Ordinary Differential Equations

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November 29, 2022

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Chapter 1

Introduction

Here we consider *ordinary differential equations*, i.e. we try and find solutions to equations involving an arbitrary one-variable function $f(x)$, it's derivatives, and functions of x, i.e. given some function F, we try and find a function f such that

$$F(x, f(x), \dots, f^{(n)}(x)) = 0.$$

It is important that n here is a finite quantity. We say the equation has *order* $n+1$. Furthermore, it is also important that the function F cannot access the properties of the function f globally, only pointwise.

Example. *An example of an ordinary differential equation is*

$$\frac{dy}{dx} + y = 0.$$

Here the function involved is $F(x, y, z) = y + z$, and the equation has order one.

Example. *The equation $(f \circ f)'(x) + f(x) = 0$ is* not *a differential equation, since $(f \circ f)'(x)$ cannot be expressed as a function of $f(x)$ and $f'(x)$.*

An interesting amount of geometry is going on here. For simplicity, consider the n'th order differential equation

$$f^{(n)}(x) = G(x, f(x), \dots, f^{(n-1)}(x))$$

Then a solution $f : I \to \mathbf{R}$ to an $n+1$ order differential equation as a *curve* in the *phase space* $\mathbf{R}^n$, i.e. a map $y : I \to \mathbf{R}^n$ satisfying the system of first order differential equations

$$\begin{aligned}
y_0' &= 1 \\
y_1' &= y_2 \\
&\vdots \\
y_{n-1}' &= y_n \\
y_n' &= F(y_0, \dots, y_{n-1})
\end{aligned}$$

The connection is obvious by setting $y_0(x) = x$, $y_1(x) = f(x)$, $y_2(x) = f'(x)$, and so on. Thus, if we define a vector field

$$X_y = (1, y_2, \dots, y_n, F(y_0, \dots, y_{n-1})),$$

then a solution to a differential equation can be viewed as a curve in phase space whose velocity vector agrees with X at all points, i.e. $y' = X_y$. Such a curve is known as a integral curve.

Example. *Consider the vector field in $\mathbf{R}^2$ given by $X_{(x,y)} = (y, -x)$. Then the curve $c(t) = (\cos(t), \sin(t))$ is an integral curve for this vector field, which corresponds to the differential equation $f'' = -f$.*

Another way of geometrically simplifying the situation is, rather than a vector field, to consider a collection of lines, i.e. a map Δ such that for each x, $\Delta_x \in \mathbf{P}(T\mathbf{R}^n_x)$ is a line in tangent space passing through x. The goal here is to find a one dimensional manifold M such that $TM_x = \Delta_x$ for each $x \in M$. The manifold M here is known as an *integral curve*, and the function Δ is known as a (one dimensional) *distribution*.

Example. *Consider the distribution Δ in $\mathbf{R}^2$ given by the two equations*

$$(xy^2 + 2xy - 1)dy + y^2 dx.$$

That is, at each point $p = (x, y)$, Δ_p is the line

$$\{(x + x_0, y + y_0) : (xy^2 + 2xy - 1)y_0 + y^2 x_0 = 0\}.$$

and we want to find a curve whose tangent at each point passes through this line. One such example is the curve M given by the equation

$$xy^2 - e^{-y} - 1 = 0.$$

By differentiating, we find that TM_p is given by the set of solutions to the equation

$$y^2 dx + (2xy + e^{-y})dy + y^2 dx = 0.$$

This equation is not equal to the other equation at all points, but it is equal when $xy^2 - e^{-y} - 1$, so M is an integral curve. In particular, a distribution in $\mathbf{R}^n$ can be given by $n-1$ linearly independant equations in $dx_1, \dots, dx_n$.

