1. Let
$$\binom{n}{m}$$
denote the number of subsets of $\{1,\dots,n\}$ which have cardinality m. These are the *binomial coefficients*. Thus $\binom{3}{2} = 3$, and $\binom{3}{0} = 1$.

   (a) Argue that for any n and m, with $n \geqslant 1$ and $m \leqslant n$,
$$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}.$$

   (b) One can define $S = \{(n,m) : n \geqslant m\}$ structurally by the following iterative method:
   - $(0,0)$ is in $S = \mathbf{N}^2$.
   - If (n,m) is in S, then $(n+1,m)$ is in S.
   - If (n,m) is in S, and $n \leqslant m+1$, then $(n,m+1)$ is in S.

   Prove using structural induction that for all $(n,m) \in S$,
$$\binom{n}{m} = \frac{n!}{m!(n-m)!},$$
   where we define $0! = 1$, and $(n+1)! = (n+1) \cdot n!$.

   (c) Argue by induction on n that for all $n \geqslant 0$, and any $x, y \in \mathbf{R}$,
$$(x+y)^n = \sum_{m=0}^{n} \binom{n}{m} x^m y^{n-m}.$$

   (d) Let
$$\binom{n}{m,k}$$
   denote the number of pairs of disjoint subsets $S_1, S_2 \subset \{1,\dots,n\}$ with $\#(S_1) = m$ and $\#(S_2) = k$. Can you find (and prove) analogous properties of the statements above for this function?

2. How many numbers below 2022 are divisible by two, three, or five.

3. How many functions from $\{1,2,3,4,5\}$ to $\{A,B,C,D\}$ are onto?

4. Recall that a graph H is *bipartite* if one can write it's vertex set as a disjoint union $V_1 \cup V_2$, where each edge in H connects an element of V_1 to an element of V_2. Let $G = (V,E)$ be an arbitrary graph with $\#(E) = K$ edges, which is not necessarily bipartite. In this problem, we will prove G has a Bipartite subgraph containing at least $K/2$ edges. One element of the proof will be a stronger version of the pidgeonhole principle:

   (Strong Pidgeonhole Principle) Given N pidgeons placed in M holes, there exists a hole containing at least $\lceil N/M \rceil$ pidgeons.

   (a) For any edge $e = (v_1, v_2) \in E$, let
$$T_e = \{W \subset V : v_1 \text{ or } v_2 \text{ is in } W, \text{ but not both}\}.$$
   Argue that $\#(T_e) = 2^{\#(V)-1}$.

   (b) Using the calculation above, show that there exists $W \subset V$ such that the number of edges between W and W^c is at least $K/2$ (Hint: The holes are the subsets of V, the pidgeons are the edges).

   (c) Conclude that there exists a subgraph of G, which is bipartite, and contains at least $K/2$ edges.