Research Statement

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I am an analyst who studies problems mainly using techniques from the field of harmonic analysis, but also some methods of combinatorics and probability theory. My research over the past few years has focused on the study of radial Fourier multiplier operators on Euclidean space, and their analogues on compact manifolds, through an understanding of the geometry and regularity of wave propagation. In addition, I have explored problems in geometric measure theory, investigating when 'structure' occurs in fractals of large dimension. Both areas of research have raised interesting questions which I plan to pursue in my postgraduate work.

During my PhD, my work on multipliers has concentrated on the relation between L^p bounds for Fourier multipliers, and L^p bounds for analogous operators on compact manifolds. My main achievement was a *transference principle* [5] between bounds for radial multipliers and bounds for multiplier operators for spherical harmonics expansions on the sphere S^d. For $d \geq 4$ and a range of L^p spaces, this principle says that the L^p boundedness of a radial Fourier multiplier implies the L^p boundedness of the multiplier on S^d with the same symbol; thus L^p bounds 'transference' from $\mathbb{R}^d$ to S^d. In addition, I completely characterized those symbols whose dilates give a uniformly bounded family of multiplier operators on $L^p(S^d)$. Both results are the first of their kind for multipliers on S^d for $p \neq 2$; more broadly, no comparable results have been established for analogous multiplier operators on any other compact manifold. More detail about this project can be found in Section 1 of this summary.

My work in geometric measure theory focuses on constructing sets of large fractal dimension avoiding certain point configurations. Before starting my PhD, I had worked with Malabika Pramanik and Joshua Zahl to construct sets with large Hausdorff dimension avoiding certain point configurations [6]. During my PhD, I continued this line of research by combining the methods of that paper with more robust probabilistic machinery to address the more difficult problem of constructing sets with large Fourier dimension avoiding configurations [4]. This method remains the only method of constructing sets of large Fourier dimension avoiding non-linear configurations, and remains the best current method for constructing sets avoiding general 'linear' point configurations when $d > 1$. This work is discussed further in Section 2.

In the near future, I hope to generalize the bounds obtained in [5] to the more general setting of multipliers associated with the Laplace-Beltrami operator on Riemannian manifolds with periodic geodesic flow. A single scale obstruction to this generalization is obtaining control over iterates of a pseudodifferential operator on M called the 'return operator'. I am also interested of obtaining bounds on manifolds whose geodesic flow has well-controlled dynamical properties, such as forming an integrable system. Related to my work in geometric measure theory, I hope to apply the probabilistic methods I exploited in the construction of sets of large Fourier dimension to construct random fractals which exhibit good $l^2 L^p$ decoupling properties. And I am interested in determining the interrelation of patterns with the study of multipliers on manifolds, in particular studying Falconer distance problems on Riemannian manifolds to local smoothing phenomena for the wave equation on manifolds. A more detailed justification for the potential of these projects can be found in Section 3 of this statement.

The remaining sections of this summary provides context and describes the results I have obtained during my PhD in further detail, finishing with a further elaboration of future work and it's feasibility given the tools I have gained from my previous work.

1 Multiplier Operators on Euclidean Space and on Manifolds

This section provides background on endpoint L^p bounds for multipliers, which motivate a more detailed discussion of my contributions to the subject.

1.1 Radial Fourier Multipliers

Multipliers have long been central objects in harmonic analysis. In his pioneering work, Fourier showed solutions to the classical equations of physics are described by Fourier multipliers, operators T defined by a function $m : \mathbb{R}^d \to \mathbb{C}$, the symbol of T, such that

$$Tf(x) = \int_{\mathbb{R}^d} m(\xi)\widehat{f}(\xi)e^{2\pi i\xi\cdot x}\,dx.$$

Of particular interest are the *radial* multipliers, whose symbol is a radial function. We denote the radial multiplier with symbol $m(\xi) = a(|\xi|)$ by T_a in the sequel. Any translation-invariant operator on $\mathbb{R}^d$ is a Fourier multiplier operator, explaining their broad applicability in areas as diverse as partial differential equations, number theory, complex variables, and ergodic theory.

In harmonic analysis, it has proved incredibly profitable to study the boundedness of Fourier multipliers with respect to various L^p norms. It seems to be one of the few tractable ways of quantifying the interactions of planar waves and those functions formed from superpositions of such waves, thus underpinning all deeper understandings of the Fourier transform. The need for a general understanding of the L^p boundedness properties of Fourier multipliers became of central interest in the 1960s, brought on by the spur of applications the Calderon-Zygmund school and their contemporaries brought to the theory. Certain conditions on a symbol were found which ensured L^p boundedness, but finding necessary and sufficient conditions on a Fourier multiplier's symbol which guarantee L^p boundedness proved to be an impenetrable, if not potentially impossible problem. No results were obtained in the past half century, aside from trivial cases where $p \in \{1, 2, \infty\}$.

