Research Statement

Jacob Denson

I am an analyst, studying problems with techniques taken mainly from harmonic analysis, but also from probability and combinatorics. My PhD research, advised by Andreas Seeger, has focused on the study of radial Fourier multiplier operators on Euclidean space, and their analogues on compact manifolds, through an understanding of the geometry and regularity of wave propagation. I have also solved problems in geometric measure theory, investigating when 'structure' occurs in random fractals of large dimension. Both projects raise interesting questions I plan to pursue in my postgraduate work.

During my PhD, my work on multipliers has concentrated on the relation between L^p bounds for Fourier multiplier operators on $\mathbb{R}^d$, and L^p bounds for analogous operators on compact manifolds, such as the multiplier operators associated with spherical harmonic expansions on S^d. My main achievement, for $d \geq 4$, and a range of L^p spaces, is a complete characterization of functions whose dilates correspond to a uniformly bounded family of multiplier operators on $L^p(S^d)$. The result implies a 'transference principle', which states that the L^p boundedness of a radial Fourier multiplier operator implies the L^p boundedness of the multiplier operator on S^d given by the same multiplier. The result for compactly supported functions m is found in [14], with the remaining part of the argument in preparation. *Both results are the first of their kind for multiplier operators on S^d for $p \neq 2$; more broadly, no comparable results have been established for analogous multiplier operators on any other compact manifold.* More detail about this project can be found in Section 1 of this statement.

My work in geometric measure theory focuses on constructing sets of large fractal dimension avoiding point configurations. Before starting my PhD, Malabika Pramanik, Joshua Zahl, and I constructed sets with large Hausdorff dimension avoiding point configurations [15]. During my PhD, I combined the methods of that paper with more robust probabilistic machinery to address the more difficult problem of constructing sets with large Fourier dimension avoiding configurations [13]. *This method remains the only method of constructing sets of large Fourier dimension avoiding nonlinear configurations, and remains the best method for avoiding general 'linear' point configurations when $d > 1$.* This work is discussed further in Section 2.

In Section 3, I discuss my plans for future research, emphasizing how my PhD work gives me the tools to succeed in these plans. *These projects include characterizing the L^p boundedness of multipliers of the Laplace-Beltrami operator on Riemannian manifolds whose geodesic flow has nice dynamical properties, obtaining $l^2(L^p)$ decoupling bounds for random fractal subsets of $\mathbb{R}$, among other more exploratory projects.*

1 Multiplier Operators on Euclidean Space and on Manifolds

Multiplier operators have long been central objects of study in harmonic analysis, with particular focus placed on the Fourier multiplier operators. Such operators T are defined by a function $m : \mathbb{R}^d \to \mathbb{C}$, the 'multiplier' of T, by setting

$$Tf(x) = \int_{\mathbb{R}^d} m(\xi)\widehat{f}(\xi)e^{2\pi i\xi\cdot x}\,dx.$$

Of particular interest are the radial Fourier multiplier operators, defined by a radial function m. For a function $a : [0,\infty) \to \mathbb{C}$, we denote the radial multiplier operator given by $m(\xi) = a(|\xi|)$ by T_a. Any operator on $\mathbb{R}^d$ commuting with translations is a Fourier multiplier, and if in addition, the operator commutes with rotates, it is a radial Fourier multiplier operator.

In harmonic analysis, it has proved incredibly profitable to study the boundedness of Fourier multiplier operators with respect to various L^p norms. It seems to be one of the few tractable ways of quantifying interactions between planar waves, thus underpinning all deeper understandings of the Fourier transform.

The L^p boundedness of a general multiplier operator became of central interest in the 1950s, brought on by the spur of applications the Calderon-Zygmund school and their contemporaries brought to the theory. Some sufficient conditions and some necessary conditions to ensure boundedness were found. But finding necessary and sufficient conditions which guarantee boundedness proved to be an impenetrable problem; such conditions are only known for L^p boundedness in simple cases where $p \in \{1,2,\infty\}$.

It thus came as a surprise when several arguments [10, 19, 22, 27] recently established necessary and sufficient conditions on a function a for a radial Fourier multiplier operator T_a to be bounded on $L^p(\mathbb{R}^d)$. Consider a decomposition $a(\rho) = \sum a_k(\rho/2^k)$, where $a_k(\rho) = 0$ for $\rho \not\in [1,2]$. For $1 \leq p \leq 2$, in order for T_a to be bounded on $L^p(\mathbb{R}^d)$, testing by Schwartz functions shows that $\sup_j \|\widehat{m}_j\|_{L^p(\mathbb{R}^d)} < \infty$ is necessary, where $m_j(\xi) = a_j(|\xi|)$. Garrigos and Seeger [19] show this is equivalent to $\sup_j C_p(a_j) < \infty$, where

$$C_p(a) = \left(\int_0^\infty \left| \langle t \rangle^{(d-1)(1/p-1/2)} \widehat{a}(t) \right|^p dt \right)^{1/p} \quad \text{and} \quad \langle t \rangle = (1+|t|^2)^{1/2}.$$

Using Bochner-Riesz operators as endpoint examples, it is natural to conjecture $\sup_j C_p(a_j) < \infty$ is not only necessary, but also sufficient to guarantee L^p boundedness for $1 < p < \frac{2d}{d+1}$. We call this conjecture the *radial multiplier conjecture*. For radial input functions this conjecture has been resolved by Garrigós and Seeger [20], though resolving this conjecture for general inputs is likely far beyond current research techniques, given that it implies the Bochner-Riesz conjecture, and thus also the restriction and Kakeya conjectures. Heo, Nazarov, and Seeger [22] have proved the conjecture for $d \geq 4$ and $1 < p < \frac{2(d-1)}{d+1}$. Cladek [10] improved this range for compactly supported a when $d = 4$ and $1 < p < 36/29$, and when $d = 3$ and $1 < p < 13/12$. Also of note is the work of Kim [27], who extended the bounds of [22] to more general 'quasi-radial multiplier operators'. Nonetheless, the full conjecture remains unresolved for all $d \geq 2$.

