



Salem Sets Avoiding Nonlinear Patterns

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Research Problem: Can Large Sets Avoid Patterns?

More specifically: If a set $X \subset \mathbb{R}^d$ has large *fractal dimension*, does it contain patterns? The main focus of this project is on the construction of counterexamples: for a given function f with domain $(\mathbb{R}^d)^n$, can we construct large sets X such that there are no distinct points $x_1, \dots, x_n \in X$ such that $f(x_1, \dots, x_n) = 0$, i.e. such that X avoids zeroes of f ? We often study functions f which vanish on the diagonal $\Delta = \{(x, \dots, x) : x \in \mathbb{R}^d\}$, which makes it difficult to avoid zeroes if X is ‘thick’, i.e. has large fractal dimension.

Here are example choices of f :

A set avoiding 3-term APs

- If $f(x_1, x_2, x_3) = (x_1 - x_2) - (x_2 - x_3)$, then sets avoiding zeroes of f do not contain three term arithmetic progressions.
- If $f(x_1, x_2, x_3) = |x_1 - x_2|^2 - |x_2 - x_3|^2$, then sets in \mathbb{R}^d avoiding zeroes of f do not contain the vertices of any isosceles triangle.

A subset of the parabola
avoiding isosceles triangles

Mainly, this project constructs large *Salem sets* avoiding zeroes of *nonlinear* functions. Here are some results taken from (D., 2021):

Theorem 1. Suppose $f : (\mathbb{R}^d)^n \rightarrow \mathbb{R}^i$ is a submersion. Then we can construct a Salem set $X \subset \mathbb{R}^d$ avoiding solutions to f with $\dim(X) = i/(n-1/2)$.

Theorem 2. Let $g : (\mathbb{R}^d)^{n-1} \rightarrow \mathbb{R}^d$ be smooth, such that $D_{x^k} g = (\partial g_i / \partial x_j^k)$ is an invertible matrix for each $1 \leq k \leq n-1$. If

$$f(x^1, \dots, x^n) = x^n - g(x^1, \dots, x^{n-1}),$$

then we can construct a Salem set $X \subset \mathbb{R}^d$ avoiding solutions to f with $\dim(X) = d/(n-3/4)$, larger than that guaranteed by Theorem 1.

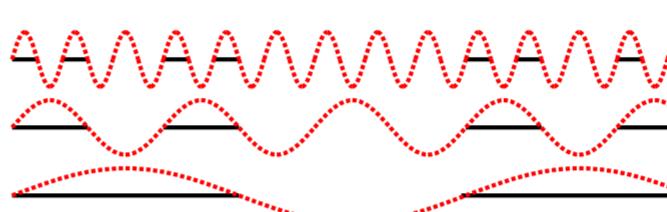
For instance, we can use these results to construct, for any smooth $\gamma : [0, 1] \rightarrow \mathbb{R}^d$, a Salem set $X \subset [0, 1]$ with dimension $4/9$ such that $\gamma(X)$ avoids vertices of isosceles triangles.

Salem Sets: Structure vs. Randomness

There are several fractal dimensions, and they differ subtly in the properties they measure:

- The *Hausdorff dimension* $\dim_H(X)$ of a set $X \subset \mathbb{R}^d$ measures the ability to distribute mass onto X in a way that does not concentrate too strongly around individual points.
- The *Fourier dimension* $\dim_F(X)$ of a set $X \subset \mathbb{R}^d$ measures the ability to distribute mass avoiding concentration near families of equally spaced points, i.e. concentration ‘at a particular frequency’, as measured quantitatively through the Fourier transform.

One always has $\dim_F(X) \leq \dim_H(X)$ for any set $X \subset \mathbb{R}^d$, but the reverse is often *not true* if the set is clustered ‘near particular frequencies’, like if X is a flat surface (clustered near frequencies travelling tangent to the hyperplane), or a Cantor set (clustered near frequencies of the form 3^n), both sets with Fourier dimension zero. On the other hand, a curved hypersurface in \mathbb{R}^d has Fourier dimension equal to $d-1$, also it’s Hausdorff dimension.



The Cantor Set correlates near frequencies of the form 3^n

We say a set X is *Salem* if $\dim_F(X) = \dim_H(X)$. *Random sets* are often almost surely Salem, since pure randomness prevents clustering at frequencies with high probability. But it is *surprisingly difficult* to control the Fourier dimension of sets when one introduces *structure* to sets, which may introduce subtle clustering near particular frequencies. In particular, the relation between *nonlinear structure* and Fourier dimension is especially

difficult to understand. For instance, even determining the Fourier dimension of the set $\{x + x^2 : x \in C\}$, where C is the Cantor set, remains an open problem.

There are many constructions of sets with large *Hausdorff dimension* avoiding zeroes of nonlinear functions f (e.g. Máté, 2017 or Fraser and Pramanik, 2018), but most constructions of large Salem sets avoiding functions f focus on linear functions f (e.g. Shmerkin, 2015 or Liang and Pramanik, 2020). Here we describe techniques to deal with the introduction of nonlinear structure to a random set via *probabilistic concentration inequalities*, and *oscillatory integrals*.

Constructing Salem Sets

We construct sets avoiding zeroes via a Cantor-type construction, i.e. iteratively defining sets $\{X_k\}$ by dissecting cubes (intervals if $d=1$) at each stage into smaller cubes, and keeping a union of smaller cubes chosen carefully so they have *good Fourier analytic properties*, and *avoid a discretized version of the pattern*, which *discretizes the problem*.

