Research Statement

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I am an analyst who studies problems mainly using techniques from harmonic analysis, but also some methods of combinatorics and probability theory. My research over the past few years has focused on the study of radial Fourier multiplier operators on Euclidean space, and their analogues on compact manifolds, through an understanding of the geometry and regularity of wave propagation. In addition, I have explored problems in geometric measure theory, investigating when 'structure' occurs in fractals of large dimension. Both areas of research have raised interesting questions which I plan to pursue in my postgraduate work.

During my PhD, my work on multipliers has concentrated on the relation between L^p bounds for Fourier multipliers, and L^p bounds for analogous operators on compact manifolds. My main achievement was a *transference principle* [5] between bounds for radial multipliers and bounds for multiplier operators for spherical harmonics expansions on the sphere S^d. For $d \geq 4$ and a range of L^p spaces, this principle says that the boundedness of a radial Fourier multiplier with symbol $a(|\xi|)$ on $L^p(\mathbb{R}^d)$ implies the boundedness of the multiplier on $L^p(S^d)$ with the same symbol a; thus L^p bounds 'transference' from $\mathbb{R}^d$ to S^d. The same methods of proof completely characterize those symbols whose dilates give a uniformly bounded family of multiplier operators on $L^p(S^d)$. Both the transference principle and the characterization of uniform boundedness are the first such results about multipliers on S^d for $p \neq 2$; more broadly, no comparable results have been established for analogous multiplier operators on any other compact manifold. More detail about this work is given in Section 1 of the summary.

My work in geometric measure theory focuses on constructing sets of large fractal dimension avoiding certain point configurations. Before starting my PhD, I had worked with Malabika Pramanik and Joshua Zahl to construct sets with large Hausdorff dimension avoiding certain point configurations [6]. During my PhD, I continued this line of research by combining the methods of that paper with more robust probabilistic machinery to address the more difficult problem of constructing sets with large Fourier dimension avoiding configurations [4]. This method remains the only method of constructing sets of large Fourier dimension avoiding non-linear configurations, and remains the best current method for constructing sets avoiding general 'linear' point configurations when $d > 1$. This work is discussed further in Section 2.

In the near future, I hope to generalize the bounds obtained in [5] to the more general setting of multipliers associated with the Laplace-Beltrami operator on Riemannian manifolds with periodic geodesic flow. One obstruction to this generalization at a 'single frequency scale' is obtaining control of iterates of a pseudodifferential operator on M called the 'return operator'. Another obstruction when 'combining frequency scales' is an endpoint refinement of the local smoothing inequality for the wave equation on M. I am also interested of obtaining bounds on manifolds whose geodesic flow has well-controlled dynamical properties, such as forming an integrable system. Related to my work in geometric measure theory, I hope to apply the probabilistic methods I exploited in the construction of sets of large Fourier dimension to construct random fractals which exhibit good l^2L^p decoupling properties. And I am interested in determining the interrelation of patterns with the study of multipliers on manifolds, in particular studying Falconer distance problems on Riemannian manifolds to local smoothing phenomena for the wave equation on manifolds. A more detailed justification for the potential of these projects can be found in Section 3 of this statement.

The remaining sections of this summary provides context and describes the results I have obtained during my PhD in further detail, finishing with a further elaboration of future work and it's feasibility given the tools I have gained from my previous work.

1 Multiplier Operators on Euclidean Space and on Manifolds

In this section, we provide background on recent developments in the characterization of L^p boundedness for radial Fourier multiplier operators and their relation to bounds for multipliers on the sphere, as well as a more detailed discussion of the bounds I was able to obtain for such operators.

1.1 Radial Fourier Multipliers

Multipliers have long been a central object in harmonic analysis. In his pioneering work, Fourier showed solutions to the classical equations of physics are described by Fourier multipliers, operators T defined by a function $m : \mathbb{R}^d \to \mathbb{C}$, the symbol of T, such that

$$Tf(x) = \int_{\mathbb{R}^d} m(\xi)\widehat{f}(\xi)e^{2\pi i\xi\cdot x}\,dx.$$

Of particular interest are the *radial* multipliers, whose symbol is a radial function. We denote the radial multiplier with symbol $m(\xi) = a(|\xi|)$ by T_a in the sequel. Any translation-invariant operator on $\mathbb{R}^d$ is a Fourier multiplier operator, explaining their broad applicability in areas as diverse as partial differential equations, number theory, complex variables, and ergodic theory.

