# Research Statement

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**My Research** focuses on problems in Harmonic analysis. In particular, I study Fourier multiplier operators on Euclidean space, and their analogues on compact manifolds, through an understanding of the geometry of wave propagation on these spaces. I also work on problems in harmonic analysis related to geometric measure theory, investigating when 'structure' occurs in fractals of large dimension. Both projects lead to interesting questions that I plan to pursue in my postgraduate work.

During my PhD, much of my work on multipliers has concentrating on relating bounds on Fourier multiplier operators on $\mathbb{R}^d$ to bounds for analogous operators on compact manifolds. I proved a 'transference principle' [2] for zonal multipliers operators on the sphere $S^d$. For $d \geq 4$ and a range of $p$, this principle shows that $L^p$ bounds for a radial Fourier multiplier with symbol $m(|\xi|)$ imply bounds for a zonal multiplier operators on $S^d$ induced by the same symbol. In the process, I also completely characterized those symbols $m$ whose dilates give a uniformly bounded family of zonal multiplier operators on $L^p(S^d)$. This is the first such characterization on $S^d$ for any $p \neq 2$, and no other such characterization, or transference principle of this form has been proved for analogous operators on any other compact manifold.

My work in geometric measure theory focuses on constructing sets of large fractal dimension avoiding certain point configurations. Together with Malabika Pramanik and Joshua Zahl, I obtained a method [3] for constructing sets avoiding configurations which have large Hausdorff dimension if the geometric configuration itself is 'low dimensional'. During my PhD, I continued this line of research by establishing several probabilistic extensions of the methods of the previous paper to address the more difficult problem of constructing sets of large *Fourier dimension* avoiding configurations [4]. This method remains the only construction method for constructing sets of large Fourier dimension avoiding nonlinear configurations, and also remains the best currently known construction method for constructing sets avoiding general 'linear' configurations when $d > 1$.

**In The Near Future**, I hope to generalize the bounds obtained in [2] to the more general setting of multipliers for eigenfunction expansions of any Laplace-Beltrami operator on a Riemannian manifold $M$ with periodic geodesic flow. A local obstruction here requires obtaining control of a pseudodifferential operator on $M$ called the 'return operator'. A global obstruction is an endpoint refinement of the local smoothing inequality for the wave equation on $M$. I am also interested in exploring what kinds of bounds for multipliers can be obtained via wave equation type techniques on manifolds whose geodesic flow has well controlled dynamical properties, such as forming an integrable system. In the study of patterns, I hope to apply the square root cancellation techniques I exploited in the construction of sets of large Fourier dimension to construct random fractals with good decoupling constants. And I am interested in determining the interrelation of patterns with the study of multipliers on manifolds, in particular studying Falconer distance type problems on manifolds by using local smoothing bounds, and exploring analogues of Fourier dimension on Riemannian manifolds.

In the remainder of this summary, I describe more formally the contributions I have made to the problems mentioned above, and finish with a further discussion of how the are feasible given the insights I have gained from work on previous problems.

# 1 Multiplier Operators On $\mathbb{R}^d$ And On Manifolds

Multipliers have been a central object in harmonic analysis since the field's inception. In his pioneering work, Fourier showed solutions to the classical equations of physics are described by Fourier multipliers, operators $T$ defined by a function $m : \mathbb{R}^d \to \mathbb{C}$, the symbol of $T$, by setting

$$Tf(x) = \int_{\mathbb{R}^d} m(|\xi|)\widehat{f}(\xi)e^{2\pi i\xi\cdot x}\, dx.$$

Of particular interest are the *radial* multipliers $T_a$, defined for a function $a : [0,\infty) \to \mathbb{C}$ as the Fourier multiplier with symbol $m(\xi) = a(|\xi|)$. Any translation-invariant operator on $\mathbb{R}^d$ is a Fourier multiplier operators, explaining their broad applicability in areas as diverse as partial differential equations, number theory, complex variables, and ergodic theory.

