Averaging over Curves

Jacob Denson

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Consider a smooth family of curves $\gamma : \mathbb{R}^2 \to \mathbb{R}$, and consider the associated averaging operator

$$Af(v,x) = \int f(x + \gamma(v,t))\phi(v,t)\ dt,$$

where $\gamma''(v,t) \neq 0$, and ϕ is smooth with compact support. We can write this operator as

$$Af(v,x) = (f * \mu_v)(x),$$

where μ_v is the Borel measure such that for any bounded, measurable g,

$$\int g(x) d\mu_v(x) = \int g(\gamma_v(t))\phi(v,t)\ dt.$$

We can then write

$$\widehat{\mu}_v(\xi) = \int e^{-2\pi i \xi \cdot x} d\mu_v(x) = \int e^{-2\pi i \xi \cdot \gamma_v(t)} \phi(t)\ dt.$$

This is an oscillatory integral, which is stationary at points t where $\xi \cdot \gamma'(v,t) = 0$. Under the assumption that $\gamma''(v,t)$ is non-vanishing, these stationary points are non-degenerate, and so provided we choose ϕ to have small support, for each $\xi \in \mathbb{R}^d$, there is at most one value of t such that $\xi \cdot \gamma'(v,t) = 0$. Let us write this value by $t_0(v,\xi)$. We can then find a smooth function $\psi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that on the domain of t_0,

$$\psi(v,\xi) = -\xi \cdot \gamma(v, t_0(v,\xi)).$$

Then the theory of stationary phase guarantees that

$$\widehat{\mu}_v(\xi) = e^{2\pi i \psi_v(\xi)} b(v,\xi),$$

where b is a symbol of order $-1/2$, with microsupport on the domain of t_0. Using the multiplication formula for the Fourier transform, we can thus write

$$Af(v,x) = \int \widehat{\mu}_v(\xi) \widehat{f}(\xi) e^{2\pi i \xi \cdot x}\ d\xi = \int b(v,\xi) e^{2\pi i [\psi(v,\xi) + \xi \cdot (x-y)]} f(y)\ d\xi\ dy.$$

This is a Fourier integral operator with phase

$$\phi(v,x,y,\xi) = \psi(v,\xi) + \xi\cdot(x-y).$$

Let's compute it's canonical relation.

We have $(\nabla_\xi\phi)(v,x,y,\xi) = \nabla_\xi\psi(v,\xi) + (x-y)$. Applying the chain rule to the definition of ψ_v, the chain rule implies that, on the microsupport of b,

$$(\nabla_\xi\psi_v)(\xi) = -\gamma(v,t_0) - (\xi\cdot\gamma'(v,t_0))(\nabla_\xi t_0) = -\gamma(v,t_0).$$

Thus the stationary points occur for values of ξ such that $x-y = \gamma(v,t_0(v,\xi))$. We then have

$$\nabla_x\phi(v,x,y,\xi) = \xi \quad \text{and} \quad \nabla_y\phi(v,x,y,\xi) = -\xi$$

and

$$\begin{aligned}\nabla_v\phi(v,x,y,\xi) &= \partial_v\psi_v(\xi)\\ &= -\xi\cdot\big(\partial_v\gamma(v,t_0(v,\xi)) + \gamma'(v,t_0(v,\xi))(\partial_v t_0)(v,\xi)\big)\\ &= -\xi\cdot\partial_v\gamma(v,t_0).\end{aligned}$$

Thus the canonical relation of the Fourier integral operator is

$$\mathcal{C} = \Big\{(v,x,y,\nu,\xi,\eta) : \nu = -\xi\cdot\partial_v\gamma(v,t_0) \text{ and } x = y + \gamma_v(t_0(\xi)) \text{ and } \xi = \eta\Big\}.$$

The projection of $\mathcal{C}$ onto the (y,η) variables give a submersion, and the projection of $\mathcal{C}$ onto (v,x) also form a submersion. For each fixed $z = (v,x)$, let

$$\Gamma_z = \Big\{(\nu,\xi)\in\mathbb{R}^3 - \{0\} : \nu = -\xi\cdot\partial_v\gamma(v,t_0)\Big\}$$

be the projection of $\mathcal{C}$ onto the (ν,ξ) variables at (v,x). The cinematic curvature condition amounts to saying that Γ_z is a conic hypersurface of dimension 2 in $\mathbb{R}^3 - \{0\}$, with one non-vanishing principal curvature.

