Characterizations of L^p Bounded Spectral Multipliers of the Laplacian on High-Dimensional Manifolds with Periodic Geodesic Flow

Jacob Denson*

November 6, 2023

Abstract

It is a well known result of Mitjagin that on a compact Riemannian manifold M, for a function $m : (0,\infty) \to \mathbb{C}$, if the spectral multiplier operators $m(\sqrt{-\Delta}/R)$ are uniformly bounded on $L^p(M)$ for $R > 0$, then the radial function $m(|\cdot|) : \mathbb{R}^d \to \mathbb{C}$ induces a bounded Fourier multiplier operator on $L^p(\mathbb{R}^d)$. In this paper, we prove the converse for manifolds in which the geodesic flow is periodic, of dimension $d \geqslant 4$ and for $(d-1)^{-1} \leqslant |1/p - 1/2| \leqslant 1/2$. In particular, this covers the case where $M = S^d$ is the sphere, in which case the spectral multipliers can be viewed as zonal convolution operators, as well as for other rank one compact symmetric spaces. In the process of proving this theorem, we find an effective characterization of the functions m for which the operators $m(\sqrt{-\Delta}/R)$ are uniformly bounded in $L^p(M)$ for the same range of p and d, which can be viewed as a variable-coefficient analogoue of the results of Heo, Nasarov and Seeger.

Let M be a compact Riemannian manifold of dimension d, let Δ be it's Laplace-Beltrami operator, and let $P = \sqrt{-\Delta}$. For any bounded function $m : (0,\infty) \to \mathbb{C}$, we can define a spectral multiplier operator $m(P)$. In this paper, we study the relation between the L^p boundedness of the dilated multipliers $m_R(P)$ for $R > 0$, where $m_R(\lambda) = m(\lambda/R)$, and the L^p boundedness of the radial Fourier multiplier operator T_m on $\mathbb{R}^d$ defined by setting

$$T_m f(x) = \int_{\mathbb{R}^d} m(|\xi|) \widehat{f}(\xi) e^{2\pi i \xi \cdot x}\, d\xi \qquad \text{for } f : \mathbb{R}^d \to \mathbb{C}.$$

In particular, we show that for any $p \in [1,\infty]$, if d is sufficiently large depending on p, then the L^p boundedness of T_m is equivalent to the uniform L^p boundedness of the operators $m_R(P)$, provided that the geodesic flow on M is periodic.

Theorem 1. *Suppose M is a compact Riemannian manifold, and the geodesic flow on M is periodic. If $d \geqslant 4$, and $1/(d-1) \leqslant |1/p - 1/2| \leqslant 1/2$, then*

$$\|T_m\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} \sim \sup_R \|m_R(P)\|_{L^p(M) \to L^p(M)},$$

where the implicit constants depend only on p and the manifold M.

*University of Madison Wisconsin, Madison, WI, jcdenson@wisc.edu

Let us describe the problem more precisely. Spectral theory guarantees the existence of a discrete set $\Lambda_M \subset [0,\infty)$, and an orthogonal decomposition $L^2(M) = \bigoplus_\lambda \mathcal{V}_\lambda$, where $\mathcal{V}_\lambda$ is a finite dimensional subspace of $C^\infty(M)$, such that $Pf = \lambda f$ for all $f \in \mathcal{V}_\lambda$. For any bounded function $m : (0,\infty) \to \mathbb{C}$, we can define a spectral multiplier operator $m(P)$ by setting

$$m(P) = \sum_{\lambda \in \Lambda_M} m(\lambda)\, \mathcal{P}_\lambda,$$

where $\mathcal{P}_\lambda$ is the orthogonal projection operator onto $\mathcal{V}_\lambda$. Our goal is to relate the L^p boundedness of the operators

$$m_R(P) = \sum_{\lambda \in \Lambda_M} m(\lambda/R)\, \mathcal{P}_\lambda,$$

and the L^p boundedness of the Fourier multiplier operator T_m defined above.

The implication

$$\|T_m\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} \lesssim \sup_R \|m_R(P)\|_{L^p(M)\to L^p(M)} \tag{1}$$

is a result of Mitjagin [5], and is known to hold for the larger range $1 \leqslant p \leqslant \infty$ and for general compact manifolds M. The result is natural, because as $R \to \infty$, the operators $m_R(P) = m(P_R)$, where $P_R = P/R$ can be viewed as $\sqrt{-\Delta_R}$, where Δ_R is the Laplace-Beltrami operator associated with the dilated metric $g_R = R^{-2} g$. As $R \to \infty$, the metric g_R gives the manifold M less curvature and more volume, and as $R \to \infty$ we might therefore naturally expect M equipped with Δ_R to behave like $\mathbb{R}^d$ equipped with the usual Laplace operator $\Delta_{\mathbb{R}^d} = \sum \partial_j^2$. Indeed, the proof of (1) in [5] essentially shows that in coordinates, $m(P_R)$ converges to $m(\Delta_{\mathbb{R}^d})$ in an appropriate sense, and since $m(\Delta_{\mathbb{R}^d})$ is just the operator T_m, equation (1) follows.

The novel result in this paper is a proof of the converse inequality to (1) under the assumption that the geodesic flow on M is periodic, i.e. a proof that

$$\sup_{R>0} \|T_R\|_{L^p(M)\to L^p(M)} \lesssim \|T_m\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} \qquad \text{if} \quad \frac{1}{d-1} \leqslant |1/p - 1/2| \leqslant \frac{1}{2}. \tag{2}$$

Equation (2) is more surprising than equation (1), since it is not obvious why L^p bounds of a spectral multiplier on a flat space should imply uniform L^p bounds for a family of spectral mulipliers on a curved space. We are not aware that any proofs of an inequality like that in (2) have yet been proved in the literature, for any manifold[1] or any value of $p \neq 2$, and it is still unclear whether one can improve (2) to a wider range of p or to a more general family of compact Riemannian manifolds. At least for values of p with $|1/p - 1/2| \leqslant 1/2d$, a very different approach is likely required than the approach given here, since it is only possible to improve the main result of [3] to this range of p. Even then, such an improvement is immensely difficult, since that result would then imply the Bochner-Riesz conjecture in it's full range, and thus the restriction and Kakeya conjecture.

The path to proving equation (2) is hinted at by the main result of [3], which states that

$$\|T_m\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} \sim C_p(m) \qquad \text{if} \quad \frac{1}{d-1} \leqslant |1/p - 1/2| \leqslant \frac{1}{2}, \tag{3}$$

[1]The only known exception to this statement is in the study of multipliers on $\mathbb{T}^d$, in which case one may use tools like the Poisson summation formula to relate multipliers on $\mathbb{T}^d$ and multipliers on $\mathbb{R}^d$

where

$$C_p(m) = \sup_{h>0} \left(\int_0^\infty \left[\langle t \rangle^{\alpha(p)} |\widehat{m}_h(t)| \right]^p \, dt \right)^{1/p}.$$

Here $\alpha(p) = (d-1)|1/p - 1/2|$, and $m_h(\lambda) = m(2^h \lambda)\chi(\lambda)$ for any fixed choice of a smooth, compactly supported function χ with support on $[1/2, 2]$ for which $\sum \chi(2^h \lambda) = 1$. Since $\sup_{R>0} C_p(m_R) \sim C_p(m)$, Theorem 1 therefore follows from the following estimate.

Proposition 2. *Suppose $1 \leqslant p \leqslant 2(d-1)/(d+1)$. Then for any function $m : (0,\infty) \to \mathbb{C}$,*

$$\|m(P)\|_{L^p(M) \to L^p(M)} \lesssim C_p(m),$$

where the implicit constant depends only on p and the manifold M.

The techniques that underlie the proof of Theorem 2 are also inspired by the proof of (3) in [3]. For instance, we will begin by studying functions m supported on a dyadic interval; for the general case, we then use variants of Littlewood-Paley and atomic decompositions of input functions to put estimates together. However, adapting these techniques to settings to the compact manifold setting requires finding more robust 'variable-coefficient' variants of such techniques. In particular, there is no tool comparable to the dilation symmetries of Euclidean space on a compact Riemannian manifold. On $\mathbb{R}^d$, we have

$$T_{m_R} = \mathrm{Dil}_{1/R} \circ T_m \circ \mathrm{Dil}_R \qquad \text{where } \mathrm{Dil}_t f(x) = f(x/t) \text{ for } t > 0.$$

This immediately gives an equivalence between the study of L^p multipliers T_m with $\mathrm{supp}(m) \subset [1/2, 2]$ and the study of L^p multipliers T_m with $\mathrm{supp}(m) \subset [R/2, 2R]$. A similar reduction is not possible on a compact manifold, and when we do a Littlewood-Paley decomposition, this will require us to consider a much more careful analysis as $R \to \infty$.

To exploit the fact that $C_p(m)$ is finite in our proof, we must relate the Fourier transform of m to the operator $m(P)$. A standard method is to apply the Fourier inversion formula to write

$$m(P/R) = \int_{-\infty}^{\infty} R\widehat{m}(Rt) e^{2\pi i t P} \, dt,$$

where

$$e^{2\pi i t P} = \sum_\lambda e^{2\pi i t \lambda} \mathcal{P}_\lambda$$

is the multiplier operator on M which, as t varies, gives solutions to the half-wave equation

$$\partial_t - iP = 0.$$

Our goal is thus to study the regularity properties of averages of the half-wave operator. We begin our proof in the next section by describing some estimates related to the half-wave equation, which we can use to reduce some of the analysis of the multipliers $m(P)$ to a geometric argument.