A similar theory of multiplier operators can be developed on the sphere $S^d$. Roughly speaking, Fourier multipliers are operators on $\mathbb{R}^d$ with $e^{2\pi i\xi\cdot x}$ as eigenfunctions. Zonal multipliers on $S^d$ are those operators with the *spherical harmonics* as eigenfunctions, i.e. the restrictions to $S^d$ of homogeneous harmonic polynomials on $\mathbb{R}^{d+1}$. Every function $f \in L^2(S^d)$ can be uniquely expanded as $\sum_{k=0}^\infty H_k f$, where $H_k f$ is a degree $k$ spherical harmonics. A *zonal multiplier* is then an operator on $S^d$ defined in terms of a function $a : \mathbb{N} \to \mathbb{C}$ by setting

$$Z_a f = \sum\nolimits_{k=0}^\infty a(k)H_k f.$$

Every rotation invariant operator on $S^d$ is a zonal multiplier, and thus such operators arise in diverse applications, including celestial mechanics, physics, and computer graphics.

In harmonic analysis, it has proved incredibly profitable to study the boundedness of Fourier multipliers with respect to the various $L^p$ norms. It seems to be one of the few tractable ways of quantifying how different types of planar waves interact with one another, thus underpinning all deeper understandings of the Fourier transform. Similarly, understanding the $L^p$ boundedness of zonal multipliers gives insight into how spherical harmonics interact with one another.

The general theory of Fourier multipliers became of central interest in the 1960s, brought on by the spur of applications the Calderon-Zygmund school and their contemporaries brought to the theory. However, completely characterizing those symbols which induce $L^p$ bounded multipliers has proved an impenetrable, if not potentially impossible problem since, aside from trivial cases where $p \in \{1, 2, \infty\}$. It thus came as a surprise in the past decade when several arguments [8, 9, 11, 10] emerged giving necessary and sufficient conditions on a symbol $a$ for the corresponding *radial* Fourier multipliers to be bounded on $L^p(\mathbb{R}^d)$. By duality, we will assume in the sequel that $1 \leq p \leq 2$. Consider a decomposition $a(\rho) = \sum a_k(\rho/2^k)$, where $a_k(\rho) = 0$ for $\rho \notin [1, 2]$. For $T_a$ to be bounded on $L^p(\mathbb{R}^d)$, testing by Schwartz functions reveals it is necessary that $\sup_j C_p(a_j) < \infty$, where

$$C_p(a) = \left( \int_0^\infty \left[ \langle t\rangle^{(d-1)(1/p-1/2)}\widehat{a}(t) \right]^p \, dt \right)^{1/p}.$$

Using Bochner-Riesz multipliers as an endpoint example, it is natural to conjecture this condition is also sufficient for $|1/p - 1/2| > 1/2d$. Heo, Nazarov, and Seeger [9] prove the condition is *sufficient* for $L^p$ boundedness for $d \geq 4$ and $|1/p - 1/2| > (d-1)^{-1}$, and Cladek [11] proves the condition is sufficient for compactly supported $a$ when $d = 4$ and $|1/p - 1/2| > 11/36$ and when $d = 3$ and $|1/p - 1/2| > 11/26$. We should also note the work of Kim [10], who extended the bounds of [9] to *quasi-radial multipliers*, i.e. Fourier multipliers with a symbol $q(\xi)$, where $q$ is a smooth, non-negative homogeneous function of order one such that $q^{-1}(1)$ has non-vanishing Gauss curvature. For input functions that are themselves radial, this conjecture has been proved [8], though resolving this conjecture for general inputs is likely far beyond current research techniques, given that it implies the Bochner-Riesz conjecture, and thus also the restriction and Kakeya conjectures.

The bound $\sup_j C_p(a_j) < \infty$ can be viewed as a condition controlling the smoothness of the functions $\{a_j\}$, but is not equivalent to the boundedness of any Sobolev or Besov norm. It is implied if the functions $\{a_j\}$ uniformly lie in the Besov space $B^{2,p}_{d(1/p-1/2)}(\mathbb{R})$, which, ignoring the logarithmic parameter $p$, says that the functions $\{a_j\}$ have $d(1/p-1/2)$ derivatives in $L^2$. One might conjecture that uniformly lying in this Besov space is sufficient to ensure $L^p$ boundedness. This conjecture is then weaker than the conjecture made in the last paragraph, and has been verified by Lee, Rogers, and Seeger [13] in the larger range $|1/p-1/2| > (d+1)^{-1}$ for $d \geq 2$.

