Characterizations of Bounded Spectral Multipliers on Manifolds with Periodic Geodesic Flow

Jacob Denson*

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Abstract

It is a well known result of Mitjagin that on a compact Riemannian manifold M, and for a function $m : (0, \infty) \to \mathbb{C}$, if the spectral multiplier operators $m(\sqrt{-\Delta}/R)$ are uniformly bounded on $L^p(M)$ for $R > 0$, then the radial function $m(|\cdot|) : \mathbb{R}^d \to \mathbb{C}$ induces a bounded Fourier multiplier operator on $L^p(\mathbb{R}^d)$. In this paper, we prove the converse for manifolds in which the geodesic flow is periodic, of dimension $d \geqslant 4$ and for $(d-1)^{-1} \leqslant |1/p-1/2| \leqslant 1/2$. In the process, we find an effective characterization of the functions m for which the operators $m(\sqrt{-\Delta}/R)$ are uniformly bounded in $L^p(M)$ for this range of p, which can be viewed as a variable-coefficient analogue of the results of Heo, Nasarov and Seeger.

1 Introduction

Let M be a compact Riemannian manifold of dimension d, let Δ be it's Laplace-Beltrami operator, and let $P = \sqrt{-\Delta}$. For any bounded function $m : (0, \infty) \to \mathbb{C}$, we can define a spectral multiplier operator $m(P)$. In this paper, we study the relation between the L^p boundedness of the dilated multipliers $m_\rho(P)$ for $\rho > 0$, where $m_\rho(\lambda) = m(\lambda/\rho)$, and the L^p boundedness of the radial Fourier multiplier operator T_m on $\mathbb{R}^d$ defined by setting

$$T_m f(x) = \int_{\mathbb{R}^d} m(|\xi|) \widehat{f}(\xi) e^{2\pi i \xi \cdot x} \, d\xi \qquad \text{for } f : \mathbb{R}^d \to \mathbb{C}.$$

In particular, we show that for any $p \in [1, \infty]$, if d is sufficiently large, then the L^p boundedness of T_m is equivalent to the uniform L^p boundedness of the operators $m_\rho(P)$, provided that the geodesic flow on M is periodic.

Theorem 1. *Suppose M is a compact Riemannian manifold, and the geodesic flow on M is periodic. If $d \geqslant 4$, and $1/(d-1) < |1/p - 1/2| \leqslant 1/2$, then*

$$\sup_\rho \|m_\rho(P)\| \sim \|T_m\|,$$

where $\|T_m\|$ is the operator norm on $L^p(\mathbb{R}^d)$, and $\|m_\rho(P)\|$ the operator norm on $L^p(M)$. The implicit constant depends only on M and p.

*University of Madison Wisconsin, Madison, WI, jcdenson@wisc.edu

One side of this inequality is already well known. A classical transplantation theorem of Mitjagin [6] states that

$$\|T_m\| \lesssim \sup_\rho \|m_\rho(P)\|. \tag{1.1}$$

Moreover, this inequality is known to hold for the larger range of exponents $1 \leqslant p \leqslant \infty$ and for general compact manifolds M (see [4] for a translation of Mitjagin's proof, originally written in German). The result is intuitive if we consider the geometry in (1.1); if we let Δ_ρ be the Laplace-Beltrami operator on M associated with the dilated metric $g_\rho = \rho^{-2} g$, and define $P_\rho = \sqrt{-\Delta_\rho}$, then $m_\rho(P) = m(P_\rho)$. As $\rho \to \infty$, the metric g_ρ gives the manifold M less curvature and more volume, and as $\rho \to \infty$ we might therefore expect M equipped with Δ_ρ to behave more and more like $\mathbb{R}^d$ equipped with the usual Laplace operator $\Delta_{\mathbb{R}^d} = \sum \partial_j^2$. Indeed, Mitjagin's proof essentially shows that in coordinates, $m_\rho(P)$ converges to $m(\Delta_{\mathbb{R}^d})$ as $\rho \to \infty$, in an appropriate sense, and since $m(\Delta_{\mathbb{R}^d})$ is just the operator T_m, (1.1) follows.

The novel result in this paper is a proof of the converse inequality to (1.1) under the assumption that the geodesic flow on M is periodic, i.e. a proof that

$$\sup_\rho \|T_\rho\| \lesssim \|T_m\| \qquad \text{if} \quad \frac{1}{d-1} \leqslant \left| \frac{1}{p} - \frac{1}{2} \right| \leqslant \frac{1}{2}. \tag{1.2}$$

Equation (1.2) is more surprising than equation (1.1), since it is not geometrically intuitive why L^p bounds of a Fourier multiplier on a flat space should imply uniform L^p bounds for a family of spectral mulipliers on a curved space. No variants of (1.2) are currently known for any manifold and any value of $p \neq 2$; the only exception is in the study of multipliers on $\mathbb{T}^d$, where one has more robust tools, like the Poisson summation formula, to relate L^p bounds for multipliers on $\mathbb{T}^d$ and L^p bounds for multipliers on $\mathbb{R}^d$ (see Section 3.6.2 of [1]). Note also that $\mathbb{T}^d$ is a Riemannian manifold with no curvature, whereas there are several curved Riemannian manifolds to which our result applies, the sphere being an elementary example, and the class of Zoll manifolds giving examples with non-constant curvature.

The path to proving (1.2) is hinted at by the main result of [2], which states that, for $(d-1)^{-1} \leqslant |1/p - 1/2| \leqslant 1/2$, $\|T_m\| \sim C_p(m)$, where

$$C_p(m) = \sup_{h>0} \left(\int_0^\infty \left[\langle t \rangle^{\alpha(p)} |\widehat{m}_h(t)| \right]^p dt \right)^{1/p}.$$

Here $\alpha(p) = (d-1)|1/p - 1/2|$, and $m_h(\lambda) = m(2^h \lambda)\chi(\lambda)$ for any fixed choice of a smooth, compactly supported function χ with support on $[1/2, 2]$ for which $\sum \chi(2^h \lambda) = 1$. Since $\sup_{\rho > 0} C_p(m_\rho) \sim C_p(m)$, and duality implies that the operator norm of $m(P)$ on $L^p(M)$ is equal to the operator norm of $m(P)$ on $L^{p'}(M)$ for $1/p + 1/p' = 1$, Theorem 1 therefore follows from the following estimate.

Proposition 2. *Suppose $1 \leqslant p \leqslant 2(d-1)/(d+1)$. Then for any function $m : (0,\infty) \to \mathbb{C}$,*

$$\|m(P)\| \lesssim C_p(m),$$

where the implicit constant depends only on p and the manifold M.

The techniques that underlie the proof of Proposition 2 are also inspired by the proof methods of [2]. As in that proof, we begin by considering a decomposition $m(\cdot) = \sum_j m_j(\cdot/2^j)$, where $\text{supp}(m_j) \subset [1/2, 2]$. We will see that the behaviour of the multipliers $M_j = m_j(P/2^j)$ is fairly benign for $j \lesssim 1$. Proposition 2 then follows if we can show that for each $j \gtrsim 10$,

$$\|M_j\| \lesssim \|\langle t\rangle^{s_p} \widehat{m}_j(t)\|_{L^p(\mathbb{R})}, \tag{1.3}$$

and also that

$$\Big\| \sum_{j \gtrsim 1} M_j \Big\| \lesssim \sup_{j \gtrsim 1} \|M_j\|. \tag{1.4}$$

We prove (1.3) by obtaining some new quasi-orthogonality estimates for averages of solutions to the half-wave equation on M, which reduces our analysis to a geometric 'density-decomposition' argument involving geodesic annuli on M. We then use variants of Littlewood-Paley theory and the theory of atomic decompositions to obtain (1.4).

2 Preliminary Setup

Let us define our problem more precisely, which will also give us a chance to introduce some notation. Spectral theory guarantees the existence of a discrete set $\Lambda_M \subset (0, \infty)$, and an orthogonal decomposition $L^2(M) = \bigoplus_{\lambda \in \Lambda_M} \mathcal{V}_\lambda$, where $\mathcal{V}_\lambda$ is a finite dimensional subspace of $C^\infty(M)$ such that $Pf = \lambda f$ for all $f \in \mathcal{V}_\lambda$. If we let $\mathcal{P}_\lambda$ be the orthogonal projection operator onto $\mathcal{V}_\lambda$, then for any bounded function $m : (0, \infty) \to \mathbb{C}$, we define $m(P)$ by setting

$$m(P) = \sum_{\lambda \in \Lambda_M} m(\lambda) \mathcal{P}_\lambda.$$

As discussed in the last section, we consider a dyadic decomposition $m(P) = \sum_j M_j$, where $M_j = \sum_\lambda m_j(\lambda/2^j)\mathcal{P}_\lambda$. To exploit the integrability of the functions $\{\widehat{m}_j\}$, we must relate the Fourier transform of a function to it's associated spectral multiplier operator. A standard method in this setting is to apply the Fourier inversion formula; given a function $h : \mathbb{R} \to \mathbb{C}$, we have

$$h(P) = \int_{-\infty}^{\infty} \widehat{h}(t) e^{2\pi i t P} \, dt,$$

where

$$e^{2\pi i t P} = \sum_\lambda e^{2\pi i t \lambda} \mathcal{P}_\lambda$$

is the multiplier operator on M which, as t varies, gives solutions to the half-wave equation $\partial_t = 2\pi i P$ on M. In our situation, we have

$$M_j = \int_{-\infty}^{\infty} 2^j \widehat{m}_j(2^j t) e^{2\pi i P t} \, dt,$$

and so our study of multipliers reduces to studying averages of the half-wave propogators.