To begin with, write

$$\varphi(\xi) = -\xi\cdot\partial_v\gamma(v,t_0).$$

Then the mean curvature of Γ_z at a point $(\varphi(\xi),\xi)$ can be written as

$$\frac{(1+\varphi_{\xi_1}^2)\varphi_{\xi_2\xi_2} - 2\varphi_{\xi_1}\varphi_{\xi_2}\varphi_{\xi_1\xi_2} + (1+\varphi_{\xi_2}^2)\varphi_{\xi_1\xi_1}}{(1+\varphi_{\xi_1\xi_1}^2+\varphi_{\xi_2\xi_2}^2)^{3/2}}.$$

We know one of the curvatures is zero because the surface is conic, and so one principal curvature is non-zero precisely when this quantity is nonzero.

We calculate that

$$\begin{aligned}\nabla_\xi\varphi &= \partial_v\gamma + (\xi\cdot\partial_v\gamma')(\nabla_\xi t_0)\\ &= \partial_v\gamma - \frac{\xi\cdot\partial_v\gamma'}{\xi\cdot\gamma''}\gamma'.\end{aligned}$$

using the fact that, because, differentiating the equation $\xi \cdot \gamma'(v, t_0) = 0$ int eh ξ variable, we find that

$$\nabla_\xi t_0 = -\frac{\gamma'(v, t_0)}{\xi \cdot \gamma''(v, t_0)}.$$

But this means that

$$\begin{aligned}
D_\xi \nabla_\xi \varphi &= \left[(\partial_v \gamma') - \frac{\xi \cdot \partial_v \gamma'}{\xi \cdot \gamma''} \gamma'' - \frac{\xi \cdot \partial_v \gamma''}{\xi \cdot \gamma''} \gamma' + \frac{(\xi \cdot \partial_v \gamma')(\xi \cdot \gamma''')}{(\xi \cdot \gamma'')^2} \gamma' \right] (\nabla_\xi t_0)^T \\
&\quad - \frac{1}{\xi \cdot \gamma''} \gamma' (\partial_v \gamma')^T + \frac{\xi \cdot \partial_v \gamma'}{(\xi \cdot \gamma'')^2} \gamma' (\gamma'')^T \\
&= -\frac{1}{\xi \cdot \gamma''} [(\partial_v \gamma')(\gamma')^T + (\gamma')(\partial_v \gamma')^T] \\
&\quad + \frac{\xi \cdot \partial_v \gamma'}{(\xi \cdot \gamma'')^2} [\gamma' (\gamma'')^T + \gamma'' (\gamma')^T] \\
&\quad + \frac{(\xi \cdot \gamma'')(\xi \cdot \partial_v \gamma'') + (\xi \cdot \partial_v \gamma')(\xi \cdot \gamma''')}{(\xi \cdot \gamma'')^3} [\gamma' (\gamma')^T].
\end{aligned}$$

TODO: Calculate quantity.

To begin with, we assume that

$$\nabla_\xi a(\xi) = -(\xi \cdot \partial_{vt}^2 \gamma(v, t_0))(\nabla_\xi t_0) - \partial_v \gamma(v, t_0).$$

Given that $\xi \cdot \gamma'(v, t_0) = 0$, we conclude that

Thus

$$\nabla_\xi a(\xi) = \frac{\xi \cdot \partial_v \gamma'}{\xi \cdot \gamma''} \gamma' - \partial_v \gamma.$$

Let us assume that γ is parameterized by arclength. If κ is the curvature, and then

$$\nabla_\xi a = \frac{\delta}{\kappa} \gamma' - \partial_v \gamma$$

so that we always have

This amounts to saying that the Hessian matrix

$$H = H(v, \xi) = \mathrm{Hess}_\xi \{ \xi \cdot \partial_v \gamma(v, t_0) \}$$

is non-zero. Using the product rule, we can write this Hessian as

$$(\partial_v \gamma')(\nabla_\xi t_0)^T + (\xi \cdot \partial_v \gamma'')(\nabla_\xi t_0)(\nabla_\xi t_0)^T + (\xi \cdot \partial_v \gamma')(H_\xi t_0).$$

