Radial Multipliers

Jacob Denson

November 21, 2022

Table Of Contents

Part I

Review of Literature

Chapter 1

General Introduction

The question of the regularity of translation-invariant operators on $\mathbf{R}^d$ has proved central to the development of modern harmonic analysis. Indeed, the regularity of scuh operators underpins any subtle understanding of the Fourier transform, since with essentially any such operator T, we can associate a tempered distribution $m : \mathbf{R}^d \to \mathbf{C}$, known as the *symbol* of T, such that for any Schwartz function $f \in \mathcal{S}(\mathbf{R}^d)$,

$$Tf(x) = \int m(\xi)\widehat{f}(\xi)e^{2\pi i\xi\cdot x}\, d\xi,$$

i.e. such that $\widehat{Tf} = m \cdot \widehat{f}$. This is why such operators are also called *Fourier multipliers*. Using the spectral calculus of unbounded operators, one can also write this operator as $m(D)$, where $D = (2\pi i)^{-1}\nabla$ is a self-adjoint normalization of the gradient operator. Thus the study of the boundedness of translation invariant operators is closely connected to the study of the interactions of the operators

$$E_\xi f(x) = \widehat{f}(\xi)e^{2\pi i\xi\cdot x},$$

which act as projections onto the common eigenspaces of the components of D, since we can write $m(D)$ as a vector-valued integral of the form

$$m(D) = \int m(\xi)E_\xi\, d\xi.$$

Thus $m(D)$ is represented as a weighted average of the operators $\{E_\xi\}$.

The study of translation invariant operators emerges from many classical questions in analysis, like that of the convergence properties of Fourier series, or in mathematical physics, through the study of the heat, wave, and Schrödinger equation. These operators also naturally have rotational symmetry, so it is natural to restrict our attention to translation-invariant operators which are also rotation-invariant. These operators are precisely those represented by symbols $m : \mathbf{R}^d \to \mathbf{C}$ which are *radial*, i.e. such that

$$m(\xi) = h(|\xi|)$$

for some function $h : [0, \infty) \to \mathbf{C}$. This is the class of *radial Fourier multipliers*. The spectral calculus again implies one can write $m(D) = h(\sqrt{-\Delta})$, where Δ is the Laplacian on $\mathbf{R}^d$. Thus the study of radial multipliers is closely connected to interactions between the spherical projection operators

$$E_\lambda f(x) = \int_{|\xi|=1} \widehat{f}(\xi) e^{2\pi i \xi \cdot x},$$

for $0 < \lambda < \infty$, which are the projections onto the eigenspaces of $\sqrt{-\Delta}$, since we then have

$$h(\sqrt{-\Delta}) = \int h(\lambda) E_\lambda.$$

Thus studying the regularity of radial Fourier multipliers allows us to understand the behaviour of weighted averages of the operators $\{E_\lambda\}$.

Stated as above, we can extend the study of radial multipliers on $\mathbf{R}^d$ to the more general setting of *geodesically complete* Riemannian manifolds X. On such a manifold we have a Laplace-Beltrami operator Δ which is an essentially self-adjoint unbounded operator on $L^2(X)$, and one can consider a spectral calculus. In particular, one can consider the study of operators of the form $h(\sqrt{-\Delta})$ for functions $h : [0, \infty) \to \mathbf{C}$. Some techniques of analyzing radial multipliers on $\mathbf{R}^d$ extend to the Riemannian case, whereas in other cases new tools are required.

This research project studies necessary and sufficient conditions to guarantee the L^p boundedness of radial multiplier operators, both in the Euclidean setting, and also in the setting of Riemannian manifolds.

1.1 Radial Multipliers on Euclidean Space

The general study of the boundedness of Fourier multipliers was initiated in the 1960s. It was quickly realized that the most fundamental estimates were those of the form

$$\|Tf\|_{L^q(\mathbf{R}^d)} \lesssim \|f\|_{L^p(\mathbf{R}^d)},$$

It is therefore natural to introduce the spaces $M^{p,q}(\mathbf{R}^d)$, consisting of all symbols m which induce a Fourier multiplier operator T bounded from $L^p(\mathbf{R}^d)$ to $L^q(\mathbf{R}^d)$. The space $M^{p,q}(\mathbf{R}^d)$ is then naturally a Banach space under the operator norm

$$\|m\|_{M^{p,q}(\mathbf{R}^d)} = \sup\left\{ \frac{\|Tf\|_{L^q(\mathbf{R}^d)}}{\|f\|_{L^p(\mathbf{R}^d)}} : f \in \mathcal{S}(\mathbf{R}^d) \right\}.$$

For notational convenience, $M^{p,p}(\mathbf{R}^d)$ is denoted by $M^p(\mathbf{R}^d)$. Duality implies that $M^{p,q}(\mathbf{R}^d)$ is isometric to $M^{q^*,p^*}(\mathbf{R}^d)$. And the fact these operators are translation invariant, together with Littlewood's Principle, implies that $M^{p,q}(\mathbf{R}^d) = 0$ unless $q \geqslant p$. Combining these two results means we can always reduce our anaysis to the case where $1 \leqslant p \leqslant 2$ and $q \geqslant p$.

It was quickly realized that some of the spaces $M^{p,q}(\mathbf{R}^d)$ are very easy to characterize. For instance, $M^{1,q}(\mathbf{R}^d)$ was characterized by virtue of the fact that the boundedness of operators with domain $L^1(\mathbf{R}^d)$ is relatively trivial, i.e. as characterized by Schur's lemma; for any symbol m, if $k = \widehat{m}$, then

$$\|m\|_{M^{1,q}(\mathbf{R}^d)} = \begin{cases} \|k\|_{L^q(\mathbf{R}^d)} & : q > 1 \\ \|k\|_{M(\mathbf{R}^d)} & : q = 1, \end{cases}.$$

where $M(\mathbf{R}^d)$ is the space of finite signed Borel measures equipped with the total variation norm. Duality also allows us to characterize the spaces $M^{p,\infty}(\mathbf{R}^d)$. The unitary nature of the Fourier transform implies that the spaces $M^{p,2}(\mathbf{R}^d)$ could also be characterized, for $1 \leqslant p \leqslant 2$, by the identity

$$\|m\|_{M^{p,2}(\mathbf{R}^d)} = \|m\|_{L^q(\mathbf{R}^d)},$$

where $q = 2p/(2-p)$. However, characterizing the remaining spaces $M^{p,q}$ for $1 < p < \infty$ and $p \in (1,2]$ and $q \in [p,\infty) - \{2\}$, proved more challenging.

In the past 60 years there has been no tractable characterization of the spaces $M^{p,q}(\mathbf{R}^d)$ for any other pair of exponents p or q.

One tool that has proved useful in studying the behaviour of multipliers outside of this range is *Littlewood-Paley* theory, which makes it natural to restrict to the study of Fourier multipliers compactly suported on dyadic annuli. We fix a smooth bump function $\phi \in C_c^\infty(\mathbf{R}^d)$ supported on $\{|\xi| \sim 1\}$, and such that $1 = \sum_j \mathrm{Dil}_{2^j} \phi$. Then given a symbol m, we define

$$m_t = (\mathrm{Dil}_{1/t} m) \cdot \phi.$$

Thus m_t describes the behaviour of the multiplier m restricted to the annulus of frequencies $|\xi| \sim t$, rescaled so that this behaviour is now lying on the annulus $|\xi| \sim 1$. Littlewood-Paley theory implies that for $1 < p, q < \infty$, then

$$\|m\|_{M^{p,q}(\mathbf{R}^d)} \sim_{p,q} \sup_{t>0} t^{d(1/p-1/q)} \|m_t\|_{M^{p,q}(\mathbf{R}^d)}.$$

In the study of multipliers we have not already characterized, it is therefore natural to restrict oneself to the study of multipliers with a compactly supported symbol; in $\mathbf{R}^d$ we can rescale, so we can assume we are working with a multiplier supported on the annulus $1/2 \leqslant |\xi| \leqslant 2$. In the sequel, we will call these *unit scale multipliers*.

One common heuristic to this theory is that the regularity of the symbol m, or equivalently, the decay of the convolution kernel k away from the origin, implies some boundedness of the symbol, viewed as a multiplier. The most well known condition of this form for $1 < p < \infty$ is the Hörmander-Mikhlin multiplier theorem, which shows that for $1 < p < \infty$, and $\varepsilon > 0$, if m is a unit scale multiplier, and k is it's convolution kernel, then TODO: Double check this exponent.

$$\|m\|_{M^p(\mathbf{R}^d)} \lesssim_{p,\varepsilon} \int k(x) \langle x \rangle^\varepsilon \, dx.$$

In light of the bound

$$\int k(x) \langle x \rangle^\varepsilon \, dx \leqslant \|k(x) \langle x \rangle^{d/2+2\varepsilon}\|_{L^2_x} \|\langle x \rangle^{-d/2-\varepsilon}\|_{L^2_x} \lesssim \|k(x) \langle x \rangle^{d/2+2\varepsilon}\|_{L^2_x},$$

we obtain a slightly weaker, but easier to use, inequality of the form

$$\|m\|_{M^p(\mathbf{R}^d)} \lesssim_\varepsilon \|m\|_{L^2_{d/2+\varepsilon}}.$$

All these results apply via a Paley-Wiener decomposition to general multipliers, i.e. with the right hand side replaced with a supremum of the quantities associated with the rescaled multipliers $\{m_t\}$.

Conversely, some control over the mass of the convolution kernel k is necessary in order to conclude that $m \in M^{p,q}(\mathbf{R}^d)$ for some exponents p and q. This is because if k is the convolution kernel corresponding to a unit scale multiplier m, and $\phi \in C_c^\infty(\mathbf{R}^d)$ has Fourier transform equal to one on the annulus $1/4 \leqslant |\xi| \leqslant 4$, then

$$\|k\|_{L^q(\mathbf{R}^d)} = \|k * \phi\|_{L^q(\mathbf{R}^d)} = \|m(D)\phi\|_{L^q(\mathbf{R}^d)} \lesssim \|m\|_{M^{p,q}(\mathbf{R}^d)}.$$

For a general, non compactly supported multiplier m, if k_t is the convolution kernel associated with the multiplier m_t, then one obtains the condition

$$\sup_{t>0} \left\{ t^{d(1/p-1/q)} \cdot \|k_t\|_{L^q(\mathbf{R}^d)} \right\} \lesssim \|m\|_{M^{p,q}(\mathbf{R}^d)},$$

One can phrase this bound in terms of the homogeneous Besov spaces $\dot{B}^{p,q}_s(\mathbf{R}^d)$, the space consisting of all distributions u on $\mathbf{R}^d$ such that the norm

$$\|u\|_{\dot{B}^{p,q}_s(\mathbf{R}^d)} = \left(\sum_{j=-\infty}^{\infty} \left(2^{js} \|P_j u\|_{L^p(\mathbf{R}^d)} \right)^q \right)^{1/q} = \|2^{js} P_j u\|_{l^q(\mathbf{Z}) L^p(\mathbf{R}^d)},$$

is finite, where $\{P_j\}$ are a family of Little-wood Paley projection operators, with P_j projecting onto a dyadic frequency band of radius 2^j. If k is the convolution kernel of a multiplier m, one can rescale the condition above to read that

$$\sup_{t>0} \left\{ t^{-d/p^*} \|P_t k\|_{L^q(\mathbf{R}^d)} \right\} \lesssim \|m\|_{M^{p,q}(\mathbf{R}^d)},$$

i.e. that

$$\|k\|_{\dot{B}^{q,\infty}_{-d/p^*}} \lesssim \|m\|_{M^{p,q}(\mathbf{R}^d)}.$$

Thus we conclude that k must satisfy some (admittedly weak) regularity assumptions to be the convolution kernel of a bounded Fourier multiplier.

Despite the lack of a complete characterization of the classes $M^{p,q}(\mathbf{R}^d)$, it is surprising that we *can* conjecture a characterization of the subspace of $M^{p,q}(\mathbf{R}^d)$ for *radial symbols* in this class, for an appropriate range of exponents. This conjecture is best phrased in terms of the result of [3], which concerned radial multipliers m whose associated operator T is bounded from the L^p norm to the L^q norm *restricted to radial functions*, i.e. such that the norm

$$\|m\|_{M^{p,q}_{\text{rad}}(\mathbf{R}^d)} = \sup \left\{ \frac{\|Tf\|_{L^q(\mathbf{R}^d)}}{\|f\|_{L^p(\mathbf{R}^d)}} : f \in \mathcal{S}(\mathbf{R}^d) \text{ and } f \text{ is radial} \right\}$$

is finite. The main result of [3] was that if $d > 1$, if $1 < p < 2d/(d+1)$, and if $p \leqslant q < 2$, then $M^{p,q}_{\text{rad}}(\mathbf{R}^d)$ is a subset of $L^1_{\text{loc}}(\mathbf{R}^d)$, and for any unit scale, integrable, radial multiplier m,

$$\|m\|_{M^{p,q}_{\text{rad}}(\mathbf{R}^d)} \sim_{p,q,d} \|k\|_{L^q(\mathbf{R}^d)}.$$

More generally, for any locally integrable radial symbol m,

$$\|m\|_{M^{p,q}_{\text{rad}}(\mathbf{R}^d)} \sim_{p,q,d} \sup_{t>0} t^{d(1/p-1/q)} \|k_t\|_{L^q(\mathbf{R}^d)} = \|k\|_{\dot{B}^{q,\infty}_{-d/p^*}}.$$

Moreover, this condition give *precisely the range* under which this characterization holds. It is natural to conjecture that the same constraint continues to hold when we remove the constraint that our inputs f are radial, i.e. that for unit scale, integrable, radial symbols m, for $d > 1$, $1 < p < 2d/(d+1)$, and for $p \leqslant q < 2$,

$$\|m\|_{M^{p,q}} \sim_{p,q,d} \|k\|_{L^q(\mathbf{R}^d)}$$

and for general locally integrable symbols m,

$$\|m\|_{M^{p,q}} \sim_{p,q,d} \|k\|_{\dot{B}^{q,\infty}_{-d/p^*}}$$

In the sequel, we call this the *radial multiplier conjecture* in $\mathbf{R}^d$. Before we move on, let us analyze the reason for the endpoints in this conjecture.

For $p = 1$, the radial multiplier conjecture is true *for compactly supported multipliers*, and one does not even need to assume that the multiplier is radial in this case. Indeed, for any (not even compactly supported) multiplier m we have

$$\|m\|_{M^{1,q}(\mathbf{R}^d)} = \|k\|_{L^q(\mathbf{R}^d)}.$$

Littlewood-Paley says this quantity is proportional to

$$\left(\sum \|P_j k\|^2_{L^q(\mathbf{R}^d)} \right)^{1/2} = \|k\|_{\dot{B}^{q,2}_0}.$$

On the other hand, for non compactly supported multipliers m the radial multiplier conjecture should say that

$$\|m\|_{M^{1,q}(\mathbf{R}^d)} \sim \|k\|_{\dot{B}_0^{q,\infty}},$$

so the result fails 'in the second order exponents', because Littlewood-Paley theory no longer applies.

To show why assuming $p \leqslant 2d/(d+1)$ is necessary, consider the multiplier

$$m(\xi) = h(|\xi|)\mathbf{I}(|\xi| \leqslant 1),$$

where $h \in C_c^\infty(\mathbf{R})$ is supported on $1/4 \leqslant r \leqslant 4$, and is equal to one for $1/2 \leqslant r \leqslant 2$. Then m is a variant of the 'ball multiplier' of Fefferman. In particular, m differs from the 'ball multiplier' by a compactly supported, smooth symbol, and thus m has all the L^p mapping properties that the ball multiplier has. In particular, it is a result of Fefferman (TODO: Cite) that the ball multiplier does not lie in $M^{p,q}(\mathbf{R}^d)$ when $d > 1$ for any values of p and q except when $p = q = 2$, so the same is true of the multiplier m given above. Now if k is the convolution kernel of m, then polar coordinates gives that

$$k(x) = \int m(\xi) e^{2\pi i \xi \cdot x} = \frac{1}{|x|^{d/2-1}} \int_{1/4}^1 r^{d/2} h(r) J_{d/2-1}(rx).$$

For $|x| \geqslant 1$, Bessel function asymptotics gives that there is some constant a, depending on d, such that

$$k(x) = |x|^{-\frac{d-1}{2}} \int_{1/4}^1 r^{\frac{d-1}{2}} h(r) \cos(r|x| + a)\, dr + O\left(|x|^{-\frac{d+1}{2}} \right).$$

Integration by parts then gives that $|k(x)| \lesssim |x|^{-\frac{d+1}{2}}$. Since k is bounded near the origin (i.e. by the Paley-Wiener theorem), we conclude that $k \in L^q(\mathbf{R}^d)$ for $q > 2d/(d+1)$. In particular, this multiplier shows the radial multiplier conjecture cannot hold for $q > 2d/(d+1)$. TODO: Is there a counterexample for the endpoint $q = 2d/(d+1)$?

Given a function h on $[0,\infty)$, we define the *d-dimensional Fourier-Bessel transform* of h as

$$\mathcal{B}_d h(r) = r^{-\frac{d-2}{2}} \int_0^\infty \rho^{d/2} h(\rho) J_{\frac{d-2}{2}}(\rho r)\, d\rho,$$

where J_α is the standard Bessel function of order α. Then if $m(\xi) = h(|\xi|)$, then we have $\{\mathcal{F}m\}(x) = \{\mathcal{B}_d h\}(|x|)$. The condition in the radial multiplier conjecture for unit scale multipliers becomes that

$$\|m\|_{M^{p,q}(\mathbf{R}^d)} \sim \left(\int_0^\infty r^{d-1} |\mathcal{B}_d h(r)|^q \, dr \right)^{1/q}.$$

We therefore that, if we let $h_t = \phi \cdot \mathrm{Dil}_{1/t} h$, then the condition in the radial multiplier conjecture becomes

$$\sup_{t>0} t^{d(1/p-1/q)} \left(\int_0^\infty r^{d-1} |\{\mathcal{B}_d h_t\}(r)|^q \right)^{1/q} < \infty.$$

One can also convert this into a statement involving the standard one-dimensional Fourier transform. One can use Bessel function asymptotics to convert this into a condition on the one-dimensional Fourier transform of h (extended to an even function on $\mathbf{R}$). Indeed, there exists a differential operator L with constant coefficients and order at most $(d-1)/2$ such that we can write

$$\begin{aligned}
\mathcal{B}_d h(r) &= r^{-\frac{d-2}{2}} \int_{1/2}^2 \rho^{\frac{d}{2}} h(\rho) \left(\cos(\rho r + a) |\rho r|^{-1/2} + O(|\rho r|^{-3/2}) \right) \, d\rho \\
&= r^{-\frac{d-1}{2}} \int_{1/2}^2 \rho^{\frac{d-1}{2}} h(\rho) \cos(\rho r + a) \, d\rho + O\left(r^{-\frac{d+1}{2}} \|h\|_{L^1} \right) \\
&= r^{-\frac{d-1}{2}} \left(L\widehat{h}(r) + \int_{-\infty}^\infty \widehat{h}(r-s) s^{-\frac{d+1}{2}} \, ds \right) + O\left(r^{-\frac{d+1}{2}} \|h\|_{L^1} \right).
\end{aligned}$$

TODO: Finish this calculation, found in Theorem 1.2 of [3]. Continuing this calculation shows that the condition in the radial multiplier condition is equivalent to

$$\sup_{t>0} t^{d(1/p-1/q)} \left(\int |\widehat{h}_t(s)|^q \langle s \rangle^{(d-1)(1-q/2)} \, ds \right)^{1/q} < \infty.$$

In particular, if h is supported at a unit scale, then the condition is that

$$\left(\int_0^\infty (1+|s|)^{(d-1)(1-q/2)} |\widehat{h}(s)|^q \, ds \right)^{1/q} < \infty.$$

Thus for most values of s, we have $|\widehat{h}(s)| \lesssim \langle s \rangle^{-(d-1)(1/q-1/2)}$. This characterization of the condition in the radial multiplier conjecture will come in handy later on when we discuss the extension of the radial multiplier conjecture to Riemannian manifolds.

We now know, by the results of [6] that the radial multiplier conjecture is true when $d \geqslant 4$ and when $1 < p < (2d-2)/(d+1)$. When $d = 4$, this was improved by [2], who showed that the conjecture is true here when $1 < p < 36/29$, where $36/29 \approx (2d-1.79)/(d+1)$. When $d = 3$, [2] also established a *restricted weak type* bound

$$\|Tf\|_{L^p(\mathbf{R}^n)} \lesssim \|f\|_{L^{p,1}(\mathbf{R}^n)}$$

when $d = 3$ and $1 < p < 13/12$, where $13/12 \approx (2d-1.66)/(d+1)$ But the radial multiplier conjecture has not yet been completely resolved in any dimension n, we do not have any strong type L^p bounds when $d = 3$, and we don't have any bounds whatsoever when $d = 2$. One goal of this research project is to investigate whether one can use modern research techniques to improve upon these bounds.

The full proof of the radial multiplier is likely far beyond current research techniques. Indeed, it remains a major open problem in harmonic analysis to determine the range of exponents for which *specific* radial Fourier multipliers are bounded in the range where the conjecture would apply, such as the Fourier multiplier on $\mathbf{R}^d$ with symbol $m_\lambda(\xi) = (1-|\xi|)_+^\lambda$, the family of *Bochner-Riesz multipliers*. The radial multiplier conjecture characterizes the range of the Bochner-Riesz multipliers, and thus the conjecture would also imply the Kakeya and restriction conjectures. All three of these results are major unsolved problems in harmonic analysis. On the other hand, the Bochner Riesz conjecture is completely resolved when $d = 2$, while in contrast, no results related to the radial multiplier conjecture are known in this dimension at all. And in any dimension $d > 2$, the range under which the Bochner-Riesz multiplier is known to hold [4] is strictly larger than the range under which the radial multiplier conjecture is known to hold, even for the restricted weak-type bounds obtained in [2]. Thus it still seems within hope that the techniques recently applied to improve results for Bochner-Riesz problem, such as broad-narrow analysis [1], the polynomial Wolff axioms [7], and methods of incidence geometry and polynomial partitioning [12] can be applied to give improvements to current results characterizing the boundedness of general radial Fourier multipliers.

Our hopes are further emboldened when we consult the proofs in [6] and [2], which reduce the radial multiplier conjecture to the study of upper bounds of quantities of the form

$$\left\| \sum_{(y,r)\in\mathcal{E}} F_{y,r} \right\|_{L^p(\mathbf{R}^n)},$$

where $\mathcal{E} \subset \mathbf{R}^n \times (0,\infty)$ is a finite collection of pairs, and $F_{y,r}$ is an oscillating function supported on a $O(1)$ neighborhood of a sphere of radius r centered at a point y. The L^p norm of this sum is closely related to the study of the tangential intersections of these spheres, a problem successfully studied in more combinatorial settings using incidence geometry and polynomial partitioning methods [13], which provides further estimates that these methods might yield further estimates on the radial multiplier conjecture.

When $d = 3$, the results of [2] are only able to obtain bounds on the L^p sums in the last paragraph when $\mathcal{E}$ is a Cartesian product of two subsets of $(0,\infty)$ and $\mathbf{R}^d$. This is why only restricted weak-type bounds have been obtained in this dimension. It is therefore an interesting question whether different techniques enable one to extend the L^p bounds of these sums when the set $\mathcal{E}$ is *not* a Cartesian product, which would allow us to upgrade the result of [2] in $d = 3$ to give strong L^p bounds. This question also has independent interest, because it would imply new results for the 'endpoint' local smoothing conjecture, which concerns the regularity of solutions to the wave equation in $\mathbf{R}^d$. Incidence geometry has been recently applied to yield results on the 'non-endpoint' local smoothing conjecture [5], which again suggests these techniques might be applied to yield the estimates needed to upgrade the result of [2] to give strong L^p-type bounds.

1.2 Multipliers on Riemannian Manifolds

Fix a geodesically complete Riemannian manifold X. We can then define operators $h(\sqrt{-\Delta})$, which are analogues to the radial multipliers studied in the Euclidean setting. Just like multiplier operators on $\mathbf{R}^n$ are crucial to an understanding of the interactions between the functions $e_\xi(x) = e^{2\pi i \xi \cdot x}$ on $\mathbf{R}^n$, understanding the operators $h(\sqrt{-\Delta})$ is crucial to understanding the interactions of eigenfunctions of the Laplace-Beltrami operator on X. We let $M^{p,q}(X, \sqrt{-\Delta})$ denote the family of all symbols $h : \mathbf{R} \to \mathbf{C}$ such that the operator $T_h = h(\sqrt{-\Delta})$ is bounded from $L^p(X)$ to $L^q(X)$, with the analogous operator norm, though, when there is no ambiguity, we will overload notation and write this space as $M^{p,q}(X)$.

To avoid technicalities, we will focus on a compact Riemannian manifold X, which are automatically geodesically complete. For such manifolds, there is a problem which prevents a direct generalization of the radial multiplier conjecture. For such a manifold, there exists $0 = \lambda_1 < \lambda_2 \leqslant \dots$ with $\lambda_i \to \infty$, and an orthonormal family of eigenfunctions $\{e_n\}$ in $C^\infty(X)$, forming a basis for $L^2(X)$, such that

$$\Delta f = \sum -\lambda_n^2 \langle f, e_n \rangle \cdot e_n.$$

Thus for any function $h : [0, \infty) \to \mathbf{C}$,

$$h\left(\sqrt{-\Delta}\right) f = \sum h(\lambda_n) \langle f, e_n \rangle \cdot e_n.$$

The operators $h(\sqrt{-\Delta})$ act as an analogue of radial multipliers on $\mathbf{R}^d$.

The study of multipliers on a Riemannian manifold has a certain technical problem, which the Euclidean case did not have. If h has compact support, this sum will be finite, and thus by the triangle inequality, trivially bounded from $L^p(X)$ to $L^q(X)$ for any exponents p and q. Thus $M^{p,q}(X)$ trivially contains all compactly supported radial multipliers. This trivializes the study of compactly supported radial multipliers in some sense, which is the complete opposite of the Euclidean case, where Littlewood-Paley allowed us to reduce the study of general multipliers to compactly supported radial multipliers. The key here is that Euclidean multipliers automatically have rescaling symmetries, whereas this is not present in the case of compact Riemannian manifolds. To get around this we add a rescaling into the definition of our operator norm, i.e. we study conditions that ensure we have a bound of the form

$$\sup_{t>0} t^{d(1/q-1/p)} \|\mathrm{Dil}_t h\|_{M^{p,q}(X)} < \infty.$$

We let $M^{p,q}_{\mathrm{Dil}}(X)$ denote the family of all multipliers for which the inequality above holds, and give it the norm induced by the quantity on the left hand side. A transference principle of Mitjagin [10] shows that if

X is a compact Riemannian manifold, and $m : \mathbf{R}^d \to \mathbf{C}$ is radial, with $m(\xi) = h(|\xi|)$, then

$$\|m\|_{M^{p,q}(\mathbf{R}^d)} \lesssim_{X,p,q} \|h\|_{M^{p,q}_{\mathrm{Dil}}(X)}.$$

Thus, in some sense, the dilation invariant Fourier multiplier problem on a compact manifold X is at least as hard as it is on $\mathbf{R}^n$. Another goal of this research project is to try and extend the radial multiplier conjecture to the setting of dilation invariant bounds for multipliers of the Laplacian on Riemannian manifolds.

As in the case $X = \mathbf{R}^d$, the study of multipliers in $M^{2,2}_{\mathrm{Dil}}(X)$ is trivial. Indeed, applying orthogonality, we calculate that

$$\begin{aligned}
\|h(\sqrt{-\Delta})f\|_{L^2(X)} &= \left\| \sum_\lambda h(\lambda) E_\lambda f \right\|_{L^2(X)} \\
&= \left(\sum_\lambda |h(\lambda)|^2 \|E_\lambda f\|_{L^2(X)}^2 \right)^{1/2} \\
&\leqslant \left(\sup_{\lambda \in \sigma(\sqrt{-\Delta})} |h(\lambda)| \right) \left(\sum_\lambda \|E_\lambda f\|_{L^2(X)}^2 \right)^{1/2} \\
&= \left(\sup_{\lambda \in \sigma(\sqrt{-\Delta})} |h(\lambda)| \right) \|f\|_{L^2(X)}.
\end{aligned}$$

Taking f to be an eigenfunction with eigenvalue λ which maximizes the value of $|h(\lambda)|$ shows this inequality is tight, i.e. we have

$$\|h\|_{M^{2,2}(X)} = \sup_{\lambda \in \sigma(\sqrt{-\Delta})} |h(\lambda)|.$$

Now applying an arbitrary dilation to h, we conclude that

$$\|h\|_{M^{2,2}_{\mathrm{Dil}}(X)} = \sup_{\lambda > 0} |h(\lambda)|.$$

Thus we have found a simple characterization of the space $M^{2,2}_{\mathrm{Dil}}(X)$.