We remark that various high powered techniques have recently been developed towards an understanding of the Bochner-Riesz conjecture, such as broad-narrow analysis, decoupling, and the polynomial method. However, these methods are difficult to apply to the two conjectures introduced above, since they are *endpoint results*. More precisely, in arguments related to the Bochner-Riesz conjecture, one allows for inequalities to have a multiplicative loss of factors of the form $R^\varepsilon$ or $\log R$, where $R$ is the frequency scale of the analysis, since the Bochner-Riesz multipliers are conjectured to be bounded on $L^p$ for an *open* interval of exponents, and so methods involving interpolation allow us to remove the multiplicative factors. But an arbitrary multiplier bounded on $L^p(\mathbb{R}^d)$ may not be bounded on $L^{p'}(\mathbb{R}^d)$ for any $p' < p$, and so such interpolation methods are not available to us. We cannot even make use of basic applications of dyadic pigeonholing in arguments related to the conjectures discussed above.

**My Main Research Goal** was to obtain analogues of the results of [9], [11], and [10] in the setting of zonal multipliers on $S^d$. Through this analogy, I proved the first transference principle from Fourier multiplier bounds to zonal multiplier bounds. Namely, for $d \geq 4$ and $|1/p-1/2| > 1/(d-1)$, if a Fourier multiplier operator $T_a$ is bounded on $L^p(\mathbb{R}^d)$, I proved that the zonal multiplier operator $Z_a$ is bounded on $L^p(S^d)$. My proof also completely characterizes those symbols whose dilates give a uniformly bounded family of zonal multiplier operators on $L^p(S^d)$. This result, in the special case when $a$ has compact support, has been submitted for publication in [2]. The method behind the general case will be submitted for publication shortly, with some of the ideas of which will be discussed in this section. In the remainder of this section, we discuss this result in more detail, emphasizing the new techniques introduced.

## Relations Between Fourier Multipliers and Zonal Multipliers

One connection which explains why analogues to the bounds for radial Fourier multipliers might be found in the study of zonal multipliers is that both classes of operators are related to the Laplace operator on their respective spaces. Namely, if $f$ is a distribution on $\mathbb{R}^d$, and $\Delta f = -\lambda^2 f$, then $f$ has Fourier support on the sphere of radius $\lambda$ centered at the origin, and so $T_a f = a(\lambda) f$. Similarly, if $\Delta$ is the Laplace-Beltrami operator on $S^d$, and $\Delta f = -\lambda(\lambda + d - 1)$, then $f$ is a spherical harmonic of degree $\lambda$, and so $Z_a f = a(\lambda) f$. Using the notation of functional calculus, we can thus write $T_a = a\big(\sqrt{-\Delta}\big)$ and $Z_a = a\big(\sqrt{\alpha^2 - \Delta}\big)$, where $\alpha = (d-1)/2$, and now the resemblance is clear. In the rest of this section, we let $P = \sqrt{\alpha^2 - \Delta}$, and let $a(P/R)$ denote the zonal multiplier with symbol $a(\cdot/R)$.

On the other hand, unlike the planar waves $e^{2\pi i \xi \cdot x}$, it is difficult to understand what a general spherical harmonic might look. Dilation symmetries on $\mathbb{R}^d$ tell us high frequency planar waves are just the dilates of low frequency planar waves; on the other hand, $S^d$ has no dilation symmetries, and high degree spherical harmonics need not look anything like low degree spherical harmonics. This could be alarming, because the transference principle we hope to prove implies a result related to dilation on $S^d$; namely, if the Fourier multiplier $T_a$ is bounded on $L^p(\mathbb{R}^d)$, then the Fourier multipliers $T_{a,R}$ with symbols $a(\cdot/R)$ are all uniformly bounded on $L^p(\mathbb{R}^d)$, and thus the transference principle we hope to prove shows that the zonal multipliers $a(P/R)$ are uniformly bounded on $L^p(S^d)$. Because of the lack of dilation symmetry on $S^d$, the behavior of the operators $a(P/R)$ might change as $R$ varies, which is discouraging.

