High Codimension Curves Can't Be Salem

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1 New Strategies

Let $U \subset \mathbb{R}^k$ be an open set, and consider a smooth immersion $\gamma : U \to \mathbb{R}^d$. For a Borel probability measure μ supported on U, and $\xi \in \mathbb{R}^d$, we let

$$I(\mu, \xi) = \int_U e^{2\pi i \xi \cdot \gamma(x)} \, d\mu(x) = \widehat{\gamma_* \mu}(\xi).$$

Our goal is to prove the following Lemma.

TODO: By a translation argument, we may assume that $\gamma : 2Q \to \mathbb{R}^d$

Lemma 1. *Let Q be a closed, axis-oriented cube, such that $2Q \subset U$. Suppose that there exists a Borel probability measure μ supported on Q such that*

$$\sup_{\xi \in \mathbb{R}^d} |\xi|^{s/2} |\widehat{\gamma_* \mu}(\xi)| < \infty.$$

Then there exists a non-negative smooth function ϕ, supported in $2Q$, such that

$$\int_U \phi(x) \, dx = 1,$$

i.e. such that the measure $\mu_\phi = \phi \, dx$ is a probability measure, and such that

$$\sup_{\xi \in \mathbb{R}^d} |\xi|^{s/2} |\widehat{\gamma_* \mu_\phi}(\xi)| < \infty.$$

Proof. Without loss of generality, assume γ gives the graph of a function, i.e.

$$\gamma(x) = (x, a(x))$$

for some smooth function a. Let $\xi_0 = e_{k+1}$.

Since γ is an immersion, for any fixed x_0, there exists a coordinate system z, defined in a neighborhood of $\gamma(x_0)$, such that

$$z(\gamma(x)) = (x, 0).$$

Consider the covector field $\omega = \xi_0 dx$. Assume without loss of generality that $\omega(0,0) = dz^{k+1}$. Then

$$\{dz^1, \dots, dz^k, \omega, dz^{k+2}, \dots, dz^n\}$$

are linearly independent covector fields in a neighborhood of $\gamma(x_0)$, so we can find a coordinate system w, such that $w(\gamma(x_0)) = (x_0, 0)$, such that $dw^j = dz^j$ for $j \neq k+1$, and such that $dw^{k+1} = \omega$. Then we actually see by the assumptions that $w^j = z^j$ for $j \neq k+1$.

$$w(\gamma(x_0)) = 0, \quad w_*(dz^j) = dw^j, \quad \text{and} \quad w_*(\omega) = dw^{k+1}.$$

These assumptions imply that for $1 \leqslant j \leqslant k$,

$$d(w^j \circ z \circ \gamma) = \gamma^* z^* dw^j = \gamma^* dz^j = dx^j.$$

Thus we actually have $w^j(z(\gamma(x))) = x^j$.

$w(\gamma(x_0)) = 0$, $w_*(dz^j) = dw^j$, and $w_*(\omega) = dw^{k+1}$. But then

$$(w \circ z)(\gamma(x_0)) = (0,)$$

Then $\{dz^1, \dots, dz^k, \xi_0 dx\}$ are linearly independent covector fields in a neighborhood of $\gamma(x_0)$, and thus there exists a coordinate system w, defined in a neighborhood of $\gamma(x_0)$, such that $(w^1, \dots, w^k) = (z^1, \dots, z^k)$, and $dw^{k+1} = \xi_0 \cdot dx$. Now, for each $v \in \mathbb{R}^k$ with $|v| < \delta$, we define a diffeomorphism A_v in a neighborhood of $\gamma(x_0)$ by setting

$$(w \circ A_v \circ w^{-1})(w^1, \dots, w^k) = (w^1, \dots, w^k) + (v, 0).$$

These diffeomorphisms are chosen precisely so that, for each x in a neighborhood of $\gamma(x_0)$,

$$A_v(\gamma(x)) = \gamma(x+v),$$

because $w(\gamma(x)) = (x, 0)$ and $w(\gamma(x+v)) = (x+v, 0)$, and so

$$w(A_v(\gamma(x))) = (x+v, 0) = w(A_v(\gamma(x+v))).$$

and also, for $|v| < \delta$,

$$DA_v(y)^T(\xi_0) = \xi_0,$$

which can be verified in the language of differential forms by noting that

$$A_v^*(\xi_0 dx) = A_v^*(dw^{k+1}) = d(w^{k+1} \circ A_v) = dw^{k+1} = \xi_0 dx,$$

i.e. so that the covector field $\xi_0 dx$ is preserved by the diffeomorphisms $\{A_v\}$.

