High Codimension Curves Can't Be Salem

April 5, 2023

Let $\gamma : I \to \mathbb{R}^3$ be a smooth, parametric curve defined on an interval $I \subset \mathbb{R}$, and let $\Gamma = \gamma(I)$ denote the parametric curve's trace. The Hausdorff dimension of Γ is equal to one, being the image of an interval under a diffeomorphism. We claim that the Fourier dimension of Γ is $2/3$, so that Γ is never a Salem set. Marstrand projection theorem variants for Fourier dimension imply that the Fourier dimension of any curve in $\mathbb{R}^d$ for $d \geqslant 3$ has Fourier dimension at most $2/3$, though I imagine similar techniques to those described here can prove the Fourier dimension of such a curve is equal to $2/d$.

Let us make the simplifying assumption that γ', γ'', and γ''' are all nonvanishing on I, and moreover, are linearly independent[1]. There exists a unique, smooth family of unit vectors $\{\xi_0(t) : t \in I\}$ in $\mathbb{R}^d$ such that

$$\xi_0(t) \cdot \gamma'(t) = \xi_0(t) \cdot \gamma''(t) = 0 \quad \text{for all } t \in I,$$

and with

$$\xi_0(t) \cdot \gamma'''(t) > 0 \quad \text{for all } t \in I.$$

It follows by taking a Taylor series in the t variable that we can guarantee that there exists $\varepsilon > 0$ such that for $0 < |t - s| < \varepsilon$, we have

$$\frac{\xi_0(t) \cdot \gamma'(s)}{(s-t)^{d-1}} > 0.$$

If we break up I into a finite union of almost disjoint union of intervals $\{I_j\}$, each with length less than $\varepsilon/3$, and set $\Gamma_j = \gamma(I_j)$, then it follows from (Ekström, Persson, Schmeling, 2015) that

$$\dim_{\mathbb{F}}(\Gamma) = \max_j \dim_{\mathbb{F}}(\Gamma_j).$$

We can therefore choose some j such that $\dim_{\mathbb{F}}(\Gamma_j) = 1$. Swapping out I for I_j, and Γ for Γ_j, we will assume in what follows that for all distinct $t, s \in I$, the smooth function ν agreeing with

$$\frac{\xi_0(t) \cdot \gamma'(s)}{(s-t)^{d-1}}$$

[1] We can probably use Sard's Theorem, or something similar, to reduce the study of any curve to one satisfying this assumption, but let's not get ahead of ourselves.

for distinct $t, s \in I$ is positive. Taking a Taylor series in the s variable, and then letting $s \to 0$ allows us to conclude that $\nu(t,t) = \xi_0(t) \cdot \gamma'''(t)$. We also consider the smooth, positive function $a(t) = (\xi_0(t) \cdot \gamma'''(t))^{1/3}$.

For a measure μ on I, a function $\gamma : I \to \mathbb{R}^3$, and $\xi \in \mathbb{R}^3$, let

$$I_\gamma(\mu, \xi) = \int e^{i\xi\cdot\gamma(t)} d\mu(t).$$

Our goal is to show that for any probability measure μ on I, and any $\varepsilon > 0$,

$$\limsup_{\xi\to\infty} |\xi|^{1/3+\varepsilon} I_\gamma(\mu, \xi) = \infty,$$

which is equivalent to proving that $\dim_{\mathbb{F}}(\Gamma) \leqslant 2/3$.

The following stationary phase result will be useful.

Lemma 1. *There exists a constant Γ such that if f is a C^1 function supported on $[-10, +10]$, then for $t \in I$, and $\lambda > 0$,*

$$I_\gamma(f, \lambda\xi_0(t)) = C\ a(t) f(t) e^{i\lambda\xi_0(t)\cdot\gamma(t)} \lambda^{-1/d} + O(\lambda^{-2/d}),$$

where the implicit constant is upper bounded by a constant multiple of $\|f\|_{L^\infty} + \|f'\|_{L^\infty}$.

Proof. This follows from one-dimensional stationary phase methods (see Erdelyi, in the discussion of Equation (4) of Section 2.9), because we have made the assumption that the function ν above is positive. □

Conversely, we can also apply the principle of nonstationary phase.