Example. *Let $X : \mathbf{R}^n \to \mathbf{R}^n$ be a vector field. Then we can define a distribution by letting Δ_x be the line lying along the vector X_x. If M is an integral curve locally parameterized by some function $y : I \to \mathbf{R}^n$, then $y'(t)$ is a constant multiple of $X_{y(t)}$ for each time t. Thus there is a function $A : I \to \mathbf{R}$, never vanishing, such that $y'(t) = A(t)X_{y(t)}$. If we can write $t = f(s)$, such that $f'(s) = A(s(t))$, then $y \circ f$ is an integral curve to X. Thus integral curves to distributions, once appropriately parameterized are equivalent to vector fields.*

To summarize, we have three problems:

- Find solutions to differential equations.
- Find curves lying tangent to vector fields.
- Find integral curves to distributions

All of these problems are (roughly) equivalent, and the goal of differential equations is to find techniques for studying them.

In general, if a vector field $X : \mathbf{R}^n \to \mathbf{R}^n$ is *locally Lipschitz*, then there is a unique integral curve passing through any point. And moreover, we can find a continuous function $\alpha : \mathbf{R}^n \times \mathbf{R}$ such that for each x_0, the function $y(t) = \alpha(x_0, t)$ is a curve with $y'(t) = X_{y(t)}$. Such a function is known as a *flow*, and gives a complete description of the integral curves of the vector field. More generally, if X is C^k, then α will also be C^k. Theoretically, α exists, but one still requires techniques to calculate α as an explicit function.

Chapter 2

Basic Techniques

2.1 Reduction to Integration

Calculus already teaches us some basic techniques for solving differential equations. The most basic differential equation $y' = f(x)$ can be solved as

$$y(x) = A + \int_0^x f(t)\,dt,$$

where A is an arbitrary constant. If we know that $y(x_0) = y_0$, for some values x_0 and y_0, then we can also write

$$y(x) = y_0 + \int_{x_0}^x f(t)\,dt.$$

This is often how a differential equation is solved; given certain *initial conditions*, which is this case are (x_0, y_0), i.e values of y at a particular timepoint, one can uniquely solve the differential equation.

2.2 Separable Equations

Slightly more generally, integration enables us to solve a *separable equation*

$$f(x)dx + g(y)dy = 0.$$

We note that if $F'(x) = f(x)$ and $G'(y) = g(y)$, then $d(F+G) = f(x)dx + g(y)dy$, and so the family of curves defined by the equation $F(x)+G(y) = C$ give a family of integral curves to the distribution.

Example. *Next, we consider a homogenous equation*

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

where P and Q are each homogenous functions of order n, i.e. for each x and y,

$$P(tx,ty) = t^n P(x,y) \quad \text{and} \quad Q(tx,ty) = t^n Q(x,y).$$

To exploit this homogeneity, we switch variables by setting $y = ux$, provided we are working where $x \neq 0$. Then $dy = xdu + udx$, and

$$\begin{aligned} P(x,y)\,dx + Q(x,y)\,dy &= x^n P(1,u)\,dx + x^n Q(1,u)(u\,dx + x\,du) \\ &= x^n[(P(1,u) + uQ(1,u))\,dx + xQ(u,1)\,du]. \end{aligned}$$

Thus the original equation is equivalent, when $x \neq 0$, to the equation

$$\frac{dx}{x} = \frac{Q(1,u)}{P(1,u) + uQ(1,u)}\,du,$$

and this equation is separable.

Example. *Consider the differential equation*

$$(a_1x + b_1y + c_1)\,dx + (a_2x + b_2y + c_2)\,dy = 0.$$

where the coefficients (a_1, b_1) and (a_2, b_2) are not constant multiples of one another. Then the lines corresponding to these coefficients have a unique intersection (x_0, y_0), i.e. such that $a_1x_0 + b_1y_0 + c_1 = a_2x_0 + b_2y_0 + c_2 = 0$. If we set $x = x_0 + h$, $y = y_0 + k$, then the equation becomes

$$(a_1h + b_1k)\,dh + (a_2h + b_2k)\,dk = 0$$

This is a homogenous differential equation, and therefore reduces to the last example. Alternatively, we can let $u = a_1x + b_1y + c_1$, and $v = a_2x + b_2y + c_2$, so that $du = a_1dx + b_1dy$ and $dv = a_2dx + b_2dy$. These two equations can then be solved for dx and dy, and substitution gives another homogenous differential equation.