Various powerful techniques have recently been developed towards an understanding of the Bochner-Riesz conjecture, such as broad-narrow analysis, decoupling, and the polynomial method. However, these methods are difficult to apply in the radial multiplier conjecture. In these methods, one allows for inequalities to have a multiplicative loss of factors of the form R^ε, where R is the frequency scale of the analysis. This multiplicative loss is negligible since the Bochner-Riesz multipliers are conjectured to be bounded on L^p for an open interval of exponents, and so interpolation-based methods allow us to remove such factors. But an arbitrary multiplier bounded on $L^p(\mathbb{R}^d)$ may not be bounded on $L^q(\mathbb{R}^d)$ for any $q < p$, and so such methods are unavailable in the study of general multipliers, partially explaining the limited range in which the conjecture has been verified. *Nonetheless, there are several related problems I am interested in studying where such methods may prove useful, which I discuss in Section 3.*

1.1 Multipliers For Spherical Harmonic Expansions on S^d

A theory of multiplier operators analogous to Fourier multiplier operators can be developed on the sphere S^d. Roughly speaking, Fourier multiplier operators are operators diagonalized by the planar waves $e^{2\pi i \xi \cdot x}$. Multipliers on S^d are operators diagonalized by the spherical harmonics, i.e. the restrictions to S^d of homogeneous harmonic polynomials on $\mathbb{R}^{d+1}$. Every function $f \in L^2(S^d)$ can be uniquely expanded as $\sum_{k=0}^\infty H_k f$, where $H_k f$ is a degree k spherical harmonic, and a multiplier for spherical harmonic expansions on S^d is then an operator defined in terms of a function $a : \mathbb{N} \to \mathbb{C}$ given by $S_a = \sum_{k=0}^\infty a(k) H_k$. For purposes of brevity, we will call such operators 'multiplier operators on S^d'. Every rotation invariant operator on S^d is a multiplier operator of this kind.

A natural question is to characterize which functions a give multiplier operators S_a bounded on $L^p(S^d)$, but the fact that the operators are described by a discrete sum makes this problem quite different from the study of radial multipliers on $\mathbb{R}^d$. A more tractable question is to determine when the operators $S_R = \sum a(k/R) H_k$ are uniformly bounded on $L^p(S^d)$. *I completely characterized such functions, for a certain range of L^p exponents and when $d \geq 4$.*

Classical methods for studying multiplier operators on S^d involve the analysis of special functions and orthogonal polynomials, e.g. in the work of Bonami and Clerc [5]. But in the 1960s, Hörmander introduced the powerful theory of Fourier integral operators to the study of such operators, which allows one to apply more modern techniques of harmonic analysis. This theory is more robust in other senses, applying to the study of multiplier operators associated with a first order self-adjoint pseudodifferential operator on a compact manifold, which we briefly outline. Given such an operator P on a manifold M with eigenvalues Λ, every function $f \in L^2(M)$ has an orthogonal decomposition $f = \sum_{\lambda \in \Lambda} f_\lambda$ where $P f_\lambda = \lambda f_\lambda$. Given $a : \Lambda \to \mathbb{C}$, we define $a(P)f = \sum_{\lambda \in \Lambda} a(\lambda) f_\lambda$. We study multiplier operators on S^d by linking them to multiplier operators of a particular pseudodifferential operator P on S^d. If Δ is the Laplace-Beltrami operator on S^d, then for any spherical harmonic f of degree k, $\Delta f = k(k+d-1)f$. Thus if $P = \sqrt{(\frac{d-1}{2})^2 - \Delta} - (\frac{d-1}{2})$, then $Pf = kf$, and so for any function $a : [0,\infty) \to \mathbb{C}$, $S_a = a(P)$.

Hörmander's idea to studying the operator $a(P)$ was to write

$$a(P) = \int \widehat{a}(t) e^{2\pi i t P}\, dt,$$

a form of the Fourier inversion formula. The multiplier operators $e^{2\pi i t P}$, as t varies, give solutions to the 'half-wave equation' $\partial_t = iP$ on M, whose solutions are related to solutions of the full wave equation $\partial_t^2 - P^2 = 0$. Thus the study of the boundedness of the operator $a(P)$ is connected to the regularity for averages of the wave equation on M, in particular to local smoothing inequalities for the wave equation.

Using this reduction, Hörmander was able to prove the L^p boundedness of Bochner-Riesz operators [24], later significantly improved by Sogge [42, 43] and Seeger and Sogge [40] for multipliers of an operator P satisfying the following assumption:

Assumption A: If $p_{\text{prin}} : T^*M \to [0,\infty)$ is the principal symbol of P, then for each $x \in M$ the 'cosphere' $S_x^* = \{\xi \in T_x^* M : p_{\text{prin}}(x,\xi) = 1\}$ has non-vanishing Gaussian curvature.