The half-wave equation on M is hyperbolic, and like other hyperbolic partial differential equations, the singularities of solutions to the half-wave propogate along characteristic curves, which is this case are the geodesics of the manifold M. For high frequency initial conditions, we should expect solutions to be concentrated near these geodesics. To obtain quantitative information following this intuition, we localize near various eigenvalue bands. Fix $\beta_0 \in C_c^\infty(\mathbb{R})$ with $\text{supp}(\beta) \subset [1/4, 4]$ and with $\beta_0(\lambda) = 1$ for $\lambda \in [1/2, 2]$. Then $\beta(t) = \beta_0(t)^2$ has the exact same properties. For $R > 0$, we define $Q_R = \beta(P/R)$. Then Q_R has range contained in the finite dimensional subspace V_R of $C^\infty(M)$ spanned by eigenfunctions of P with eigenvalues in $[R/4, 4R]$. Since P is elliptic, it is often a useful heuristic that elements of V_R have similar properties to a function on $\mathbb{R}^d$ with Fourier support on the annulus $\{\xi : R/4 \leqslant |\xi| \leqslant 4R\}$. In particular, the uncertainty principle tells us the kernel of the operator $Q_R \circ e^{2\pi i t P} \circ Q_R$ should be smooth, and locally constant at the scale $1/R$.

The symbol calculus allows us to introduce the operators $\{Q_R\}$ to our multipliers, writing

$$M_j = Q_j \circ M_j \circ Q_j = \int_{\mathbb{R}} 2^j \widehat{m}_j(2^j t)(Q_j \circ e^{2\pi i t P} \circ Q_j)\, dt.$$

We only have explicit formulas defining solutions to the half-wave equation for small times; this is why we must exploit the fact that the manifold M has periodic geodesic flow. Normalizing the metric on M appropriately, we may assume that the geodesic flow has period one. It follows that $e^{2\pi i(t+n)P} = e^{2\pi i t P}$ for any $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. We may then write

$$M_j = \int_{-1/2}^{1/2} b_j(t)(Q_j \circ e^{2\pi i t P} \circ Q_j)\, dt,$$

where $b_j : [-1/2, 1/2] \to \mathbb{C}$ is the periodization

$$b_j(t) = \sum_{n \in \mathbb{Z}} 2^j \widehat{m}_j(2^j(t+n)).$$

We then split our analysis of these multipliers into two regimes. In the first regime, over times $|t| \leqslant \varepsilon_M$, we perform a further wave packet decomposition at a frequency scale 2^j, whereas we do not perform this further wave packet decomposition over times $|t| > \varepsilon_M$, since we have better estimates on b_j over these times. We summarize this decomposition in the following lemma, whose proof we relegate to the appendix.

Lemma 3. *Let $\mathcal{T}_j = \mathbb{Z}/2^j \cap [-\varepsilon_M, \varepsilon_M]$. Then we can write*

$$b_j = \sum_{t_0 \in \mathcal{T}} b^I_{j,t_0} + b^{II}_j,$$

such that the following properties hold:

- *supp*$(b^I_{j,t_0}) \subset [t_0 - 2/2^j, t_0 + 2/2^j]$ *and supp*$(b^{II}_j) \subset [-1/2, 1/2] \smallsetminus [-\varepsilon_M/2, \varepsilon_M/2]$.
- *We have*

$$\left(\sum_{t_0 \in \mathcal{T}_j} \left[\|b^I_{j,t_0}\|_{L^1(\mathbb{R})} \langle 2^j t_0 \rangle^{\alpha(p)} \right]^p \right)^{1/p} \lesssim 2^{-j/p'} C_p(m)$$

and

$$\|b^{II}_j\|_{L^p(\mathbb{R})} \lesssim 2^{-j(1/p' + \alpha(p))} C_p(m).$$

We can thus write

$$M_j = M_j^I + M_j^{II} = \left(\sum_{t_0 \in \mathcal{T}_j} M_{j,t_0}^I \right) + M_j^{II}$$

where

$$M_{j,t_0}^I = \int b_{j,t_0}^I(t)(Q_j \circ e^{2\pi i t P} \circ Q_j)\, dt \quad \text{and} \quad M_j^{II} = \int b_j^{II}(t)(Q_j \circ e^{2\pi i t P} \circ Q_j)\, dt.$$

Given the comparably better bounds for the function b_j^{II} (we have an extra multiplicative factor $2^{-j\alpha(p)}$ to work with), we will be able to obtain bounds on M_j^{II} simply by applying Hölder's inequality, which reduces our analysis to an endpoint local smoothing inequality. On the other hand, we will obtain control over the operator M_j^I by understanding the interactions of functions of the form $f_{x_0,t_0} = M_{j,t_0}^I u_{x_0}$, where $u_{x_0} : M \to \mathbb{C}$ is a function with $\text{supp}(u_0)$ contained in $B(x_0, 2/2^j)$, the radius $2/2^j$ geodesic ball centered at some point $x_0 \in M$. We begin our analysis of these interactions in the next section.

3 Estimates For High-Frequency Wave Packets

The discussion at the end of the introduction motivated us to consider functions obtained by taking averages of the wave equation over a local set of times, with initial conditions localized to a particular frequency. In this section, we obtain pointwise bounds and orthogonality estimates for such functions, which we summarize in the following proposition.

Proposition 4. *For any compact Riemannian manifold M, there exists a small geometric constant $\varepsilon_M > 0$ such that for $R \geqslant 1/\varepsilon_M$, the following estimates hold:*

- *(Pointwise Estimates) Fix any $|t_0| \leqslant \varepsilon_M$ and $x_0 \in M$. Consider any two functions $c : \mathbb{R} \to \mathbb{C}$ and $u : M \to \mathbb{C}$ with*

$$\textit{supp}(c) \subset [t_0 - 2/R, t_0 + 2/R] \quad \textit{and} \quad \textit{supp}(u) \subset B(x_0, 2/R).$$

  If we define $f : M \to \mathbb{C}$ by setting

$$f = \int c(t)(Q_R \circ e^{2\pi i t P} \circ Q_R)\{u\}\, dt.$$

  Then for any $K \geqslant 0$, and $x \in M$,

$$|f(x)| \lesssim_M \|c\|_{L^1(\mathbb{R})} \|u\|_{L^1(M)} \frac{R^d}{(R d_g(x, x_0))^{\frac{d-1}{2}}} \Big\langle R \big| |t_0| - d_g(x, x_0) \big| \Big\rangle^{-K}.$$

- *(Quasi-Orthogonality Estimates) Fix $t_0, t_1 \in \mathbb{R}$ with $|t_0 - t_1| \leqslant \varepsilon_M$, and $x_0, x_1 \in M$. Consider any two pairs of functions $c_0, c_1 : \mathbb{R} \to \mathbb{C}$ and $u_0, u_1 : M \to \mathbb{C}$ with*

$$\textit{supp}(c_j) \subset [t_j - 2/R, t_j + 2/R] \quad \textit{and} \quad \textit{supp}(u_j) \subset B(x_j, 2/R).$$

Define two functions

$$f_j = \int c_j(t)(Q_R \circ e^{2\pi i t P} \circ Q_R)\{u_j\}\, dt.$$

Then for any $K \geqslant 0$,

$$|\langle f_0, f_1 \rangle| \lesssim_K \left(\prod_j \|c_j\|_{L^1(\mathbb{R})} \|u_j\|_{L^1(M)} \right) \frac{R^d}{(R d_g(x_0, x_1))^{\frac{d-1}{2}}} \Big\langle R \big| |t_0 - t_1| - d_g(x_1, x_0) \big| \Big\rangle^{-K}.$$

Suppose c, c_0, c_1, u, u_0, and u_1 are all L^1 normalized. The pointwise estimate tell us that the function f is concentrated on a geodesic annulus of radius t_0 centered at x_0, with thickness $1/R$, and on this annulus it has height at most $R^{\frac{d+1}{2}} |t_0|^{-\frac{d-1}{2}}$. The quasi-orthogonality estimate tells us that the two functions f_0 and f_1 are only significantly correlated with one another if the annuli on which f_0 and f_1 are externally or internally tangent to one another, and then the inner product $\langle f_0, f_1 \rangle$ has magnitude at most $R^{\frac{d+1}{2}} |t_0 - t_1|^{\frac{d-1}{2}}$. This estimate is then an analogue of Lemma 3.3 of [2], though with different exponents because here we are using the half wave equation to define our functions f_j, whereas in [3] the functions are simply defined by taking a smooth functions adapted to the respective annuli.