Fortunately, whatever differences the operators $a(P/R)$ have as $R$ varies are not as relevant to the study of $L^p$ boundedness as one might at first think. This is because zonal multipliers only fail to be bounded on $L^p(S^d)$ because of 'high frequency behavior'; a zonal multiplier whose symbol is compactly supported is bounded on all of the $L^p$ spaces. And there are various heuristics that tell us that the Laplacian on $S^d$ begins to behave more and more similar to the Laplacian on $\mathbb{R}^d$ when restricted to high frequency eigenfunctions. For instance, on $S^d$, the operator $P/R$ can be written as $\sqrt{\alpha^2/R + \Delta_R}$, where $\Delta_R$ is the Laplacian associated with the metric $g_R = R^2 g$ on $S^d$. As $R \to \infty$, the metric $g_R$ gives $S^d$ less curvature and more volume, and so we might imagine that, as $R \to \infty$, the operator $P/R$ behaves more and more like $\sqrt{-\Delta}$ on $\mathbb{R}^d$, and thus multipliers of $P/R$ behave more and more like radial Fourier multipliers, i.e. we might imagine that the equation $T_a = \lim_{R\to\infty} a(P/R)$ holds in a certain heuristic sense.

The last equation also leads us to believe that if the operators $a(P/R)$ are uniformly bounded on $L^p(S^d)$, then $T_a$ is bounded on $L^p(\mathbb{R}^d)$. This is true for all $1 \leq p \leq \infty$, a classical transference result of Mitjagin [12]. On the other hand, the transference principle we prove is more unusual: the boundedness of a limit point need not necessitate uniform bounds on the limiting sequence. This might explain why the principle is more difficult to prove. Indeed, Mitjagin's argument Mitjagin's argument generalizes to show that for any compact manifold $M$, and any elliptic, self-adjoint pseudodifferential operator $P$, the uniform boundedness of the multiplier operators $a(P/R)$ on $L^p(M)$ implies the boundedness of the Fourier multiplier $T$ on $L^p(\mathbb{R}^d)$ whose symbol is given by the principal symbol of $P$. Until our new transference principle, no analogous transference principle has been shown for any $P$ and any exponent $p \neq 2$, excluding trivial cases, nor any characterization of functions $a$ such that the operators $\{a(P/R)\}$ are uniformly bounded on $L^p(M)$ for any $p \neq 2$.

### Bounding Band Limited Zonal Multipliers

To discuss the techniques I developed which lead to the aforementioned characterization, let us begin by summarizing the rough methodology by which the arguments in [9, 11, 10] are able to obtain these bounds. We begin by describing the *band limited* part of the argument:

> *Take a decomposition $T_a = \sum_{j \in \mathbb{Z}} T_j$, where $T_j$ is the multiplier operator with symbol $a_j(\cdot/2^j)$. Our goal is the band limited bound $\|T_j f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$, uniformly in $j$ We then write $T_j f = k * f$, where $k$ is the Fourier transform of $a_j(\cdot/2^j)$. We then write $k = \sum k_\tau$ and $f = \sum f_\theta$, where the functions $\{k_\tau\}$ are supported on disjoint thickness $2^{-j}$ annuli, and the functions $\{f_\theta\}$ are supported on disjoint side length $2^{-k}$ cubes, then $T_j f = \sum_{\tau,\theta} k_\tau * f_\theta$. Using the fact that the Fourier transform is unitary, that the function $f$ is band limited, and some Bessel function estimates, one can argue that the inner product $\langle k_\tau * f_\theta, k_{\tau'} * f_{\theta'} \rangle$ is negligible unless the annulus of radius $\tau$ centered at $\theta$ is near tangent to the annulus of radius $\tau'$ centered at $\theta'$.*
>
> *Using the inner product estimate, together with an argument for counting incidences, one can show that the $L^2$ norm of a sum $\sum_{(\tau,\theta)\in\mathcal{E}} k_\tau * f_\theta$ is well behaved if $\mathcal{E}$ is suitably 'sparse', and interpolation with a trivial $L^1$ estimate yields an $L^p$ estimate on the sum. Conversely, if the set $\mathcal{E}$ is clustered, then $\sum_{(\tau,\theta)\in\mathcal{E}} k_\tau * f_\theta$ will be concentrated on only a few annuli, and so we can also get good $L^p$ estimates. But then we can estimate $\|Tf\|_{L^p(\mathbb{R}^d)} = \|\sum k_\tau * f_\theta\|_{L^p(\mathbb{R}^d)}$ by either approach, depending on whether a sparse part of the sum dominates, or whether a clustered part of the sum dominates.*

My paper [2] proves an analogue of this argument for zonal multiplier operators on $S^d$ for $d \geq 4$ and $|1/p - 1/2| > 1/(d-1)$, in particular, obtaining the aforementioned transference principle in this range.