If, at each stage of the construction, we choose a large $N > 0$, subdivide each cube into smaller sidelength $1/N^{1/s}$ cubes, and take N of these cubes from each of the original intervals, then iteration for a fixed s should yield a set with Hausdorff dimension s . Since the subdivided set ‘lives at a scale $1/N^{1/s}$ ’, the uncertainty principle tells us to care about frequencies $|\xi| \lesssim N^{1/s}$.

And indeed, obtaining a Salem set reduces to verifying the following exponential sum square root cancellation bound can be obtained:

Lemma. For arbitrarily large $N > 0$, there exists an N element subset S of $[0, 1]^d$ such that for any $\xi \in \mathbb{Z}^d$ with $|\xi| \lesssim N^{1/s}$

$$\left| \frac{1}{N} \sum_{x \in S} e^{2\pi i \xi \cdot x} \right| \lesssim N^{-1/2},$$

and for distinct $y_1, \dots, y_n \in S$, $|f(y_1, \dots, y_n)| \gtrsim N^{-1/s}$ (S contains no ‘near zeroes’).

Let us illustrate how the problem becomes harder as we increase s , i.e. we try and construct larger Salem sets. To do this, pick $10N$ points $\{x_1, \dots, x_{10N}\}$ uniformly at random from $[0, 1]^d$. There are roughly $O(N^n)$ tuples (y_1, \dots, y_n) , where each y_i is taken from the points x_i . Each tuple has probability $O(N^{-i/s})$ of forming a near zero of f , since the zero set of f is a $dn-i$ dimensional hypersurface in $(\mathbb{R}^n)^d$. Thus we expect there to be roughly $O(N^{n-i/s})$ tuples formed from the points $\{x_i\}$ which give near zeroes.

- If $s \leq i/n$, we expect no tuples will give near zeroes, so setting $S = \{x_1, \dots, x_N\}$ will satisfy the constraints of the Lemma with high probability. Easy!
- If $s > i/n$, we expect there to be tuples giving near zeroes. So we let S be the set of points from the set $\{x_i\}$ which remain after *pruning*, i.e. after removing any point x_i which equals y_n for some tuple (y_1, \dots, y_n) forming a near zero of f . If $s \leq i/(n-1)$, then we will prune at most $O(N^{n-i/s}) \ll 10N$ points, which means we can still guarantee S contains N points. For $s > i/(n-1)$, we cannot guarantee S contains any points, so $i/(n-1)$ is the limiting dimension we can expect.

For $s \leq i/n$, the selection process above is completely random, and so the square root cancellation property is almost automatic. But the pruning we must perform for $s > i/n$ is *structured*, i.e. it removes points clustered near zeroes of f , which may cause subtle problems with the Fourier dimension / square root cancellation.

Dealing With Pruning

It suffices to show the ‘pruning inequality’

$$\left| \frac{1}{N} \sum_{x_k \text{ pruned}} e^{2\pi i \xi \cdot x_k} \right| \lesssim N^{-1/2}$$

is true, since the random points exhibit square root cancellation before they are pruned. For $s \leq i/(n-1/2)$, we can guarantee $O(N^{n-i/s}) = O(N^{1/2})$ points have been pruned, so the pruning inequality follows trivially from the triangle inequality. For $s > i/(n-1/2)$ we work harder. We now assume $f(x) = x^n - g(x^1, \dots, x^{n-1})$ as in Theorem 2, so $i=d$ now. The left hand side of the pruning bound can be viewed as a very nonlinear function $F_\xi(x) = F_\xi(x_1, \dots, x_{10N})$ of the initial uniformly random points chosen. The theory of *probabilistic concentration inequalities* gives various tools guaranteeing that $|F_\xi(x) - \mathbb{E}[F_\xi(x)]|$ is bounded with high probability provided the maximum ‘influence’ of each variable x_i on F is not too large. Since we remove the points corresponding to the last coordinate of (y_1, \dots, y_n) , these points have ‘a little too much influence’, but this can be dealt with because these variables are ‘linear’ in f (because of the extra structure assumed in Theorem 2), so we obtain $|F_\xi(x) - \mathbb{E}[F_\xi(x)]| \lesssim N^{-1/2}$ with high probability for $s \leq d/(n-1)$.

Finally, we can use some inclusion-exclusion bounds to reduce the study of $\mathbb{E}[F_\xi(x)]$ to an oscillatory integral and apply non-stationary phase. But the inclusion-exclusion bounds obtained only work for $s \leq d/(n-3/4)$ (one must understand the ‘exclusion’ in more detail past this range), which is why Theorem 2 only obtains a Salem set of dimension $d/(n-3/4)$ rather than dimension $d/(n-1)$.

What's Next

Here are some problems to improve the results described in this poster:

- Can one improve the inclusion-exclusion bounds in the analysis of pruned sets to improve the dimension $d/(n-3/4)$ in Theorem 2 to $d/(n-1)$, the best possible bound we can expect purely via pruning random points?
- Is there a nontrivial concentration argument for general f as in Theorem 1?
- Can we use modern ‘square root cancellation methods’, e.g. decoupling, to construct more structured Salem sets?

Unfortunately, I could not attend the workshop this weekend. But I'd be very excited to hear from you if you have questions, or want to talk more about pattern avoidance. Contact me via email at jcdenson@wisc.edu so we can correspond / arrange a time to talk online!