In harmonic analysis, it has proved incredibly profitable to study the boundedness of Fourier multipliers with respect to various L^p norms. It seems to be one of the few tractable ways of quantifying how different types of planar waves interact with one another, thus underpinning all deeper understandings of the Fourier transform. The need for an understanding of the L^p boundedness properties of a general Fourier multiplier became of central interest in the 1960s, brought on by the spur of applications the Calderon-Zygmund school and their contemporaries brought to the theory. Necessary conditions on a symbol to ensure the corresponding Fourier multiplier was bounded on L^p were found, but finding necessary and sufficient conditions which guarantee L^p boundedness proved to be an impenetrable, if not potentially impossible problem. No results were obtained in the past half century, aside from trivial cases where $p \in \{1, 2, \infty\}$.

It thus came as a surprise when recently several arguments [2, 8, 10, 12] emerged giving necessary and sufficient conditions on a symbol a for the *radial* Fourier multiplier T_a to be bounded on $L^p(\mathbb{R}^d)$. Consider a decomposition $a(\rho) = \sum a_k(\rho/2^k)$, where $a_k(\rho) = 0$ for $\rho \notin [1, 2]$. For $1 \leq p \leq 2$, in order for T_a to be bounded on $L^p(\mathbb{R}^d)$, testing by Schwartz functions reveals it is necessary that $\sup_j C_p(a_j) < \infty$, where

$$C_p(a) = \left(\int_0^\infty \left[(1 + |t|)^{(d-1)(1/p-1/2)}\widehat{a}(t) \right]^p\,dt \right)^{1/p},$$

Duality implies the boundedness of T_a on $L^p(\mathbb{R}^d)$ is equivalent to it's boundedness on $L^{p'}(\mathbb{R}^d)$ when $1/p+1/p' = 1$, and so for $2 \leq p \leq \infty$ it is natural to define $C_p(a) = C_{p'}(a)$. Using Bochner-Riesz multipliers as endpoint examples, it is natural to conjecture the condition $\sup_j C_p(a_j) < \infty$ is not only necessary, but also *sufficient* to guarantee L^p boundedness for $|1/p - 1/2| > 1/2d$. For radial input functions this conjecture is true [8], though resolving this conjecture for general inputs is likely far beyond current research techniques, given that it implies the Bochner-Riesz conjecture, and thus also the restriction and Kakeya conjectures.

Heo, Nazarov, and Seeger [10] have proved the conjecture for $d \geq 4$ and $|1/p - 1/2| > (d-1)^{-1}$. Cladek [2] improved the range of the conjecture for compactly supported a. She proved the result when $d = 4$ and $|1/p-1/2| > 11/36$ and when $d = 3$ and $|1/p-1/2| > 11/26$. Also of note is the work of Kim [12] also extended the bounds of [10] to *quasi-radial multipliers*, Fourier multipliers with a symbol $q(\xi)$ which is smooth, non-negative, homogeneous of order one, and whose level sets are hypersurfaces of non-vanishing Gauss curvature. Nonetheless, the full conjecture remains unsolved for all $d \geq 2$.

Often bounds on multipliers are obtained by assuming smoothness properties of the symbol a. The bound $\sup_j C_p(a_j) < \infty$ can be viewed in some sense as such a condition, but is not equivalent to the boundedness of any Sobolev, Besov, or Triebel-Lizorkin norm. However, the bound is implied if the functions $\{a_j\}$ uniformly lie in the Besov space $B_p(\mathbb{R}) := B^{d(1/p-1/2)}_{2,p}(\mathbb{R})$, which roughly speaking says that the functions $\{a_j\}$ have $d(1/p-1/2)$ derivatives in L^2. One could conjecture that for $|1/p-1/2| > 1/2d$, the operator T_a is bounded on $L^p(\mathbb{R}^d)$ if $\sup_j \|a_j\|_{B_p(\mathbb{R})} < \infty$. This conjecture is weaker than the last, and only gives necessary, not sufficient, conditions for boundedness, but whose proof might be more accessible to current techniques. Indeed, this weaker conjecture has been verified by Lee, Rogers, and Seeger [15] to be true for all $d \geq 2$ in the larger Stein-Tomas range $|1/p - 1/2| > (d+1)^{-1}$.

We remark that various high powered techniques have recently been developed towards an understanding of the Bochner-Riesz conjecture, such as broad-narrow analysis, decoupling, and the polynomial method. However, these methods are difficult to apply when studying the two conjectures introduced above, since they are *endpoint results*. More precisely, in arguments related to the Bochner-Riesz conjecture, one allows for inequalities to have a multiplicative loss of factors of the form R^ε or $\log R$, where R is the frequency scale of the analysis. This is negligible to the analysis, since the Bochner-Riesz multipliers are conjectured to be bounded on L^p for an *open* interval of exponents, and so methods involving interpolation between L^p spaces allow us to remove these multiplicative factors when making conclusions. But an arbitrary multiplier bounded on $L^p(\mathbb{R}^d)$ may not be bounded on $L^{p'}(\mathbb{R}^d)$ for any $p' < p$, and so such methods are unavailable to us in the study of these conjectures, explaining the limited range in which the conjectures have been verified.