Note that when $P = \sqrt{(\frac{d-1}{2})^2 - \Delta} - (\frac{d-1}{2})$ on S^d, the principal symbol is the Riemannian metric on T^*S^d, the cospheres are ellipses, and so Assumption A is satisfied. These bounds were obtained by introducing the approach, which works within the Stein-Tomas range, of reducing the problem to $L^2(M) \to L^p(M)$ bounds for spectral projection operators on M. Recently, Kim [28] adapted Sogge's approach to obtain certain necessary conditions ensuring $a(P)$ is bounded on $L^p(M)$ for operators P satisfying Assumption A on the scale of Besov spaces. But these bounds are far from a complete characterization of boundedness; for instance, they do not imply the boundedness of the wave multipliers $(1+P)^{-(d-1)(1/p-1/2)} e^{2\pi i P}$ on $L^p(M)$. *The main goal of my research project was to find a complete characterizations of boundedness, which would be the first such result in the literature.*

1.2 My Contributions To The Study of Multipliers

As mentioned above, the main goal of my PhD research into multipliers was to obtain analogues of the arguments of [10, 22, 27] for multiplier operators on S^d, i.e. proving that for $P = ((\frac{d-1}{2})^2 - \Delta)^{1/2}$ on S^d, then the operators $\{a(P/R)\}$ are uniformly bounded on $L^p(S)^d$ if and only if $\sup_j C_p(a_j) < \infty$. I obtained such analogues, and more generally applying to multipliers for a range of different operators P that satisfy Assumption A and the following additional assumption:

Assumption B: The eigenvalues of P are contained in an arithmetic progression.

All eigenvalues of the operator P above are positive integers, so this assumption is satisfied on S^d. The assumption also holds more generally for multipliers on the 'rank one symmetric spaces' $\mathbb{RP}^d$, $\mathbb{CP}^d$, $\mathbb{HP}^d$, and $\mathbb{OP}^2$, i.e. operators diagonalized by analogous functions to the spherical harmonics on these spaces. It is very difficult to completely remove Assumption B, for reasons involving the inability to understand the large time behavior of the wave equation on compact manifolds. Nonetheless, I discuss in Section 3 potential methods for obtaining bounds under weaker assumptions. Under Assumption A and Assumption B, in [14] I proved a 'single scale' analogue of the bound of Heo, Nazarov and Seeger.

Theorem. [14] *Suppose P is an elliptic pseudodifferential operator of order one on a compact manifold M satisfying Assumptions A and B. Then for a function a with $supp(a) \subset [1,2]$, and for $1 < p < 2(\frac{d-1}{d+1})$, uniformly in $R > 0$,*

$$\sup_{R>0} \|a(P/R)\|_{L^p(M) \to L^p(M)} \sim C_p(a).$$

In a paper to be submitted for publication shortly, I provide further arguments justifying that for an arbitrary function a, the operator $a(P)$ is bounded on $L^p(M)$ if $\sup_j C_p(a_j) < \infty$, thus obtaining a complete analogue of the argument of [22] for multiplier operators on S^d. The argument involves using an L^∞ atomic decomposition à la the decompositions of Chang and Fefferman [9], and then controlling the interactions between different frequency scales using certain inner product estimates.

An important corollary of this result is a *transference principle* between Fourier multiplier operators and multiplier operators on S^d. Since the condition $\sup_j C_p(a_j)$ is necessary for T_a to be bounded on $L^p(\mathbb{R}^d)$, we conclude that for $|1/p - 1/2| > 1/d$, if T_a is bounded on $L^p(\mathbb{R}^d)$, then the multiplier $a(P)$ is bounded on $L^p(M)$. Thus bounds 'transfer' from $\mathbb{R}^d$ to S^d. Aside from the study of Fourier multipliers on $\mathbb{R}^d$, this is the first transference principle of this kind. *There are no results in the literature for any $p \neq 2$, any other compact manifold M, and any operator P which guarantee that $a(P)$ is bounded on $L^p(M)$ if T_a is bounded on $L^p(\mathbb{R}^d)$.*

Another important corollary is a characterization of the functions a which give convergent eigenfunction expansion in L^p, i.e. that a function a satisfies $\lim_{R\to\infty} a(P/R)f = f$ for all $f \in L^p(M)$, where the limit is taken in $L^p(M)$, if and only if $a(0) = 1$ and $\sup_j C_p(a_j) < \infty$. This immediately follows from the theorem above by the uniform boundedness principle. *As with the transference principle above, these results are the first of their kind for $p \neq 2$ and any other compact manifold M.*

The proof in [14] is an adaption of the argument of [22] for bounding radial Fourier multiplier operator. That argument involves writing a radial multiplier operator as a convolution $Tf = k * f$. We consider a decomposition $k = \sum k_\tau$ and $f = \sum f_\theta$, where the functions $\{k_\tau\}$ are supported on disjoint annuli supported at the origin, and the functions $\{f_\theta\}$ are supported on disjoint cubes, and thus we can write $Tf = \sum_{\tau,\theta} k_\tau * f_\theta$. Estimates guarantee that the inner products $\langle k_\tau * f_\theta, k_{\tau'} * f_{\theta'} \rangle$ are negligible unless the annulus of radius τ centered at θ is near tangent to the annulus of radius τ' centered at θ'. These inner product estimates are combined with a 'sparse incidence argument' which when interpolated with a stopping time argument, yields the required L^p bounds. The main difficulty in adapting this approach is the difficulty in obtaining analogous inner product estimates, and handling the case where τ is large (i.e. handling the long time behaviour of the wave equation). We conclude this discussion by describing the two main techniques I obtained to resolve these problems.

