Detangling a Twisted Form in L^4

Jacob Denson and Jacob Fiedler

(after Polona Durcik, 2015)

University of Wisconsin Madison

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The form Λ

Define $\mathbf{F}$ to be the following entanglement of four functions F_1, F_2, F_3, and F_4 on $\mathbb{R}^2$:

$$\mathbf{F}(x, x', y, y') := F_1(x, y) F_2(x', y) F_3(x, y') F_4(x', y')$$

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And consider the quadrilinear form

$$\Lambda(F_1, F_2, F_3, F_4) := \int_{\mathbb{R}^2} \widehat{\mathbf{F}}(\xi, -\xi, \eta, -\eta)m(\xi, \eta)d\xi d\eta,$$

where $m : \mathbb{R}^2 \to \mathbb{C}$ obeys the symbol estimates $|\partial^\alpha m(\xi, \eta)| \lesssim (|\xi| + |\eta|)^{-|\alpha|}$ for sufficiently large α.

Main theorem

Durcik's main result in this paper is the following:

Theorem

The quadrilinear form Λ satisfies

$$|\Lambda(F_1, F_2, F_3, F_4)| \lesssim \|F_1\|_{L^4(\mathbb{R}^2)} \|F_2\|_{L^4(\mathbb{R}^2)} \|F_3\|_{L^4(\mathbb{R}^2)} \|F_4\|_{L^4(\mathbb{R}^2)}$$

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We also note that Durcik was able to generalize the above estimate for this form in a subsequent paper.

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$$|\Lambda(F_1, F_2, F_3, F_4)| \lesssim \|F_1\|_{L^{p_1}(\mathbb{R}^2)}\|F_2\|_{L^{p_2}(\mathbb{R}^2)}\|F_3\|_{L^{p_3}(\mathbb{R}^2)}\|F_4\|_{L^{p_4}(\mathbb{R}^2)}$$

whenever $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} = 1$ *and* $2 < p_i \leq \infty$ *for all* i.

The twisted paraproduct

A special case of this quadrilinear form is the so-called 'twisted paraproduct' introduced by Demeter and Thiele and defined as follows:

$$T(F_1, F_2, F_3) := \Lambda(F_1, F_2, F_3, 1).$$

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This form had to be treated differently than the others in their work because it exhibits certain "modulation invariance". For instance,

$$T(f(y)F_1, F_2, F_3) = T(F_1, f(y)F_2, F_3)$$

We note that Kovac was able to prove L^p bounds for this form.

The bilinear Hilbert transform

The twisted paraproduct is closely related to the bilinear Hilbert transform, which in the one dimensional case is defined as

$$H(f,g)(x) = \int f(x+t)g(x+\beta t)\frac{dt}{t}$$

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where the integral is the principal value integral. In the two dimensional case, we have

$$H(F_1,F_2)(\vec{x}) = \int_{\mathbb{R}^2} F_1(\vec{x}+A_1(t,s))F_2(\vec{x}+A_2(t,s))K(t,s)dtds$$

Where K is a Calderon-Zygmund kernel and A_i are matricies, at least one of which is nonsingular. As we will briefly discuss, the bilinear Hilbert transform has applications to ergodic theory.

The triangular Hilbert transform

Some further motivation for studying these entangled forms comes from the "triangular Hilbert" transform. If we let

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Alternatively, up to a constant, we can write

$$\Lambda_\Delta(G_1, G_2, G_3) = -\int_{\mathbb{R}^3} \frac{G_1(x, y)G_2(y, z)G_3(z, x)}{x + y + z} dxdydz$$

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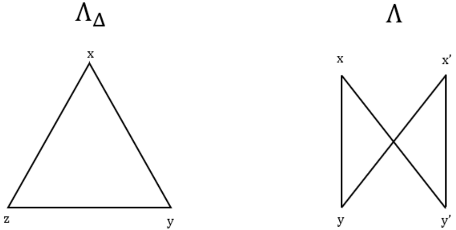

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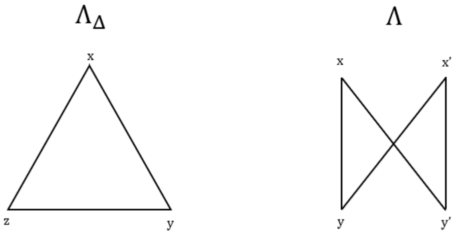

We note that boundedness of the triangular Hilbert transform implies boundedness of the two dimensional bilinear Hilbert transform in some cases, so any improved understanding of entanglement is helpful.