1 Estimates

2 BLAH

The idea behind the proof of the result of [3] is to reduce the analysis of radial multiplier operators to the study of the L^p norms of sums of the form

$$\left\| \sum_{(x_0,t_0)\in\mathbb{R}^d\times(0,\infty)} S_{x_0,t_0} \right\|_{L^p(\mathbb{R}^d)},$$

where for each $x_0 \in \mathbb{R}^d$ and $t_0 \in (0,\infty)$, the function S_{x_0,t_0} is supported on an annulus of radius t_0 and center x_0. We are able to translate this geometric argument to the variable coefficient setting, by replacing the condition that S_{x_0,t_0} is localized to an annulus of radius t_0 and center x_0, with the condition that S_{x_0,t_0} is obtained from the output of the *half-wave equation* $\partial_t + \Delta = 0$ on S^d at time t_0, with an input localized near the point $x_0 \in S^d$. For small times, this output can be well understood by approximating solutions to the half-wave equation by the *Lax-Hörmander Parametrix*, which describes the solutions in terms of certain oscillatory integrals.

using variable-coefficient generalizations of techniques that have been used to obtain the boundedness of radial Fourier multiplier operators of the form $h(|\cdot|)$. In particular, we prove that certain conditions on the Fourier transform of h, which are necessary for the operators $\{h(P/R)\}$ to be uniformly bounded on $L^p(S^d)$, are actually *sufficient* for a restricted range of exponents p. To state this condition, define

$$s_q = (d-1)\left|\frac{1}{q} - \frac{1}{2}\right| \quad \text{for } 1 \leqslant q \leqslant \infty,$$

and define

The condition that $C_p(h)$ is finite is necessary for the operators

$$\{h(P/R) : R > 0\}$$

to be uniformly bounded on $L^p(S^d)$. The main result of this paper is that, for a restricted range of p, this condition is also *sufficient*.

Theorem 3. *Suppose $1 \leqslant p < 2(d-1)/(d+1)$. Then for all functions $f \in L^p(S^d)$, for all functions h with supp$(h) \subset \{t : 1/2 \leqslant |t| \leqslant 2\}$, and for all $R > 0$,*

$$\|h(P/R)f\|_{L^p(S^d)} \lesssim C_p(h)\|f\|_{L^p(S^d)},$$

where the implicit constant depends only on d.

Remarks 4.

1. *Theorem 2 immediately implies by duality that for $(d-3)/2(d-1) < p < \infty$,*

$$\|h(P/R)\|_{L^p(S^d)\to L^p(S^d)} \lesssim \left(\int_0^\infty \left[\langle t\rangle^{s_{p'}} |\widehat{h}(t)| \right]^{p'} dt \right)^{1/p'} \quad \textit{uniformly for } R > 0,$$

where $1/p + 1/p' = 1$.

2. *By a transference principle of Mitjagin [5], the uniform boundedness of the operators* $\{h(P/R) : R > 0\}$ *on* $L^p(S^d)$ *implies that the Fourier multiplier operator on* $\mathbb{R}^d$ *with symbol* $h(|\cdot|)$ *is bounded in* $L^p(\mathbb{R}^d)$. *As discussed in [3], this can only be true if* $C_p(h) < \infty$, *so the finiteness of* $C_p(h)$ *is necessary for the operators* $\{h(P/R)\}$ *to be uniformly bounded in* R. *The boundedness of* $C_p(h)$ *has already been proved sufficient provided the inputs functions* $f \in L^p(\mathbb{R}^d)$ *are restricted to* zonal *functions [1]. The main novelty of the result of this paper is that we can extend this bound to general* $f \in L^p(\mathbb{R}^d)$.

3. *If* $1 \leqslant p < 2d/(d+1)$ *and* $C_p(h) < \infty$, *then [3] implies that the operator* $h(|\cdot|)$ *is bounded as a Fourier multiplier operator on* $L^p(\mathbb{R}^d)$ *with*

$$\|h(|\cdot|)\|_{M^p(\mathbb{R}^d)} \lesssim C_p(h).$$

Interpolation and duality (see Section 2.5.5 of [2] for more details) implies that the operator is also a Fourier multiplier operator on $L^2(\mathbb{R}^d)$, *and so we conclude that*

$$\|h\|_{L^\infty(\mathbb{R})} = \|h(|\cdot|)\|_{M^2(\mathbb{R}^d)} \lesssim C_p(h).$$

4. *The projection operators* $\{\mathcal{P}_\lambda\}$ *are each individually smoothing, though not uniformly as* $\lambda \to \infty$. *They thus individually satisfy bounds of the form* $\|\mathcal{P}_\lambda f\|_{L^p(S^d)} \lesssim_\lambda \|f\|_{L^p(M)}$ *for all* $1 \leqslant p \leqslant \infty$. *It thus follows trivially from the triangle inequality, and that there are finitely many eigenvalues for* P *in* $[0, 100]$, *that for any* $R \leqslant 100$,

$$\begin{aligned}\|M_R f\|_{L^p(S^d)} &\leqslant \sum_\lambda |h(\lambda/R)| \|\mathcal{P}_\lambda f\|_{L^p(S^d)} \\ &\leqslant \|h\|_{L^\infty(0,200)} \textstyle\sum_{\lambda \in [0,200]} \|\mathcal{P}_\lambda f\|_{L^p(S^d)} \\ &\lesssim \|h\|_{L^\infty(0,\infty)} \|f\|_{L^p(S^d)}.\end{aligned}$$

Thus, in the analysis that follows, we will always assume that $R \geqslant 100$.

5. *Because* h *is a unit scale multiplier, if we fix a smooth bump function* β *supported on* $\{1/4 \leqslant |t| \leqslant 4\}$, *and equal to one on* $\{1/2 \leqslant |t| \leqslant 2\}$, *set* $\beta_R(\lambda) = \beta(\lambda/R)$, *and set* $Q_R = \beta(P/R)$, *then*

$$h(P/R) = Q_R \circ h(P/R) \circ Q_R.$$

By including the operators $\{Q_R\}$ *in our analysis, we essentially reduce our analysis to the study to inputs and outputs lying in the range of the operators* Q_R, *which is equal to the finite dimensional subspace* V_R *of* $C^\infty(S^d)$ *spanned by eigenfunctions of* P *with eigenvalue in* $R/4 \leqslant \lambda \leqslant 4R$. *Since* P *is positive-semidefinite and self-adjoint, it is often useful to use the heuristic that an element of* V_R *should behave like a function on* $\mathbb{R}^d$ *with Fourier support on* $\{R/4 \leqslant |\xi| \leqslant 4R\}$.

A particular application of this heuristic is an analogue of Bernstein's inequality on $\mathbb{R}^d$ *(see [7], Proposition 5.1), but for functions on a Riemannian manifold lying in* V_R. *This analogue states that for* $1 < r < \infty$, *uniformly for* $R \geqslant 100$ *and* $f \in V_R$ *we have*

$$\|f\|_{L^r_s(S^d)} \lesssim_{r,s} R^s \|f\|_{L^r(S^d)}. \tag{4}$$

See Section 3.3 of [6] for a proof.

Another useful inequality follows from the fact that the family of functions $\{\beta_R\}$ *form a uniformly bounded subset of the Fréchet space* $\mathcal{S}^0$*, i.e. satisfying estimates of the form*

$$|\partial_\lambda^n \{\beta_R\}(\lambda)| \lesssim_n \langle \lambda \rangle^{-n} \quad \textit{uniformly in } R > 0.$$

It follows that the operators $\{Q_R\}$ *are pseudodifferential operators of order zero, uniformly bounded as operators on* $L^r_s(M)$ *for all* $1 < r < \infty$*, i.e. satisfying*

$$\|Q_R f\|_{L^r_s(S^d)} \lesssim_{r,s} \|f\|_{L^r_s(S^d)}. \tag{5}$$

See Corollary 4.3.2 of [6] for more details.

Consider a cover

$$\{|t| < 100/R\} \cup \{50/R < |t| < 1/100\} \cup \{1/200 < |t| < \infty\}$$

of $\mathbb{R}$, and find a smooth partition of unity $\chi_{I,R}$, $\chi_{II,R}$, and $\chi_{III,R}$ adapted to these sets. Without loss of generality, we may assume all three functions are even, that $\chi_{I,R}(t) = \chi_I(Rt)$, for some smooth, compactly supported function χ_I adapted to the open set $\{|t| < 1\}$, and also assume that $\chi_{III,R} = \chi_{III}$ is independent of R. Given this partition of unity, we now write

$$h(P/R) = I_R + II_R + III_R,$$

where, for $\Pi \in \{I, II, III\}$, the operators

$$\Pi_R = \int \chi_{\Pi,R}(t) R\widehat{h}(Rt) e^{2\pi i t P}\, dt$$

isolate the study of $h(P/R)$ to the behaviour of the half-wave propogators on three different time intervals. The remainder of the argument will consist of separately obtaining L^p boundedness for each of the three operators $Q_R \circ \Pi_R \circ Q_R$, since then the triangle inequality gives the L^p boundedness of

$$(Q_R \circ I_R \circ Q_R) + (Q_R \circ II_R \circ Q_R) + (Q_R \circ II_R \circ Q_R) = h(P/R).$$

The study of the operators $\{I_R\}$ will reduce to a study of pseudodifferential operators, we will be able to apply the endpoint local smoothing inequality of [4] to control the operators $\{III_R\}$, and the study of the operators $\{II_R\}$ will be given by generalizations of the methods of [3] to a variable coefficient setting.