Giving that the argument above exploits convolution on $\mathbb{R}^d$, one might expect that we might use 'zonal convolution' in our argument, i.e. an analogue of convolution on $S^d$. It is likely one can use this approach to obtain $L^p$ bounds under assumptions on the integrability of the zonal convolution kernel. However, the lack of a dilation symmetry on $S^d$ means that the zonal convolution kernel for $a(P/R)$ is likely unrelated to the convolution kernel for $a(P)$ as $R$ varies, so based on our previous discussion it is likely difficult to obtain a transference principle using this technique. Instead, we follow an approach due to Hörmander BLAH, successfully used in several other problems on manifolds BLAH, and use the Fourier inversion formula to write

$$Z_j f = \int_{-\infty}^{\infty} 2^j \widehat{a}_j(2^j t) e^{2\pi i t P} f \, dt,$$

where $P = \sqrt{\alpha^2 - \Delta}$ is as above, and $e^{2\pi i t P}$ are the wave propagator operators which, as $t$ varies, give solutions to the wave equation $\partial_t^2 = \Delta - \alpha^2$ with zero velocity initial conditions, or equivalently, solutions to $\partial_t = P$, the 'half-wave equation'. Given a general input $f$, we perform a decomposition analogous to the method above, writing $f = \sum f_\theta$ and $T_j = \sum T_\tau$, where the functions $\{f_\theta\}$ are supported on disjoint sets of diameter $2^{-k}$, and $T_\tau = \int b_\tau(t) e^{2\pi i t P} \, dt$, where $2^j \widehat{a}_j(2^j t) = \sum_\tau b_\tau(t)$ for a family of functions $\{b_\tau\}$ is supported on disjoint side length $2^{-j}$ intervals. We can thus write $T_j f = \sum T_\tau f_\theta$.

The behavior of the wave equation is closely tied to the behavior of geodesics on $S^d$. In particular, for high frequency inputs we have a near explicit representation of the wave propagator operators for $|t| < 1/2$, in a coordinate system, by an oscillatory integral

$$(e^{2\pi i t P} f)(x) \approx \int_{\mathbb{R}^d} a(t, x, y, \xi) e^{2\pi i [\phi(x,y,\xi) + t|\xi|_y]} f(y) \, d\xi \, dy,$$

where $a$ is a symbol of order 0, $|\xi|_y = (\sum g^{jk}(y) \xi_j \xi_k)^{1/2}$ is obtained from the Riemannian metric of $S^d$, and $\phi$ solves the eikonal equation $|(\nabla_x \phi)(x, y, \xi)|_x = |\xi|_y$ subject to the constraint that $\phi(x, y, \xi) = 0$ for $(x - y) \cdot \xi = 0$.

Oscillatory integral representations of the operators $\{e^{2\pi i t P}\}$ *can* be obtained for $e^{2\pi i t P}$ simply by the fact they form a semigroup, and so for $n - 1 \leq t < n$ we can write $e^{2\pi i t P} = (e^{2\pi i (t/n) P})^n$ as the composition of $n$ oscillatory integrals. The theory of phase reduction for Fourier integral operators can help us reduce this composition, but it is difficult to control this quantity quantitatively.

simply by the fact that they are obtained by repeated compositions of the propogators with $|t| < 1/2$, and qualitative understanding of their behavior can be understand from the general theory of *Fourier integral operators*, but obtaining good control on $e^{2\pi i t P}$ for large $t$ becomes difficult.

