Research Statement

Jacob Denson

I am an analyst, studying problems with techniques taken mainly from harmonic analysis, but also probability theory and incidence geometry. My PhD research, advised by Andreas Seeger, has focused on the study of radial Fourier multiplier operators on Euclidean space, and their analogues on compact manifolds, through an understanding of the geometry and regularity of wave propagation. I have also solved problems in geometric measure theory, investigating when 'structure' occurs in random fractals of large dimension. Both projects raise interesting questions I plan to pursue in my postgraduate work.

During my PhD, my work on multipliers has concentrated on relating L^p bounds for Fourier multiplier operators on $\mathbb{R}^d$ to L^p bounds for analogous operators on compact manifolds, such as the multiplier operators associated with spherical harmonic expansions on S^d. My main achievement, for $d \geq 4$, and a range of L^p spaces, is a complete characterization of functions whose dilates correspond to a uniformly bounded family of multiplier operators on $L^p(S^d)$. The result implies a 'transference principle', that the L^p boundedness of a radial Fourier multiplier operator implies the L^p boundedness of the multiplier operator on S^d given by the same symbol. The result for compactly supported functions m is found in [15], with the remaining part of the argument in preparation. *Both the characterization and transference principle are the first of their kind for multiplier operators on S^d for $p \neq 2$; more broadly, no comparable results have been established for analogous multiplier operators on any other compact manifold.* More detail about this project can be found in Section 1 of this statement.

My work in geometric measure theory focuses on constructing sets of large fractal dimension avoiding point configurations. Before starting my PhD, Malabika Pramanik, Joshua Zahl, and I constructed sets with large Hausdorff dimension avoiding point configurations [13, 16]. During my PhD, I combined the methods of that paper with more robust probabilistic machinery to address the more difficult problem of constructing sets with large Fourier dimension avoiding configurations [14]. *This method remains the only method of constructing sets of large Fourier dimension avoiding nonlinear configurations, and remains the best method for avoiding general 'linear' point configurations when $d > 1$.* This work is discussed further in Section 2.

In Section 3, I discuss my plans for future research, emphasizing how my PhD work gives me the tools to succeed in these plans. *The more concrete projects include improving the range of current bounds for radial Fourier multiplier operators, characterizing the L^p boundedness of more general multipliers on compact manifolds, and obtaining $l^2(L^p)$ decoupling bounds for random fractal subsets of $\mathbb{R}$, among several more exploratory projects involving extremizers and o-minimality.*

1 Multiplier Operators on Euclidean Space and on Manifolds

1.1 Radial Fourier Multiplier Operators

Fourier Multiplier operators have long been central objects of study in harmonic analysis. Such operators T are defined by a function $m : \mathbb{R}^d \to \mathbb{C}$, the 'multiplier' of T, by setting

$$Tf(x) = \int_{\mathbb{R}^d} m(\xi)\widehat{f}(\xi)e^{2\pi i\xi\cdot x}\,dx.$$

Of particular interest are the radial Fourier multiplier operators, defined by a radial function m. For a function $a : [0,\infty) \to \mathbb{C}$, we denote the radial multiplier operator given by $m(\xi) = a(|\xi|)$ by T_a. Any operator on $\mathbb{R}^d$ commuting with translations is a Fourier multiplier, and if in addition, the operator commutes with rotates, it is a radial Fourier multiplier operator.

In harmonic analysis, it has proved incredibly profitable to study the boundedness of Fourier multiplier operators with respect to various L^p norms. It seems to be one of the few tractable ways of quantifying the degree to which planar waves interact with one another, thus underpinning all deeper understandings of the Fourier transform. The L^p boundedness of a general multiplier operator became of central interest in the 1950s, brought on by the spur of applications the Calderon-Zygmund school and their contemporaries brought to the theory. Some sufficient conditions and some necessary conditions to ensure boundedness were found. But finding necessary and sufficient conditions which guarantee boundedness proved to be an impenetrable problem; such conditions for L^p boundedness are only known in simple cases where $p \in \{1,2,\infty\}$.

It thus came as a surprise when several arguments [9, 20, 22, 25] recently established necessary and sufficient conditions on a function a for a radial Fourier multiplier operator T_a to be bounded on $L^p(\mathbb{R}^d)$. Consider a decomposition $a(\rho) = \sum a_j(\rho/2^k)$, where $a_j(\rho) = 0$ for $\rho \not\in [1,2]$. For $1 \leq p \leq 2$, in order for T_a to be bounded on $L^p(\mathbb{R}^d)$, testing by Schwartz functions shows that $\sup_j \|\widehat{m}_j\|_{L^p(\mathbb{R}^d)} < \infty$ is necessary, where $m_j(\xi) = a_j(|\xi|)$. Garrigos and Seeger [20] show this is equivalent to $\sup_j C_p(a_j) < \infty$, where

$$C_p(a) = \left(\int_0^\infty \left| \langle t \rangle^{(d-1)(1/p-1/2)} \widehat{a}(t) \right|^p dt \right)^{1/p} \quad \text{and} \quad \langle t \rangle = (1+|t|^2)^{1/2}.$$

Using Bochner-Riesz operators as endpoint examples, it is natural to conjecture $\sup_j C_p(a_j) < \infty$ is not only necessary, but also sufficient to guarantee L^p boundedness for $1 < p < \frac{2d}{d+1}$. We call this conjecture the *radial multiplier conjecture*. For radial input functions this conjecture has been resolved by Garrigós and Seeger [20], though resolving this conjecture for general inputs is likely far beyond current research techniques, given that it implies the Bochner-Riesz conjecture, and thus also the restriction and Kakeya conjectures. Heo, Nazarov, and Seeger [22] proved the conjecture for $d \geq 4$ and $1 < p < \frac{2(d-1)}{d+1}$. Cladek [9] improved this range for compactly supported a when $d = 4$ and $1 < p < 36/29$, and when $d = 3$ and $1 < p < 13/12$. Also of note is the work of Kim [25], who extended [22] to more general 'quasi-radial multiplier operators'. Nonetheless, the full conjecture remains unresolved for all $d \geq 2$.