Let us start by obtaining analogues to the inner product estimates. A natural approaches is to use the Lax-Hörmander parametrix for the wave equation, which reduces our inner product estimates for small τ to a bound for oscillatory integrals. But the phase of this integral that arises from this parametric non-explicit, given in terms of a solution to an eikonal equation on M. One novel approach I made in [14] was making the observation that if Assumption A holds, then P gives an implicit geometric structure to the manifold M, turning it into a *Finsler manifold*. The phase of the oscillatory integral occurring from the Lax-Hörmander parametrix is then directly related to the length of certain geodesics on this Finsler manifold, and using the Finsler analogue of the second variation formula for geodesics, I was able to obtain the required inner product estimates that occur in [22] for small τ. Such estimates apply to multipliers of an arbitrary pseudodifferential operator P satisfying Assumption A, and likely have applications in other problems.

The inner product estimates above are sufficient to obtain bounds for small τ, but for large τ this approach fails, as the Lax-Hörmander parametrix breaks down past the injectivity radius of the manifold M, preventing us from applying a direct analogue of the arguments of [22]. Similar problems emerge in other approaches to the study of multipliers on manifolds. This was the impetus for Sogge's method of studying Bochner-Riesz multipliers in the Tomas Stein ranges, which reduces the problem to the study of $L^p \to L^2$ bounds for spectral projection operators on M, used in [43] and [28]. However, we cannot use this method in this problem, since the method initially involves using the estimate $\|a(P)f\|_{L^p(M)} \lesssim \|a(P)f\|_{L^2(M)}$, which is too inefficient to fully characterize the required L^p estimates. I was able to work around this problem by reducing the required bounds to certain $L^p_x L^p_t$ estimates for the wave equation on the manifold. Such an argument behaves somewhat like Sogge's spectral projection argument, but does not involve a switch to L^2 norms, avoiding the problems of such an approach. The catch is that $L^p_x L^p_t$ estimates for the wave equation, related to the phenomenon of local smoothing on manifolds, are not as well understand as spectral projectors. This is why we must assume the rather strict Assumption B, which makes such estimates feasible. *In future work I hope investigate ways to weaken Assumption B, and I discuss several possible feasible ways to weaken this assumption in Section 3*.

2 Configuration Avoidance

How large must a set $X \subset \mathbb{R}^d$ be before it must contain a certain point configuration, such as three points forming a triangle congruent to a given triangle, or four points forming a parallelogram? Problems of this flavor have long been studied in combinatorics, such as when X is restricted to a discrete set, such as the grid $\{1,\dots,N\}^d$. In the last 50 years, analysts have also begun studying analogous problems for infinite subsets $X \subset \mathbb{R}^d$, where the size of X is measured via a suitable 'fractal dimension', one of various different numerical statistics which measure how 'spread out' X is in space. The most common fractal dimension in use is the Hausdorff dimension of a set X, but we also consider the Fourier dimension as a refinement of Hausdorff dimension which takes into account more subtle behavior of X related to its correlation with the planar waves $e^{2\pi i \xi \cdot x}$ for $\xi \in \mathbb{R}^d$.

Unlike many other problems in harmonic analysis, such as the Kakeya conjecture, we often do not have good expected lower bounds for the dimension at which configurations must appear. For instance, we do not know for $d > 2$ how large the Hausdorff dimension a set $X \subset \mathbb{R}^d$ must be before it contains all three vertices of an isosceles triangle, the threshold being somewhere between $d/2$ and $d-1$. Similarly, for a fixed angle $\theta \in (0,\pi)$, we do not know how large the Hausdorff dimension of X must be contains three distinct points A, B, and C which when connected determine an angle ABC equal to θ. Depending on whether $\cos^2\theta$ is rational or irrational, the results of Máthe [35] and Harangi, Keleti, Kiss, Maga, Máthe, Mattila, and Strenner [21] imply the threshold is respectively, either between $d/4$ and $d-1$, or between $d/8$ and $d-1$. We should not even necessarily expect currently known lower bounds to be the 'correct bounds' in these problems, as we do with other problems in harmonic analysis, such as the restriction conjecture and the Falconer distance problem; until recently, certain results due to Łaba and Pramanik [30] seemed to imply that subsets of $[0,1]$ of Fourier dimension one must necessarily contain an arithmetic progression of length three, but Shmerkin has shown this need not be the case [41].

Given that we do not have good lower bounds with which to make definite conjectures, it is of interest to find general methods that we can use to produce counterexamples in these types of problems. That is, we wish to find methods with which to construct sets with large fractal dimension that do not contain certain point configurations. My research in geometric measure theory has so far focused on finding these types of methods using quantitative methods from probability theory.

2.1 A Review of Fractal Dimension and Configuration Avoidance

It is most natural in our context to define fractal dimension of a set in terms of the properties of measures supported on that set. The Hausdorff dimension of a set $X \subset \mathbb{R}^d$ is the least upper bound of the quantities s for which there exists a finite Borel measure μ supported on X which satisfies the 'ball condition' that $\mu(B_r) \lesssim r^s$ for all $r > 0$ and all radius r balls $B_r \subset \mathbb{R}^d$; the Fourier dimension is the least such s for which there exists μ with the decay estimate $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-s/2}$. Intuitively, for large s the ball condition implies that μ has mass 'spread out', and the Fourier decay condition implies that the support of μ is uncorrelated with the waves $e^{2\pi i \xi \cdot x}$ when ξ is large, which implies the mass of μ must be spread out. The Hausdorff dimension is always larger than the Fourier dimension, but this inequality is often strict, the Fourier decay capturing more subtle information about the set X, and this means it is often much harder to construct sets with large Fourier dimension avoiding configurations than sets with large Hausdorff dimension.