3 Analysis of I_R

Let us analyze

$$I_R = \int \chi_I(Rt) R\widehat{h}(Rt) e^{2\pi i t P}\, dt.$$

We are analyzing inputs to I_R coming from the composition of a general element of $C^\infty(S^d)$ with Q_R, which heuristically localizes the 'frequency support' of this function to a band of frequencies with magnitude $\sim R$. Thus, by uncertainty principle heuristics, such functions are locally constant at a scale $1/R$. The half-wave equation propogates a majority of the mass of it's input at a unit speed, and since the operators $\{I_R\}$ are obtained by averaging the half-wave equation over times $\lesssim 1/R$, we should expect that the behaviour of the operators $\{I_R\}$ to behave in a localized manner. In fact, the following analysis will show that the operators $\{I_R\}$ are pseudodifferential operators, which will allow us to conclude these operators are uniformly bounded in $L^p(S^d)$.

Lemma 5. *For all $f \in C^\infty(S^d)$,*

$$\|I_R f\|_{L^p(S^d)} \lesssim \|h\|_{L^\infty(\mathbb{R})} \|f\|_{L^p(S^d)} \lesssim C_p(h) \|f\|_{L^p(S^d)},$$

where the implicit constant is uniformly bounded in $R \geqslant 1$ and h, for $1 < p < \infty$. Thus in particular,

$$\|(Q_R \circ I_R \circ Q_R) f\|_{L^p(S^d)} \lesssim C_p(h) \|f\|_{L^p(S^d)}.$$

Proof. Let a be the inverse Fourier transform of the function $t \mapsto \chi_I(t)\widehat{h}(t)$. Then $I_R = a(P/R)$. If ψ denotes the inverse Fourier transform of χ_I, then we can write

$$a(\lambda) = \int h(\alpha)\psi(\lambda - \alpha)\, d\alpha.$$

The fact that h is a unit scale multiplier, and ψ is Schwartz, implies that

$$|\partial^\alpha a(\lambda)| \lesssim_{\alpha,N} \|h\|_{L^\infty(\mathbb{R})} \langle \lambda \rangle^{-N}.$$

If we set $a_R(\lambda) = a(\lambda/R)$, then

$$|\partial^\alpha a_R(\lambda)| \lesssim_\alpha \|h\|_{L^\infty(\mathbb{R})} \langle \lambda \rangle^{-|\alpha|},$$

with an implicit constant independent of R for $R \geqslant 1$. Thus the family of symbols $\{a_R : R \geqslant 1\}$ form a uniformly bounded subset of the Frećhet space $\mathcal{S}^0(\mathbb{R})$ of order zero symbols, and so the operators I_R are pseudodifferential operators of order zero, and uniformly bounded in the $L^p(S^d)$ norm for all $1 < p < \infty$, which yields the required claim. □

4 Analysis of III_R

We now show the uniform boundedness of the operators $\{III_R\}$ on $L^p(S^d)$ in the range of p we are considering in this problem, by a reduction to an endpoint local smoothing inequality. This might seem unintuitive, since the operators III_R are obtained by averaging the wave equation over large times $|t| \gtrsim 1$, whereas local smoothing gives bounds for averages of the wave equation over times $|t| \lesssim 1$. We are able to reduce large times to small times by exploiting the *periodicity* of the half-wave equation on the sphere.

Lemma 6. *Fix $1 < p < 2d/(d+1)$, let q be the Hölder conjugate to p, and let $I = [-1/2, 1/2]$. Suppose that the sharp local smoothing inequality*

$$\left\|e^{2\pi i t P} f\right\|_{L^q(S^d) L^q_t(I)} \lesssim \|f\|_{L^q_{s_q - 1/q}(S^d)}$$

holds for all $f \in C^\infty(S^d)$. Then the operators $\{III_R\}$ satisfy a bound

$$\|(III_R \circ Q_R)f\|_{L^p(S^d)} \lesssim C_p(h)\|f\|_{L^p(S^d)},$$

with the implicit constant uniformly bounded in R. In particular,

$$\|(Q_R \circ III_R \circ Q_R)f\|_{L^p(S^d)} \lesssim C_p(h)\|f\|_{L^p(S^d)},$$

Proof. For each R, the *class* of operators of the form $\{III_R\}$ formed from a multiplier h satisfying the hypothesis of Theorem 3 is closed under taking adjoints. Indeed, if III_R is obtained from h, then III_R^* is obtained from the multiplier $\overline{h}$. Because of this self-adjointness, if we can prove that for any multiplier h satisfying the assumptions of the theorem, the operators $\{III_R\}$ are uniformly bounded in $L^q(S^d)$, where q is the Hölder conjugate to p, then it follows by duality that for any such h, it is also true that the operators $\{III_R\}$ are uniformly bounded back in the original $L^p(S^d)$ norm. In this argument we will prove such L^q estimates, because we will exploit *local smoothing* inequalities, which tend to work better with large Lebesgue exponents, precisely because Lebesgue norms with large exponents are more sensitive to functions with sharp peaks, something explicitly prevented by obtaining control over the smoothness of a function.

We begin by noting that for a pair of Hölder conjugates p and q, $s_q = s_p$. Using the periodicity of the wave equation on S^d, i.e. that

$$e^{2\pi i(t+n)P} = e^{2\pi itP} \qquad \text{for } n \in \mathbb{Z} \text{ and } t \in \mathbb{R},$$

we can write

$$III_R = \int_{-1/2}^{1/2} H_R(t)e^{2\pi itP}\,dt,$$

where

$$H_R(t) = \sum_l \chi_{III}(t)R\widehat{h}(R(t+l)) = \sum_l H_{R,l}(t).$$

Now

$$\begin{aligned}
&\left(\sum_{l\neq 0}\left(|Rl|^{s_q}\|H_{R,l}\|_{L^p[-1/2,1/2]}\right)^p\right)^{1/p}\\
&\sim R\left(\int_{-1/2}^{1/2}\sum_l\left(|R(t+l)|^{s_q}|\widehat{h}(R(t+l))|\right)^p\,dt\right)^{1/p}\\
&\sim R\left(\int_{|t|\geqslant 1/2}\left(|Rt|^{s_q}|\widehat{h}(Rt)|\right)^p\,dt\right)^{1/p}\\
&\lesssim R^{1/q}C_p(h).
\end{aligned}$$

and

$$
\begin{aligned}
\|H_{R,0}\|_{L^p[-1/2,1/2]} &= \left(\int_{-1/2}^{1/2} |\chi_{III}(t) R\widehat{h}(Rt)|^p \right)^{1/p} \\
&\leqslant \left(\int_{1/200 \leqslant |t| \leqslant 1/2} |R\widehat{h}(Rt)|^p \right)^{1/p} \\
&= R^{1/q} \left(\int_{R/3}^{R/2} |\widehat{h}(t)|^p \right)^{1/p} \\
&\lesssim R^{1/q - s_q} C_p(h).
\end{aligned}
$$

Since the family of functions $\{H_{R,l}\}$ could in general be chosen arbitrarily, they can be quite correlated, and so we should expect Hölder's inequality should be efficient, in the worst case. Thus we conclude that, provided $p < 2d/(d+1)$, so that $q > 2d/(d-1)$, and thus

$$
qs_q = (d-1)(q/2-1) > 1,
$$

so we can use Hölder's inequality to conclude that

$$
\begin{aligned}
\|H_R\|_{L^p[-1/2,1/2]} &\leqslant \sum_l \|H_{R,l}\|_{L^p[-1/2,1/2]} \\
&= \|H_{R,0}\|_{L^p[-1/2,1/2]} + \sum_{l \neq 0} \left(|Rl|^{s_q} \|H_{R,l}\|_{L^p[-1/2,1/2]} \right) |Rl|^{-s_q} \\
&\lesssim R^{-s_q + 1/q} C_p(h) + (R^{1/q} C_p(h)) \left(\sum_{l \neq 0} |Rl|^{-s_q q} \right)^{1/q} \\
&= R^{-s_q + 1/q} C_p(h) \left(1 + \left(\sum_{l \neq 0} |l|^{-s_q q} \right)^{1/q} \right) \\
&= R^{-s_q + 1/q} C_p(h) \left(1 + \left(\sum_{l \neq 0} |l|^{-s_q q} \right)^{1/q} \right) \\
&\lesssim_p R^{-s_q + 1/q} C_p(h).
\end{aligned}
$$

A further application of Hölder's inequality shows that

$$
\begin{aligned}
|III_R| &= \left| \int_{-1/2}^{1/2} H_R(t) e^{2\pi i t P} \, dt \right| \\
&\lesssim C_p(h) R^{-s_q + 1/q} \left(\int_{-1/2}^{1/2} |e^{2\pi i P}|^q \, dt \right)^{1/q}.
\end{aligned}
$$

Applying the endpoint local smoothing inequality, we conclude that

$$
\begin{aligned}
\|(III_R \circ Q_R) f\|_{L^q(M)} &\lesssim C_p(h) R^{-s_q + 1/q} \|e^{2\pi i P}(Q_R f)\|_{L^q_t L^q_x} \\
&\lesssim C_p(h) R^{-s_q + 1/q} \|Q_R f\|_{L^q_{s_q - 1/q}(M)},
\end{aligned}
$$

Applying Bernstein's inequality gives

$$\|Q_R f\|_{L^q_{s_q-1/q}(M)} \lesssim R^{s_q-1/q}\|f\|_{L^q(M)}.$$

Thus we conclude that

$$\|(III_R \circ Q_R)f\|_{L^q(M)} \lesssim C_p(h)\|f\|_{L^q(M)}.$$

We have therefore bounded III_R uniformly in R. $\square$

Corollary 1.2 of [4] establishes that the sharp local smoothing inequality holds for $p < 2(d-1)/(d+1)$, which covers the range of parameters studied in this paper. Thus we have obtained uniform bounds on the operators $\{III_R\}$.