Lemma 2. *Suppose that if f is a C^1 function supported on an interval of length L, ξ is a unit vector in $\mathbb{R}^d$, and and $|\xi \cdot \gamma'(t)| \geqslant \varepsilon$ for all $t \in I$. Then*

$$I_\gamma(f, \lambda\xi) \lesssim_\gamma \frac{L}{\lambda} \left(\frac{\|f'\|_{L^\infty}}{\varepsilon} + \frac{\|f\|_{L^\infty}}{\varepsilon^2} \right).$$

Proof. We integrate by parts, calculating that

$$\begin{aligned}
\left| \int e^{i\lambda\xi\cdot\gamma(t)} f(t)\ dt \right| &= \frac{1}{\lambda} \left| \int \frac{d}{dt} \left\{ e^{i\lambda\xi\cdot\gamma(t)} \right\} \frac{f(t)}{\xi\cdot\gamma'(t)}\ dt \right| \\
&= \frac{1}{\lambda} \left| \int e^{i\lambda\xi\cdot\gamma(t)} \left(\frac{f'(t)}{\xi\cdot\gamma'(t)} - \frac{f(t)}{(\xi\cdot\gamma'(t))^2} (\xi\cdot\gamma''(t)) \right)\ dt \right| \\
&\lesssim_\gamma \frac{L}{\lambda} \left(\frac{\|f'\|_{L^\infty}}{\varepsilon} + \frac{\|f\|_{L^\infty}}{\varepsilon^2} \right).
\end{aligned}$$

□

Lemma 3. *Let $\gamma_M(t) = (t, t^2, t^3)$ be the parameterization of the moment curve $\Gamma_M = \gamma_M(\mathbb{R})$. For any $\varepsilon \in (0, 1/100)$, if t_0 is a fixed time, ξ_0 is one of the vectors orthogonal to both $\gamma_M'(t_0)$ and $\gamma_M''(t_0)$, $\lambda \gtrsim_\varepsilon 1$, then*

$$\sup_{|\xi-\lambda\xi_0|\leqslant\varepsilon\lambda} |\xi|^{1/3} |I_{\gamma_M}(\mu, \lambda\xi)| \lesssim_\varepsilon 1.$$

Proof. Fix $\delta > 0$ and $\lambda \geqslant 1$, and suppose there was a probability measure μ compactly supported on some interval I such that

$$\sup_{|\xi - \lambda\xi_0| \leqslant \lambda\varepsilon} |\xi|^{1/3} |I_{\gamma_M}(\mu, \xi)| \leqslant \delta.$$

Define a linear transformation

$$A_h = \begin{pmatrix} 1 & 0 & 0 \\ 2h & 1 & 0 \\ 3h^2 & 3h & 1 \end{pmatrix}.$$

Then $A_h \gamma_M(t) = \gamma_M(h) + \gamma_M(t+h)$ for all $t, h \in \mathbb{R}$. If $\gamma_{M,h}(t) = \gamma_M(t+h)$, we thus have

$$\begin{aligned} I_{\gamma_{M,h}}(\mu, \xi) &= \int e^{i\xi\cdot\gamma(t+h)} d\mu(t) \\ &= e^{-i\xi\cdot\gamma(h)} \int e^{i\xi\cdot A_h\gamma(t)} d\mu(t)\ dt \\ &= e^{-i\xi\cdot\gamma(h)} \int e^{i(A_h^T\xi)\cdot\gamma(t)} d\mu(t)\ dt \\ &= e^{-i\xi\cdot\gamma(h)} I_{\gamma_M}(\mu, A_h^T\xi). \end{aligned}$$

If we consider an L^1 normalized smooth bump function $\phi : \mathbb{R} \to \mathbb{R}$ adapted to $\{|h| \leqslant \varepsilon/2\}$, and define a smooth function $f = \phi * \mu$, then

$$I_{\gamma_M}(f, \lambda\xi_0) = \int \phi(h) I_{\gamma_{M,h}}(\mu, \lambda\xi_0)\ dh = \int \phi(h) e^{-i\xi\cdot\gamma_M(h)} I(\mu, \lambda A_h^T\xi_0)\ dh.$$