We consider a model problem for pattern avoidance; given a fixed function $f : (\mathbb{R}^d)^n \to \mathbb{R}^m$, we ask how large must the dimension of a set X be to guarantee that there exists $x_1,\dots,x_n \in X$ such that $f(x_1,\dots,x_n) = 0$. We focus on finding lower bounds for this problem, constructing sets X with large Hausdorff or Fourier dimension such that X avoids the zeroes of f, in the sense that for any distinct points $x_1,\dots,x_n \in X$, $f(x_1,\dots,x_n) \neq 0$. This model has been considered in various contexts:

(A) If $m = 1$, and f is a polynomial of degree n with rational coefficients, Máthe [35] constructs a set with Hausdorff dimension d/n avoiding the zeroes of f.

(B) If f is a C^1 submersion, Fraser and Pramanik [17] constructs a set with Hausdorff dimension $m/(n-1)$ avoiding the zeroes of f.

(C) If the zero set $f^{-1}(0)$ has Minkowski dimension at most s, I, together with my Master's thesis advisors Malabika Pramanik and Joshua Zahl [15] constructed sets of Hausdorff dimension $(dn - s)/(n-1)$ avoiding the zeroes of f.

(D) If f can be factored as $f = g \circ T$, where $T : (\mathbb{R}^d)^n \to \mathbb{R}^l$ is a full-rank, rational coefficient linear transformation, and $g : \mathbb{R}^l \to \mathbb{R}^m$ is a C^1 submersion, then I [14] have constructed a set with Hausdorff dimension m/l avoiding the zero sets of f.

Notice that the above four methods only construct sets with large Hausdorff dimension avoiding patterns. They say nothing about constructing sets with large Fourier dimension, which in general is a much harder problem involving a delicate interplay between 'randomness' and 'structure'. Most 'structured' sets have low Fourier dimension, and so most methods for constructing sets with large Fourier dimension require making certain 'random choices' which on average do not correlate with any particular planar wave. Structure must be added to some degree to avoid containing a given configuration, but adding too much structure will likely add a high degree of correlation of your sets with certain planar waves, resulting in your set having Fourier dimension zero. Certain results have been obtained, however, for linear functions f:

(E) If $f(x_1, \dots, x_n) = a_1 x_1 + \dots + a_n x_n$ with $\sum a_j = 0$, Pramanik and Liang [33] construct $X \subset [0,1]$ with Fourier dimension 1 avoiding the zeroes of f. This generalizes a construction of Shmerkin [41], who proved the result in the special case where $f(x_1, x_2, x_3) = (x_3 - x_1) - 2(x_2 - x_1)$ detects arithmetic progressions of length 3.

(F) Körner constructed subsets $X \subset [0,1]$ with Fourier dimension $(k-1)^{-1}$ such that for any integers $m_0, \dots, m_k$, and any distinct $x_1, \dots, x_k \in X$, $a_0 \neq a_1 x_1 + \dots + a_n x_n$.

The focus on linear functions is natural, since the Fourier transform behaves in a predictable way with respect to linearity, whereas the relation of the Fourier transform with respect to other nonlinear phenomena is much more subtle, and more poorly understood. *The main goal of my research project was to find constructions of sets with large Fourier dimension avoiding the zeroes of a nonlinear functions f.*

2.2 My Contributions To The Study Of Configurations

It seems more difficult to adapt methods (A) and (D) above to construct sets with positive Fourier dimension, since the constructions involve constructing X at each spatial scale by choosing a good family of intervals, and then considering a large union of translates of the intervals along an arithmetic progression. Such a construction gives intervals correlated with waves of frequency complementing the spacing of the arithmetic progression, and so gives a set with Fourier dimension zero. On the other hand, methods (B) and (C) involve mostly pigeonholing arguments, so they seem the most likely to be able to be adapted to the Fourier dimension setting. I was able to adapt some of the ideas of these methods to obtain such a result. *In future work, I hope to explore adaptions of methods (A) and (D) to the Fourier dimension by ideas of* [41].

For simplicity, I focused on the case when $m = d$ and when the function f was C^1 and full rank, an assumption also made in [17]. Then by the implicit function theorem, after possibly rearranging indices, we can locally write $f(x_1, \dots, x_n) = x_1 - g(x_2, \dots, x_n)$ for a function $g : (\mathbb{R}^d)^{n-1} \to \mathbb{R}^d$. In [13], under the assumption that g was a submersion in each variable $x_2, \dots, x_n$, I was able to modify the construction of [17] to construct sets with Fourier dimension $d/(n - 3/4)$ avoiding the zeroes of f. Under the further assumption that we can write $g(x_2, \dots, x_n) = ax_2 + h(x_3, \dots, x_n)$ for $a \in \mathbb{Q}$, I was able to construct sets with Fourier dimension $d/(n-1)$ avoiding the zeroes of f, recovering the Hausdorff dimension bound of [17] in the Fourier dimension setting.