It thus came as a surprise when several recent arguments [2, 8, 10, 12] emerged giving necessary and sufficient conditions on a symbol a for the *radial* Fourier multiplier T_a to be bounded on $L^p(\mathbb{R}^d)$. Consider a decomposition $a(\rho) = \sum a_k(\rho/2^k)$, where $a_k(\rho) = 0$ for $\rho \notin [1, 2]$. For $1 \leq p \leq 2$, in order for T_a to be bounded on $L^p(\mathbb{R}^d)$, testing by Schwartz functions reveals it is necessary that $\sup_j C_p(a_j) < \infty$, where

$$C_p(a) = \left(\int_0^\infty \left[(1 + |t|)^{(d-1)(1/p-1/2)} \widehat{a}(t) \right]^p \, dt \right)^{1/p}.$$

Duality implies the boundedness of T_a on $L^p(\mathbb{R}^d)$ is equivalent to it's boundedness on $L^{p'}(\mathbb{R}^d)$ when $1/p+1/p' = 1$, and so for $2 \leq p \leq \infty$ it is natural to define $C_p(a) = C_{p'}(a)$. Using Bochner-Riesz multipliers as endpoint examples, it is natural to conjecture the condition $\sup_j C_p(a_j) < \infty$ is not only necessary, but also *sufficient* to guarantee L^p boundedness for $|1/p - 1/2| > 1/2d$. For radial input functions this conjecture is true [8], though resolving this conjecture for general inputs is likely far beyond current research techniques, given that it implies the Bochner-Riesz conjecture, and thus also the restriction and Kakeya conjectures.

Heo, Nazarov, and Seeger [10] have proved the conjecture for $d \geq 4$ and $|1/p - 1/2| > (d-1)^{-1}$. Cladek [2] improved the range of the conjecture for compactly supported a. She proved the result when $d = 4$ and $|1/p-1/2| > 11/36$ and when $d = 3$ and $|1/p-1/2| > 11/26$. Also of note is the work of Kim [12], who extended the bounds of [10] to *quasi-radial multipliers*, Fourier multipliers with a symbol which is homogeneous, smooth, and non-vanishing, and whose level sets are hypersurfaces of non-vanishing Gauss curvature. Nonetheless, the full conjecture remains unsolved for all $d \geq 2$.

It is a classic heuristic that the *smoothness* of the symbol of a multiplier determines it's L^p boundedness. The bound $\sup_j C_p(a_j) < \infty$ can be viewed in some sense as such a condition, but is not equivalent to the boundedness of any Sobolev, Besov, or Triebel-Lizorkin norm. However, we do have bounds of the form $\|a\|_{A_p(\mathbb{R})} \lesssim C_p(a) \lesssim \|a\|_{B_p(\mathbb{R})}$, where

$$A_p(\mathbb{R}) := B^{(d-1)(1/p-1/2)}_{p,\infty}(\mathbb{R}) \quad \text{and} \quad B_p(\mathbb{R}) := B^{d(1/p-1/2)}_{2,p}(\mathbb{R}).$$

Thus $C_p(a) < \infty$ is a property slightly stronger than saying a has $(d-1)(1/p-1/2)$ derivatives in L^p, but which is implied if a has $d(1/p-1/2)$ derivatives in L^2. One could conjecture that for $|1/p - 1/2| > 1/2d$, the operator T_a is bounded on $L^p(\mathbb{R}^d)$ if $\sup_j \|a_j\|_{B_p(\mathbb{R})} < \infty$. This conjecture only gives necessary, not sufficient, conditions for boundedness, but the proof might be more accessible to current available techniques. Indeed, this weaker conjecture has been verified by Lee, Rogers, and Seeger [15] to be true for all $d \geq 2$ in the larger Stein-Tomas range $|1/p - 1/2| > (d+1)^{-1}$.

We remark that various high powered techniques have recently been developed towards an understanding of the Bochner-Riesz conjecture, such as broad-narrow analysis, decoupling, and the polynomial method. However, these methods are difficult to apply when studying the two conjectures introduced above, since they are *endpoint results*. More precisely, in arguments related to the Bochner-Riesz conjecture and related results, one allows for inequalities to have a multiplicative loss of factors of the form R^ε or $\log R$, where R is the frequency scale of the analysis. This is negligible to the analysis, since the Bochner-Riesz multipliers are conjectured to be bounded on L^p for an *open* interval of exponents, and so methods involving interpolation between L^p spaces allow us to remove these multiplicative factors when making conclusions. But an arbitrary multiplier bounded on $L^p(\mathbb{R}^d)$ may not be bounded on $L^{p'}(\mathbb{R}^d)$ for any $p' < p$, and so such methods are unavailable to us in the study of these conjectures, explaining the limited range in which the conjectures have been verified.

1.2 Multipliers For Spherical Harmonic Expansions on S^d

A theory of multiplier operators analogous to Fourier multipliers can be developed on the sphere S^d. Roughly speaking, Fourier multipliers are operators on $\mathbb{R}^d$ with $e^{2\pi i \xi \cdot x}$ as eigenfunctions. Zonal multipliers on S^d are those operators with the *spherical harmonics* as eigenfunctions, i.e. the restrictions to S^d of homogeneous harmonic polynomials on $\mathbb{R}^{d+1}$. Every function $f \in L^2(S^d)$ can be uniquely expanded as $\sum_{k=0}^\infty H_k f$, where $H_k f$ is a degree k spherical harmonics. A multiplier for spherical harmonic expansions on S^d is then an operator defined in terms of a function $a : \mathbb{N} \to \mathbb{C}$ by setting

$$Z_a f = \sum_{k=0}^\infty a(k) H_k f.$$

For purposes of brevity, we will call such operators 'multipliers on S^d' in the sequel. Every rotation invariant operator on S^d is a multiplier, and thus such operators arise in diverse applications, including celestial mechanics, physics, and computer graphics.