The remainder of this section is devoted to a proof of Proposition 4. Since R is fixed, we will write Q_R as Q in the sequel. For both estimates, we want to consider the operators in coordinates, so we can use the *Lax-Hörmander Parametrix* to understand the wave propogators in terms of various oscillatory integrals. Start by covering M by a finite family of open sets $\{V_\alpha\}$, chosen such that for each α, there is a coordinate chart U_α such that the neighborhood $N(V_\alpha, 0.5)$ is contained in U_α. Let $\{\eta_\alpha\}$ be a partition of unity subordinate to $\{V_\alpha\}$. It will be convenient to define $V_\alpha^* = N(V_\alpha, 0.1)$. The next Lemma allows us to approximate the operator Q, and the propogators $e^{2\pi i t P}$ with operators which have more explicit representations in the coordinate system $\{U_\alpha\}$, by an error term which is negligible to the results of Proposition 4.

Lemma 5. *For each α, and $|t| \leqslant 1/100$, there exists Schwartz operators Q_α and $W_\alpha(t)$, each with kernel supported on $U_\alpha \times V_\alpha^*$, such that the following properties hold:*

- *For $f \in L^1(M)$ with $supp(f) \subset V_\alpha^*$,*

  $$supp(Q_\alpha f) \subset N(supp(f), 0.1) \quad and \quad supp(W_\alpha(t) f) \subset N(supp(f), 0.1).$$

  Moreover, for all $N \geqslant 0$,

  $$\|(Q - Q_\alpha) f\|_{L^\infty(M)} \lesssim_N R^{-N} \|f\|_{L^1(M)}$$

  and

  $$\left\| \Big(Q_\alpha \circ \Big(e^{2\pi i t P} - W_\alpha(t) \Big) \circ Q_\alpha \Big) \{f\} \right\|_{L^\infty(M)} \lesssim_N R^{-N} \|f\|_{L^1(M)}.$$

- *In the coordinate system of U_α, Q_α is a pseudodifferential operator of order zero given by a symbol $\sigma(x, \xi)$, where*

  $$supp(\sigma) \subset \{\xi \in \mathbb{R}^d : R/2 \leqslant |\xi| \leqslant 2R\},$$

and σ satisfies derivative estimates of the form

$$|\partial_x^\beta \partial_\xi^\kappa \sigma(x,\xi)| \lesssim_{\beta,\kappa} R^{-|\kappa|}.$$

- *In the coordinate system U_α, the operator $W_\alpha(t)$ has a kernel $W_\alpha(t,x,y)$ with an oscillatory integral representation*

$$W_\alpha(t,x,y) = \int s(t,x,y,\xi) e^{2\pi i [\phi(x,y,\xi) + t|\xi|_y]}\, d\xi,$$

where s has compact support in it's x and y coordinates, with

$$supp_\xi(s) \subset \{\xi \in \mathbb{R}^d : R/2 \leqslant |\xi| \leqslant 2R\},$$

where s satisfies derivative estimates of the form

$$|\partial_{t,x,y}^\beta \partial_\xi^\kappa s| \lesssim_{\beta,\kappa} R^{-|\kappa|},$$

and where $|\cdot|_y$ denotes the norm on $\mathbb{R}^n_\xi$ induced by the Riemannian metric on S^d on the contangent space $T_y^* S^d$.

We relegate the proof of Lemma 4 to the appendix, the proof being a fairly technical calculation involving the calculus of Fourier integral operators. Let us now proceed with the proof of the pointwise bounds in Proposition 4 using this lemma. Given $u : M \to \mathbb{C}$, write $u = \sum u_\alpha$, where $u_\alpha = \eta_\alpha u$. Lemma 5 implies that if we define

$$f_\alpha = \int c(t)(Q_\alpha \circ W_\alpha(t) \circ Q_\alpha)\{u_\alpha\}\, dt,$$

then

$$\left\| f - \sum\nolimits_\alpha f_\alpha \right\|_{L^\infty(M)} \lesssim_N R^{-N} \|f\|_{L^1(M)}.$$

This error is negligible to the bounds we want to obtain in Proposition 4. We will bound each of the functions $\{f_\alpha\}$ separately from one another, applying the triangle inequality to get the main pointwise bounds.

To obtain the pointwise bounds, it suffices to expand out the implicit integrals in the definition of f_α, writing, in the coordinate system U_α,

$$\begin{aligned} f_\alpha(x) = \int c(t)\sigma(x,\eta) & e^{2\pi i \eta\cdot(x-y)} \\ & s(t,y,z,\xi) e^{2\pi i [\phi(y,z,\xi) + tp(z,\xi)]} \\ & \quad \sigma(z,\theta) e^{2\pi i \theta \cdot (z-w)} (\eta_\alpha u)(w) \\ & \quad\quad dt\, dy\, dz\, dw\, d\theta\, d\xi\, d\eta. \end{aligned}$$

The integral looks highly complicated, but can be simplified considerably by noticing that most variables are highly localized. To begin with, we note that since s is smooth and compactly supported in all it's variables, so s should roughly behave like a linear combination of tensor products of it's variables. Using Fourier series, we can write

$$s(t,y,z,\xi) = \sum_{n \in \mathbb{Z}^d} s_{n,1}(y) s_{n,2}(t,z,\xi),$$

where $s_{n,1}(y) = e^{2\pi i n \cdot y}$, and where

$$|\partial_{t,z}^\alpha \partial_\xi^\kappa \{s_{n,2}\}| \lesssim_{\alpha,k,N} |n|^{-N} R^{-|\kappa|}.$$

If we write $a_n(x,\xi) = a_{n,1}(x,R\xi) a_{n,2}(R\xi)$, where

$$a_{n,1}(x,\xi) = \int \sigma(x,\eta) s_{n,1}(y) e^{2\pi i [\eta \cdot (x-y) - \phi(x,x_0,\xi)]}\, dy\, d\eta$$

and

$$a_{n,2}(\xi) = \int c(t) s_{n,2}(t,z,\xi) \sigma(z,\theta)(\eta_\alpha u)(w) e^{2\pi i [\phi(y,z,\xi) + tp(z,\xi) + \theta \cdot (z-w) - t_0 |\xi|_{x_0}]}\, dt\, dz\, dw\, d\theta$$

then

$$f_\alpha(x) = R^d \sum_{n \in \mathbb{Z}^d} \int a_n(x,\xi) e^{2\pi i R[\phi(x,x_0,\xi) + t_0 |\xi|_{x_0}]}\, d\xi.$$

We have $\operatorname{supp}(a_n) \subset \{|\xi| \sim 1\}$ and

$$|(\nabla_\xi^\kappa a_n)(x,\xi)| \lesssim_{\kappa,N} |n|^{-N} \|c\|_{L^1(\mathbb{R})}.$$

To obtain an efficient upper bound on this oscillatory integral, it will be convenient to change coordinate systems in a way better respecting the Riemannian metric at x_0, i.e. finding a smooth family of diffeomorphisms $\{F_{x_0} : S^{d-1} \to S^{d-1}\}$ such that $|F_{x_0}|_{x_0} = 1$. We can choose this function such that $F_{x_0}(-x) = -F_{x_0}(x)$. Then if $a_n'(x,\rho,\eta) = a_n(x,\rho F_{x_0}(\eta)) JF_{x_0}(\eta)$, then a change of variables gives that

$$R^d \int a_n(x,\xi) e^{2\pi i R[\phi(x,x_0,\xi) + t_0 |\xi|_{x_0}]} = R^d \int_0^\infty \rho^{d-1} \int_{|\eta|=1} a_n'(x,\rho,\eta) e^{2\pi i R\rho[\phi(x,x_0,F_{x_0}(\eta)) + t_0]}\, d\eta\, d\rho.$$