The spaces $M^{1,q}_{\mathrm{Dil}}(X)$ are a little more tricky, since we do not have a precise theory of the Fourier transform in the setting of general Riemannian manifolds. To take a look at these bounds, we recall that $L^1 \to L^q$ bounds of an operator are characterized by Schur's test. If $\{e_n\}$ is a $C^\infty(X)$ basis of eigenfunctions on X, with $\Delta e_n = -\lambda_n^2 e_n$, then

$$h(\sqrt{-\Delta})f(x) = \sum h(\lambda_n)\langle f, e_n \langle e_n(x) = \int \left(\sum_n h(\lambda_n) e_n(x) \overline{e_n(y)} \right) f(y)\, dy.$$

Thus the kernel of $h(\sqrt{-\Delta})$ is precisely $K(x,y) = \sum_n h(\lambda_n) e_n(x) \overline{e_n(y)}$, and we conclude by Schur's test that

$$\|h\|_{M^{1,q}(X)} = \left\| \sum_n h(\lambda_n) e_n(x) \overline{e_n(y)} \right\|_{L^\infty_y L^q_x}.$$

In the case $X = \mathbf{R}^n$, the analogous kernel is $K(x,y) = \int_{\mathbf{R}^d} h(|\xi|) e^{2\pi i \xi \cdot x} \overline{e^{2\pi i \xi \cdot y}}$, which can be explicitly reduced to $K(x,y) = \mathcal{B}_d h(|x-y|)$, and the condition of being contained in $M^{1,q}(\mathbf{R}^d)$ then becomes that

$$\left(\int r^{d-1} |\mathcal{B}_d h(r)|^q\, dr \right)^{1/q} < \infty.$$

If h is compactly supported, then this condition is equivalent to the condition that

$$\left(\int (1+|t|)^{(d-1)(1-q/2)} |\widehat{h}(t)|^q\, dt \right)^{1/q} < \infty.$$

In the general setting we do not have quite as nice a formula, but we can still *force* the Fourier transform into the equation to see if it can be used to understand these quantities (which will be necessary for studying the radial multiplier conjecture). We thus write

$$\begin{aligned}
\sum_n h(\lambda_n) e_n(x) \overline{e_n(y)} &= \sum_n \left(\int \widehat{h}(t) e^{2\pi i t \lambda_n} e_n(x) \overline{e_n(y)}\, dt \right) \\
&= \int \widehat{h}(t) \left(\sum_n e^{2\pi i t \lambda_n} e_n(x) \overline{e_n(y)} \right) dt \\
&= \int \widehat{h}(t) W_t(x,y)\, dt,
\end{aligned}$$

where $W_t(x,y) = \sum_n e^{2\pi i t \lambda_n} e_n(x)\overline{e_n(y)}$ is the kernel of the *half-wave propogator* $e^{2\pi i t\sqrt{-\Delta}}$ on X. The connection between radial multipliers on X and the Fourier transform of their symbol is therefore closely related to the study of the half-wave equation $\partial_t = \sqrt{-\Delta}$ on X. We are therefore looking for an inequality of the form

$$\left\| \int \widehat{h}(t) W_t(x,y)\, dt \right\|_{L^\infty_y L^q_x} \lesssim \left(\int (1+|t|)^{(d-1)(1-q/2)} |\widehat{h}(t)|^q \, dt \right)^{1/q}$$

to hold. TODO: What techniques can we use to obtain this bound? TODO: Can we come up with a proof of this bound in the model case $X = \mathbf{R}^d$, i.e. an alternate proof of the characterization of $L^1 \to L^q$ boundedness?

Directly translating the assumptions of the radial multiplier conjecture to this setting yields the following statement: If $h : [0,\infty) \to \mathbf{R}$ is a function, and we define

$$A_{p,q}(h) = \sup_{t>0} t^{d(1/p-1/q)} \left(\int |\widehat{h}_t(s)|^q (1+|s|)^{(d-1)(1-q/2)} \, ds \right)^{1/q},$$

then for what values of p and q is is true that the inequality

$$\|h\|_{M^{p,q}_{\mathrm{Dil}}(X)} \lesssim A_{p,q}(h)$$

still holds. Mitjagin's result implies that we require $1 < p < 2d/(d+1)$ and $p \leqslant q < 2$, and we conjecture that, perhaps under appropriate assumptions on X, we can achieve similar ranges of exponents as have been obtained for the Euclidean radial multiplier conjecture.

On general compact manifolds, there are difficulties arising from a generalization of the radial multiplier conjecture, connected to the fact that analogues of the Kakeya / Nikodym conjecture are false in this general setting [9]. But these problems do not arise for constant curvature manifolds, like the sphere. The sphere also has over special properties which make it especially amenable to analysis, such as the fact that solutions to the wave equation on spheres are periodic. Best of all, there are already results which achieve the analogue of [3] on the sphere. Thus it seems reasonable that current research techniques can obtain interesting results for radial multipliers on the sphere, at least in the ranges established in [6] or even [2].

1.3 Summary

In conclusion, the results of [6] and [2] indicate three lines of questioning about radial Fourier multiplier operators, which current research techniques place us in reach of resolving. The first question is whether we can extend the range of exponents upon which the conjecture of [3] is true, at least in the case $d = 2$ where Bochner-Riesz has been solved. The second is whether we can use more sophisticated arguments to prove the L^p sum bounds obtained in [2] when $d = 3$ when the sums are no longer Cartesian products, thus obtaining strong L^p characterizations in this settiong, as well as new results about the endpoint local smoothing conjecture. The third question is whether we can generalize these bounds obtained in these two papers to study radial Fourier multipliers on the sphere.

Chapter 2

Distributions on Manifolds

How do we work with distributions on $\mathbf{R}^d$? We first identify a vector space of test functions, say, the space $\mathcal{D}(\mathbf{R}^d)$ of smooth, compactly supported functions, the space $\mathcal{E}(\mathbf{R}^d)$ of all smooth functions, or the space $\mathcal{S}(\mathbf{R}^d)$ of Schwartz functions. The distributions are then formally defined as the dual space of this class of test functions. To actually work with these distributions, we find an explicit way to represent them, via a bilinear pairing; for $\mathcal{D}(\mathbf{R}^d)$, the bilinear pairing $\mathcal{E}(\mathbf{R}^d) \times \mathcal{D}(\mathbf{R}^d) \to \mathbf{C}$ given by

$$(\phi, \psi) \mapsto \int_{\mathbf{R}^d} \phi(x)\psi(x)\, dx.$$

This pairing allows us to naturally identify certain distributions in $\mathcal{D}(\mathbf{R}^d)^*$ as elements of $\mathcal{E}(\mathbf{R}^d)$ via this pairing, and actually, *all distributions* are weak limits of such distributions. Thus we can intuitively study elements of $\mathcal{D}(\mathbf{R}^d)^*$ as if they behaved like elements of $\mathcal{E}(\mathbf{R}^d)$, at least when integrated against elements of $\mathcal{D}(\mathbf{R}^d)$. Reversing this pairing allows us to think of elements of $\mathcal{E}(\mathbf{R}^d)^*$ as elements of $\mathcal{D}(\mathbf{R}^d)$, and the pairing $\mathcal{S}(\mathbf{R}^d) \times \mathcal{S}(\mathbf{R}^d) \to \mathbf{C}$ allows us to think of tempered distributions of $\mathcal{S}(\mathbf{R}^d)$ as if they themselves were elements of $\mathcal{S}(\mathbf{R}^d)$.

The bilinear pairings here often behave well with respect to natural operations on the respective spaces. For instance, we have integration by parts identities

$$(\partial^\alpha \phi, \psi) = (-1)^{|\alpha|}(\phi, \partial^\alpha \psi)$$

for all the pairings above. And for the pairing of $\mathcal{S}(\mathbf{R}^d)$, we have the multiplication formula

$$(\phi,\psi) = \left(\mathcal{F}\phi, \overline{\mathcal{F}\psi}\right)$$

where $\mathcal{F}$ is the Fourier transform. By taking weak limits, this allows us to extend the derivative and Fourier transform operations for distributions. In particular, we can take the derivatives of elements of $\mathcal{D}(\mathbf{R}^d)^*$, $\mathcal{E}(\mathbf{R}^d)^*$ and $\mathcal{S}(\mathbf{R}^d)^*$, and we can take the Fourier transform of elements of $\mathcal{S}(\mathbf{R}^d)^*$. By taking vector pairings, i.e. by setting

$$(v,w) = \int (v \cdot w)\, dx$$

where v and w are vector fields, we can define vector-valued distributions, and the theory above extends analogously.

We can also defined other more technical operations on distributions u as if they were functions, provided that we have information about their *wavefront sets* $\mathrm{WF}(u)$, which is a subset of $\mathbf{R}^d_x \times \mathbf{R}^d_\xi$, conic in the variable ξ, which give information about the position and 'direction' of the singularities of u. As an example, if $f : \mathbf{R}^n_x \to \mathbf{R}^m_y$ is smooth, and $u \in \mathcal{D}(\mathbf{R}^m)$ is a distribution such that $\mathrm{WF}(u)$ does not contain any point of the form (y,η) for which there exists $x \in f^{-1}(y)$ with $Df(x)^t\eta = 0$, then we can define f^*u, which can be interpreted as a weak limit of a sequence $\{f^*u_n\}$, where $u_n \in \mathcal{E}(\mathbf{R}^m)$ converges weakly to u in a way respecting the wavefront set of u; the advantage of this is that f^*u_n is just $u_n \circ f$. If $\phi \in \mathcal{E}(\mathbf{R}^m)$, then the chain rule

$$[\nabla_x(f^*\phi)](x) = Df(x)^T f^*(\nabla_y\phi)(x)$$

can be written as

$$(\nabla_x(f^*\phi), v) = (f^*(\nabla_y\phi), Df \cdot v),$$

which allows us to extend the chain rule to the pullback of distributions.

Now let's do the same thing on a Riemannian manifold M^d. Here we also have natural spaces of test functions, i.e. the spaces $\mathcal{E}(M)$ and $\mathcal{D}(M)$ of smooth functions, the latter of which specified to have compact support. Here the natural pairing $\mathcal{E}(M) \times \mathcal{D}(M) \to \mathbf{C}$ is given by integration against the *volume measure* on the manifold M, i.e. the pairing is given by

$$(\phi,\psi) \mapsto \int_M \phi(x)\psi(x)\, dV(x).$$

This pairing allows us to identify $\mathcal{E}(M)^*$ and $\mathcal{D}(M)^*$ with weak limits of elements of $\mathcal{D}(M)$ and $\mathcal{E}(M)$ respectively. It will also be convenient to consider *vector-valued distributions*, i.e. the dual spaces of the spaces $\mathcal{E}(\Gamma(TM))$ and $\mathcal{D}(\Gamma(TM))$ of smooth vector fields, the latter of which limited to have compact support. These spaces have a pairing given by

$$(X,Y) \mapsto \int_M \langle X,Y\rangle_g \, dV.$$

Thus we can identify the dual spaces $\mathcal{E}(\Gamma(TM))^*$ and $\mathcal{D}(\Gamma(TM))^*$ with weak limits of vector fields.

On a Riemannian manifold, the natural derivative operators to consider are the *gradient operator*, which, for a given function $f \in \mathcal{E}(M)$, gives a smooth vector field $\nabla_g f \in \Gamma(TM)$, which has the property that for any other smooth vector field $X \in \Gamma(TM)$,

$$X(f) = \langle X, \nabla_g f\rangle_g.$$

We also have a *divergence operator*, which associates with any smooth vector field $X \in \Gamma(TM)$ a smooth function $\nabla_g \cdot X$ such that the integration by parts identity

$$(X, \nabla_g f) = -(\nabla_g \cdot X, f)$$

holds. This formula gives us a way to interpret the gradient $\nabla_g u \in \mathcal{D}(\Gamma(TM))^*$ of a general distribution $u \in \mathcal{D}(M)^*$, by testing the gradient against a general vector field. Similarily, we can consider the divergence $\nabla_g \cdot X \in \mathcal{D}(M)^*$ of a distributional vector fields $X \in \mathcal{D}(\Gamma(TM))^*$. Combining these operators gives us the *Laplace-Beltrami operator* $\Delta_g f = \nabla_g \cdot \nabla_g f$, which we can now consider as a map from $\mathcal{D}(M)^*$ to itself, and from $\mathcal{E}(M)^*$ to itself. We note that in coordinates, we have

$$\nabla_g f = \sum_{i,j} \frac{\partial f}{\partial x^i} g^{ij} \frac{\partial}{\partial x^j},$$

and the divergence is given by

$$\nabla_g \cdot X = |g|^{-1/2} \sum_i \frac{\partial}{\partial x^i} \left\{ |g|^{1/2} X^i \right\},$$

where $|g|$ is the determinant of the matrix with coefficients $\{g_{ij}\}$, which we can roughly think of as the volume of the unit ball in the metric.

Every distribution u has an associated wavefront set $\mathrm{WF}(u)$, which forms a conic subset of T^*M. We can consider the pullback of a distribution along a smooth map $f : (M, g) \to (N, g')$, to find the chain rule for this pullback, we note that for any vector field Y on N, we can calculate that for any $p \in M$,

$$\langle (\nabla_g f^* \phi)_p, X_p \rangle_g = \langle (f^* \nabla_{g'} \phi)_p, f_* X_p \rangle_{g'}.$$

Thus

$$(\nabla_g f^* \phi, X) = (f^*(\nabla_{g'} \phi), f_* X),$$

where we view the elements of the pairing on the right hand side as elements of $\Gamma(f^*(TN))$.

Chapter 3

The Lax Parametrix

Here we discuss a method due to Lax, which constructs a *parametrix* for the equation $\partial_t - 2\pi i T$ over small times, i.e. for $|t| \lesssim 1$, which is expressed as a *Fourier integral operator*. The equation $\partial_t = 2\pi i T$ is a pseudodifferential variant of a basic hyperbolic partial differential equation, so based on the type of parametrices one can construct in the hyperbolic setting, one might hope to find a parametrix defined by an oscillatory integral of the form

$$Sf(x,t) = S(t)f(x) = \int s(t,x,y,\xi) e^{2\pi i \Phi(t,x,y,\xi)} f(y)\, dy\, d\xi,$$

such that:

- $\Phi(t,x,y,\xi) = \phi(x,y,\xi) + tp(y,\xi)$, where ϕ is smooth away from $\xi = 0$, homogeneous of degree one, and $\phi(x,y,\xi) \approx (x-y)\cdot\xi$ in the sense that on the support of s,

  $$\partial_\xi^\beta \{\phi(x,y,\xi) - (x-y)\cdot\xi\} \lesssim_\beta |x-y|^2 |\xi|^{1-\beta}.$$

  In particular, this implies that $\phi(x,y,\xi) = 0$ when $(x-y)\cdot\xi = 0$.

- s is a symbol of order zero, supported on $|x-y| \lesssim 1$ and on $|\xi| \geqslant 1$, in such a way that

  $$|\nabla_\xi \phi(x,y,\xi)| \gtrsim |x-y| \quad \text{and} \quad |\nabla_x \phi(x,y,\xi)| \gtrsim |\xi|$$

  for $(x,y) \in \operatorname{supp}_x(s) \times \operatorname{supp}_y(s)$.

If $(\partial_t - 2\pi i T) \circ S = 0$ is a smoothing operator on $(-\varepsilon, \varepsilon) \times M$ and $S(0)$ differs from the identity operator by a smoothing operator, then $S - e^{2\pi i t T}$ is smoothing for $|t| \leqslant \varepsilon$. The construction of the parametrix S will therefore give us much more information about the behaviour of the propogators $e^{2\pi i t T}$ over small times.

To find a choice of ϕ and s which gives us this parametrix, let us start by determining what properties these functions should satisfy. Let us fix a coordinate system (x, U), where $x(U)$ is a precompact subset of $\mathbf{R}^n$. Let us assume that in these coordinates, T has symbol $a(x, \xi)$. Then the kernel of $(\partial_t - 2\pi i T) \circ S$ in this coordinate system is

$$\int (\partial_t + 2\pi i T(x, D)) \left\{ s(t, \cdot, y, \xi) e^{2\pi i \Phi(t, \cdot, y, \xi)} \right\} \, d\xi.$$

If we set

$$s'(t, x, y, \xi) = e^{-2\pi i \Phi(t, x, y, \xi)} (\partial_t - 2\pi i T(x, D)) \left\{ s(t, \cdot, y, \xi) e^{2\pi i \Phi(t, \cdot, y, \xi)} \right\}$$

then the kernel is

$$\int s'(t, x, y, \xi) e^{2\pi i \Phi(t, x, y, \xi)} \, d\xi.$$

Provided that s' is a symbol of order $-\infty$ for $0 < |t| \leqslant \varepsilon$, integration by parts shows that $(\partial_t + 2\pi i T) \circ S$ is smoothing, and so we will try to choose ϕ and s so as to obtain such a result.

In our discussion of pseudodifferential operators, we have already discussed an asymptotic formula for s', namely, if

$$r_{x,y}(z) = \nabla_x \phi(x, z, \xi) \cdot (x - z) - \{\phi(x, y, \xi) - \phi(z, y, \xi)\}.$$

then for any $N > 0$, if $a \sim \sum_{k=-\infty}^{1} a_k$, where a_k is homogeneous of degree

k, and if $\xi_\phi = \nabla_x \Phi(t,x,y,\xi) = \nabla_x \phi(x,y,\xi)$,

$$\begin{aligned}
&s'(t,x,y,\xi) \\
&\quad = \underbrace{\big(p(y,\xi) - a(x,\xi_\phi)\big) \cdot s(t,x,y,\xi)}_{\text{symbols of order 1}} \\
&\qquad + \underbrace{\partial_t s(t,x,y,\xi)}_{\text{symbol of order 0}} \\
&\qquad - \sum_{1 \leqslant |\beta| < N} \underbrace{\frac{2\pi i}{\beta! \cdot (2\pi i)^\beta} \cdot \partial_\xi^\beta a(x,\xi_\phi) \partial_z^\beta \{e^{2\pi i r_{x,y}(z)} s(t,z,y,\xi)\}|_{z=y}}_{\text{symbols of order } 1 - \lceil |\beta|/2 \rceil} \\
&\qquad + R_N(t,x,y,\xi).
\end{aligned}$$

where, because $|\nabla_x \Phi(t,x,y,\xi)| \gtrsim |\xi|$ on the support of s,

$$\langle \xi \rangle^{t - \lceil N/2 \rceil} R_N \in L^\infty((-\varepsilon,\varepsilon) \times U \times U \times \mathbf{R}^d).$$

It is simple to establish estimates of the form

$$|\partial_x^\alpha \partial_y^\beta \partial_\xi^\lambda s'(t,x,y,\xi)| \lesssim \langle \xi \rangle^{N_{\alpha\beta\lambda}}.$$

Thus if we can justify that $|s'(t,x,y,\xi)| \lesssim_N \langle \xi \rangle^{-N}$ for all $N > 0$, then it will follow that s' is a symbol of order $-\infty$. We now determine the propoerties of the symbol s and the symbol ϕ which will give us these estimates.

To begin with, let us specify the function ϕ. In order to guarantee that s' is a symbol of order zero, the expansion above shows that $(p(y,\xi) - p(x,\xi_\phi)) \cdot s(t,x,y,\xi)$ must be a symbol of order zero. This will be true if we can pick ϕ such that, on the support of s, and for $|\xi| \gtrsim 1$,

$$p(x, \nabla_x \phi(x,y,\xi)) = p(y,\xi).$$

This is an example of an *Eikonal equation*, e.g. an equation of the form

$$q(z, \nabla_z \psi(z)) = 0$$

for some function $q(z,\zeta)$. In our case, $z = (x,y,\xi)$, so $\zeta = (\zeta_x, \zeta_y, \zeta_\xi)$, and so

$$q(z,\zeta) = p(x,\zeta_x) - p(y,\xi).$$

Let us make some further remarks we desire about our choice of function ϕ:

- We want ϕ to be homogeneous and smooth away from the origin. If we solve the equation for all $|\xi| = 1$, and then extend ϕ such that for $\lambda > 0$ and $|\xi| = 1$,

$$\phi(x, y, \lambda\xi) = \lambda\phi(x, y, \xi)\psi(\lambda),$$

where ψ is smooth, equal to one for $|\lambda| \geqslant 3/4$, and vanishing for $|\lambda| \leqslant 1/2$, then ϕ will satisfy the equation for all $|\xi| \gtrsim 1$. This means that

$$p(x, \nabla_x\phi(x, y, \xi)) - p(y, \xi)$$

is smooth and supported on $|\xi| \lesssim 1$, which implies it is a symbol of order $-\infty$, which suffices for our construction. Thus it suffices to solve the equation for $|\xi| = 1$.

- Since ϕ is smooth away from the origin and homogeneous, the equation

$$|\partial_\xi^\beta \{\phi(x, y, \xi) - (x - y) \cdot \xi\}| \lesssim_\beta |x - y|^2 \langle\xi\rangle^{1-\beta}$$

holds if, for $|\xi| = 1$, we have $\phi(x, y, \xi) = 0$ whenever $(x-y)\cdot\xi = 0$, and $\nabla_x\phi(x, y, \xi) = \xi$ whenever $x = y$. Thus we have some *initial conditions* for our Eikonal equation.

The second condition constitutes a type of initial condition for ϕ, since it specifies it's behaviour on a hypersurface, a kind of Cauchy condition, and thus we should expect these are close to the conditions that give unique solutions to the equation. And the following Lemma indeed shows that there is a unique function ϕ, defined for $|x - y| \lesssim 1$ and $|\xi| = 1$ with these properties.

Lemma 3.1. *Let Z be a smooth manifold, and let $q(z, \zeta)$ be a real-valued, smooth function defined locally around a point $(z_0, \zeta_0) \in T^*Z$. Let S be a smooth hypersurface in Z passing through z_0 with conormal vector ζ_S at z_0, such that*

$$\frac{\partial q}{\partial \zeta_S}(z_0, \zeta_0) = \lim_{t \to 0} \frac{q(z_0, \zeta_0 + t\zeta_S) - q(z_0, \zeta_0)}{t}$$

is nonzero. Suppose that ψ is any smooth function defined on S locally about z_0, such that $d\psi(z_0)$ agrees with the action of ζ_0 on $T_{x_0}S$. Then there exists a unique smooth function ϕ defined in a neighborhood of z_0, which agrees with ψ on S, satisfies the Eikonal equation $q(z, \nabla_z\phi(z)) = 0$, and has $\nabla_z\phi(z_0) = \zeta_0$.

Proof. TODO: See Sogge, Theorem 4.1.1. □

In our case,

$$Z = \{(x,y,\xi) : |\xi| = 1\}.$$

We have $z_0 = (x_0, x_0, \xi_0)$, $\zeta_0 = (\xi, \xi, 0)$, and

$$S = \{(x,y,\xi) : |\xi| = 1 \quad \text{and} \quad (x-y)\cdot\xi = 0\}.$$

The conormal vector ξ_S of S at z_0 is a multiple of $(\xi_0, -\xi_0, 0)$, and so by homogeneity,

$$\frac{\partial q}{\partial \xi_S} = \lim_{t\to 0} \frac{p(x_0,(1+t)\xi_0) - p(x_0,\xi_0)}{t} = p(x_0,\xi_0),$$

which is nonvanishing because T is elliptic. If we define ψ equal to zero on S, then $d\psi = 0$, which agrees with the action of ζ_0 on S. Thus the theorem applies local uniqueness and existence to solutions to the Eikonal equation, and by compactness of Z we can patch such solutions together to find a solution defined for all $|x-y| \lesssim 1$.

We therefore conclude that there exists a unique choice of ϕ such that, if s has small enough support, $s'(t,x,y,\xi)$ is a symbol of order zero. Next, let us see what constraints are forced on us in order to ensure that $S(0)$ differs from the identity by a smoothing operator. The kernel of U is precisely

$$\int s(0,x,y,\xi) e^{2\pi i \phi(x,y,\xi)}\, d\xi.$$

We now show that this operator is actually a *pseudodifferential operator* of order zero, and determine it's symbol up to first order.

To do this, we write $\phi_\alpha(x,y,\xi) = (1-\alpha)\phi(x,y,\xi) + \alpha(x-y)\cdot\xi$. Let U_α be the operator with kernel

$$\int s(0,x,y,\xi) e^{2\pi i \phi_\alpha(x,y,\xi)}\, d\xi.$$

Assume the support of s is close enough to the diagonal such that

$$|\nabla_\xi \phi_\alpha(x,y,\xi)| \gtrsim |x-y|$$

on the support of s. Then $\partial_\alpha^n U_t$ has kernel

$$\int (2\pi i)^n (\phi_1 - \phi_0)^n s(0,x,y,\xi) e^{2\pi i \phi_\alpha(x,y,\xi)}\, d\xi.$$

This is an oscillatory integral defined by a symbol of order n. However, when $t = 1$, the fact that $\phi(x,y,\xi) \approx (x-y)\cdot\xi$, together with the formula for converting pseudodifferential operators with compound symbols into standard Kohn-Nirenberg type symbols shows that $\partial_\alpha^n U_1$ is actually a pseudodifferential operator of order $-n$. Integration by parts, similarily, shows that $\partial_\alpha^n U_t$ is defined by an oscillator integral against a symbol of order $-n$. But this means that if we define a pseudodifferential operator by the asymptotic formula

$$V \sim \sum \frac{(-1)^n}{n!}\partial_\alpha^n U_1,$$

then $U - V$ is smoothing. Indeed, for any n, by Taylor's formula we have

$$U = \sum_{k=0}^{n-1} \frac{(-1)^n}{n!}\partial_\alpha^n U_1 + \frac{(-1)^n}{n!}\int_0^1 \alpha^{n-1}\partial_\alpha^n U_\alpha \, d\alpha$$

The integral here is an oscillatory integral defined against a symbol of order $-n$, and thus taking $n \to \infty$ verifies the claim.

It is an important remark that reversing this argument shows that *any* pseudodifferential operator can be written in the form above for the particular choice of ϕ we have given. This is a special case of the *equivalence of phase functions* theorem. This in particular guarantees that we can choose a symbol $I(x,y,\xi)$ of order zero such that $U - 1$ is smoothing if and only if $s(0,x,y,\xi) - I(x,y,\xi)$ is a symbol of order $-\infty$. The symbol I can be chosen to be vanishing for $|x-y| \gtrsim 1$, since the difference will be a smoothing pseudodifferential operator.

Next, the quantity

$$\begin{aligned}
&\partial_t s(t,x,y,\xi) \\
&\quad + \sum_{k=1}^{d} \partial_\xi^k a(x,\xi_\phi)\partial_x^k s(t,x,y,\xi) \\
&\quad + \left(a_0(x,\xi_\phi) + \frac{1}{2\pi}\sum_{|\beta|=2} \partial_\xi^\beta p(x,\xi_\phi)\partial_x^\beta \phi(x,y,\xi) \right) s(t,x,y,\xi).
\end{aligned}$$

must be a symbol of order -1. But because the coefficients of this equation are smooth, and all derivatives are bounded, it follows from the general theory of transport equations that there exists a unique smooth, function s_0 defined for $|t| \leqslant \varepsilon$, which is a symbol of order zero, such that $s_0(0,x,y,\xi) = I(x,y,\xi)$, s_0 vanishes for $|x-y| \gtrsim 1$, and satisfies the transport equation

$$\begin{aligned}\partial_t s_0(t,x,y,\xi)\\ &+ \sum_{k=1}^{d} \partial_\xi^k a(x,\xi_\phi) \partial_x^k s_0(t,x,y,\xi)\\ &+ \left(a_0(x,\xi_\phi) + \frac{1}{2\pi} \sum_{|\beta|=2} \partial_\xi^\beta p(x,\xi_\phi) \partial_x^\beta \phi(x,y,\xi) \right) s_0(t,x,y,\xi) = 0.\end{aligned}$$

We have thus justified that the quantity

$$R_0(t,x,y,\xi) = e^{-2\pi i \Phi(t,x,y,\xi)} (\partial_t - 2\pi i T)(s_0(t,\cdot,y,\xi) e^{2\pi i \Phi(t,x,y,\xi)})$$

is a symbol of order -1. Now we come to a quirk of this parametrix, which does not occur in the study of hyperbolic partial differential equations. Since the operator $P(x,D)$ is only *pseudolocal* rather than completely local, the remainder term R_0 is *not* necessarily supported on a neighborhood of the origin. To fix this, we now successively define the terms $\{s_k\}$ for $k < 0$, which are symbols of order $-k$, such that $s_k(0,x,y,\xi) = 0$, and

$$TODO: SPECIFY REQUIRED EQUATION.$$

Again, solutions exist for small time periods. And this implies that $e^{-2\pi i \Phi(t,x,y,\xi)}(\partial_t + 2\pi i T)((s_0 + \dots + s_{-k}) e^{2\pi i \Phi(t,x,y,\xi)})$ is a symbol of order $-k$ (TODO: Is It), and we can continue the calculation to complete the argument.