Classic methods for studying multipliers on S^d involve the analysis of special functions and orthogonal polynomials. But in the 1960s Hörmander introduced the powerful theory of Fourier integral operators to the study of such multipliers, which allows one to apply more modern techniques of harmonic analysis the theory. This theory is more robust in other senses, applying to the study of multipliers of a first order self-adjoint pseudodifferential operator on a compact manifold, which we briefly outline. Given such an operator P on a manifold M, if Λ is the set of eigenvalues of P, then every function $f \in L^2(M)$ has an orthogonal decomposition $f = \sum_{\lambda \in \Lambda} f_\lambda$ where $P f_\lambda = \lambda f_\lambda$. For any symbol $a : \Lambda \to \mathbb{C}$, we define

$$a(P) f = \sum_{\lambda \in \Lambda} a(\lambda) f_\lambda.$$

We note that if Δ is the Laplace-Beltrami operator on S^d, then for any spherical harmonic f of degree k, $Pf = k(k+d-1)f$. Thus if $P = \sqrt{\alpha^2 - \Delta}$, where $\alpha = (d-1)/2$, then $Pf = kf$, and so any multiplier Z_a on S^d can also be written using the notation above as $a(P)$. In this general setup, Hörmander's idea was to use the Fourier inversion formula to write

$$a(P) = \int \widehat{a}(t) e^{2\pi i t P}\, dt,$$

The multiplier operators $e^{2\pi i t P}$, as t varies, give solutions to the *half-wave equation* $\partial_t = iP$ on the manifold M, whose solutions allow one to describe solutions to the full wave equation $\partial_t^2 - P^2 = 0$. Thus the study of the boundedness of the operator $a(P)$ is connected to the regularity of the wave equation on M.

Using this reduction, Hörmander was able to prove the L^p boundedness of Bochner-Riesz multipliers [11]. Sogge [21, 22] significantly improved these bounds, introducing the approach, which works within the Stein-Tomas range, of reducing the problem to $L^2(M) \to L^p(M)$ bounds for spectral projection operators on M. Recently, Kim [13] adapted Sogge's approach to analyze multipliers of an operator P satisfying the following assumption:

Assumption A: If $p_{\text{prin}} : T^*M \to [0, \infty)$ is the principal symbol of P, then for each $x \in M$ the 'cosphere' $S^*_x = \{\xi \in T^*_x M : p_{\text{prin}}(x, \xi) = 1\}$ has non-vanishing Gaussian curvature.

Kim proved that under Assumption A, in the Stein-Tomas range $|1/p - 1/2| > (d+1)^{-1}$, if $\sup_j \|a_j\|_{B_p(\mathbb{R})} < \infty$, then the operator $a(P)$ is bounded on $L^p(M)$, thus obtaining an analogue of the result of Lee, Rogers and Seeger in this setting. In particular, Assumption A is satisfied when $P = \sqrt{\alpha^2 - \Delta}$ on S^d, since the cospheres of P are ellipses, and so Kim's result applies to multipliers on S^d. However, there are no results in the literature which show that an operator $a(P)$ is bounded on $L^p(M)$ if $\sup_j C_p(a_j) < \infty$, for any p and any manifold M. *The main goal of my research project was to remedy this.*"

1.3 My Contributions To The Study of Multipliers

As mentioned above, the main goal of my PhD research into multipliers was to obtain analogues of the arguments of [2, 10, 12] for multipliers on S^d, i.e. proving that for some range of p and all functions a, if $\sup_j C_p(a_j) < \infty$, then Z_a is bounded on $L^p(S^d)$. I was able to able to obtain such analogues. Moreover, the argument is somewhat robust, applying to multipliers for a range of different operators P that satisfy Assumption A and the following additional assumption:

Assumption B: The eigenvalues of P are contained in an arithmetic progression.

When $P = \sqrt{\alpha^2 - \Delta}$ on S^d, all eigenvalues are positive integers, so assumption B is satisfied. The assumption also holds more generally for multipliers on the *rank one symmetric spaces* $\mathbb{RP}^d$, $\mathbb{CP}^d$, $\mathbb{HP}^d$, and $\mathbb{OP}^2$, i.e. operators diagonalized by analogous functions to the spherical harmonics on these spaces. Nonetheless, Assumption B is less natural than Assumption A, and I hope to obtain bounds under weaker assumptions in future work. Under Assumption A and Assumption B, in [5] I proved a 'single scale' analogue of the bound of Heo, Nazarov and Seeger; for a function a supported on $[1, 2]$, and for $|1/p - 1/2| > 1/d$, I prove that uniformly in R,

$$\|a(P/R)f\|_{L^p(M)} \lesssim C_p(a)\|f\|_{L^p(M)}.$$

In a paper to be submitted for publication shortly, I provide further arguments justifying that for an arbitrary function a, the operator $a(P)$ is bounded on $L^p(M)$ if $\sup_j C_p(a_j) < \infty$, thus obtaining a complete analogue of the argument of [10] for multipliers on S^d.