Using this oscillatory integral representation and the stationary phase formula, we can obtain a substitute for the Bessel estimates used in the original argument, justifying that $\langle T_\tau f_\theta, T_{\tau'} f_{\theta'} \rangle$ is negligible unless the geodesic annulus of radius $\tau$ and center $\theta$ is near tangent to the geodesic annulus of radius $\tau'$ and center $\theta'$, provided that $\tau$ is bounded away from 1. Here we use the stationary phase formula to obtain good bounds on the oscillatory integrals that emerge. However, one subtlety is showing that, restricted to values in $|\xi| = 1$, the critical points of the function $\phi(x, y, \cdot)$ are appropriately non-degenerate. Using the Hamilton-Jacobi approach to the study of the eikonal equation, one can identify the quantity $\phi(x, y, \xi)$ with the signed distance from the hyperplane $\{x' : (x' - y) \cdot \xi = 0\}$ to the point $x$ with respect to the Riemannian metric. I came up with a geometric argument involving the second variation formula for geodesics, which justifies that the function $\phi$ has only two stationary points, and each is non-degenerate, with the Hessian at each point having magnitude proportional to $d_g(x, y)^{d-1}$.

After this, the argument for radial Fourier multipliers generalizes quite directly to the case of $S^d$, since the incidence properties required for annuli on $\mathbb{R}^d$ are roughly analogous to the properties of geodesic annuli on $S^d$.

### Combining Dyadic Pieces With Atomic Decompositions

- Next, we consider a decomposition $f = \sum f_k$, where the Fourier transform of $f_k$ is supported on $|\xi| \sim 2^k$, then we can write $Tf = \sum T_k f_k$. Bounds on $T_k$ have been controlled by the previous argument, and we are now left with the job of 'recombining scales'. To obtain this bound, we consider a decomposition of each of the functions $f_k$ into '$L^\infty$ atoms'. Morally speaking, we are able to write $f_k = \sum A_{k,\theta}$, where $\theta$ runs over a family of dyadic boxes, each with side length exceeding $2^{-k}$, which morally we should think of as disjoint, and where $A_{k,\theta}$ is a 'molecule', which decomposes as $A_{k,\theta} = \sum a_{k,\theta,j}$, where $a_{k,\theta,j}$ is an '$L^\infty$ atom' on $\theta$, in the sense that $\|a_{k,j,\theta}\|_{L^\infty(\mathbb{R}^d)} \lesssim |\theta|^{-1/p} \|a_{k,j,\theta}\|_{L^p(\mathbb{R}^d)}$ and satisfy a square function estimate, which reduces proving the bound $\|Tf\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$ to proving a bound of the form

$$\left\| \sum u_{k,j,\theta} \right\| \lesssim \left( \sum\nolimits_j \left\| \left( \sum\nolimits_{k,\theta} |a_{k,j,\theta}|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)}^p \right)^{1/p},$$

  where $u_{k,j,\theta} = T_k a_{k,j,\theta}$. Unlike the previous argument, we thus have to deal with the interaction between different frequency scales, but here we do have an additional square root cancellation to help obtain the bound.

The argument I obtained also follows this pattern, but we must introduce several new techniques when adapting the method the method to the study of spherical harmonics.

The uniformity in $k$ actually follows immediately if we can prove the bound for $k = 0$ because of the dilation symmetry on $\mathbb{R}^d$, and the fact that we have uniform control over the functions $\{a_k\}$. In the analysis of the Fourier multiplier $T_0$, the support of the symbol implies

The bound is then obtained by a geometric argument involving incidences of annuli. Once this is obtained, a square function bound implies control over $\sum T_k$.

First off, their assumptions are about uniform control over dyadic pieces of the symbol $a$. More precisely, if $a$ is dyadically decomposed as a sum $a(\rho) = \sum_{k \in \mathbb{Z}} a_k(\rho/2^k)$, where $a_k$ has support on $[1, 2]$, then the necessary and sufficient condition for boundedness is that the quantities $C_p(a_k)$ are uniformly bounded in $k$, where

First, a general symbol $a$ is dyadically decomposed as a sum $a(\rho) = \sum_{k \in \mathbb{Z}} b_k(\rho/2^k)$, where each of the functions $b_k$ is supported on the interval $[1, 2]$. If we set $a_k(\rho) = a_k(2^k \rho)$, then $T_a = \sum T_{a_k}$.n

The proofs begin by establishing bounds of the form $\|T_{a_k} f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$, uniformly in $k$. By applying a dilation symmetry, it suffices without loss of generality to look at $a_0$.