For each fixed ρ, we claim that the phase has exactly two stationary points in the η variable, at the values $\pm\eta_0$, where x_1 lies on the geodesic passing through x_0 tangent to the vector $\eta_0^\sharp$ (here we are using the musical isomorphism to map the cotangent vector η_0 to a tangent vector). Moreover, at these values,

$$\phi(x_1, x_0, F_{x_0}(\pm\eta_0)) = \pm d_g(x_1, x_0),$$

and the Hessian at $\pm\eta_0$ is (positive / negative) definite, with each eigenvalue having magnitude exceeding a constant multiple of $d_g(x_1,x_0)$. It follows from the principle of stationary phase, that the integral above can be written as

$$\frac{R^d}{[R d_g(x_1,x_0)]^{\frac{d-1}{2}}} \sum_\pm \int_0^\infty \rho^{\frac{d-1}{2}} a_{n,\pm}''(x,\rho) e^{2\pi i R\rho[t_0 \pm d_g(x_1,x_0)]}\, d\rho,$$

where $a_{n,\pm}''$ is supported on $|\rho| \sim 1$, and

$$|\partial_\rho^m a_{n,\pm}''| \lesssim_K |n|^{-K} \|c\|_{L^1(\mathbb{R})}.$$

Integrating by parts in the ρ variable if $\pm d_g(x_1,x_0)+t_0$ is large, and then taking in absolute values, we conclude that

$$\left|\int a_n(x,\xi)e^{2\pi iR[\phi(x_1,x_0,\xi)+t_0|\xi|_{x_0}]}\right| \lesssim_{K_1,K_2} |n|^{-K_1}\frac{\|c\|_{L^1(\mathbb{R})}}{(Rd_g(x_1,x_0))^{\frac{d-1}{2}}}\sum_{\pm}\langle R|t_0 \pm d_g(x_1,x_0)|\rangle^{-K_2}.$$

Taking $K_1 \geqslant d+1$ and $K_2 = K$, and then summing in the n variable, we conclude that

$$\begin{aligned}|f_\alpha(x)| &= \left|R^d\sum_n\int a_n(x,\xi)e^{2\pi iR[\phi(x_1,x_0,\xi)+t'|\xi|_{x_0}]}\right|\\ &\lesssim_K \|c\|_{L^1(\mathbb{R})}\frac{R^d}{(Rd_g(x_1,x_0))^{\frac{d-1}{2}}}\sum_{\pm}\langle R|t_0\pm d_g(x_1,x_0)|\rangle^{-K}.\end{aligned}$$

Thus we have proved the bounds required.

The quasi-orthogonality arguments are obtained by a largely analogous method. One major difference is that we can use the self-adjointness of the operators Q, and the unitary group structure of $\{e^{2\pi itP}\}$, to write

$$\begin{aligned}\langle f_0,f_1\rangle &= \int c_0(t)c_1(s)\Big\langle (Q\circ e^{2\pi itP}\circ Q)\{u_0\},(Q\circ e^{2\pi isP}\circ Q)\{u_1\}\Big\rangle\\ &= \int c_0(t)c_1(s)\Big\langle (Q^2\circ e^{2\pi i(t-s)P}\circ Q^2)\{u_0\},u_1\Big\rangle\\ &= \int c(t)\Big\langle (Q^2\circ e^{2\pi itP}\circ Q^2)\{u_0\},u_1\Big\rangle,\end{aligned}$$

where $c(t)=\int c_0(u)c_1(u-t)\,du$ is essentially the convolution of the functions, satisfying

$$\|c\|_{L^1(\mathbb{R})}\lesssim \|c_0\|_{L^1(\mathbb{R})}\|c_1\|_{L^1(\mathbb{R})} \quad\text{and}\quad \operatorname{supp}(c)\subset[(t_0-t_1)-4/R,(t_0-t_1)+4/R].$$

After this, one proceeds exactly as in the proof of the pointwise estimate. We write the inner product as

$$\sum_\alpha\int c(t)\Big\langle (Q^2\circ e^{2\pi itP}\circ Q^2)\{\eta_\alpha u_0\},u_1\Big\rangle.$$

Then we use Lemma 5 too replace $Q^2\circ e^{2\pi itP}\circ Q^2$ with $Q_\alpha^2\circ W_\alpha(t)\circ Q_\alpha^2$ using Lemma 5, modulo a negligible error. The integral

$$\sum_\alpha\int c(t)\Big\langle (Q_\alpha^2\circ W_\alpha(t)\circ Q_\alpha^2)\{\eta_\alpha u_0\},u_1\Big\rangle$$

is then only non-zero if both the supports of u_0 and u_1 are compactly contained in U_α. Thus we can switch to the coordinate system of U_α, in which we can express the inner product by oscillatory integrals of the exact same kind as those occuring in the pointwise estimate. Integrating away any the highly localized variables, and then applying stationary phase, we obtain the required estimate.

4 Regime I: Density Arguments For Dyadic Pieces

In this section, we begin obtaining estimates for the operator M_j^I. Given a general input $u : M \to \mathbb{C}$, we consider a maximal $1/2^j$ separated subset $\mathcal{X}_j$ of M, then consider a decomposition $u = \sum_{x_0 \in \mathcal{X}_j} u_{x_0}$, where u_{x_0} is supported on $B(x_0, 2/2^j)$, such that for all $r \in [1, \infty]$,

$$\|u\|_{L^r(M)} \sim \left(\sum_{x_0 \in \mathcal{X}_j} \|u_{x_0}\|_{L^r(M)}^r \right)^{1/r}.$$

If we set $f_{x_0,t_0} = M_{j,t_0}^I \{u_{x_0}\}$, then

$$\|M_j^I u\|_{L^p(M)} = \left\| \sum f_{x_0,t_0} \right\|_{L^p(M)}.$$

In this section, we use the quasi-orthogonality estimates of the last section to obtain L^2 estimates on partial sums of the functions $\{f_{x_0,t_0}\}$, under a density assumption on the set of indices we are summing over. To obtain bounds on $\| \sum f_{x_0,t_0} \|_{L^p(M)}$, we will later perform a *density decomposition* to break up $\mathcal{X}_j \times \mathcal{T}_j$ into a low and high density piece, and the methods of this section, appropriately interpolated, will be used to control the L^p norm of the low density piece.

Proposition 6. *Fix $u \geqslant 1$. Consider a set $\mathcal{E} \subset \mathcal{X} \times \mathcal{T}$. Write*

$$\mathcal{E} = \bigcup_{k=0}^{\infty} \mathcal{E}_k,$$

where $\mathcal{E}_k = \{(x,t) \in \mathcal{E} : |t| \sim 2^{k-j}\}$. Suppose that each of the sets $\mathcal{E}_k$ has density type $(2^j u, 2^{k-j})$, i.e. so that for any set $B \subset \mathcal{X} \times \mathcal{T}$ with $diam(B) \leqslant 2^{k-j}$,

$$\#(\mathcal{E}_k \cap B) \leqslant 2^j u \; diam(B).$$

Then

$$\left\| \sum_k \sum_{(x_0,t_0) \in \mathcal{E}_k} 2^{k\frac{d-1}{2}} f_{x_0,t_0} \right\|_{L^2(S^d)} \lesssim 2^{jd} \log(u) u^{\frac{2}{d-1}} \sum_k 2^{k(d-1)} \# \mathcal{E}_k.$$

Proof. Write $F = \sum F_k$, where

$$F_k = 2^{k\frac{d-1}{2}} \sum_{(x_0,t_0) \in \mathcal{E}} f_{x_0,t_0}.$$

Applying Cauchy-Schwartz, we have

$$\|F\|_{L^2(M)}^2 \lesssim \log(u) \left(\sum_{k \leqslant 10 \log(u)} \|F_k\|_{L^2(M)}^2 + \| \sum_{k \geqslant 10 \log(u)} F_k\|_{L^2(M)}^2 \right).$$

Without loss of generality, increasing the implicit constant, we can assume that $\{k : \mathcal{E}_k \neq \varnothing\}$ is 10-separated, and that all values of t with $(x,t) \in \mathcal{E}$ are positive (the case where all values of t being negative being treated analogously, and then combined with the positive values trivially using the triangle inequality). Thus if F_k and $F_{k'}$ are both nonzero, then $k = k'$ or $|k - k'| \geqslant 10$. For $k \geqslant k' + 10$, let us estimate $\langle F_k, F_{k'} \rangle$. We can decompose this inner product into a sum of quantities of the form $2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} \langle f_{x_0,t_0}, f_{x_1,t_1} \rangle$, where $t_0 \sim 2^{k-j}$ and $t_1 \sim 2^{k'-j}$. Now consider the two sets

$$\mathcal{G}_{x_0,t_0,\mathrm{low}} = \{(x_1, t_1) \in \mathcal{E}_{k'} : |d_g(x_0, x_1) - (t_0 - t_1)| \lesssim 2^{k'+10-j}\}$$

and for $l \geqslant k' + 10$, consider the set

$$\mathcal{G}_{x_0,t_0,l} = \{(x_1, t_1) \in \mathcal{E}_{k'} : |d_g(x_0, x_1) - (t_0 - t_1)| \sim 2^{l-j}\}.$$