An important corollary of this result is a *transference principle* between Fourier multipliers and multipliers on S^d. Since the condition $\sup_j C_p(a_j)$ is necessary for T_a to be bounded on $L^p(\mathbb{R}^d)$, we conclude that for $|1/p - 1/2| > 1/d$, if T_a is bounded on $L^p(\mathbb{R}^d)$, then the multiplier $a(P)$ is bounded on $L^p(M)$. Aside from the study of Fourier multipliers on $\mathbb{R}^d$, this is the first transference principle of this kind. There are no results in the literature for any $p \neq 2$, any other compact manifold M, and any operator P which guarantee that $a(P)$ is bounded on $L^p(M)$ if T_a is bounded on $L^p(\mathbb{R}^d)$.

Another corollary is a characterization of the symbols a such that multipliers of the form $\{a(P/R) : R > 0\}$ are uniformly bound on $L^p(M)$. If $\sup_j C_p(a_j) < \infty$, then the results above imply that the operators $a(P/R)$ are uniformly bounded on $L^p(M)$, because the quantity $\sup_j C_p(a_j)$ changes by at most a constant when we dilate a by a factor of R. The converse follows from a classic result of Mitjagin [19]. The uniform boundedness principle implies that a function a satisfies $\lim_{R\to\infty} a(P/R)f = f$ for all $f \in L^p(M)$, where the limit is taken in $L^p(M)$, if and only if $a(0) = 1$ and $\sup_j C_p(a_j) < \infty$. As for the transference principle above, these results are the first of their kind for any $p \neq 2$ and any other compact manifold M.

As mentioned above, [5] only covers a 'single frequency scale' analogue of the results of [10]. The argument in [10] for combining scales involves decomposing inputs using an L^∞ atomic decomposition inspired by the decompositions of Chang and Fefferman [1], and then controlling the interactions between different frequency scales using certain inner product estimates. We have obtained analogues of these inner product estimates, discussed below, and the atomic decomposition method generalizes to an arbitrary compact manifold, and so we expect to submit a paper describing these methods, and obtaining the full method above, shortly.

I would like to finish this section by emphasizing two novel methods I introduced in [5]. Before this, I would like to briefly describe the approach of [10] for bounding radial multipliers, which heavily inspired the method of [5]:

Let T be a radial multiplier. Then we can write $Tf = k * f$, where k is the Fourier transform of the symbol of a. We write $k = \sum k_\tau$ and $f = \sum f_\theta$, where the functions $\{k_\tau\}$ are supported on disjoint annuli supported at the origin, and the functions $\{f_\theta\}$ are supported on disjoint cubes. Then $Tf = \sum_{\tau,\theta} k_\tau * f_\theta$. Using the Fourier transform and Bessel functions, we can justify that the inner product $\langle k_\tau * f_\theta, k_{\tau'} * f_{\theta'} \rangle$ is negligible unless the annulus of radius τ centered at θ is near tangent to the annulus of radius τ' centered at θ' at some point. Combining this inner product estimate with a 'sparse incidence argument' for such annuli, one can show that the L^2 norm of a sum $\sum_{(\tau,\theta)\in\mathcal{E}} k_\tau * f_\theta$ is well behaved if $\mathcal{E}$ is a suitably sparse set. Interpolation with a trivial L^1 estimate yields an L^p estimate on the sum. Conversely, if the set $\mathcal{E}$ is clustered, then $\sum_{(\tau,\theta)\in\mathcal{E}} k_\tau * f_\theta$ will be concentrated on only a few annuli, and so we can also get good L^p estimates simply using pointwise estimates. But then $\|Tf\|_{L^p(\mathbb{R}^d)} = \|\sum k_\tau * f_\theta\|_{L^p(\mathbb{R}^d)}$ can be estimated, depending on whether a sparse or clustered part of the sum dominates.

The main difficulty of adapting this approach to the study of the multipliers $a(P)$ is in obtaining the analogous inner product estimates on manifolds. The natural approach here is to use the Lax-Hörmander parametrix for the wave equation, which reduces our inner product estimates *for small* τ to a bound for oscillatory integrals. One problem is that the phase of this integral is non-explicit, given by a solution to a partial differential equation on M. One novel approach I made in [5] was making the observation that if Assumption A holds, then P gives an implicit geometric structure to the manifold M, turning it into a *Finsler manifold*. The phase of the oscillatory integral occurring from the Lax-Hörmander parametrix is then directly related to the length of certain geodesics on this Finsler manifold, and using the Finsler analogue of the second variation formula for geodesics, I was able to obtain the required inner product estimates that occur in [10] for small τ. Such estimates apply to multipliers of an arbitrary pseudodifferential operator P satisfying Assumption A, and likely have applications in other problems.