Various powerful techniques have recently been developed towards an understanding of the Bochner-Riesz conjecture, such as broad-narrow analysis, decoupling, and the polynomial method. However, these methods are difficult to apply in the radial multiplier conjecture. In these methods, one allows for inequalities to have a multiplicative loss of factors of the form R^ε, where R is the frequency scale of the analysis. This multiplicative loss is negligible since the Bochner-Riesz multipliers are conjectured to be bounded on L^p for an open interval of exponents, and so interpolation-based 'ε-removal' methods allow us to remove such factors. But an arbitrary multiplier bounded on $L^p(\mathbb{R}^d)$ may not be bounded on $L^q(\mathbb{R}^d)$ for any $q < p$, so such methods are unavailable in the study of general multipliers, partially explaining the limited range in which the conjecture has currently been verified. *Nonetheless, I have several ideas for improving the range of these conjectures, which I discuss in Section 3.*

1.2 Multipliers For Spherical Harmonic Expansions on S^d

A theory of multiplier operators analogous to Fourier multiplier operators can be developed on the sphere S^d. Roughly speaking, Fourier multiplier operators are operators diagonalized by the planar waves $e^{2\pi i \xi \cdot x}$. Multipliers on S^d are operators diagonalized by the spherical harmonics, i.e. the restrictions to S^d of homogeneous harmonic polynomials on $\mathbb{R}^{d+1}$. Every function $f \in L^2(S^d)$ can be uniquely expanded as $\sum_{k=0}^\infty H_k f$, where $H_k f$ is a degree k spherical harmonic, and a multiplier for spherical harmonic expansions on S^d is then an operator defined in terms of a function $a : \mathbb{N} \to \mathbb{C}$ given by $S_a = \sum_{k=0}^\infty a(k) H_k$. For purposes of brevity, we will call such operators 'multiplier operators on S^d'. Every rotation invariant operator on S^d is a multiplier operator of this kind.

A natural question is to characterize which functions a give multiplier operators S_a bounded on $L^p(S^d)$, but the fact that the operators are described by a discrete sum makes this problem quite different from the study of radial multipliers on $\mathbb{R}^d$. A more tractable question is to determine when the operators $S_R = \sum a(k/R) H_k$ are uniformly bounded on $L^p(S^d)$, since for large $R > 0$ these operators place more and more emphasis on high-frequency eigenfunctions, and heuristic evidence suggests such eigenfunctions can be understood using the geometrical techniques that have proved so useful to the study of analogous operators on $\mathbb{R}^d$. *I completely characterized those functions a which give rise to a uniformly* *bounded family of operators $\{S_R\}$ on a certain range of L^p spaces, and when $d \geq 4$.*

Classical methods for studying multiplier operators on S^d involve the analysis of special functions and orthogonal polynomials, e.g. in the work of Bonami and Clerc [4]. But in the 1960s, Hörmander introduced the powerful theory of Fourier integral operators to the study of such operators, which allows one to apply more modern techniques of harmonic analysis. This theory is more robust in other senses, applying to the study of multiplier operators associated with a first order self-adjoint pseudodifferential operator on a compact manifold, which we briefly outline. Given such an operator P on a manifold M with eigenvalues Λ, every function $f \in L^2(M)$ has an orthogonal decomposition $f = \sum_{\lambda \in \Lambda} f_\lambda$ where $Pf_\lambda = \lambda f_\lambda$. Given $a : \Lambda \to \mathbb{C}$, we define $a(P)f = \sum_{\lambda \in \Lambda} a(\lambda) f_\lambda$. We study multiplier operators on S^d by linking them to multiplier operators of a particular pseudodifferential operator P on S^d. If Δ is the Laplace-Beltrami operator on S^d, then for any spherical harmonic f of degree k, $\Delta f = k(k+d-1)f$. Thus if $P = (s_d^2 - \Delta) - s_d$, with $s_d = \frac{d-1}{2}$, then $Pf = kf$ for all degree k spherical harmonics f, and so for any function $a : [0,\infty) \to \mathbb{C}$, $S_a = a(P)$.

Hörmander's idea to studying the operator $a(P)$ was to use functional calculus to write

$$a(P) = \int \widehat{a}(t) e^{2\pi i t P}\, dt,$$

a form of the Fourier inversion formula. The multiplier operators $e^{2\pi i t P}$, as t varies, give solutions to the half-wave equation $\partial_t = iP$ on M. Thus the study of the boundedness of the operator $a(P)$ is connected to the regularity for averages of this wave equation on M, and in particular, to local smoothing inequalities for the wave equation.

Using this reduction, Hörmander [23] proved uniform bounds in $L^p(M)$ for the Bochner-Riesz multipliers $(1 - P^2/R^2)_+^\delta$ on M, later significantly improved by Sogge [38, 39] and Seeger and Sogge [36] for multipliers of an operator P satisfying the following assumption:

Assumption A: If $p_{\text{prin}} : T^*M \to [0,\infty)$ is the principal symbol of P, then for each $x \in M$ the 'cosphere' $S_x^* = \{\xi \in T_x^*M : p_{\text{prin}}(x,\xi) = 1\}$ has non-vanishing Gaussian curvature.

Note that when $P = (s_d^2 - \Delta)^{1/2} - s_d$ on S^d, the principal symbol is the Riemannian metric on T^*S^d, the cospheres are ellipses, and so Assumption A is satisfied. These bounds are obtained via the method of reducing the problem to $L^2(M) \to L^p(M)$ bounds for spectral projection operators on M, a method that allows one to uniformly bound the Bochner-Riesz multipliers on $L^p(M)$ in the Tomas-Stein range. Recently, Kim [26] adapted Sogge's approach to obtain certain necessary conditions ensuring $a(P)$ is bounded on $L^p(M)$ on the scale of Besov spaces. But these bounds are far from a complete characterization of boundedness; for instance, they do not imply the boundedness of the wave multipliers $(1+P)^{-(d-1)(1/p-1/2)} e^{2\pi i P}$ on $L^p(M)$. *The main goal of my research project was to find a complete characterizations of boundedness, which would be the first such result in the literature.*.

1.3 My Contributions To The Study of Multipliers

The main goal of my PhD research into multipliers was to obtain analogues of the arguments of [9, 22, 25] in the setting of compact manifolds, in particular for the operator $P = (s_d^2 - \Delta)^{1/2} - s_d$ on S^d which would characterize multiplier operators for spherical harmonics on S^d. I obtained such analogues for a range of different operators P that satisfy Assumption A and the following additional assumption:

Assumption B: The eigenvalues of P are contained in an arithmetic progression.

All eigenvalues of the operator P above are positive integers, so this assumption is satisfied on S^d. The assumption also holds more generally for multipliers on the 'rank one symmetric spaces' $\mathbb{RP}^d$, $\mathbb{CP}^d$, $\mathbb{HP}^d$, and $\mathbb{OP}^2$. The necessity of Assumption B comes from our inability to understand the large time behavior of the wave equation on compact manifolds precisely enough. *Nonetheless, I discuss in Section 3 potential methods for obtaining results under weaker assumptions*. Under Assumption A and Assumption B, in [15] I proved a 'single scale' analogue of the bound of Heo, Nazarov and Seeger.