Let us use the density properties of $\mathcal{E}$ to control the size of these index sets. First, note that for any $(x_0, t_0) \in \mathcal{E}_k$ and $(x_1, t_1) \in \mathcal{E}_{k'}$, $t_0 - t_1$ lies in a radius $O(2^{k'-j})$ interval centered at t_0:

- Let us first estimate interactions between the functions S_{x_0,t_0} and S_{x_1,t_1} with $(x_1, t_1) \in \mathcal{G}_{x_0,t_0,\mathrm{low}}$. If $(x_1, t_1) \in \mathcal{G}_{x_0,t_0,\mathrm{low}}$, then x_1 must lie in a width $O(2^{k'-j})$ and radius $O(2^{k-j})$ annulus centered at x_0. Thus $\mathcal{G}_{x_0,t_0,\mathrm{low}}$ is covered by $O(2^{(k-k')(d-1)})$ balls of radius $2^{k'-j}$. The density properties of $\mathcal{E}_{k'}$ implies that

  $$\#\mathcal{G}_{x_0,t_0,l} \lesssim 2^j u \, 2^{(k-k')(d-1)}(2^{k'-j}) = u 2^{(k-k')(d-1)+k'}.$$

  Together with Proposition 4, we conclude that

  $$2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} \sum_{(x_1,t_1) \in \mathcal{G}_{x_0,t_0,\mathrm{low}}} |\langle f_{x_0,t_0}, f_{x_1,t_1} \rangle| \lesssim_M 2^{jd} 2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} \left(u 2^{(k-k')(d-1)+k'} \right) \left(2^{-k\frac{d-1}{2}} \right).$$

  We can now sum over $\log(u) \lesssim k' \leqslant k - 10$ and $(x_0, t_0) \in \mathcal{E}_k$ to find

  $$2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} \sum_{(x_0,t_0) \in \mathcal{E}_k} \sum_{k' \leqslant k-10} \sum_{(x_1,t_1) \in \mathcal{G}_{x_0,t_0,\mathrm{low}}} |\langle f_{x_0,t_0}, f_{x_1,t_1} \rangle| \lesssim 2^{jd} 2^{k(d-1)} \#\mathcal{E}_k.$$

- Next, let's estimate interactions between the functions f_{x_0,t_0} and f_{x_1,t_1} with $(x_1, t_1) \in \mathcal{G}_{x_0,t_0,l}$ with $k' + 10 \leqslant l \leqslant k - 5$. If $(x_1, t_1) \in \mathcal{G}_{x_0,t_0,l}$, then x_1 must lie in one of two geodesic annuli centered at x_0, each width $O(2^{l-j})$ and radii $O(2^{k-j})$. Thus $\mathcal{G}_{x_0,t_0,l}$ is covered by $O(2^{(l-k')} 2^{(k-k')(d-1)})$ balls of radius $2^{k'-j}$, and the density of $\mathcal{E}_{k'}$ implies that

  $$\#\mathcal{G}_{x_0,t_0,l} \lesssim 2^j u \, 2^{(l-k')} 2^{(k-k')(d-1)} 2^{k'-j} = u 2^l 2^{(k-k')(d-1)}.$$

  Together with Proposition 4, we conclude that

  $$2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} \sum_{(x_1,t_1) \in \mathcal{G}_{x_0,t_0,l}} |\langle f_{x_0,t_0}, f_{x_1,t_1} \rangle| \lesssim_N 2^{jd} 2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} \left(u 2^l 2^{(k-k')(d-1)} \right) \left(2^{-k\frac{d-1}{2}} 2^{-lN} \right).$$

  Picking $N > 1$, we can sum over $k' + 10 \leqslant l \leqslant k - 5$, $\log(u) \lesssim k' \leqslant k - 10$, and $(x_0, t_0) \in \mathcal{E}_k$ to find

  $$\sum_{(x_0,t_0) \in \mathcal{E}_k} \sum_{k' \leqslant k-10} \sum_{k'+10 \leqslant l \leqslant k-5} \sum_{(x_1,t_1) \in \mathcal{G}_{x_0,t_0,l}} 2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} |\langle f_{x_0,t_0}, f_{x_1,t_1} \rangle| \lesssim 2^{jd} 2^{k(d-1)} \#\mathcal{E}_k.$$

- Now let's estimate the interactions between the functions f_{x_0,t_0} and f_{x_1,t_1} with $(x_1,t_1) \in \mathcal{G}_{x_0,t_0,l}$, for $k+10 \leqslant l \leqslant j$, then x_1 must lie in a geodesic ball of radius $O(2^{l-j})$ centered at x_0. Such a ball is covered by $O(2^{(l-k')d})$ balls of radius $2^{k'-j}$, and the density of $\mathcal{E}_{k'}$ implies that
$$\#\mathcal{G}_{x_0,t_0,l} \lesssim 2^j u\, 2^{(l-k')d}(2^{k'-j}) = u2^{(l-k')d}2^{k'}.$$
Together with Proposition 4, we conclude that
$$2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}} \sum_{(x_1,t_1)\in\mathcal{G}_{x_0,t_0,l}} |\langle f_{x_0,t_0}, f_{x_1,t_1}\rangle| \lesssim_N 2^{jd}2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}}\left(u2^{(l-k')d}2^{k'}\right)\left(2^{-lN}\right).$$
Picking $N > d$, we can sum over $k-5 \leqslant l \leqslant 10j$, $\log(u) \lesssim k' \leqslant k-10$, and $(x_0,t_0) \in \mathcal{E}_k$ to conclude that
$$2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}} \sum_{(x_0,t_0)\in\mathcal{E}_k} \sum_{k'\leqslant k-10} \sum_{k-5\leqslant l\leqslant 10j} \sum_{(x_1,t_1)\in\mathcal{G}_{x_0,t_0,l}} |\langle f_{x_0,t_0}, f_{x_1,t_1}\rangle| \lesssim 2^{jd}.$$

Putting these three bounds together, we conclude that
$$\sum_{\log(u)\lesssim k'<k} |\langle F_k, F_{k'}\rangle| \lesssim 2^{jd}\sum_k 2^{k(d-1)}\#\mathcal{E}_k.$$
In particular, we have
$$\|F\|^2_{L^2(M)} \lesssim \log(u)\left(\sum_k \|F_k\|^2_{L^2(M)} + 2^{jd}\sum_k 2^{k(d-1)}\#\mathcal{E}_k\right).$$
Next, let us fix some parameter a, and decompose $[2^{k-j}, 2^{k+1-j}]$ into the disjoint union of length u^a intervals
$$I_{k,\mu} = [2^{k-j} + (\mu-1)u^a2^{-j}, 2^{k-j} + \mu u^a 2^{-j}] \quad \text{for } 1 \leqslant \mu \leqslant 2^k/u^a,$$
and thus considering a further decomposition $\mathcal{E}_k = \bigcup \mathcal{E}_{k,\mu}$ and $F_k = \sum F_{k,\mu}$. As before, increasing the implicit constant in the Proposition, we may assume without loss of generality that the set $\{\mu : \mathcal{E}_{k,\mu} \neq \varnothing\}$ is 10-separated. We now estimate
$$\sum_{\mu\geqslant\mu'+10} |\langle F_{k,\mu}, F_{k,\mu'}\rangle|.$$
For $(x_0,t_0) \in \mathcal{E}_{k,\mu}$ and $l \geqslant 1$, define
$$\mathcal{H}_{x_0,t_0,l} = \left\{(x_1,t_1) \in \mathcal{E}_{k,\mu'} : \max(d_g(x_0,x_1), t_0-t_1) \sim 2^l u^a 2^{-j}\right\}.$$
Then $\bigcup_{l\geqslant 1}\mathcal{H}_{x_0,t_0,l}$ covers $\bigcup_{\mu\geqslant\mu'+10}\mathcal{E}_{k,\mu'}$. The density properties of $\mathcal{E}_{k,\mu'}$ imply that provided that $l \leqslant k - a\log_2 u + 10$ (so that $2^l u^a 2^{-j} \leqslant 2^{k-j}$),
$$\#\mathcal{H}_{x_0,t_0,l} \lesssim (2^j u)(2^l u^a/2^j) = u^{a+1}2^l$$

For $(x_1, t_1) \in \mathcal{H}_{x_0,t_0,l}$, we claim that

$$2^{k(d-1)}|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim 2^{jd}2^{k(d-1)}(2^l u^a)^{-\frac{d-1}{2}}.$$