Theorem. [13] *Suppose that $g : [0,1]^{d(n-1)} \to \mathbb{R}^d$ is a function such that for each $k \in \{0,\dots,n-2\}$, the $d \times d$ matrix $D_k g = (\partial g_i / \partial x_{dk+j})_{i,j=1}^d$ is invertible. Then there exists a Salem set $X \subset [0,1]^d$ of dimension $d/(n-3/4)$ such that for all distinct $x_1,\dots,x_n \in X$, $x_1 \neq f(x_2,\dots,x_n)$. If, in addition, $g(x_2,\dots,x_n) = ax_2 + h(x_3,\dots,x_n)$ for some $a \in \mathbb{Q}$, then there exists a Salem set $X \subset [0,1]^d$ of dimension $d/(n-1)$ such that for all distinct $x_1,\dots,x_n \in X$, $x_1 \neq f(x_2,\dots,x_n)$.*

As with most of the other approaches discussed above, we construct a set X avoiding zeroes via a 'Cantor-type construction'. Fix a parameter α. We iteratively define a nested family of sets $\{X_k\}$, each a union of cubes of some fixed length l_k, and define $X = \bigcap_k X_k$. The set X_{k+1} is obtained from X_k by partitioning X_k each sidelength l_k cube into N^d sidelength l_{k+1} cubes, where $N := l_k / l_{k+1}$, and letting X_{k+1} be formed from the union of a subcollection of these cubes. The construction in [15] and [13] is very simple: To construct X_{k+1} from X_k, we start by taking a set S by taking $\sim N^\alpha$ points uniformly at random from the centers of the sidelength l_{k+1} cubes in the partition of each sidelength l_k cube in X_k. Some points from this set will form near zeroes of the function f; we let

$$S_{\text{bad}} = \{x \in S : |f(x,x_2,\dots,x_n)| \leq 10 l_{k+1} \text{ for some } x_2,\dots,x_n \in S\},$$

and define X_{k+1} to be the union of all sidelength l_{k+1} cubes centered at points in $S - S_{\text{bad}}$. The set X will then avoid the zeroes of the function f. Provided that $\alpha \leq (nd-s)/(n-1)$, we have with high probability that $\#(S_{\text{bad}}) \ll \#(S)$, and so with high probability, at each stage of the construction X_k is a union of $\sim l_k^{-\alpha}$ cubes of sidelength α; it is therefore natural to expect the set X almost surely has Hausdorff dimension α, and indeed, in [15] this is shown to be the case.

Simply counting the number of cubes at each scale is not sufficient to obtain a Fourier dimension bound. In [13], I made the observation that the core feature of constructions that yield Fourier dimension bounds is that they must involve a square root cancellation bound. More precisely, if we denote the centers of the sidelength l_k cubes forming X_k by $\{x_1,\dots,x_M\}$, and if for all $1 \lesssim |\xi| \lesssim N$ the square root cancellation bound

$$\left| \frac{1}{M} \sum_{j=1}^{M} e^{2\pi i \xi \cdot x_j} \right| \lessapprox M^{-1/2} \tag{1}$$

holds at all scales, then the resulting set X will have Fourier dimension agreeing with its Hausdorff dimension. Indeed, consider the probability measure $\mu_k = M^{-1} \sum_{j=1}^M \chi_j$ supported on X_k, wjere χ_j is a smooth bump function adapted to the cube centered at x_j. Then for $|\xi| \lesssim 1/l_k$, since $M \sim l_k^{-\alpha}$ with high probability, (1) implies that $|\widehat{\mu}_k| \lessapprox M^{-1/2} \lessapprox |\xi|^{-\alpha/2}$. On the other hand, the uncertainty principle implies that $\widehat{\mu}_k$ decays rapidly for $|\xi| \gtrsim 1/l_k$, and so $\widehat{\mu}_k$ has the appropriate Fourier decay required. Taking weak limits of the measures $\{\mu_k\}$, we find that $|\widehat{\mu}(\xi)| \lessapprox |\xi|^{-\alpha/2}$ has the right Fourier decay to justify that X has Fourier dimension α with high probability.

We obtain the required square root cancellation bounds using the theory of *concentration of measure* for high dimensional probability, which determines when a sum of random variables has square root cancellation away from the mean with high probability. If we are taking a sum of independent random variables, often Hoeffding's inequality gives sharp concentration bounds. But in our construction, the random points $\{x_j,\dots,x_M\}$ are not chosen independently from one another; the points in the initial set S are independent, but not the points in the set $S - S_{\text{bad}}$. There are certain standard tools to handle this problem, such as McDiarmid or Azuma's inequality, though in this setting they fail to ensure square root cancellation unless $\alpha \leq d/n$, whereas we want to choose the larger value $\alpha = d/(n-1)$. In [13], I found a novel way to interlace Hoeffding and McDiarmid's inequality to ensure square root cancellation away from the mean occurs with high probability for $\alpha \leq d/(n-1)$.

After ensuring square root cancellation of the mean, the final problem is to show that the mean of $M^{-1} \sum e^{2\pi i \xi \cdot x_j}$ has square root cancellation, which proved to be the most inefficient aspect of the argument. This mean can be written as an oscillatory integral, though in M variables, and so usual techniques in the theory of oscillatory integrals fail to handle this bound since they are usually *dimension dependent*, and we need bounds uniform in M. Instead, I was able to use an inclusion exclusion argument, together with a Whitney decomposition of the thickened zero set of the function f to obtain the required bounds. This is the least optimal part of the argument, being the main reason why we obtain a Fourier dimension of $d/(n-3/4)$ in general rather than $d/(n-1)$; however, if f satisfies a weak linearity a slight modification of the random construction ensures that the mean of $M^{-1}\sum e^{2\pi i \xi \cdot x_j}$ is always zero, yielding the large Fourier dimension bound $d/(n-1)$ in this case.