Example. *If we consider the 'parallel case' of the previous problem, i.e.*

$$(a_1x + b_1y + c_1)\,dx + A(a_1x + b_1y + c_2)\,dy$$

Then substituting $u = a_1x + b_1y$ gives a separable equation in x and u, or in y and u.

2.3 Exact Differential Equation

More general than the separable case is the case where we have

$$\omega = f(x,y)dx + g(x,y)dy = 0$$

where $d\omega = 0$. This occurs if

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0.$$

In this case, there exists a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = f \quad \text{and} \quad \frac{\partial F}{\partial y} = g.$$

It then follows that $F(x,y) = 0$ gives a family of integral curves for the differential equation.

2.4 Integrating Factors

There is a trick which can be used to reduce non-exact differential equation to exact differential equations. Given a differential form ω, there may exist a function f such that $f\omega$ is exact, and therefore integral curves can be found.

Example. *The equation $(y^2 + y)dx - xdy = 0$ is not exact. Nonetheless, if we multiply by $1/y^2$, we obtain the equation $(1 + 1/y)dx - (x/y^2)dy = 0$, which is exact. In fact, the equation is $d(x/y + x) = 0$, so the integral curves are given by $x/y + x = C$.*

Theoretically, if one can find

Chapter 3

Vector Fields on $\mathbf{RP}^n$

On $\mathbf{R}^n$, local flows related to a smooth vector field v may fail to extend to global flows because solutions approach ∞ in finite time. However, often we may be able to embed $\mathbf{R}^n$ in a compact manifold K, and extend v to a smooth vector field on K. Since K is compact, all local flows extend to global flows, and thus we can consider a global flow on $\mathbf{R}^n$ which 'passes through ∞' in finite time.

For instance, recall that the space $\mathbf{RP}^n$ is the compact quotient space of $\mathbf{R}^{n+1} - \{0\}$ by the group action of $\mathbf{R}^\times$ by scaling, so that x is identified with λx for any $\lambda \neq 0$. The quotient structure gives it a natural topological structure, which can also be identified with the topology which makes the projection maps on each of the coordinate systems

$$x_i : [x] \mapsto (x^1/x^i, \dots, \widehat{x^i/x^i}, \dots, x^{n+1}/x^i)$$

defined on $U_i = \{[x] : x_i \neq 0\}$, homeomorphisms. It is a smooth manifold if we consider the x_i as diffeomorphisms.

Example. *The classic example of a vector field which cannot be extended to a global flow is $v(x) = x^2$ on $\mathbf{R}$, which has a flow*

$$\varphi_t(x) = \frac{x}{1 - xt}$$

Which has a singularity at $t = x^{-1}$. Note, however, that if we write this map in projective coordinates, then we find $\varphi_t[x : y] = [x : y - tx]$. In this formulation, it is easy to see that each map φ_t can be extended uniquely to a smooth map *from* $\mathbf{RP}^1$ *to* $\mathbf{RP}^1$*, and the group equation still holds.*

$$\varphi_{t+s}[x:y] = [x:y-x(t+s)] = [x:(y-sx)-tx] = \varphi_t(\varphi_s[x:y])$$

An alternate way to see this is to let $y = 1/x$ *denote the inverse coordinate system on projective space. We then calculate that for* $y \neq 0, \infty$*, that*

$$v(y) = x^2 \partial_x(y) = -x^2 y^2 = -\partial_y$$

and v *can be uniquely extended to a smooth vector field on* $\mathbf{RP}^1$ *by defining* $v(\infty) = \partial_y$*, and therefore generates a global flow on* $\mathbf{RP}^1$ *because* $\mathbf{RP}^1$ *is compact. This technique is not general, however. If we consider the vector field* $v(x) = x^3 \partial_x$*, then we find that* $v(y) = -y^{-1}\partial_y$*, which cannot be extended to a smooth vector field at* $y = 0$*. This is because solutions approach infinity 'too fast' – we find the flows take the form*

$$\varphi_t(y) = \sqrt{y^2 - 2t}$$

And these solutions approach $y = 0$ *with infinite slope.*

Sometimes the geometry of projective space provides an enlightening viewpoint on a particular differential equation.