Indeed, for such tuples we have

$$d_g(x_0,x_1) \gtrsim 2^l u^a/2^j \quad \text{or} \quad |d_g(x_0,x_1) - (t_0 - t_1)| \gtrsim 2^l u^a/2^j,$$

and the estimate follows from Proposition 4 in either case. Since $d \geqslant 4$,

$$\begin{aligned}\sum_{1 \leqslant l \leqslant k - a\log_2 u + 10} \sum_{(x_1,t_1)\in\mathcal{H}_{x_0,t_0,l}} 2^{k(d-1)}|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| &\lesssim 2^{jd} \sum_{1 \leqslant l \leqslant k - a\log_2 u + 10} (2^{k(d-1)})(2^l u^a)^{-\frac{d-1}{2}}(u^{a+1}2^l)\\ &\lesssim 2^{jd} \sum_{1 \leqslant l \leqslant k - a\log_2 u + 10} 2^{k(d-1)}2^{-l\frac{d-3}{2}}u^{1-a\left(\frac{d-3}{2}\right)}\\ &\lesssim 2^{jd}2^{k(d-1)}u^{1-a\left(\frac{d-3}{2}\right)}.\end{aligned}$$

For $l > k - a\log_2 u + 10$, a tuple (x_1, t_1) lies in $\mathcal{H}_{x_0,t_0,l}$ if and only if $d_g(x_0, x_1) \sim 2^l u^a/2^j$, since we always have

$$|t_0 - t_1| \lesssim 2^k/2^j \ll 2^l u^a/2^j.$$

We conclude from Proposition 4 that

$$2^{k(d-1)}|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim_M 2^{jd}2^{k(d-1)}(2^l u^a)^{-M}.$$

Now $\mathcal{H}_{x_0,t_0,l}$ is covered by $O((2^{l-k}u^a)^d)$ balls of radius $2^k/2^j$, and the density properties of $\mathcal{E}_k$ imply that

$$\#\mathcal{H}_{x_0,t_0,l} \lesssim (2^j u)(2^{l-k}u^a)^d(2^k/2^j) \lesssim u^{1+ad}2^{ld}2^{-k(d-1)}.$$

Thus, picking $M > \max(d, 1 + ad)$, we conclude that

$$\begin{aligned}\sum_{l \geqslant k - a\log_2 u + 10} \sum_{(x_1,t_1)\in\mathcal{H}_{x_0,t_0,l}} 2^{k(d-1)}|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| &\lesssim 2^{jd} \sum_{l \geqslant k - a\log_2 u + 10} (2^{k(d-1)})(2^l u^a)^{-M}u^{1+ad}2^{ld}2^{-k(d-1)}\\ &\lesssim 2^{jd}.\end{aligned}$$

Putting these two bounds together, and then summing over the tuples (x_0, t_0), we conclude that

$$\sum_{\mu \geqslant \mu' + 10} |\langle F_{k,\mu}, F_{k,\mu'}\rangle| \lesssim 2^{jd}\left(1 + 2^{k(d-1)}u^{1-a\left(\frac{d-3}{2}\right)}\right)\#\mathcal{E}_{k,\mu}.$$

Now summing in μ, we conclude that

$$\|F_k\|^2_{L^2(S^d)} \lesssim \sum_\mu \|F_{k,\mu}\|^2_{L^2(S^d)} + 2^{jd}\left(1 + 2^{k(d-1)}u^{1-a\left(\frac{d-3}{2}\right)}\right)\#\mathcal{E}_k.$$

The functions in the sum defining $F_{k,\mu}$ are highly coupled, and it is difficult to use anything except Cauchy-Schwartz to break them apart. Since $\#(\mathcal{T} \cap I_{k,\mu}) \sim u^a$, if we set $F_{k,\mu} = \sum_{t\in\mathcal{T}\cap I_{k,\mu}} F_{k,\mu,t}$, then we find

$$\|F_{k,\mu}\|^2_{L^2(S^d)} \lesssim u^a \sum_{t\in\mathcal{T}\cap I_{k,\mu}} \|F_{k,\mu,t}\|^2_{L^2(S^d)}.$$

Fortunately, since $\mathcal{X}$ is 1-separated, the functions in $F_{k,\mu,t}$ are quite orthogonal to one another, and so

$$\|F_{k,\mu,t}\|_{L^2(S^d)}^2 \lesssim 2^{jd}2^{k(d-1)}\#(\mathcal{E}_k \cap (S^d \times \{t\})).$$

But this means that

$$u^a \sum_t \|F_{k,\mu,t}\|_{L^2(S^d)}^2 \lesssim 2^{jd}2^{k(d-1)}u^a\#\mathcal{E}_{k,\mu}.$$

and so

$$\begin{aligned}\|F_k\|_{L^2(S^d)}^2 &\lesssim \sum_\mu \|F_{k,\mu}\|_{L^2(S^d)}^2 + 2^{jd}\left(1 + 2^{k(d-1)}u^{1-a\left(\frac{d-3}{2}\right)}\right)\#\mathcal{E}_k \\ &\lesssim 2^{jd}\left(2^{k(d-1)}u^a + (1 + 2^{k(d-1)}u^{1-a\left(\frac{d-3}{2}\right)}\right)\#\mathcal{E}_k.\end{aligned}$$

Picking $a = 2/(d-1)$, we conclude that

$$\|F_k\|_{L^2(S^d)}^2 \lesssim 2^{jd}2^{k(d-1)}u^{\frac{2}{d-1}}\#\mathcal{E}_k.$$

Thus, returning to our bound for F, we conclude that

$$\|F\|_{L^2(S^d)}^2 \lesssim 2^{jd}\log(u)u^{\frac{2}{d-1}}\sum_k 2^{k(d-1)}\#\mathcal{E}_k.$$

Together with Proposition BLAH of the last section, this completes the proof of the L^2 density bound. $\square$

5 Regime I: Density Decompositions

In this section, we describe the *density decomposition* argument, which will enable us to prove L^p bounds for the operator M_j^I. We fix a function u. Then

$$M_j u = M_j\{Q_j u\}.$$

We can then use a partition of unity to write

$$Q_j u = \sum_{x_0 \in \mathcal{X}_j} u_{x_0},$$

where u_{x_0} is supported on $B(x_0, 2/2^j)$, and, by Bernstein's inequality,

$$\left(\sum_{x_0 \in \mathcal{X}_j} \|u_{x_0}\|_{L^1(M)}^p\right)^{1/p} \lesssim 2^{-jd/p'}\left(\sum_{x_0 \in \mathcal{X}_j} \|u_{x_0}\|_{L^p(M)}^p\right)^{1/p} \lesssim 2^{-jd/p'}\|Q_j u\|_{L^p(M)} \lesssim 2^{-jd/p'}\|u\|_{L^p(M)}.$$

Define

$$\mathcal{X}_{j,l} = \{x_0 \in \mathcal{X}_j : 2^{l-1} < \|u_{x_0}\|_{L^1(M)} \leqslant 2^l\}$$

and let

$$\mathcal{T}_{j,r} = \{t_0 \in \mathcal{T}_j : 2^{r-1} < \|b_{j,t_0}\|_{L^1(M)} \leqslant 2^r\}.$$

Then

$$\left(\sum_l 2^{lp}\#\mathcal{X}_{j,l}\right)^{1/p} \lesssim 2^{-j/p'}\|u\|_{L^p(M)}.$$

Define functions $f_{x_0,t_0} = M^I_{j,t_0}u_{x_0}$. Our computation would be complete if we could show that for any function $c : \mathcal{X}_j \times \mathcal{T}_j \to \mathbb{C}$,

$$\left\|\sum_{l,r} 2^{-(l+r)} \sum_{x_0\in\mathcal{X}_{j,l}} \sum_{t_0\in\mathcal{T}_{j,r}} t_0^{\frac{d-1}{2}} c(x_0,t_0) f_{x_0,t_0}\right\|_{L^p(M)} \lesssim 2^{j(\alpha(p)-1+d/p')} \left(\sum |c(x_0,t_0)|^p t_0^{d-1}\right)^{1/p}. \quad (5.1)$$

Indeed, if we then set $c(x_0,t_0) = 2^{l+r}t_0^{-\frac{d-1}{2}}$ for $x_0 \in \mathcal{X}_{j,l}$ and $t_0 \in \mathcal{T}_{j,r}$, then we find that

$$\begin{aligned}\|M^I_j\{Q_j u\}\|_{L^p(M)} &= \left\|\sum f_{x_0,t_0}\right\|_{L^p(M)}\\ &\lesssim 2^{j(\alpha(p)-1+d/p')}\left(\sum\left[\|u_{x_0}\|_{L^1(M)}\|b_{j,t_0}\|_{L^1(\mathbb{R})}t_0^{\alpha(p)}\right]^p\right)^{1/p}\\ &\lesssim 2^{j(\alpha(p)-1+d/p')}[2^{-j[\alpha(p)+1/p']}C_p(m)][2^{-jd/p'}\|u\|_{L^p(M)}]\\ &\lesssim C_p(m)\|u\|_{L^p(M)}.\end{aligned}$$