Ergodic averages

Let X be a probability space and let $T, S : X \to X$ be commuting measure-preserving transformations on X. For $f, g \in L^\infty(X)$, one can investigate the almost everywhere convergence of the averages

$$\frac{1}{N}\sum_{n=1}^{N} f(T^n x) g(S^{-n} x) \quad \text{as } N \to \infty.$$

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Using a paraproduct estimate, Demeter and Thiele showed convergence of a related family of averages, including

$$\frac{1}{N^2}\sum_{n=1}^{N}\sum_{m=1}^{N} f(T^n S^m x) g(T^{-n} S^m x) \tag{1}$$

Bounding oscillation of ergodic averages

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Idea: attempt to bound (a weighted) version of the oscillation of the terms in a manner that essentially only depends on the L^2 norms of f and g. If done correctly, we will obtain a.e. convergence of the average.

Bounding oscillation of ergodic averages

What is one way to prove convergence of averages like those above?

Idea: attempt to bound (a weighted) version of the oscillation of the terms in a manner that essentially only depends on the L^2 norms of f and g. If done correctly, we will obtain a.e. convergence of the average. For instance, Demeter shows

$$\Big\|\Big(\sum_{j=1}^{J-1}\sup_{k\in[u_j,u_{j+1})}|W_k(f,g)(x)-W_{u_{j+1}}(f,g)(x)|^2\Big)^{\frac{1}{2}}\Big\|_{L^{1,\infty}(X)} \lesssim J^{\frac{1}{4}}\|f\|_{L^2(X)}\|g\|_{L^2(X)},$$

(where W is the weighted average and J is a term related to the oscillation) implies a.e. convergence.

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- Next, move from $\mathbb{Z}^2$ to the probability space X by using the functions F on $\mathbb{Z}^2$ which are of the form $F(n, m) = f(T^n S^m x)$ for some $x \in X$.

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Using this transfer principle, for instance, it suffices to prove an inequality for the oscillation of

$$\int_{\mathbb{R}^2} F_1(x+t, y+s) F_2(x-t, y+s) \Psi_k(t) \Phi_k(t) dtds$$

to obtain convergence of the second ergodic average.

Key tools in proving the main theorem

The proof of the main theorem proceeds first through a decomposition of the symbol into suitable pieces which we attempt to bound uniformly. These pieces are highly symmetric, and we can use a combination of Cauchy-Schwarz an a certain ‘telescoping identity’ to gradually detangle the form.

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(B) Exploiting cancellation using the aforementioned telescoping identity, which for intuition’s sake behaves like a multilinear variant of an integration by parts.

(C) Using monotonicity to replacing arbitrary functions with concrete functions (e.g. Gaussians).

Decomposing the symbol I

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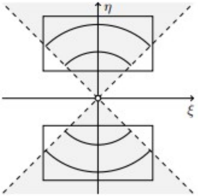

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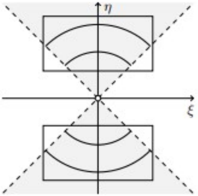

We define θ to be such that $\hat{\theta}$ is smooth, real, radial, and supported in some annulus. Normalizing $\hat{\theta}$ and defining $m_t(\xi, \eta) := m(\xi, \eta)\hat{\theta}(t\xi, t\eta)$ allows us to write

$$m(\xi, \eta) = \int_0^\infty m_t(\xi, \eta) \frac{dt}{t}.$$

Decomposing the symbol II

Using a technical lemma, we obtain the existence of sufficiently nice functions ν_1, ν_2 that satisfy

$$m_t(\xi, \nu) = m_t(\xi, \nu)\hat{\nu}_1(t\xi)\hat{\nu}_2(t\eta)$$

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Using Fourier inversion, we obtain

$$m_t(\xi, \eta) = \left(\int_{\mathbb{R}^2} \mu_t(u, v) e^{2\pi i u t \xi} e^{2\pi i v t \eta} \right) \hat{\nu}_1(t\xi)\hat{\nu}_2(t\eta)$$

with $\mu_t(u, v) := t^2 \hat{m}(tu, tv)$.

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with $\mu_t(u, v) := t^2 \hat{m}(tu, tv)$.It is not hard to show that

$$|\mu_t(u, v)| \lesssim (1 + |u|)^{-12}(1 + |v|)^{-12}$$

Decomposing the symbol III

Now, define

$$\widehat{\varphi}_{t,u}(\xi) := (1+|u|)^{-5}\hat{\nu}_1(t\xi)^{1/2}e^{\pi iut\xi}$$

$$\widehat{\psi}_{t,v}(\xi) := \hat{\nu}_2(t\xi)^{1/2}e^{\pi ivt\eta}$$

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Using the properties of the ν_i, we have,

$$m(\xi,\eta) = \int \mu_t(u,v)(1+|u|)^{10}\widehat{\varphi}_{t,u}(\xi)^2\widehat{\psi}_{t,v}(\xi)^2\, dudv\frac{dt}{t}$$

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Note that $\mu_t(u,v)(1+|u|)^{10}$ decays rapidly. So, it suffices to establish bounds on

$$\int \widehat{\varphi}_{t,u}(\xi)^2\widehat{\psi}_{t,v}(\xi)^2 \, \widehat{\mathbf{F}}(\xi,-\xi,\eta,-\eta) \, \frac{dt}{t} d\xi d\eta$$

uniformly in u and v.