Chapter 4

The Hadamard Parametrix

The Hadamard parametrix gives an alternate expression as a wave to invert the wave equation on a Riemannian manifold, though one requires much tighter control of the geometry of the underlying Riemannian manifold. We begin with summarizing facts about the fundamental solution of the standard wave equation on $\mathbf{R}^{d+1}$, before moving onto fundamental solutions of the wave equation on $\mathbf{R}^{d+1}$ with an arbitrary *constant coefficient* Riemannian metric, and then we move to constructing a parametrix on an arbitrary Riemannian manifold.

4.1 Euclidean Case

Let's begin with the standard wave equation on $\mathbf{R}^d$, equipped with the standard metric. We are therefore concerned with constructing a fundamental solution for the D'Alembertian operator $\Box = \partial_t^2 - \Delta_x$ on $\mathbf{R}^{d+1}$. One choice of fundamental solution for $\Box$ is the *forward fundamental solution*, defined for $d \geqslant 2$ by the equation

$$E_+(x,t) = c_d \frac{H(t)}{\mathrm{Im}(|x|^2 - (t+i0)^2)^{\frac{d-1}{2}}}$$

where H is the heaviside step function, and

$$c_d = \frac{2}{(n+1)A_{n+1}},$$

and where A_{n+1} denotes the surface area of S^n. We have

$$\operatorname{Supp}(E_+) = \{(x,t) : t \geqslant 0 \text{ and } |x| \leqslant t\},$$

Furthermore, we have

$$\operatorname{Sing\,Supp}(E_+) = \{(x,t) : t \geqslant 0 \text{ and } |x| = t\},$$

i.e. E_+ has singular support on the *forward light cone*. When d is odd, E_+ actually has *support* on the forward light cone; this is *Huygen's principle*. For $d = 2$, we have

$$E_+(t,x) = \frac{H(t)H(t^2 - |x|^2)}{2\pi(t^2 - |x|^2)^{1/2}},$$

where the right hand side is locally integrable, and thus defines a distribution. For $d = 3$, we have

$$E_+(t,x) = \frac{H(t)\delta(t^2 - |x|^2)}{2\pi} = \frac{H(t)\delta(t - |x|)}{4\pi t}.$$

When $d \geqslant 4$, the forward fundamental solution becomes a distribution of higher order, i.e. becoming more singular on the forward light cone. For $d = 2$ the equation above no longer applies, but we have the simpler formula

$$E_+(t,x) = \frac{H(t)H(t^2 - |x|^2)}{2}.$$

It is interesting to note that E_+ is the *unique* fundamental solution supported on the interior of the forward light cone.

Lemma 4.1. *If v is a fundamental solution of the D'Alembertian, supported on the interior of the forward light cone, then $v = E_+$.*

Proof. If $u = v - E_+$, then u is supported on the interior of the forward light cone and $\Box u = 0$. But this means that

$$u = \delta * u = \Box E_+ * u = E_+ * \Box u = E_+ * 0 = 0,$$

where these convolutions are well defined precisely because of the support of all the quantities involved. $\square$

The reflection of the forward fundamental solution about the origin $t = 0$ is another fundamental solution to the wave equation, which we denote by E_-. It is supported on the interior of the backward light cone, and called the *backward fundamental solution*. Taking convex combinations of these fundamental solutions gives a plethora of other fundamental solutions, like the solution

$$E_{\text{FW}} = \frac{E_+ + E_-}{2},$$

the *Feynman-Wheeler* fundamental solution.

Using the Fourier transform, we can write

$$E_+(x,t) = \frac{H(t)}{2\pi} \int \frac{\sin(2\pi t|\xi|)}{|\xi|} e^{2\pi i \xi \cdot x}\, d\xi.$$

Modifying this formula gives Fourier expressions for the backward fundamental solution, and the Feynman-Wheeler fundamental solution. We also have a solution of the form

$$E_F(t,x) = \frac{1}{4\pi i} \int e^{2\pi i(\xi \cdot x + |t\xi|)}\, \frac{d\xi}{|\xi|},$$

the *Feynman fundamental solution*. This fundamental solution has support on the entirety of $\mathbf{R}^{d+1}$, which is counterintuitive given the finite propogation speed of the wave equation.

Let us use these fundamental solutions to solve the Cauchy problem for the wave equation.

Theorem 4.2. *Suppose $f,g \in C^\infty(\mathbf{R}^d)$, and $F \in C^\infty(\mathbf{R}^d \times [0,\infty))$. Then the Cauchy problem*

$$\Box u(x,t) = F(x,t)$$

with $u(0,t) = f(x)$ and $\partial_t u(x,0) = g(x)$ has a unique solution in $C^\infty(\mathbf{R}^d \times [0,\infty))$, and we can write this solution as

$$u(t) = \partial_t E_+(t) * f + E_+(t) * g + \int_0^t E_+(t-s) * F(s)\, ds.$$

If we assume $f,g \in \mathcal{S}(\mathbf{R}^d)$, and $F \in C^\infty(\mathbf{R}_t, \mathcal{S}(\mathbf{R}^d_x))$, then we can also write

$$u(x,t) = \cos(2\pi t\sqrt{-\Delta}) f + \frac{\sin(2\pi t\sqrt{-\Delta})}{\sqrt{-\Delta}} g + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s)\, ds.$$

Proof. The Fourier multiplier formula follows from the Fourier expression of E_+. The expression above is well defined since $\partial_t E_+(t)$ and $E_+(t)$ are smooth functions of t valued in $\mathcal{E}^*(\mathbf{R}^d)$. Now

$$\Box E_+(t) = \Box \partial_t E_+(t) = 0$$

for $t > 0$, so it is clear that $v(t) = \partial_t E_+(t) * f + E_+(t) * g$ solves the wave equation for $t > 0$. Since $(E_+)(0+) = 0$, and $(\partial_t E_+)(0+) = \delta_0$, $v(t)$ has the required initial conditions, so would solve the equation provided there were no forcing term. Thus it suffices to show that

$$w(t) = \int_0^t E_+(t-s) * F(s)\, ds$$

solves the equation $\Box w = F$, with vanishing initial conditions. Now since $E_+(0+) = 0$,

$$\partial_t w(t) = \int_0^t (\partial_t E_+)(t-s) * F(s)\, ds$$

and since $\partial_t E_+(0+) = \delta_0$,

$$\begin{aligned}
\partial_t^2 w(t) &= F(t) + \int_0^t (\partial_t^2 E_+)(t-s) * F(s)\, ds. \\
&= F(t) + \int_0^t (\Delta E_+)(t-s) * F(s)\, ds \\
&= F(t) + \Delta w(t).
\end{aligned}$$

Thus $\Box w = F$. It is clear from the above formulas that $w(0+) = \partial_t w(0+) = 0$. Thus we proved the *existence* of solutions to the wave equation. Uniqueness follows from our uniqueness argument that E_+ is the unique fundamental solution, since it suffices to show that there is no nonzero $u \in C^\infty(\mathbf{R}^d \times [0,\infty))$ with $\Box u = 0$ and with vanishing initial conditions. $\Box$

In order to construct fundamental solutions to the wave equation on a Riemannian manifold, it is helpful to note that we can find constants $\{a_\nu : \nu \geqslant 1\}$ such that if

$$E_\nu(x,t) = a_\nu H(t) \operatorname{Im}(|x|^2 - (t+i0)^2)^{\nu - \frac{d-1}{2}}$$

then $\Box E_\nu = \nu E_{\nu-1}$, where $E_0 = E_+$. As $\nu \to \infty$, these distributions become less and less singular on the forward light cone. We also have a Fourier representation formula; for each ν, E_ν is a finite linear combination of terms of the form

$$t^j H(t) \int \frac{e^{2\pi i(x\cdot\xi + t|\xi|)}}{|\xi|^{\nu+k+1}}\, d\xi$$

where $j+k=\nu$. In particular, we see from this formula that E_ν is a Fourier integral of order at most $d/4-\nu-1$.

4.2 Constant Coefficient Metric

Now let's move on to the case of the wave equation on $\mathbf{R}^{d+1}$, where $\mathbf{R}^d$ is equipped with a different metric $g = \sum g_{ij} dx^i dx^j$, although one with constant coefficients. The Laplace-Beltrami operator then becomes

$$\Delta_g f = \sum_{i,j} g^{ij} \partial_{i,j} f = \nabla_x \cdot \nabla_g f.$$

Let us put the coefficients of the metric in a positive definite $d \times d$ matrix G, i.e. with $G_{ij} = g_{ij}$.

To find a fundamental solution here, it is easiest to try and find a change of coordinates where the Laplace-Beltrami operator behaves like a usual Laplace operator. So suppose that $f = h \circ T$, for some invertible linear operator $T : \mathbf{R}^d \to \mathbf{R}^d$ with coefficients T_{ij}, and some smooth function h. The chain rule implies that

$$\nabla_g f = (G^{-1}T^t)(\nabla_x h \circ T)$$

and thus

$$\Delta_g f = \nabla_x \cdot \nabla_g f = \sum_{i,j} (TG^{-1}T^t)_{i,j} (\partial_{i,j} h \circ T).$$

If we choose T such that $TG^{-1}T^t$ is the identity matrix, then we obtain that

$$\Delta_g f = \Delta h \circ T,$$

This happens precisely when $G = T^t T$. Abusing notation slightly for functions u on $\mathbf{R}^{d+1}$, with $u = v \circ T$, we have

$$\Box_g u = \Box v \circ T.$$

Taking weak limits allows us to find a fundamental solution. In particular, taking weak limits tells us that if E is the forward fundamental solution to $\Box$ on $\mathbf{R}^d$, then $E_+ = T^*E$ is the forward fundamental solution to the wave equation $\Box_g$. We have

$$\begin{aligned} E_+(x,t) &= E(Tx,t) \\ &= c_d \cdot \frac{H(t)}{\operatorname{Im}(|Tx|^2 - (t+i0)^2)^{\frac{d-1}{2}}} \\ &= c_d \cdot \frac{H(t)}{\operatorname{Im}(|x|_g^2 - (t+i0)^2)^{\frac{d-1}{2}}}, \end{aligned}$$

Note that we do not need to introduce an extra integrating factor from changing coordinates because we are working with distributions tested against the *volume measure* $dV_g = |g|^{1/2}dx$ on $\mathbf{R}^d$, i.e. for $\phi \in \mathcal{D}(\mathbf{R}^d)$, the fundamental solution we have constructed satisfies the equation

$$\int\int E_+(x,t)\Box_g(x,t)\phi \, dV(x)\, dt = \phi(0,0).$$

Analogous to the behaviour of the forward fundamental solution with respect to the standard metric, the fundamental solution here has support on the interior of the light cone, i.e. the set

$$\{(x,t) \in \mathbf{R}^{d+1} : t \geqslant 0 \text{ and } |x|_g \leqslant t\}.$$

We also have a Fourier transform representation, i.e. as

$$E_+(x,t) = \frac{H(t)}{2\pi} \int \frac{\sin(2\pi t|\xi|_g)}{|\xi|_g} e^{2\pi i \xi \cdot x}\, d\xi$$

where $|\xi|_g$ is given by the metric on the *cotangent* space, i.e. the metric given with the matrix of coefficients given by G^{-1} rather than G itself.

4.3 General Riemannian Metric

Let us now address the wave equation on a general Riemannian manifold. Because we have a non constant coefficient equation, we cannot expect to obtain a fundamental solution, instead only a *parametrix*, which gives a solution *modulo smoothing terms*.

To begin with, it will be helpful to work in *normal coordinates*. These are a natural set of coordinates that can be chosen centered at any point p_0 on a Riemannian manifold (M^d, g). To obtain this coordinate system, we take *geodesics* emerging from p_0. More precisely, if B is a suitably small enough open ball centered at the origin in $T^*_{p_0}M$, then the geodesic exponential map $\exp : B \to M$ is a diffeomorphism, with inverse thus giving a coordinate system $x : U \to B$. For $a \in B$, let $G(a)$ be the $d \times d$ positive definite matrix giving the coefficients of the metric g at the point $x^{-1}(a)$. Since we are working in normal coordinates, the matrix satisfies the identity $G(a)a = G(0)a$ for $a \in B$, which roughly means that the metric behaves like the Euclidean metric as long as we are measuring directions radiating outward from the origin.

In general, we can break the Laplace-Beltrami operator into two terms, i.e. writing

$$\Delta_g = \nabla_x \cdot \nabla_g f + a \cdot \nabla_x f = L_g f + R_g f.$$

The first term is a second order differential operator in f with 'constant-coefficients' (once we switch to taking gradients with respect to the metric rather than the coordinate system), and the second term is a first order differential equation with a non-constant coefficient vector

$$a = |g|^{-1/2} \nabla_g \{|g|^{1/2}\}.$$

This second term proves to be a problem with constructing a fundamental solution due to the non-constant coefficients. If we could make them disappear, then we claim that we could construct a fundamental solution to the wave equation for small times. Indeed, *suppose we are working in normal coordinates*, and we let

$$E_+(x,t) = c_d \frac{H(t)}{\operatorname{Im}\left(|x|^2_{g(0)} - (t+i0)^2 \right)^{\frac{d-1}{2}}},$$

where $g(0)$ is the constant-coefficient metric on B agreeing with g at the origin. Then we have seen that E is a fundamental solution to $\Box_{g(0)}$ with respect to the volume form $dV_{g(0)}$, i.e. so that we have

$$\Box_{g(0)} E = \delta.$$

We claim that for sufficiently small times $|t| \lesssim 1$, we also have that, with respect to the volume form $dV_{g(0)}$,

$$(\partial_t^2 - L_g)E_+(x,t) = \delta.$$

Thus with respect to the volume form dV_g, we also have

$$(\partial_t^2 - L_g)E_+(x,t) = \delta$$

Disregarding the operator R_g, we have essentially constructed a fundamental solution to $\Box_g$ for small times.

To check this claim, let us perform some calculations with a function of the form $f(x) = F(|x|_{g(0)}^2)$, where $F \in C^\infty(\mathbf{R})$. We claim that

$$\Delta_{g(0)} f = L_g f.$$

Taking weak limits of this equation gives the required claim above. This follows from the more general equation that

$$\nabla_{g(0)} f = \nabla_g f,$$

Since then

$$\Delta_{g(0)} f = \nabla_x \cdot \nabla_{g(0)} f = \nabla_x \cdot \nabla_g f = L_g f.$$

The reason this equation holds is precisely because F only depends on the *radial direction* of the point x, which has nice properties in normal coordinates.

Lemma 4.3. *If we are working in a normal coordinate system, then*

$$\nabla_{g(0)} f = \nabla_g f.$$

Proof. We calculate using the chain rule that

$$\begin{aligned}\nabla_{g(0)} f &= G(0)^{-1}(\nabla_x f)\\ &= 2F'(|x|_{g(0)}^2)x\end{aligned}$$

and (now using the normal coordinate equation)

$$\begin{aligned}\nabla_g f(x) &= G(x)^{-1}(\nabla_x f)\\ &= 2F'(|x|_{g(0)}^2)[G(x)^{-1}G(0)]x\\ &= 2F'(|x|_{g(0)}^2)[G(x)^{-1}G(x)]x\\ &= 2F'(|x|_{g(0)}^2)x.\end{aligned}$$

Comparing calculations shows we have equality. $\square$

More generally, our calculations imply that if we define

$$E_\nu(x,t) = c_d H(t) \mathrm{Im}\Big(|x|_{g(0)}^2 - (t+i0)^2 \Big)^{\nu - \frac{d-1}{2}},$$

then $(\partial_t^2 - L_g) E_0 = \delta$, and $(\partial_t^2 - L_g) E_\nu = \nu E_{\nu-1}$. We also have that for $\nu \geqslant 1$,

$$(\partial_t^2 - L_g) E_\nu = ((-1/2) E_{\nu-1}) x.$$

We have now essentially performed the calculations to construct a parametrix to Δ_g.

Suppose $\alpha_0 \in C^\infty(B)$. Suppose $E_0(x,t) = F_0(|x|_g^2, t)$. Then we can expand out $\Box_g$ to yield that

$$\begin{aligned}
\Box_g\{\alpha_0 E_0\} &= (\partial_t^2 - L_g)\{\alpha_0 E_0\} - R_g\{\alpha_0 E_0\} \\
&= \alpha_0(0)\delta - \nabla_x \cdot \{(\nabla_g \alpha_0) E_0\} - \langle \nabla_g \alpha_0, \nabla_g E_0 \rangle_g - R_g\{\alpha_0 E_0\} \\
&= \alpha_0(0)\delta - \Delta_g \alpha_0 \cdot E_0 - 2\langle \nabla_g \alpha_0, \nabla_g E_0 \rangle_g - R_g\{\alpha_0 E_0\} \\
&= \alpha_0(0)\delta - \langle \alpha_0 a + 2\nabla_g \alpha_0, \nabla_g E_0 \rangle_g - (\Delta_g \alpha_0 + a \cdot \nabla_x \alpha_0) \cdot E_0.
\end{aligned}$$

We write

$$\begin{aligned}
\langle \alpha_0 a + 2\nabla_g \alpha_0, \nabla_g E_0 \rangle_g &= 2F_0'(|x|_g^2, t) \langle \alpha_0 a + 2\nabla_g \alpha_0, x \rangle_g \\
&= 2F_0'(|x|_g^2, t) \left(\langle a, x \rangle_g \cdot \alpha_0 + 2\langle \nabla_g \alpha_0, x \rangle \right) \\
&= 2F_0'(|x|_g^2, t) \left(\langle a, x \rangle_{g(0)} \cdot \alpha_0 + 2x \cdot \nabla_x \alpha_0 \right) \\
&= 2F_0'(|x|_g^2, t) \left(\rho \cdot \alpha_0 + 2\nabla_x \alpha_0 \cdot x \right).
\end{aligned}$$

If we choose α_0 such that $\alpha_0(0) = 1$, and $\rho \alpha_0 + 2\nabla_x \alpha_0 \cdot x = 0$, then we therefore conclude that

$$\Box_g\{\alpha_0 E_0\} = \delta - (\Delta_g \alpha_0 + a \cdot \nabla_x \alpha_0) \cdot E_0 = \delta - c_0 E_0.$$

Thus we have a fundamental solution with a remainder $c_0 E_0$. We deal with this additional term using the fact that $\Box_g\{E_1\} = E_0$. We choose a smooth function α_1 so that

$$\Box_g\{\alpha_0 E_0 + \alpha_1 E_1\} = \delta - c_1 E_1,$$

and we consider this process, noting that, locally in time, the $\{E_\nu\}$ become more and more regular as ν increases. We will be able to choose $\{\alpha_\nu\}$ for all ν such that

$$\Box_g\{\alpha_0 E_0 + \alpha_1 E_1 + \cdots + \alpha_n E_n\} = \delta - c_{n+1}E_{n+1},$$

and thus we obtain an arbitrarily good approximation to a fundamental solution. One obtains a similar equation for $\Box_g\{\alpha_\nu E_\nu\}$ to the one we calculated above for all ν (see Sogge, Chapter 2), and moreover, we have

$$\alpha_0(x) = |g(x)|^{-1/4}|g(0)|^{1/4}$$

and for $\nu \geqslant 1$,

$$\alpha_\nu(x) = \alpha_0(x)\int_0^1 t^{\nu-1}\frac{\Delta_g \alpha_{\nu-1}(tx)}{\alpha_0(tx)}\,dt.$$

By induction on ν, all these equations are smooth.

Since all the quantities here depend smoothly on the metric g, and the metric smoothly varies as we vary the point p_0 we started with, we can perform this construction in a smoothly varying way for all points p_0 on the manifold M. If we let x and y vary over a compact set $K \subset M$, we thus obtain a function

$$K_n(x,t;y) = \sum_{\nu=0}^n \alpha_\nu(x,y)E_\nu(d_g(x,y),t).$$

which is well defined for $|t| \lesssim_K 1$. We then have

$$\Box_g K_n(x,y,t) = \delta - (\Delta_g \alpha_{n+1})E_{n+1}(d_g(x,y),t) = \delta - c_{n+1}E_{n+1}(d_g(x,y),t).$$

TODO: How do we get a solution operator from this parametrix. These operators should be FIOs of order $-1/4$, and for a fixed time, operators of order 0. The remainder should be an FIO of order $-1/4-n-1$, and for a fixed time, of order $-n-1$.

But from the perspective of regularity, a finite expansion is normally enough. We can express $E_n(d_g(x,y),t)$ as a finite linear combination of terms of the form

$$t^j H(t)\int \frac{e^{2\pi i(d_g(x,y)\cdot\xi + t|\xi|)}}{|\xi|^{\nu+k+1}}\,d\xi.$$

In particular, this implies that for each n, S_n is a Fourier integral operator of order $-5/4$, and if we fix the time, we get a Fourier integral operator of order -1.

$$\frac{\sin(t|\xi|)}{|\xi|}$$

Assuming we have localized the support of f to a compact set K contained in a particular coordinate system (y, V), and if $F(x,\xi)$ denotes the Fourier transform of $y \mapsto c_n(x,y) f(y) |g(y)|^{1/2}$, then

$$\int c_n(x,y) E_n(d_g(x,y), t) f(y)\, dV_g(y)$$

is a linear combination of terms of the form

$$t^j H(t) \int \frac{F(x,\xi)}{|\xi|^{\nu+k+1}} e^{2\pi i (d_g(x,y)\cdot\xi + t|\xi|)}\, d\xi$$

Now we have

$$\|F|\xi|^s\|_{L^2(\mathbf{R}^d)} \lesssim \|f\|_{\dot{H}^s(\mathbf{R}^d)}$$

Thus, modulo an operator that becomes more and more regular as $n \to \infty$, locally in x and t, we have found operators which approximate solution operators to the wave equation.

TODO: USE HADAMARD PARAMETRIX TO SHOW WAVE EQUATION HAS FINITE PROPOGATION SPEED. SOGGE Chapter 2.

4.4 Explicit Hadamard Parametrix For The Sphere

TODO

Consider geodesic normal coordinates for the sphere in $\mathbf{R}^{d+1}_x$, centered at the south pole. Without loss of generality, to compute this metric we may assume our sphere is the locus given by the equation

$$(y_0 - 1)^2 + y_1^2 + \dots + y_d^2 = 1$$

and that the south pole is the origin. The metric in these coordinates is precisely the restriction of the metric $\sum dy_i^2$ on $T\mathbf{R}^{d+1}$ to S^d. To work out the geodesic normal coordinates $x = (x_1, \dots, x_d)$ centered at the south pole. Then

$$y_0 = 1 - \cos(r) \quad \text{and} \quad (y_1, \dots, y_d) = \sin(r) \cdot \frac{x}{|x|}$$

It will be simpler to work in polar coordinates r and θ, where θ is a unit vector in $\mathbf{R}^d$. Then

$$y_0 = 1 - \cos(r) \quad \text{and} \quad (y_1, \ldots, y_d) = \sin(r)\theta.$$

Thus

$$dy_0 = \sin(r)\, dr$$

and for $1 \leqslant i \leqslant d$,

$$dy_i = \cos(r)\theta_i\, dr + \sin(r) d\theta_i.$$

Thus $g^2 = dr^2 + \sin(r)^2 \sum_i (d\theta_i)^2$. The Laplace-Beltrami operator in these coordinates is thus

$$\Delta_g = \frac{1}{\sin(r)^{d-2}} \frac{\partial}{\partial r} \left\{ \sin(r)^{d-2} \frac{\partial}{\partial r} \right\} + \frac{1}{\sin r)^2} \Delta_\theta.$$

And we have

$$|g| = \sin(r)^{2(d-1)}$$

Thus, using the terminology of Sogge, Chapter 2,

$$a^r(x) = -\frac{(d-1)}{\sin(r)}$$

$$\rho = -(d-1)\frac{r}{\sin(r)}$$

$$\alpha_0(r) = \frac{c}{\sin(x)^{\frac{d-1}{2}}}$$

$$-\frac{(d-1)}{2}\cot(r)\alpha_0 = \partial_r \alpha_0$$

The highest order term in the Hadamard Parametrix for the Laplace-Beltrami operator is

$$E_0(t, x) = E_+(t, r)$$

$$g_{rr} = 1$$

$$g_{\theta_i \theta_i} = \sin(r)^2$$

$$g^{rr} = 1$$

$$g^{\theta_i\theta_i} = 1/\sin(r)^2$$

$$g^2 = \left(\cos^2(r)(4d(\sin(r)-1)^2+1)\right) dr^2 + \dots$$

$$g^2 = \left(2\cos(r)\cos\left(\frac{r}{2}+\frac{\pi}{4}\right)\right)^2 dr^2 + \dots$$

and for $1 \leqslant i \leqslant d$,

$$dy_i = (2|x|\cos|x|(1-\sin|x|) - \sin|x|(2-\sin|x|))\frac{x_i(x\cdot dx)}{|x|^3} 2\cos|x|(1-\sin|x|)\frac{x_i(x\cdot dx)}{|x|^2} - \frac{\sin|x|(2 -}{}$$

it will help to work with polar normal coordinates $x = r\cdot\theta$, where $r > 0$ and $\theta \in S^{d-1}$. Then $y_0 = \sin(r)$ and $(y_1,\dots,y_d) = \sin(r)(2-\sin(r))\theta$

point at a height $y_0 = a$ is given by the curve $\gamma(t) = (a,)$

$$\int_0^t$$

Consider the geodesic normal coordinates $y = (y_1,\dots,y_d)$ centered at the south pole.

What is the metric $G = \{g\}$ in these coordinates. In the normal (x,y,z) coordinates, the metric

Chapter 5

Notes on Bochner-Riesz

The goal of this section is to compare and contrast approaches to understanding the Bochner-Riesz conjecture on Euclidean space and on compact Riemannian manifolds, in order to reflect on the differences in understanding multipliers on $\mathbf{R}^d$ vs on a compact manifold X before we attack the more general multiplier problem in this setting. We define the Riesz multipliers via symbols $r_\rho^\delta : [0,\infty] \to [0,\infty)$, defined for $\rho > 0$ and a real number δ by setting, for $\tau > 0$,

$$r_\rho^\delta(\tau) = (1 - \tau/\rho)_+^\delta.$$

Here $s_+ = \max(s,0)$. The resulting radial multipliers on $\mathbf{R}^n$, and on a compact Riemannian manifold X, will be denoted by

$$R_\rho^\delta = r_\rho^\delta\left(\sqrt{-\Delta}\right).$$

The goal of the Bochner-Riesz conjecture is to determine bounds on the operators $\{R_\rho^\delta\}$ invariant under dilation of the symbol.

5.1 Euclidean Case

Let's review a reduction of Bochner-Riesz to Tomas Stein:

- First, we can *rescale the problem*. If $r^\delta = r_1^\delta$, then

$$r_\rho^\delta(\lambda) = r^\delta(\lambda/\rho).$$

Thus if $R^\delta = R^\delta_1$, then $R^\delta_\rho = R^\delta \circ \mathrm{Dil}_{1/\rho}$, and so the operators $\{R^\delta_\rho\}$ are uniformly bounded from L^p to L^p for all ρ if and only if R^δ is bounded from L^p to L^p.