1.2 Multipliers For Spherical Harmonic Expansions on S^d

A theory of multiplier operators analogous to Fourier multipliers can be developed on the sphere S^d. Roughly speaking, Fourier multipliers are operators on $\mathbb{R}^d$ with $e^{2\pi i \xi \cdot x}$ as eigenfunctions. Zonal multipliers on S^d are those operators with the *spherical harmonics* as eigenfunctions, i.e. the restrictions to S^d of homogeneous harmonic polynomials on $\mathbb{R}^{d+1}$. Every function $f \in L^2(S^d)$ can be uniquely expanded as $\sum_{k=0}^\infty H_k f$, where $H_k f$ is a degree k spherical harmonics. A multiplier for spherical harmonic expansions on S^d is then an operator defined in terms of a function $a : \mathbb{N} \to \mathbb{C}$ by setting

$$Z_a f = \sum\nolimits_{k=0}^\infty a(k) H_k f.$$

For purposes of brevity, we will call such operators 'multipliers on S^d' in the sequel. Every rotation invariant operator on S^d is a multiplier, and thus such operators arise in diverse applications, including celestial mechanics, physics, and computer graphics.

Classic methods for studying multipliers on S^d involve the analysis of special functions and orthogonal polynomials. But in the 1960s Hörmander introduced the powerful theory of Fourier integral operators to the study of such multipliers, which allows one to apply more modern techniques of harmonic analysis the theory. This theory is more robust in other senses, and in particular allows for the study of the much more general setup of multiplier operators of a general first order self-adjoint pseudodifferential operator P on a manifold M, which we briefly outline. Given such an operator P, if Λ is the set of eigenvalues for P, then every function $f \in L^2(M)$ has an orthogonal decomposition $f = \sum_{\lambda \in \Lambda} f_\lambda$ where $P f_\lambda = \lambda f_\lambda$. For any symbol $a : \Lambda \to \mathbb{C}$, we define a multiplier operator using the orthogonal decomposition above by setting

$$a(P) f = \sum\nolimits_{\lambda \in \Lambda} a(\lambda) f_\lambda.$$

We note that if Δ is the Laplace-Beltrami operator on S^d, then $Pf = k(k+d-1)f$ for any spherical harmonic f of degree k. Thus if $P = \sqrt{\alpha^2 - \Delta}$, where $\alpha = (d-1)/2$, then $Pf = kf$

for a degree k harmonic f, and so any multiplier Z_a on S^d can also be written as $a(P)$. In this general setup, Hörmander's idea was to use the Fourier inversion formula to write

$$a(P) = \int \widehat{a}(t) e^{2\pi i t P}\, dt,$$

The multiplier operators $e^{2\pi i t P}$, as t varies, give solutions to the *half-wave equation* $\partial_t = iP$ on the manifold M, whose solutions allow one to describe solutions to the full wave equation $\partial_t^2 - P^2 = 0$. Thus the study of the boundedness of the operators $a(P)$ is connected to the regularity of the wave equation on M.

Using this reduction, Hörmander [11] was able to prove L^p boundedness of the analogues of the Bochner-Riesz multipliers in this setting. Sogge [21,22] improved these bounds, introducing the approach, which works within the Stein-Tomas range, of reducing the problem to certain $L^2(M) \to L^p(M)$ bounds for spectral projection operators on M. Recently, Kim [13] adapted Sogge's approach to analyze multipliers of an operator P satisfying the following assumption:

Assumption A: If $p_{\text{prin}} : T^*M \to [0, \infty)$ is the principal symbol of P, then for each $x \in M$ the 'cosphere' $S_x^* = \{\xi \in T_x^*M : p_{\text{prin}}(x, \xi) = 1\}$ has non-vanishing Gaussian curvature.

Kim proved that under Assumption A, in the Stein-Tomas range $|1/p - 1/2| > (d+1)^{-1}$, if $\sup_j \|a_j\|_{B_p(\mathbb{R})} < \infty$, then the operator $a(P)$ is bounded on $L^p(M)$, thus obtaining an analogue of the result of Lee, Rogers and Seeger on compact manifolds. In particular, Assumption A is satisfied when $P = \sqrt{\alpha^2 - \Delta}$ on S^d, since the cospheres of P are ellipses, and so Kim's result applies to multipliers on S^d. Note, however that there are no results in the literature for any exponent p, and any manifold M and operator P, which show that an operator $a(P)$ is bounded on $L^p(M)$ if $\sup_j C_p(a_j) < \infty$. *The main goal of my research project was to remedy this.*

1.3 My Contributions To The Study of Multipliers

As mentioned above, the main goal of my PhD research into multipliers was to see if we could obtain analogues of the arguments of [2,10,12] for multipliers on S^d, i.e. proving that for some range of p and all functions a, if $\sup_j C_p(a_j) < \infty$, then the multiplier Z_a is bounded on $L^p(S^d)$. I was able to able to obtain such analogues. Moreover, our argument is somewhat robust, applying to multipliers for a range of different operators P. Namely, we assume P satisfies Assumption A, and in addition, satisfies the following additional assumption:

Assumption B: The eigenvalues of P are contained in an arithmetic progression.

