Thesis Summary

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My thesis project, advised by Andreas Seeger, studies connections between radial multipliers on $\mathbb{R}^d$ and multipliers of spherical harmonic expansions on the sphere S^d, using methods of Fourier integral operators. For a function $a : [0,\infty) \to \mathbb{C}$, we define a radial Fourier multiplier operator T_a and a multiplier operator S_a on S^d by setting

$$T_a f(x) = \int_{\mathbb{R}^d} a(|\xi|)\widehat{f}(\xi)e^{2\pi i\xi\cdot x}\,dx \quad \text{and} \quad S_a f = \textstyle\sum_{k=0}^\infty a(k)H_k f.$$

Here H_k is the orthogonal projection operator onto the space of degree k spherical harmonics on S^d.

There is some evidence that the boundedness of the operator T_a on $L^p(\mathbb{R}^d)$ and the *uniform* boundedness of the operators $\{S_{a_R} : R > 0\}$ on $L^p(S^d)$ are connected, where $a_R(\cdot) = a(\cdot/R)$ are dilates of a. Indeed, Mitjagin [6] proved that $\|T_a\|_{L^p\to L^p} \lesssim \sup_R \|S_{a_R}\|_{L^p\to L^p}$ for all $1 \leq p \leq \infty$. The result is intuitive, since, very roughly speaking, one locally has $T_a = \lim_{R\to\infty} S_{a_R}$ because one can view the dilation of a instead as a dilation of the metric on S^d, which becomes flatter and flatter as $R \to \infty$. Mitjagin's result follows by 'taking operator norms on each side of the equation'. The reverse inequality $\sup_R \|S_{a_R}\|_{L^p\to L^p} \lesssim \|T_a\|_{L^p\to L^p}$ is less intuitive, much more difficult to establish, and there is some evidence the inequality does not hold in general for all L^p. Indeed, it was unknown whether the reverse inequality was true for all $p \neq 2$. In my thesis, I will establish this inequality for a range of L^p spaces on S^d. More precisely, I prove the following:

- I proved the inequality $\sup_R \|S_{a_R}\|_{L^p\to L^p} \lesssim \|T_a\|_{L^p\to L^p}$ for $1 < p < \frac{2(d-1)}{(d+1)}$ and $\frac{2(d-1)}{(d-3)} < p < \infty$, thus proving the first known *transference principle* from $\mathbb{R}^d$ to S^d for any $p \neq 2$, and more generally, the first transference principle from $\mathbb{R}^d$ to analogous operators on any compact manifold for $p \neq 2$.

- Consider a decomposition $a(\rho) = \sum_{j\in\mathbb{Z}} a_j(\rho/2^j)$, where $a_j(\rho) = 0$ for $\rho \notin [1,2]$. Heo, Nazarov, and Seeger [4] showed that for $1 < p < \frac{2(d-1)}{(d+1)}$, $\|T_a\|_{L^p\to L^p} \sim \sup_j C_p(a_j)$, where

$$C_p(a) = \left(\int_0^\infty \left|\langle t\rangle^{(d-1)(1/p-1/2)}\widehat{a}(t)\right|^p dt\right)^{1/p} \quad \text{and} \quad \langle t\rangle = (1+|t|^2)^{1/2}.$$

  I proved $\sup_R \|S_{a_R}\|_{L^p\to L^p} \sim \sup_j C_p(a_j)$ for $1 < p < \frac{2(d-1)}{(d+1)}$, thus obtaining an analogue of the results of [4] for multipliers on S^d. This is the first characterization of the uniform boundedness of the operators S_{a_R} for any $p \neq 2$ and any $d \geq 2$.

The proofs of these results, for functions a with *compact support*, can be found in [3], with a paper extending these results to the general case in preparation. In the remainder of this summary I give a brief description of the methods by which we obtain these results.