The inner product estimates above are sufficcient to obtain bounds for small τ, but for large τ this approach fails as the Lax-Hörmander parametrix breaks down past the *injectivity radius* of the manifold M, preventing us from applying a direct analogue of the arguments of [10]. Similar problems emerge in other approaches to the study of multipliers on manifolds. This was the impetus for Sogge's method, which reduced bounds for Bochner-Riesz multipliers to the study of $L^p \to L^2$ bounds for spectral projection operators on M, used in [22] and [13]. However, we cannot use this method in this problem, since the method intially involves using the estimate $\|a(P)f\|_{L^p(M)} \lesssim \|a(P)f\|_{L^2(M)}$, which is too inefficient in our problem; Sobolev embedding heuristics suggest this inequality incurs a lost of $1/p - 1/2$ derivatives, which is fine for obtaining bounds under the assumption that the functions a_j uniformly have $d(1/p - 1/2)$ derivatives in L^2, as in [13], but not for $\sup_j C_p(a_j) < \infty$, when the functions a_j are only guaranteed to have $(d-1)(1/p - 1/2)$ derivatives in L^p.

I was able to work around this problem by reducing the required bounds to certain $L^p_x L^p_t$ estimates for the wave equation on the manifold. Such an argument behaves somewhat like Sogge's spectral projection argument, but does not involve a switch to L^2 norms, avoiding the problems of such an approach. The catch is that $L^p_x L^p_t$ estimates for the wave equation, related to the phenomenon of local smoothing on manifolds, are not as well understand as spectral projectors. This is why we must assume the rather strict Assumption B, which implies that the propagators $e^{2\pi i t P}$ are periodic in t, simplifying these estimates. As discussed later, in future work I hope investigate ways to weaken Assumption B.

2 Configuration Avoidance

How large must a set $X \subset \mathbb{R}^d$ be before it must contain a certain point configuration, such as three points forming a triangle congruent to a given triangle, or four points forming a parallelogram? Problems of this flavor have long been studied in combinatorics, such as when X is restricted to a discrete set, such as the grid $\{1, \dots, N\}^d$. In the last 50 years, analysts have also begun studying analogous problems for infinite subsets $X \subset \mathbb{R}^d$, where the size of X is measured via a suitable *fractal dimension*, one of various different numerical statistics which measure how 'spread out' X is in space. The most common fractal dimension in use is the *Hausdorff dimension* of a set X, but we also consider the *Fourier dimension* as a refinement of Hausdorff dimension which takes into account more subtle behavior of X related to it's correlation with the planar waves $e^{2\pi i \xi \cdot x}$ for $\xi \in \mathbb{R}^d$.

Unlike many other problems in harmonic analysis, we often do not have good expected *lower* bounds for the dimension at which configurations must appear. For instance, we do not know for $d > 2$ how large the Hausdorff dimension a set $X \subset \mathbb{R}^d$ must be before it contains all three vertices of an isosceles triangle, the threshold being somewhere between $d/2$ and $d-1$. Similarly, for a fixed angle $\theta \in (0, \pi)$, we do not know how large the Hausdorff dimension of X must be contains three distinct points A, B, and C which when connected determine an angle ABC equal to θ. If $\cos^2\theta$ is rational, the results of Máthe [18] and Harangi, Keleti, Kiss, Maga, Máthe, Mattila, and Strenner [9] imply the threshold is somewhere between $d/4$ and $d-1$. If $\cos^2\theta$ is irrational, the threshold is somewhere between $d/8$ and $d-1$. We should not even necessarily expect currently known lower bounds to be the 'correct bounds' in these problems, as we do with other problems in harmonic analysis, such as the restriction conjecture and the Falconer distance problem; Until recently, certain results due to Łaba and Pramanik [14] seemed to imply that subsets of $[0, 1]$ of Fourier dimension one must necessarily contain an arithmetic progression of length three, but Schmerkin has shown this need not be the case [20].

Given that we do not have good lower bounds with which to make definite conjectures, it is of interest to find general methods that we can use to produce counterexamples in these types of problems. That is, we wish to find methods with which to construct sets with large fractal dimension that *do not* contain certain point configurations. My research in geometric measure theory has so far focused on finding these types of methods.

2.1 A Review of Hausdorff Dimension and Configuration Avoidance

Let us consider a model problem for pattern avoidance; given a fixed function $f : (\mathbb{R}^d)^n \to \mathbb{R}^m$, how large must the dimension of a set X be to guarantee that there exists $x_1, \dots, x_n \in X$ such that $f(x_1, \dots, x_n) = 0$. We focus on finding lower bounds for this problem, constructing sets X with large Hausdorff or Fourier dimension such that X *avoids the zeroes* of f, in the sense that for any distinct points $x_1, \dots, x_n \in X$, $f(x_1, \dots, x_n) \neq 0$. This model has been considered in various contexts:

(A) If $m = 1$, and f is a polynomial of degree n with rational coefficients, Máthe [18] constructs a set with Hausdorff dimension d/n avoiding the zeroes of f.

(B) If f is a C^1 submersion, Fraser and Pramanik [7] constructs a set with Hausdorff dimension $m/(n-1)$ avoiding the zeroes of f.

(C) If the zero set $f^{-1}(0)$ has Minkowski dimension at most s, I, together with my Master's thesis advisors Malabika Pramanik and Joshua Zahl [6] constructed sets of Hausdorff dimension $(dn - s)/(n - 1)$ avoiding the zeroes of f.