When $P = \sqrt{\alpha^2 - \Delta}$ on S^d, all eigenvalues are positive integers, so assumption B is satisfied for multipliers on S^d. The assumption also holds more generally for multipliers on the *rank one symmetric spaces* $\mathbb{RP}^d$, $\mathbb{CP}^d$, $\mathbb{HP}^d$, and $\mathbb{OP}^2$, i.e. operators diagonalized by analogous functions to the spherical harmonics on these spaces. Nonetheless, Assumption B is less natural than Assumption A, and I hope to obtain bounds under weaker assumptions in future work. Under Assumption A and Assumption B, in [5] I proved a 'single scale' version of this bound, i.e. proving that if a is supported on $[1, 2]$, and $|1/p - 1/2| > 1/d$ then, uniformly in j,

$$\|a(P/2^j) f\|_{L^p(M)} \lesssim C_p(a) \|f\|_{L^p(M)}.$$

In a paper to be submitted for publication shortly, I provide further arguments justifying that for an arbitrary function a, the operator $a(P)$ is bounded on $L^p(M)$ if $\sup_j C_p(a_j) < \infty$, thus obtaining a complete analogue of the argument of [10] for multipliers on S^d.

This result has several important corollaries. Firstly, it implies a *transference principle* between Fourier multipliers and multipliers on S^d. Since the condition $\sup_j C_p(a_j)$ is necessary for T_a to be bounded on $L^p(\mathbb{R}^d)$, we conclude that for $|1/p - 1/2| > 1/d$, if T_a is bounded on $L^p(\mathbb{R}^d)$, then the multipliers $a(P)$ is bounded on $L^p(M)$. Aside from the study of Fourier multipliers on $\mathbb{R}^d$, this is the first transference principle of this kind. There are no results in the literature for any $p \neq 2$, any other compact manifold M, and any operator P which guarantee that $a(P)$ is bounded on $L^p(M)$ if T_a is bounded on $L^p(\mathbb{R}^d)$.

Another corollary is a characterization of the symbols a such that multipliers of the form $\{a(P/R) : j > 0\}$ are uniformly bound on $L^p(M)$ for $|1/p - 1/2| > (d-1)^{-1}$. If $\sup_j C_p(a_j) < \infty$, then the results above imply that the operators $a(P/R)$ are uniformly bounded on $L^p(M)$, because the quantity $\sup_j C_p(a_j)$ changes by at most a constant when we dilate a by a factor of R. The converse follows from a classic result of Mitjagin [19], so we have proved necessary and sufficient conditions for the operators $\{a(P/2^j) : j > 0\}$ to be uniformly bounded on $L^p(M)$. By the uniform boundedness principle, this result also classifies all functions a such that $a(P/2^j)f$ converges in L^p to f as $j \to \infty$ for all $f \in L^p(M)$. As before, these results are the first for any $p \neq 2$ and any other compact manifold M.

I was able to obtain the results above by two main innovations. In order to justify these innovations, let us briefly describe the approach of [10] for bounding radial multipliers:

Let T be a radial multiplier. Then we can write $Tf = k * f$, where k is the Fourier transform of the symbol of a. We write $k = \sum k_\tau$ and $f = \sum f_\theta$, where the functions $\{k_\tau\}$ are supported on disjoint annuli supported at the origin, and the functions $\{f_\theta\}$ are supported on disjoint cubes. Then $Tf = \sum_{\tau,\theta} k_\tau * f_\theta$. Some calculations involving Bessel functions allow us to argue that the inner product $\langle k_\tau * f_\theta, k_{\tau'} * f_{\theta'} \rangle$ is negligible unless the annulus of radius τ centered at θ is near tangent to the annulus of radius τ' centered at θ'. Combining this inner product estimate with a 'sparse incidence argument' for such annuli, one can show that the L^2 norm of a sum $\sum_{(\tau,\theta)\in\mathcal{E}} k_\tau * f_\theta$ is well behaved if $\mathcal{E}$ is suitably 'sparse'. Interpolation with a trivial L^1 estimate yields an L^p estimate on the sum. Conversely, if the set $\mathcal{E}$ is clustered, then $\sum_{(\tau,\theta)\in\mathcal{E}} k_\tau * f_\theta$ will be concentrated on only a few annuli, and so we can also get good L^p estimates simply using pointwise estimates. But then we can estimate $\|Tf\|_{L^p(\mathbb{R}^d)} = \|\sum k_\tau * f_\theta\|_{L^p(\mathbb{R}^d)}$ by either approach, depending on whether a sparse part or a clustered part of the sum dominates.