Theorem. [15] *Suppose P is an elliptic pseudodifferential operator of order one on a compact manifold M, and the operator P satisfies Assumptions A and B. Then for a function a with $supp(a) \subset [1,2]$, and for $1 < p < 2(\frac{d-1}{d+1})$, uniformly in $R > 0$,*

$$\sup_{R>0} \|a(P/R)\|_{L^p(M) \to L^p(M)} \sim C_p(a).$$

In a paper to be submitted for publication shortly, I provide further arguments justifying that for an arbitrary function a, the operator $a(P)$ is bounded on $L^p(M)$ if $\sup_j C_p(a_j) < \infty$, thus obtaining a complete analogue of the argument of [22] for multiplier operators on S^d. The argument involves using an L^∞ atomic decomposition à la the decompositions of Chang and Fefferman [8] to control the interactions between different frequency scales.

An important corollary of the above theorem is a *transference principle* between Fourier multiplier operators and multiplier operators on S^d. Since the condition $\sup_j C_p(a_j)$ is necessary for T_a to be bounded on $L^p(\mathbb{R}^d)$, we conclude that for $|1/p - 1/2| > 1/d$, if T_a is bounded on $L^p(\mathbb{R}^d)$, then the multiplier $a(P)$ is bounded on $L^p(M)$. Thus bounds 'transfer' from $\mathbb{R}^d$ to S^d. Aside from the study of Fourier multipliers on $\mathbb{R}^d$, this is the first transference principle of this kind. *There are no results in the literature for any $p \neq 2$, any other compact manifold M, and any operator P which guarantee that $a(P)$ is bounded on $L^p(M)$ if T_a is bounded on $L^p(\mathbb{R}^d)$.*

Another important corollary is a characterization of the functions a which give convergent eigenfunction expansion in L^p, i.e. that a function a satisfies $\lim_{R\to\infty} a(P/R)f = f$ for all $f \in L^p(M)$, where the limit is taken in $L^p(M)$, if and only if $a(0) = 1$ and $\sup_j C_p(a_j) < \infty$. This immediately follows from the theorem above by the uniform boundedness principle. *As with the transference principle above, these results are the first of their kind for $p \neq 2$ and any other compact manifold M.*

The proof in [15] is an adaption of the argument of [22] for bounding radial Fourier multiplier operators. That argument involves writing a radial multiplier operator as a convolution $Tf = k * f$. We consider a decomposition $k = \sum k_\tau$ and $f = \sum f_\theta$, where the functions $\{k_\tau\}$ are supported on disjoint annuli supported at the origin, and the functions $\{f_\theta\}$ are supported on disjoint cubes, and thus we can write $Tf = \sum_{\tau,\theta} k_\tau * f_\theta$. Estimates guarantee that the inner products $\langle k_\tau * f_\theta, k_{\tau'} * f_{\theta'} \rangle$ are negligible unless the annulus of radius τ centered at θ is near tangent to the annulus of radius τ' centered at θ'. These inner product estimates are combined with a 'sparse incidence argument' for counting circle tangencies, which when interpolated with a stopping time argument, yields the required L^p bounds. The main difficulty in adapting this approach is the difficulty in obtaining analogous inner product estimates, and handling the case where τ is large (i.e. handling the long time behavior of the wave equation). We conclude this discussion by describing the two main techniques I developed to resolve these difficulties.

Let us start with the inner product estimates. A natural approach is to use the Lax-Hörmander parametrix for the wave equation, which reduces our inner product estimates for small τ to a bound for certain oscillatory integrals. But the phase of this integral that arises is non-explicit, given in terms of a solution to an eikonal equation on M. One novel aapproach I made in [15] was making the observation that if Assumption A holds, then P gives M an implicit geometric structure, turning it into a *Finsler manifold*. The phase of the oscillatory integral occurring from the Lax-Hörmander parametrix is then directly related to a problem about estimating the length of geodesics on this Finsler manifold, and using the Finsler analogue of the second variation formula, we obtain the required inner product estimates that occur in [22] for small τ. These inner product estimates apply to multipliers of an arbitrary pseudodifferential operator P satisfying Assumption A.

The inner product estimates above are sufficient to obtain bounds for small τ, but for large τ this approach fails, as the Lax-Hörmander parametrix breaks down past the injectivity radius of the manifold M, preventing us from applying direct analogues of the arguments in [22]. Similar problems emerge in other approaches to the study of multipliers on manifolds. This was the impetus for Sogge's method, used in [39] and [26], of studying Bochner-Riesz multipliers in the Tomas Stein ranges, which reduces the problem to the study of $L^p \to L^2$ bounds for spectral projection operators on M. We cannot use this method, since the method initially involves the use of appropriately localized estimates of the form $\|a(P)f\|_{L^p} \lesssim \|a(P)f\|_{L^2}$ in order to effectively apply orthogonality. This is too inefficient for the general multiplier operators we consider since additional regularity on a is necessary to eventually recover L^p bounds from L^2 bounds.

I was able to work around this problem by reducing the required bounds to certain $L^p_x L^p_t$ estimates for the wave equation on the manifold. Such an argument behaves somewhat like Sogge's spectral projection argument, but does not involve a switch to L^2, avoiding the problems with Sogge's approach. The catch is that $L^p_x L^p_t$ estimates for the wave equation, related to the phenomenon of local smoothing on manifolds, are not as well understand as spectral projectors. This is why we must assume the rather strict Assumption B, making such estimates feasible. *In future work I hope investigate ways to weaken Assumption B, and I discuss some feasible scenarios to weaken this assumption in Section 3.*

2 Configuration Avoidance

How large must a set $X \subset \mathbb{R}^d$ be before it must contain a certain point configuration, such as three points forming a triangle congruent to a given triangle, or four points forming a parallelogram? Problems of this flavor have long been studied in combinatorics, such as when X is restricted to a discrete set such as the grid $\{1,\dots,N\}^d$. In the last 50 years, analysts have also begun studying analogous problems for infinite subsets $X \subset \mathbb{R}^d$, where the size of X is measured via a suitable 'fractal dimension', one of various different numerical statistics which measure how 'spread out' X is in space. The most common fractal dimension in use is the Hausdorff dimension of a set X, but we also consider the Fourier dimension as a refinement of Hausdorff dimension which takes into account more subtle behavior of X related to its correlation with the planar waves $e^{2\pi i \xi \cdot x}$ for $\xi \in \mathbb{R}^d$.