Consider a smooth, non-negative bump function ψ on $\mathbb{R}^d$, which is equal to one on a neighborhood of $\gamma(x_0)$. For small v, consider the measure $\mu_v = \text{Trans}_v \mu$. We calculate using the multiplication formula that

$$\begin{aligned}
\widehat{\gamma_* \mu_v}(\lambda \xi_0) &= \int_U e^{2\pi i \lambda \xi_0 \cdot \gamma(x+v)}\, d\mu(x) \\
&= \int_U e^{2\pi i \lambda \xi_0 \cdot A_v(\gamma(x))}\, d\mu(x) \\
&= \int_{\mathbb{R}^d_y} e^{2\pi i \lambda \xi_0 \cdot A_v(y)}\, d(\gamma_* \mu)(y).
\end{aligned}$$

Note that $\nabla_y \{\xi_0 \cdot A_v(y)\} = A_v(y)^T \xi_0 = \xi_0$, so that

$$\begin{aligned} \xi_0 \cdot A_v(y) &= \xi_0 \cdot A_v(\gamma(x_0)) + \xi_0 \cdot (y - \gamma(x_0)) \\ &= \xi_0 \cdot [\gamma(x_0 + v) - \gamma(x_0)] + \xi_0 \cdot y. \end{aligned}$$

Thus

$$\widehat{\gamma_* \mu_v}(\lambda \xi_0) = e^{2\pi i \lambda \xi_0 \cdot [\gamma(x_0+v) - \gamma(x_0)]} \widehat{\gamma_* \mu}(\lambda \xi_0).$$

Write $\phi = \xi_0 \cdot A_v(y) - \eta \cdot y$. Then

$$\nabla_y \phi = DA_v(y)^T \xi_0 - \eta = \xi_0 - \eta$$

is independent of y. Thus we can write

$$\phi = c(\xi_0, v, \eta) + (\xi_0 - \eta) \cdot y.$$

Then

$$|I(y, \lambda, \nu)| |\widehat{\psi}(\eta - \xi_0)|$$

We can upper bound the magnitude of I using nonstationary phase, i.e. because we can write

$$I(\eta, v, \lambda) = \int_{\mathbb{R}^d_y} \psi(y) e^{2\pi i \lambda \phi(y, \eta, v)}\, dy,$$

where

$$\phi(y, \eta, v) = [\xi_0 \cdot A_v(y) - \eta \cdot y].$$

Then $\nabla_y \phi(y, \eta, v) = DA_v(y)^T \xi_0 - \eta = \xi_0 - \eta$, i.e. so that we actually have

$$\phi(y, \eta, v) = c(\xi_0, v) + (\xi_0 - \eta) \cdot y.$$

But this means that

$$I(\eta, v, \lambda) = c(\xi_0, v) \widehat{\psi}$$

where

$$I(\eta, v, \lambda) = \int_{\mathbb{R}^d_y} \psi(y) e^{2\pi i \lambda [\xi_0 \cdot A_v(y) - \eta \cdot y]}\, dy = \int_{\mathbb{R}^d_y} \psi(y) e^{2\pi i \lambda \phi(y; \eta, v)}\, dy.$$

We calculate that

$$\nabla_y \phi(y; \eta, \lambda, v) = DA_v(y)^T \xi_0 - \eta.$$

Our choice of diffeomorphisms $\{A_v\}$ implies that $DA_v(y)^T \xi_0 = \xi_0$ for all y. Thus

$$\nabla_y \phi(y; \eta, \lambda, v) = \xi_0 - \eta.$$

Thus we can apply integration by parts to conclude that

$$|I(\eta, v, \lambda)| \lesssim_N \lambda^{-N} |\xi_0 - \eta|^{-N}.$$

Thus we conclude that

$$\begin{aligned}
\lambda^d \int_{|\eta-\xi_0|\geqslant\lambda^{-\alpha}} & I(\eta,v,\lambda)\widehat{\gamma_*\mu}(\lambda\eta)\, d\eta \\
&\lesssim_N \lambda^{d-N} \int_{|\eta-\xi_0|\geqslant\lambda^{-\alpha}} |\xi_0-\eta|^{-N} \lesssim 1 \\
&= \lambda^{d-N} \int_{\lambda^{-\alpha}}^{\infty} t^{d-1-N}\, dt \\
&\lesssim \lambda^{(1-\alpha)(d-N)}.
\end{aligned}$$

If $\alpha = 1 - [s/2(N-d)]$, we obtain that this integral is $O(\lambda^{-s/2})$. Taking N arbitrarily large allows us to pick α arbitrarily close to one. Then

$$\begin{aligned}
\lambda^d \int_{|\eta-\xi_0|\leqslant\lambda^{1-\varepsilon/d}} & I(\eta,v,\lambda)\widehat{\gamma_*\mu}(\lambda\eta)\, d\eta \\
&\leqslant \lambda^d \int_{|\eta-\xi_0|\leqslant\lambda^{1-\varepsilon/d}} \lambda^{-s/2} \\
&= \lambda^{d-(1-\varepsilon/d)d-s/2} = \lambda^{\varepsilon-s/2}.
\end{aligned}$$