(D) If f can be factored as $f = g \circ T$, where $T : (\mathbb{R}^d)^n \to \mathbb{R}^l$ is a full-rank, rational coefficient linear transformation, and $g : \mathbb{R}^l \to \mathbb{R}^m$ is a C^1 submersion, then I [5] have constructed a set with Hausdorff dimension m/l avoiding the zero sets of f.

Notice that the above four methods only construct sets with large *Hausdorff dimension* avoiding patterns. They say nothing about constructing sets with large Fourier dimension, which in general is a much harder problem involving a delicate interplay between 'randomness' and 'structure'. Most 'structured' sets have low Fourier dimension,and so most methods for constructing sets with large Fourier dimension require making certain 'random choices' which on average do not correlate with any particular planar wave. Structure must be added to some degree to avoid containing a given configuration, but adding too much structure will likely add a high degree of correlation of your sets with certain planar waves, resulting in your set having Fourier dimension zero. Certain results have been obtained, however, for *linear* functions f:

(E) If $f(x_1, \dots, x_n) = a_1 x_1 + \dots + a_n x_n$ with $\sum a_j = 0$, Pramanik and Liang [17] construct a set $X \subset [0, 1]$ with Fourier dimension $\dim_{\mathbb{F}}(X) = 1$ avoiding the zeroes of f. This generalizes a construction of Schmerkin [20], who proved the result in the special case where $f(x_1, x_2, x_3) = (x_3 - x_1) - 2(x_2 - x_1)$ detects arithmetic progressions of length 3.

(F) Körner constructed subsets $X \subset [0, 1]$ with Fourier dimension $(k - 1)^{-1}$ such that for any integers $m_0, \dots, m_k$, and any distinct $x_1, \dots, x_k \in X$, $a_0 \neq a_1 x_1 + \dots + a_n x_n$.

The focus on linear functions is natural, since the Fourier transform behaves in a predictable way with respect to linearity. On the other hand, the understanding of the Fourier transform with respect to other nonlinear phenomena is poorly understood. *The main goal of my research project was to find constructions of sets with large Fourier dimension avoiding the zeroes of a nonlinear functions f.*

2.2 My Contributions To The Study Of Configurations

It seems very difficult, if not impossible to adapt methods (A) and (D) above to construct sets with positive Fourier dimension, since the constructions involve constructing X at each spatial scale by choosing a good family of intervals, and then considering a large union of translates of the intervals along an arithmetic progression. This ensures a spread out family of intervals, and thus a set with large Hausdorff dimension. But it is not good for ensuring Fourier decay, since a function concentrated near an arithmetic progression must have a large Fourier coefficient at frequencies complementing the spacing of this progression. On the other hand, methods (B) and (C) involve mostly pigeonholing arguments, so they seem the most likely to be able to be adapted to the Fourier dimension setting. I was able to adapt some of the ideas of these methods to obtain such a result.

For simplicity, I focused on the case when $m = d$ and when the function f was C^1 and full rank, as assumed in [7]. Then by the implicit function theorem, after possibly rearranging indices, we can locally write $f(x_1, \dots, x_n) = x_1 - g(x_2, \dots, x_n)$ for a function $g : (\mathbb{R}^d)^{n-1} \to \mathbb{R}^d$. In [4], under the assumption that g was a submersion in each variable $x_2, \dots, x_n$, I was able to modify the construction of [7] to construct sets with Fourier dimension $d/(n-3/4)$ avoiding the zeroes of f. Under the further assumption that we can write $g(x_2, \dots, x_n) = ax_2 + h(x_3, \dots, x_n)$ for $a \in \mathbb{Q}$, I was able to construct sets with Fourier dimension $d/(n-1)$ avoiding the zeroes of f, recovering the Hausdorff dimension bound of [7] in the Fourier dimension setting.

As with most of the other approaches discussed above, we construct a set X avoiding zeroes via a 'Cantor-type construction'. Fix a parameter α. We iteratively define a nested family of sets $\{X_k\}$, each a union of cubes of some fixed length l_k, and define $X = \bigcap_k X_k$. The set X_{k+1} is obtained from X_k by partitioning X_k each sidelength l_k cube into N^d sidelength l_{k+1} cubes, where $N := l_k/l_{k+1}$, and letting X_{k+1} be formed from the union of a subcollection of these cubes. The construction in [6] and [4] is very simple: To construct X_{k+1} from X_k, we start by taking a set S by taking $\sim N^\alpha$ points uniformly at random from the centers of the sidelength l_{k+1} cubes in the partition of each sidelength l_k cube in X_k. Some points from this set will form near zeroes of the function f; we let

$$S_{\text{bad}} = \{x \in S : |f(x, x_2, \dots, x_n)| \leq 10 l_{k+1} \text{ for some } x_2, \dots, x_n \in S\},$$

and define X_{k+1} to be the union of all sidelength l_{k+1} cubes centered at points in $S - S_{\text{bad}}$. The set X will then avoid the zeroes of the function f. Provided that $\alpha \leq (nd-s)/(n-1)$, we have with high probability that $\#(S_{\text{bad}}) \ll \#(S)$, and so with high probability, at each stage of the construction X_k is a union of $\sim l_k^{-\alpha}$ cubes of sidelength α; it is therefore natural to expect the set X almost surely has Hausdorff dimension α, and indeed, in [6] this is shown this is the case.