Unlike many other problems in harmonic analysis, such as the Kakeya conjecture, we often do not have good expected lower bounds for the dimension at which configurations must appear. For instance, we do not know for $d > 2$ how large the Hausdorff dimension a set $X \subset \mathbb{R}^d$ must be before it contains all three vertices of an isosceles triangle, the threshold being somewhere between $d/2$ and $d-1$. Similarly, for a fixed angle $\theta \in (0,\pi)$, we do not know how large the Hausdorff dimension of X must be before it contains three distinct points A, B, and C which when connected determine an angle ABC equal to θ. Depending on whether $\cos^2\theta$ is rational or irrational, the results of Máthe [32] and Harangi, Keleti, Kiss, Maga, Máthe, Mattila, and Strenner [21] imply the threshold is respectively, either between $d/4$ and $d-1$, or between $d/8$ and $d-1$. We should not even necessarily expect currently known lower bounds to be the 'correct bounds' in these problems, as we do with other problems in harmonic analysis, such as the restriction conjecture and the Falconer distance problem; until recently, certain results due to Łaba and Pramanik [28] seemed to suggest that subsets of $[0,1]$ of Fourier dimension one must necessarily contain an arithmetic progression of length three, but Shmerkin has shown this need not be the case [37].

Given that we do not have good lower bounds with which to make definite conjectures, it is of interest to find sets with large fractal dimension that do not contain certain point configurations. My research in geometric measure theory has so far focused on these constructions, using quantitative methods from high dimensional probability theory.

2.1 A Review of Fractal Dimension and Configuration Avoidance

It is natural to define fractal dimension of a set in terms of the properties of measures supported on that set. The Hausdorff dimension of a set $X \subset \mathbb{R}^d$ is the least upper bound of the quantities s for which there exists a finite Borel measure μ supported on X which satisfies the 'ball condition' that $\mu(B_r) \lesssim r^s$ for all $r > 0$ and all radius r balls $B_r \subset \mathbb{R}^d$; the Fourier dimension is the least such s for which there exists μ with the decay estimate $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-s/2}$. Intuitively, for large s the ball condition implies that μ has mass 'spread out', and the Fourier decay also implies this, by forcing μ to be uncorrelated with the waves $e^{2\pi i \xi \cdot x}$ when ξ is large. The Hausdorff dimension is always larger than the Fourier dimension, but this inequality is often strict, the Fourier decay capturing more subtle information about the set X, and this means it is often much harder to construct sets with large Fourier dimension avoiding configurations than sets with large Hausdorff dimension avoiding configurations.

We consider a model problem for pattern avoidance; given a fixed function $f : (\mathbb{R}^d)^n \to \mathbb{R}^m$, we construct sets $X \subset \mathbb{R}^d$ of large dimension which avoid the zeroes of f, in the sense that for any distinct points $x_1,\dots,x_n \in X$, $f(x_1,\dots,x_n) \neq 0$. This model has been considered in various contexts:

(A) If $m = 1$, and f is a polynomial of degree n with rational coefficients, Máthe [32] constructs a set with Hausdorff dimension d/n avoiding the zeroes of f.

(B) If f is a C^1 submersion, Fraser and Pramanik [18] constructs a set with Hausdorff dimension $m/(n-1)$ avoiding the zeroes of f.

(C) If $f^{-1}(0)$ has Minkowski dimension at most s, together with my Master's Thesis advisors Malabika Pramanik and Joshua Zahl [16], I constructed sets of Hausdorff dimension $(dn-s)/(n-1)$ avoiding the zeroes of f.

(D) If f can be factored as $f = g \circ l$, where $l : (\mathbb{R}^d)^n \to \mathbb{R}^k$ is a full-rank, rational coefficient linear transformation, and $g : \mathbb{R}^k \to \mathbb{R}^m$ is a C^1 submersion, then in work with Pramanik and Zahl described in my Master's Thesis [15], we constructed sets of Hausdorff dimension m/k avoiding the zeroes of f.

Notice that the above four methods only construct sets with large Hausdorff dimension avoiding patterns. They say nothing about the Fourier dimension of the sets they construct, which in general is a much harder question, involving a delicate interplay between 'randomness' and 'structure'. Most 'structured' sets have low Fourier dimension, and so most methods for constructing sets with large Fourier dimension require making certain 'random choices' which on average do not correlate with any particular planar wave. Structure must be added to increase the dimension of the set beyond trivial results, but adding too much structure can lead to high correlation with planar waves, and thus decimate the Fourier dimension of the resulting set. Certain results have been obtained, however, for linear functions f:

(E) If $f(x_1,\dots,x_n) = a_1x_1 + \cdots + a_nx_n$ with $\sum a_j = 0$, Pramanik and Liang [31] construct $X \subset [0,1]$ with Fourier dimension 1 avoiding the zeroes of f, generalizing a construction of Shmerkin [37], who proved the result in the special case where $f(x_1,x_2,x_3) = (x_3 - x_1) - 2(x_2 - x_1)$, a function detecting arithmetic progressions of length 3.

(F) Körner constructed subsets $X \subset [0,1]$ with Fourier dimension $(k-1)^{-1}$ such that for any integers $m_0,\dots,m_k$, and any distinct $x_1,\dots,x_k \in X$, $a_0 \neq a_1x_1 + \cdots + a_nx_n$.

The focus on linear functions is natural, since the Fourier transform behaves in a predictable way with respect to linearity, whereas the relation of the Fourier transform with respect to other nonlinear phenomena is much more subtle, and more poorly understood. *The main goal of my research on configurations during my PhD was to push past this linearity, finding constructions of sets with large Fourier dimension avoiding the zeroes of nonlinear functions.*

2.2 My Contributions To The Study Of Configurations

It seems more difficult to adapt methods (A) and (D) above to construct sets with positive Fourier dimension, since the constructions involve constructing X at each spatial scale by choosing a good family of intervals, and then considering a large union of translates of the intervals along an arithmetic progression. Such a construction gives intervals correlated with waves of frequency complementing the spacing of the arithmetic progression, and so results in the construction of a set with Fourier dimension zero. On the other hand, methods (B) and (C) involve mostly pigeonholing arguments, so they seem the most likely to be able to be adapted to the Fourier dimension setting. I was able to adapt some of the ideas of these methods to obtain such a result. *In future work, I hope to explore adaptions of methods (A) and (D) to the Fourier dimension by ideas of* [37].