Thus

$$\begin{aligned}
H_\xi t_0 &= -\frac{\gamma'' (\nabla_\xi t_0)^T}{\xi \cdot \gamma''} + \frac{\gamma' (\gamma'' + (\xi \cdot \gamma''') \nabla_\xi t_0)^T}{|\xi \cdot \gamma''|^2} \\
&= \frac{\gamma'' (\gamma')^T + \gamma' (\gamma'')^T}{|\xi \cdot \gamma''|^2} - \frac{\xi \cdot \gamma'''}{(\xi \cdot \gamma'')^3} \gamma' (\gamma')^T.
\end{aligned}$$

Let us assume for simplicity that γ is given by an arclength parameterization. We therefore compute that

$$(H_\xi t_0)\{\xi\} = \frac{1}{\xi \cdot \gamma''}\gamma',$$

and so

$$H\{\xi\} = \frac{\xi \cdot \partial_v \gamma'}{\xi \cdot \gamma''}\gamma'.$$

We also calculate that

$$(H_\xi t_0)\{\gamma'\} = \frac{1}{|\xi \cdot \gamma''|^2}\gamma'' - \frac{\xi \cdot \gamma'''}{(\xi \cdot \gamma'')^3}\gamma'$$

and so

$$H\{\gamma'\} = \frac{-1}{\xi \cdot \gamma''}\partial_v \gamma' + \frac{(\xi \cdot \partial_v \gamma'')}{|\xi \cdot \gamma''|^2}\gamma' + \frac{\xi \cdot \partial_v \gamma'}{|\xi \cdot \gamma''|^2}\gamma'' - \frac{(\xi \cdot \gamma''')(\xi \cdot \partial_v \gamma')}{(\xi \cdot \gamma'')^3}\gamma'.$$

Thus H has rank zero if and only if

$$\xi \cdot \partial_v \gamma' = 0 \quad \text{and} \quad (\xi \cdot \partial_v \gamma'')\gamma' = (\xi \cdot \gamma'')\partial_v \gamma'.$$

Since ξ is a multiple of $\partial_v \gamma'$ and of γ'' because of our arclength parameterization, this holds if and only if

$$\partial_v \gamma' = 0 \quad \text{and} \quad \gamma'' \cdot \partial_v \gamma'' = 0.$$

If $c(v,t)$ is now an arbitrary curve parameterization, and we define

$$L(v,t) = \int_0^t |c'(v,s)|\, ds$$

and then set $\gamma(v,t) = c(v, L^{-1}(v,t))$, then γ is an arc length parameterization. We have

$$\partial_v \gamma = \partial_v c + c' \int_0^t |c(v,s)|$$

If $c(v,t)$ is now an arbitrary curve parameterization, and we define $\gamma(v,t) = c(v, L^{-1}(v,t))$

Example. Let

$$\gamma(v,t) = v(\cos(t/v), \sin(t/v)).$$

be the arclength parameterization inducing the spherical averaging function. Then

$$\gamma' = (-\sin(t/v), \cos(t/v)),$$

and

$$\gamma'' = (-1/v)(\cos(t/v), \sin(t/v)).$$

We also have

$$\partial_v \gamma' = (t/v^2)(\cos(t/v), -\sin(t/v))$$

This is non-vanishing away from $t = 0$. But we also have

$$\partial_v \gamma'' = (1/v^2)(\cos(t/v), -\sin(t/v)) - (t/v^3)(\sin(t/v), \cos(t/v)).$$

For $t = 0$, $\partial_v \gamma''$ is equal to $(1/v^2)(1, 0)$, whereas γ'' is equal to $(-1/v)(1, 0)$. These vectors are not orthogonal to one another, i.e. their dot product is $-1/v^3$, so the cinematic curvature condition is satisfied.

For simplicity, we assume γ gives an arclength parameterization, i.e. so that γ' and γ'' are orthogonal to one another. Since

$$H_\xi t_0 = \frac{\gamma'(v, t_0)\,(\gamma''(v, t_0) + (\xi \cdot \gamma''(v, t_0))\nabla_\xi t_0)^T}{|\xi \cdot \gamma''(v, t_0)|^2} - \frac{\gamma''(v, t_0)(\nabla_\xi t_0)^T}{\xi \cdot \gamma''(v, t_0)}$$

we get that

$$(H_\xi t_0)\{\gamma'(v, t_0)\} = 0.$$

Thus we conclude that

$$\begin{aligned} H(v, \xi)\{\gamma'(v, t_0)\} &= -\frac{(\partial_v \gamma') + (\xi \cdot \partial_v \gamma'')(\nabla_\xi t_0)}{\xi \cdot \gamma''(v, t_0)} \\ &= -\frac{(\xi \cdot \gamma'')(\partial_v \gamma') - (\xi \cdot \partial_v \gamma'')\gamma'}{|\xi \cdot \gamma''(v, t_0)|^2}. \end{aligned}$$