5 Analysis of II_R: Density Decompositions

It finally remains to analyze the operator $Q_R \circ II_R \circ Q_R$, where

$$II_R = \int \chi_{II}(t) R\widehat{h}(Rt) e^{2\pi i t P}\, dt$$

is obtained by integrating the wave propogators over times $100/R \leqslant |t| \leqslant 0.01$ respectively. To prevent notation from growing too cumbersome later on, let us eschew uses of the subscript R in our operators in this section, e.g. writing II_R as

$$II = \int b(t) e^{2\pi i t P}\, dt,$$

where $b(t) = \chi_{II}(t) R\widehat{h}(Rt)$. We then have

$$\|b(t)\langle t\rangle^{s_p}\|_{L^p(\mathbb{R})} \lesssim R^{1-1/p-s_p} C_p(h).$$

Bounding II requires a more subtle analysis of the geometric behaviour of the wave-propogator operators, and we will begin by converting our problem in coordinates on S^d, where the kernels have more explicit representations in oscillatory integrals.

We will employ some restricted weak type bounds, together with interpolation, to obtain L^p estimates on the operators $Q \circ II \circ Q$. We thus introduce a set $E \subset S^d$ and try to obtain $L^{p,\infty}$ bounds on the function $S = (Q \circ II_W \circ Q)\{E\}$. Given that Q already acts, heuristically, by localizing the behaviour of it's inputs to the frequency R, despite the choice of the set E, the uncertainty principle implies $Q\{E\}$ should be locally constant at a scale $1/R$, and so it is natural to discretize at this scale. Consider a maximal $1/2R$ separated subset $\mathcal{X}$ of S^d. Then break E down into a disjoint union of sets $\{E_{x_0} : x_0 \in \mathcal{X}\}$, where for $x_0 \in \mathcal{X}$, the set E_{x_0} is supported on the geodesic ball of radius $1/R$ about x_0. Similarily, let $\mathcal{T}$ be all points in the lattice $\mathbb{Z}/10R$ lying in the set $\{100/R \leqslant |t| \leqslant 1\}$, and write

$$b = \sum_{t \in \mathcal{T}} u(t) b_t,$$

where for each $t \in \mathcal{T}$, $u(t) = \|b\|_{L^\infty[t-10/R, t+10/R]}$, and b_t is a smooth function, compactly supported on the sidelength $1/R$ interval centered at t, satisfying

$$|\partial^\alpha b_t| \lesssim_\alpha R^{|\alpha|},$$

with implicit constants uniform in b and t. By the Plancherel-Polya theorem,

$$\|u(t)\langle t\rangle^{s_p}\|_{l^p(\mathcal{T})} \lesssim R^{1-s_p}.$$

We can then write

$$S = \sum |E_{x_0}| S_{x_0,t_0} \quad \text{where} \quad S_{x_0,t_0} = \int |E_{x_0}|^{-1} b_{t_0}(t)(Q \circ e^{2\pi i t P} \circ Q)\{E_{x_0}\}\, dt.$$

Our computation would be complete if we could show that for any coefficients $\{c(x_0,t_0) : x_0 \in \mathcal{X}, t_0 \in \mathcal{T}\}$,

$$\left\| \sum_{x_0,t_0} c(x_0,t_0) t_0^{\frac{d-1}{2}} S_{x_0,t_0} \right\|_{L^p(S^d)} \lesssim R^{s_p - 1 + d(1-1/p)} \left(\sum_{x_0,t_0} |c(x_0,t_0)|^p t_0^{d-1} \right)^{1/p}.$$

Indeed, we set $c(x_0,t_0) = |E_{x_0}| u(t_0) t_0^{-\frac{d-1}{2}}$ and apply Hölder's inequality, then the inequality above gives exactly that

$$\|S\|_{L^p(S^d)} \lesssim C_p(h) |E|^{1/p},$$

For $p = 1$, this follows from applying the triangle inequality, and using the pointwise estimates

$$|S_{x_0,t_0}(x)| \lesssim_M \frac{R^{d-1}}{(R d_g(x,x_0))^{\frac{d-1}{2}}} \Big\langle R \big| t_0 - d_g(x,x_0) \big| \Big\rangle^{-M}.$$

Applying interpolation, for $p > 1$ we need only prove a restricted weak type version of this inequality. In other words, we can restrict c to be the indicator function of a set $\mathcal{E}$, and take $L^{p,\infty}$ norms on the left hand side. If we write $\mathcal{E} = \bigcup_k \mathcal{E}_k$, where $\mathcal{E}_k$ is the set of $(x,t) \in \mathcal{E}$ with $|t| \sim 2^k/R$, then the inequality reads that

$$\left\| \sum_{k=1}^\infty 2^{k\frac{d-1}{2}} \sum_{(x_0,t_0) \in \mathcal{E}_k} S_{x_0,t_0} \right\|_{L^{p,\infty}(S^d)}^p \lesssim R^{(d-1)p-d} \left(\sum_{k=1}^\infty 2^{k(d-1)} \# \mathcal{E}_k \right).$$

This is equivalent to showing that for any $\lambda > 0$,

$$\left| \left\{ x : \left| \sum_k 2^{k\frac{d-1}{2}} S_{x_0,t_0}(x) \right| \geqslant \lambda \right\} \right| \lesssim \lambda^{-p} R^{(d-1)p-d} \sum_k 2^{k(d-1)} \# \mathcal{E}_k.$$

The case $\lambda \lesssim R^{d-1}$ follows from the L^1 boundedness we've already proved, so we may assume $\lambda \gtrsim R^{d-1}$ in the sequel.

To obtain this bound, we employ the method of density decompositions, introduced in [3]. Let

$$A = \left(\frac{\lambda}{R^{d-1}} \right)^{(d-1)(1-p/2)} \log \left(\frac{\lambda}{R^{d-1}} \right)^{O(1)}.$$

Then for each k, consider the collection $\mathcal{B}_k(\lambda)$ of all balls B with radius at most $2^k/R$ such that $\#\mathcal{E}_k \cap B \geqslant RA\text{rad}(B)$. Applying the Vitali covering lemma, we can find a disjoint family of balls $\{B_1, \dots, B_N\}$ in $\mathcal{B}_k$ such that the balls $\{B_1^*, \dots, B_N^*\}$ obtained by dilating the balls by 5 cover $\bigcup \mathcal{B}_k(\lambda)$. Then

$$\sum \text{rad}(B_j) \leqslant R^{-1}A^{-1}\#\mathcal{E}_k,$$

and the set $\widehat{\mathcal{E}}_k = \mathcal{E}_k - \bigcup \mathcal{B}_k(\lambda)$ has density type $(RA, 2^k/R)$. Then we conclude that, using the quasi-orthogonality estimates below,

$$\left\| \sum_k \sum_{(x_0,t_0)\in\widehat{\mathcal{E}}_k} 2^{k\frac{d-1}{2}} S_{x_0,t_0} \right\|_{L^2(S^d)}^2 \lesssim_p R^{d-2}\log(A)A^{\frac{2}{d-1}} \sum_k 2^{k(d-1)}\#\mathcal{E}_k.$$

Appling Chebyshev's inequality, and utilizing the choice of A above, we conclude that

$$\begin{aligned}\left| \left\{ x : \left| \sum_k \sum_{(x_0,t_0)\in\widehat{\mathcal{E}}_k} 2^{k\frac{d-1}{2}} S_{x_0,t_0}(x) \right| \geqslant \lambda/2 \right\} \right| &\lesssim R^{d-2}\log(A)A^{\frac{2}{d-1}} \sum_k 2^{k(d-1)}\#\mathcal{E}_k \\ &\lesssim \lambda^{-p} R^{(d-1)p-d} \sum_k 2^{k(d-1)}\#\mathcal{E}_k.\end{aligned}$$

Conversely, we exploit the clustering of the sets $\mathcal{E}_k - \widehat{\mathcal{E}}_k$ to bound

$$\left| \left\{ x : \left| \sum_k \sum_{(x_0,t_0)\in\mathcal{E}_k - \widehat{\mathcal{E}}_k} 2^{k\frac{d-1}{2}} S_{x_0,t_0}(x) \right| \geqslant \lambda/2 \right\} \right|$$

That is, we have found balls $B_1^* < \dots, B_N^*$, each with radius $O(2^k/R)$, such that

$$\sum \text{rad}(B_j) \leqslant R^{-1}A^{-1}\#\mathcal{E}_k.$$

Let (x_j, t_j) denote the center of the ball B_j. Then the function

$$\sum_{(x_0,t_0)\in B_j} S_{x_0,t_0}$$

has mass concentrated on the geodesic annulus $\text{Ann}_j \subset S^d$ with radius t_j and thickness $O(\text{rad}(B_j))$, a set with measure $(2^k/R)^{d-1}\text{rad}(B_j)$. For $(x_0, t_0) \in B_j$, we calculate using the pointwise bounds that

$$\begin{aligned}\int_{\text{Ann}_j^c} |S_{x_0,t_0}(x)|\, dx &\lesssim R^{d-1} \int_{\text{rad}(B_j)\lesssim|t_j - d_g(x,x_0)|\leqslant 1} \langle R|t_0 - d_g(x,x_0)|\rangle^{-M} \\ &\lesssim R^{d-1} \int_{\text{rad}(B_j)\leqslant|t_j - s|\leqslant 1} s^{d-1}\langle R|t_0 - s|\rangle^{-M}\, ds \\ &\lesssim 2^{k(d-1)} R^{d-1} (R\text{rad}(B_j))^{-M}.\end{aligned}$$

Because the set of points in $\mathcal{E}_k$ is $1/R$ separated, there can only be at most $O(R\text{rad}(B_j))^{d+1}$ values of (x_0, t_0), and so applying the triangle inequality gives that the sum of the L^1 norm outside of Ann_j is

$$\lesssim 2^{k(d-1)} R^{d-1} (R\,\text{rad}(B_j))^{d+1-M}$$

Note that since $\#\mathcal{E}_k \cap B_j \geqslant RA\text{rad}(B_j)$, and because $\mathcal{E}_k$ is $1/R$ discretized,

$$\text{rad}(B_j) \geqslant (A/R)^{\frac{1}{d-1}},$$

and this, together with Markov's inequality, is enough to justify the required bound. Conversely, since $1 < p < 2(d-1)/(d+1)$, we have

$$\begin{aligned}\sum |\text{Ann}_j| &\lesssim (2^k/R)^{d-1} \sum_j \text{rad}(B_j) \\ &\lesssim (2^k/R)^{d-1} R^{-1} (L/R^{d-1})^{-(d-1)(1-p/2)} log(L/R^{d-1})^{O(1)} \\ &\lesssim \lambda^{-p} R^{(d-1)p-d} 2^{k(d-1)} \#\mathcal{E}_k,\end{aligned}$$

Summing over k completes the analysis.