- We now perform a *spatial decomposition*. Let k^δ be the convolution kernel corresponding to the operator R^δ. We break up the effects of the operator spatially into dyadic annuli, i.e. writing

$$k^\delta(x) = \sum_{j=0}^\infty k^\delta_j(2^j x),$$

  where k^δ_0 is supported on $|x| \leqslant 2$, and all of the other kernels k^δ_j are supported on the annuli $\{1/2 \leqslant |x| \leqslant 1\}$, and can be written as

$$k^\delta_j(x) = \phi \cdot \mathrm{Dil}_{1/2^j} k^\delta$$

  for some $\phi \in C_c^\infty$ supported on the annulus $\{1/2 \leqslant |x| \leqslant 2\}$ and equal to one on the annulus $\{3/4 \leqslant |x| \leqslant 3/2\}$. We analyze each of the convolution kernels separately and then collect up each of the bounds we obtain by applying the triangle inequality. Thus we let R^δ_j be the operator with convolution kernel k^δ_j. Provided we can obtain a bound of the form

$$\|R^\delta_j f\|_{L^p(\mathbf{R}^d)} \lesssim 2^{-\varepsilon j} \|f\|_{L^p(\mathbf{R}^d)}$$

  for some $\varepsilon > 0$, and some implicit constant uniform in j, we can sum up the bounds using the triangle inequality to bound R^δ.

- Spatial localization means that the operators $\{R^\delta_j\}$ are *local*, i.e. for any function f, the support of $R^\delta_j f$ is contained in a $O(1)$ neighborhood of the support of f. A decomposition argument, thus implies that it suffices to obtain a bound of the form

$$\|R^\delta_j f\|_{L^p(\mathbf{R}^d)} \lesssim 2^{-\varepsilon j} \|f\|_{L^p(\mathbf{R}^d)}$$

  for functions f *supported on balls of radius* 1, since the general bound will follow from this.

- We *reduce to L^2 bounds*: Now that f is suported on a ball of radius 1, R_j^δ is supported on a ball of radius $O(1)$, and so for $p \leqslant 2$ we have

$$\|R_j^\delta f\|_{L^p(\mathbf{R}^d)} \lesssim \|R_j^\delta f\|_{L^2(\mathbf{R}^d)}.$$

  Thus it suffices to obtain a bound of the form $\|R_j^\delta f\|_{L^2(\mathbf{R}^d)} \lesssim \|f\|_{L^p(\mathbf{R}^d)}$. Switching from the L^p norm to the L^2 norm is the most inefficient part of the proof, but it enables us to apply more powerful tools which we only have in $L^2(\mathbf{R}^d)$. Getting around this reduction is key to improving the currently known Bochner-Riesz bounds.

- We reduce the problem to Tomas-Stein. Since we are now in $L^2(\mathbf{R}^d)$, we can apply Plancherel. If ψ_j^δ is the Fourier transform of k_j^δ, then we obtain that

$$\|R_j^\delta f\|_{L^2(\mathbf{R}^d)} = \|\psi_j^\delta \cdot \widehat{f}\|_{L^2(\mathbf{R}^d)}.$$

  A stationary phase calculation shows that ψ_j^δ has the majority of its mass on an annulus of radius 2^j and width $O(1)$, and has magnitude $O(2^{-j\delta})$ there, i.e.

$$|\psi_j^\delta(\xi)| \lesssim_N 2^{-\delta j} \langle 2^j - |\xi| \rangle^{-N}.$$

  Thus by Tomas-Stein, if R_S denotes the restriction operator to the unit sphere S, we find that

$$\begin{aligned}
\|\psi_j^\delta \cdot \widehat{f}\|_{L^2(\mathbf{R}^d)} &\lesssim_N 2^{-\delta j} \left(\int_0^\infty \langle 2^j |1-r| \rangle^{-2N} \int_{|\xi|=1} |\widehat{f}(r\xi)|^2 \, d\sigma \, r^{d-1} dr \right)^{1/2} \\
&\lesssim 2^{-\delta j} \left(\int_0^\infty \langle 2^j |1-r| \rangle^{-2N} \|R_S \circ \mathrm{Dil}_r f\|_{L^2(S^{n-1})}^2 \frac{dr}{r} \right)^{1/2} \\
&\lesssim 2^{-\delta j} \|f\|_{L^p(\mathbf{R}^d)} \left(\int_0^\infty \langle 2^j |1-r| \rangle^{-2N} r^{2d/p-1} dr \right)^{1/2} \\
&\lesssim 2^{-\delta j} \|f\|_{L^p(\mathbf{R}^d)}.
\end{aligned}$$

  This bound is summable in j, which yields the required result.

Let us end our discussion of the Euclidean case by expanding on the computation of the inequality

$$|\psi_j^\delta(\xi)| \lesssim_N 2^{-j\delta} \langle 2^j - |\xi| \rangle^{-N}.$$

Before this, let's see why the result is *intuitive*. The function ψ_j^δ is obtained by localizing the frequency multiplier m^δ on the spatial side and then rescaling. Thus our result is intuitively saying that the phase-portrait of the multiplier is concentrated on a neighborhood of the set

$$\{(x,\xi) : |\xi| \leqslant 1 \text{ and } ||\xi| - 1| = 1/|x|\}.$$

This makes sense, since the 'high frequency' components of m^δ should be distributed near the boundary of the unit ball, since this is where the symbol becomes singular; that the spatial part should be inversely proportional to the distance to the boundary can be detected by taking derivatives of m in the frequency variable, i.e. noting that if $||\xi| - 1| \sim 1/2^j$, then

$$|\nabla^N m^\delta(\xi)| \lesssim_{N,\delta} (1 - |\xi|)^{\delta - N} \sim 2^{-j\delta} 2^{jN}.$$

And we see the derivative grows in N as a power of 2^j, which is inversely propoertional to $||\xi| - 1|$. Working more precisely, we have

$$\psi_j^\delta = 2^{-jd} \left[\widehat{\phi} * \mathrm{Dil}_{2^j} m^\delta \right].$$

The function is the average of $\widehat{\phi}$ over a ball of radius $O(2^j)$ so we immediately obtain a bound by using the rapid decay of $\widehat{\phi}$, thus obtaining that

$$|\psi_j^\delta(\xi)| \lesssim_N \langle 2^j - |\xi| \rangle^{-N}.$$

Thus we see that ψ_j^δ has the majority of it's support on the ball of radius 2^j. But we can do much better than this using the fact that $\widehat{\phi}$ is *oscillatory*, since ϕ is supported away from the origin, and m^δ is *mostly* smooth. More precisely, $\widehat{\phi}$ oscillates at frequencies ~ 1, so we should expect integration by parts to yield useful decay on a quantity $\widehat{\phi} * \mathrm{Dil}_{2^j} f$ if we had a bound $|\nabla^N f| \ll 2^{Nj}$ for large $N > 0$. This is true of m^δ away from a thickness $O(2^{-j})$ annulus containing the unit ball. Thus we are motivated to define $m^\delta = a_j^\delta + b_j^\delta$, where

$$a_j^\delta(\xi) = m^\delta(\xi)\eta(2^j(1 - |\xi|)) \quad \text{and} \quad b_j^\delta(\xi) = m_j^\delta(\xi)(1 - \eta(2^j(1 - |\xi|)))$$

where $\eta(t)$ is supported on $|t| \leqslant 1$ and equal to one for $|t| \leqslant 1/2$. The function b_j^δ is therefore supported on $|\xi| \leqslant 1 - 1/2^{j+1}$. For $N > 0$, we have

$$|\nabla^N m_j^\delta(\xi)| \lesssim_{N,\delta} (1 - |\xi|)^{\delta - N}.$$

By the product rule, $\nabla^N b_j^\delta$ is a sum of derivatives of m_j^δ and of derivatives of $1-\eta(2^j(|\xi|-1))$. The support of any derivative of the latter is supported on $|\xi| \geqslant 1 - 1/2^j$. Thus we have

$$|\nabla^N b_j^\delta(\xi)| \lesssim_N (1-|\xi|)^{\delta-N}\mathbf{I}(|\xi| \leqslant 1-1/2^{j+1}) + 2^{j(N-\delta)}\mathbf{I}(1-1/2^j \leqslant |\xi| \leqslant 1).$$

Since ϕ is supported away from the origin, we may antidifferentiate $\widehat{\phi}$ arbitrarily many times without any singular behaviour emerging. But now averaging the Nth antiderivative of $\widehat{\phi}$, which is rapidly decaying, with the Nth derivative of $\mathrm{Dil}_{2^j} b_j^\delta$, which is rapidly decaying outside of an annulus of width 1 and radius 2^j, we find that

$$|(\widehat{\phi} * \mathrm{Dil}_{2^j} b_j^\delta)(\xi)| \lesssim 2^{j(d-\delta)}\langle 2^j - \xi\rangle^{-N}.$$

The multiplier a_j^δ is not so smooth, but it is supported on a very thin annulus of radius 1 and thickness $O(2^{-j})$, and m^δ has magnitude at most $2^{-\delta j}$ on this annulus, which gives that

$$|(\widehat{\phi} * \mathrm{Dil}_{2^j} a_j^\delta)(\xi)| \lesssim 2^{-j\delta}\int_{||\eta|-2^j|\leqslant 1} |\widehat{\phi}(\xi-\eta)|\, d\eta \lesssim_N 2^{j(d-\delta)}\langle 2^j - \xi\rangle^{-N}.$$

Putting these results together gives the required bound.

5.2 Manifold Case

The analogue of the Tomas Stein theorem on a compact Riemannian manifold X is a result due to Sogge, so let's see if we can obtain a result for compact manifolds using similar techniques:

- The first problem is that on a compact Riemannian manifold we do not have a rescaling symmetry which we can use to reduce the study of the Bochner-Riesz multipliers R_ρ^δ to the case $\rho = 1$. Thus we must analyze a general multiplier of the form R_ρ^δ for all $\rho > 0$. The case of small ρ is easily dealt with using the triangle inequality, so we may assume that $\rho \gtrsim 1$ in what follows.

- Now we try and reduce to Sogge's spectral cluster bounds, which are analogous to the Tomas-Stein bounds in $\mathbf{R}^d$. If we are able to justify that $K_{\rho,j}^{\delta}$ behaves like a spectral band projection operator, as in the Euclidean setting, we'd be able to apply this bound. Plancherel does not quite have an analogy to the L^2 setting on a manifold. But we can instead use the wave operator and it's parametrices, i.e. that

$$\begin{aligned} R_{\rho}^{\delta} &= \sum_{\lambda} r^{\delta}(\lambda/\rho) E_{\lambda} \\ &= \rho \int_0^{\infty} \widehat{r^{\delta}}(\rho t) e^{2\pi i t\sqrt{-\Delta}}\, dt \\ &= c_{\delta} \cdot \rho^{-\delta} \int_0^{\infty} e^{2\pi i \rho t} (t + i0)^{-\delta-1} e^{2\pi i t \sqrt{-\Delta}}\, dt. \end{aligned}$$

The singularity in the definition of this integral occurs at $t = 0$, so the operator should, for large t, be relatively well behaved.

- Since we expect the function is well behaved for large t, let's bound these terms so we may reduce to controlling the integral over $t \lesssim 1$. Fix $\alpha \in C_c^{\infty}(\mathbf{R})$ equal to one in a neighborhood of zero, and consider the behaviour of R_{ρ}^{δ} for large t, i.e. the operator

$$R_{\rho}^{\delta} = c_{\delta} \cdot \rho^{-\delta} \int_0^{\infty} (1 - \alpha(t)) \cdot e^{-2\pi i \rho t} t^{-\delta-1} e^{2\pi i t\sqrt{-\Delta}}\, dt.$$

If ψ is the inverse Fourier transform of $c_{\delta} t^{-\delta-1}(1 - \alpha(t))$, then ψ is bounded and rapidly decreasing because all of the derivatives of it's Fourier transform are smooth and integrable. We thus can revert back to the multiplier setting and write

$$R_{\rho}^{\delta} = \rho^{-\delta} \sum_{\lambda} \psi(\lambda - \rho) E_{\lambda}.$$

The rapid decay here means we can be fairly lazy in controlling this operator, for instance, employing the Sobolev embedding bound

$$\|E_{\lambda} f\|_{L^2(X)} \lesssim \langle \lambda \rangle^{d(1/p-1/2)-1/2} \|f\|_{L^p(X)}$$

and the triangle inequality, using the rapid decay to obtain that

$$\|R_{\rho}^{\delta} f\|_{L^p(X)} \lesssim \langle \rho \rangle^{-[\delta - d(1/p-1/2)+1/2]} \|f\|_{L^p(X)},$$

which is better than what we need. Thus we now need only bound the operator

$$\tilde{R}^\delta_\rho = c_\delta \cdot \rho^{-\delta} \int_0^\infty \alpha(t) e^{2\pi i \rho t} (t + i0)^{-\delta-1} e^{2\pi i t\sqrt{-\Delta}}\, dt.$$

The advantage of doing this is because we only have understanding of the wave operator through Fourier integral operators (through the Lax parametrix) for times $t \lesssim 1$.

- We now 'spatially localize' as in the Euclidean case, though things look different here since we are dealing with the wave equation. We choose β such that

$$1 = \eta + \sum_{j=1}^\infty \mathrm{Dil}_{2^j} \beta.$$

We then write

$$R^\delta_\rho = \sum_{j=0}^{O(\log \rho)} R^\delta_{\rho,j}$$

where for $j > 0$

$$R^\delta_{\rho,j} = c_\delta \cdot \rho^{-\delta} \int_0^\infty \alpha(t) (\mathrm{Dil}_{2^j} \beta)(\rho t) e^{-2\pi i \rho t} t^{-\delta-1} e^{2\pi i \sqrt{-\Delta}}\, dt,$$

and

$$\begin{aligned} R^\delta_{\rho,0} &= c_\delta \cdot \rho^{-\delta} \int_0^\infty \alpha(t) \eta(\rho t) e^{-2\pi i \rho t} t^{-\delta-1} e^{2\pi i \sqrt{-\Delta}}\, dt \\ &= c_\delta \cdot \rho^{-\delta} \int_0^\infty \eta(\rho t) e^{-2\pi i \rho t} t^{-\delta-1} e^{2\pi i \sqrt{-\Delta}}\, dt, \end{aligned}$$

where the last identity follows because the result the support of the integral is on $t \lesssim 1/\rho$, and we are assuming ρ is large so that α may be assumed equal to one on the support of the integral. Thus R^δ_ρ is an integral over $t \sim 2^j/\rho$. This is analogous to the spatial decomposition we performed in the Euclidean setting, except now we have the wave equation involved, and the 'pseudolocal' finite speed of propogation for the wave equation now must substitute for the explicit spatial localization we obtained in the Euclidean decomposition.

- Despite the singularity that occurs at the origin, the case $j = 0$ is simplest to deal with. If we define

$$m(\lambda) = c_\delta(\widehat{\eta} * r_\delta)$$

then $R^\delta_{\rho,0}$ is a multiplier operator with symbol

$$m_\rho(\lambda) = \rho^{-\delta}\mathrm{Dil}_\rho m.$$

We have estimates of the form

$$|\nabla^N m(\lambda)| \lesssim_N \langle\lambda\rangle^{-M}.$$

Thus

$$|\nabla^N m_\rho(\lambda)| \lesssim_N \rho^{-\delta-N}\langle\lambda/\rho\rangle^{-M}.$$

In particular, taking $M = N$ and $M = 0$ yields that

$$|\nabla^N m_\rho(\lambda)| \lesssim_N \rho^{-\delta}\langle\lambda\rangle^{-N}.$$

Thus $\{\rho^\delta m_\rho\}$ are a uniformly bounded family of symbols of order zero. Thus (TODO: Review estimates for multipliers given by a symbol) we can obtain that

$$\|m_\rho(\sqrt{-\Delta})f\|_{L^p(X)} \lesssim \rho^{-\delta}\|f\|_{L^p(X)} \lesssim \|f\|_{L^p(X)}.$$

TODO: Check there isn't an error here since the $\rho^{-\delta}$ terms helps us out, but shouldn't our bounds be scale invariant?

- Now we deal with the $j > 0$ terms, and we must use the pseudolocal finite speed of propogation of the wave equation as a substitute for explicit localization. Since we have localized to times $t \lesssim 1$. We deal with this by using the Lax parametrix for the wave equation, but first we must ensure the remainder terms from employing the parametrix are well behaved. For $t \lesssim 1$, we can write $e^{2\pi i t\sqrt{-\Delta}} = Q(t) + R(t)$, where $Q(t)$ is a Fourier integral operator supported on a $O(1)$ neighborhood of the diagonal $\Delta = \{(x,x) : x \in X\}$, and with kernel given in coordinates by

$$(x,y) \mapsto \int e^{2\pi i[\phi(x,y,\xi)+t|\xi|]} q(t,x,y,\xi)\, d\xi$$

where q is a symbol of order zero, and ϕ is homogeneous of order one in ξ, with $\phi(x,y,\xi) \approx (x-y)\cdot\xi$, in the sense that

$$|\nabla_\xi^N[\phi(x,y,\xi) - (x-y)\cdot\xi]| \lesssim_N |x-y|^2|\xi|^{1-N}$$

for all $N > 0$. The operators $\{R(t)\}$ are smoothing, i.e. with a joint kernel A uniformly in $C^\infty([-1,1]\times X\times X)$. Thus we write

$$\begin{aligned} R^\delta_{\rho,j} &= c_\delta \cdot \rho^{-\delta} \int_0^\infty \alpha(t)(\mathrm{Dil}_{2^j}\beta)(\rho t) e^{-2\pi i \rho t} t^{-\delta-1}(Q(t)+R(t))\, dt \\ &= R^\delta_{\rho,j,Q} + R^\delta_{\rho,j,R}. \end{aligned}$$

Let's control the $R(t)$ term. Computing the integral of the kernel defining $R^\delta_{\rho,j,R}$ leads to a term of the form

$$c_\delta 2^j \rho^{-1-\delta}(\widehat{\alpha A} * \mathrm{Dil}_{\rho/2^j}\widehat{\beta} * r^\delta)(\rho).$$

The function αA is smooth and compactly supported in the t variable, so it's Fourier transform is rapidly decaying. The same is true of $\widehat{\beta}$, except it is rescaled so we can imagine the majority of it's mass occurs on $|\lambda| \lesssim \rho/2^j$. Finally, r^δ is concentrated on $|\lambda| \lesssim 1$. Thus the kernel is pointwise bound from above by a constant times

$$2^j \rho^{-1-\delta} \int_{\rho - O(1)}^{\rho + O(1)} (\widehat{\alpha A} * \mathrm{Dil}_{\rho/2^j}\widehat{\beta})(\lambda)\, d\lambda.$$

Taking advantage of the oscillation of $\widehat{\beta}$, and the smoothness of $\widehat{\alpha A}$, i.e. integrating by parts, one can shows that for $|\lambda - \rho| \lesssim 1$

$$|(\widehat{\alpha A} * \mathrm{Dil}_{\rho/2^j}\widehat{\beta})(\lambda)| \lesssim_{N,M} (\rho/2^j)^N \cdot \rho^{-M} \cdot (\rho/2^j) = \rho^{1+N-M} 2^{-(N+1)j},$$

Taking $N = M$ gives that the kernel is bounded above by

$$2^j \rho^{-1-\delta}\left(\rho 2^{-(N+1)j}\right) = 2^{-Nj}.$$

But now trivial estimates, e.g. using Schur's lemma implies that

$$\|R^\delta_{\rho,j,R} f\|_{L^p(X)} \lesssim_N \rho^{-\delta} 2^{-Nj} \|f\|_{L^p(X)},$$

a bound that can be summed in j by taking, e.g. $N = 1$. Thus we are now reduced to the study of the oscillatory integral operators $R^\delta_{\rho,j,Q}$.

- Now let's localize. First off, the condition that $K_{\rho,j}$ is supported on the diagonal, and the compactness of X, means we need only prove the result restricted to a single coordinate chart. Let $K_{\rho,j,t}$ be the kernel of the operator $R^\delta_{\rho,j,Q}$. Intuitively, the wave equation travels at unit speed, so, since $R^\delta_{\rho,j,Q}$ involves the wave equation localized to times $t \sim 2^j/\rho$, we should expect this kernel to be localized to $|x-y| \lesssim 2^j/\rho$. In fact, we will show that the restricted kernel

$$K'_{\rho,j,t}(x,y) = K_{\rho,j,t}(x,y) \cdot \mathbf{I}(|x-y| \geqslant 2^{j(1+\varepsilon)}/\rho)$$

has $L^\infty_y L^1_x$ and $L^\infty_x L^1_y$ bounds of the form $O_{\varepsilon,N}(2^{-jN})$, so that Schur's lemma implies that if we write $(R^\delta_{\rho,j,Q})'$ as the operator with kernel $K'_{\rho,j,t}$, then

$$\|(R^\delta_{\rho,j,Q})' f\|_{L^p(X)} \lesssim_N 2^{-jN} \|f\|_{L^p(X)}.$$

This reduces us to proving localized estimates of the following form: for some $\varepsilon > 0$, and for any function f supported on a ball of radius $2^j/\rho$, we have a bound

$$\|R^\delta_{\rho,j,Q} f\|_{L^p(O(2^j/\rho))} \lesssim 2^{-j\varepsilon} \|f\|_{L^p(X)}.$$

Notice the localization we get here is slightly weaker than in the Euclidean setting (the operators are localized to balls of radius $O(2^{j(1+\varepsilon)}/\rho)$ for any $\varepsilon > 0$ rather than localized to balls of radius $O(2^j/\rho)$) which means our bounds here need the slightly greater decay in j (the $O(2^{-j\varepsilon})$ bound above) rather than a bound independent of j.

To prove the bounds for the restricted kernel $K'_{\rho,j,t}$ above, we just apply the principle of nonstationary phase to the integral representation, which says that for $|x-y| \gtrsim 2^{j(1+\varepsilon)}/\rho$ we have, taking the Fourier inversion formula in the t variable,

$$\begin{aligned} K'_{\rho,j,t} &= c_\delta \rho^{-\delta} \int_0^\infty \int \alpha(t)(\mathrm{Dil}_{2^j}\beta)(\rho t)(t+i0)^{-\delta-1} q(t,x,y,\xi) e^{2\pi i[\phi(x,y,\xi)+t|\xi|-\rho t]}\, d\xi\, dt \\ &= \int a^\delta_{\rho,j}(x,y,\xi,|\xi|-\rho) e^{2\pi i \phi(x,y,\xi)}\, d\xi, \end{aligned}$$

where

$$a^\delta_{\rho,j}(x,y,\xi,\cdot) = c_\delta 2^j \rho^{-1-\delta} (\widehat{\alpha q(\cdot,x,y},\xi) * \mathrm{Dil}_{\rho/2^j}\beta * r^\delta * q_\cdot(x,y,\xi))$$

and therefore TODO satisfies estimates of the form

$$|\nabla_t^n \nabla_\xi^m a_{\rho,j}^\delta| \lesssim_{n,m,N} 2^{-j\delta}(2^j/\rho)^n \langle 2^j \tau/\rho\rangle^{-N} \langle \xi \rangle^{-m}$$

Nonstationary phase TODO thus gives the required bounds.

- It now suffices to show that for some $\varepsilon > 0$, and for any function f supported on a ball B of radius $2^j/\rho$, we have a bound

  $$\|R_{\rho,j,Q}^\delta f\|_{L^p(O(1)\cdot B)} \lesssim 2^{-j\varepsilon} \|f\|_{L^p(X)}.$$

  Since we are localized, we can now, like in the Euclidean case, reduce to an L^2 bound, i.e. writing

  $$\|R_{\rho,j,Q}^\delta f\|_{L^p(O(1)\cdot B)} \lesssim (2^j/\rho)^{d(1/p-1/2)} \|R_{\rho,j,Q}^\delta f\|_{L^2(O(1)\cdot B)}.$$

  It now suffices to note TODO that $R_{\rho,j,Q}^\delta$ is a Fourier multiplier operator with symbol which is pointwise bouned by $O_N(2^{-j\delta}\langle 2^j \tau/\rho\rangle^{-N})$, so we can now TODO apply Sogge's version of Tomas Stein on manifolds summed over geometric intervals to yield the required bounds.

Chapter 6

Heo, Nazarov, and Seeger: Initial Radial Conjecture Results

In this chapter we give a description of the techniques of Heo, Nazarov, and Seeger's 2011 paper *Radial Fourier Multipliers in High Dimensions* [11]. One of the main goals of this paper is to verify the radial multiplier conjecture in $\mathbf{R}^d$ for $d \geqslant 4$, and $1 < p < p_d$, where $p_d = 2(d-1)/(d+!)$, i.e. that if $m \in L^\infty(\mathbf{Z})$ is a radial function, $d \geqslant 4$, and $\eta \in \mathcal{S}(\mathbf{R}^d)$ is nonzero, then

$$\|m\|_{M^p(\mathbf{R}^d)} \sim \sup_{t>0} t^{d/p} \|T_m(\text{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)} \quad \text{for } p \in \left(1, \frac{2(d-1)}{d+1}\right),$$

where the implicit constant depends on p and η. We have

$$\sup_{t>0} t^{d/p} \|T_m(\text{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)} \sim \sup_{t>0} \frac{\|T_m(\text{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)}}{\|\text{Dil}_t \eta\|_{L^p(\mathbf{R}^d)}}.$$

Thus we find that the boundedness of T_m on $\mathcal{S}(\mathbf{R}^d)$ is equivalent to it's boundedness on the family of inputs $\{\text{Dil}_t \eta\}$. If we make the assumption that m is compactly supported, then the assumption is equivalent to the fact that the convolution kernel k associated with m is in $L^p(\mathbf{R}^n)$.

Another consequence of the techniques of this paper is that an 'endpoint' result for local smoothing. Namely, the techniques of the paper imply that if $d \geqslant 4$, and $q > 2 + 4/(d-3)$, then

$$\frac{1}{2L} \int_{-L}^{L} \|e^{it\sqrt{-\Delta}} f\|_{L^q(\mathbf{R}^d)}^q \, dt \lesssim_{q,d} \|(I - L^2\Delta)^{\alpha/2} f\|_{L^q(\mathbf{R}^d)}^q,$$

where $\alpha = d(1/2 - 1/q) - 1/2$. TODO: Why is this an 'endpoint result', i.e. is it because it works for an arbitrarily L, rather than a unit time interval like local smoothing normally deals with?

6.1 Discretized Reduction

It is obvious that

$$\|m\|_{M^p(\mathbf{R}^d)} \gtrsim_\eta \sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)},$$

so it suffices to show that

$$\|m\|_{M^p(\mathbf{R}^d)} \lesssim_\eta \sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \eta)\|_{L^p(\mathbf{R}^d)},$$

We will show this via a discrete convolution inequality, which can also be used to prove local smoothing results for the wave equation.

Let σ_r be the surface measure for the sphere of radius r centered at the origin in $\mathbf{R}^d$. Also fix a nonzero, radial, compactly supported function $\psi \in \mathcal{S}(\mathbf{R}^d)$ whose Fourier transform is non-negative, and vanishes to high order at the origin. Given $x \in \mathbf{R}^d$ and $r \geqslant 1$, define $\chi_{xr} = \mathrm{Trans}_x(\sigma_r * \psi)$, which we view as a smooth function oscillation on a thickness ≈ 1 annulus of radius r centered at x. Our goal is to prove the following inequality.

Lemma 6.1. *For any $a : \mathbf{R}^d \times [1,\infty) \to \mathbf{C}$, and $1 \leqslant p < p_d$,*

$$\left\| \int_{\mathbf{R}^d} \int_1^\infty a(x,r) \chi_{x,r}\, dx\, dr \right\|_{L^p(\mathbf{R}^d)} \lesssim \left(\int_{\mathbf{R}^d} \int_1^\infty |a(x,r)|^p r^{d-1} dr dx \right)^{1/p}.$$

The implicit constant here depends on p, d, and ψ.