Combining these calculations allows us to conclude that

$$|\widehat{\gamma_*\mu_v}(\lambda\xi_0)| \lesssim_\varepsilon \lambda^{\varepsilon-s/2}.$$

We start with some basic techniques from the study of differential manifolds. Write the standard coordinates of $\mathbb{R}^k$ by $(x^1,\dots,x^k)$, and the standard coordinates of $\mathbb{R}^d$ by $(y^1,\dots,y^d)$. Applying implicit function theorem type techniques (see Theorem 10 of Spivak, Vol 1, Chapter 2), for any $x_0 \in \mathbb{R}^k$, we can find a coordinate system z defined in a neighborhood of $\gamma(x_0)$ such that

$$z(\gamma(x)) = (x,0).$$

Set $w^j(x) = z^j(x)$ for $1 \leqslant j \leqslant k$, and let $w^{k+1}(x) = x \cdot \xi_0$. Then $dw^{k+1} = \xi_0 dx$, and $\{dw^1,\dots,dw^{k+1}\}$ are linearly independent at $\gamma(x_0)$, so we can extend these functions to a coordinate system w defined in a neighborhood of $\gamma(x_0)$. Now we consider a family of diffeomorphisms $\{A_v\}$ defined in a neighborhood of $\gamma(x_0)$, and for small $v \in \mathbb{R}^k$, such that

$$(w \circ A_v \circ w^{-1})(w^1,\dots,w^d) = (w^1,\dots,w^d) + (v,0).$$

Then $\{A_v\}$ is chosen precisely so that for x in a neighborhood of x_0,

$$A_v(\gamma(x)) = \gamma(x+v),$$

and also,

$$A_v^*(\xi_0 dx) = A_v^*(dw^{k+1}) = d(w^{k+1}\circ A_v) = dw^{k+1} = \xi_0 dx.$$

Thus the covector field $\xi_0 dx$ is preserved by the family of diffeomorphisms $\{A_v\}$.

$\square$

if and only if there exists a smooth function $\phi : U \to \mathbb{R}$, supported on a compact subset of U, such that if $\nu = \gamma_*(\phi\, dx)$, then

$$|\widehat{\nu}(\xi)| \lesssim |\xi|^{-s/2}.$$

We do this by using stationary phase to show that 'translates' of μ continue to have good FOurier decay estimates, which allows us to show that a convolution of μ with a smooth, compactly supported

Let $\gamma : \mathbb{R} \to \mathbb{R}^d$ be an immersed curve. We study the problem of finding $\xi_0, \xi_1 \in \mathbb{R}^d$ and a smooth map $A : \mathbb{R}^d \to \mathbb{R}^d$, such that $A(\gamma(t)) = \gamma(t+1)$ for all $t \in \mathbb{R}$, and such that $A^*(\xi_0\, dx) = \xi_1\, dx$.

This is certainly possible for some γ, but *not all* γ. Indeed, if $\gamma(t) = (t, t^2, t^3)^T$, then

$$\gamma(t+1) = (t+1, (t+1)^2, (t+1)^3)^T = (1,1,1)^T + t \cdot (1,2,3)^T + t^2 \cdot (0,1,3)^T + t^3 \cdot (0,0,1)^T.$$

Thus if

$$A(x,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{pmatrix} (x,y,z)^T + (1,1,1)^T.$$

Then for any ξ_0, we can pick

$$\xi_1 = \xi_0 \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{pmatrix},$$

and the result will work.

Let's investigate conditions to guarantee ξ_0 is a candidate solution to this problem? If $T(t) = t+1$, then

$$A \circ \gamma = \gamma \circ T.$$

Thus $\gamma^*(A^*\omega) = T^*(\gamma^*\omega)$. But

$$\gamma^*(A^*\omega) = \gamma^*(\xi_1\, dx) = \xi_1 \cdot \gamma'(t)\, dt$$

and

$$T^*(\gamma^*\omega) = T^*(\xi_0 \cdot \gamma'(t)\, dt) = \xi_0 \cdot \gamma'(t+1)\, dt.$$

Thus we conclude that we must have $\xi_1 \cdot \gamma'(t) = \xi_0 \cdot \gamma'(t+1)$ for all t. In the case of the moment curve, if $\xi_j = (a_j, b_j, c_j)$, then

$$a_1 + 2b_1 t + 3c_1 t^2 = a_0 + 2(t+1)b_0 + 3(t+1)^2 c_0$$

which allows us to conclude that

$$\xi_1 = \xi_0 \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{pmatrix}.$$

In general, we *cannot* expect a pair ξ_0 and ξ_1 to exist as above.

Is it instead possible to change coordinates so that $A^*(\xi_0 dx) = \xi(x)dx$, where ξ is an *injective* function (i.e. so that the oscillatory integrals below have no stationary points).