6 Analysis of II_R: Quasi-Orthogonality

Our first goal will be to understand how orthogonal the functions $\{S_{x_0,t_0}\}$ are to one another, which will give L^2 estimates for S, that can be interpolated with L^1 estimates to obtain the required L^p estimates. The rest of this section will be devoted to proving the following inner product estimate, which, together with a density decomposition argument, introduced in [3], can be used to obtain L^2 estimates, which we can then interpolate to obtain L^p estimates for the function S.

Lemma 7.

$$|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim_M \frac{R^{d-2}}{(Rd_g(x_0,x_1))^{\frac{d-1}{2}}} \sum_{\pm} \left\langle R\big|(t_0 - t_1) \pm d_g(x_1, x_0)\big|\right\rangle^{-M}.$$

Let us proceed with the proof. To begin with, we can use the self-adjointness of the operators Q, the semigroup structure of $\{e^{2\pi i t P}\}$, and the fact that multipliers commute, to write

$$\begin{aligned}\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle &= \int \frac{b_{t_0}(t)\overline{b_{t_1}(s)}}{|E_{x_0}||E_{x_1}|} \left\langle (Q \circ e^{2\pi i t P} \circ Q)\{E_{x_0}\}, (Q \circ e^{2\pi i s P} \circ Q)\{E_{x_1}\}\right\rangle \,dt\,ds \\ &= \int \frac{b_{t_0}(t)\overline{b_{t_1}(s)}}{|E_{x_0}||E_{x_1}|} \left\langle (Q^2 \circ e^{2\pi i (t-s) P} \circ Q^2)\{E_{x_0}\}, E_{x_1}\right\rangle \\ &= \int \frac{c_{t_0,t_1}(t)}{|E_{x_0}||E_{x_1}|} \left\langle (Q^2 \circ e^{2\pi i t P} \circ Q^2)\{E_{x_0}\}, E_{x_1}\right\rangle,\end{aligned}$$

where

$$c_{t_0,t_1}(t) = \int b_{t_0}(u)\overline{b_{t_1}}(u-t)\,dt,$$

is the convolution of b_{t_0} with the reflection of $\overline{b_{t_1}}$ about the y-axis. Thus c_{t_0,t_1} is supported on the length $2/R$ interval centered at $t_0 - t_1$, and has L^1 norm $O(1/R^2)$ by Young's convolution inequality.

We next perform a decomposition of the inner product into various coordinate systems. Cover S^d by a finite family of sets $\{V_\alpha\}$, chosen such that for each V_α, there is a coordinate chart U_α such that the neighbourhood $N(V_\alpha, 0.5)$ is contained in U_α. Let $\{\eta_\alpha\}$ be a partition of unity subordinate to $\{V_\alpha\}$. It will also be convenient to define $V_\alpha^* = N(V_\alpha, 0.1)$. We can then write

$$\langle S_{t_0,x_0}, S_{t_1,x_1} \rangle = \sum_\alpha \int \frac{c_{t_0,t_1}(t)}{|E_{x_0}||E_{x_1}|} \big\langle (Q^2 \circ e^{2\pi i t P} \circ Q^2)\{\eta_\alpha E_{x_0}\}, E_{x_0} \big\rangle \, dt.$$

We will bound each of the terms on the right separately from one another, by working with each inner product in the coordinate systems $\{U_\alpha\}$.

The next Lemma allows us to approximate the operator Q, and the propogators $e^{2\pi i t P}$, with operators which have more explicit representations in the coordinate system U_α, by an error term which is negligible to our analysis. It utilizes the *Lax-Hörmander parametric* for the half-wave equation over small times, which expresses $e^{2\pi i t P}$ in coordinates as a Fourier integral operator.

Lemma 8. *For each α, and $|t| \leqslant 1/100$, there exists Schwartz operators Q_α and $W_\alpha(t)$, each with kernel supported on $U_\alpha \times V_\alpha^*$, such that the following properties hold:*

- *For $f \in C^\infty(S^d)$ with supp$(f) \subset V_\alpha^*$,*

  $$\text{supp}(Q_\alpha f) \subset N(\text{supp}(f), 0.1) \quad \text{and} \quad \text{supp}(W_\alpha(t) f) \subset N(\text{supp}(f), 0.1).$$

  Moreover,

  $$\|(Q^2 - Q_\alpha) f\|_{L^2(S^d)} \lesssim_N R^{-N} \|f\|_{L^2(S^d)}$$

  and

  $$\left\| \Big(Q_\alpha \circ \Big(e^{2\pi i t P} - W_\alpha(t) \Big) \circ Q_\alpha \Big) \{f\} \right\|_{L^2(S^d)} \lesssim_N R^{-N} \|f\|_{L^2(S^d)}.$$

- *In the coordinate system of U_α, Q_α is a pseudodifferential operator of order zero given by a symbol $\sigma(x,\xi)$, where*

  $$\text{supp}(\sigma) \subset \{|\xi| \sim R\},$$

  and σ satisfies derivative estimates of the form

  $$|\partial_x^\beta \partial_\xi^\kappa \sigma(x,\xi)| \lesssim_{\beta,\kappa} R^{-|\kappa|}.$$

- *In the coordinate system U_α, the operator $W_\alpha(t)$ has a kernel $W_\alpha(t,x,y)$ with an oscillatory integral representation*

  $$W_\alpha(t,x,y) = \int s(t,x,y,\xi) e^{2\pi i [\phi(x,y,\xi) + t|\xi|_y]} \, d\xi,$$

where s has compact support in it's x and y coordinates, with

$$supp(s) \subset \{|\xi| \sim R\},$$

where s satisfies derivative estimates of the form

$$|\partial_{t,x,y}^{\beta} \partial_{\xi}^{\kappa} s| \lesssim_{\beta,\kappa} R^{-|\kappa|},$$

and where $|\cdot|_y$ denotes the norm on $\mathbb{R}^n_\xi$ induced by the Riemannian metric on S^d on the contangent space $T_y^* S^d$.

Thus, ignoring errors negligible to our analysis, we need only analyze

$$\left| \int \frac{c_{t_0,t_1}(t)}{|E_{x_0}||E_{x_1}|} \left\langle (Q_\alpha \circ W_\alpha(t) \circ Q_\alpha)\{\phi_\alpha E_{x_0}\}, E_{x_1} \right\rangle du \right|.$$

The behaviour of all operations in this expression are now completely localized to U_α for inputs supported on V_α^*; in particular, this expression is equal to zero unless E_{x_0} and E_{x_1} are both compactly contained in U_α, So we can now naturally work with the kernels of the operators in coordinates to upper bound the inner product, which will complete the required estimate of the inner product.

Lemma 9. *Let c be an integrable function supported on the length $10/R$ interval centered at a value t^* with $|t| \leqslant 1/100$. Then*

$$\begin{aligned} &\left| \int \frac{c(t)}{|E_{x_0}||E_{x_1}|} \left\langle (Q_\alpha \circ W_\alpha(t) \circ Q_\alpha)\{\phi_\alpha E_{x_0}\}, E_{x_1} \right\rangle dt \right| \\ &\quad \lesssim_M R^d \frac{\|c\|_{L^1(\mathbb{R})}}{(R d_g(x_1,x_0))^{\frac{d-1}{2}}} \sum_{\pm} \left\langle R \big| t^* \pm d_g(x_1,x_0) \big| \right\rangle^{-M} \end{aligned}$$

Proof. We write the integral as

$$\begin{aligned} \int \frac{c(t)}{|E_{x_0}||E_{x_1}|} &(\eta_\alpha E_{x_1})(w) \sigma(w,\theta) e^{2\pi i \theta \cdot (w-x)} \\ &s(t,x,y,\xi) e^{2\pi i [\phi(x,y,\xi) + t|\xi|_y]} \sigma(y,\eta) e^{2\pi i \eta \cdot (y-z)} E_{x_0}(z) \\ &dt\, dx\, dy\, dz\, dw\, d\theta\, d\xi\, d\eta. \end{aligned}$$

The integral looks complicated, but can be simplified considerably by noticing that all the spatial variables are highly localized. To begin with, we use the fact that s is smooth and compactly supported in all it's variables, so s should roughly behave like a linear combination of tensor products; using Fourier series, we can write

$$s(t,x,y,\xi) = \sum_{n \in \mathbb{Z}^d} s_{n,1}(x) s_{n,2}(t,y,\xi),$$

where $s_{n,1}(x) = e^{2\pi i n \cdot x}$, and where

$$|\partial_{t,y}^{\alpha} \partial_{\xi}^{\kappa} \{s_{n,2}\}| \lesssim_{\alpha,\kappa,N} |n|^{-N} R^{-|\kappa|}$$