For simplicity, I focused on the case when $m = d$ and when the function f was C^1 and full rank, an assumption also made in [18]. Then by the implicit function theorem, after possibly rearranging indices, we can locally write $f(x_1,\dots,x_n) = x_1 - g(x_2,\dots,x_n)$ for a function $g : (\mathbb{R}^d)^{n-1} \to \mathbb{R}^d$. In [14], under the assumption that g was a submersion in each variable $x_2,\dots,x_n$, I was able to modify the construction of [18] to obtain Fourier dimension bounds.

Theorem. [14] *Suppose that $g : [0,1]^{d(n-1)} \to \mathbb{R}^d$ is a function such that for each $k \in \{0,\dots,n-2\}$, the $d \times d$ matrix $D_k g = (\partial g_i / \partial x_{dk+j})_{i,j=1}^d$ is invertible. Then there exists a Salem set $X \subset [0,1]^d$ of dimension $d/(n-3/4)$ such that for all distinct $x_1,\dots,x_n \in X$, $x_1 \neq f(x_2,\dots,x_n)$. If, in addition, $g(x_2,\dots,x_n) = ax_2 + h(x_3,\dots,x_n)$ for some $a \in \mathbb{Q}$, then there exists a Salem set $X \subset [0,1]^d$ of dimension $d/(n-1)$ such that for all distinct $x_1,\dots,x_n \in X$, $x_1 \neq f(x_2,\dots,x_n)$.*

As with most of the other approaches discussed above, we construct our sets X via a 'Cantor-type construction'. Fix a parameter α. We iteratively define a nested family of sets $\{X_k\}$, each a union of cubes of some fixed length l_k, and define $X = \bigcap_k X_k$. The set X_{k+1} is obtained from X_k by partitioning X_k each sidelength l_k cube into N^d sidelength l_{k+1} cubes, where $N := l_k / l_{k+1}$, and defining X_{k+1} as a subcollection of these cubes. The construction in [14] quite simple: To construct X_{k+1} from X_k, we start by taking a set S by taking $\sim N^\alpha$ points uniformly at random from the centers of the sidelength l_{k+1} cubes in the partition of each sidelength l_k cube in X_k. Some points from this set will form near zeroes of the function f; we let

$$S_{\text{bad}} = \{x \in S : |f(x,x_2,\dots,x_n)| \leq 10 l_{k+1} \text{ for some } x_2,\dots,x_n \in S\},$$

and define X_{k+1} to be the union of all sidelength l_{k+1} cubes centered at points in $S - S_{\text{bad}}$. The set X will then avoid the zeroes of the function f. Provided that $\alpha \leq (nd-s)/(n-1)$, we have with high probability that $\#(S_{\text{bad}}) \ll \#(S)$, and so with high probability, at each stage of the construction X_k is a union of $\sim l_k^{-\alpha}$ cubes of sidelength α; it is therefore natural to expect the set X almost surely has Hausdorff dimension α, and indeed, in [16] this is shown to be the case. The challenge is to show the construction has good Fourier analytic properties with high prrobability.

Simply counting the number of cubes at each scale is not sufficient to obtain a Fourier dimension bound. One observation made in [14] is that the core requirement is a square root cancellation bound. More precisely, if we denote the centers of the sidelength l_k cubes forming X_k by $\{x_1,\dots,x_M\}$, and if for all $1 \lesssim |\xi| \lesssim N$ the square root cancellation bound

$$\left| \frac{1}{M} \sum_{j=1}^{M} e^{2\pi i \xi \cdot x_j} \right| \lessapprox M^{-1/2} \tag{1}$$

holds at all scales, then the resulting set X will have Fourier dimension agreeing with its Hausdorff dimension. Indeed, consider the probability measure $\mu_k = M^{-1} \sum_{j=1}^M \chi_j$ supported on X_k, where χ_j is a smooth bump function adapted to the cube centered at x_j. Then for $|\xi| \lesssim 1/l_k$, since $M \sim l_k^{-\alpha}$ with high probability, (1) implies that $|\widehat{\mu}_k| \lessapprox M^{-1/2} \lesssim |\xi|^{-\alpha/2}$. On the other hand, the uncertainty principle implies that $\widehat{\mu}_k$ decays rapidly for $|\xi| \gtrsim 1/l_k$, and so $\widehat{\mu}_k$ has the appropriate Fourier decay required. Taking weak limits of the measures $\{\mu_k\}$, we find that $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-\alpha/2}$ has the right Fourier decay to justify that X has Fourier dimension α with high probability.

We obtain the required square root cancellation bounds using the theory of *concentration of measure* from high dimensional probability theory, which determines when a sum of random variables has square root cancellation away from the mean with high probability. If we are taking a sum of independent random variables, often Hoeffding's inequality gives sharp concentration bounds. But in our construction, the random points $\{x_1,\dots,x_M\}$ are not chosen independently from one another; the points in the initial set S are independent, but not the points in the set $S - S_{\text{bad}}$. There are certain standard tools to handle this problem, such as McDiarmid or Azuma's inequality, though applied directly in this setting they only guarantee a Fourier decay with $\alpha = d/n$ rather than the larger rate $\alpha = d/(n-1)$. In [14], I found a novel way to interlace Hoeffding and McDiarmid's inequality to ensure square root cancellation away from the mean occurs with high probability for the larger decay rate.

The final problem is then to show that the mean of $M^{-1} \sum e^{2\pi i \xi \cdot x_j}$ has square root cancellation. This proved to be the most inefficient aspect of the argument. We can write the mean as an oscillatory integral, though in M variables, and so usual techniques in the theory of oscillatory integrals fail to handle this bound since they are usually *dimension dependent*, and we need uniform bounds in M. Instead, I was able to use an inclusion exclusion argument, together with a Whitney decomposition of the thickened zero set of the function f to obtain the required bounds. This is the least optimal part of the argument, and is the main reason a Fourier dimension of $d/(n-3/4)$ is obtained in general rather than $d/(n-1)$; however, if f satisfies a weak linearity a slight modification of the random construction ensures that the mean of $M^{-1}\sum e^{2\pi i\xi\cdot x_j}$ is always zero, yielding the large Fourier dimension bound $d/(n-1)$ in this case.

3 Future Lines of Research

In this section, I discuss three plans for future projects in detail which I strongly believe should produce reliable results in the near future. Afterwards, we conclude with some more exploratory projects.