3 Future Lines of Research

In this section, I discuss two concrete plans for future projects in detail which I strongly believe should produce reliable results in the near future. Before this, I briefly discuss some more exploratory projects whose methodology is not quite as clearly defined:

Weighted Estimates of Fractal Type: At the beginning of Section 1, we discussed the radial multiplier conjecture, and that no methods involving an 'ε loss' can be used in this conjecture, because of the lack of the ability to use interpolation arguments. However, if we weaken the conjecture to allow for a ε of room, we can allow for this loss. Namely, such methods may apply to prove that a Fourier multiplier T_a is bounded on $L^p(\mathbb{R}^d)$ if $\sup_j C_{p-\varepsilon}(a_j) < \infty$ for some $\varepsilon > 0$. I hope to pursue whether more modern techniques such as decoupling and polynomial partitioning can be used to prove such bounds, especially in the case $d = 2$ in which the radial multiplier conjecture is completely unknown.

One possible direction that I believe may yield results are suggested by several results on 'weighted estimates of fractal type' for extension operators, such as the work of Du and Zhang [16] and Ortiz [37]. These estimates are dual to the 'sparse incidence bounds' discussed in Section 1.2, and thus finding analogues of 'sparse decompositions' in this dual setting may allow one to apply these weighted estimates to achieve improved bounds.

Multilinear Problems Associated With Radial Multipliers: Finding a dual argument to the sparsity argument normally used in the recent characterization of boundedness for radial Fourier multipliers also allows us to prove results directly in the range $2 \leq p \leq \infty$, which also allows for us to use certain *multilinear to linear arguments* in our proofs to prove new results. In particular, I believe one can use work on the k-linear restriction conjecture to the cone done by Barceló [1], Wolff [45], Ou and Wang [38], Beltran and Saari [3], and Gao, Liu, Miao and Xi [18] to obtain new results.

In addition, I am also interested in exploring whether methods for obtaining endpoint estimates for linear radial Fourier multipliers discussed in Section 2 can be used in the setting of *bilinear radial Fourier multipliers*, combining time-frequency methods from recent work by Bernicot and Grafakos [4] and Liu and Wang [34] on the bilinear Bochner-Riesz problem.

Vector Field Methods For The Wave Equation on Black Hole Spacetimes In my PhD work, I mainly worked on obtaining L^p control over linear wave equations. Recently, I have started exploring methods for studying the long time behaviour of nonlinear wave equations. The 'vector field method' has proved a key tool in this setting for quite some time, capturing the dispersive nature of waves by using vector fields on the background space-time to captur approximate symmetries of an equation. The method has proven a powerful tool in the study of highly nonlinear hyperbolic equations. Recently, Ifrim and Tataru [26] have developed a new method called 'wave packet testing' complementing the vector field method, to obtain global control on various nonlinear dispersive equations, first applied to the nonlinear Schrödinger equation [25], and later applied successfully for many other dispersive equations such as water waves [26], Maxwell-Dirac systems [23], and modified KdV equations [36].

Notably, Ifrim and Tataru's methods have not been applied to the study of wave equations on black-hole spacetimes. For some time, obtaining sharp decay estimates of solutions to the linear wave-equation on high-dimensional Schwarzschild black hole spacetimes proved difficult, until resolved by Schlue [39]. But Schlue's method requires modification for spacetimes with dynamic backgrounds, which introduce a nonlinearity into the problem. I believe Ifram and Tataru's methods could apply the necessary control to obtain results.

Analysis of Highly Degenerate Oscillatory Integrals via o-Minimality: Together with Johnsrude, Sandberg, and de Oliveira Andrade, I recently wrote a study guide on the recent use of 'o-minimality' in a paper of Basu, Guo, Zhang, and Zorin-Kranich [2] to the study of oscillatory integrals, which we plan to put onto the arXiv shortly. The method gives optimal decay for highly degenerate oscillatory integrals with an algebraic phase, and *I hope to find new applications of this theory in the study of Fourier integral operators, such as the study of the spectral theory of highly degenerate Hörmander Sums of Squares operators*. Such results have applications to the study of multipliers of sub-Laplacians on highly degenerate algebraic Lie groups.

We now discuss two more concrete projects I strongly believe will produce results in the near future:

3.1 Multipliers Associated With Periodic Geodesic Flow

The results I discussed in Section 2 for multiplier operators on S^d generalize to multipliers of an arbitrary first order, elliptic pseudodifferential operator P on a compact manifold M, provided that P satisfied Assuptions A and B of that section. Assumption A relates to the curvature of the principal symbol, and this assumption cannot really be weakened without significantly changing the character of the results, which heavily depend on this curvature. On the other hand, Assumption B arises as an artifact of the methods of our proof. We can likely obtain similar bounds while weakening this assumptions; for instance, Kim [28] obtained bounds on the scale of Besov spaces only under Assumption A.

It is likely very difficult that we can complete remove Assumption B using current research methods while still recovering the results of [14]. This limitation follows from our current inability to understand the large time behavior of wave equations on compact manifolds precisely enough. If we were able to follow the method of [14], which reduced the large time argument to a smoothing inequality for the wave equation, then the results of that paper would follow for another operator P if we could prove

$$\left\| \left(\int_k^{k+1} |e^{2\pi i t P} f|^{p'} \, dt \right)^{1/p'} \right\|_{L^{p'}(M)} \lesssim k^{\delta} \|f\|_{L^p_{d(1/p-1/2)-1/p'}(M)} \tag{2}$$

for some $\delta < (d-1)(1/p-1/2) - 1/p'$.