? Consider the constant vector field $\omega = \xi_0\, dx$ for some fixed $\xi_0 \in \mathbb{R}^d$. Is it possible for $A^*\omega$ to be a constant vector field, i.e. for there to exist ξ_1 such that $A^*\omega = \xi_1\, dx$?

For some γ, this is possible.

Consider the covector field ω on $\mathbb{R}^d$ which is a constant vector field in the usual coordinates, i.e. $\omega = \xi_0\, dx$. If i is the inclusion map of the curve into $\mathbb{R}^d$, then $i = A \circ i$, which implies that

$$i^* = i^* \circ A^*.$$

Is it possible for $DA_h(x)$

2 Explicit Example

Let $\gamma(t) = (t, \sin(t), e^t)$, and set $\xi_0 = (0,0,1)$. We try and apply the strategy in the last section to this particular example. Set

$$(x', y', z') = C(x,y,z) = (x, y - \sin(x), z - e^x).$$

Then $C(\gamma(t)) = (t,0,0)$, i.e. the image of γ is the x'-axis. Since $z = z' + e^x = z' + e^{x'}$, we conclude that

$$dz = dz' + e^{x'} dx'$$

Our goal is to choose a *new* set of coordinates (x,y,z), which map the x'-axis to the x-axis, but such that $dz' + e^{x'} dx'$ is a covector field in (x,y,z) with constant coefficients.

We need to choose (x,y,z) such that when $y' = z' = 0$, $y = z = 0$ and $x = x'$. At least up to second order, we should probably guess a solution is of the form

$$x = x' + y'v_1 + z'w_1 \quad y = y'v_2 + z'w_2 \quad z = y'v_3 + z'w_3.$$

When $y' = z' = 0$, we thus have

$$dx = dx' + v_1 dy' + w_1 dz' \quad dy = v_2 dy' + w_2 dz' \quad dz = v_3 dy' + w_3 dz'.$$

Inverting this equation, assuming that $v_2 w_3 - w_2 v_3 = 1$, we conclude that

$$dx' = dx + (v_3 w_1 - v_1 w_3) dy + (v_1 w_2 - v_2 w_1) dz$$

$$dy' = w_3 dy - w_2 dz \quad \text{and} \quad dz' = -v_3 dy + v_2 dz.$$

Thus

$$\begin{aligned} e^{x'} dx' + dz' &= e^x dx' + dz' \\ &= e^x(dx + (v_3 w_1 - v_1 w_3) dy + (v_1 w_2 - v_2 w_1) dz) \\ &\quad + (-v_3 dy + v_2 dz) \\ &= e^x dx + [e^x (v_3 w_1 - v_1 w_3) - v_3] dy + [e^x (v_1 w_2 - v_2 w_1) + v_2] dz. \end{aligned}$$

In particular, if $x = x'$, $y = e^x y'$ and $z = e^x y' + e^{-x} z'$, then for $y' = z' = 0$,

$$dx = dx' \quad dy = e^x dy' \quad dz = e^x dy' + e^{-x} dz'$$

and so $dy' = e^{-x} dy$ and $dz' = -e^x dy + e^x dz$, so

$$e^x dx' + dz' = e^x (dx - dy + dz).$$

TODO: Can we always guarantee a vector times a scalar multiple of x?

If we assume that $v_2 w_3 - v_3 w_2 = 1$, then for $y' = z' = 0$,

$$dz' + e^{x'} dx' = (e^x - (w_1 v_2 - w_2 v_1)) dx - w_2 dy + v_2 dz.$$

$$x = x' + e^x z'$$

$$y = y' + z'$$
$$z = y' + 2z'$$
$$dx = dx' + e^x dz'$$
$$dy = dy' + dz'$$
$$dz = dy' + 2dz'$$

Then

$$dx' = dx + e^x dy - e^x dz$$
$$dy' = 2dy - dz$$
$$dz' = -dy + dz$$

Thus

In particular, if, for $y' = z' = 0$,

$$x = x' + e^x z'$$

$$y = y'$$
$$z = z'$$

Then $dz' + e^{x'} dx' = dz + e^x(dx - e^x dx')$ $dx = dx' + e^x dz'$

This is an invertible set of linear equations, so for any (a, b, c), there exists a smooth family of functions $v_j(x', 0, 0)$ and $w_j(x', 0, 0)$ such that $dz' + e^{x'} dx' = adx + bdy + cdz$ when $y' = z' = 0$. Now how do we extend this

Define

$$C(x, y, z) = (x, y - \sin(x), z - e^x).$$

Then $C(\gamma(t)) = (t, 0, 0)$ for all t, and

$$C^{-1}(x, y, z) = (x, y + \sin(x), z + e^x).$$

If we pull back ξ_0 by C^{-1}, then because

$$DC^{-1}(x, y, z) = \begin{pmatrix} 1 & 0 & 0 \\ \cos(x) & 1 & 0 \\ e^x & 0 & 1 \end{pmatrix},$$