Simply counting the number of cubes at each scale is not sufficient to obtain a Fourier dimension bound. In [4], I made the observation that the core feature of constructions that yield Fourier dimension bounds is that they must involve a *square root cancellation bound.* More precisely, if we denote the centers of the sidelength l_k cubes forming X_k by $\{x_1, \dots, x_M\}$, then for all $1 \lesssim |\xi| \lesssim N$ then the resulting set X will have Fourier dimension agreeing with it's Hausdorff dimension if the square root cancellation bound

$$\left| \frac{1}{M} \sum_{j=1}^{M} e^{2\pi i \xi \cdot x_j} \right| \lessapprox M^{-1/2} \tag{1}$$

holds at all scales. Indeed, consider the probability measure $\mu_k = M^{-1} \sum_{j=1}^M \chi_j$ supported on X_k, wjere χ_j is a smooth bump function adapted to the cube centered at x_j. Then for $|\xi| \lesssim 1/l_k$, since $M \sim l_k^{-\alpha}$ with high probability, (1) implies that $|\widehat{\mu}_k| \lessapprox M^{-1/2} \lesssim |\xi|^{-\alpha/2}$. On the other hand, the uncertainty principle implies that $\widehat{\mu}_k$ decays rapidly for $|\xi| \gtrsim 1/l_k$, and so $\widehat{\mu}_k$ has the appropriate Fourier decay required. Taking weak limits of the measures $\{\mu_k\}$, we find that $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-\alpha/2}$ has the right Fourier decay to justify that X has Fourier dimension α.

The necessity for square root cancellation bounds explains why random techniques often play a core role in the construction of sets with large Fourier dimension, since the phenomena of square root cancellation occurs in a plethora of random constructions, and probabilists have established many tools in the theory of *concentration of measure* to determine when a sum of random variables has square root cancellation away from the mean *with high probability.* If we are taking a sum of independent random variables, often Hoeffding's inequality gives sharp bounds ensuring square root cancellation. But in this case the random points $\{x_j, \dots, x_M\}$ are *not* chosen independently from one another. The initial set of points chosen to form the set S in the construction above are taken uniformly at random, but the points in the set $S - S_{\text{bad}}$ are no longer independent from one another. There are certain standard tools to handle this problem, such as McDiarmid or Azuma's inequality, though in this setting they fail to ensure square root cancellation unless α, which is not large enough for our purposes. In [4], I found a novel way to interlace Hoeffding and McDiarmid's inequality together to ensure square root cancellation away from the mean occurs with high probability for $\alpha \leq 1/(n-1)$.

After ensuring square root cancellation of the mean, the final problem is to show that the mean of $M^{-1} \sum e^{2\pi i \xi \cdot x_j}$ has square root cancellation, which proved to be the most inefficient aspect of the argument. This mean can be written as an oscillatory integral, though in M variables, and so usual techniques in the theory of oscillatory integrals fail to handle this bound since they are usually *dimension dependent*, and we need bounds uniform in M. Instead, I was able to use an inclusion exclusion argument, together with a Whitney decomposition of the thickened zero set of the function f to obtain the required bounds. This is the least optimal part of the argument, yielding a Fourier dimension of $d/(n-3/4)$ rather than $d/(n-1)$; however, if f satisfies a weak linearity a slight modification of the random construction ensures that the mean of $M^{-1} \sum e^{2\pi i \xi \cdot x_j}$ is always zero, yielding the large Fourier dimension bound $d/(n-1)$ in this case. I am interested in determining whether techniques in the theory can yield the dimension$d/(n-1)$ bound in general, though I do not think this is a good research project to pursue immediately given the availability of techniques available at the time.

3 Future Lines of Research

Given the context from the previous two sections, we finish this summary by describing in more detail several problems I believe may be accessible given the techniques I have used to solve previous problems.

3.1 Multipliers Associated With Periodic Geodesic Flow

In Section 2, I discussed that the results I were able to obtain for multipliers on S^d generalized to multipliers of an arbitrary first order, elliptic, self-adjoint pseudodifferential operator P on a compact manifold M, provided that P satisfied two assumptions. Assumption A relates to the curvature of the principal symbol, and this assumption cannot really be weakened without significantly changing the character of the results, which heavily depend on this curvature. On the other hand, Assumption B arises as an artifact of the methods of our proof. We can likely obtain similar bounds while weakening this assumptions; for instance, Kim [13] obtained bounds on the scale of Besov spaces without any assumptions other than Assumption A.

