Large Salem Sets Avoiding Polynomial Patterns

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Abstract

TODO

Adapting the discrete strategy of (TODO) to the continuous setting, and together with the translation dimension boosting argument of Schmerkin, we prove that there exists a Salem set $E \subset [0,1]$ such that for any distinct $x_1, x_2, x_3, x_4 \in E$, $x_1 - x_2 \neq (x_3 - x_4)^2$.

We construct E as follows. Fix a squarefree integer k, and consider a family of subsets $R_n \subset \{0, \dots, k-1\}$ for each $n \geqslant 0$. Define

$$E = \left\{ \sum_{n=1}^{\infty} a_n k^{-n} : a_{2n} \in R_n \text{ for all } n \geqslant 1 \right\}.$$

We claim that E avoids patterns if $\{R_n\}$ are chosen suitably well. Let us suppose that there exists $x_1, x_2, x_3, x_4 \in E$ such that $x_1 - x_2 = (x_3 - x_4)^2$. Write

$$x_1 = \sum_{n=1}^{\infty} a_n k^{-n}, \quad x_2 = \sum_{n=1}^{\infty} b_n k^{-n}, \quad x_3 = \sum_{n=1}^{\infty} c_n k^{-n}, \quad \text{and} \quad x_4 = \sum_{n=1}^{\infty} d_n k^{-n}.$$

Then

$$\sum_{n=1}^{\infty} (a_n - b_n) k^{-n} = \left(\sum_{n=1}^{\infty} (c_n - d_n) k^{-n} \right)^2.$$

Let i be the first index such that $a_i \neq b_i$, and let j be the first index where $c_i \neq d_i$. Then

$$\left| \sum (a_n - b_n) k^{-n} - (a_i - b_i) k^{-i} \right| \leqslant k^{-i},$$

and

$$\left| \left(\sum (c_n - d_n) k^{-n} \right)^2 - (c_j - d_j)^2 k^{-2j} \right| \leqslant (2k-1) k^{-2j}.$$

Thus

$$|(a_i - b_i) k^{-i} - (c_j - d_j)^2 k^{-2j}| \leqslant k^{-i} + (2k-1) k^{-2j}.$$

If $2j > i$, then $|(a_i - b_j) k^{2j-i} - (c_j - d_j)^2| \leqslant k^{2j-i} + (2k-1)$.

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