Example. *Consider the differential equation* $\ddot{u} + \alpha u = 0$*, as* α *ranges over* $\mathbf{R}$*. This corresponds to the two dimensional first order system specified by the vector field* $v(u,w) = (w, -\alpha u)$*. This means that on the integral curves defined by this vector field,*

$$-\alpha u du = w dw$$

so the integral curves lie on the level curves to $w^2 + \alpha u^2$*. For* $\alpha > 0$*, this value is always positive, and defines an ellipse. Since* v *does not vanish on any ellipse of a positive radius, we see these ellipses must describe the integral curves. For* $\alpha < 0$*, the level curves of* $w^2 + \alpha u^2$ *describe hyperbolas not passing through the origin, so these hyperbolas are the integral curves. For* $\alpha = 0$*, the integral curves are easily seen to be the lines parallel to the* x *axis. Switching to the coordinates* $x = u/w$*,* $y = 1/w$*, we find that for* $y \neq 0$*,*

$$\begin{aligned} v(x,y) &= (w\partial_u(x) - \alpha u \partial_w(x), w\partial_u(y) - \alpha u \partial_w(y)) \\ &= (1 + \alpha u^2/w^2, \alpha u/w^2) = (1 + \alpha x^2, \alpha x y) \end{aligned}$$

This function can be extended to a smooth vector field on the whole of $\mathbf{RP}^2$ *by defining* $v(x,0) = (1 + \alpha x^2, 0)$*.*

Chapter 4

Linear Differential Equations

A linear differential equation is one of the form $Lf = 0$, where

$$(Lf)(x) = a_n(x)(D^n f)(x) + \cdots + a_0(x)f(x) = 0,$$

where the $a_0, \dots, a_n$ are functions, and D is the differentiation operator, where we assume $a_n(x) \neq 0$ for all x so no singularities are formed.

What makes linear differential equations tractable is that the family of solutions to the equation spans a subspace of the space of all differentiable equations. Indeed, if $Lf = 0$ and $Lg = 0$, then $L(af + bg) = 0$, for any constants a and b. Moreover, this subspace is *finite dimensional*. For each $b_0, \dots, b_{n-1} \in \mathbf{R}$, the theory of uniqueness and existence implies that there exists a unique differentiable function f such that $Lf = 0$, and $f(0) = b_0$, $f'(0) = b_1, \dots, f^{(n-1)}(0) = b_{n-1}$. Thus the set of solutions to a degree n differential equation is n dimensional.

Thus our goal is to find n solutions $\{f_1, \dots, f_n\}$ to the differential equation which are linearly independent, since then any solution can be written as a linear combination of the differential equations. One such trick is to consider the Wronskian determinant

$$W(f_1, \dots, f_n)(x) = \det \begin{pmatrix} f_1(x) & \dots & f_n(x) \\ f_1'(x) & \dots & f_n'(x) \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{pmatrix}$$

If there exists a point x_0 such that $W(f_1, \dots, f_n)(x_0) \neq 0$, then the vectors in the matrix span the set of all initial parameters at x_0, and therefore

$\{f_1, \dots, f_n\}$ is a basis for the set of solutions. In particular, this also means $W(f_1, \dots, f_n)(x_0) \neq 0$ everywhere. Conversely, if the functions do not form a basis, then $W(f_1, \dots, f_n) = 0$.