For $p = 1$, this inequality follows simply by applying the triangle inequality, and applying the pointwise estimates of Proposition 4. Applying interpolation, for $p > 1$ we need only prove a restricted weak type version of this inequality. In other words, we can restrict c to be the indicator function of a set $\mathcal{E} \subset \mathcal{X}_j \times \mathcal{T}_j$, and take $L^{p,\infty}$ norms on the left hand side. If we write $\mathcal{E} = \bigcup_k \mathcal{E}_{k,l,r}$, where $\mathcal{E}_{k,l,r}$ is the set of (x,t) in $\mathcal{E} \cap (\mathcal{X}_{j,l} \times \mathcal{T}_{j,r})$ with $|t| \sim 2^k/R$, then the inequality reads that

$$\left\|\sum_{l,r}\sum_{k=1}^{\infty} 2^{k\frac{d-1}{2}} \sum_{(x_0,t_0)\in\mathcal{E}_k} 2^{-(l+r)} f_{x_0,t_0}\right\|^p_{L^{p,\infty}(S^d)} \lesssim 2^{j[(d-1)p-d]}\sum_{k=1}^{\infty} 2^{k(d-1)}\#\mathcal{E}_k.$$

This is equivalent to showing that for any $\lambda > 0$,

$$\left|\left\{x : \left|\sum_{k,l,r} 2^{k\frac{d-1}{2}} f_{x_0,t_0}(x)\right| \geqslant \lambda\right\}\right| \lesssim \lambda^{-p}2^{j[(d-1)p-d]}\sum_k 2^{k(d-1)}\#\mathcal{E}_k.$$

The case $\lambda \lesssim 2^{j(d-1)}$ follows from the L^1 boundedness we've already proved. To prove the inequality when $\lambda \gtrsim 2^{j(d-1)}$, we employ the method of density decompositions introduced in [2]. Let

$$A = \left(\frac{\lambda}{R^{d-1}}\right)^{(d-1)(1-p/2)} \log\left(\frac{\lambda}{R^{d-1}}\right)^{O(1)}.$$

Then for each k, consider the collection $\mathcal{B}_k(\lambda)$ of all balls B with radius at most $2^k/R$ such that $\#\mathcal{E}_k \cap B \geqslant RA\text{rad}(B)$. Applying the Vitali covering lemma, we can find a disjoint family of balls $\{B_1,\dots,B_N\}$ in $\mathcal{B}_k$ such that the balls $\{B_1^*,\dots,B_N^*\}$ obtained by dilating the balls by 5 cover $\bigcup\mathcal{B}_k(\lambda)$. Then

$$\sum \text{rad}(B_j) \leqslant R^{-1}A^{-1}\#\mathcal{E}_k,$$

and the set $\widehat{\mathcal{E}}_k = \mathcal{E}_k - \bigcup \mathcal{B}_k(\lambda)$ has density type $(RA, 2^k/R)$. Then we conclude that, using the quasi-orthogonality estimates below,

$$\left\| \sum_k \sum_{(x_0,t_0)\in\widehat{\mathcal{E}}_k} 2^{k\frac{d-1}{2}} S_{x_0,t_0} \right\|_{L^2(S^d)}^2 \lesssim_p R^{d-2} \log(A) A^{\frac{2}{d-1}} \sum_k 2^{k(d-1)} \#\mathcal{E}_k.$$

Appling Chebyshev's inequality, and utilizing the choice of A above, we conclude that

$$\begin{aligned} \left| \left\{ x : \left| \sum_k \sum_{(x_0,t_0)\in\widehat{\mathcal{E}}_k} 2^{k\frac{d-1}{2}} S_{x_0,t_0}(x) \right| \geqslant \lambda/2 \right\} \right| &\lesssim R^{d-2} \log(A) A^{\frac{2}{d-1}} \sum_k 2^{k(d-1)} \#\mathcal{E}_k \\ &\lesssim \lambda^{-p} R^{(d-1)p-d} \sum_k 2^{k(d-1)} \#\mathcal{E}_k. \end{aligned}$$

Conversely, we exploit the clustering of the sets $\mathcal{E}_k - \widehat{\mathcal{E}}_k$ to bound

$$\left| \left\{ x : \left| \sum_k \sum_{(x_0,t_0)\in\mathcal{E}_k - \widehat{\mathcal{E}}_k} 2^{k\frac{d-1}{2}} S_{x_0,t_0}(x) \right| \geqslant \lambda/2 \right\} \right|$$

That is, we have found balls $B_1^* < \dots, B_N^*$, each with radius $O(2^k/R)$, such that

$$\sum \mathrm{rad}(B_j) \leqslant R^{-1} A^{-1} \#\mathcal{E}_k.$$

Let (x_j, t_j) denote the center of the ball B_j. Then the function

$$\sum_{(x_0,t_0)\in B_j} S_{x_0,t_0}$$

has mass concentrated on the geodesic annulus $\mathrm{Ann}_j \subset S^d$ with radius t_j and thickness $O(\mathrm{rad}(B_j))$, a set with measure $(2^k/R)^{d-1}\mathrm{rad}(B_j)$. For $(x_0,t_0) \in B_j$, we calculate using the pointwise bounds that

$$\begin{aligned} \int_{\mathrm{Ann}_j^c} |S_{x_0,t_0}(x)|\, dx &\lesssim R^{d-1} \int_{\mathrm{rad}(B_j)\lesssim|t_j - d_g(x,x_0)|\leqslant 1} \langle R|t_0 - d_g(x,x_0)|\rangle^{-M} \\ &\lesssim R^{d-1} \int_{\mathrm{rad}(B_j)\leqslant|t_j-s|\leqslant 1} s^{d-1} \langle R|t_0 - s|\rangle^{-M}\, ds \\ &\lesssim 2^{k(d-1)} R^{d-1} (R\mathrm{rad}(B_j))^{-M}. \end{aligned}$$

Because the set of points in $\mathcal{E}_k$ is $1/R$ separated, there can only be at most $O(R\mathrm{rad}(B_j))^{d+1}$ values of (x_0, t_0), and so applying the triangle inequality gives that the sum of the L^1 norm outside of Ann_j is

$$\lesssim 2^{k(d-1)} R^{d-1} (R\, \mathrm{rad}(B_j))^{d+1-M}$$

Note that since $\#\mathcal{E}_k \cap B_j \geqslant RA\mathrm{rad}(B_j)$, and because $\mathcal{E}_k$ is $1/R$ discretized,

$$\mathrm{rad}(B_j) \geqslant (A/R)^{\frac{1}{d-1}},$$

and this, together with Markov's inequality, is enough to justify the required bound. Conversely, since $1 < p < 2(d-1)/(d+1)$, we have

$$\begin{aligned}\sum |\mathrm{Ann}_j| &\lesssim (2^k/R)^{d-1} \sum_j \mathrm{rad}(B_j) \\ &\lesssim (2^k/R)^{d-1} R^{-1} (L/R^{d-1})^{-(d-1)(1-p/2)} log(L/R^{d-1})^{O(1)} \\ &\lesssim \lambda^{-p} R^{(d-1)p-d} 2^{k(d-1)} \# \mathcal{E}_k,\end{aligned}$$

Summing over k completes the analysis.

6 Regime II: Local Smoothing

We now how to bound the operators $\{M_j^{II}\}$, by a reduction to an endpoint local smoothing inequality, namely, the inequality that

$$\|e^{2\pi i t P} f\|_{L^q(M) L_t^q[-1/2,1/2]} \lesssim \|f\|_{L^q_{\alpha(p')-1/p'}}. \tag{6.1}$$

This inequality is proved in Corollary 1.2 of [5] for $p < 2(d-1)/(d+1)$, which covers the range of parameters studied in this paper. Alternatively, (5.1), which we proved for a general function c, can be used to prove (6.1), by a generalization of the method of Section 10 of [2]. This completes the proof of (1.3).

Proposition 7. *We have*

$$\|M_j^{II}\{Q_j u\}\|_{L^p(M)} \lesssim C_p(m) \|f\|_{L^p(M)}.$$

Proof. For each j, the *class* of operators of the form $\{M_j^{II}\}$ formed from a multiplier m with $C_p(m) < \infty$ is closed under taking adjoints. Indeed, if M_j^{II} is obtained from m, then $(M_j^{II})^*$ is obtained from the multiplier $\overline{m}$. Because of this self-adjointness, if we can prove that for any multiplier m satisfying the assumptions of the theorem,

$$\|M_j^{II}\{Q_j u\}\|_{L^{p'}(M)} \lesssim C_p(m) \|f\|_{L^{p'}(M)},$$

then the result will follow. We do this because it is easier to exploit local smoothing inequalities in this setting, which tend to give better results when large Lebesgue exponents are involved, precisely because Lebesgue norms with large exponents are more sensitive to functions with sharp peaks, something explicitly prevented by obtaining control over the smoothness of a function.