The Setup

Goal

Show $$\left|\int \widehat{\mathbf{F}}(\xi,-\xi,\eta,-\eta)\widehat{\varphi}_t(\xi)^2\widehat{\psi}_t(\xi)^2\ (dt/t)\ d\xi\ d\eta\right| \lesssim 1.$$

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Switch Fourier Multiplier to Convolution Operator

Bound above is equivalent to $| \int_0^\infty \Lambda_t \; dt/t| \lesssim 1$*, where*

$$\Lambda_t = \int \mathbf{F}(x, y, x', y') \varphi_t(\tilde{x} - x) \varphi_t^-(\tilde{x} - x') \psi_t(\tilde{y} - y) \psi_t^-(\tilde{y} - y').$$

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New Notation

Write $\Lambda_t = \Lambda_{\varphi_t, \varphi_t^-, \psi_t, \psi_t^-}$, *where*

$$\Lambda_{a,b,c,d} = \int \mathbf{F}(x, y, x', y') a(\tilde{x} - x) b(\tilde{x} - x') c(\tilde{y} - y) d(\tilde{y} - y').$$

Disentangling with Cauchy-Schwarz

Separate out y and y'

$$
\begin{aligned}
\Lambda_t &= \int F_1(x,y)F_2(x',y)F_3(x',y')F_4(x,y') \\
&\qquad \varphi_t(\tilde{x}-x)\varphi_t^-(\tilde{x}-x')\,\psi_t(\tilde{y}-y)\psi_t^-(\tilde{y}-y') \\
&= \int \left(\int F_1(x,y)F_2(x',y)\psi_t(\tilde{y}-y)\,dy\right) \\
&\qquad \left(\int F_4(x,y')F_3(x',y')\psi_t^-(\tilde{y}-y')\,dy'\right) \\
&\qquad\qquad \varphi_t(\tilde{x}-x)\varphi_t^-(\tilde{x}-x')\,dx\,dx'\,d\tilde{x}\,d\tilde{y}.
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Remark

Cauchy-Schwarz is efficient (All F_j are equal in the worst case).

Cauchy Schwarz

$$\left|\int ABCD\right| \leq \left(\int AA|C|\right)^{\frac{1}{2}} \left(\int BB|D|\right)^{\frac{1}{2}}.$$

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$$A = \int F_1(x,y)F_2(x',y)\psi_t(\tilde{y}-y)\,dy$$

$$B = \int F_4(x,y')F_3(x',y')\psi_t(\tilde{y}-y')\,dy'$$

and

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Result

Since $\Lambda_t = \int ABCD$, *we obtain that*

$$|\Lambda_t| \leq \Lambda_{|\varphi_t|,|\varphi_t^-|,\psi_t,\psi_t}(F_1,F_2,F_2,F_1)^{1/2}$$
$$\Lambda_{|\varphi_t|,|\varphi_t^-|,\psi_t^-,\psi_t^-}(F_4,F_3,F_3,F_4)^{1/2}.$$

Multilinear Integration By Parts

Remark

Want to do the same trick with the x and x' variables, but we can't without losing cancellation since we'd have to take absolute values of ψ_t and ψ_t^-. Can 'juggle' the cancellation by integrating by parts in t.

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Integration by Parts Identity

If $-t\partial_t|\widehat{\rho}_i|^2 = |\widehat{\sigma}_i(t\tau)|^2$, *then*

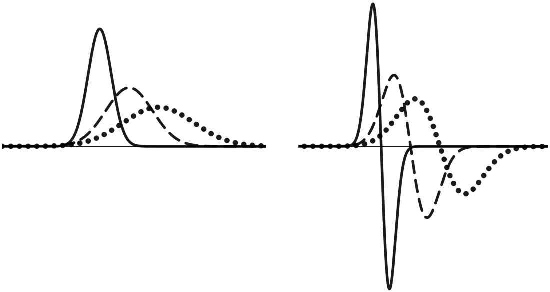
$$\int \Lambda_{\sigma_1,\sigma_1,\rho_2,\rho_2}\,(dt/t) = -\int \Lambda_{\rho_1,\rho_1,\sigma_2,\sigma_2}\,(dt/t).$$

$$+\,|\widehat{\rho}_1(0)|^2|\widehat{\rho}_2(0)|^2 \int_{\mathbb{R}^2} F_1F_2F_3F_4$$