If we define $a_n(\xi) = a_{n,1}(R\xi)a_{n,2}(R\xi)$, where

$$a_{n,1}(\xi) = |E_{x_1}|^{-1} \int (\eta_\alpha E_{x_1})(w)\sigma(w,\theta)s_{n,1}(x)e^{2\pi i[\theta\cdot(w-x)+(x-x_1)\cdot\xi]}\, d\theta\, dw\, dx$$

and

$$a_{n,2}(\xi) = |E_{x_0}|^{-1} \int c(t)s_{n,2}(t,y,\xi)\sigma(y,\zeta)E_{x_0}(z)e^{2\pi i[(\phi(t^*,x_0,\xi)-\phi(t,y,\xi))+\zeta\cdot(y-z)]}\, d\zeta\, dt\, dy\, dz,$$

then, rescaling, we can write the required integral as

$$R^d \sum_{n\in\mathbb{Z}^d} \int a_n(\xi)e^{2\pi iR[\phi(x_1,x_0,\xi)+t^*|\xi|_{x_0}]}\, d\xi.$$

Notice that $\mathrm{supp}(a_n) \subset \{|\xi| \sim 1\}$, and

$$|(\nabla_\xi^\kappa a_n)(\xi)| \lesssim_{\kappa,N} |n|^{-N}\|c\|_{L^1(\mathbb{R})}.$$

To obtain an efficient upper bound on this oscillatory integral, it will be convenient to change coordinate systems in a way better respecting the Riemannian metric at x_0, i.e. finding a smooth family of diffeomorphisms $\{F_{x_0} : S^{d-1} \to S^{d-1}\}$ such that $|F_{x_0}|_{x_0} = 1$. We can choose this function such that $F_{x_0}(-x) = -F_{x_0}(x)$. Then if $\tilde{a}_n(\rho,\eta) = a_n(\rho F_{x_0}(\eta))JF_{x_0}(\eta)$, then a change of variables gives that

$$\int a_n(\xi)e^{2\pi iR[\phi(x_1,x_0,\xi)+t^*|\xi|_{x_0}]} = \int_0^\infty \rho^{d-1} \int_{|\eta|=1} \tilde{a}_n(\rho,\eta)e^{2\pi iR\rho[\phi(x_1,x_0,F_{x_0}(\eta))+t^*]}\, d\eta\, d\rho.$$

For each fixed ρ, we claim that the phase has exactly two stationary points in the η variable, at the values $\pm\eta_0$, where x_1 lies on the geodesic passing through x_0 tangent to the vector $\eta_0^\sharp$ (here we are using the musical isomorphism to map the cotangent vector η_0 to a tangent vector). Moreover, at these values,

$$\phi(x_1,x_0,F_{x_0}(\pm\eta_0)) = \pm d_g(x_1,x_0),$$

and the Hessian at $\pm\eta_0$ is (positive / negative) definite, with each eigenvalue having magnitude exceeding a constant multiple of $d_g(x_1,x_0)$. It follows from the principle of stationary phase, that the integral above can be written as

$$\frac{1}{(Rd_g(x_1,x_0))^{\frac{d-1}{2}}} \sum_\pm \int_0^\infty \rho^{\frac{d-1}{2}} f_{n,\pm}(\rho)e^{2\pi iR\rho[t^*\pm d_g(x_1,x_0)]}\, d\rho,$$

where $f_{n,\pm}$ is supported on $|\rho| \sim 1$, and

$$|\partial_\rho^m f_{n,\pm}| \lesssim_{m,N} |n|^{-N}\|c\|_{L^1(\mathbb{R})}.$$

Integrating by parts in the ρ variable if $\pm d_g(x_1,x_0)+t^*$ is large, and then taking in absolute values, we conclude that

$$\left| \int a_n(\xi)e^{2\pi iR[\phi(x_1,x_0,\xi)+t^*|\xi|_{x_0}]} \right| \lesssim_{N,M} |n|^{-N} \frac{\|c\|_{L^1(\mathbb{R})}}{(Rd_g(x_1,x_0))^{\frac{d-1}{2}}} \sum_\pm \langle R|t^* \pm d_g(x_1,x_0)|\rangle^{-M}.$$

Taking $N \geqslant d+1$, and summing in the n variable, we conclude that

$$\left|\sum_n \int a_n(\xi) e^{2\pi i R[\phi(x_1,x_0,\xi)+t'|\xi|_{x_0}]}\right| \lesssim_M \frac{\|c\|_{L^1(\mathbb{R})}}{(Rd_g(x_1,x_0))^{\frac{d-1}{2}}} \sum_{\pm} \langle R|t^* \pm d_g(x_1,x_0)|\rangle^{-M}.$$

But this is precisely an estimate for the quantity we wished to estimate. $\square$

Now applying this Lemma with $c = c_{t_0,t_1}$, and then summing in α, we complete the proof of Lemma 7.

7 Analysis of II_R: L^2 Estimates

Lemma 7 of the last section implies two functions S_{x_0,t_0} and S_{x_1,t_1} can only be correlated in L^2 if $d_g(x_0,x_1) \approx |t_0 - t_1|$. We now exploit this geometry to obtain some L^2 estimates for sums of the functions S_{x_0,t_0}.

Lemma 10. *Fix $u \geqslant 1$. Consider a set $\mathcal{E} \subset \mathcal{X} \times \mathcal{T}$. Write*

$$\mathcal{E} = \bigcup_{k=0}^{\infty} \mathcal{E}_k,$$

where $\mathcal{E}_k = \{(x,t) \in \mathcal{E} : |t| \sim 2^k/R\}$. Suppose that each of the sets $\mathcal{E}_k$ has density type $(Ru, 2^k/R)$, i.e. for any set $B \subset \mathcal{X} \times \mathcal{T}$ with diam$(B) \leqslant 2^k/R$,

$$\#(\mathcal{E}_k \cap B) \leqslant Ru \; diam(B).$$

Then

$$\left\|\sum_k \sum_{(x_0,t_0)\in\mathcal{E}_k} 2^{k\frac{d-1}{2}} S_{x_0,t_0}\right\|_{L^2(S^d)}^2 \lesssim R^{d-2}\log_2(u) u^{\frac{2}{d-1}} \sum_k 2^{k(d-1)} \#\mathcal{E}_k.$$

Write $F = \sum F_k$, where

$$F_k = 2^{k\frac{d-1}{2}} \sum_{(x_0,t_0)\in\mathcal{E}} S_{x_0,t_0}.$$

Applying Cauchy-Schwartz, we have

$$\|F\|_{L^2(S^d)}^2 \lesssim \log_2(u) \left(\sum_{k \lesssim \log_2(u)} \|F_k\|_{L^2(S^d)}^2 + \| \sum_{k \gtrsim \log_2(u)} F_k\|_{L^2(S^d)}^2 \right).$$

Without loss of generality by increasing the implicit constant, we can assume that $\{k : \mathcal{E}_k \neq \varnothing\}$ is 10-separated, and that all values of t with $(x,t) \in \mathcal{E}$ are positive (the case where all values of t being negative being treated analogously). Thus if F_k and $F_{k'}$ are both nonzero, then $k = k'$ or $|k-k'| \geqslant 10$. For $k \geqslant k'+10$, let us estimate $\langle F_k, F_{k'}\rangle$. We can decompose this inner product into a sum of quantities of the form $2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} \langle S_{x_0,t_0}, S_{x_1,t_1}\rangle$, where $t_0 \sim 2^k/R$ and $t_1 \sim 2^{k'}/R$. Now consider the two sets

$$\mathcal{G}_{x_0,t_0,\text{low}} = \{(x_1,t_1) \in \mathcal{E}_{k'} : |d_g(x_0,x_1) - (t_0 - t_1)| \lesssim 2^{k'+10}/R\}$$

and for $l \geqslant k' + 10$, consider the set

$$\mathcal{G}_{x_0,t_0,l} = \{(x_1,t_1) \in \mathcal{E}_{k'} : |d_g(x_0,x_1) - (t_0 - t_1)| \sim 2^l/R\}.$$

Let us use the density properties of $\mathcal{E}$ to control the size of these index sets. First, note that for any $(x_0,t_0) \in \mathcal{E}_k$ and $(x_1,t_1) \in \mathcal{E}_{k'}$, $t_0 - t_1$ lies in a radius $O(2^{k'}/R)$ interval centered at t_0:

- Let us first estimate interactions between the functions S_{x_0,t_0} and S_{x_1,t_1} with $(x_1,t_1) \in \mathcal{G}_{x_0,t_0,\mathrm{low}}$. If $(x_1,t_1) \in \mathcal{G}_{x_0,t_0,\mathrm{low}}$, then x_1 must lie in a width $O(2^{k'}/R)$ and radius $O(2^k/R)$ annulus centered at x_0. Thus $\mathcal{G}_{x_0,t_0,\mathrm{low}}$ is covered by $O(2^{(k-k')(d-1)})$ balls of radius $2^{k'}/R$. The density properties of $\mathcal{E}_{k'}$ implies that

  $$\#\mathcal{G}_{x_0,t_0,l} \lesssim Ru\ 2^{(k-k')(d-1)}(2^{k'}/R) = u2^{(k-k')(d-1)+k'}.$$

  Together with Lemma 7, we conclude that

  $$2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}} \sum_{(x_1,t_1)\in\mathcal{G}_{x_0,t_0,\mathrm{low}}} |\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim_M R^{d-2}2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}}\left(u2^{(k-k')(d-1)+k'}\right)\left(2^{-k\frac{d-1}{2}}\right).$$