How does Lemma 6.1 prove the required result? Suppose $m : \mathbf{R}^d \to \mathbf{C}$ is a radial multiplier, so we can consider it's convolution kernel $k : \mathbf{R}^d \to \mathbf{C}$, which is also radial. Let $k(x) = b(|x|)$ for some function $b : [0,\infty) \to \mathbf{C}$. If we set $a(x,r) = g(x)b(r)$ for any function $g : \mathbf{R}^d \to \mathbf{C}$, then the function

$$F(x) = \int_{\mathbf{R}^d} \int_1^\infty a(x',r) \chi_{x',r}\, dx'\, dr,$$

is equal to $k * \psi * g$. In this setting, Lemma 6.1 says that

$$\|k * \psi * g\|_{L^p(\mathbf{R}^d)} \lesssim \|k\|_{L^p(\mathbf{R}^d)} \|g\|_{L^p(\mathbf{R}^d)}.$$

The left hand side is a Fourier multiplier operator applied to g, with symbol equal to $\widehat{\psi} \cdot m$, which is clearly related to the bounds we want to show. In particular, if m is compactly supported away from the origin, let's say, on the annulus $1/2 \leqslant |\xi| \leqslant 2$. If we chose ψ so that $\widehat{\psi}$ is nonvanishing on the annulus $1/4 \leqslant |\xi| \leqslant 2$, then the multiplier $1/\widehat{\psi}$ is smooth on the support of m, and so satisfies $L^p \to L^p$ bounds for all $1 < p < \infty$ restricted to functions with Fourier support on m. In particular, we conclude that m is bounded from L^p to L^p if it's Fourier transform lies in $L^p(\mathbf{R}^d)$. We can then use other tools (Hardy space technology and the like) to study more general multipliers that aren't compactly supported.

To prove Lemma 6.1, it suffices to prove the following discretized estimate where we replace integrals with sums.

Theorem 6.2. *Fix a finite family of pairs $\mathcal{E} \subset \mathbf{R}^d \times [1,\infty)$, which is* discretized *in the sense that for any (x_1, r_1) and (x_2, r_2) in $\mathcal{E}$, one either has $x_1 = x_2$, or $|x_1 - x_2| \geqslant 1$, and one either has $r_1 = r_2$, or $|r_1 - r_2| \geqslant 1$. Then for any $a : \mathcal{E} \to \mathbf{C}$ and $1 \leqslant p < 2(d-1)/(d+1)$,*

$$\left\| \sum_{(x,r)\in\mathcal{E}} a(x,r) \chi_{x,r} \right\|_{L^p(\mathbf{R}^d)} \lesssim \left(\sum_{(x,r)\in\mathcal{E}} |a(x,r)|^p r^{p-1} \right)^{1/p},$$

where the implicit constant depends on p, d, and ψ, but most importantly, is independent of $\mathcal{E}$.

Proof of Lemma 6.1 from Lemma 6.2. For any $a : \mathbf{R}^d \times [1,\infty) \to \mathbf{C}$, if we consider the vector-valued function $\mathbf{a}(x,r) = a(x,r)\chi_{x,r}$, then

$$\int_{\mathbf{R}^d} \int_1^\infty \mathbf{a}(x,r)\, dr\, dx = \int_{[0,1)^d} \int_0^1 \sum\nolimits_{n\in\mathbf{Z}^d} \sum\nolimits_{m>0} \mathrm{Trans}_{n,m} \mathbf{a}(x,r)\, dr\, dx$$

Minkowski's inequality thus implies that

$$\begin{aligned}
\left\| \int_{\mathbf{R}^d} \int_1^\infty \mathbf{a}(x,r)\, dr\, dx \right\|_{L^p(\mathbf{R}^d)} &\leqslant \int_{[0,1)^d} \int_0^1 \left\| \sum_{n \in \mathbf{Z}^d} \sum_{m > 0} \mathrm{Trans}_{n,m} \mathbf{a}(x,r) \right\|_{L^p(\mathbf{R}^d)} dr\, dx \\
&\lesssim \int_{[0,1)^d} \int_0^1 \left(\sum_{n \in \mathbf{Z}^d} \sum_{m>0} |a(x-n, r+m)|^p r^{d-1} \right)^{1/p} dr\, dx \\
&\leqslant \left(\int_{[0,1)^d} \int_0^1 \sum_{n \in \mathbf{Z}^d} \sum_{m>0} |a(x-n,r+m)|^p r^{d-1}\, dr\, dx \right)^{1/p} \\
&= \left(\int_{\mathbf{R}^d} \int_1^\infty |a(x,r)|^p r^{d-1} dr dx \right)^{1/p}. \qquad \square
\end{aligned}$$

Lemma 6.2 is further reduced by considering it as a bound on the operator $a \mapsto \sum_{(x,r)\in\mathcal{E}} a(x,r)\chi_{x,r}$. In particular, applying real interpolation, it suffices for us to prove a restricted strong type bound. Given any discretized set $\mathcal{E}$, let $\mathcal{E}_k$ be the set of $(x,r) \in \mathcal{E}$ with $2^k \leqslant r < 2^{k+1}$. Then Lemma 6.2 is implied by the following Lemma.

Lemma 6.3. *For any* $1 \leqslant p < 2(d-1)/(d+1)$ *and* $k \geqslant 1$,

$$\left\| \sum_{(x,r)\in\mathcal{E}} \chi_{x,r} \right\|_{L^p(\mathbf{R}^d)} \lesssim_p \left(\sum_{k \geqslant 1} 2^{k(d-1)} \#(\mathcal{E}_k) \right)^{1/p}.$$

Remark. Note that if $2^k \leqslant r \leqslant 2^{k+1}$, then because $\|\chi_{x,r}\|_{L^p(\mathbf{R}^d)} \sim 2^{k(d-1)/p}$, we can write this as

$$\left\| \sum_{(x,r)\in\mathcal{E}} \chi_{x,r} \right\|_{L^p(\mathbf{R}^d)} \lesssim_p \left(\sum_{(x,r)\in\mathcal{E}} \|\chi_{x,r}\|^p_{L^p(\mathbf{R}^d)} \right)^{1/p}.$$

Thus we are proving a kind of $l^p L^p$ decoupling for the functions $\chi_{x,r}$. This is strictly weaker than an $l^2 L^p$ decoupling bound. TODO: Could we possibly get an $l^2 L^p$ decoupling bound here?

6.2 Density Decomposition

To control these sums, we apply a 'density decomposition', somewhat analogous to a Calderon Zygmund decomposition, which will enable us to obtain L^2 bounds. We say a 1-separated set $\mathcal{E}$ in $\mathbf{R}^d \times [R, 2R)$ is of *density type*

(u,R) if

$$\#(B \cap \mathcal{E}) \leqslant u \cdot \text{diam}(B)$$

for each ball B in $\mathbf{R}^{d+1}$ with diameter $\leqslant R$.

Theorem 6.4. *For any 1-separated set $\mathcal{E}_k \subset \mathbf{R}^d \times [2^k, 2^{k+1})$, we can consider a disjoint union $\mathcal{E}_k = \bigcup_{m=1}^\infty \mathcal{E}_k(2^m)$ with the following properties:*

- *For each m, $\mathcal{E}_k(2^m)$ has density type $(2^m, 2^k)$.*
- *If B is a ball of radius $r \leqslant 2^k$ containing at least $2^m \cdot r$ points of $\mathcal{E}_k$, then*

$$B \cap \mathcal{E}_k \subset \bigcup_{m' \geqslant m} \mathcal{E}_k(2^{m'}).$$

- *For each m, there are disjoint balls $\{B_i\}$, with radii $\{r_i\}$, each at most 2^k, such that*

$$\sum_i r_i \leqslant \frac{\#(\mathcal{E}_k)}{2^m}$$

such that $\bigcup B_i^*$ covers $\bigcup_{m' \geqslant m} \mathcal{E}_k(2^{m'})$, where B_i^* denotes the ball with the same center as B_i but 5 times the radius.

Proof. Define a function $M : \mathcal{E}_k \to [0,\infty)$ by setting

$$M(x,r) = \sup\left\{ \frac{\#(\mathcal{E}_k \cap B)}{\text{rad}(B)} : (x,r) \in B \text{ and } \text{rad}(B) \leqslant 2^k \right\}.$$

We can establish a kind of weak L^1 estimate for M using a Vitali type argument. Let

$$\widehat{\mathcal{E}}_k(2^m) = \{(x,r) \in \mathcal{E}_k : M(x,r) \geqslant 2^m\}.$$

We can therefore cover $\widehat{\mathcal{E}}_k(2^m)$ by a family of balls $\{B\}$ such that $\#(\mathcal{E}_k \cap B) \geqslant 2^m \text{rad}(B)$. The Vitali covering lemma allows us to find a disjoint subcollection of balls $B_1, \dots, B_N$ such that $B_1^*, \dots, B_N^*$ covers $\widehat{\mathcal{E}}_k(2^m)$. We find that

$$\#(\mathcal{E}_k) \geqslant \sum_i \#(B_i \cap \mathcal{E}_k) \geqslant 2^m \sum_i \text{rad}(B_i),$$

Setting $\mathcal{E}_k = \widehat{\mathcal{E}}_k(2^m) - \bigcup_{k' > k} \widehat{\mathcal{E}}_{k'}(2^m)$ thus gives the required result. $\square$

To prove Lemma 6.3, we perform a decomposition of $\mathcal{E}_k$ for each k, into the sets $\mathcal{E}_k(2^m)$, and then define $\mathcal{E}^m = \bigcup_{k \geqslant 1} \mathcal{E}_k^m$. For appropriate exponents, we will prove L^p bounds on the functions

$$F^m = \sum_{(x,r) \in \mathcal{E}^m} \chi_{x,r}$$

which are exponentially decaying in m, i.e. that

$$\|F^m\|_{L^p(\mathbf{R}^d)} \lesssim m \cdot 2^{-m(1/p - 1/p_d)} \left(\sum_k 2^{k(d-1)} \#(\mathcal{E}_k) \right)^{1/p}.$$

Thus summing in m using the triangle inequality gives a bound on $F = \sum_m F^m$, in the range $1 < p < p_d$, i.e. that

$$\|F\|_{L^p(\mathbf{R}^d)} \lesssim \left(\sum_k 2^{k(d-1)} \#(\mathcal{E}_k) \right)^{1/p},$$

proving Lemma 6.3. To get the bound on F^m, we interpolate being an L^2 bound for F^m, and an L^0 bound (i.e. a bound on the measure of the support of F^m). First, we calculate the support of F^m.

Lemma 6.5. *For each k,*

$$|supp(F_k^m)| \lesssim 2^{-m} 2^{k(d-1)} \#\mathcal{E}_k.$$

Thus we have

$$|supp(F^m)| \leqslant \sum_k |supp(F_k^m)| \lesssim \sum_k 2^{-m} 2^{k(d-1)} \#\mathcal{E}_k.$$

Proof. We recall that for each k and m, we can find disjoint balls $B_1, \dots, B_N$ with radii $r_1, \dots, r_N \leqslant 2^k$ such that

$$\sum_{i=1}^N r_i \leqslant 2^{-m} \#\mathcal{E}_k,$$

where $\mathcal{E}_k(2^m)$ is covered by the expanded balls $B_1^* \cup \dots \cup B_N^*$. If we write

$$F_{k,i}^m = \sum_{(x,r) \in \mathcal{E}_k(2^m) \cap B_i^*} \chi_{x,r},$$

then $\operatorname{supp}(F_k^m) \subset \bigcup_i \operatorname{supp}(F_{k,i}^m)$. For each $(x,r) \in B_i^* \cap \mathcal{E}_k(2^m)$, the support of $\chi_{x,r}$, an annulus of thickness $O(1)$ and radius r, is contained in an annulus of thickness $O(r_i)$ and radius $O(2^k)$ with the same centre as B_i. Thus we conclude that

$$|\operatorname{supp}(F_{k,i}^m)| \lesssim r_i 2^{k(d-1)},$$

and it follows that

$$|\operatorname{supp}(F_k^m)| \leqslant \sum_i r_i 2^{k(d-1)} \leqslant 2^{-m} 2^{k(d-1)} \# \mathcal{E}_k. \qquad \square$$

From interpolation, it therefore suffices to prove the following L^2 estimate on the function F^m.

Lemma 6.6. *Suppose $\mathcal{E} = \bigcup_k \mathcal{E}_k$ is a one-separated set, where $\mathcal{E}_k \subset \mathbf{R}^d \times [2^k, 2^{k+1})$ is a set of density type $(2^m, 2^k)$. Then*

$$\left\| \sum_{(x,r)\in\mathcal{E}} \chi_{x,r} \right\|_{L^2(\mathbf{R}^d)} \lesssim \sqrt{m} \cdot 2^{\frac{m}{d-1}} \left(\sum_k 2^{k(d-1)} \#(\mathcal{E}_k) \right)^{1/2}.$$

Note that this bound gets worse and worse as m grows, whereas the support bound gets better and better, since annuli are concentrating in a small set, which is bad from the perspective of constructive interference, but absolutely fine from the perspective of a support bound. Interpolation gives a bound exponentially decaying in m for $1 < p < p_d$.

6.3 L^2 Bounds

Proving 6.6 is where the weak-orthogonality bounds from Lemma 6.7 come into play. Indeed, we can write the inequality as

$$\left\| \sum_{(x,r)\in\mathcal{E}} \chi_{x,r} \right\|_{L^2(\mathbf{R}^d)} \lesssim \sqrt{m} \cdot 2^{\frac{m}{d-1}} \left(\sum_{(x,r)\in\mathcal{E}} \|\chi_{x,r}\|_{L^2(\mathbf{R}^d)}^2 \right)^{1/2},$$

and if we had perfect orthogonality, or even almost orthogonality, then we could replace the $\sqrt{m} \cdot 2^{\frac{m}{d-1}}$ term with a constant.

To prove this L^2 bound, we require an analysis of the interference patterns of the functions $\chi_{x,r}$, which are supported on various annuli, but oscillate on these annuli. We will use almost orthogonality principles to understand these interference patterns which work the best now we have reduced our analysis to L^2 bounds.

Lemma 6.7. *For any $N > 0$, $x_1, x_2 \in \mathbf{R}^d$ and $r_1, r_2 \geqslant 1$,*

$$|\langle \chi_{x_1,r_1}, \chi_{x_2,r_2} \rangle| \lesssim_N (r_1 r_2)^{(d-1)/2} (1 + |r_1 - r_2| + |x_1 - x_2|)^{-(d-1)/2} \sum_{\pm,\pm} (1 + ||x_1 - x_2| \pm r_1 \pm r_2|)^{-N}.$$

In particular,

$$|\langle \chi_{x_1,r_1}, \chi_{x_2,r_2} \rangle| \lesssim \left(\frac{r_1 r_2}{|(x_1, r_1) - (x_2, r_2)|} \right)^{(d-1)/2}$$

Remark. Suppose $r_1 \leqslant r_2$. Then Lemma 6.7 implies that χ_{x_1,r_1} and χ_{x_2,r_2} are roughly uncorrelated, except when $|x_1 - x_2|$ and $|r_1 - r_2|$ is small, and in addition, one of the following two properties hold:

- $r_1 + r_2 \approx |x_1 - x_2|$, which holds when the two annuli are 'approximately' externally tangent to one another.
- $r_2 - r_1 \approx |x_1 - x_2|$, which holds when the two annuli are 'approximately' internally tangent to one another.

Heo, Nazarov, and Seeger do not exploit the tangency information, though utilizing the tangencies seems important to improve the results they obtain. Laura Cladek's paper exploits this tangency information, to some extent, to obtain the improved result in her paper.

Proof. We write

$$\begin{aligned}
\langle \chi_{x_1 r_1}, \chi_{x_2 r_2} \rangle &= \langle \widehat{\chi}_{x_1 r_1}, \widehat{\chi}_{x_2 r_2} \rangle \\
&= \int_{\mathbf{R}^d} \widehat{\sigma_{r_1} * \psi}(\xi) \cdot \overline{\widehat{\sigma_{r_2} * \psi}(\xi)} e^{2\pi i (x_2 - x_1) \cdot \xi} \, d\xi \\
&= (r_1 r_2)^{d-1} \int_{\mathbf{R}^d} \widehat{\sigma}(r_1 \xi) \overline{\widehat{\sigma}(r_2 \xi)} |\widehat{\psi}(\xi)|^2 e^{2\pi i (x_2 - x_1) \cdot \xi} \, d\xi.
\end{aligned}$$

Define functions A and B such that $B(|\xi|) = \widehat{\sigma}(\xi)$, and $A(|\xi|) = |\widehat{\psi}(\xi)|^2$. Then

$$\langle \chi_{x_1,r_1}, \chi_{x_2,r_2} \rangle = C_d (r_1 r_2)^{d-1} \int_0^\infty s^{d-1} A(s) B(r_1 s) B(r_2 s) B(|x_2 - x_1| s) \, ds.$$

Using well known asymptotics for the Fourier transform for the spherical measure, we have

$$B(s) = s^{-(d-1)/2} \sum_{n=0}^{N-1} (c_{n,+} e^{2\pi i s} + c_{n,-} e^{-2\pi i s}) s^{-n} + O_N(s^{-N}).$$

But now substituting in, assuming $A(s)$ vanishes to order $100N$ at the origin, we conclude that

$$\begin{aligned}
\langle \chi_{x_1 r_1}, \chi_{x_2 r_2} \rangle &= C_d \left(\frac{r_1 r_2}{|x_1 - x_2|} \right)^{(d-1)/2} \sum_{n,\tau} c_{n,\tau} r_1^{-n_1} r_2^{-n_2} |x_2 - x_1|^{-n_3} \\
&\qquad \left\{ \int_0^\infty A(s) s^{-(d-1)/2} s^{-n_1 - n_2 - n_3} e^{2\pi i (\tau_1 r_1 + \tau_2 r_2 + \tau_3 |x_2 - x_1|) s} \, ds \right\} \\
&\lesssim_N \left(\frac{r_1 r_2}{|x_1 - x_2|} \right)^{\frac{d-1}{2}} \left(1 + \frac{1}{|x_1 - x_2|^N} \right) \sum_\tau (1 + |\tau_1 r_1 + \tau_2 r_2 + \tau_3 |x_2 - x_1||)^{-5N} \\
&\lesssim_N \left(\frac{r_1 r_2}{|x_1 - x_2|} \right)^{\frac{d-1}{2}} \left(1 + \frac{1}{|x_1 - x_2|^N} \right) \sum_\tau (1 + |\tau_1 \tau_3 r_1 + \tau_2 \tau_3 r_2 + |x_2 - x_1||)^{-5N}.
\end{aligned}$$

This gives the result provided that $1 + |x_1 - x_2| \geqslant |r_1 - r_2|/10$ and $|x_1 - x_2| \geqslant 1$. If $1 + |x_1 - x_2| \leqslant |r_1 - r_2|/10$, then the supports of χ_{x_1, r_1} and χ_{x_2, r_2} are disjoint, so the inequality is trivial. On the other hand, if $|x_1 - x_2| \leqslant 1$, then the bound is trivial by the last sentence unless $|r_1 - r_2| \leqslant 10$, and in this case the inequality reduces to the simple inequality

$$\langle \chi_{x_1, r_1}, \chi_{x_2, r_2} \rangle \lesssim_N (r_1 r_2)^{(d-1)/2}.$$

But this follows immediately from the Cauchy-Schwartz inequality. $\square$

The exponent $(d-1)/2$ in Lemma 6.7 is too weak to apply almost orthogonality directly to obtain L^2 bounds on $\sum_{(x,r) \in \mathcal{E}_k} \chi_{xr}$ on it's own, but together with the density decomposition assumption we will be able to obtain Lemma 6.6.

Proof of Lemma 6.6. Without loss of generality, we may assume that the k such that $\mathcal{E}_k \neq \varnothing$ is 10-separated. Write

$$F = \sum_{(x,r) \in \mathcal{E}} \chi_{x,r}$$

and $F_k = \sum_{(x,r)\in\mathcal{E}_k} \chi_{x,r}$. First, we deal with $F_{\lesssim m} = \sum_{k\leqslant 10m} F_k$ trivially, i.e. writing

$$\|F\|_{L^2(\mathbf{R}^d)} \lesssim m^{1/2}\left(\sum_{k\leqslant 10m}\|F_k\|^2_{L^2(\mathbf{R}^d)} + \|\sum_{k>10m} F_k\|_{L^2(\mathbf{R}^d)}\right)^{1/2}.$$

We then decompose

$$\|\sum_{k>10m} F_k\|^2_{L^2(\mathbf{R}^d)} \leqslant \sum_{k>10m}\|F_k\|^2_{L^2(\mathbf{R}^d)} + 2\sum_{k'>k>10m}|\langle F_k, F_{k'}\rangle|.$$

Let us analyze $\langle F_k, F_{k'}\rangle$. The term will become a sum of the form $\langle \chi_{x,r}, \chi_{y,s}\rangle$, where $r \sim 2^k$ and $s \sim 2^{k'}$. Because of our assumption of being 10-separated, we have $r \leqslant s/2^{10}$. If $\langle \chi_{x,r}, \chi_{y,s}\rangle \neq 0$, then since the support of $\chi_{y,s}$ is an annulus of radius s centered at y, with thickness $O(1)$, and $\chi_{x,r}$ has support on an annulus of radius r centered at x, with thickness $O(1)$, the fact that r is comparatively smaller than s implies that (x,r) must be contained in the annulus of radius s centered at y, with thickness $O(2^k)$. Such an annulus is covered by $O(2^{(k'-k)(d-1)})$ balls of radius 2^k. Each ball can only contain 2^{k+m} points (x,r), and so there can be at most

$$O(2^{k'(d-1)}2^{-k(d-1)}2^{k+m}) = O(2^{k'(d-1)-k(d-2)+m}).$$

pairs $(x,r)\in\mathcal{E}_k$ for which $\langle \chi_{x,r}, \chi_{y,s}\rangle \neq 0$. For such pairs we have

$$|\langle \chi_{x,r}, \chi_{y,s}\rangle| \lesssim \left(\frac{2^k 2^{k'}}{2^{k'}}\right)^{\frac{d-1}{2}} = 2^{\frac{k(d-1)}{2}}.$$

Thus we conclude that

$$|\langle F_k, \chi_{y,s}\rangle| \lesssim 2^{-k(\frac{d-3}{2})+k'(d-1)+m}.$$

Summing over $10m < k < k'$, we conclude that since $d \geqslant 4$,

$$\sum_{10m<k<k'}|\langle F_k, \chi_{y,s}\rangle| \lesssim 2^{k'(d-1)+m}\sum_{10m<k<k'}2^{-k\frac{d-3}{2}} \lesssim 2^{k'(d-1)+m}2^{-5m} \lesssim 2^{k'(d-1)}.$$

But this means that

$$\sum_{10m<k<k'}|\langle F_k, F_{k'}\rangle| \lesssim 2^{k'(d-1)}\cdot \#(\mathcal{E}_{k'}).$$

This means that

$$\|\sum_{k>10m} F_k\|^2_{L^2(\mathbf{R}^d)} \lesssim \sum_{k>10m} \|F_k\|^2_{L^2(\mathbf{R}^d)} + \sum_{k'} 2^{k'(d-1)}\#(\mathcal{E}_{k'}),$$

and it now suffices to deal with estimates the $\|F_k\|_{L^2(\mathbf{R}^d)}$, i.e. the interactions of functions supported on radii of comparable magnitude. To deal with these, we further decompose the radii, writing $[2^k, 2^{k+1})$ as the disjoint union of intervals $I_{k,\mu} = [2^k + (\mu-1)2^{am}, 2^k + \mu 2^{am}]$, for some a to be chosen later. These interval induces a decomposition $\mathcal{E}_k = \bigcup_\mu \mathcal{E}_{k,\mu}$. Again, incurring a constant loss at most, we may assume that the μ such that $\mathcal{E}_{k,\mu} \neq \varnothing$ are 10 separated. We write $F_k = \sum F_{k,\mu}$, and we have

$$\|F_k\|^2_{L^2(\mathbf{R}^d)} = \sum_\mu \|F_{k,\mu}\|^2_{L^2(\mathbf{R}^d)} + \sum_{\mu<\mu'} |\langle F_{k,\mu}, F_{k,\mu'}\rangle|.$$

We now consider $\chi_{x,r}$ and $\chi_{y,s}$ with $r \in I_{k,\mu}$ and $s \in I_{k',\mu}$. Then we must have $|x-y| \lesssim 2^k$ and $2^{am} \leqslant |r-s| \lesssim 2^k$, and so we have

$$\begin{aligned}|\sum_{\mu<\mu'} \langle F_{k,\mu}, \chi_{y,s}\rangle| &\lesssim 2^{k(d-1)} \sum_{\substack{(x,r)\in\mathcal{E}_k \\ 2^{am} \leqslant |(x,r)-(y,s)| \lesssim 2^k}} |(x,r)-(y,s)|^{-\frac{d-1}{2}} \\ &\lesssim 2^{k(d-1)} \sum_{am \leqslant l \leqslant k} 2^{-l(d-1)/2} \#\{(x,r)\in\mathcal{E}_k : |(x,r)-(y,s)| \sim 2^l\}.\end{aligned}$$

Using the density assumption,

$$\#\{(x,r)\in\mathcal{E}_k : |(x,r)-(y,s)| \sim 2^l\} \lesssim 2^{l+m}$$

and so we obtain that, again using the assumption that $d \geqslant 4$,

$$|\sum_{\mu<\mu'} \langle F_{k,\mu}, \chi_{y,s}\rangle| \lesssim 2^{k(d-1)} 2^{m(1-a(d-3)/2)}.$$

Now summing over all (y,s), we obtain that

$$|\sum_{\mu<\mu'} \langle F_{k,\mu}, F_{k,\mu'}\rangle| \lesssim 2^{k(d-1)} 2^{m(1-a(d-3)/2)} \#(\mathcal{E}_{k,\mu'}).$$

and now summing over μ' gives that

$$\|F_k\|^2_{L^2(\mathbf{R}^d)} \lesssim \sum_\mu \|F_{k,\mu}\|^2_{L^2(\mathbf{R}^d)} + 2^{k(d-1)} 2^{m(1-a(d-3)/2)} \#\mathcal{E}_k,$$

which is a good enough bound if we pick a to be large enough. Now we are left to analyze $\|F_{k,\mu}\|_{L^2(\mathbf{R}^d)}$, i.e. analyzing interactions between annuli which have radii differing from one another by at most $O(2^{am})$. Since the family of all possible radii are discrete, the set $\mathcal{R}_{k,\mu}$ of all possible radii has cardinality $O(2^{am})$. We do not really have any orthogonality to play with here, so we just apply Cauchy-Schwartz, writing $F_{k,\mu} = \sum_{r \in \mathcal{R}_{k,\mu}} F_{k,\mu,r}$, to write

$$\|F_{k,\mu}\|_{L^2(\mathbf{R}^d)}^2 \lesssim 2^{am} \sum_r \|F_{k,\mu,r}\|_{L^2(\mathbf{R}^d)}^2.$$

Recall that $\chi_{x,r} = \mathrm{Trans}_x(\sigma_r * \psi)$, where ψ is a compactly supported function whose Fourier transform is non-negative and vanishes to high order at the origin. In particular, we now make the additional assumption that $\psi = \psi_\circ * \psi_\circ$ for some other compactly function $\psi_\circ$ whose Fourier transform is non-negative and vanishes to high order at the origin. Then we find that $F_{k,\mu,r}$ is equal to the convolution of the function

$$A_r = \sum_{(x,r) \in \mathcal{E}} \mathrm{Trans}_x \psi_\circ$$

with the function $\sigma_r * \psi_\circ$. Using the standard asymptotics for the Fourier transform of σ_r, i.e. that for $|\xi| \geqslant 1$,

$$|\widehat{\sigma}_r(\xi)| \lesssim r^{d-1}(1 + r|\xi|)^{-\frac{d-1}{2}},$$

and since $|\widehat{\psi_\circ}(\xi)| \lesssim_N |\xi|^N$, we get that if $r \geqslant 1$, then for $|\xi| \leqslant 1/r$,

$$|\widehat{\sigma}_r(\xi)\widehat{\psi_\circ}(\xi)| \lesssim_N r^{d-1-N}$$

and for $|\xi| \geqslant 1/r$,

$$|\widehat{\sigma}_r(\xi)\widehat{\psi_\circ}(\xi)| \lesssim_N r^{\frac{d-1}{2}} |\xi|^{-N}.$$

Thus in particular,the L^∞ norm of the Fourier transform of $\sigma_r * \psi_\circ$ is $O(r^{(d-1)/2})$. Now the functions $\psi_\circ$ are compactly supported, so since the set of x such that $(x,r) \in \mathcal{E}$ is one-separated, we find that

$$\|A_r\|_{L^2(\mathbf{R}^d)} \lesssim \#\{x : (x,r) \in \mathcal{E}\}^{1/2}.$$

But this means that

$$\|F_{k,\mu,r}\|_{L^2(\mathbf{R}^d)} = \|A_r * (\sigma_r * \psi_\circ)\|_{L^2(\mathbf{R}^d)} \lesssim r^{\frac{d-1}{2}} \#\{x : (x,r) \in \mathcal{E}\}^{1/2}.$$

Thus we have that

$$\|F_{k,\mu}\|_{L^2(\mathbf{R}^d)}^2 = 2^{am} \cdot \#\mathcal{E}_{k,\mu} \cdot 2^{k(d-1)}.$$

Summing over μ gives that

$$\|F_k\|_{L^2(\mathbf{R}^d)}^2 = 2^{k(d-1)} \#\mathcal{E}_k (2^{am} + 2^{m(1-a(d-3)/2)}).$$

Picking $a = 2/(d-1)$ optimizes this bound, giving

$$\|F_k\|_{L^2(\mathbf{R}^d)} \lesssim 2^{m/(d-1)} 2^{k(d-1)/2} (\#\mathcal{E}_k)^{1/2}.$$

Plugging this into the estimates we got for F gives the required bound. $\square$

Chapter 7

Cladek: Improvements Using Incidence Geometry

The results of Heo, Nazarov, and Seeger only apply when $d \geqslant 4$. Cladek found a method to get an initial radial multiplier conjecture result in $\mathbf{R}^3$, and an improvmeent of the bounds obtained by Heo, Nazarov, and Seeger when $d = 3$. The idea is to exploit the fact that one need only prove a version of 6.2 for a set $\mathcal{E} = \mathcal{E}_X \times \mathcal{E}_R$, where $\mathcal{E}_X$ is a one-separated family of points, and $\mathcal{E}_R$ are a family of radii. One can then exploit this Cartesian product structure when analyzing functions of the form

$$F = \sum_{(x,r)\in\mathcal{E}} \chi_{x,r},$$

in particular, improving upon the result of [6].