Then the L^∞ norm of f and f' is $O_\varepsilon(1)$, and $f(t_0) \gtrsim_\varepsilon 1$, so we conclude that

$$I_{\gamma_M}(f, \lambda\xi_0) = C\ a(t_0) f(t_0) e^{i\lambda\xi_0} \lambda^{-1/3} + O_\varepsilon(\lambda^{-2/3}).$$

In particular, we conclude that for $\lambda \gtrsim_\varepsilon 1$,

$$|I_{\gamma_M}(f, \lambda\xi_0)| \gtrsim C_\varepsilon \lambda^{-1/3}.$$

Now $|A_h^T\xi_0 - \xi_0| \leqslant 4h|\xi_0|$ for $|h| \leqslant 1/100$, we know by assumption that $|I(\mu, \lambda A_h^T\xi_0)| \leqslant \delta\lambda^{-1/3}$. But this means we conclude that

$$\lambda^{-1/3} \lesssim_\varepsilon \delta\lambda^{-1/3},$$

and thus that $\delta \gtrsim_\varepsilon 1$, completing the proof. $\square$

For any measure μ on I, we fix $\delta > 0$, and consider a family of $O(\delta^{-1})$ points $\mathcal{T}$ such that the length δ intervals $\{I_t : t \in \mathcal{X}_\delta\}$ with center t cover $[0,1]$, and for each t, the middle third of the interval I_t is disjoint from $I_{t'}$ for $t \neq t'$. Consider a smooth partition of unity $\{\chi_t\}$ adapted to these intervals. For each $t \in \mathcal{T}$, define $\mu_t = \chi_t\mu$. For any $t \in \mathcal{T}$, consider the degree three polynomial curve $\gamma_t : \mathbb{R} \to \mathbb{R}^d$ given by

$$\gamma_t(s) = \gamma(t) + \gamma'(t)(s-t) + \frac{\gamma''(t)}{2}(s-t)^2 + \frac{\gamma'''(t)}{6}(s-t)^3.$$

then for any $t' \in I_t$, $|\gamma(t') - \gamma_t(t')| \lesssim \delta^4$. This means that the deviations between γ and γ_t, once localized to a δ neighborhood of t, should be undetectable for frequencies with magnitude $O(\delta^{-4})$, i.e. for $|\xi| \lesssim \delta^{-4}$, we should expect to have

$$I_\gamma(\mu, \xi) \approx \sum_t I_{\gamma_t}(\mu_t, \xi).$$

If we let B_t be the matrix with columns $\delta^{-1}\gamma'(t)$, $\delta^{-2}\gamma''(t)/2$, and $\delta^{-3}\gamma'''(t)/6$, then

$$\gamma_t(s) - \gamma(t) = B_t \gamma_M(\delta(s-t)).$$

Thus if ν_t is the dilation of $\mathrm{Trans}_{-t}\mu_t$ by a factor $1/\delta$, then

$$\begin{aligned}
I_{\gamma_t}(\mu_t, \xi) &= \int e^{i\xi\cdot\gamma_t(s)} d\mu_t(s) \\
&= \int e^{i\xi\cdot[\gamma(t) + B_t\gamma_M((s-t)/\delta)]}\, d\mu_t(s) \\
&= e^{i\xi\cdot\gamma(t)} \int e^{i(B_t^T\xi)\cdot\gamma_M((s-t)/\delta)} d\mu_t(s) \\
&= e^{i\xi\cdot\gamma(t)} I_{\gamma_M}(\nu_t, B_t^T\xi).
\end{aligned}$$

Thus we get

$$I_\gamma(\mu, \xi) \approx \sum_t e^{i\xi\cdot\gamma(t)} I_{\gamma_M}(\nu_t, B_t^T\xi).$$

for $|\xi| \ll \delta^{-4}$. We now consider an L^1 normalized, smooth bump function ϕ supported on a width δ interval about the origin, and define $f_t = \nu_t * \phi$. We have seen that

$$I_{\gamma_M}(f_t, B_t^T\xi) = \int \phi(h) e^{-i\xi\cdot\gamma_M(h)} I(\nu_t, A_h^T B_t^T \xi)\, dh.$$

Suppose (THIS IS THE CHEAT) we can find a matrix C_h such that $A_h^T B_t^T \xi = B_t^T C_h \xi$. Then

$$\sum_t I_{\gamma_M}(f_t, B_t^T\xi) = \sum_t \int \phi(h) e^{-i\xi\cdot\gamma_M(h)} I(\nu_t, B_t^T C_h \xi) \approx \int \phi(h) e^{-i\xi\cdot\gamma_M(h)} I_\gamma(\mu, C_h\xi)\, dh.$$

Then C_0 is the identity matrix, and so we can imagine that $|C_h\xi| \sim |\xi|$ for small h.