Description of Methods

Classical methods for studying multiplier operators on S^d involve the analysis of special functions and orthogonal polynomials, e.g. in the work of Bonami and Clerc [1]. However, it is tough to combine this approach with more modern harmonic analysis methods. In the 1960s, Hörmander made a breakthrough by introducing the theory of Fourier integral operators, where more modern techniques may be applied. Note that the pseudodifferential operator $P = \sqrt{(\frac{d-1}{2})^2 - \Delta} - (\frac{d-1}{2})$ on S^d satisfies $Pf = kf$ for any spherical harmonic f of degree k, since $\Delta f = k(k+d-1)f$. Thus we may write $S_{a_R} = a_R(P)$, using the language of functional calculus. Hörmander proposed using the Fourier inversion formula to write

$$a_R(P) = \int_{\mathbb{R}} \widehat{a}_R(t)e^{2\pi itP}\,dt = \int_{\mathbb{R}} R\widehat{a}(Rt)e^{2\pi itP}\,dt.$$

The operators $\{e^{2\pi itP}\}$, as t varies, give solutions to the 'half-wave equation' $\partial_t = iP$ on M. Thus the study of the boundedness of the operator $a(P)$ is connected to the regularity for averages of the wave equation on M, in particular to local smoothing inequalities. To obtain control over these averages, we exploit the fact that the operators $\{e^{2\pi itP}\}$ have *cinematic curvature*, and that the operators $\{e^{2\pi itP}\}$ are 1-periodic because all eigenvalues of P are integers.

For $|t| < 1/2$, the Lax-Hörmander parametrix approximates $e^{2\pi itP}$ by an oscillatory integral with a phase Φ related to an eikonal equation on S^d. This oscillatory integral reveals the underlying *dynamics* of the wave equation; the operator $e^{2\pi itP}$ acts on wave packets localized in phase space T^*S^d by transporting them along the geodesic flow on T^*S^d. Using this intuition, for functions f_0 and f_1 spatially supported near $x_0, x_1 \in S^d$, one should expect $\langle e^{2\pi it_0P} f_0, e^{2\pi it_1P} f_1\rangle$ is negligible unless the radius t_0 annulus centered at x_0 is near tangent to the radius t_1 annulus centered at x_1. Obtaining sharp control over *how negligible* is difficult given the non-explicit phase Φ. However, I obtained such control by taking a geometric interpretation of the eikonal equation defining Φ, and using the second variation formula for geodesics on S^d to obtain new nondegeneracy estimates for critical points of the phase Φ. Generalizations of these bounds for other pseudodifferential operators P are also obtained in [3] by generalizing this method to geodesics on *Finsler manifolds*.

Once the appropriate inner product estimates are obtained, our problem reduces to counting near tangencies of a family of annuli. We obtain suitable estimates for the number of tangencies when the annuli we are considering are suitably sparse. Combining these estimates with a 'density decomposition' of an arbitrary family of annuli into subsets of different sparsity, using a stopping time argument akin to the Calderón-Zygmund decomposition, we obtain the appropriate L^p bounds.

The approach above fails as $t \to \pm 1/2$, since the Lax-Hörmander parametrix becomes degenerate past the injectivity radius of the manifold S^d. This is a common problem in the study of multipliers on manifolds. Sogge [8] introduced the method of reducing bounds past the injectivity radius to the study of $L^p \to L^2$ discrete restriction bounds, but such methods are not sharp enough for our purpose. I found an alternate method to reduce the required bounds for $|t| \geq 1/2$ to $L^p_x L^p_t$ localized estimates for the wave equation on S^d, and thus to the local smoothing estimates of Lee and Seeger [5].

Using the methods above, for a general function $a(\rho) = \sum_{j\in\mathbb{Z}} a_j(\rho/2^j)$, one can individually bound the L^p norm of the operators $a_j(P/R2^j)$. To combine scales, we decompose a general input in $L^p(S^d)$ into L^∞ atoms, à la the decompositions of Chang and Fefferman [2]. By refining the tangency estimates we obtain for annuli of large radius one can then control the interactions of $a_R(P)$ applied to different atoms by a square function introduced by Peetre [7], and these square functions are bounded on $L^p(S^d)$, from which we conclude that the operators $a_R(P)$ are uniformly bounded on $L^p(S^d)$.

References

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