7.1 Result in 3 Dimensions

As in [6], Cladek first performs a density decomposition, i.e. writing

$$F = \sum F_k^m$$

where

$$F_k^m = \sum_{(x,r)\in\mathcal{E}_k(2^m)} \chi_{x,r}.$$

Cladek then interpolates between an L^0 bound and an L^2 bound on the resulting functions. The L^0 bound is exactly the same bound used in [6].

Theorem 7.1. *For the function F, we have*

$$|supp(F_k^m)| \lesssim 2^{-m} 4^k \# \mathcal{E}_k$$

and thus

$$|supp(F^m)| \lesssim \sum_k 2^{-m} 4^k \# \mathcal{E}_k.$$

The L^2 bound is improved upon, which is what allows us to obtain a new result in three dimensions.

Lemma 7.2. *Suppose $\mathcal{E} = \bigcup_k \mathcal{E}_k$ is a one-separated set, where $\mathcal{E}_k \subset \mathbf{R}^d \times [2^k, 2^{k+1})$ is a set of density type $(2^m, 2^k)$. Then*

$$\left\| \sum_{(x,r)\in\mathcal{E}} \chi_{x,r} \right\|_{L^2(\mathbf{R}^d)} \lesssim_\varepsilon 2^{[(11/13)+\varepsilon]m} \sum_k 4^k \# \mathcal{E}_k.$$

Interpolation thus yields that for a set of density type 2^m as in this Lemma,

$$\| \sum_{(x,r)\in\mathcal{E}} \chi_{x,r} \|_{L^p(\mathbf{R}^d)} \lesssim_\varepsilon 2^{-m(1/p - 12/13 - \varepsilon)} (\sum_k 4^k \# \mathcal{E}_k)^{1/p}.$$

If $1 < p < 13/12$, this sum is favorable in m, and may be summed without harm to prove the radial multiplier conjecture for unit scale radial multipliers in this range.

Proof of Lemma 7.2. Write

$$F_k = \sum_{(x,r)\in\mathcal{E}_k} \chi_{x,r}.$$

As before, we can throw away terms for $k \leqslant 10m$, i.e. obtaining that

$$\| \sum F_k \|_{L^2(\mathbf{R}^d)} \lesssim m^{1/2} \left(\sum_k \|F_k\|_{L^2(\mathbf{R}^d)}^2 + \sum_{10m < k < k'} |\langle F_k, F_{k'} \rangle| \right)^{1/2}.$$

Our proof thus splits into two cases: where the radii are incomparable, and where the radii are comparable.

TODO: □

7.2 Results in 4 Dimensions

TODO

Chapter 8

Mockenhaupt, Seeger, and Sogge: Exploiting Wave-Equation Periodicity

The main goal of the paper *Local Smoothing of Fourier Integral Operators and Carleson-Sjölin Estimates* is to prove local regularity theorems for a class of Fourier integral operators in $I^\mu(Z, Y; \mathcal{C})$, where Y is a manifold of dimension $n \geqslant 2$, and Z is a manifold of dimension $n+1$, which naturally arise from the study of wave equations. A consequence of this result will be a local smoothing result for solutions to the wave equation, i.e. that if $2 < p < \infty$, then there is δ depending on p and n, such that if $T : Y \to Y \times \mathbf{R}$ is the solution operator to the wave equation, and Y is a compact manifold whose geodesics are periodic, then T is continuous from from $L^p_c(Y)$ to $L^p_{\alpha,\text{loc}}(Y \times \mathbf{R})$ for $\alpha \leqslant -(n-1)|1/2 - 1/p| + \delta$. Such a result is called local smoothing, since if we define $Tf(t,x) = T_t f(x)$, then the operator T_t is, for each t, a Fourier integral operator of order zero, with canonical relation

$$\mathcal{C}_t = \{(x, y; \xi, \xi) : x = y + t\widehat{\xi}\},$$

where $\widehat{\xi} = \xi/|\xi|$ is the normalization of ξ. Standard results about the regularity of hyperbolic partial differential equations show that each of the operators T_t is continuous from $L^p_c(Y)$ To $L^p_{\alpha,\text{loc}}(Y \times \mathbf{R})$ for $\alpha \leqslant -(n-1)|1/2 - 1/p|$, and that this bound is sharp. Thus T is *smoothing* in the t variable, so that for any $f \in L^p$, the functions $T_t f$ 'on average' gain a regularity of δ over the worst case regularity at each time. The local smoothing conjecture states that this result is true for any $\delta < 1/p$.

The class of Fourier integral operators studied are those satisfying the following condition: as is standard, the canonical relation $\mathcal{C}$ is a conic Lagrangian manifold of dimension $2n+1$. The fact that $\mathcal{C}$ is Lagrangian implies $\mathcal{C}$ is locally parameterized by $(\nabla_\zeta H(\zeta,\eta), \nabla_\eta H(\zeta,\eta), \zeta, \eta)$, where H is a smooth, real homogeneous function of order one. If we assume $\mathcal{C} \to T^*Y$ is a submersion, then $D_\xi[\nabla_\eta H(\zeta,\eta)]$ has full rank, which implies $D_\eta[\nabla\xi H(\zeta,\eta)] = (D_\xi[\nabla_\eta H(\zeta,\eta)])^T$ has full rank, and thus the projection $\mathcal{C} \to T^*Z$ is an immersion. We make the further assumption that the projection $\mathcal{C} \to Z$ is a submersion, from which it follows that for each z in the image of this projection, the projection of points in $\mathcal{C}$ onto T_z^*Z is a conic hypersurface Γ_z of dimension n. The final assumption we make is that all principal curvatures of Γ_z are non-vanishing.

Remark. The projection properties of $\mathcal{C}$ imply that, in $T^*(Z \times Y)$, there exists a smooth phase ϕ defined on an open subset of $Z \times T^*Y$, homogeneous in T^*Y, such that locally we can write $\mathcal{C}$ as $(z, \nabla_z\phi(z,\eta), \nabla_\eta\phi(z,\eta), \eta)$ for $\eta \neq 0$. Then, working locally on conic sets,

$$\Gamma_z = \{(\nabla_z \phi(z,\eta))\},$$

and the curvature condition becomes that the Hessian $H_{\eta\eta}\langle \nabla_z\phi, \nu\rangle$ has constant rank $n-1$, where ν is the normal vector to Γ_z. This is a natural homogeneous analogue of the Carleson-Sjölin condition for non-homogeneous oscillatory integral operators, i.e. the Carleson-Sjölin condition is allowed to assume $H_{\eta\eta}\phi$ has rank n, which cannot be possible in our case, since ϕ is homogeneous here. An approach using the analytic interpolation method of Stein or the Strichartz / Fractional Integral approach generalizes the Carleson-Sjölin theorem to show that for any smooth, non-homogeneous phase function $\Phi : \mathbf{R}^{n+1} \times \mathbf{R}^n \to \mathbf{R}$, and any compactly supported smooth amplitude a on $\mathbf{R}^{n+1} \times \mathbf{R}^n$. Consider the operators

$$T_\lambda f(z) = \int a(z,y) e^{2\pi i \lambda \Phi(z,y)} f(y)\, dy.$$

If the associated canonical relation $\mathcal{C}$, if $\mathcal{C}$ projects submersively onto $T^*\mathbf{R}^n$, so that for each $z \in \mathbf{R}^{n+1}$ in the image of the projection map $\mathcal{C}$, the set $S_z \subset \mathbf{R}^{n+1}$ obtained from the inverse image of the projection of $\mathcal{C} \to Z$ at z is a n dimensional hypersurface with k non-vanishing curvatures. Then for $1 \leqslant p \leqslant 2$,

$$\|T_\lambda f\|_{L^q(\mathbf{R}^{n+1})} \lesssim \lambda^{-(n+1)/q} \|f\|_{L^p(\mathbf{R}^n)}.$$

where $q = p^*(1 + 2/k)$.

Remark. We can also see these assumptions as analogues in the framework of cinematic curvature, splitting the z coordinates into 'time-like' and 'space-like' parts. Working locally, because $\mathcal{C} \to T^*Y$ is a submersion, we can consider coordinates $z = (x,t)$ so that, with the phase ϕ introduced above, $D_x(\nabla_\eta \phi)$ has full rank n, and that $\partial_t \phi(x,t,\eta) \neq 0$. Then for each $z = (x,t)$, we can locally write $\partial_t \phi(x,t,\eta) = q(x,t,\nabla_x \phi(x,t,\eta))$, homogeneous in η, and then

$$\mathcal{C} = \{(x,t,y;\xi,\tau,\eta) : (x,\xi) = \chi_t(y,\eta), \tau = q(x,t,\xi)\},$$

where χ_t is a canonical transformation. Our curvature conditions becomes that $H_{\xi\xi} q$ has full rank $n-1$. This is the cinematic curvature condition introduced by Sogge.

Under these assumptions, the paper proves that any Fourier integral operator T in $I^{\mu-1/4}(Z,Y;\mathcal{C})$ maps $L^2_c(Y)$ to $L^q_{\text{loc}}(Z)$ if

$$2\left(\frac{n+1}{n-1}\right) \leqslant q < \infty \quad \text{and} \quad \mu \leqslant -n(1/2 - 1/q) + 1/q.$$

and maps $L^p_c(Y)$ to $L^p_{\text{loc}}(Z)$ if

$$p > 2 \quad \text{and} \quad \mu \leqslant -(n-1)(1/2 - 1/p) + \delta(p,n).$$

If we introduce time and space variables locally as in the remark above, any operator in $I^{\mu-1/4}(Z,Y;\mathcal{C})$ can be written locally as a finite sum of operators of the form

$$Tf(x) = \int_{-\infty}^{\infty} T_t f(x),$$

where

$$T_t f(x) = \int a(t,x,\eta) e^{2\pi i \phi(x,t,y,\eta)} f(y)\, dy\, d\eta.$$

is a Fourier integral operator whose canonical relation is a locally a canonical graph, then the general theory implies that each of the maps T_t maps $L^2_c(Y)$ to $L^q_{\text{loc}}(X)$ if

$$2 \leqslant q \leqslant \infty \quad \text{and} \quad \mu \leqslant -n(1/2 - 1/q)$$

so that here we get local smoothing of order $1/q$, and also maps $L^p_c(Y)$ to $L^p_{\text{loc}}(X)$ if

$$1 < p < \infty \quad \text{and} \quad \mu \leqslant -(n-1)|1/p - 1/2|$$

so we get $\delta(p,n)$ smoothing. A consequence of the smoothing, via Sobolev embedding, is a maximal theorem result for the operator T_t, i.e. that for any finite interval I, the operator

$$Mf = \sup_{t \in I} |T_t f|$$

maps $L^p_c(Y)$ to $L^p_{\text{loc}}(X)$ if $\mu < -(n-1)(1/2 - 1/p) - (1/p - \delta(p,n))$. If the local smoothing conjecture held, we would conclude that, except at the endpoint T^* has the same $L^p_c(Y)$ to $L^p_{\text{loc}}(X)$ mapping properties as each of the operators T_t. We also get square function estimates, such that for any finite interval I, if we consider

$$Sf(x) = \left(\int_I |T_t f(x)|^2 \, dt \right)^{1/2},$$

then for

$$2\frac{n+1}{n-1} \leqslant q < \infty \quad \text{and} \quad \mu \leqslant -n(1/2 - 1/q) + 1/2,$$

the operator S is bounded from $L^2_c(Y)$ to $L^q_{\text{loc}}(X)$.

Our main reason to focus on this paper is the results of the latter half of the paper applying these techniques to radial multipliers on compact manifolds with periodic geodesics. Thus we consider a compact Riemannian manifold M, such that the geodesic flow is periodic with minimal period $2\pi \cdot \Pi$. We consider $m \in L^\infty(\mathbf{R})$, such that $\sup_{s>0} \|\beta \cdot \text{Dil}_s m\|_{L^2_\alpha(\mathbf{R})} = A_\alpha$ is finite for some $\alpha > 1/2$ and some $\beta \in C^\infty_c(\mathbf{R})$. We define a 'radial multiplier' operator

$$Tf = \sum_\lambda m(\lambda) E_\lambda f$$

where E_λ is the projection of f onto the space of eigenfunctions for the operator $\sqrt{-\Delta}$ on M with eigenvalue λ. We can also write this operator as $m(\sqrt{-\Delta})$. Then the wave propogation operator $e^{2\pi i t \sqrt{-\Delta}}$ is periodic of period Π. The Weyl formula tells us that the number of eigenvalues of $\sqrt{-\Delta}$ which are smaller than λ is equal to $V(M) \cdot \lambda^n + O(\lambda^{n-1})$.

Theorem 8.1. *Let $m \in L^2_\alpha(\mathbf{R})$ be supported on $(1,2)$, and assume $\alpha > 1/2$, then for $2 \leqslant p \leqslant 4$, $f \in L^p(M)$, and for any integer k,*

$$\left\| \sup_{2^k \leqslant \tau \leqslant 2^{k+1}} |Dil_\tau m(\sqrt{-\Delta}) f| \right\|_{L^p(M)} \lesssim_\alpha \|m\|_{L^2_\alpha(M)} \|f\|_{L^p(M)}.$$

Proof. To understand the radial multipliers we apply the Fourier transform, writing

$$T_\tau f = (\mathrm{Dil}_\tau m)(\sqrt{-\Delta}) f = m(\sqrt{-\Delta}/\tau) f = \int_{-\infty}^{\infty} \tau \widehat{m}(t\tau) e^{2\pi i t \sqrt{-\Delta}} f \, dt.$$

If we define $\beta \in C_c^\infty((1/2, 8))$ such that $\beta(s) = 1$ for $1 \leqslant s \leqslant 4$, and set $L_k f = \mathrm{Dil}_{2^k} \beta(\sqrt{-\Delta}) f$, then for $2^k \leqslant \tau \leqslant 2^{k+1}$

$$T_\tau f = (\mathrm{Dil}_\tau m)(\sqrt{-\Delta}) f = (\mathrm{Dil}_\tau m \cdot \mathrm{Dil}_{2^k} \beta)(\sqrt{-\Delta}) = T_\tau L_k f.$$

so Cauchy-Schwartz implies that

$$\begin{aligned}
|T_\tau f(x)| &= \left| \int_{-\infty}^{\infty} \tau \widehat{m}(\tau) e^{2\pi i t \sqrt{-\Delta}} L_k f(x) \, dt \right| \\
&\leqslant \|m\|_{L^2_\alpha(M)} \left(\int_{-\infty}^{\infty} \frac{\tau}{(1 + |t\tau|^2)^\alpha} |e^{2\pi i t \sqrt{-\Delta}} L_k f(x)|^2 \right)^{1/2} \\
&\leqslant \|m\|_{L^2_\alpha(M)} \left(\int_{-\infty}^{\infty} \frac{2^k}{(1 + |2^k t|^2)^\alpha} |e^{2\pi i t \sqrt{-\Delta}} L_k f(x)|^2 \right)^{1/2}
\end{aligned}$$

Because of periodicity, if we set $w_k(t) = 2^k / (1 + |2^k t|^2)^\alpha$, it suffices to prove that for $\alpha > 1/2$,

$$\left\| \left(\int_0^\Pi w_k(t) |e^{2\pi i t \sqrt{-\Delta}} L_k f(x)|^2 \, dt \right)^{1/2} \right\|_{L^p(M)} \lesssim_{\alpha,p} \|f\|_{L^p(M)}.$$

This is a weighted combination of the wave propogators, roughly speaking, assigning weight 2^k for $t \lesssim 1/2^k$, and assigning weight $1/t$ to values $t \gtrsim 1/2^k$.

For a fixed $0 < \delta$, we can split this using a partition of unity into a region where $t \gtrsim \delta$ and a region where $t \lesssim \delta$, where δ is independent of k.

For each t, the wave propogation $e^{2\pi it\sqrt{-\Delta}}$ is a Fourier integral operator of order zero (we have an explicit formula for small t, and the composition calculus for Fourier integral operators can then be used to give a representation of the propogation operators for all times t, such that the symbols of these operators are locally uniformly bounded in S^0). Thus the square function estimate above can be applied in the region where $t \gtrsim \delta$, because the weighted square integral above has weight $O_\delta(1)$ uniformly in k.

Next, we move onto the region $t \lesssim 1/2^k$. The symbol of the operator $e^{2\pi it\sqrt{-\Delta}}$

Finally we move onto the region $1/2^k \lesssim t \lesssim \delta$. On this region we have $w_k(t) \sim 1/t$, which hints we should try using dyadic estimates. In particular, suppose that for $\gamma \leqslant \delta$, we have a family of dyadic estimates of the form

$$\left\| \left(\int_\gamma^{2\gamma} |e^{2\pi it\sqrt{-\Delta}} L_k f|^2 \, dt \right)^{1/2} \right\|_{L^p(M)} \lesssim \gamma^{1/2}(1+\gamma 2^k)^\varepsilon \cdot \|f\|_{L^p(M)}.$$

Summing over the $O(k)$ dyadic numbers between $1/2^k$ and δ gives

$$\left\| \left(\int_{1/2^k \lesssim t \lesssim \delta} |e^{2\pi it\sqrt{-\Delta}} L_k f|^2 \, \frac{dt}{t} \right)^{1/2} \right\|_{L^p(M)} \lesssim 2^{\varepsilon k} \|f\|_{L^p(M)}$$

If we were able to obtain this inequality for some $\varepsilon > 0$, then we could bound

that for all $0 < \gamma < \Pi/2$

If we localize near $t \lesssim 1/2^k$ by multiplying by $\phi(2^k t)$ for some compactly supported smooth ϕ supported on $|t| \lesssim 1$, then for t on the support of $\phi(2^k t)$ we have a weight proportional to 2^k, and rescaling shows that it suffices to bound the quantities

$$\left\| \left(\int \phi(t) |e^{2\pi i(t/2^k)\sqrt{-\Delta}} L_k f(x)|^2 \, dt \right)^{1/2} \right\|$$

the family of functions

$$\left\| \left(\int |\phi(t) e^{2\pi i(t/2^k)\sqrt{-\Delta}} L_k f(x)|^2 \, Dt \right)^{1/2} \right\|_{L^p_x} \lesssim \sup \|e^{2\pi i(t/2^k)\sqrt{-\Delta}} L_k f\|_{L^p_x}$$

$a_k(t) = 2^{-k/2}\widehat{\phi}(t/2^k)\beta(\tau/2^k)$
it suffices to uniformly bound quantities of the form

$$\left\| \left(\int 2^k \phi(2^k t) |e^{2\pi i \sqrt{-\Delta}} L_k f(x)|^2 \, dt \right)^{1/2} \right\|_{L^p(M)} \lesssim_{\alpha,p} \|f\|_{L^p(M)}$$

We now apply a dyadic decomposition to deal with the smaller values of t. Let us assume for simplicity of notation that $\delta < 1$, and then consider a partition of unity $1 = \sum_{j=1}^{\infty} \phi(2^j t)$ for $0 \leqslant t \leqslant 1$, and such that ϕ is localized near $1/4 \leqslant t \leqslant 2$, then our goal is to bound the quantities

$$\left\| \left(\int_{-\infty}^{\infty} \phi(2^j t) \frac{2^k}{(1 + |2^k t|^2)^\alpha} |A_t L_k f(x)|^2 \, dt \right)^{1/2} \right\|_{L^p(M)},$$

which are each proportional to

$$s$$

$\square$

Chapter 9

Lee and Seeger: Decomposition Arguments For Estimating Fourier Integral Operators

Let's now discuss a paper [8] entitled *Lebesgue Space Estimates For a Class of Fourier Integral Operators Associated With Wave Propogation*. In this paper, Lee and Seeger prove a variable coefficient version of the result of Heo, Nazarov, and Seeger, i.e. generalizing that result as it applies to sharp local smoothing on $\mathbf{R}^d$ to the local smoothing of Fourier integral operators satisfying the cinematic curvature condition.

We consider a localized Fourier integral operator $T : \mathcal{D}(Y) \to \mathcal{D}^*(Z)$ of order $\mu - 1/4$, where $\dim(Y) = d$ and $\dim(Z) = d+1$, with a canonical relation $\mathcal{C}$ (which must be a $2d+1$ dimensional submanifold of $T^*Z \times T^*Y$ by virtue of the fact it is Lagrangian), satisfying the following properties:

- The projection map $\pi_{T^*Y} : \mathcal{C} \to T^*Y$ is a submersion. It follows that around any point $(z_0, y_0; \zeta_0, \eta_0)$ we can choose coordinate systems y on Y and $z = (x,t)$ on Z centered at z_0 and y_0 such that $\zeta_0 = dx_1$, $\eta_0 = dy_1$, and the tangent plane to $\mathcal{C}$ at this point is given by

  $$dx = dy \quad \text{and} \quad d\xi = d\eta \quad \text{and} \quad d\tau = 0.$$

  In particular, it follows that $\pi_Z : \mathcal{C} \to Z$ is a submersion, and we can locally find a function $\phi(z,\eta)$, homogeneous in η, such that, locally,

  $$\mathcal{C} = \{(z, \nabla_\eta \phi(z,\eta); \nabla_z \phi(z,\eta), \eta)\}.$$

By assumption on the tangent space of $\mathcal{C}$,

$$\nabla_\eta \phi(0, e_1) = 0 \quad \text{and} \quad \nabla_z \phi(0, e_1) = e_1.$$

The equivalence of phase theorem implies we can find a symbol $a(x,t,y,\eta)$ of order μ such that, after appropriately localizing the operator T, we have

$$Tf(x,t) = \int a(x,t,y,\eta) e^{2\pi i[\phi(x,t,\eta) - y\cdot\eta]} f(y)\, d\eta\, dy.$$

- The last assumption implies that for each z_0, $\Sigma_{z_0}\pi_Z^{-1}(z_0)$ is a d dimensional submanifold of $\mathcal{C}$. Moreover, our choice of coordinates makes it easy to see that the natural map $\Sigma_{z_0} \to T^*_{z_0}Z$ is an immersion, whose image is the immersed hypersurface Γ_{z_0} of $T^*_{z_0}$. Indeed, the tangent plane to Σ_{z_0} at the point above is given in coordinates by

  $$dx = dy = dt = d\tau = 0 \quad \text{and} \quad d\xi = d\eta.$$

  And this is projected injectively to the plane defined by $d\tau = 0$ in $T^*_{z_0}Z$. Our other assumption we make about $\mathcal{C}$ is an assumption on *cinematic curvature*. We assume that for each z_0, the hypersurface Σ_{z_0} is a cone with l nonvanishing principal curvatures, for some $1 \leqslant l \leqslant d-1$. Since

  $$\Sigma_{z_0} = \{(z_0; \nabla_z \phi(z_0, \eta_0)\}.$$

  The projection assumptions imply that the $(d+1)\times d$ matrix $D_\eta \nabla_z \phi$ has full rank, and the curvature assumptions imply that the Hessian matrix $H_\eta\{\partial\phi/\partial t\}$ has rank at least l in a neighborhood of our initial point.

Given these assumptions, the following result is obtained.

Theorem 9.1. *If*

$$l \geqslant 3 \quad \text{and} \quad \frac{2l}{l-2} < q < \infty \quad \text{and} \quad \mu \leqslant \frac{d}{q} - \frac{d-1}{2},$$

then T maps $L^q(Y)$ into $L^q(Z)$.

If we take $l = d-1$, we get the full assumption of 'cinematic curvature' and we can use this to get results about local smoothing of the wave equation on compact Riemannian manifolds, which recovers the local smoothing result of Heo, Nazarov, and Seeger obtained in their paper on radial Fourier multipliers.