Now how

Suppose $a : [0, \infty) \to \mathbb{C}$ is compactly supported on the interval $[1/2, 2]$. Then the radial function $k(x) = \int a(|\xi|) e^{2\pi i \xi \cdot x}\, d\xi$ is the a convolution kernel for the radial multiplier operator $T_a$, i.e. $T_a f = k * f$ for all inputs $f$. The bounds on radial multipliers obtained in BLAH are then of the form $\|T_a f\|_{L^p(\mathbb{R}^d)} \lesssim \|k\|_{L^p(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}$. Dilation symmetry then immediately implies BLAH DYADIC RESULT, and then some methods of atomic decompositions and Littlewood Paley theory can be combined to obtain a tight result. On a sphere, we *can* define the zonal convolution kernel $k$ corresponding to a zonal multiplier $Z_a$, such that

$$(Z_a f)(x) = \int_{S^d} k(y \cdot x) f(y)\, dy.$$

If $a$ is supported on $[1/2, 2]$, then weighted $L^p$ bounds on $k$ can be used to imply bounds on $Z_a$, but these bounds will not scale, and it is difficult to determine how the bounds scale under dilations since there is no relation between the zonal convolution kernel $k$ corresponding to $a$, and the convolution kernel corresponding to the dilations of $a$.

A fix is obtained by writing $Z_a f$ using the *cosine transform* of $a$, i.e. in terms of

$$\widehat{a}(t) = \int_0^\infty a(\rho) e^{2\pi i \rho t}\, d\rho.$$

The cosine transform *does* scale under dilations, and functional calculus and the Fourier inversion formula allows us to write $Z_a$ in terms of $\widehat{a}$, i.e. by setting

$$Z_a f = a(P) f = \int_{-\infty}^{\infty} \widehat{a}(t) e^{2\pi i t P}\, dt, \tag{1}$$

where $P = \sqrt{\alpha^2 - \Delta}$, and $e^{2\pi i t P} f = e^{2\pi i t k} f$ for a spherical harmonic $f$ of degree $k$. The operators $u(t) = e^{2\pi i t P} f$ give solutions to the half-wave equation $\partial_t u = P u$. We can understand the geometric behaviour of the half-wave equation by using the theory of *Fourier integral operators*.

. Indeed, if $f$ is a spherical harmonic of degree $k$, then $Z_a f = a(k) f$ and by the Fourier inversion formula $\int \widehat{a}(t) e^{2\pi i t P} f\ dt = \int \widehat{a}(t) e^{2\pi i t k} f\ dt = a(k) f(t)$. For any function $f$, the functions $u(t) = e^{2\pi i t P} f$ give a solution to the wave equation $\partial_t^2 u(t) = P u(t)$ with $u(0) = f$ and $\partial_t u(0) = 0$.

The bounds on radial multipliers obtained in BLAH depend on bounds on the convolution kernel corresponding to the multiplier.

The main goal of my research project on multipliers is to understand

deconstructive interference between a family of planar waves, or spherical harmonics of different degrees. Necessary and sufficient conditions for a Fourier multiplier operator to be bounded on $L^1(\mathbb{R}^d)$ or $L^\infty(\mathbb{R}^d)$ were quickly realized.

## 2 Pattern Avoidance

How large must a set be before it must contain a certain point configuration? Problems of this flavor have long been studied in various areas of combinatorics. In the last 50 years, analysts have also begun studying analogous problems for infinite subsets $X \subset \mathbb{R}^d$, where the size of $X$ is measured in terms of a suitable *fractal dimension*, often *Hausdorff dimension*, but also sometimes *Fourier dimension*, the latter of which tending to imply more structure than the former.

Several definite conjectures on problems about the *density* of certain point configurations in sets have been raised, such as the Falconer distance problem. But there are relatively few definite conjectures about the dimension a set requires before it must contain *at least one* family of points fitting a certain kind of configuration. For instance, we do not know for $d > 2$ how large the Hausdorff dimension a set $X \subset \mathbb{R}^d$ must be before it contains all three vertices of an isosceles triangle, the threshold being somewhere between $d/2$ and $d - 1$.