Examples of Pairs of ρ and σ

Example Pairs are given by

$$\rho(t,x) = t^{-1}e^{-(x/t)^2} \quad \text{and} \quad \sigma(t,x) = -(4\pi^{1/2}x/t^2)e^{-2\pi(x/t)^2}.$$



Proof of Identity

Write

$$
\begin{aligned}
-|\widehat{\rho}_1(0)|^2|\widehat{\rho}_2(0)|^2 &= \int_0^\infty \partial_t\{|\widehat{\rho}_1(t\xi)|^2|\widehat{\rho}_2(t\eta)|^2\}\,dt \\
&= \int_0^\infty t\partial_t\{|\widehat{\rho}_1(t\xi)|^2\}|\widehat{\rho}_2(t\eta)|^2\,(dt/t) \\
&\quad + \int_0^\infty |\widehat{\rho}_1(t\xi)|^2 t\partial_t\{|\widehat{\rho}_2(t\eta)|^2\,(dt/t).
\end{aligned}
$$

Now multiply by $\widehat{\mathbf{F}}(\xi,-\xi,\eta,-\eta)$*, and integrate in* ξ *and* η*.*

Using the Identity

Integration by Parts Identity

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$$+\,|\widehat{\rho}_1(0)|^2|\widehat{\rho}_2(0)|^2\int_{\mathbb{R}^2} F_1F_2F_3F_4$$

Problems

We cannot use this to directly bound

$$|\Lambda_{|\varphi_t|,|\varphi_t^-|,\psi_t,\psi_t}(F_1,F_2,F_2,F_1)|.$$

Monotonicity

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We have

$$\Lambda_{a,a',b,b} = \int \left(\int F_1(x,y) F_2(x',y) b(\tilde{y} - y)\, dy \right)^2 a(\tilde{x} - x) a'(\tilde{x} - x'),$$

and so $\Lambda_{a,a',b,b}$ is monotonic in a and a'.

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Replacing General Functions With Gaussians

Let $G_{t,\alpha} = \alpha^{-1} \exp(-(x/\alpha)^2)$ *be normalized Gaussians. For appropriate fast decaying constants* C *and* C',

$$|\varphi_t| + |\varphi_t^-| \leq |\varphi_t| + |\varphi_t^-| \leq \int_1^\infty C(\alpha) G_{t,\alpha}.$$

$$\text{Thus} \quad \Lambda_{|\varphi_t|,|\varphi_t^-|,\psi_t,\psi_t} \leq \int_1^\infty C'(\alpha) \Lambda_{G_{t,\alpha},G_{t,\alpha},\psi_t,\psi_t}.$$

It suffices to prove uniform estimates in α.

Back to Cauchy Schwarz

Apply Integration By Parts

$$\Lambda_{G_t,G_t,\psi_t,\psi_t} = c\int F_1^2 F_2^2 - \Lambda_{g_t,g_t,\Psi_t,\Psi_t}.$$

Notice that the oscillating term has been moved to the third and fourth term.

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Cauchy Schwarz

We can now use Cauchy-Schwarz while preserving oscillation.

$$\begin{aligned} &\Lambda_{g_t,g_t,\Psi_t,\Psi_t}(F_1, F_2, F_2, F_1) \\ &\qquad \leq \Lambda_{g_t,g_t,|\Psi_t|,|\Psi_t|}(F_1, F_1, F_1, F_1)^{1/2} \\ &\qquad\qquad \Lambda_{g_t,g_t,|\Psi_t|,|\Psi_t|}(F_2, F_2, F_2, F_2)^{1/2}. \end{aligned}$$

A Final Integration By Parts

Monotonicity

Use monotonicity to switch the Ψ_t values with a Gaussian, i.e. so that

$$\Lambda_{g_t, g_t, |\Psi_t|, |\Psi_t|}(F_1, F_1, F_1, F_1) \lesssim \Lambda_{g_t, g_t, G'_t, G'_t}(F_1, F_1, F_1, F_1).$$

A Final Integration By Parts

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$$\Lambda_{g_t,g_t,|\Psi_t|,|\Psi_t|}(F_1,F_1,F_1,F_1) \lesssim \Lambda_{g_t,g_t,G_t',G_t'}(F_1,F_1,F_1,F_1).$$

Apply Integration By Parts

We can now perform a final integration by parts to write

$$\Lambda_{g_t,g_t,G_t',G_t'}(F_1,F_1,F_1,F_1) = c\int F_1^4 - \Lambda_{G_t,G_t,g_t',g_t'}(F_1,F_1,F_1,F_1).$$

But both the Λ terms here are positive, i.e. they cannot cancel one another out.

$$\Lambda_{g_t,g_t,G_t',G_t'}(F_1,F_1,F_1,F_1) \lesssim \int F_1^4 = \|F_1\|_{L^4(\mathbb{R}^2)}^4.$$