I doubt current research techniques allow us to completely removing Assumption B while still recovering the results of [5], a limitation of our current inability to understand the large time behavior of wave equations on compact manifolds. If we were able to follow the method of [3], which reduced the large time argument to a smoothing inequality for the wave equation, then the results of that paper would follow for another operator P if we could prove

$$\left\| \left(\int_k^{k+1} |e^{2\pi i t P} f|^{p'} \, dt \right)^{1/p'} \right\|_{L^{p'}(M)} \lesssim k^\delta \|f\|_{L^p_{d(1/p-1/2)-1/p'}(M)} \tag{2}$$

for some $\delta < (d-1)(1/p - 1/2) - 1/p'$. If P satisfies assumption B, then after rescaling, we may assume without loss of generality that all eigenvalues of P are integers, so that $e^{2\pi i k P} = I$ is the identity for all k, and then (2) holds for all $|1/p - 1/2| > (d-1)^{-1}$ and with $\delta = 0$ by the local smoothing inequality of Lee and Seeger [16].

Whether this bound is true in other contexts remains unknown. The next simplest case to consider would be when the operator P has the property that $e^{2\pi i k P}$ is *close* to the identity for all k. This happens precisely when the *Hamiltonian flow* on T^*M given by the vector field $H = (\partial p_{\text{prin}}/\partial\xi, -\partial p_{\text{prin}}/\partial x)$ is periodic, where p_{prin} is the principal symbol of P. Indeed, the theory of propagation of singularities in the study of Fourier integral operators then tells us that the operator $R = e^{2\pi i P}$ is a pseudodifferential operator of order zero, and it's principal symbol is related to an invariant of the flow known as the Maslov index. The operator has been studied a little by spectral theorists, and there it is known as the *return operator*. If we are able to justify bounds of the form

$$\|R^k f\|_{L^p_{d(1/p-1/2)-1/p'}} \lesssim k^\delta \|f\|_{L^p_{d(1/p-1/2)-1/p'}},$$

or a frequency localized variation of this bound, then the local smoothing inequality of Lee and Seeger yields (2). Such bounds are of interest since they cover all the operators $P = \sqrt{-\Delta}$, where Δ is the Laplace-Beltrami operator on a Riemannian manifold with periodic geodesic flow. They are even of interest on the sphere, since our method only allows us to tell when multipliers of the form $a(\sqrt{\alpha^2 - \Delta}/2^j)$ are uniformly bounded on $L^p(S^d)$, whereas these bounds would allow us to tell when the multipliers $a(\sqrt{-\Delta}/R)$ are uniformly bounded on $L^p(S^d)$.

3.2 Decoupling On Random Fractals

One major development in harmonic analysis over the past decade has been a greater understanding of the phenomenon of *decoupling*, or *Wolff-type estimates*. Given a family of almost disjoint subsets $\mathcal{E}_\delta = \{E_{\delta,j}\}$ of $\mathbb{R}^d$ parameterized by a small constant $\delta > 0$, $L^p(l^2)$ decoupling discusses bounds of the form

$$\Big\| \sum f_j \Big\|_{L^p(\mathbb{R}^d)} \leq D_p(\delta) \Big(\sum_j \|f_j\|_{L^p(\mathbb{R}^d)}^2 \Big)^{1/2},$$

where the Fourier transforms of the functions f_j lie on the sets E_j, and where $D_p(\delta)$ denotes the best constant under which this equation holds for all such $\{f_j\}$. A genuine *decoupling inequality* results when $D_p(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$ for all $\varepsilon > 0$, or the stronger logarithmic bound $D_p(\delta) \lesssim \log(\delta)$.

The relations between such constants when the families $\mathcal{E}_\delta$ are supported on transverse tubes associated with curved surfaces have been heavily studied (BLAH). More poorly understood is the analysis of decoupling on *fractal sets*. Fix a set X obtained from a Cantor like construction $\{X_j\}$ as discussed in Section 2.2. Define $\mathcal{E}_k$ to be the set of all side length l_k cubes which form X_k. Then consider the decoupling constants $D_p(k)$ associated with these sets. Some analysis has been done for fractal subsets of the parabola in the presence of certain arithmetic information on the fractal sets . But a general understanding is limited, especially for subsets of $\mathbb{R}$, in which case no 'curvature' in a standard sense can be present.

Just as decoupling bounds for surfaces are connected to restriction theory on that surface, decoupling bounds for fractals are connected to the study of fractal restriction estimates

$$\|\widehat{g\mu}\|_{L^p(\mathbb{R}^d)} \lesssim \|g\|_{L^q(\mathbb{R}^d,\mu)}.$$

Indeed,

3.3 Radial Multiplier Bounds And 'Fractal Weighted Estimates'

The main proof of the radial Fourier multiplier estimates in [2, 10, 12] involve a 'density decomposition'. We take a discretized set $\mathcal{E} \subset \mathbb{R}^{1+d}$, and use a Calderon-Zygmund type 'stopping argument' to decompose $\mathcal{E}$ into sets $\mathcal{E}_k$, where $\mathcal{E}_k$ has the property that $\#\{\mathcal{E}_k \cap B\} \leq 2^k \text{rad}(B)$ for all sufficiently small balls B. If one establishes L^2 estimates for such sparse sets, then this method allows one to obtain general L^p estimates for all sums using the decomposition method.

Recently I have seen various results in the literature on various problems which seem 'dual' to the sparse bounds above. In particular, BLAH has done work on fractal weighted estimates for the extension problem on the parabola, and BLAH has done work on fractal wiehgted estimates for the wave equation

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