The pushforward of the constant covector field ξ_0 is equal to

$$e^x \, dx + dz.$$

Our goal is to 'flatten' this covector field, while fixing x axis. Suppose we have

If we set

$$B(x, y, z) = (x + ye^x, y, z).$$

Then

$$B^{-1}(x, y, z) = (x, y, z)$$

$$
\begin{aligned}
x' &= x + yv_1 + zw_1 \\
y' &= y + yv_2 + zw_2 \\
z' &= z + yv_3 + zw_3
\end{aligned}
$$

then $y = z = 0$,

$$
\begin{aligned}
dx' &= dx + v_1 dy + w_1 dz \\
dy' &= dy + v_2 dy + w_2 dz \\
dz' &= dz + v_3 dy + w_3 dz
\end{aligned}
$$

$$
\begin{pmatrix} 1 & v_1 & w_1 \\ 0 & 1+v_2 & w_2 \\ 0 & v_3 & 1+w_3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a + bv_1 + cw_1 \\ (1+v_2)b + cw_2 \\ v_3 b + (1+w_3)c \end{pmatrix}
$$

possible when $y = z = 0$. So now, given $v(x,0,0)$ and $w(x,0,0)$, can we extend these solutions in such a way that the covector field remains constant. Set

$$
\begin{aligned}
v(x,y,z) &= v(x,0,0) \\
w(x,y,z) &= w(x,0,0)
\end{aligned}
$$

$a + bv_1 + cw_1 = e^x$

Then C is a diffeomorphism from a neighborhood of the origin to a neighborhood of the origin, with

$$
DC(x,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ -\cos(x) & 1 & 0 \\ -e^x & 0 & 1 \end{pmatrix}.
$$

Thus

$$
DC(x,y,z)^T \xi_0 = \begin{pmatrix} 1 & -\cos(x) & -e^x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -e^x \\ 0 \\ 1 \end{pmatrix}.
$$

Note that

$$
C^{-1}(x,y,z) = (x, y + \sin(x), z).
$$

If we define

$$
\begin{aligned}
A_h(x,y,z) &= C^{-1}\left(C(x,y,z) + (h,0,0)\right) \\
&= C^{-1}\left((x+h, y - \sin(x), z)\right) \\
&= (x+h, y - \sin(x) + \sin(x+h), z),
\end{aligned}
$$

then $A_h(\gamma(t)) = \gamma(t+h)$, and

$$
DA_h(x,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ \cos(x+h) - \cos(x) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

and

$$
DA_h(x,y,z)^T \xi_0 = \begin{pmatrix} 1 & \cos(x+h) - \cos(x) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},
$$

which is independent of x.

3 Old Strategy

Let $\gamma : I \to \mathbb{R}^3$ be a smooth, parametric curve defined on an interval $I \subset \mathbb{R}$, and let $\Gamma = \gamma(I)$ denote the parametric curve's trace. The Hausdorff dimension of Γ is equal to one, being the image of an interval under a diffeomorphism. We claim that the Fourier dimension of Γ is $2/3$, so that Γ is never a Salem set. Marstrand projection theorem variants for Fourier dimension imply that the Fourier dimension of any curve in $\mathbb{R}^d$ for $d \geqslant 3$ has Fourier dimension at most $2/3$, though I imagine similar techniques to those described here can prove the Fourier dimension of such a curve is equal to $2/d$.

Let us make the simplifying assumption that γ', γ'', and γ''' are all nonvanishing on I, and moreover, are linearly independent[1]. There exists a unique, smooth family of unit vectors $\{\xi_0(t) : t \in I\}$ in $\mathbb{R}^d$ such that

$$\xi_0(t) \cdot \gamma'(t) = \xi_0(t) \cdot \gamma''(t) = 0 \quad \text{for all } t \in I,$$

and with

$$\xi_0(t) \cdot \gamma'''(t) > 0 \quad \text{for all } t \in I.$$

It follows by taking a Taylor series in the t variable that we can guarantee that there exists $\varepsilon > 0$ such that for $0 < |t - s| < \varepsilon$, we have

$$\frac{\xi_0(t) \cdot \gamma'(s)}{(s-t)^{d-1}} > 0.$$

If we break up I into a finite union of almost disjoint union of intervals $\{I_j\}$, each with length less than $\varepsilon/3$, and set $\Gamma_j = \gamma(I_j)$, then it follows from (Ekström, Persson, Schmeling, 2015) that

$$\dim_{\mathbb{F}}(\Gamma) = \max_j \dim_{\mathbb{F}}(\Gamma_j).$$

We can therefore choose some j such that $\dim_{\mathbb{F}}(\Gamma_j) = 1$. Swapping out I for I_j, and Γ for Γ_j, we will assume in what follows that for all distinct $t, s \in I$, the smooth function ν agreeing with

$$\frac{\xi_0(t) \cdot \gamma'(s)}{(s-t)^{d-1}}$$

for distinct $t, s \in I$ is positive. Taking a Taylor series in the s variable, and then letting $s \to 0$ allows us to conclude that $\nu(t,t) = \xi_0(t) \cdot \gamma'''(t)$. We also consider the smooth, positive function $a(t) = (\xi_0(t) \cdot \gamma'''(t))^{1/3}$.

