\to\infty} \sin(2\pi n\alpha)$ exists.

43 Bonus Day: Spring 2021 Final Qualifying Exam

127. (3) For a Lebesgue measurable subset E of $\mathbb{R}$, denote $\mathbf{1}_E$ the indicator function of E (i.e. $\mathbf{1}_E(x) = 1$ for $x \in E$ and $\mathbf{1}_E(x) = 0$ for $x \in E^c$).

    Let $\{E_n : n \in \mathbb{N}\}$ be a family of Lebesgue measurable subsets of $\mathbb{R}$ with finite measure and let f be a measurable function such that
$$\lim_{n\to\infty} \int_{\mathbb{R}} |f(x) - \mathbf{1}_{E_n}(x)|dx = 0.$$
    Prove that f is the indicator function of a measurable set.

128. (6) Let $f : \mathbb{R} \to \mathbb{R}$ be a compactly supported function that satisfies the Holder condition with exponent $\beta \in (0, 1)$, i.e., there exists a constant $A < \infty$ such that $\forall x, y \in \mathbb{R} : |f(x) - f(y)| \leqslant A|x - y|^\beta$. Consider the function g defined by
$$g(x) = \int_{-\infty}^{\infty} \frac{f(y)}{|x-y|^\alpha} dy$$
    where $\alpha \in (0, \beta)$.

    (a) Prove that g is a continuous function at 0.

    (b) Prove that g is differentiable at 0. (Hint: Try the dominated convergence theorem).

129. ((Spring 2021) points) A real valued function f defined on $\mathbb{R}$ belongs to the space $C^{1/2}(\mathbb{R})$ if and only if
$$\sup_{x\in\mathbb{R}} |f(x)| + \sup_{x\neq y} \frac{|f(x) - f(y)|}{\sqrt{|x-y|}} < \infty.$$
    Prove that a function $f \in C^{1/2}(\mathbb{R})$ if and only if there exists a constant C so that for every $\varepsilon > 0$, there is a bounded function $\varphi \in C^\infty(\mathbb{R})$ such that
$$\sup_{x\in\mathbb{R}} |f(x) - \varphi(x)| \leqslant C\varepsilon^{1/2} \quad \text{and} \quad \sup_{x\in\mathbb{R}} \varepsilon^{1/2}|\varphi'(x)| \leqslant C.$$

130. (Fall 2021) Let $f \in L^1(\mathbb{R})$ satisfy $\int_a^b f(x)dx = 0$ for any two rational numbers a and b, $a < b$. Does it follow that $f(x) = 0$ for almost every x?

131. (Spring 2020) Show that $\int_0^\infty \frac{\sin(x)}{x^{2/3}} dx$ converges. Determine whether the integral
$$\int_1^\infty \frac{\sin x}{\sin(x) + x^{2/3}} dx$$
    converges or not. Hint: use Taylor expansion.

132. (Fall 2021) Let $\{f_n\}$ be a sequence of monotonic functions on $[0, 1]$ converging to a function f in measure (with respect to the Lebesgue measure). Show that f coincides almost everywhere with a monotonic function f_0 and that $f_n(x) \to f_0(x)$ at every point of continuity of f_0.

133. (Fall 2020) Let $f : [a, b] \to \mathbb{R}$ be a continuous function that is strictly increasing. Prove that the inverse function f^{-1} is absolutely continuous on $[f(a), f(b)]$ if and only if
$$m(E) = 0 \quad \text{where} \quad E := \left\{x \in (a, b) : f'(x) = 0\right\}.$$

44 Questions that need solutions

134. (Fall 2017) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function that has continuous partial derivatives in $\mathbb{R}^2$. Define $\chi : \mathbb{R}^3 \to \{0, 1\}$ by $\chi(x) = 1$ if $x_3 > g(x_1, x_2)$, and $\chi(x) = 0$ otherwise. Compute the derivatives $\partial\chi/\partial x^i$ for $i = 1, 2, 3$.

135. Let $\alpha \in (0, 1)$, and for $f \in C[0, 1]$, and $x \in [0, 1]$, define

$$(T_\alpha f)(x) = \int_0^1 \sin(x - y)|x - y|^{-\alpha} f(y)\, dy.$$

   (a) Prove that T_α extends to a bounded operator on $L^2[0, 1]$.
   (b) For which $\alpha \in (0, 1)$ is $T_\alpha : L^2[0, 1] \to L^2[0, 1]$ a compact operator?