  We can now sum over $\log_2(u) \lesssim k' \leqslant k-10$ and $(x_0,t_0) \in \mathcal{E}_k$ to find

  $$2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}} \sum_{(x_0,t_0)\in\mathcal{E}_k}\sum_{k'\leqslant k-10}\sum_{(x_1,t_1)\in\mathcal{G}_{x_0,t_0,\mathrm{low}}} |\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim R^{d-2}2^{k(d-1)}\#\mathcal{E}_k.$$

- Next, let's estimate interactions between the functions S_{x_0,t_0} and S_{x_1,t_1} with $(x_1,t_1) \in \mathcal{G}_{x_0,t_0,l}$ with $k'+10 \leqslant l \leqslant k-5$. If $(x_1,t_1) \in \mathcal{G}_{x_0,t_0,l}$, then x_1 must lie in one of two geodesic annuli centered at x_0, each width $O(2^l/R)$ and radii $O(2^k/R)$. Thus $\mathcal{G}_{x_0,t_0,l}$ is covered by $O(2^{(l-k')}2^{(k-k')(d-1)})$ balls of radius $2^{k'}/R$, and the density of $\mathcal{E}_{k'}$ implies that

  $$\#\mathcal{G}_{x_0,t_0,l} \lesssim Ru\ 2^{(l-k')}2^{(k-k')(d-1)}2^{k'}/R = u2^l2^{(k-k')(d-1)}.$$

  Together with Lemma 7, we conclude that

  $$2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}} \sum_{(x_1,t_1)\in\mathcal{G}_{x_0,t_0,l}} |\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim_M R^{d-2}2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}}\left(u2^l2^{(k-k')(d-1)}\right)\left(2^{-k\frac{d-1}{2}}2^{-lM}\right).$$

  Picking $M > 1$, we can sum over $k'+10 \leqslant l \leqslant k-5$, $\log_2(u) \lesssim k' \leqslant k-10$, and $(x_0,t_0) \in \mathcal{E}_k$ to find

  $$\sum_{(x_0,t_0)\in\mathcal{E}_k}\sum_{k'\leqslant k-10}\sum_{k'+10\leqslant l\leqslant k-5}\sum_{(x_1,t_1)\in\mathcal{G}_{x_0,t_0,l}} 2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}}|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim R^{d-2}2^{k(d-1)}\#\mathcal{E}_k.$$

- Now let's estimate the interactions between the functions S_{x_0,t_0} and S_{x_1,t_1} with $(x_1,t_1) \in \mathcal{G}_{x_0,t_0,l}$, for $k+10 \leqslant l \leqslant \log_2 R$, then x_1 must lie in a geodesic ball of radius $O(2^l/R)$ centered at x_0. Such a ball is covered by $O(2^{(l-k')d})$ balls of radius $2^{k'}/R$, and the density of $\mathcal{E}_{k'}$ implies that

  $$\#\mathcal{G}_{x_0,t_0,l} \lesssim Ru\ 2^{(l-k')d}(2^{k'}/R) = u2^{(l-k')d}2^{k'}.$$

Together with Lemma 7, we conclude that

$$2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}}\sum_{(x_1,t_1)\in\mathcal{G}_{x_0,t_0,l}}|\langle S_{x_0,t_0},S_{x_1,t_1}\rangle| \lesssim_M R^{d-2}2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}}\left(u2^{(l-k')d}2^{k'}\right)\left(2^{-lM}\right).$$

Picking $M > d$, we can sum over $k-5 \leqslant l \lesssim \log R$, $\log_2(u) \lesssim k' \leqslant k-10$, and $(x_0,t_0)\in\mathcal{E}_k$ to conclude that

$$2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}}\sum_{(x_0,t_0)\in\mathcal{E}_k}\sum_{k'\leqslant k-10}\sum_{k-5\leqslant l\lesssim\log R}\sum_{(x_1,t_1)\in\mathcal{G}_{x_0,t_0,l}}R^{d-2}|\langle S_{x_0,t_0},S_{x_1,t_1}\rangle| \lesssim R^{d-2}.$$

Putting these three bounds together, we conclude that

$$\sum_{\log_2(u)\lesssim k'<k}|\langle F_k,F_{k'}\rangle| \lesssim R^{d-2}\sum_k 2^{k(d-1)}\#\mathcal{E}_k.$$

In particular, we have

$$\|F\|^2_{L^2(S^d)} \lesssim \log_2(u)\left(\sum_k\|F_k\|^2_{L^2(S^d)} + R^{d-2}\sum_k 2^{k(d-1)}\#\mathcal{E}_k\right).$$

Next, let us fix some parameter a, and decompose $[2^k/R, 2^{k+1}/R]$ into the disjoint union of length u^a intervals

$$I_{k,\mu} = [2^k/R + (\mu-1)u^a/R, 2^k/R + \mu u^a/R] \quad \text{for } 1 \leqslant \mu \leqslant 2^k/u^a,$$

and thus considering a further decomposition $\mathcal{E}_k = \bigcup \mathcal{E}_{k,\mu}$ and $F_k = \sum F_{k,\mu}$. As before, increasing the implicit constant in the Lemma, we may assume without loss of generality that the set $\{\mu : \mathcal{E}_{k,\mu} \neq \varnothing\}$ is 10-separated. We now estimate

$$\sum_{\mu\geqslant\mu'+10}|\langle F_{k,\mu},F_{k,\mu'}\rangle|.$$

For $(x_0,t_0)\in\mathcal{E}_{k,\mu}$ and $l \geqslant 1$, define

$$\mathcal{H}_{x_0,t_0,l} = \left\{(x_1,t_1)\in\mathcal{E}_{k,\mu'} : \max(d_g(x_0,x_1),t_0-t_1) \sim 2^l u^a/R\right\}.$$

Then $\bigcup_{l\geqslant 1}\mathcal{H}_{x_0,t_0,l}$ covers $\bigcup_{\mu\geqslant\mu'+10}\mathcal{E}_{k,\mu'}$. The density properties of $\mathcal{E}_{k,\mu'}$ imply that provided that $l \leqslant k - a\log_2 u + 10$ (so that $2^l u^a/R \leqslant 2^k/R$),

$$\#\mathcal{H}_{x_0,t_0,l} \lesssim (Ru)(2^l u^a/R) = u^{a+1}2^l$$

For $(x_1,t_1)\in\mathcal{H}_{x_0,t_0,l}$, we claim that

$$2^{k(d-1)}|\langle S_{x_0,t_0},S_{x_1,t_1}\rangle| \lesssim R^{d-2}2^{k(d-1)}(2^l u^a)^{-\frac{d-1}{2}}.$$

Indeed, for such tuples we have

$$d_g(x_0,x_1) \gtrsim 2^l u^a/R \quad \text{or} \quad |d_g(x_0,x_1)-(t_0-t_1)| \gtrsim 2^l u^a/R,$$

and the estimate follows from Lemma 7 in either case. Since $d \geqslant 4$,

$$\begin{aligned}\sum_{1 \leqslant l \leqslant k - a\log_2 u + 10} \sum_{(x_1,t_1) \in \mathcal{H}_{x_0,t_0,l}} 2^{k(d-1)} |\langle S_{x_0,t_0}, S_{x_1,t_1} \rangle| &\lesssim R^{d-2} \sum_{1 \leqslant l \leqslant k - a\log_2 u + 10} (2^{k(d-1)})(2^l u^a)^{-\frac{d-1}{2}} (u^{a+1} 2^l) \\ &\lesssim R^{d-2} \sum_{1 \leqslant l \leqslant k - a\log_2 u + 10} 2^{k(d-1)} 2^{-l\frac{d-3}{2}} u^{1-a\left(\frac{d-3}{2}\right)} \\ &\lesssim R^{d-2} 2^{k(d-1)} u^{1-a\left(\frac{d-3}{2}\right)}.\end{aligned}$$

For $l > k - a\log_2 u + 10$, a tuple (x_1, t_1) lies in $\mathcal{H}_{x_0,t_0,l}$ if and only if $d_g(x_0, x_1) \sim 2^l u^a / R$, since we always have

$$|t_0 - t_1| \lesssim 2^k / R \ll 2^l u^a / R.$$

We conclude from Lemma 7 that

$$2^{k(d-1)} |\langle S_{x_0,t_0}, S_{x_1,t_1} \rangle| \lesssim_M R^{d-2} 2^{k(d-1)} (2^l u^a)^{-M}.$$

Now $\mathcal{H}_{x_0,t_0,l}$ is covered by $O((2^{l-k} u^a)^d)$ balls of radius $2^k/R$, and the density properties of $\mathcal{E}_k$ imply that

$$\#\mathcal{H}_{x_0,t_0,l} \lesssim (Ru)(2^{l-k} u^a)^d (2^k/R) \lesssim u^{1+ad} 2^{ld} 2^{-k(d-1)}.$$

Thus, picking $M > \max(d, 1 + ad)$, we conclude that

$$\begin{aligned}\sum_{l \geqslant k - a\log_2 u + 10} \sum_{(x_1,t_1) \in \mathcal{H}_{x_0,t_0,l}} 2^{k(d-1)} |\langle S_{x_0,t_0}, S_{x_1,t_1} \rangle| &\lesssim R^{d-2} \sum_{l \geqslant k - a\log_2 u + 10} (2^{k(d-1)})(2^l u^a)^{-M} u^{1+ad} 2^{ld} 2^{-k(d-1)} \\ &\lesssim R^{d-2}.\end{aligned}$$