3.1 Extending The Range of Endpoint Radial Multiplier Bounds

At the beginning of Section 1, we discussed the radial multiplier conjecture. Recall that the conjecture stated a Fourier multiplier T_a was bounded on $L^p(\mathbb{R}^d)$ if and only if $\sup_j C_p(a_j) < \infty$, where $a(\rho) = \sum a_j(2^{-j}\rho)$ is a dyadic decomposition of a, and the quantities C_p are weighted L^p norms of the Fourier transform of a. In that section we discussed that no methods involving an 'ε loss' can be used in this conjecture, because of the inability to use certain interpolation arguments. However, if we weaken the conjecture to allow for an ε of room, we can allow for this loss. Namely, such methods may apply to prove that a Fourier multiplier T_a is bounded on $L^p(\mathbb{R}^d)$ if $\sup_j C_{p-\varepsilon}(a_j) < \infty$ for some $\varepsilon > 0$. I hope to pursue whether more modern techniques in restriction theory, such as decoupling and polynomial partitioning, can be used to prove such bounds, especially in the case $d = 2$ in which no such results are known. Below I describe several potential methods that may yield improved bounds.

One possible direction is suggested by several results on 'weighted estimates of fractal type' for extension operators, such as the work of Du and Zhang [17] and Ortiz [33], the latter of which directly involves counting tangencies of annuli as in [22]. These estimates are morally dual to the 'sparse incidence bounds' discussed in Section 1.2. Thus finding analogues of 'sparse decompositions' in this dual setting may allow one to apply the techniques from these papers to achieve improved bounds.

Finding a dual argument to the sparsity bounds used in recent characterization of boundedness for radial Fourier multipliers also allows us to prove results directly in the range $2 \leq p \leq \infty$. This allows for us to use certain multilinear to linear arguments to prove bounds. In particular, I believe one may be able to use work on the k-linear restriction conjecture to the cone, such as by Barceló [1], Wolff [41], Ou and Wang [34], Beltran and Saari [3], and Gao, Liu, Miao and Xi [19], to obtain new results.

Another direction is suggested recent work of Zahl [42], who uses a line-sphere correspondence developed in early doctoral work of Sophus Lie to translate problems about counting circle tangencies in $\mathbb{R}^3$ to counting incidences of complex lines on a quadric surface in $\mathbb{C}^3$ which can be identified with the Heisenberg group. These lines satisfy the Wolff axiom, and so it might be hoped that techniques from Kakeya theory in $\mathbb{R}^3$ might be adapted in the 'sparse setting' to yield better estimates than were obtained by [9]. When $d \neq 3$, the Lie-Sphere correspondence can still be used, mapping spheres in $\mathbb{R}^d$ to points in a quadric surface Q in $\mathbb{P}^{d+2}$. Two circles which intersect tangentially map to two points orthogonal with respect to a bilinear form B on $\mathbb{R}^{d+3}$. The problem is thus reduced to counting pairs of orthogonal vectors, which is certainly not as well studied as incidences of lines, though projection arguments can reduce the problem to counting incidences between points and hyperplanes.

3.2 Multipliers Associated With Periodic Geodesic Flow

The results I discussed in Section 2 for multiplier operators on S^d generalized to multipliers of an arbitrary first order, elliptic pseudodifferential operator P on a compact manifold M, provided that P satisfied Assumptions A and B of that section. Assumption A relates to the curvature of the principal symbol of P, and this assumption cannot really be weakened without significantly changing the character of the results. On the other hand, Assumption B arises as an artifact of the methods of our proof. We can likely obtain similar bounds while weakening this assumptions; for instance, Kim [26] obtained bounds on the scale of Besov spaces only under Assumption A.

It is likely very difficult that we can complete remove Assumption B using current research methods while still recovering the results of [15]. This limitation follows from our current inability to understand the large time behavior of wave equations on compact manifolds precisely enough. If we were able to follow the method of [15], which reduced the large time argument to a smoothing inequality for the wave equation, then the results of that paper would follow for another operator P if we could prove

$$\left\| \left(\int_k^{k+1} |e^{2\pi i t P} f|^{p'} \, dt \right)^{1/p'} \right\|_{L^{p'}(M)} \lesssim k^\delta \|f\|_{L^p_{d(1/p-1/2)-1/p'}(M)} \tag{2}$$

for some $\delta < d(1/p - 1/2) - 1/2$.

If P satisfies assumption B, then after rescaling, we may assume without loss of generality that all eigenvalues of P are integers, so that $e^{2\pi i k P} = I$ is the identity for all k, and then (2) holds with $\delta = 0$ for all $|1/p - 1/2| > (d-1)^{-1}$ by the local smoothing inequality of Lee and Seeger [30]. Whether this bound is true in other contexts remains unknown. The next simplest case to consider would be when the operator P has the property that $e^{2\pi i k P}$ is *close* to the identity for all k. This happens precisely when the *Hamiltonian flow* on T^*M given by the vector field $H = (\partial p_{\text{prin}}/\partial \xi, -\partial p_{\text{prin}}/\partial x)$ is periodic, where p_{prin} is the principal symbol of P. Indeed, results of Colin de Verdiére [10] related to the theory of propagation of singularities of Fourier integral operators then tell us that the operator $R = e^{2\pi i P}$ is a pseudodifferential operator of order zero, and its principal symbol is related to an invariant of the flow known as the Maslov index. The operator has been studied a little by spectral theorists, and there it is known as the *return operator*. If we are able to justify bounds of the form

$$\|R^k f\|_{L^p} \lesssim k^\delta \|f\|_{L^p},$$

or a frequency localized variation of this bound, then the local smoothing inequality of Lee and Seeger yields (2). Such bounds are of interest since they cover all the operators $P = \sqrt{-\Delta}$, where Δ is the Laplace-Beltrami operator on a Riemannian manifold with periodic geodesic flow. They are even of interest on the sphere, since our method only allows us to tell when multipliers of the form $a(P/R)$ are uniformly bounded on $L^p(S^d)$, where $P = \sqrt{(\frac{d-1}{2})^2 - \Delta}$ whereas these bounds would allow us to tell when the multipliers $a\big(\sqrt{-\Delta}/R\big)$ are uniformly bounded on $L^p(S^d)$.

If such a project is successful, it then becomes of interest to determine whether one can obtain such bounds on manifolds whose geodesic flow has well-controlled dynamical properties, for instance, forming an integrable dynamical system; One model class of such integrable systems are the ellipsoids, or more generally, the convex surfaces of revolution. Results of Colin de Verdière [11], which prove these manifolds are quantum completely integrable, may help in this case, allowing one to reduce the study of $\sqrt{-\Delta}$ to a family of commuting pseudodifferential operators with integer spectrum.