For a measure μ on I, a function $\gamma : I \to \mathbb{R}^3$, and $\xi \in \mathbb{R}^3$, let

$$I_\gamma(\mu, \xi) = \int e^{i\xi \cdot \gamma(t)} d\mu(t).$$

Our goal is to show that for any probability measure μ on I, and any $\varepsilon > 0$,

$$\limsup_{\xi \to \infty} |\xi|^{1/3+\varepsilon} I_\gamma(\mu, \xi) = \infty,$$

which is equivalent to proving that $\dim_{\mathbb{F}}(\Gamma) \leqslant 2/3$.

The following stationary phase result will be useful.

[1] We can probably use Sard's Theorem, or something similar, to reduce the study of any curve to one satisfying this assumption, but let's not get ahead of ourselves.

Lemma 2. *There exists a constant* Γ *such that if* f *is a* C^1 *function supported on* $[-10, +10]$*, then for* $t \in I$*, and* $\lambda > 0$*,*

$$I_\gamma(f, \lambda\xi_0(t)) = C\ a(t)f(t)e^{i\lambda\xi_0(t)\cdot\gamma(t)}\lambda^{-1/d} + O(\lambda^{-2/d}),$$

where the implicit constant is upper bounded by a constant multiple of $\|f\|_{L^\infty} + \|f'\|_{L^\infty}$*.*

Proof. This follows from one-dimensional stationary phase methods (see Erdelyi, in the discussion of Equation (4) of Section 2.9), because we have made the assumption that the function ν above is positive. $\square$

Conversely, we can also apply the principle of nonstationary phase.

Lemma 3. *Suppose that if* f *is a* C^1 *function supported on an interval of length* L*,* ξ *is a unit vector in* $\mathbb{R}^d$*, and and* $|\xi \cdot \gamma'(t)| \geqslant \varepsilon$ *for all* $t \in I$*. Then*

$$I_\gamma(f, \lambda\xi) \lesssim_\gamma \frac{L}{\lambda}\left(\frac{\|f'\|_{L^\infty}}{\varepsilon} + \frac{\|f\|_{L^\infty}}{\varepsilon^2}\right).$$

Proof. We integrate by parts, calculating that

$$\begin{aligned}
\left|\int e^{i\lambda\xi\cdot\gamma(t)} f(t)\ dt\right| &= \frac{1}{\lambda}\left|\int \frac{d}{dt}\left\{e^{i\lambda\xi\cdot\gamma(t)}\right\}\frac{f(t)}{\xi\cdot\gamma'(t)}\ dt\right| \\
&= \frac{1}{\lambda}\left|\int e^{i\lambda\xi\cdot\gamma(t)}\left(\frac{f'(t)}{\xi\cdot\gamma'(t)} - \frac{f(t)}{(\xi\cdot\gamma'(t))^2}(\xi\cdot\gamma''(t))\right)\ dt\right| \\
&\lesssim_\gamma \frac{L}{\lambda}\left(\frac{\|f'\|_{L^\infty}}{\varepsilon} + \frac{\|f\|_{L^\infty}}{\varepsilon^2}\right).
\end{aligned}$$

$\square$

Lemma 4. *Let* $\gamma_M(t) = (t, t^2, t^3)$ *be the parameterization of the moment curve* $\Gamma_M = \gamma_M(\mathbb{R})$*. For any* $\varepsilon \in (0, 1/100)$*, if* t_0 *is a fixed time,* ξ_0 *is one of the vectors orthogonal to both* $\gamma_M'(t_0)$ *and* $\gamma_M''(t_0)$*,* $\lambda \gtrsim_\varepsilon 1$*, then*

$$\sup_{|\xi - \lambda\xi_0| \leqslant \varepsilon\lambda} |\xi|^{1/3}|I_{\gamma_M}(\mu, \lambda\xi)| \lesssim_\varepsilon 1.$$

Proof. Fix $\delta > 0$ and $\lambda \geqslant 1$, and suppose there was a probability measure μ compactly supported on some interval I such that

$$\sup_{|\xi - \lambda\xi_0| \leqslant \lambda\varepsilon} |\xi|^{1/3}|I_{\gamma_M}(\mu, \xi)| \leqslant \delta.$$

Define a linear transformation

$$A_h = \begin{pmatrix} 1 & 0 & 0 \\ 2h & 1 & 0 \\ 3h^2 & 3h & 1 \end{pmatrix}.$$