The main difficulty of adapting this approach to the study of the multipliers $a(P)$ is in obtaining the analogous inner product estimates on manifolds. The natural approach here is to use the Lax-Hörmander parametrix for the wave equation, which reduces our inner product estimates *for small* τ to a bound for oscillatory integrals. One problem is that the phase of this integral is non-explicit, given by a solution to a partial differential equation on M. In [5], I made the observation that if Assumption A holds, P gives an implicit geometric structure to the manifold M, making it into a *Finsler manifold*. Finsler manifolds are like Riemannian manifolds, though instead of a smooth choice of inner products being fixed on the tangent spaces of the manifold, in Finsler geometry a smooth choice of strictly convex vector space norms are given on the tangent spaces of the manifold. The phase of the oscillatory integral occurring from the Lax-Hörmander parametrix is then directly related to *geodesics* on this Finsler manifold, and using the Finsler analogue of the second variation formula for geodesics, I was able to obtain the correct analogues of the inner product estimates that occur in [10]. Such estimates apply to multipliers of an arbitrary pseudodifferential operator P satisfying Assumption A for *small* τ, and likely have applications in other problems.

A second difficult occurs because the inner product estimates obtained only work for small τ, because the Lax-Hörmander parametrix is only established up to the *injectivity radius* of the manifold M. For large τ, this argument breaks down, and so applying a direct analogue of the arguments of [10] is not possible. Similar problems emerge in other methods of bounding multipliers on manifolds. This was the impetus for Sogge's introduction of his method for reducing bounds for Bochner-Riesz multipliers to the study of $L^p \to L^2$ bounds for spectral projection operators, used in [22] and [13]. This method involves initially using the estimate $\|a(P)f\|_{L^p(M)} \lesssim \|a(P)f\|_{L^2(M)}$, which holds on compact manifolds for $1 \leq p \leq 2$, and then recovering L^p bounds later by using the $L^p \to L^2$ bounds for spectral projections. But this method cannot be used in our situation, since working in L^2 involves an unacceptable loss in derivatives that makes it impossible to recover the $\sup_j C_p(a_j)$ bound.

A solution I found to this difficulty was instead of reducing the problem to $L^p \to L^2$ bounds for the spectral projection operators, we could instead reduce the problem to studying local $L^p_x L^p_t$ estimates for the wave equation on the manifold. Such an argument behaves somewhat like Sogge's spectral projection argument, but allows us to remain in L^p so that we do not lose any derivatives in our argument. The catch is that $L^p_x L^p_t$ estimates for the wave equation, related to the phenomenon of local smoothing, are not as well understand as spectral projectors. This is why we must make Assumption B, which implies that the propagators $e^{2\pi i t P}$ are periodic in t, simplifying these estimates.

The italicized argument above describes the unit scale argument of [10] that we generalized in [5]. The remaining argument in [10] involves decomposing forming an L^∞ atomic decomposition, ala the approach of Chang and Fefferman [1], for a general input f in $L^p(\mathbb{R}^d)$, and then combining scales by using inner product estimates $\langle k_\tau * f_\theta, k_{\tau'} * f_{\theta'} \rangle$. Since we have obtained a generalization of these inner product estimates already in the manifold case in [5], and Peetre square function and it's resulting atomic decomposition have direct analogues on compact manifolds.

2 Configuration Avoidance

How large must a set $X \subset \mathbb{R}^d$ be before it must contain a certain point configuration, such as three points forming a triangle congruent to a given triangle, or four points forming a parallelogram? Discrete problems of this flavor have long been studied in combinatorics, for instance, such as when X is restricted to be a subset of the grid $\{1, \dots, N\}^d$. In the last 50 years, analysts have also begun studying analogous problems for infinite subsets $X \subset \mathbb{R}^d$, where the size of X is measured via a suitable *fractal dimension*, various different numerical statistics which measure how 'spread out' X is in space. The most common fractal dimension in use is the *Hausdorff dimension* of a set X, but we also consider the *Fourier dimension* as a refinement of Hausdorff dimension which also takes into account more subtle behavior of X related to it's correlation with the planar waves $e^{2\pi i \xi \cdot x}$ for $\xi \in \mathbb{R}^d$.