Theorem 9.2. *Consider a finite interval I, as well as*

$$d \geqslant 4 \quad \textit{and} \quad \frac{2(d-1)}{d-3} < q < \infty.$$

If M is a compact Riemannian manifold, and $\alpha = (d-1)/2 - d/q$, then

$$\|e^{it\sqrt{-\Delta}} f\|_{L^q_t(I) L^q_x(M)} \lesssim_I \|f\|_{L^q_\alpha(M)}.$$

Proof. For any compact time interval I, the Lax parametrix construction allows one, for any suitably small coordinate system, to find a phase function

$$\phi(x,y,\xi) \approx (x-y)\cdot\xi$$

and a symbol a of order zero such that

$$\operatorname{supp}(a) \subset \{(x,t,y,\eta) : |x-y| \lesssim 1 \text{ and } |\eta| \gtrsim 1\},$$

such that if $\Phi(t,x,y,\eta) = \phi(x,y,\eta) + t|\eta|_g$, then, modulo smoothing operators, for $|t| \lesssim 1$ we have

$$(e^{it\sqrt{-\Delta}} f)(x) = \int a(x,t,y,\eta) e^{2\pi i \Phi(t,x,y,\eta)} f(y)\, d\eta\, dy.$$

Now define

$$Tf(x,t) = \int a(x,t,y,\xi) e^{2\pi i \Phi(t,x,y,\xi)} f(y)\, d\xi\, dy,$$

and set

$$Sf = T\{(1-\Delta)^{-\alpha/2} f\}.$$

Then

$$Sf(x) = \int \left[a(x,t,y',\eta)(1+|\xi|^2)^{-\alpha/2} \right] e^{2\pi i (\Phi(t,x,y',\eta) + \xi\cdot(y'-y))} f(y)\, d\eta\, dy'\, dy.$$

Then S is a Fourier integral operator of order $\mu - 1/4$, where $\mu = -\alpha$. The canonical relation of S (as with T) is

$$\mathcal{C} = \{(x,t,y;\eta,\omega,\eta) : x = \exp_y(t\xi/|\xi|) \text{ and } \omega = |\xi|_g\}.$$

One immediately sees that the projection condition is satisfied, and if we are working on a coordinate system localized smaller than the injectivity radius of M, for each $z_0 = (x_0, t_0)$,

$$\Gamma_{z_0} = \{(\xi,\omega) : \omega = |\xi|_g\}$$

is a spherical cone, and thus has $d-1$ nonvanishing principal curvatures. Applying the result, we see that for

$$d \geqslant 4 \quad \text{and} \quad \frac{2(d-1)}{d-3} < q < \infty \quad \text{and} \quad \alpha \geqslant \frac{d-1}{2} - \frac{d}{q}.$$

the operator S maps $L^q(M)$ into $L^q(M \times \mathbf{R})$, which is equivalent to T mapping $L^q_\alpha(M)$ into $L^q(M \times \mathbf{R})$. $\square$

9.1 Frequency Localization and Discretization

Let us describe the idea of the proof. Let $K(z,y)$ denote the kernel of T, i.e.

$$K(x,t,y) = \int a(x,t,y,\eta) e^{2\pi i[\phi(x,t,\eta) - y\cdot\eta]}\, d\eta.$$

Without loss of generality, we may assume that a is supported on $|\eta| \geqslant 1$, since integrals over small frequencies give a smoothing operator. We perform a frequency decomposition, writing

$$K(z,y) = \sum_{i=1}^\infty 2^{i\mu} K_i(z,y)$$

where

$$K_i(z,y) = \int \chi_i(z,y,2^{-i}\eta) e^{2\pi i[\phi(x,t,\eta) - y\cdot\eta]}\, d\eta$$

for some family of functions $\{\chi_i\}$ supported on a common compact subset of $Z \times Y \times \Xi$, and satisfy estimates of the form $|\nabla^N \chi_i| \lesssim_N 1$, for all $N \geqslant 0$, uniformly in i. We can set

$$\chi_i(z,y,\eta) = 2^{-i\mu} a(z,y,2^i\eta)\chi(\eta),$$

since then χ_i is supported on $|\eta| \sim 1$ and by virtue of the fact a is a symbol, we have the required estimates. By performing another decomposition, we may assume Ξ is an arbitrarily small neighborhood of e_1, such that for $z \in Z$ and $\eta \in \Xi$,

$$\nabla_z \phi(z,\eta) \approx e_1 \quad \text{and} \quad D_z \nabla_\eta \phi(z,\eta) \approx \begin{pmatrix} I_d \\ 0 \end{pmatrix}$$

and $H_\eta\{\partial\phi/\partial t\}$ has rank at least l. In this section, we analyze each of these operators separately. If we write T_i for the operator with kernel K_i, then here we will prove that

$$\|T_i f\|_{L^p} \lesssim 2^{i\left(\frac{d-1}{2} - \frac{d}{q}\right)} \|f\|_{L^p}.$$

for $q > 2l/(l-2)$. In Seeger, Sogge, and Stein, it is proved that

$$\|T_i f\|_{L^\infty} \lesssim 2^{i\frac{d-1}{2}} \|f\|_{L^\infty}.$$

By interpolation, it thus suffices to prove a restricted weak type inequality of the form

$$\|T_i \chi_E\|_{L^{q_l,\infty}} \lesssim 2^{i(d/l-1/2)} |E|^{1/q_l}$$

where

$$q_l = 2 + 4/(l-2).$$

By duality, it suffices to show that for

$$p_l = 2 - 4/(l+2),$$

we have

$$\|T_i^* \chi_E\|_{L^{p_l,\infty}} \lesssim 2^{i(2d-l)/2l} |E|^{1/p_l},$$

which is equivalent to prove that for $t > 0$, the measure of the set

$$\{y \in M : |T_i^* \chi_E(y)| \geqslant t\}$$

is bounded by $O(t^{-p_l} 2^{i(2d-l)/(l+2)} |E|)$. The operator T_i^* has kernel

$$K_i^*(y,z) = \overline{K_i(z,y)} = \int \chi_i(z,y,2^{-i}\eta) e^{2\pi i (y\cdot\eta - \phi(z,\eta))}\, dz$$

so we still have a Fourier integral operator, but with a reversed canonical relation. We will obtain these bounds by proving an analogous discretized result at a scale $1/2^i$.

We consider $\mathcal{Z}_i = 2^{-i}\mathbf{Z}^{d+1} \cap [-\varepsilon^2, \varepsilon^2]^{d+1}$, for some small constant $\varepsilon > 0$. For each $z \in \mathcal{Z}_i$ we consider a function $a_{i,z}$ supported on frequencies $|\eta| \sim 2^i$ which make an angle $O(\varepsilon^2)$ with e_1, so that

$$|\partial_\eta^\alpha a_{i,z}(\eta)| \leqslant 2^{-i|\alpha|}$$

for $|\alpha| \lesssim 1$. Set

$$S_{i,z}(y) = \int a_{i,z}(\eta) e^{2\pi i (y\cdot\eta - \phi(z,\eta))}\, d\eta.$$

Our job is to understand the sums $\sum S_{i,z}$.

Lemma 9.3. *For each $\mathcal{E} \subset \mathcal{Z}_i$, the measure of the set of y such that*

$$|\sum_{z\in\mathcal{E}} S_{i,z}(y)| \geqslant t$$

is

$$\lesssim 2^{i\frac{dl-2}{(l+2)}} t^{-p_l} \cdot \#(\mathcal{E}).$$

How does one reduce our problem to this setting? First, suppose we assume that

$$\chi_i(z, y, 2^{-i}\xi) = \eta_i(z, 2^{-i}\xi) \cdot \chi_\circ(y)$$

for some $\chi_0 \in C_c^\infty(\mathbf{R}^d)$ supported on a small neighborhood of the origin, and where η_i is supported on a set of diameter $O(\varepsilon^2)$ near $(z, \xi) = (0, e_1)$ and with uniformly bounded derivatives in i up to a suitably high order. Then one may set

$$a_{i,z}(\xi) = 2^{(i+1)d} \int_{Q_z} \eta_i(z, 2^{-i}\xi) e^{2\pi i(\phi(z,\xi) - \phi(z,\xi))}.$$

The phase function has derivatives $O(2^{-i})$, which gives the required results. To get a general result, we apply a Fourier series, writing

$$\chi_i(z, y, 2^{-i}\xi) = \sum_{\nu\in\mathbf{Z}^d} c_{k,\nu}\eta_{i,\nu}(z, 2^{-i}\xi) e^{2\pi i y\cdot\eta}$$

where the constants, and the derivatives of $\eta_{i,\nu}$, are rapidly decaying in ν.

9.2 L^1 Estimates

To understand the individual pieces S_z, we consider a maximal $2^{-i/2}$ separated set Θ covering the unit sphere, and perform a further decomposition

$$a_z(\eta) = \sum_{\theta \in \Theta} a_{z,\theta}(\eta),$$

where $a_{z,\theta}$ is supported in a cone with aperture $O(2^{-i/2})$ centered at θ. Then $a_{z,\theta}$ is roughly speaking, supported on a set with length $2^{i/2}$ tangent to the radial direction, and with length 2^i in the radial direction. Thus differentiating in the radial direction no longer leads to quite as good derivative estimates, namely, if $u_1, \dots, u_M$ are unit vectors tangent to θ, and $M + N \lesssim 1$, then

$$(\theta \cdot \nabla_\eta)^N \prod_{i=1}^{M} (u_i \cdot \nabla_\eta)\{a_{z,\theta}\} \lesssim 2^{-kN - kM/2}.$$

The decomposition of a_z of course leads to a decomposition $S_z = \sum S_{z,\theta}$.

Now because each component of $\nabla_\eta \phi$ is homogeneous of degree 0, Euler's homogeneous function theorem says that

$$H_\eta \phi(x,\eta) \cdot \eta = 0.$$

Integration by parts (TODO: How? Also is there a typo?) yields that

$$|S_{z,\theta}(y)| \lesssim 2^{i\frac{d+1}{2}} \left(1 + 2^i |(\nabla_\xi \phi(z,\theta) - y) \cdot \theta| + 2^{k/2} |\Pi_{\theta^\perp}(\nabla_\xi \phi(z,\theta) - y)|\right)^{-O(1)}.$$

Roughly speaking, this inequality says that, roughly speaking, $S_{z,\theta}$ has magnitude $2^{i\frac{d+1}{2}}$, and is concentrated on a tube centered at $\nabla_\xi \phi(z,\theta)$, with thickness 2^{-i} in the radial direction, and thickness $2^{-i/2}$ in the tangential direction to θ. In particular, we find that

$$\|S_{z,\theta}\|_{L^1} \lesssim 1.$$

The triangle inequality (probably the best we can do in general in the L^1 setting) implies that

$$\|S_z\|_{L^1} \lesssim 2^{i(d-1)/2}.$$

This is the bound we will use in L^1.

To get more interesting bounds in other L^p spaces, we look at the orthogonality of the functions $\{S_z\}$. On the Fourier side of things, we have

$$\widehat{S}_z(\eta) = a_z(\eta) e^{-2\pi i \phi(z,\eta)}.$$

Thus by Parsevel, we have

$$\langle S_z, S_w \rangle = \langle \widehat{S_z}, \widehat{S_w} \rangle = \int a_z(\eta) a_w(\eta) e^{2\pi i [\phi(z,\eta) - \phi(w,\eta)]}.$$

TODO: Expand on rest of argument.

9.3 Adapting the Argument to Fourier Multipliers

Let $T = m(-\sqrt{\Delta})$ be a radial multiplier on $\mathbf{R}^n$, i.e. such that

$$Tf(x) = \int m(|\xi|) e^{2\pi i \xi \cdot (x-y)} f(y)\, d\xi\, dy.$$

If m is a symbol, then we can interpret T directly as a Pseudodifferential Operator. But Heo, Nazarov, and Seeger's result discuss families of multipliers m that are not even necessarily smooth, but do satisfy certain integrability conditions. To fix this, we assume a priori that we have applyied a decomposition argument, so we may assume m is compactly supported away from the origin. Then (by Paley-Wiener) $\widehat{m}$ is a smooth symbol of some finite order satisfying some integrability properties, which indicates how we might apply the theory of Fourier integral operators, i.e. by taking the Fourier transform of m, we get that

$$Tf(x) = \int \widehat{m}(\rho) e^{2\pi i [\rho|\xi| + \xi \cdot (x-y)]} f(y)\, d\rho\, d\xi\, dy.$$

This is 'almost' a Fourier integral operator, except the phase is not smooth unless $\widehat{m}$ is supported away from the origin (fixed by a decomposition argument), and the phase is non-homogeneous. To fix the non-homogeneity, we just isolate the operator in ρ, writing

$$Tf(x) = \int_{-\infty}^{\infty} \widehat{m}(\rho) T_\rho f(x)\, d\rho,$$

where

$$T_\rho f(x) = e^{2\pi i \rho \sqrt{-\Delta}} f(x) = \int e^{2\pi i [\rho|\xi| + \xi \cdot (x-y)]} f(y)\, d\xi\, dy$$

is the propogation operator for the half-wave equation $\partial_t u = \sqrt{-\Delta} \cdot u$. It has phase $\phi(x,y,\xi) = \rho|\xi| + \xi \cdot (x-y)$, and thus we have a stationary frequency value when $x = y - \rho\widehat{\xi}$, where $\widehat{\xi} = \xi/|\xi|$ is the normalization of ξ. This has canonical relation

Chapter 10

Beltran, Hickman, and Sogge: Decoupling for Fourier Integral Operators

The paper we now discuss extends the theory of decoupling, which was originally used to establish local smoothing for the wave equation on Euclidean space, to the setting of more general FIOs. Here we attempt to study L^p to L^p estimates for Fourier integral operators given by

$$Tf(x,t) = \int_{\mathbf{R}^d} e^{2\pi i \phi(x,t;\xi)} b(x,t;\xi)(1+|\xi|^2)^{\mu/2} \widehat{f}(\xi)\, d\xi$$

where b is a compactly supported symbol of order zero, compactly supported in x and t, and ϕ is a phase function, homogeneous of degree one in the ξ variable. We let

$$K(x,t;y) = \int e^{2\pi i[\phi(x,t;\xi) - y\cdot\xi]} b(x,t;\xi)(1+|\xi|^2)^{\mu/2}\, d\xi$$

denote the kernel. Since $\nabla_\xi \phi(x,t;\xi)$ is homogeneous of degree zero in ϕ, the sets

$$\Sigma_{(x,t)} = \{\nabla_\xi \phi(x,t,\xi) : \xi \in \mathbf{R}^n\} \subset \mathbf{R}^n_y$$

are usually manifolds of dimension $n-1$. They are related to the singular support of K.

To study the L^p behaviour of T, we break up the behaviour of the operator dyadically in the ξ variable, thus setting

$$T = T_{\leqslant 1} + \sum_{n=1}^{\infty} T_n,$$

where, for a given $\lambda > 0$, we let T^λ be an operator with kernel K^λ given by

$$K^\lambda(x,t;\xi) = \int e^{2\pi i[\phi(x,t;\xi) - y\cdot\xi]} b(x,t;\xi)(1+|\xi|^2)^{\mu/2} \beta(\xi/\lambda)\, d\xi.$$

It can be verified that $T_{\leqslant 1}$ is a pseudo-differential operator of order 0, and is therefore bound on L^p for all $1 < p < \infty$. It therefore suffices to show that as $\lambda \to \infty$,

$$\|T^\lambda f\|_{L^p(\mathbf{R}^d)} \lesssim \lambda^{-\varepsilon} \|f\|_{L^p(\mathbf{R}^d)}$$

so that we may sum in n in the expansion of T via the triangle inequality to obtain an L^p bound for the original operator.

For large λ, the principle of stationary phase tells us we should expect K^λ to be concentrated in the set

$$\{(x,t;y) : |\nabla_\xi \phi(x,t;\xi) - y| \leqslant 1/\lambda \text{ for some } \xi\},$$

since the phase oscillates to a significant degree for $|\nabla\phi(x,t;\xi) - y| \gtrsim 1/\lambda$, roughly a $1/\lambda$ neighborhood of the singular support of K. Also we have $\|K^\lambda\|_{L^\infty_x L^\infty_y} \lesssim \lambda^{\mu+d}$ trivially by taking in absolute values. This gives the crude estimate that $\|K_n\|_{L^\infty_x L^1_y} \lesssim \lambda^{\mu+d-1}$. Thus we obtain by Schur's Lemma that

$$\|T^\lambda f\|_{L^1(\mathbf{R}^{d+1})} \lesssim \lambda^{\mu+d-1} \|f\|_{L^1(\mathbf{R}^d)}.$$

We will get a much better bound by a more sophisticated decomposition of the kernels $\{K^\lambda\}$.

For a given λ, let $\{\xi^\lambda_\nu\}$ be a maximal, $\lambda^{-1/2}$ separated subset of the unit sphere in $\mathbf{R}^n$, where ν ranges over some set Θ^λ with $\#(\Theta^\lambda) \sim \lambda^{(d-1)/2}$. Let

$$\Gamma^\lambda_\nu = \{\xi \in \mathbf{R}^d_\xi : |\xi \cdot \xi^\lambda_\nu| \geqslant (1 - c\lambda^{-1/2}) \cdot |\xi|\}$$

for some suitably small constant $c > 0$. Let $\{\chi^\lambda_\nu\}$ be a smooth partition of unity, homogeneous of degree zero, adapted to the Γ^λ_ν. We thus have

$$|D^\alpha \chi^\lambda_\nu(\xi)| \lesssim_\alpha \lambda^{|\alpha|/2} |\xi|^{1-\alpha}.$$

We thus consider operators T_ν^λ with kernels K_ν^λ given by

$$K_\nu^\lambda(x,t;y) = \int e^{2\pi i(\phi(x,t;\xi)-y\cdot\xi)} b_\nu^\lambda(x,t;\xi)(1+|\xi|^2)^{\mu/2}$$

where

$$b_\nu^\lambda(x,t;\xi) = b(x,t;\xi)\beta(\xi/\lambda)\chi_\nu^\lambda(\xi).$$

Stationary phase again tell us that $K_\nu^\lambda(x,t;y)$ satisfies the bounds

$$|K_\nu^\lambda(x,t;y)| \lesssim_N \frac{\lambda^{\mu+(d+1)/2}}{\langle \lambda|\pi_{\xi_\nu^\lambda}(y-\nabla_\xi \phi(x,t,\xi_\nu^\lambda)| + \lambda^{1/2}|\pi_{\xi_\nu^\lambda}^\perp(y-\nabla_\xi\phi(x,t,\xi_\nu^\lambda))|\rangle^N}.$$

This bound immediately yields via Schur's Lemma that for all $1 \leqslant p \leqslant \infty$,

$$\|K_\nu^\lambda\|_{L^\infty_{x,t}L^1_y} \lesssim \lambda^\mu,$$

and thus that

$$\|T_\nu^\lambda f\|_{L^\infty(\mathbf{R}^{d+1})} \lesssim \lambda^\mu \|f\|_{L^\infty(\mathbf{R}^d)},$$

a much better bound than was obtained trivially than from the global sum.

We might hope to then combine this still fairly trivial bound with a square function estimate of the form

$$\|T_\nu^\lambda f\|_{L^p(\mathbf{R}^{d+1})} \lesssim_\varepsilon \lambda^\varepsilon \|S^\lambda f\|_{L^p(\mathbf{R}^{d+1})}$$

where

$$S^\lambda f = \left(\sum_\nu |T_\nu^\lambda f|^2 \right)^{1/2},$$

which in some sense, captures the orthogonality of the operators $\{T_\nu^\lambda\}$. This then yields that for $p \geqslant 2$, that

$$\begin{aligned}
\|T_\nu^\lambda f\|_{L^p_{x,t}} &\lesssim_\varepsilon \lambda^\varepsilon \|T_\nu^\lambda f\|_{L^p_{x,t} l^2_\nu} \\
&\leqslant \lambda^\varepsilon \|T_\nu^\lambda f\|_{L^p_{x,t} l^p_\nu} \\
&= \lambda^\varepsilon \|T_\nu^\lambda f\|_{l^p_\nu L^p_{x,t})} \\
&\lesssim \lambda^\varepsilon \lambda^{\mu+(d-1)/2} \#(\Theta^\lambda)^{1/p} \\
&= \lambda^{\varepsilon+\mu+(d-1)/p},
\end{aligned}$$

thus giving bounds for $\mu > (d-1)/2$, i.e., the non-endpoint local smoothing.

Wolff noticed that the non-endpoint local smoothing results could be obtained with a weaker bound than a square function estimate, namely, an l^p *decoupling inequality* of the form

$$\|T^\lambda f\|_{L^p(\mathbf{R}^{d+1})} \lesssim \lambda^{\alpha(p)+\varepsilon} \|T_\nu^\lambda f\|_{l_\nu^p L_{x,t}^p},$$

where if $2 \leqslant p \leqslant 2(d+1)/(d-1)$, then

$$\alpha(p) = (d-1)|1/p - 1/2|,$$

and for $2(d+1)/(d-1) \leqslant p < \infty$,

$$\alpha(p) = (d-1)|1/p - 1/2| - 1/p.$$

The L^p norm of the localized pieces is much easier to estimate. For instance, we have

$$\|T_\nu^\lambda f\|_{L_{x,t}^\infty} \lesssim \lambda^\mu \|f\|_{L^\infty},$$

and thus

$$\|T_\nu^\lambda f\|_{l_\nu^\infty L_{x,t}^\infty} \lesssim \lambda^\mu \|f\|_{L^\infty}.$$

On the other hand, we have an L^2 energy conservation estimate of the form

$$\|T_\nu^\lambda f\|_{L_{x,t}^2} \lesssim \|T_\nu^\lambda f\|_{L_t^\infty L_x^2} \lesssim \lambda^\mu \|f_\nu^\lambda\|_{L^2}$$

where f_ν^λ is the localization of f_ν^λ on the Fourier side to the support of χ_ν^λ. This immediately yields via Parseval's inequality and orthogonality that

$$\|T_\nu^\lambda f\|_{l_\nu^2 L_{x,t}^2} \lesssim \lambda^\mu \|f_\nu^\lambda\|_{l_\nu^2 L_x^2} \lesssim \lambda^\mu \|f\|_{L_x^2}.$$

Interpolation thus yields that for $2 \leqslant p \leqslant \infty$,

$$\|T_\nu^\lambda f\|_{l_\nu^p L_{x,t}^p} \lesssim \lambda^\mu \|f\|_{L^p},$$

and thus that, together with Wolff's decoupling inequality,

$$\|T^\lambda f\|_{L^p(\mathbf{R}^{d+1})} \lesssim_\varepsilon \lambda^{\alpha(p)+\mu+\varepsilon} \|f\|_{L^p(\mathbf{R}^d)},$$

and thus we get boundedness of T for $\mu < \alpha(p)$, which gives $1/p$ degrees of local smoothing.

Part II

Attempts To Solve Problems

Chapter 11

Relations to Local Smoothing

Let us now try and prove certain special cases of the radial multiplier conjecture on the sphere S^n. Thus we fix a symbol h, and study operators of the form

$$T_R = h\left(\sqrt{-\Delta}/R\right) = \sum h(\lambda/R)E_\lambda,$$

where E_λ is the projection operator onto the eigenspace corresponding to the eigenvalue λ. In particular, we wish to characterize the boundedness properties of the operators $T_{h,R}$, in terms of appropriate control of the Fourier transform of the function h. For simplicity, let us assume that $\text{supp}(h)$ is contained in $1/2 \leqslant \lambda \leqslant 2$, with the hope that things will generalize to non compactly supported values using the appropriate dyadic decomposition technology. For an exponent $1 \leqslant p < 2d/(d+1)$, we then assume that the quantity

$$A_p(h) = \left(\int_0^\infty (1+|s|)^{(d-1)(1-p/2)}|\widehat{h}(s)|^p\, ds\right)^{1/p}$$

is finite, which is a necessary condition for the multiplier $h(\sqrt{-\Delta})$ to be bounded on $L^p(\mathbf{R}^d)$ or $L^{p^*}(\mathbf{R}^d)$, and thus by a transference principle of Mitjagin, necessary for the family of operators $\{T_R : R > 0\}$ to be uniformly bounded in R on $L^p(S^n)$ or $L^{p^*}(S^n)$. The triangle inequality gives uniform boundedness for $R \leqslant 1$, so we may assume in what follows that $R \geqslant 1$.

Our goal is to show that, uniformly in R, we have

$$\|T_R f\|_{L^p} \lesssim \|f\|_{L^p}.$$

Since T_R is a multiplier with symbol supported on $R/2 \leqslant \lambda \leqslant 2R$, we may assume that f is also supported on this frequency range, i.e. is in the span of eigenfunctions to $\sqrt{-\Delta}$ with eigenvalue $R/2 \leqslant \lambda \leqslant 2R$. In particular, this implies that we have derivative estimates of the form

$$\|f\|_{L^p_\alpha} \lesssim_\alpha (1+R)^\alpha \|f\|_{L^p},$$

for any $\alpha \geqslant 0$, i.e. a kind of Bernstein's inequality.

To exploit the fact that $A_p(h)$ is finite, we apply the Fourier transform to the sum defining T_R, writing $T_R f$ as the vector valued integral

$$T_R f = \int_{-\infty}^{\infty} R\widehat{h}(Rt) e^{2\pi i t\sqrt{-\Delta}} f\, dt.$$

where $\{e^{2\pi i t\sqrt{-\Delta}}\}$ are the solution operators to the half wave equation

$$\frac{\partial}{\partial t} = 2\pi i \sqrt{-\Delta}.$$

Using the periodicity of the wave equation, we write

$$T_R f = \int_{-1/2}^{1/2} \left(\sum_{l=-\infty}^{\infty} R \cdot \widehat{h}(R(t+l)) \right) e^{2\pi i t\sqrt{-\Delta}} f\, dt$$

Applying Hölder's inequality, we have

$$\left| \sum_{l=-\infty}^{\infty} \widehat{h}(R(t+l)) \right| \lesssim \left(\sum_{l=-\infty}^{\infty} |\widehat{h}(R(t+l))|^p \langle R(t+l) \rangle^{(d-1)(1-p/2)} \right)^{1/p}$$
$$\left(\sum_{l=-\infty}^{\infty} \langle R(t+l) \rangle^{-(d-1)(p^*/2-1)} \right)^{1/p^*}.$$

If $d(t,\mathbf{Z}) = s$ for some $s \in [0,1/2]$, then since $1 < p < 2d/(d+1)$, using the local constancy of $\widehat{h}$ given that h is compactly supported, we have

$$\left(\sum_{l=-\infty}^{\infty} \langle R(t+l) \rangle^{-(d-1)(p^*/2-1)} \right)^{1/p^*} \sim \langle Rs \rangle^{-(d-1)(1/2-1/p^*)}.$$

Write

$$T_R f = \sum_{k=0}^{O(\log R)} T_{R,k} f,$$

where

$$T_{R,0} f = \int \left(\tilde{\eta}(Rt) \sum_{l=-\infty}^{\infty} R \cdot \widehat{h}(R(t+l)) \right) e^{2\pi i t\sqrt{-\Delta}} f \, dt$$

and

$$T_{R,k} f = \int \eta(Rt/2^k) \left(\sum_{l=-\infty}^{\infty} R \cdot \widehat{h}(R(t+l)) \right) e^{2\pi i t\sqrt{-\Delta}} f \, dt.$$

If α is the inverse Fourier transform of $\tilde{\eta}$, then $T_{R,0}$ corresponds to a multiplier operator $m_{R,0}(\sqrt{-\Delta})$, where

$$m_{R,0}(\lambda) = R^{-1} \sum_n h(n/R) \alpha \left(\frac{\lambda - n}{R} \right).$$

But

$$D^\beta m_{R,0}(\lambda) = R^{-1-\beta} \sum_n h(n/R) D^\beta \alpha \left(\frac{\lambda - n}{R} \right),$$

so that

$$|D^\beta m_{R,0}(\lambda)| \lesssim_N R^{-\beta} \|h\|_{L^\infty} \langle \lambda/R \rangle^{-N} \lesssim \lambda^{-\beta} \|h\|_{L^\infty}.$$

Thus $m_{R,0}$ is a symbol of order zero, uniformly bounded by the L^∞ norm of h, and so for $1 < p < \infty$,

$$\|T_{R,0} f\|_{L^p(M)} \lesssim_p \|h\|_{L^\infty} \|f\|_{L^p(M)}.$$

A similar analysis gives that for $k > 0$,

$$\|T_{R,k} f\|_{L^p(M)} \lesssim_p 2^k \|h\|_{L^\infty} \|f\|_{L^p(M)}$$

so we could reasonably use this bound for $k \lesssim 1$. On the other hand, Hölder's inequality implies that

$$|T_{R,k} f| \lesssim A_p(h) R^{1/p^*} 2^{-k(d-1)(1/2-1/p^*)} \left(\int_{|t| \sim 2^k/R} |e^{2\pi i t\sqrt{-\Delta}} f|^{p^*} \, dt \right)^{1/p^*}.$$

Let us analyze $T_{R,k}$ first. Suppose the endpoint local smoothing conjecture held at the particular value p^* we were consider, at all scales, i.e. so that

$$\left\| \left(\int_{|t|\sim 2^k/R} |e^{2\pi i t\sqrt{-\Delta}} f|^{p^*} \right)^{1/p^*} \right\|_{L^{p^*}(M)} \lesssim (2^k/R)^{1/p^*} \|f\|_{L^{p^*}_{\alpha_{p^*}}(M)}$$

where $\alpha_{p*} = (d-1)(1/2 - 1/p^*) - 1/p^*$. Then, using the fact that

$$\|f\|_{L^{p^*}_{\alpha_{p^*}}} \lesssim R^{\alpha_{p^*}} \|f\|_{L^{p^*}(M)},$$

we conclude that

$$\begin{aligned} \|T_{R,k} f\|_{L^{p^*}(M)} &\lesssim A_p(h) R^{1/p^*} 2^{-k(d-1)(1/2-1/p^*)} (2^k/R)^{1/p^*} R^{\alpha_{p^*}} \|f\|_{L^{p^*}(M)} \\ &\lesssim A_p(h) R^{\alpha_{p^*}} 2^{-k\alpha_{p^*}} \|f\|_{L^{p^*}(M)}. \end{aligned}$$

This bound is summable in k, resulting in a bound uniform in R, provided that

$$k \geqslant \log_2(R) - O(1).$$

Thus we are left to analyze values k with $O(1) \leqslant k \leqslant \log_2(R)$, which is equivalent, for instance, to analyzing times $100/R \leqslant |t| \leqslant 1/100$.

On the other hand, if we *increase* the exponent in the integrability condition $A_p(h)$ by a ε, i.e. assuming the quantity

$$A_{p,\varepsilon}(h) = \left(\int |\widehat{h}(t)|^p (1+|t|)^{(d-1)(1-p/2)+\varepsilon} \right)^{1/p}$$

is finite, then the strategy above leads us to study the wave equation over a very very small set of times, i.e. $100/R \leqslant |t| \leqslant R^{-O(\varepsilon)}$.