Thus we conclude that cinematic curvature occurs if and only if

$$(\xi \cdot \gamma'')(\partial_v \gamma') \neq (\xi \cdot \partial_v \gamma'')\gamma',$$

Since $\partial_v \gamma'$ is orthogonal to γ' under the assumption that γ is an arc-length parameterization, and the fact that γ'' points in the same direction as ξ, we conclude that cinematic curvature occurs if and only if $\gamma'' \neq 0$, or if

$$\gamma'(v, t_0) + \xi \cdot \gamma''(v, t_0(v, \xi))[\nabla_\xi t_0(v, \xi)] = 0$$

$D_\xi \partial_v \psi_v$ is non-vanishing. By the chain rule, we calculate that this quantity vanishes precisely when

$$\partial_v \gamma_v(t_0) = -[\xi \cdot \partial^2_{v,t} \gamma_v(t_0)](\nabla_\xi t_0).$$

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By the chain rule, this will hold if $\nabla_\xi t_0$ is non-zero at ξ, and $\partial^2_{v,t}\gamma_v$ is non-zero at $t_0(\xi)$. But $\nabla_\xi t_0 \neq 0$ using the fact that $\gamma''_v \neq 0$, so the cinematic curvature condition holds under the assumption that $\gamma''_v \neq 0$, and $\partial^2_{v,t}\gamma \neq 0$. Thus the cinematic curvature condition is satisfied under the assumption that each of the curve you are averaging over has non-vanishing curvature, and if $\partial^2_{v,t}\gamma \neq 0$, i.e. the tangent vectors of γ change as we vary v.

Suppose we specify the curve as $\{x : \Phi(v,x) = 0\}$. Then the normal the curve at a point $x \in \mathbb{R}^2$ is given by $(\nabla_x \Phi)(v,x)$. Then the canonical relation can be written as the five dimensional conic surface generated by the four dimensional manifold

$$\Big\{(v,x,y,\nu,\xi,\eta) : \Phi(v,x-y) = 0, \xi = \eta = (\nabla_x \Phi)(v,x-y), \nu = (\partial_v \Phi)(v,x-y)\Big\}.$$

For a fixed (v,x), the conic surface $\Gamma_{(v,x)}$ is generated by the curve

$$\Big\{(\nu,\xi) : \xi = (\nabla_x \Phi)(v,x-y) \text{ and } \nu = (\partial_v \Phi)(v,x-y) \text{ for some } y \text{ with } \Phi(v,x-y) = 0\Big\}.$$

We can write $y = x - \gamma(t)$, and then the curve is precisely

$$\Big\{(\nu,\xi) : \xi = (\nabla_x \Phi)(v,\gamma(v,t)) \text{ and } \nu = (\partial_v \Phi)(v,\gamma(v,t)) \text{ for some } t\Big\}.$$

Define

$$c(t) = \left(\Phi_{x_1}(v,\gamma), \Phi_{x_2}(v,\gamma), \Phi_v(v,\gamma)\right).$$

Then

$$c'(t) = \big(\Phi_{x_1x_1}\gamma_1' + \Phi_{x_1x_2}\gamma_2', \Phi_{x_1x_2}\gamma_1' + \Phi_{x_2x_2}\gamma_2', \Phi_{x_1v}\gamma_1' + \Phi_{x_2v}\gamma_2'\big)$$

$$c'(t) = \big((D_x\nabla_x\Phi)\{\gamma'\}, \nabla_x\{\partial_v\Phi\}\cdot\gamma'\big)$$

and

$$c''(t) = ((D_x\nabla_x\Phi)\{\gamma''\}+, s)$$

This curve has the required curvature if the function $t \mapsto (\nabla_x \Phi(v,\gamma(t))$

$$\mathcal{C} = \Big\{(v,x,y,\nu,\xi,\eta) : \nu = -\xi\cdot\Big\}$$

are averaging over a smooth multiple of the surface measure on the curve in $\mathbb{R}^2$ $\Phi(v,x) = 0$.