We can now argue that $\dim_{\mathbb{F}}(\Gamma) \leqslant 2/3$. Suppose that instead, we could choose μ such that

$$\limsup_{\xi\to\infty} |\xi|^{2/3+\varepsilon} |I_\gamma(\mu, \xi)| < \infty.$$

Then for any $\xi \in \mathbb{R}^d$, the right hand side of the identity above satisfies estimates of the form

$$\left| \int \phi(h) e^{-i\xi\cdot\gamma_M(h)} I_\gamma(\mu, C_h\xi)\, dh \right| \lesssim |\xi|^{-1/3-\varepsilon}.$$

For $|\xi| \sim \delta^{-4}$, we get that this quantity is $\lesssim \delta^{4/3+\varepsilon}$. On the other hand, the left hand side is a sum of quantities to which we can apply stationary and nonstationary phase. If we choose $c > 0$ small enough, depending on γ, then because of the linear independence of γ', γ'', and γ''', if, for $t_0 \in \mathcal{T}$, we set $\xi = \xi_0(t_0)$, then for any $t \neq t_0$, and any $t' \in I_t$, $|\xi \cdot \gamma'(t')| \geqslant c\delta$. This implies that the principle of nonstationary phase can be applied to the quantity $I_\gamma(\nu_t, B_t^T \xi)$. For each t_0, the function f_{t_0} has L^∞ norm at most $O(\delta^{-1}\nu_{t_0}(\mathbb{R}))$, and f'_{t_0} has L^∞ norm bounded by $O(\delta^{-2}\nu_{t_0}(\mathbb{R}))$. Applying the principle of nonstationary phase, for $t \neq t_0$ we conclude that

$$|I_{\gamma_M}(f_t, B_t^T \xi)| \lesssim \delta^{-2}\nu_{t_0}(\mathbb{R})|\xi|^{-1}.$$

Summing over $t \neq t_0$ gives that

$$\sum_{t \neq t_0} |I_{\gamma_M}(f_t, B_t^T \xi)| \lesssim \delta^{-2}|\xi|^{-1}.$$

If we take $|\xi| \sim \delta^{-4}$, this quantity is $O(\delta^2)$. On the other hand, we have $f_{t_0}(t_0) \gtrsim \delta^{-1}\nu_{t_0}(\mathbb{R})$, and so the principle of stationary phase we calculated at the beginning of our argument shows that

$$|I_{\gamma_M}(f_{t_0}, B_{t_0}^T \xi)| \gtrsim \delta^{-1}\nu_{t_0}(\mathbb{R})|\xi|^{-1/3}$$

so for $|\xi| \sim \delta^{-4}$, we get that this quantity is $\gtrsim \delta^{1/3}\nu_{t_0}(\mathbb{R})$. Since $\sum_t \nu_t(\mathbb{R}) = \mu(\mathbb{R}) = 1$, the pigeonhole principle implies we can pick some t_0 such that $\nu_{t_0}(\mathbb{R}) \gtrsim \delta$. But then the quantity above is $\gtrsim \delta^{4/3}$. But putting these bounds together gives that

$$|\sum_t I_{\gamma_M}(f_t, B_t^T \xi)| \geqslant |I_{\gamma_M}(f_{t_0}, B_{t_0}^T \xi)| - \sum_{t \neq t_0} |I_{\gamma_M}(f_t, B_t^T \xi)| \gtrsim \delta^{4/3}.$$

But we therefore conclude that $\delta^{4/3} \lesssim \delta^{4/3+\varepsilon}$, which gives a contradiction if δ is taken appropriately small.