Putting these two bounds together, and then summing over the tuples (x_0, t_0), we conclude that

$$\sum_{\mu \geqslant \mu' + 10} |\langle F_{k,\mu}, F_{k,\mu'} \rangle| \lesssim R^{d-2} \left(1 + 2^{k(d-1)} u^{1-a\left(\frac{d-3}{2}\right)}\right) \#\mathcal{E}_{k,\mu}.$$

Now summing in μ, we conclude that

$$\|F_k\|_{L^2(S^d)}^2 \lesssim \sum_{\mu} \|F_{k,\mu}\|_{L^2(S^d)}^2 + R^{d-2} \left(1 + 2^{k(d-1)} u^{1-a\left(\frac{d-3}{2}\right)}\right) \#\mathcal{E}_k.$$

The functions in the sum defining $F_{k,\mu}$ are highly coupled, and it is difficult to use anything except Cauchy-Schwartz to break them apart. Since $\#(\mathcal{T} \cap I_{k,\mu}) \sim u^a$, if we set $F_{k,\mu} = \sum_{t \in \mathcal{T} \cap I_{k,\mu}} F_{k,\mu,t}$, then we find

$$\|F_{k,\mu}\|_{L^2(S^d)}^2 \lesssim u^a \sum_{t \in \mathcal{T} \cap I_{k,\mu}} \|F_{k,\mu,t}\|_{L^2(S^d)}^2.$$

Fortunately, since $\mathcal{X}$ is 1-separated, the functions in $F_{k,\mu,t}$ are quite orthogonal to one another, and so

$$\|F_{k,\mu,t}\|_{L^2(S^d)}^2 \lesssim R^{d-2} 2^{k(d-1)} \#(\mathcal{E}_k \cap (S^d \times \{t\})).$$

But this means that

$$u^a \sum_t \|F_{k,\mu,t}\|_{L^2(S^d)}^2 \lesssim R^{d-2} 2^{k(d-1)} u^a \#\mathcal{E}_{k,\mu}.$$

and so

$$\|F_k\|_{L^2(S^d)}^2 \lesssim \sum_\mu \|F_{k,\mu}\|_{L^2(S^d)}^2 + R^{d-2}\left(1 + 2^{k(d-1)}u^{1-a\left(\frac{d-3}{2}\right)}\right)\#\mathcal{E}_k$$
$$\lesssim R^{d-2}\left(2^{k(d-1)}u^a + (1 + 2^{k(d-1)}u^{1-a\left(\frac{d-3}{2}\right)}\right)\#\mathcal{E}_k.$$

Picking $a = 2/(d-1)$, we conclude that

$$\|F_k\|_{L^2(S^d)}^2 \lesssim R^{d-2}2^{k(d-1)}u^{\frac{2}{d-1}}\#\mathcal{E}_k.$$

Thus, returning to our bound for F, we conclude that

$$\|F\|_{L^2(S^d)}^2 \lesssim R^{d-2}\log_2(u)u^{\frac{2}{d-1}}\sum_k 2^{k(d-1)}\#\mathcal{E}_k.$$

This completes the proof of the L^2 density bound.

8 Combining Dyadic Pieces

Consider a general function $h : (0,\infty) \to \mathbb{C}$, and suppose that we can write

$$h(\lambda) = \sum_{k\in\mathbb{Z}} h_k(\lambda/2^k),$$

where h_k has support on $\{|\lambda| \sim 1\}$, and for each k,

$$\left(\int_0^\infty \left[\langle t\rangle^{s_p}|\widehat{h}_k(t)|\right]^p\, dt\right)^{1/p} \leqslant C_p(h),$$

Then

$$\widehat{h}(t) = \sum(2^kR)\widehat{h}_k(2^kRt),$$

and we have can write

$$h(P/R) = \sum_k \int (2^kR)\widehat{h}_k(2^kRt)e^{2\pi itP}\, dt.$$

If $b_{k,0}(t) = (2^kR)\widehat{h}_k(2^kRt)\chi(t)$,

$$\left\|\langle 2^kRt\rangle^{s_p}b_{k,0}\right\|_{L^p[-1/2,1/2]} = (2^kR)^{1-1/p}\|\langle t\rangle^{s_p}\widehat{h}_k(t)\| = (2^kR)^{1-1/p}C_p(h).$$

If $b_{k,n}(t) = (2^kR)\widehat{h}_k(2^kR(t+n))\chi(t)$ then

$$\left\|\langle 2^kRt\rangle^{s_p}\sum_{n\neq 0} b_{k,n}\right\|_{L^p} = \left\|\langle 2^kRt\rangle^{s_p}\left(\sum_{n\neq 0}|2^kRn|^{-p's_p}\right)^{1/p'}\left(\sum_{n\neq 0}|2^kRn|^{ps_p}|b_{k,n}|^p\right)^{1/p}\right\|_{L^p}$$
$$= \left\|\langle 2^kRt\rangle^{s_p}(2^kR)^{-s_p}\left(\sum_{n\neq 0}|2^kRn|^{ps_p}|b_{k,n}|^p\right)^{1/p}\right\|_{L^p}$$
$$= (2^kR)^{1-1/p}C_p(h).$$

Thus we should expect just have to deal with small times?

$$\begin{aligned}
&\left(\int_{-1/2}^{1/2}\left|\langle 2^kRt\rangle^{s_p}\sum_n (2^kR)\widehat{h}_k(2^kR(t+n))\right|^p\,dt\right)^{1/p}\\
&\quad= (2^kR)^{1-1/p}\left(\int_{-2^kR/2}^{2^kR/2}\left|\langle t\rangle^{s_p}\sum_n \widehat{h}_k(t+2^kRn)\right|^p\,dt\right)^{1/p}\\
&\quad\lesssim (2^kR)^{1-1/p}\left(\int\left|\langle t\rangle^{s_p}|\widehat{h}_k(t)|\right|^p+\left|\langle t\rangle^{s_p}\left(\sum_{n\neq 0}(2^kRn)^{-s_pp'}\right)^{1/p'}\left(\sum_{n\neq 0}(2^kRn)^{s_pp}|\widehat{h}_k(t+2^kRn)|^p\right)^{1/p}\right|^p\right)^{1/p}\\
&\lesssim (2^kR)^{1-1/p}\int\left(\sum_n\langle t+2^kRn\rangle^{ps_p}|\widehat{h}_k(t+2^kRn)|^p\right)^{1/p}\\
&\lesssim C_p(h)(2^kR)^{1-1/p}
\end{aligned}$$

Thus

$$T=\sum T_s$$

where

$$T_s=\int_{-1/2}^{1/2}(2^sR)\sum_n \widehat{h}_s(2^sR(t+n))e^{2\pi itP}.$$

so that, by the results of the previous sections,

$$\|h_k(P/R)f\|_{L^p(M)}\lesssim C_p(h)\|f\|_{L^p(M)}\quad\text{uniformly in } R>0,$$

where the implicit constants are also independent of the functions $\{h_k\}$. Now

$$H_{k,R}(t)=\sum_l \chi_{III}(t)(2^kR)\widehat{h}_k(2^kR(t+l))=\sum_l H_{k,R,l}(t).$$

Then we conclude that

$$\left(\sum_{l\neq 0}\left(|2^kRl|^{s_q}\|H_{R,k,l}\|^p_{L^p[-1/2,1/2]}\right)\right)^{1/p}\lesssim (2^kR)^{1/q}C_p(h)$$

and

$$\|H_{R,k,0}\|^p_{L^p[-1/2,1/2]}\lesssim (2^kR)^{1/q-s_q}C_p(h),$$

so that by Hölder

$$\|H_{k,R}\|_{L^p[-1/2,1/2]}\lesssim (2^kR)^{-s_q+1/q}C_p(h).$$

Thus by the local smoothing inequality,

$$\|\int H_{k,R}(t)e^{2\pi itP}\{\chi(P/R2^k)f\}\|_{L^p(M)}\lesssim C_p(h)\|\chi(P/R2^k)f\|_{L^p(M)}$$

Now

$$\|\sum_k H_{k,R}\chi(P/R2^k)f\|_{L^p(M)}\|$$

Let us write

$$h_s = h_{s,\mathrm{Low}} + h_{s,\mathrm{High}},$$

where $\widehat{h}_{s,\mathrm{High}}$ is supported on $|t| \geqslant T$. Then

$$h_{s,\mathrm{High}}(P/R)$$

$$H_{s,R}(t) = \sum \chi_{III}(t)R\widehat{h}_s(Rt)$$

$$h_s(\cdot/R2^s)$$

$$k_s(t) = (R2^s)\widehat{h}_s(2^s\cdot)$$

$$H_{s,R}(t) = \sum \chi_{III}(t)(R2^s)\widehat{h}_s(2^st)$$

Then

References

[1] Sriram Alladi, *A multiplier theorem for ultraspherical polynomials*, Ph.D. thesis, University of Wisconsin-Madison, 2015.

[2] Loukas Grafakos, *Classical Fourier analysis*, Springer, 2014.

[3] Yaryong Heo, Fedor Nazarov, and Andreas Seeger, *Radial Fourier multipliers in high dimensions*, Acta Math. **206** (2011 (ArXiv 2009)), no. 1, 55–92.

[4] Sanghyuk Lee and Andreas Seeger, *Lebesgue space estimates for a class of Fourier integral operators associated with wave propogation*, Math. Nachr. **286** (2013), no. 7, 745–755.

[5] Boris S. Mitjagin, *Divergenz von spektralentwicklungen in L_p-räumen*, Internat. Ser. Numer. Math. **25** (1974), 521–530.

[6] Christopher D. Sogge, *Fourier integrals in classical analysis*, Cambridge University Press, 2017.

[7] Thomas H. Wolff, *Lectures in harmonic analysis*, The American Mathematical Society, 2003.