Then $A_h\gamma_M(t) = \gamma_M(h) + \gamma_M(t+h)$ for all $t, h \in \mathbb{R}$. If $\gamma_{M,h}(t) = \gamma_M(t+h)$, we thus have

$$\begin{aligned}
I_{\gamma_{M,h}}(\mu, \xi) &= \int e^{i\xi\cdot\gamma(t+h)} d\mu(t) \\
&= e^{-i\xi\cdot\gamma(h)} \int e^{i\xi\cdot A_h\gamma(t)} d\mu(t)\ dt \\
&= e^{-i\xi\cdot\gamma(h)} \int e^{i(A_h^T\xi)\cdot\gamma(t)} d\mu(t)\ dt \\
&= e^{-i\xi\cdot\gamma(h)} I_{\gamma_M}(\mu, A_h^T\xi).
\end{aligned}$$

If we consider an L^1 normalized smooth bump function $\phi : \mathbb{R} \to \mathbb{R}$ adapted to $\{|h| \leqslant \varepsilon/2\}$, and define a smooth function $f = \phi * \mu$, then

$$I_{\gamma_M}(f, \lambda\xi_0) = \int \phi(h) I_{\gamma_{M,h}}(\mu, \lambda\xi_0)\, dh = \int \phi(h) e^{-i\xi\cdot\gamma_M(h)} I(\mu, \lambda A_h^T \xi_0)\, dh.$$

Then the L^∞ norm of f and f' is $O_\varepsilon(1)$, and $f(t_0) \gtrsim_\varepsilon 1$, so we conclude that

$$I_{\gamma_M}(f, \lambda\xi_0) = C\; a(t_0) f(t_0) e^{i\lambda\xi_0} \lambda^{-1/3} + O_\varepsilon(\lambda^{-2/3}).$$

In particular, we conclude that for $\lambda \gtrsim_\varepsilon 1$,

$$|I_{\gamma_M}(f, \lambda\xi_0)| \gtrsim C_\varepsilon \lambda^{-1/3}.$$

Now $|A_h^T\xi_0 - \xi_0| \leqslant 4h|\xi_0|$ for $|h| \leqslant 1/100$, we know by assumption that $|I(\mu, \lambda A_h^T \xi_0)| \leqslant \delta\lambda^{-1/3}$. But this means we conclude that

$$\lambda^{-1/3} \lesssim_\varepsilon \delta\lambda^{-1/3},$$

and thus that $\delta \gtrsim_\varepsilon 1$, completing the proof. □

For any measure μ on I, we fix $\delta > 0$, and consider a family of $O(\delta^{-1})$ points $\mathcal{T}$ such that the length δ intervals $\{I_t : t \in \mathcal{X}_\delta\}$ with center t cover $[0,1]$, and for each t, the middle third of the interval I_t is disjoint from $I_{t'}$ for $t \neq t'$. Consider a smooth partition of unity $\{\chi_t\}$ adapted to these intervals. For each $t \in \mathcal{T}$, define $\mu_t = \chi_t \mu$. For any $t \in \mathcal{T}$, consider the degree three polynomial curve $\gamma_t : \mathbb{R} \to \mathbb{R}^d$ given by

$$\gamma_t(s) = \gamma(t) + \gamma'(t)(s-t) + \frac{\gamma''(t)}{2}(s-t)^2 + \frac{\gamma'''(t)}{6}(s-t)^3.$$

then for any $t' \in I_t$, $|\gamma(t') - \gamma_t(t')| \lesssim \delta^4$. This means that the deviations between γ and γ_t, once localized to a δ neighborhood of t, should be undetectable for frequencies with magnitude $O(\delta^{-4})$, i.e. for $|\xi| \lesssim \delta^{-4}$, we should expect to have

$$I_\gamma(\mu, \xi) \approx \sum_t I_{\gamma_t}(\mu_t, \xi).$$

If we let B_t be the matrix with columns $\delta^{-1}\gamma'(t)$, $\delta^{-2}\gamma''(t)/2$, and $\delta^{-3}\gamma'''(t)/6$, then

$$\gamma_t(s) - \gamma(t) = B_t \gamma_M(\delta(s-t)).$$

Thus if ν_t is the dilation of $\mathrm{Trans}_{-t}\mu_t$ by a factor $1/\delta$, then

$$\begin{aligned}
I_{\gamma_t}(\mu_t, \xi) &= \int e^{i\xi\cdot\gamma_t(s)} d\mu_t(s) \\
&= \int e^{i\xi\cdot[\gamma(t) + B_t\gamma_M((s-t)/\delta)]}\, d\mu_t(s) \\
&= e^{i\xi\cdot\gamma(t)} \int e^{i(B_t^T\xi)\cdot\gamma_M((s-t)/\delta)} d\mu_t(s) \\
&= e^{i\xi\cdot\gamma(t)} I_{\gamma_M}(\nu_t, B_t^T\xi).
\end{aligned}$$