Unlike many other problems in harmonic analysis, we often do not have good expected *lower* bounds for the dimension at which configurations must appear. For instance, we do not know for $d > 2$ how large the Hausdorff dimension a set $X \subset \mathbb{R}^d$ must be before it contains all three vertices of an isosceles triangle, the threshold being somewhere between $d/2$ and $d-1$. Similarly, for a fixed angle $\theta \in (0, \pi)$, we do not know how large the Hausdorff dimension of X must be contains three distinct points A, B, and C which when connected determine an angle ABC equal to θ. If $\cos^2\theta$ is rational, the results of Máthe [18] and Harangi, Keleti, Kiss, Maga, Máthe, Mattila, and Strenner [9] imply the threshold is somewhere between $d/4$ and $d-1$. If $\cos^2\theta$ is irrational, the threshold is somewhere between $d/8$ and $d-1$. We should not even necessarily expect currently known lower bounds to be the 'correct bounds' in these problems, as we do with other problems in harmonic analysis, such as the restriction conjecture and the Falconer distance problem; Until recently, certain results due to Łaba and Pramanik [14] seemed to imply that subsets of $[0, 1]$ of Fourier dimension one must necessarily contain an arithmetic progression of length three, but Schmerkin has shown this need not be the case [20].

Given that we do not have good lower bounds with which to make definite conjectures, it is of interest to find general methods that we can use to produce counterexamples in these types of problems. That is, we wish to find methods with which to construct sets with large fractal dimension that *do not* contain certain point configurations. My research in geometric measure theory has so far focused on finding these types of methods.

2.1 Hausdorff Dimension and Configuration Avoidance

Let us consider a good model problem for pattern avoidance; given a fixed function $f : (\mathbb{R}^d)^n \to \mathbb{R}^m$, how large must the Hausdorff or Fourier dimension of a set X be to guarantee that there exists $x_1, \dots, x_n \in X$ such that $f(x_1, \dots, x_n) = 0$. We focus on finding lower bounds for this problem, constructing sets X with large Hausdorff or Fourier dimension such that X *avoids the zeroes* of f, in the sense that for any distinct points $x_1, \dots, x_n \in X$, $f(x_1, \dots, x_n) \neq 0$. This general model has been considered in various contexts:

(A) If $m = 1$, and f is a polynomial of degree n with rational coefficients, Máthe [18] constructs a set with Hausdorff dimension d/n avoiding the zeroes of f.

(B) If f is a C^1 submersion, Fraser and Pramanik [7] constructs a set with Hausdorff dimension $m/(n-1)$ avoiding the zeroes of f.

(C) If the zero set $f^{-1}(0)$ has Minkowski dimension at most s, I, together with my Master's thesis advisors Malabika Pramanik and Joshua Zahl [6] constructed sets of Hausdorff dimension $(dn - s)/(n-1)$ avoiding the zeroes of f.

(D) If f can be factored as $f = g \circ T$, where $T : (\mathbb{R}^d)^n \to \mathbb{R}^l$ is a full-rank, rational coefficient linear transformation, and $g : \mathbb{R}^l \to \mathbb{R}^m$ is a C^1 submersion, then I [5] have constructed a set with Hausdorff dimension m/l avoiding the zero sets of f.

Notice that the above four methods only construct sets with large *Hausdorff dimension* avoiding patterns. They say nothing about constructing sets with large Fourier dimension, which in general is much harder. Most 'structured' sets have low Fourier dimension. For instance, the standard middle thirds Cantor set has Fourier dimension zero, because the intervals at stage n of the usual construction of the Cantor set lie on arithmetic progressions of length $1/3^n$, and are thus highly correlated with planar waves of frequency 3^n. Constructing sets with large Fourier dimension often requires making certain 'random choices' which on average do not correlate with any particular planar wave. Thus constructions of sets with large Fourier dimension involve a delicate interplay between 'randomness' and 'structure'. 'Random' sets often have Fourier dimension and Hausdorff dimension that agree with one another. On the other hand, most 'structured' . Constructing sets with large Fourier dimension avoiding configurations thus often requires a delicate balancing act between adding randomness and structure to the structure. Structure must be added to some degree to avoid containing a given configuration, but adding too much structure will result in your set likely having Fourier dimension zero. Construction of sets with large Fourier dimension have been considered, but only for *linear* functions f:

(E) If $f(x_1, \dots, x_n) = a_1 x_1 + \dots + a_n x_n$ with $\sum a_j = 0$, Pramanik and Liang [17] construct a set $X \subset [0, 1]$ with Fourier dimension $\dim_{\mathbb{F}}(X) = 1$ avoiding the zeroes of f. This generalizes a construction of Schmerkin [20], who proved the result in the special case where $f(x_1, x_2, x_3) = (x_3 - x_1) - 2(x_2 - x_1)$ detects arithmetic progressions of length 3.

(F) Körner constructed subsets $X \subset [0, 1]$ with Fourier dimension $(k-1)^{-1}$ such that for any integers $m_0, \dots, m_k$, and any distinct $x_1, \dots, x_k \in X$, $a_0 \neq a_1 x_1 + \dots + a_n x_n$.

