A Characterization of L^p Boundedness For Spectral Multipliers

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Fourier Multipliers on $\mathbb{R}^d$

Given $a : \mathbb{R}^d \to \mathbb{C}$, consider the Fourier Multiplier

$$F_a g(x) = \int_{\mathbb{R}^d} a(\xi) \hat{g}(\xi) e^{i\xi \cdot x} \, d\xi$$

Spectral Multipliers on S^d

For $f : S^d \to \mathbb{C}$, define $\Delta f = \Delta_{\mathbb{R}^{d+1}} \tilde{f}$

Then Every f has expansion $f = \sum_\lambda f_\lambda$

s.t. $\Delta f_\lambda = -\lambda^2 f_\lambda$ and $\|f\|_{L^2}^2 = \sum_\lambda \|f_\lambda\|_{L^2}^2$

For $m : [0,\infty) \to \mathbb{C}$ define $T_m g = \sum m(\lambda) g_\lambda$

Main Question

How does T_m relate to F_m?

Correctly Posed Problem

Main Question

How does T_m relate to F_m?

On S^d

If $m \in L^\infty$ and $\mathrm{supp}(m)$ is compact, T_m is trivially bounded on $L^p(S^d)$ for all $p \in [1,\infty]$

On $\mathbb{R}^d$

Ball Multiplier F_m ($m = \mathbb{1}_{[0,1]}$) is unbounded on $L^p(\mathbb{R}^d)$ for $p \neq 2$

Correct Formulation

Relation between F_m and uniform boundedness of $\sup_R T_{m_R}$ where $m_R(\lambda) = m(\lambda/R)$

Correct Formulation

Relation between F_m and uniform boundedness of T_{m_R} where $m_R(\lambda) = m(\lambda/R)$

Caveat: Currently only have complete proof when $\mathrm{supp}(m)$ is compact

Main Theorem (D., In Preparation)

If $1 < p < \frac{2(d-1)}{d+1}$, $\|F_m\|_{L^p \to L^p} \sim \|T_m\|_{L^p \to L^p}$

Result holds for any M with periodic geodesic flow

No other result giving Transference from $\mathbb{R}^d$ to M for any M and p

Main Theorem (D., In Preparation)

If $1 < p < \frac{2(d-1)}{d+1}$, $\|F_m\|_{L^p \to L^p} \sim \sup_R \|T_{m_R}\|_{L^p \to L^p}$

Transference from S^d to $\mathbb{R}^d$ (Mitjagin, 1974) implies that for all $1 \leq p \leq \infty$

$$\|F_m\| \lesssim \sup_R \|T_{m_R}\|$$

"Geometrically $F_m = \lim_{R \to \infty} T_{m_R}$"

There are reasons to believe **Main Theorem** Fails when extended to all M and p

Relation To Radial Multiplier Characterizations

Very Difficult, If Not Impossible Problem

Find a simple characterization of a s.t. $\|F_a\|_{L^p \to L^p} < \infty$

Radial Multiplier (Super) Conjecture

If a is radial, $\operatorname{supp}(m)$ is compact, $1 < p < \frac{2d}{d+1}$ then

$$\|F_a\|_{L^p \to L^p} \sim \|\hat{a}\|_{L^p}$$

(Heo–Nazarov–Seeger, 2011) True if $1 < p < \frac{2(d-1)}{d+1}$

Main Theorem (D., In Preparation)

If $1<p<\frac{2(d-1)}{d+1}$, $\|F_m\|_{L^p\to L^p} \sim \sup_R \|T_{m_R}\|_{L^p\to L^p}$

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Proof of Main Theorem follows by proving analog of HNS

$$\sup_R \|T_{m_R}\| \lesssim \|\widehat{m(|\cdot|)}\|_{L^p(\mathbb{R}^d)}$$

$$\|\widehat{m(|\cdot|)}\|_{L^p} \lesssim \|F_m\| \lesssim \sup_R \|T_{m_R}\| \lesssim \|\widehat{m(|\cdot|)}\|_{L^p(\mathbb{R}^d)}$$

$F_m(\delta) = \widehat{m(|\cdot|)} * \delta = \widehat{m(|\cdot|)}$

Mitjagin

COROLLARIES OF MAIN RESULT

Main Theorem (D., In Preparation)

If $1<p<\frac{2(d-1)}{d+1}$, $\|F_m\|_{L^p\to L^p} \sim \sup_R \|T_{m_R}\|_{L^p\to L^p}$

Characterization of L^p Boundedness If $1<p<\frac{2(d-1)}{d+1}$ and supp(m) is compact

$$\sup_R \|T_{m_R}\|_{L^p\to L^p} \sim \|\widehat{m(|\cdot|)}\|_{L^p(\mathbb{R}^d)}$$

Partial Sums: If $1<p<\frac{2(d-1)}{d+1}$ and supp(m) is compact

$\lim_{R\to\infty} m(\sqrt{\lambda}/R) f_\lambda = f$ for all $f\in L^p(S^d)$

iff $m(0)=1$ and $\|\widehat{m(|\cdot|)}\|_{L^p}<\infty$.