Nonhomogenous equations can be viewed by very similar methods. Consider a differential equation of the form $Lf = g$, for some function g. If we can find a *single* function f_0 such that $Lf_0 = g$, then any other solution is given by $f_0 + b$, where $Lb = 0$. Thus we need only find a single non-homogenous solution, and then solve the homogenous form $Lf = 0$ of the equation.

4.1 Homogenous Linear Differential Equations with Constant Coefficients

A homogenous differential equation is one of the form

$$a_n D^n f + \cdots + a_0 f = 0,$$

where D is the differential operator, and $a_0, \dots, a_n$ are constants. These are perhaps the simplest general class of differential equations to solve. The first idea is to see a correspondence between these operators and polynomials. We note that if L and S are two linear differential operators with constant coefficients, then these operators actually commute, i.e. $L \circ S = S \circ L$. Since these operators also form an algebra under composition, and have as a basis $\{1, D, D^2, \dots\}$, the space of linear differential operators with constant coefficients is actually isomorphic to $\mathbf{C}[D]$.

The advantage of this result is that we can apply the fundamental theorem of algebra. Thus we can write any homogenous differential equation as

$$L = (D - \alpha_1)^{m_1} \dots (D - \alpha_n)^{m_n}$$

where $\alpha_1, \dots, \alpha_n \in \mathbf{C}$. Now we have to do some calculation. For each k, the solution space to $(D - \alpha_k)^{m_k} f = 0$ has dimension m_k, and is given by $\{e^{\alpha_k x}, \dots, x^{m_k - 1} e^{\alpha_k x}\}$. Furthermore, the solution spaces to each $(D - \alpha_k)^{m_k}$ are disjoint to one another. Thus the solution space to L breaks down into the direct sum of the solution spaces $(D - \alpha_k)^{m_k}$. And so we can solve each of the individual differential equations, and then work our way back up.

Remark. If we are working over the real numbers, we cannot completely break polynomials into linear factors, instead having to deal with factors of the form $((D-\alpha)^2+\beta^2)^m$. This solution space is spanned by the real vector space

$$\{e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x), \dots, x^{m-1}e^{\alpha x}\cos(\beta x), x^{m-1}e^{\alpha x}\sin(\beta x)\}.$$

and so we obtain sines and cosines in our decomposition as well as exponentials. Of course, the two representations of the solution space are connected by Euler's formula $e^{\beta i x} = \cos(\beta x) + \sin(\beta x)$.

4.2 The Method of Annihilation

It is difficult to find explicit solutions to non-homogenous differential equations with constant coefficients. However, when the 'constant' term of the differential equation is also given as a solution to a differential equation, the method of annihilation enables one to obtain an explicit solution.

Let L_0 be a linear differential operator with constant coefficients, and suppose we wish to solve the equation $L_0 y = f$, for some function f, and suppose in addition that there exists a differential operator L_1 with constant coefficients such that $L_1 f = 0$. Then $L_1(L_0 y) = L_1(f) = 0$, so y is a solution to the homogenous equation $(L_1 \circ L_0)(y) = 0$, with constant coefficients. One can obtain the complete span of solutions to this equation, and then calculate coefficients to obtain a particular solution to $L_0 y = f$.

Example. *Consider the differential equation $L_0 y = e^{5x}$, where $L_0 = D^2+2D+1 = (D+1)^2$. We note that if $f(x) = e^{5x}$, then $L_1 f = 0$ where $L_1 = D-5$. Thus $L_0 L_1 y = 0$, and $L_0 L_1 = (D-5)(D+1)^2$. Thus there must be constants A, B, and C such that*

$$y = Ae^{5x} + Be^{-x} + Cxe^{-x}.$$

Without loss of generality, we can set $B = C = 0$, since $L_0(e^{-x}) = L_0(xe^{-x}) = 0$. Since $L_0(e^{5x}) = 36e^{5x}$, we can set $A = 1/36$. Thus a general solution to the differential equation is given by $y = e^{5x}/36 + Be^{-x} + Cxe^{-x}$.

Bibliography