The focus on linear functions is natural, since the Fourier transform behaves in a predictable way with respect to linearity. On the other hand, the understanding of the Fourier transform with respect to other nonlinear phenomena is poorly understood. The main goal of my research project was to find constructions of sets with large Fourier dimension for *nonlinear* f.

2.2 My Contributions To The Study Of Configurations

It seems very difficult, if not impossible to adapt methods (A) and (D) above to construct sets with positive Fourier dimension, since the constructions involve constructing X at each spatial scale by choosing a good family of intervals, and then considering a large union of translates of the intervals along an arithmetic progression. This ensures a spread out family of intervals, and thus a set with large Hausdorff dimension. But it is not good for ensuring Fourier decay, since a function concentrated near an arithmetic progression must have a large Fourier coefficient at frequencies complementing the spacing of this progression. On the other hand, methods (B) and (C) involve mostly pigeonholing arguments, so they seem the most likely to be able to be adapted to the Fourier dimension setting. I was able to adapt some of the ideas of these methods to obtain such a result.

For simplicity, I focused on the case when $m = d$ and when the function f was C^1 and full rank, as assumed in [7]. Then by the implicit function theorem, after possibly rearranging indices, we can locally write $f(x_1, \dots, x_n) = x_1 - g(x_2, \dots, x_n)$ for a function $g : (\mathbb{R}^d)^{n-1} \to \mathbb{R}^d$. In [4], under the assumption that g was a submersion in each variable $x_2, \dots, x_n$, I was able to modify the construction of [7] to construct sets with Fourier dimension $d/(n-1/2)$ avoiding the zeroes of f. Under the further assumption that we can write $g(x_2, \dots, x_n) = ax_2 + h(x_3, \dots, x_n)$ for $a \in \mathbb{Q}$, I was able to construct sets with Fourier dimension $d/(n-1)$ avoiding the zeroes of f, recovering the Hausdorff dimension bound of [7] in the Fourier dimension setting.

TODO: Discuss Probabilistic Methods Involved In Proof.

3 Future Lines of Research

Given the context from the previous two sections, we finish this summary by describing in more detail several problems I believe may be accessible given the techniques I have used to solve previous problems.

3.1 Multipliers Associated With Periodic Geodesic Flow

In Section 2, I discussed that the results I were able to obtain for multipliers on S^d generalized to multipliers of an arbitrary first order, elliptic, self-adjoint pseudodifferential operator P on a compact manifold M, provided that P satisfied two assumptions. Assumption A relates to the curvature of the principal symbol, and this assumption cannot really be weakened without significantly changing the character of the results, which heavily depend on this curvature. On the other hand, Assumption B arises as an artifact of the methods of our proof. We can likely obtain similar bounds while weakening this assumptions; for instance, Kim [13] obtained bounds on the scale of Besov spaces without any assumptions other than Assumption A.

I doubt current research techniques allow us to completely removing Assumption B while still recovering the results of [5], a limitation of our current inability to understand the large time behavior of wave equations on compact manifolds. If we were able to follow the method of [3], which reduced the large time argument to a smoothing inequality for the wave equation, then the results of that paper would follow for another operator P if we could prove

$$\left\| \left(\int_k^{k+1} |e^{2\pi itP} f|^{p'} \, dt \right)^{1/p'} \right\|_{L^{p'}(M)} \lesssim k^\delta \|f\|_{L^p_{d(1/p-1/2)-1/p'}(M)} \tag{1}$$

for some $\delta < (d-1)(1/p - 1/2) - 1/p'$. If P satisfies assumption B, then after rescaling, we may assume without loss of generality that all eigenvalues of P are integers, so that $e^{2\pi ikP} = I$ is the identity for all k, and then (1) holds for all $|1/p - 1/2| > (d-1)^{-1}$ and with $\delta = 0$ by the local smoothing inequality of Lee and Seeger [16].

Whether this bound is true in other contexts remains unknown. The next simplest case to consider would be when the operator P has the property that $e^{2\pi ikP}$ is *close* to the identity for all k. This happens precisely when the *Hamiltonian flow* on T^*M given by the vector field $H = (\nabla_\xi p_{\text{prin}}, -\nabla_x p_{\text{prin}})$ is periodic, where p_{prin} is the principal symbol of P. Indeed, the theory of propagation of singularities in the study of Fourier integral operators then tells us that the operator $R = e^{2\pi i P}$ is a pseudodifferential operator of order zero, and it's principal symbol is related to an invariant of the flow known as the Maslov index. The operator has been studied a little by spectral theorists, and there it is known as the *return operator*. If we are able to justify bounds of the form

$$\|R^k f\|_{L^p_{d(1/p-1/2)-1/p'}} \lesssim k^\delta \|f\|_{L^p_{d(1/p-1/2)-1/p'}}$$

then the local smoothing inequality of Lee and Seeger yields (1). Such bounds are of interest because they include the class of all operators $P = \sqrt{-\Delta}$, where Δ is the Laplace-Beltrami operator on a Riemannian manifold with periodic geodesic flow. They are even of interest on the sphere, since our method only allows us to tell when multipliers of the form $a(\sqrt{\alpha^2 - \Delta}/2^j)$ are uniformly bounded on $L^p(S^d)$, whereas these bounds would allow us to tell when the multipliers $a(\sqrt{-\Delta}/R)$ are uniformly bounded on $L^p(S^d)$.