3.3 Subpolynomial Decoupling On Random Fractals

One major development in harmonic analysis in the past decade has been a greater understanding of the phenomenon of *decoupling*, or *Wolff-type estimates*. Given a family of almost disjoint subsets $\mathscr{E}_\delta$ of $\mathbb{R}^d$ parameterized by $\delta > 0$, $L^p(l^2)$ decoupling discusses bounds of the form

$$\left\| \sum f_j \right\|_{L^p(\mathbb{R}^d)} \leq D_p(\delta) \Big(\sum_j \|f_j\|_{L^p(\mathbb{R}^d)}^2 \Big)^{1/2},$$

where the Fourier transforms of the functions f_j are supported on distinct subsets of $\mathscr{E}_\delta$, and $D_p(\delta)$ denotes the best constant under which this equation holds for all such $\{f_j\}$. After [12], we say a *subpolynomial decoupling inequality* results when one can prove that $D_p(\delta) \lesssim_\varepsilon (1/\delta)^\varepsilon$ for all $\varepsilon > 0$.

Much work has been carried out for $d \geq 2$, when the sets $\mathscr{E}_\delta$ are $\delta \times \delta^{1/2}$ caps associated with partitions of neighborhoods of curves and surfaces, and the decoupling inequalities are obtained by virtue of curvature and torsion properties. But the analysis of decoupling on *fractal sets* is still poorly understood.

Consider a sequence of integers $n(i)$, and a set X obtained from a Cantor-like construction $\{X_i\}$ as in Section 2.2, where X_i is a union of a family of cubes $\mathcal{Q}_i$ with some fixed sidelength $\delta := \delta_i$. We let $\mathcal{E}_\delta = \{Q \cap C_{i+n(i)} : Q \in \mathcal{Q}_i\}$, and ask for which Cantor type constructions $\{X_i\}$ and for which sequences $\{n(i)\}$ do we obtain a subpolynomial decoupling inequality for the families $\{\mathcal{E}_\delta\}$.

Some analysis has been done in this setting, but no subpolynomial decoupling bounds have been established for any fractal set. Chang, de Dios Pont, Greenfeld, Jamneshan, Li, and Madrid have obtain such results for self-similar Cantor sets with good numerical properties [7], but none of the bounds obtained give subpolynomial decoupling inequalities in the above sense. Decoupling inequalities for random fractal sets have been obtained by Łaba and Wang [29], see also Łaba [27]; these are also not subpolynomial decoupling inequalities, but the bounds they obtained were sufficient for their applications to the study of $L^p \to L^2$ fractal restriction bounds. In fact, in the range of p they were considering, subpolynomial decoupling is not possible; one can see using the local constancy property and Khintchine type heuristics that if X is chosen sufficiently randomly, and $\#\mathcal{Q}_i \gtrsim \delta_i^{-s}$ for each i, then subpolynomial $L^p(l^2)$ decoupling is impossible for any choice of $\{n(i)\}$ unless $2 \leq p \leq 2d/s$. Unlike for curves and surfaces, it is unclear what the canonical 'scale' of the problem is; decoupling for curves and surfaces is possible because the $\delta \times \delta^{1/2}$ caps along these shapes point in different directions. But it is unclear what the canonical choice of $\{n(i)\}$ should be so that the sets $C_{i+n(i)}$ do not interact significantly.

I believe the techniques related to my results in [14] can be applied to obtaining random decoupling inequalities. Methods from the theory of concentration of measure have been applied by Bourgain [5] and Talagrand [40] in order to prove the existence of $\Lambda(p)$ sets, in particular, the method of majorizing measures and selection processes. One might view $\Lambda(p)$ sets as a kind of discrete variant of sets upon which decoupling bounds hold, so it is likely to believe these methods generalize to the continuous setting. Using these methods, I hope to obtain an analogue of the proof of $l^2(L^p)$ decoupling for the paraboloid found in [6], i.e. establishing an analogue of multilinear Kakeya for the sets $\mathcal{E}_\delta$, and then applying an induction on scales to obtain a subpolynomial fractal decoupling inequality.

3.4 More Exploratory Projects

We now discuss some exploratory projects, whose methodology is not quite as clearly defined:

Fourier Dimension of High Codimension Hypersurfaces Via Extremizer Methods: A curved hypersurface in $\mathbb{R}^d$ has Fourier dimension $d-1$. On the other hand, the Fourier dimension of higher codimension hypersurfaces in $\mathbb{R}^d$ is unknown; stationary-phase methods bound the Fourier decay of smooth measures, but it is unknown if such decay is best possible among all measures, even for non-degenerate curves in $\mathbb{R}^3$; in that case smooth measures decay at a rate of $|\xi|^{-2/3}$, but it is difficult to rule out the existence of non-smooth measures decaying at a faster rate. A potential solution may lie in the theory of extremizers. Indeed, there are many such results in the area that characterize the smoothness of extremizers. If we are able to obtain such results for Fourier decay, we can rule out non-smooth fast decaying measures, and thus find the best possible Fourier decay of measures on surfaces.

Analysis of Highly Degenerate Oscillatory Integrals via o-Minimality: Together with Johnsrude, Sandberg, and de Oliveira Andrade, I recently wrote a study guide on the recent use of 'o-minimality' in a paper of Basu, Guo, Zhang, and Zorin-Kranich [2] to the study of oscillatory integrals, which we plan to put onto the arXiv shortly. The method gives optimal decay for highly degenerate oscillatory integrals with an algebraic phase. I hope to find new applications of this theory in the study of Fourier integral operators. One application may lie in the study of convolution operators on highly degenerate algebraic Lie groups, from which arise oscillatory integrals with a degenerate phase.

References

[1] Bartolomé Barceló, *The restriction of the Fourier transform to some curves and surfaces*, Studia Math. **84** (1986), no. 1, 39–69. MR871845

[2] Saugata Basu, Shaoming Guo, Ruixiang Zhang, and Pavel Zorin-Kranich, *A stationary set method for estimating oscillatory integrals*, arXiv 2021.