It is not clear

, for instance, how large the Hausdorff dimension a set $X \subset \mathbb{R}^d$ must have before it contains the vertices of at least one isosceles triangle, or, for a particular angle $\theta \in [0, \pi]$, how large $X$ must be before it contains three points $A$, $B$, and $C$ which when connected form an angle $\theta$; the only case here that is fully resolved is when $\theta \in \{0, \pi\}$, or when $\theta = \pi/2$ and $d$ is even: when $\theta = 0$ and $\theta = \pi$, the threshold is $d - 1$, when $\theta = \pi/2$, the threshold is somewhere between $d/2$ and $\lceil d/2 \rceil$, for rational $\theta$ the threshold is somewhere between $d/4$ and $d - 1$, and when $\theta$ is irrational the threshold is somewhere between $d/8$ and $d - 1$.

Until recently, certain results [5] seemed to indicate that subsets of $[0, 1]$ of Fourier dimension one must necessarily contain an arithmetic progression of length three, but this has proved not to be the case [6].

The ability to form definite conjectures depends on the ability to produce counterexamples for certain problems. In this case, counterexamples take the form of constructing sets with large fractal dimension that *do not* contain certain point configurations. My research in geometric measure theory has so far focused on this type of problem.

During my MSc, my advisors and I found a construction that produces sets $X$ with large Hausdorff dimension that avoid a particular configuration, given that the particular configuration is 'small' [3]. More precisely, let us suppose we are looking at configurations of $k$ points in $\mathbb{R}^d$. The set of all tuples of points that fit a given configuration can be identified with a subset $C$ of $(\mathbb{R}^d)^k$.

Provided that the Minkowski dimension of $C$ is at most $\beta$, we constructed a set $X$ with Hausdorff dimension $(dk - \beta)/(k - 1)$ such that if $x_1, \dots, x_k$ are distinct points in $X$, then $(x_1, \dots, x_k) \notin C$. In particular, for a Lipschitz function $f : (\mathbb{R}^d)^k \to \mathbb{R}^d$, we construct a set $X$ with Hausdorff dimension $d/k$ such that for distinct $x_0, \dots, x_k \in X$, $x_0 \neq f(x_1, \dots, x_k)$, recovering the main result of [7].

During my PhD, I decided to investigate whether

. My research in geometric measure theory so far has been on trying to produce such counterexamples. In BLAH, Pramanik and Fraser. In BLAH, I rephrased their argument in probabilistic terms

of a set must be before the set of all distances

Several definite conjectures on problems of this kind have been established since the project begun, such as the Falconer distance problem or Kakeya conjecture, where the point configuration in mind are points lying at a certain distance from one another, or line segments pointing in other directions. For other

The natural fractal dimension used to measure the size of a set $X$ is often the Hausdorff dimension $\dim_{\mathbb{H}}(X)$ of $X$. But sometimes the *Fourier dimension* $\dim_{\mathbb{F}}(X)$ proves useful, which measures the best possible decay that the Fourier transform of measures supported on $X$ can have; if $\alpha < \dim_{\mathbb{F}}(X)$, then there exists a nonzero measure $\mu$ on $X$ such that $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-\alpha}$ for all $\xi \in \mathbb{R}^d$. The Fourier dimension thus, morally speaking, measures how uncorrelated the set $X$ is with the Fourier characters $e_\xi(x) = e^{2\pi i \xi \cdot x}$.

## 3 Future Lines of Research

The work I have conducted naturally suggests several **future problems**.

- Analyzing the 'return time operator' to extend results on expansions of spherical harmonics to the study of the Laplace-Beltrami operator on $S^d$.
- Determining whether our methods extend to other manifolds whose geodesic flow is simpler to understand, such as integrable systems.
- Analyzing whether local smoothing bounds
- Constructing Random Salem Sets which satisfy a Decoupling Bound.
- Determining the relation between certain 'fractal weighted estimates' for the wave equation on $\mathbb{R}^d$ and the 'density decomposition' of multiplier bounds.

In fact, this resemblance opens up a whole new world of families of operators. Given an arbitrary elliptic self-adjoint first order classical pseudo-differential operator $P$

This method is highly robust and depends very little that we are working on the sphere; pretty much the only property we end up using is that the wave equation $\partial_t u = Pu$ has *periodic solutions.*

The natural analogue of the study of radial multipliers on $\mathbb{R}^d$ is the study of multipliers of a Laplace-Beltrami operator on a Riemannian manifold. The natural analogue of the study of quasiradial multipliers on $\mathbb{R}^d$ is the study of multipliers of an operator associated with a *Finsler geometry* on the manifold.

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