$$\boxed{\sup_R \| T_{m_R} \| \lesssim \| \widehat{m(|\cdot|)} \|_{L^p}}$$

Prove using

- <u>Fourier Inversion</u>: $T_{m_R} = \int R\hat{m}(Rt)\, e^{2\pi i t\sqrt{-\Delta}}\, dt$
- <u>Lax–Hörmander Parametrix</u> for High-Frequency inputs

$$e^{2\pi i t\sqrt{-\Delta}}(x,y) \approx \int a(x,y,t,\zeta)\, e^{2\pi i \Phi(x,y,t,\zeta)}\, d\zeta$$

- Stationary Phase + <u>Rausch Triangle Comparison Theorem</u> applied to parametrix reduces question to incidence problem for geodesic annuli
- incidence results using <u>density decomposition</u>.

$$\boxed{\text{Thanks For Listening!}}$$

Better Posed Problem

Define $m_R(\lambda) = m(\lambda/R)$

(1) When is $\sup_R \|T_{m_R}\|_{L^p\to L^p} < \infty$?

(Mitjagin, 1974) implies $\|F_m\|_{L^p\to L^p} \lesssim \sup_R \|T_{m_R}\|_{L^p\to L^p}$.

"Geometrically $F_m = \lim_{R\to\infty} T_{m_R}$"

> **More Subtle Question**
>
> For which m do we know (2) $\sup_R \|T_{m_R}\|_{L^p\to L^p} \lesssim \|F_m\|_{L^p\to L^p}$

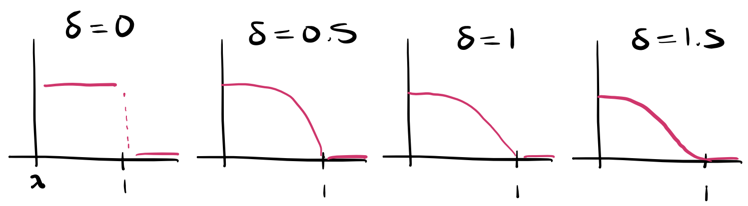
By UBP, if $m(0) = m(0+) = 1$, (1) holds iff

$$\lim_{R\to\infty} \sum m(\lambda/R) f_\lambda = f \quad \text{in } L^p(M)$$

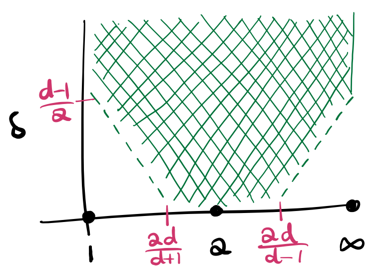
for all $f \in L^p(M)$.

Bochner Riesz

Let $m^\delta(\lambda) = (1-\lambda^2)_+^\delta$.



> **Conjecture**
>
> F_{m^δ} is bounded on $L^p(\mathbb{R}^d)$ iff $(p,\delta) \in$



Note L^p boundedness implies $L^{p'}$ boundedness by duality.

w.l.o.g. will assume $1 \le p \le 2$ in later slides

$$\|T_{m^\delta} f\|_{L^p(M)} = \left\|\sum m^\delta(\lambda) f_\lambda\right\|_{L^p(M)}$$
$$\leq \sum m^\delta(\lambda) \, \|f_\lambda\|_{L^p(M)}$$
$$\leq \max_{0\leq\lambda\leq 1} \|f_\lambda\|_{L^p(M)}$$
$$\lesssim \|f\|_{L^p(M)}$$

Analogues of Basic $\mathbb{R}^d$ Results

More Subtle Question

For which m do we know (2) $\sup_R \|T_{m_R}\|_{L^p \to L^p} \lesssim \|F_m\|_{L^p\to L^p}$

- When $p=2$, $\sup_R \|T_{m_R}\|_{L^2\to L^2} = \sup_\lambda |m(\lambda)|$

  $\|F_m\|_{L^2\to L^2} = \|m\|_{L^\infty} = \text{ess. } \sup_\lambda |m(\lambda)|$

  So (2) only holds if $\sup_\lambda |m| \sim \text{ess. } \sup_\lambda |m|$.

- When $|\partial_\lambda^\alpha m(\lambda)| \lesssim \lambda^{-\alpha}$ for $|\alpha| \leq d/2+1$, (2) holds

  (Seeger / Sogge, 1989)

Main Theorem

Theorem (D., In Preparation)

If M has periodic geodesic flow, and $1 < p < \frac{2(d-1)}{d+1}$,

$$\sup_R \| T_m \|_{L^p \to L^p} \sim \| F_m \|_{L^p \to L^p}$$

Corollary Define $m_j(\lambda) = m(2^j \lambda) \chi(\lambda)$ for $\chi \in C_c^\infty(\mathbb{R}_+)$

$\sup_R \| T_m \|_{L^p \to L^p} < \infty$ iff $C_p(m) = \sup_j \int |\widehat{m}_j(t)|^p (1+|t|)^{(d-1)(1-p/2)} dt$ is finite. Thus complete classification of m for which (2) holds.

Conjecture: Theorem extends to $1 < p < \frac{2d}{d+1}$.