Thus we get

$$I_\gamma(\mu,\xi) \approx \sum_t e^{i\xi\cdot\gamma(t)} I_{\gamma_M}(\nu_t, B_t^T\xi).$$

for $|\xi| \ll \delta^{-4}$. We now consider an L^1 normalized, smooth bump function ϕ supported on a width δ interval about the origin, and define $f_t = \nu_t * \phi$. We have seen that

$$I_{\gamma_M}(f_t, B_t^T\xi) = \int \phi(h) e^{-i\xi\cdot\gamma_M(h)} I(\nu_t, A_h^T B_t^T \xi)\, dh.$$

Suppose (THIS IS THE CHEAT) we can find a matrix C_h such that $A_h^T B_t^T\xi = B_t^T C_h \xi$. Then

$$\sum_t I_{\gamma_M}(f_t, B_t^T\xi) = \sum_t \int \phi(h) e^{-i\xi\cdot\gamma_M(h)} I(\nu_t, B_t^T C_h\xi) \approx \int \phi(h) e^{-i\xi\cdot\gamma_M(h)} I_\gamma(\mu, C_h\xi)\, dh.$$

Then C_0 is the identity matrix, and so we can imagine that $|C_h\xi| \sim |\xi|$ for small h.

We can now argue that $\dim_{\mathbb{F}}(\Gamma) \leqslant 2/3$. Suppose that instead, we could choose μ such that

$$\limsup_{\xi\to\infty} |\xi|^{2/3+\varepsilon} |I_\gamma(\mu,\xi)| < \infty.$$

Then for any $\xi \in \mathbb{R}^d$, the right hand side of the identity above satisfies estimates of the form

$$\left| \int \phi(h) e^{-i\xi\cdot\gamma_M(h)} I_\gamma(\mu, C_h\xi)\, dh \right| \lesssim |\xi|^{-1/3-\varepsilon}.$$

For $|\xi| \sim \delta^{-4}$, we get that this quantity is $\lesssim \delta^{4/3+\varepsilon}$. On the other hand, the left hand side is a sum of quantities to which we can apply stationary and nonstationary phase. If we choose $c > 0$ small enough, depending on γ, then because of the linear independence of γ', γ'', and γ''', if, for $t_0 \in \mathcal{T}$, we set $\xi = \xi_0(t_0)$, then for any $t \neq t_0$, and any $t' \in I_t$, $|\xi\cdot\gamma'(t')| \geqslant c\delta$. This implies that the principle of nonstationary phase can be applied to the quantity $I_\gamma(\nu_t, B_t^T\xi)$. For each t_0, the function f_{t_0} has L^∞ norm at most $O(\delta^{-1}\nu_{t_0}(\mathbb{R}))$, and f_{t_0}' has L^∞ norm bounded by $O(\delta^{-2}\nu_{t_0}(\mathbb{R}))$. Applying the principle of nonstationary phase, for $t \neq t_0$ we conclude that

$$|I_{\gamma_M}(f_t, B_t^T\xi)| \lesssim \delta^{-2}\nu_{t_0}(\mathbb{R})|\xi|^{-1}.$$

Summing over $t \neq t_0$ gives that

$$\sum_{t\neq t_0} |I_{\gamma_M}(f_t, B_t^T\xi)| \lesssim \delta^{-2}|\xi|^{-1}.$$

If we take $|\xi| \sim \delta^{-4}$, this quantity is $O(\delta^2)$. On the other hand, we have $f_{t_0}(t_0) \gtrsim \delta^{-1}\nu_{t_0}(\mathbb{R})$, and so the principle of stationary phase we calculated at the beginning of our argument shows that

$$|I_{\gamma_M}(f_{t_0}, B_{t_0}^T\xi)| \gtrsim \delta^{-1}\nu_{t_0}(\mathbb{R})|\xi|^{-1/3}$$

so for $|\xi| \sim \delta^{-4}$, we get that this quantity is $\gtrsim \delta^{1/3}\nu_{t_0}(\mathbb{R})$. Since $\sum_t \nu_t(\mathbb{R}) = \mu(\mathbb{R}) = 1$, the pigeonhole principle implies we can pick some t_0 such that $\nu_{t_0}(\mathbb{R}) \gtrsim \delta$. But then the quantity above is $\gtrsim \delta^{4/3}$. But putting these bounds together gives that

$$|\sum_t I_{\gamma_M}(f_t, B_t^T\xi)| \geqslant |I_{\gamma_M}(f_{t_0}, B_{t_0}^T\xi)| - \sum_{t\neq t_0} |I_{\gamma_M}(f_t, B_t^T\xi)| \gtrsim \delta^{4/3}.$$

But we therefore conclude that $\delta^{4/3} \lesssim \delta^{4/3+\varepsilon}$, which gives a contradiction if δ is taken appropriately small.