If P satisfies assumption B, then after rescaling, we may assume without loss of generality that all eigenvalues of P are integers, so that $e^{2\pi i k P} = I$ is the identity for all k, and then (2) holds for all $|1/p - 1/2| > (d-1)^{-1}$ and with $\delta = 0$ by the local smoothing inequality of Lee and Seeger [32]. Whether this bound is true in other contexts remains unknown. The next simplest case to consider would be when the operator P has the property that $e^{2\pi i k P}$ is *close* to the identity for all k. This happens precisely when the *Hamiltonian flow* on T^*M given by the vector field $H = (\partial p_{\text{prin}}/\partial \xi, -\partial p_{\text{prin}}/\partial x)$ is periodic, where p_{prin} is the principal symbol of P. Indeed, results of Colin de Verdiére [11] related to the theory of propagation of singularities of Fourier integral operators then tell us that the operator $R = e^{2\pi i P}$ is a pseudodifferential operator of order zero, and its principal symbol is related to an invariant of the flow known as the Maslov index. The operator has been studied a little by spectral theorists, and there it is known as the *return operator*. If we are able to justify bounds of the form

$$\|R^k f\|_{L^p_{d(1/p-1/2)-1/p'}} \lesssim k^{\delta} \|f\|_{L^p_{d(1/p-1/2)-1/p'}},$$

or a frequency localized variation of this bound, then the local smoothing inequality of Lee and Seeger yields (2). Such bounds are of interest since they cover all the operators $P = \sqrt{-\Delta}$, where Δ is the Laplace-Beltrami operator on a Riemannian manifold with periodic geodesic flow. They are even of interest on the sphere, since our method only allows us to tell when multipliers of the form $a(P/R)$ are uniformly bounded on $L^p(S^d)$, where $P = \sqrt{(\frac{d-1}{2})^2 - \Delta}$ whereas these bounds would allow us to tell when the multipliers $a\big(\sqrt{-\Delta}/R\big)$ are uniformly bounded on $L^p(S^d)$.

3.2 Genuine Decoupling On Random Fractals

One major development in harmonic analysis in the past decade has been a greater understanding of the phenomenon of *decoupling*, or *Wolff-type estimates*. Given a family of almost disjoint subsets $\mathscr{E}_\delta$ of $\mathbb{R}^d$ parameterized by $\delta > 0$, $L^p(l^2)$ decoupling discusses bounds of the form

$$\Big\| \sum f_j \Big\|_{L^p(\mathbb{R}^d)} \leq D_p(\delta) \Big(\sum_j \|f_j\|_{L^p(\mathbb{R}^d)}^2 \Big)^{1/2},$$

where the Fourier transforms of the functions f_j are supported on distinct subsets of $\mathscr{E}_\delta$, and $D_p(\delta)$ denotes the best constant under which this equation holds for all such $\{f_j\}$. After [12], we say a *genuine decoupling inequality* results when one can prove that $D_p(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$ for all $\varepsilon > 0$.

Much work has been carried out for $d \geq 2$, and when the sets $\mathscr{E}_\delta$ are δ caps associated with partitions of δ-neighborhoods of curves and surfaces, and the decoupling inequalities are obtained by virtue of the curvature and torsion properties of the shapes they are associated with. But the analysis of decoupling on *fractal sets* is still poorly understood. Consider a sequence of integers $n(i)$, and a set X obtained from a Cantor-like construction $\{X_i\}$ as in Section 2.2, where X_i is a union of a family of cubes $\mathscr{Q}_i$ with some fixed sidelength $\delta := \delta_i$. We let $\mathscr{E}_\delta = \{Q \cap C_{i+n(i)} : Q \in \mathscr{Q}_i\}$, and ask for which Cantor type constructions $\{X_i\}$ and for which sequences $\{n(i)\}$ do we obtain a genuine decoupling inequality for the families $\{\mathscr{E}_\delta\}$.

Some analysis has been done in this setting, but no genuine decoupling bounds have been established for any fractal set. Chang, de Dios Pont, Greenfeld, Jamneshan, Li, and Madrid have obtain such results for self-similar Cantor sets with good numerical properties [8], but none of the bounds obtained give genuine decoupling inequalities in the above sense. Decoupling inequalities for random fractal sets have been obtained by Łaba and Wang [31], see also Łaba [29]; these are also not genuine decoupling inequalities, but the bounds they obtained were sufficient for their applications to the study of $L^p \to L^2$ fractal restriction bounds. In fact, in the range of p they were considering, genuine decoupling is not possible; one can see by taking counter examples using the local constancy property and Khintchine type heuristics that if X is chosen sufficiently randomly, and $\#\mathscr{Q}_i \gtrsim \delta_i^{-s}$ for each i, then genuine $L^p(l^2)$ decoupling is impossible for any choice of $\{n(i)\}$ unless $2 \leq p \leq 2d/s$.

I believe the techniques related to my results in [13] can be applied to obtaining random decoupling inequalities. Methods from the theory of concentration of measure have been applied by Bourgain [6] and Talagrand [44] in order to prove the existence of $\Lambda(p)$ sets, in particular, the method of majorizing measures and selection processes. One might view $\Lambda(p)$ sets as a kind of discrete variant of sets upon which decoupling bounds hold, so it is likely to believe these methods generalize to the continous setting. Using these methods, I hope to obtain an analogue of the proof of $l^2(L^p)$ decoupling for the paraboloid found in [7], i.e. establishing an analogue of multilinear Kakeya for the sets $\mathscr{E}_\delta$, and then apply an induction on scales to obtain a genuine fractal decoupling inequality.

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