[3] David Beltran and Olli Saari, *$L^p - L^q$ local smoothing estimates for the wave equation via k-broad Fourier restriction*, J. Fourier Anal. Appl. **28** (2022), no. 5, Paper No. 76, 29. MR4487774

[4] Aline Bonami and Jean-Louis Clerc, *Sommes de Cesàro et multiplicateurs des développements en harmoniques sphériques*, Trans. Amer. Math. Soc. **183** (1973), 223–263. MR338697

[5] Jean Bourgain, *Bounded orthogonal systems and the $\Lambda(p)$-set problem*, Acta Math. **162** (1989), no. 3-4, 227–245. MR989397

[6] Jean Bourgain and Ciprian Demeter, *A study guide for the l^2 decoupling theorem*, Chinese Ann. Math. Ser. B **38** (2017), no. 1, 173–200. MR3592159

[7] Alan Chang, Jaume de Dios Pont, Rachel Greenfeld, Asgar Jamneshan, Zane Kun Li, and José Madrid, *Decoupling for fractal subsets of the parabola*, Math. Z. **301** (2022), no. 2, 1851–1879. MR4418339

[8] Sun-Yung A. Chang and Robert Fefferman, *A continuous version of duality of H^1 with BMO on the bidisc*, Ann. of Math. (2) **112** (1980), no. 1, 179–201. MR584078

[9] Laura Cladek, *Radial Fourier multipliers in $\mathbb{R}^3$ and $\mathbb{R}^4$*, Anal. PDE **11** (2018), no. 2, 467–498. MR3724494

[10] Yves Colin de Verdière, *Sur le spectre des opérateurs elliptiques à bicaractéristiques toutes périodiques*, Comment. Math. Helv. **54** (1979), no. 3, 508–522. MR543346

[11] ———, *Spectre conjoint d'opérateurs pseudo-différentiels qui commutent. II. Le cas intégrable*, Math. Z. **171** (1980), no. 1, 51–73. MR566483

[12] Ciprian Demeter, *Fourier restriction, decoupling, and applications*, Cambridge Studies in Advanced Mathematics, 2019.

[13] Jacob Denson, *Cartesian products avoiding patterns*, Master's Thesis, 2019.

[14] ———, *Large Salem sets avoiding nonlinear configurations*, arXiv 2021.

[15] ———, *Multipliers for spherical harmonic expansions*, arXiv 2024.

[16] Jacob Denson, Malabika Pramanik, and Joshua Zahl, *Large sets avoiding rough patterns*, Harmonic analysis and applications, 2021, pp. 59–75. MR4238597

[17] Xiumin Du and Ruixiang Zhang, *Sharp L^2 estimates of the Schrödinger maximal function in higher dimensions*, Ann. of Math. (2) **189** (2019), no. 3, 837–861. MR3961084

[18] Robert Fraser and Malabika Pramanik, *Large sets avoiding patterns*, 2018, pp. 1083–1111. MR3785600

[19] Chuanwei Gao, Bochen Liu, Changxing Miao, and Yakun Xi, *Improved local smoothing estimate for the wave equation in higher dimensions*, J. Funct. Anal. **284** (2023), no. 9, Paper No. 109879, 48. MR4551611

[20] Gustavo Garrigós and Andreas Seeger, *Characterizations of Hankel multipliers*, Math. Ann. **342** (2008), no. 1, 31–68. MR2415314

[21] Viktor Harangi, Tamás Keleti, Gergely Kiss, Péter Maga, András Máthé, Pertti Mattila, and Balázs Strenner, *How large dimension guarantees a given angle?*, Monatsh. Math. **171** (2013), no. 2, 169–187. MR3077930

[22] Yaryong Heo, Fëdor Nazarov, and Andreas Seeger, *Radial Fourier multipliers in high dimensions*, Acta Math. **206** (2011), no. 1, 55–92. MR2784663

[23] Lars Hörmander, *On the Riesz means of spectral functions and eigenfunction expansions for elliptic differential operators*, Some Recent Advances in the Basic Sciences, Vol. 2 (Proc. Annual Sci. Conf., Belfer Grad. School Sci., Yeshiva Univ., New York, 1965-1966), 1969, pp. 155–202. MR257589

[24] Mihaela Ifrim and Daniel Tataru, *The lifespan of small data solutions in two dimensional capillary water waves*, Arch. Ration. Mech. Anal. (2017).

[25] Jongchon Kim, *Endpoint bounds for quasiradial Fourier multipliers*, Ann. Mat. Pura Appl. (4) **196** (2017), no. 3, 773–789. MR3654932

[26] ———, *Endpoint bounds for a class of spectral multipliers in compact manifolds*, Indiana Univ. Math. J. **67** (2018), no. 2, 937–969. MR3798862

[27] Izabella Łaba, *Maximal operators and decoupling for $\Lambda(p)$ Cantor measures*, Ann. Fenn. Math. **46** (2021), no. 1, 163–186. MR4277805

[28] Izabella Łaba and Malabika Pramanik, *Arithmetic progressions in sets of fractional dimension*, Geom. Funct. Anal. **19** (2009), no. 2, 429–456. MR2545245

[29] Izabella Łaba and Hong Wang, *Decoupling and near-optimal restriction estimates for Cantor sets*, Int. Math. Res. Not. IMRN **9** (2018), 2944–2966. MR3801501

[30] Sanghyuk Lee and Andreas Seeger, *Lebesgue space estimates for a class of Fourier integral operators associated with wave propogation*, Math. Nachr. **286** (2013), no. 7, 745–755.

[31] Yiyu Liang and Malabika Pramanik, *Fourier dimension and avoidance of linear patterns*, Adv. Math. **399** (2022), Paper No. 108252, 50. MR4384610

[32] András Máthé, *Sets of large dimension not containing polynomial configurations*, Adv. Math. **316** (2017), 691–709. MR3672917

[33] Alexander Ortiz, *A sharp weighted Fourier extension estimate for the cone in $\mathbb{R}^3$ based on circle tangencies*, arXiv 2024.

[34] Yumeng Ou and Hong Wang, *A cone restriction estimate using polynomial partitioning*, J. Eur. Math. Soc. (JEMS) **24** (2022), no. 10, 3557–3595. MR4432906

[35] Volker Schlue, *Decay of linear waves on higher-dimensional Schwarzschild black holes*, Anal. PDE (2013).

[36] A. Seeger and C. D. Sogge, *Bounds for eigenfunctions of differential operators*, Indiana Univ. Math. J. **38** (1989), no. 3, 669–682. MR1017329

[37] Pablo Shmerkin, *Salem sets with no arithmetic progressions*, Int. Math. Res. Not. IMRN **7** (2017), 1929–1941. MR3658188

[38] Christopher D. Sogge, *Oscillatory integrals and spherical harmonics*, Duke Math. J. **53** (1986), no. 1, 43–65. MR835795

[39] ———, *On the convergence of Riesz means on compact manifolds*, Ann. of Math. (2) **126** (1987), no. 3, 439–447.

[40] Michel Talagrand, *Sections of smooth convex bodies via majorizing measures*, Acta Math. **175** (1995), no. 2, 273–300. MR1368249

[41] Thomas Wolff, *A sharp bilinear cone restriction estimate*, Ann. of Math. (2) **153** (2001), no. 3, 661–698. MR1836285

[42] Joshua Zahl, *Sphere tangencies, line incidences and Lie's line-sphere correspondence*, Math. Proc. Cambridge Philos. Soc. **172** (2022), no. 2, 401–421. MR4379071