3.2 Decoupling On Random Fractals

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3.3 Radial Multiplier Bounds And 'Fractal Weighted Estimates'

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References

[1] Sun-Yung A. Chang and Robert Fefferman, *A continuous version of duality of H^1 with BMO on the bidisc*, Ann. of Math. (2) **112** (1980), no. 1, 179–201. MR584078

[2] Laura Cladek, *Radial Fourier multipliers in $\mathbb{R}^3$ and $\mathbb{R}^4$*, Anal. PDE **11** (2018), no. 2, 467–498. MR3724494

[3] Jacob Denson, *Cartesian products avoiding patterns*, Master's Thesis, 2019.

[4] ______, *Large Salem sets avoiding nonlinear configurations*, 2021.

[5] ______, *Multipliers of spherical harmonics*, 2024.

[6] Jacob Denson, Malabika Pramanik, and Joshua Zahl, *Large sets avoiding rough patterns*, Harmonic analysis and applications, 2021, pp. 59–75. MR4238597

[7] Robert Fraser and Malabika Pramanik, *Large sets avoiding patterns*, 2018, pp. 1083–1111. MR3785600

[8] Gustavo Garrigós and Andreas Seeger, *Characterizations of Hankel multipliers*, Math. Ann. **342** (2008), no. 1, 31–68. MR2415314

[9] Viktor Harangi, Tamás Keleti, Gergely Kiss, Péter Maga, András Máthé, Pertti Mattila, and Balázs Strenner, *How large dimension guarantees a given angle?*, Monatsh. Math. **171** (2013), no. 2, 169–187. MR3077930

[10] Yaryong Heo, Fëdor Nazarov, and Andreas Seeger, *Radial Fourier multipliers in high dimensions*, Acta Math. **206** (2011), no. 1, 55–92. MR2784663

[11] Lars Hörmander, *On the Riesz means of spectral functions and eigenfunction expansions for elliptic differential operators*, Some Recent Advances in the Basic Sciences, Vol. 2 (Proc. Annual Sci. Conf., Belfer Grad. School Sci., Yeshiva Univ., New York, 1965-1966), 1969, pp. 155–202. MR257589

[12] Jongchon Kim, *Endpoint bounds for quasiradial Fourier multipliers*, Ann. Mat. Pura Appl. (4) **196** (2017), no. 3, 773–789. MR3654932

[13] ______, *Endpoint bounds for a class of spectral multipliers in compact manifolds*, Indiana Univ. Math. J. **67** (2018), no. 2, 937–969. MR3798862

[14] Izabella Łaba and Malabika Pramanik, *Arithmetic progressions in sets of fractional dimension*, Geom. Funct. Anal. **19** (2009), no. 2, 429–456. MR2545245

[15] Sanghyuk Lee, Keith M. Rogers, and Andreas Seeger, *Square functions and maximal operators associated with radial Fourier multipliers*, Advances in analysis: the legacy of Elias M. Stein, 2014, pp. 273–302. MR3329855

[16] Sanghyuk Lee and Andreas Seeger, *Lebesgue space estimates for a class of Fourier integral operators associated with wave propogation*, Math. Nachr. **286** (2013), no. 7, 745–755.

[17] Yiyu Liang and Malabika Pramanik, *Fourier dimension and avoidance of linear patterns*, Adv. Math. **399** (2022), Paper No. 108252, 50. MR4384610

[18] András Máthé, *Sets of large dimension not containing polynomial configurations*, Adv. Math. **316** (2017), 691–709. MR3672917

[19] Boris S. Mitjagin, *Divergenz von Spektralentwicklungen in L_p-Räumen*, Linear operators and approximation, II (Proc. Conf., Math. Res. Inst., Oberwolfach, 1974), 1974, pp. 521–530. MR410438

[20] Pablo Shmerkin, *Salem sets with no arithmetic progressions*, Int. Math. Res. Not. IMRN **7** (2017), 1929–1941. MR3658188

[21] Christopher D. Sogge, *Oscillatory integrals and spherical harmonics*, Duke Math. J. **53** (1986), no. 1, 43–65. MR835795

[22] ———, *On the convergence of Riesz means on compact manifolds*, Ann. of Math. (2) **126** (1987), no. 3, 439–447.