Using Hölder's inequality, we find that

$$\begin{aligned}|M_j^{II}| &= \left| \int_{-1/2}^{1/2} b_j^{II} e^{2\pi i t P} \, dt \right| \\ &\leqslant \|b_j^{II}\|_{L^p(\mathbb{R})} \left(\int_{-1/2}^{1/2} |e^{2\pi i t P}|^{p'} \right)^{1/p'} \\ &\lesssim 2^{-j(1/p'+\alpha(p))} C_p(m) \left(\int_{-1/2}^{1/2} |e^{2\pi i t P}|^{p'} \right)^{1/p'}.\end{aligned}$$

Applying the endpoint local smoothing inequality, we conclude that

$$\begin{aligned}\|M_j^{II}\{Q_j u\}\|_{L^{p'}(M)} &\lesssim C_p(m)2^{j(1/p'-\alpha(p'))}\|e^{2\pi iP}(Q_j u)\|_{L_t^{p'}L_x^{p'}} \\ &\lesssim C_p(m)2^{j(1/p'-\alpha(p'))}\|Q_j u\|_{L^q_{\alpha(p')-1/p'}(M)},\end{aligned}$$

Applying Bernstein's inequality gives

$$\|Q_j u\|_{L^q_{\alpha(p')-1/p'}(M)} \lesssim 2^{j(\alpha(p')-1/p')}\|u\|_{L^p(M)}.$$

Thus we conclude that

$$\|M_j^{II}\{Q_j f\}\|_{L^{p'}(M)} \lesssim C_p(m)\|u\|_{L^{p'}(M)}. \qquad \square$$

7 Combining Dyadic Estimates

As of the last section, we have now completed the argument justifying that the operators M_j are each separately bounded on $L^p(M)$, with operator norm given in inequality (1.3). Our goal now is to prove (1.4), which will enable us to sum in j to obtain the required general bound

$$\|m(P)u\|_{L^p(M)} \lesssim C_p(m)\|u\|_{L^p(M)}.$$

This argument is made comparatively easier by the fact that the functions $\{M_j\}$ are supported on different dyadic intervals, which means we can apply variants of Littlewood-Paley theory and other square function estimates. We use these tools to obtain an atomic decomposition for our input functions, inspired by the calculations in [7], which will yield the required bound.

Lemma 8. *Consider the coordinate charts $\{U_\alpha\}$ and $\{V_\alpha\}$ introduces in Section 3. Then we can write*

$$u_j = \sum_\alpha \sum_s \sum_{W \in \mathcal{W}_{\alpha,s}} a_{\alpha,j,s,W}.$$

For each s, $\mathcal{W}_{\alpha,s}$ is a union of almost disjoint dyadic cubes in the coordinate system U_α, and the following properties hold:

- *If W has sidelength 2^l, then $a_{\alpha,j,s,W}$ is only nonzero if $l \geqslant -s$.*
- *For each such dyadic cube W, $a_{\alpha,j,s,W}$ is supported on the inverse image of W. We have*

$$\left(\sum_j \sum_\alpha \sum_{W \in \mathcal{W}_{\alpha,s}} \|a_{\alpha,j,s,W}\|_{L^2(M)}\right)^{1/2} \lesssim 2^s|\Omega_s|,$$

- *For any assignment $\{j(\alpha,W)\}$ for each $W \in \mathcal{W}_{\alpha,s}$, and any $0 \leqslant r \leqslant 2$, we have*

$$\left(\sum_\alpha \sum_{W \in \mathcal{W}_{\alpha,s}} |W|\|a_{\alpha,j(\alpha,W),W,s}\|^p_{L^\infty(M)}\right)^{1/p} \lesssim 2^s|\Omega_s|.$$

Using this lemma, we write

$$m(P) = \sum M_j u_j = \sum_{\alpha,j,s} \sum_{W \in \mathcal{W}_{\alpha,s}} M_j a_{\alpha,j,s,W}.$$

We will find it convenient to reorder this sum as

$$m(P) = \sum_{\alpha,j,s,l} \sum_{W \in \mathcal{W}_{\alpha,s,l}} M_j a_{\alpha,j,s,W},$$

where $\mathcal{W}_{\alpha,s,l}$ are the cubes in $\mathcal{W}_{\alpha,s}$ with sidelength 2^{l-j}. We then write $M_j = M_{j,l}^{\text{Short}} + M_{j,l}^{\text{Long}}$, where

$$M_{j,l}^{\text{Short}} = \int b_j(t)\psi(2^{j-l}t)(Q_j \circ e^{2\pi itP} \circ Q_j)\ dt$$

and

$$M_{j,l}^{\text{Long}} = \int b_j(t)\Big(1 - \psi(2^{j-l}t)\Big)(Q_j \circ e^{2\pi itP} \circ Q_j)\ dt.$$

We bound the short and long range interactions separately.

The function $M_{j,l}^{\text{Short}}\{a_{\alpha,j,s,W}\}$ is then concentrated on W^*. Since the sets $\{W^*\}$ have bounded overlap, for each fixed s, we have a square root cancellation bound

$$\|\sum_{\alpha,j,l} M_{j,l}^{\text{Short}}\{a_{\alpha,j,s,W}\}\|_{L^2(M)} \lesssim \left(\sum_{\alpha,j,s,l} \|M_{j,l}^{\text{Short}}\{a_{\alpha,j,s,W}\}\|_{L^2(M)}^2\right)^{1/2}.$$

But our calculations in previous sections should already have shown (via interpolation between $L^p(M)$ and $L^{p'}(M)$) that

$$\|M_{j,l}^{\text{Short}}\{a_{\alpha,j,s,W}\}\|_{L^2(M)} \lesssim C_p(M)\|a_{\alpha,j,s,W}\|_{L^2(M)},$$

and so we have

$$\|\sum_{\alpha,j,l} M_{j,l}^{\text{Short}}\{a_{\alpha,j,s,W}\}\|_{L^2(M)} \lesssim C_p(m)\left(\sum_{\alpha,j,s,l} \|a_{\alpha,j,s,W}\|_{L^2(M)}^2\right)^{1/2} \lesssim C_p(m)2^s|\Omega_s|^{1/2}.$$

Applying Hölder's inequality, since our inputs are supported on Ω_s, we have

$$\|\sum_{\alpha,j,l} M_{j,l}^{\text{Short}}\{a_{\alpha,j,s,W}\}\|_{L^p(M)} \lesssim C_p(m)2^s|\Omega_s|^{1/p}.$$

Summing in s, we have

$$\|\sum_{\alpha,j,s,l} M_{j,l}^{\text{Short}}\{a_{\alpha,j,s,W}\}\|_{L^p(M)} \lesssim C_p(m)\sum_s 2^s|\Omega_s|^{1/p} \lesssim C_p(m)\|Su\|_{L^p(M)} \lesssim C_p(m)\|u\|_{L^p(M)}.$$

Thus we've obtained bounds for the short range interactions.

Next, we consider the long range interactions. Applying Minkowski's inequality, we can write

$$\|\sum_{\alpha,j,l} M_{j,l}^{\text{Long}}\{a_{\alpha,j,s,W}\}\|_{L^p(M)} \lesssim \sum_{l \geqslant 0}\left(\sum_j \|\sum_{l(W)=l-j} M_{j,l}^{\text{Long}}\{A_{j,W}\}\|_{L^p(M)}^p\right)^{1/p}.$$

Suppose we can prove that there exists $\varepsilon > 0$ such that

$$\| \sum_{l(W)=l-j} M_{j,l}^{\text{Long}}\{A_{j,W}\}\|_{L^p(M)} \lesssim C_p(m)2^{-l\varepsilon} \left(\sum_{l(W)=l-j} |W| \|A_{j,W}\|_{L^\infty(M)}^p \right)^{1/p}.$$

Lemma BLAH then implies that this quantity is bounded by $C_p(m)2^{-l\varepsilon}2^s|\Omega_s|$, and summing in j gives that

$$\| \sum_{\alpha,j,l} M_{j,l}^{\text{Long}}\{a_{\alpha,j,s,W}\}\|_{L^p(M)} \lesssim C_p(m)2^{-l\varepsilon}2^s|\Omega_s|.$$

Summing in s then gives

$$\| \sum_{\alpha,j,l} M_{j,l}^{\text{Long}}\{a_{\alpha,j,s,W}\}\|_{L^p(M)} \lesssim C_p(m)\|Su\|_{L^p(M)} \lesssim C_p(m)\|u\|_{L^p(M)},$$

Putting these two bounds together gives $\|m(P)u\|_{L^p(M)} \lesssim C_p(m)\|u\|_{L^p(M)}$, which completes the proof.

8 Appendix

TODO

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