Let's explore the analysis over these very very small set of times. Fix $\varepsilon > 0$, let $\phi \in C_c^\infty((0,\infty))$ and equal to one for $|t| \lesssim 1$, and consider an operator of the form

$$T_R f = \int \phi(R^\varepsilon t) \left(\sum_{l=-\infty}^{\infty} R\widehat{h}(R(t+l)) \right) e^{2\pi i t\sqrt{-\Delta}} f.$$

Let a_R denote the inverse Fourier transform of $t \mapsto \phi(R^\varepsilon t)(\sum_l R\widehat{h}(R(t+l)))$. Then

$$a_R(\lambda) = R^{-\varepsilon} \sum_{\omega \in \mathbf{Z}} h\left(\frac{\lambda - \omega}{R}\right) \widehat{\phi}(\omega/R^\varepsilon)$$

In particular, for $N \geqslant 0$,

$$D^N a_R(\lambda) = R^{-(N+1)\varepsilon} \sum_{\omega \in \mathbf{Z}} h\left(\frac{\lambda - \omega}{R}\right) D^N \widehat{\phi}(\omega/R^\varepsilon).$$

Thus we conclude that

$$|D^N a_R(\lambda)| \lesssim_N R^{-N\varepsilon} \|h\|_{L^\infty}$$

In particular, if $|\lambda| \sim R$, then $|D^N a_R(\lambda)| \lesssim_N |\lambda|^{-N\varepsilon}$. In particular, if $\psi \in C_c^\infty(\mathbf{R})$ is supported on the annulus $|\lambda| \sim 1$, then the functions $\tilde{a}_R(\lambda) = \psi(\lambda/R) a_R(\lambda)$ uniformly lie in some symbol class $\mathcal{S}_\varepsilon^0(\mathbf{R})$, i.e. nonstandard symbols of order zero. We also have

$$|D^N a_R(\lambda)| \lesssim_N R^{-(N+1)\varepsilon} \left(\sup_{R'} \|\mathrm{Dil}_{R'} h\|_{l^1(\mathbf{Z})} \right),$$

which implies that under the assumption that the supremum in the inequality above is finite, then $\tilde{a}_R$ lies unformly in the symbol class $\mathcal{S}_\varepsilon^{-\varepsilon}(\mathbf{R})$. In the Euclidean setting, a bounded family of operators in $\mathcal{S}_\varepsilon^m(\mathbf{R}^d)$ will be uniformly bounded in $L^p(\mathbf{R}^d)$ for $m = -(1-\varepsilon)d/2$, so we should expect the condition is only sufficient if $\varepsilon \geqslant 1 - 2/(d+2)$.

Chapter 12

Attempt Using Decoupling

Let us try and attack our problem using decoupling. Fix a function $h : (0,\infty) \to \mathbf{R}$, which is compactly supported on $\{1 < \lambda < 2\}$, and use this function to induce a family of radial multiplier operators on the sphere S^d, of the form

$$T_R f = \sum h(\lambda/R) \langle f, e_\lambda \rangle e_\lambda.$$

Our goal is to obtain bounds of the form

$$\sup_{R>0} \|T_R f\|_{L^p(S^d)} \lesssim \|f\|_{L^p(S^d)},$$

under the assumptions that the Fourier transform

$$\widehat{h}(t) = 2\int_0^\infty h(\lambda) \cos(2\pi\lambda t)\, dt$$

lies in L^q. Without loss of generality, as in the last chapter, because of the support of h, for a given R, we may assume that our inputs f are a sum of eigenfunctions on the sphere with eigenvalue between R and $2R$. We rewrite

$$T_R f = \int_{-\infty}^{\infty} R\widehat{h}(Rt) e^{2\pi i t\sqrt{-\Delta}} f\, dt.$$

Using the fact that the wave equation is periodic of period one on S^{d-1}, we have

$$T_R f = \int_{-1/2}^{1/2} H_R(t) e^{2\pi i t\sqrt{-\Delta}} f\, dt,$$

where

$$H_R(t) = R \sum_{l=-\infty}^{\infty} \widehat{h}(R(t+l)) = \sum_{R<l<2R} h(l/R) e^{2\pi i l t}.$$

In coordinates, modulo a smoothing operator, whose behaviour is negligible, we can write the kernel of $e^{2\pi i t\sqrt{-\Delta}}$ as

$$\int a(x,t,y,\xi) e^{2\pi i \Phi(t,x,y,\xi)}\, d\xi,$$

where a is a symbol of order zero with

$$\mathrm{supp}(a) \subset \{(x,t,y,\xi) : |x-y| \lesssim 1 \text{ and } |\xi| \gtrsim 1\},$$

and $\Phi(t,x,y,\xi) \approx (x-y)\cdot\xi + t|\xi|_g$. We can thus write the kernel of $T_R f$ as

$$\int_{-1/2}^{1/2} \int H_R(t) a(x,t,y,\xi) e^{2\pi i \Phi(t,x,y,\xi)}\, d\xi.$$

We now perform a decomposition of $T_R f$ in the frequency range, writing

$$T_R f = T_R^{\leqslant 0} + \sum_{n=1}^{\infty} T_R^n$$

where T_R^λ has kernel

$$\int_{-1/2}^{1/2} \int H_R(t) a(x,t,y,\xi) \beta(\xi/\lambda) e^{2\pi i \Phi(t,x,y,\xi)}\, d\xi\, dt,$$

and $T_R^{\leqslant 0}$ is supported on $|\xi| \leqslant 1$, and thus has the right L^p bounds simply by applying the triangle inequality.

We now try and apply the decoupling result of Beltran, Hickman, and Sogge; if we cover the unit sphere in $\mathbf{R}^d_\xi$ by $O(\lambda^{-(d-1)/2})$ points Θ_λ, consider an appropriate partition of unity $\{\chi_\lambda^\nu : \nu \in \Theta_\lambda\}$, and thus write

$$T_R^\lambda = \sum T_R^{\lambda,\nu},$$

where $T_R^{\lambda,\nu}$ has kernel

$$\begin{aligned}\int_{100/R \leqslant |t| \leqslant 1/100} \int H_R(t) a(x,t,y,\xi) \beta(\xi/\lambda) \chi_\lambda^\nu(\xi) e^{2\pi i \Phi(t,x,y,\xi)}\, d\xi\, dt \\ = \int H_R(t) a_{\lambda,\nu}(x,t,y,\xi) e^{2\pi i \Phi(t,x,y,\xi)}.\end{aligned}$$

Let us suppose that, using the techniques of their paper, we can show that

$$\|T_R^\lambda f\|_{L^{p^*}(\mathbf{R}^d)} \lesssim_{p,\varepsilon} \lambda^{\alpha(p^*)+\varepsilon} \left(\sum_\nu \|T_{R,\nu}^\lambda f\|_{L^{p^*}(\mathbf{R}^d)}^{p^*} \right)^{1/p^*}.$$

Thus it suffices to analyze the behaviour of each of the operators $T_{R,\nu}^\lambda$ separately. For each fixed t, energy conservation implies that

$$\begin{aligned} \|T_R^\lambda f\|_{L^2(\mathbf{R}^d)} &\lesssim \left(\int_{100/R \leqslant |t| \leqslant 1/100} H_R(t)\, dt \right) \|f_\nu^\lambda\|_{L^2(\mathbf{R}^d)} \\ &\lesssim \int_{100/R \leqslant |t| \leqslant 1/100} R \langle Rs \rangle^{-(d-1)(1/2-1/p)} \|f_\nu^\lambda\|_{L^2(\mathbf{R}^d)} \\ &\lesssim \|f_\nu^\lambda\|_{L^2(\mathbf{R}^d)}. \end{aligned}$$

Thus L^2 orthogonality implies that

$$\left(\sum_\nu \|T_{R,\nu}^\lambda f\|_{L^2(\mathbf{R}^d)}^2 \right)^{1/2} \lesssim \|f\|_{L^2(\mathbf{R}^d)}.$$

On the other hand, to obtain an interpolation at L^∞, we must understand the operator

$$\sup_\nu \|T_{R,\nu}^\lambda f\|_{L^\infty(\mathbf{R}^d)}$$

A stationary phase argument shows that, if we write

$$T_{R,\nu}^\lambda f = \int_{-1/2}^{1/2} H_R(t) T_{R,\nu}^{\lambda,t} f,$$

then the kernel $K_{R,\nu}^{\lambda,t}$ of $T_{R,\nu}^{\lambda,t}$ satisfies estimates of the form

$$|K_{R,\nu}^{\lambda,t}(x,t;y)| \lesssim_N \frac{\lambda^{(d+1)/2}}{\left\langle \lambda |\pi_\nu \nabla_\xi \Phi(x,y,t,\nu)| + \lambda^{1/2} |\pi_\nu^\perp \nabla_\xi \Phi(x,y,t,\nu)| \right\rangle^N}.$$

Here we have $\nabla_\xi \Phi(x,y,t,\nu) = (x-y) + t\nu + O(|x-y|)$. Thus we conclude that, for a fixed x, this kernel has the majority of it's support on a cap centered at the point $x + t\nu$, with thickness $1/\lambda$ in the direction ν, and thickness $1/\lambda^{1/2}$ in directions tangential to ν. But this implies that the kernel of $K^\lambda_{R,\nu}$ is essentially supported on a $1/\lambda^{1/2}$ neighborhood of the line $\{t\nu : 100/R \leqslant |t| \leqslant 1/100\}$, and moreover, on that line we have

$$|K^\lambda_{R,\nu}(x;t\nu)| \lesssim \int_{t-1/\lambda}^{t+1/\lambda} R\langle Rs\rangle^{-(d-1)(1/2-1/p^*)}\lambda^{(d+1)/2}\,ds$$

For $\lambda \geqslant R$, since $|t| \geqslant 100/R$, we get that

$$|K^\lambda_{R,\nu}(x;t\nu)| \lesssim \lambda^{-1}R^{1-(d-1)(1/2-1/p^*)}t^{-(d-1)(1/2-1/p^*)}.$$

These same estimates hold replacing $t\nu$ with $t\nu+v$ for some v perpendicular to ν with $|v| \leqslant \lambda^{-1/2}$. Thus we get that

$$\int |K^\lambda_{R,\nu}(x;y)|\,dy \lesssim 1.$$

For $\lambda \leqslant R$, and $|t| \leqslant 10/\lambda$, we get that

$$|K^\lambda_{R,\nu}(x;t\nu)| \lesssim \lambda^{(d+1)/2}$$

and for $|t| \geqslant 10/\lambda$, we get that

$$|K^\lambda_{R,\nu}(x;t\nu)| \lesssim \lambda^{(d-1)/2}R^{1-(d-1)(1/2-1/p^*)}t^{-(d-1)(1/2-1/p^*)}.$$

Integrating these results gives that

$$\int |K^\lambda_{R,\nu}(x;y)|\,dy \lesssim 1,$$

the same bound as was obtained for $\lambda \geqslant R$. Thus Schur's Lemma gives

$$\sup_\nu \|T^\lambda_{R,\nu}f\|_{L^\infty} \lesssim \|f\|_{L^\infty}.$$

Interpolating gives that

$$\left(\sum \|T^\lambda_{R,\nu}f\|^{p^*}_{L^{p^*}}\right)^{1/p^*} \lesssim \|f\|_{L^{p^*}},$$

and thus that

$$\|T^\lambda_R f\|_{L^{p^*}} \lesssim_\varepsilon \lambda^{\alpha(p^*)+\varepsilon}\|f\|_{L^{p^*}}.$$

Chapter 13

Trying to Use Hadamard Parametrix

Let
$$T_R f = \sum_\lambda m(\lambda/R)\langle f, e_\lambda\rangle e_\lambda.$$
If we write
$$M(t) = \int_0^\infty m(\lambda)\cos(2\pi t\lambda)\, d\lambda,$$
then the inverse formula implies that
$$m(\lambda/R) = \int M(t)\cos\left(\frac{2\pi\lambda t}{R}\right)\, dt.$$
Thus we have
$$T_R f = \int M(t)\cos\left(\frac{2\pi\sqrt{-\Delta}t}{R}\right) f\, dt$$
Local smoothing implies that we need only analyze an integral of the form
$$T_R f = R\int \eta(Rt)M(Rt)\cos\left(2\pi\sqrt{-\Delta}\cdot t\right) f\, dt,$$
where η has support on an arbitrarily small, but fixed portion of the origin. Localizing the operator by a partition of unity, and then applying the Hadamard parametrix, we should expect to control the kernel $T_R f$ by a finite sum of functions of the form
$$K_R(x,y) = R\int\int \frac{c(x,y)\eta(Rt)M(Rt)}{|\xi|^\nu} e^{2\pi i(\xi\cdot d_g(x,y)+t|\xi|)}\, d\xi\, dR,$$

where a is smooth and compactly supported, and d_g denotes the geodesic distance on the manifold. Let us perform a frequency decomposition, writing $K_R = K_{R,0} + 2^{k(d-\nu)} \sum_{k=1}^{\infty} K_{R,k}$, where

$$K_{R,0}(x,t;y) = R \int M(Rt) \frac{c(x,y)\eta(Rt)\psi_0(\xi)}{|\xi|^\nu} e^{2\pi i(\xi \cdot d_g(x,y) + t|\xi|)}\, d\xi\, dt,$$

and for $k \geqslant 1$,

$$K_{R,k}(x,t;y) = 2^{k(\nu-d)} R \int \int \frac{a(x,y)\eta(Rt)M(Rt)\tilde{\psi}(\xi/2^k)}{|\xi|^\nu} e^{2\pi i(\xi \cdot d_g(x,y) + t|\xi|)}\, d\xi\, dt.$$

Rescaling, and setting $a_\nu(x,t,y,\xi) = a(x,y)\eta(t)\tilde{\psi}(\xi)/|\xi|^\nu$ gives that

$$K_{R,k}(x,t;y) = \int M(t) \int a_\nu(x,t,y,\xi) e^{2\pi i 2^k(\xi \cdot d_g(x,y) + (t/R)|\xi|)}\, d\xi\, dt.$$

Similarily,

$$K_{R,0}(x,t;y) = \int M(t) a_{\nu,0}(x,y,t,\xi) e^{2\pi i(\xi \cdot d_g(x,y) + (t/R)|\xi|)}\, d\xi\, dt.$$

Our goal is to obtain some L^p estimates on this operator that are summable in k, thus obtaining bounds of the form

$$\|T_{R,k} f\|_{L^p} \lesssim 2^{k(\nu-d-\varepsilon)}$$

for some $\varepsilon > 0$, which hold uniformly in R. The operator $T_{R,0}$ should not be an issue since one can just take in absolute values to obtain the required result.

Stationary phase tells us that the majority of the mass of the kernel $K_{R,k}$ should be concentrated on points $(x,t;y)$ where $|d_g(x,y) - t/R| \lesssim 2^{-k}$, a geodesic annulus of radius t/R, and thickness 2^{-k}. If we are to try a decoupling result, let us split this annulus into a family Θ_k of sectors of aperture $2^{-k/2}$ with the finite itnersection property. We should require a set of sectors $\Theta_{R,k}$ with $\#(\Theta_k) \lesssim 2^{k(d-1)/2}$. If we consider a partition of unity $\{\chi_\theta\}$ localizing the operator to these sectors, we can therefore write $T_{R,k} = \sum_\theta T_{R,k,\theta}$. Let us suppose a Wolff-type decoupling bound held for this operators, i.e. that

$$\|T_{R,k} f\|_{L^p} \lesssim_\varepsilon 2^{k(\alpha(p)+\varepsilon)} \left(\sum_\theta \|T_{R,k,\theta} f\|_{L^p}^p \right)^{1/p}.$$

Let us thus analyze a particular one of these operators $T_{R,k,\theta}$, which has kernel

$$K_{R,k,\theta}(x,t;y) = \int\int M(t) a_{\nu,\theta}(x,y,t,\xi) e^{2\pi i 2^k (\xi \cdot d_g(x,y) + (t/R)|\xi|)}\, d\xi\, dt,$$

where $a_{\nu,\theta} = a_\nu \cdot \chi_\theta$. For each t, and x, nonstationary phase tells us that the mass of the kernel $K_{R,k,\theta}(x,t;\cdot)$ should be concentrated on a cap of long thickness 2^{-k} and short thicknesses $2^{-k/2}$, containing in the intersection of the sector θ and the 2^{-k} neighborhood of the annulus of radius t/R centered at y. We should expect that on this cap the kernel should have amplitude equal to $O(M(t)2^{-k(d-1)})$. Thus we have

$$\|T_{R,k,\theta} f\|_{L^\infty(\mathbf{R}^d)} \lesssim A_p(m) 2^{-3k(d+1)/2} \|f\|_{L^\infty}$$

Chapter 14

Attempt Using Heo-Nazarov-Seeger Technique

Suppose h is a radial multiplier with support on $\{1 \leqslant \lambda \leqslant 2\}$, and let

$$b(t) = 2\int_0^\infty h(\lambda)\cos(2\pi\lambda).$$

If $B_R(t) = R\sum_l b(R(t+l))$, our goal is to prove uniform L^p bounds on the radial multiplier operator

$$T_R = \int_1^2 B_R(t) e^{2\pi i t\sqrt{-\Delta}}.$$

We may assume our input is a linear combination of eigenfunctions with eigenvalue between R and $2R$. If we reduce to local coordinates, we can write

$$e^{2\pi i t\sqrt{-\Delta}} f(x) = \int_{\mathbf{R}^d} a(x,t,y,\xi) e^{2\pi i(\phi(x,y,\xi) - t|\xi|_g)} f(y)\, d\xi,$$

where a is a compactly supported symbol of order zero, and where $\phi(x,y,\xi) \approx (x-y)\cdot\xi$ (this is only up to a smoothing operator, whose behaviour is irrelevant for the purposes of our argument since for such operator trivial L^p estimates hold). Thus we have

$$T_R f(x) = \int_{\mathbf{R}^d}\int_{\mathbf{R}^d}\int_1^2 B_R(t) a(x,t,y,\xi) e^{2\pi i(\phi(x,y,\xi)-t)} f(y)\, dt\, dy\, dx.$$

Now let $\eta \in \mathcal{S}(\mathbf{R}^d)$ be a Schwartz function vanishing to high order at the origin, and consider the operator

$$\tilde{T}_R f(x) = \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \int_1^2 B_R(t) a(x,t,y,\xi) \eta(\xi) e^{2\pi i (\phi(x,y,\xi) - t|\xi|_g)} f(y)\, dt\, dy\, dx.$$

Then $\tilde{T}_R f = T_R \circ$

$$\left\| \int_{\mathbf{R}^d} \int_1^2 B_R(t) a(x,t,y,\xi) e^{2\pi i (\phi(x,y,\xi) - t|\xi|_g)} f(y)\, d\xi\, dy\, dt \right\|.$$

Part III

Papers I Don’t Understand Yet

Chapter 15

Seeger: Singular Convolution Operators in L^p Spaces

Let $m : \mathbf{R}^d \to \mathbf{C}$ be the symbol for a Fourier multiplier operator $m(D)$. If the resulting operator $m(D)$ was bounded from $L^p(\mathbf{R}^d)$ to $L^p(\mathbf{R}^d)$ with operator norm A, then the operator would also be bounded 'at all scales'. That is, if we consider a littlewood Paley decomposition, i.e. taking

$$f = \sum_{i=0}^{\infty} f_i$$

where $\widehat{f_i} = \eta_i \widehat{f}$ is supported on $2^i \leqslant |\xi| \leqslant 2^{i+1}$ for $i \geqslant 1$, and $|\xi| \leqslant 2$ for $i = 0$, then we would have estimates of the form

$$\|m(D)f_i\|_{L^p(\mathbf{R}^d)} \lesssim \|f_i\|_{L^p(\mathbf{R}^d)} \lesssim \|f\|_{L^p(\mathbf{R}^d)}, \tag{15.1}$$

where the implicit constant is uniform in i. The main focus of the paper in question is to determine whether a uniform bound of the form (15.1) implies $m(D)$ is bounded. More precisely, is it true that

$$\|m\|_{M^p(\mathbf{R}^d)} \lesssim_p \sup_{i \geqslant 0} \|m_i\|_{M^p(\mathbf{R}^d)}, \tag{15.2}$$

where $m_i = \eta_i m$.

The Hilbert transform H is a Fourier multiplier with symbol $m(\xi) = \text{sgn}(\xi)$. For each $i > 0$, $m_i(\xi) = \eta_i \text{sgn}(\xi)$, so that

$$K_i(x) = \widehat{\eta_i \text{sgn}(\xi)} = 2^i H\eta(2^i x).$$

Thus

$$\|K_i\|_{L^1(\mathbf{R})} = \|H\eta\|_{L^1(\mathbf{R})}.$$

TODO

It is clear that (15.2) is true for $p = 2$, since in this case the bound is equivalent to an inequality of the form

$$\|m\|_{L^\infty(\mathbf{R}^d)} \lesssim \sup_{i \geqslant 0} \|m_i\|_{L^\infty(\mathbf{R}^d)},$$

which is true because the supports of the symbols $\{m_i\}$ are almost all pairwise disjoint. On the other hand, (15.2) does not hold when $p = 1$ or $p = \infty$, which makes sense, since Littlewood-Paley runs into all kinds of problems for these values of p. Arguing more precisely, the condition would be equivalent to showing that for any $K : \mathbf{R}^d \to \mathbf{C}$,

$$\|K\|_{L^1(\mathbf{R}^d)} \lesssim \sup_{i \geqslant 0} \|K * \widehat{\eta}_i\|_{L^1(\mathbf{R}^d)}.$$

If

$$K_N(x) = \int_{|\xi| \leqslant 2^N} e^{2\pi i \xi \cdot x}\, d\xi$$

is the Dirichlet kernel, then $\|K_N\|_{L^1(\mathbf{R})} \sim N$. On the other hand, for $i \leqslant N - 1$, we have $K_N * \widehat{\eta}_i = \widehat{\eta}_i$, so that

$$\|K_N * \widehat{\eta}_i\|_{L^1(\mathbf{R})} = \|\widehat{\eta}_i\|_{L^1(\mathbf{R})} \lesssim 1.$$

For $i \geqslant N + 1$, we have $K_N * \widehat{\eta}_i = 0$, so that

$$\|K_N * \widehat{\eta}_i\|_{L^1(\mathbf{R})} = 0 \lesssim 1.$$

For $i = N$, we have

$$(K_N * \widehat{\eta_N})(x) = 2^N \int_0^1 \eta(\xi) e^{2\pi i 2^N (\xi \cdot x)} + \int_1^2 \eta(-\xi) e^{-2\pi i 2^N (\xi \cdot x)}\, d\xi$$

$$\int |K_N * \widehat{\eta}_i|$$

whereas one

$$K_N * \widehat{\eta}_i = \begin{cases} \widehat{\eta}_i & : i \lesssim N \\ 0 & : i \gtrsim N \end{cases},$$

and so $\|K_N * \widehat{\eta}_i\|_{L^1(\mathbf{R})} \lesssim 1$ uniformly in N and i. We can then use Baire category techniques to find a kernel K not in $L^1(\mathbf{R})$, but such that $\|K * \eta_i\|_{L^1(\mathbf{R})} \lesssim 1$, uniformly in i.

The result actually fails for $2 < p < \infty$, due to an examples of Triebel. For simplicity, let's work in $\mathbf{R}$. If we fix a bump function $\phi \in C_c^\infty(\mathbf{R})$ supported in $[-1,1]$, and set

$$m_N(\xi) = \sum_{k=N}^{2N} e^{2\pi i(2^k \xi)} \phi(\xi - 2^k),$$

then $m_N(\xi)\eta_i(\xi) = m_{N,i}(\xi)$, where $m_{N,i}(\xi) = e^{2\pi i(2^k\xi)}\phi(\xi - 2^k)$, and so $K_{N,i}(x) = \widehat{m_{N,i}}(x) = e^{2\pi i 2^k(x-2^k)}\widehat{\phi}(x - 2^k)$, hence

$$\|m_{N,i}(D)f\|_{L^p(\mathbf{R}^d)} = \|K_{N,i} * f\|_{L^p(\mathbf{R}^d)} \leqslant \|\widehat{\phi}\|_{L^1(\mathbf{R})}\|f\|_{L^p(\mathbf{R})} \lesssim \|f\|_{L^p(\mathbf{R})}.$$

On the other hand, the operator norm of $m_N(D)$ from $L^p(\mathbf{R})$ to $L^p(\mathbf{R})$ is actually $\gtrsim_p N^{|1/p-1/2|}$, and thus not bounded uniformly in N, so Baire category shows things don't work so well here.

This paper shows that one *can* get uniform bounds assuming an additional, very weak smoothness condition, which rules out the example m_N above. Under the most simple assumptions, if (15.1) holds, and $\|m_i\|_{\Lambda^\varepsilon} \lesssim 2^{-ik}$, where Λ^ε is the ε-Lipschitz norm, then $\|m(D)f\|_{L^r(\mathbf{R}^d)} \lesssim \|f\|_{L^r(\mathbf{R}^d)}$ whenever $|1/r - 1/2| < |1/p - 1/2|$. Under slightly stronger smoothness assumptions, we can actually conclude $\|m(D)f\|_{L^p(\mathbf{R}^d)} \lesssim \|f\|_{L^p(\mathbf{R}^d)}$.

To prove the result, we rely on Littlewood-Paley theory and the Fefferman-Stein sharp maximal function. Without loss of generality we may assume that $2 < p < \infty$. We will actually show that if for all i and $\omega \geqslant 0$,

$$\int_{|x|\geqslant\omega} |K_i(x)|\, dx \leqslant B(1 + 2^i\omega)^{-\varepsilon},$$

consistent with the fact that, if m_i was smooth, the uncertainty principle would say that K_i would live on a ball of radius $1/2^i$. We will then prove that $\|m(D)f\|_{L^p(\mathbf{R}^d)} \leqslant A\widetilde{\log}(B/A)^{|1/2-1/p|}$. Our goal is to show that if

$$S^\# f(x) = \sup_{x\in Q} ⨍_Q \left(\sum_{i=0}^\infty \left| m_i(D)f(y) - ⨍_Q m_i(D)f(z)\, dz \right|^2 \right)^{1/2} dy,$$

then $\|S^\# f\|_{L^p(\mathbf{R}^d)} \lesssim A\widetilde{\log}(B/A)^{1/2-1/p}\|f\|_{L^p(\mathbf{R}^d)}$. It then follows by Littlewood-Paley theory implies

$$\begin{aligned}
\|m(D)f\|_{L^p(\mathbf{R}^d)} &\lesssim_p \left\| \left(\sum_{k=0}^\infty |m_i(D)f|^2 \right)^{1/2} \right\|_{L^p(\mathbf{R}^d)} \\
&\leqslant \left\| M \left[\left(\sum_{k=0}^\infty |m_i(D)f|^2 \right)^{1/2} \right] \right\|_{L^p(\mathbf{R}^d)} \\
&\lesssim \left\| S^\# \left(\sum_{k=0}^\infty |m_i(D)f|^2 \right)^{1/2} \right\|_{L^p(\mathbf{R}^d)} \\
&\lesssim A\widetilde{\log}(B/A)^{1/2-1/p}.
\end{aligned}$$

To bound $S^\#$, we linearize using duality, picking Q_x for each x, and a family of functions $\chi_i(x,y)$ such that $\left(\sum |\chi_i(x,y)|^2\right)^{1/2} \leqslant 1$, such that

$$S^\# f(x) \approx \fint_{Q_x} \sum_{i=0}^\infty \left(m_i(D)f(y) - \fint_{Q_x} m_i(D)f(z)\, dz \right) \chi_i(x,y)\, dy.$$

Thus $S^\# f = S_1 f + S_2 f$, where if Q_x has sidelength $2^{l(x)}$,

$$S_1 f(x) = \fint_{Q_x} \sum_{|i+l(x)| \leqslant \tilde{\log}(B/A)} \left(m_i(D)f(y) - \fint_{Q_x} m_i(D)f(z)\, dz \right) \chi_i(x,y)\, dy$$

and

$$S_2 f(x) = \fint_{Q_x} \sum_{|i+l(x)| \geqslant \tilde{\log}(B/A)} \left(m_i(D)f(y) - \fint_{Q_x} m_i(D)f(z)\, dz \right) \chi_i(x,y)\, dy.$$

If $|i+l(x)| \lesssim 1$, then the uncertainty principle tells us that $m_i(D)f$ is roughly constant on squares on radius Q_x, up to some small error, so that we should expect

$$\left| m_i(D)f(y) - \fint_{Q_x} m_i(D)f(z)\, dz \right| \lesssim \left| \fint_{Q_x} m_i(D)f(z)\, dz \right|.$$

Thus it is natural to use the bound, $|S_1 f(x)| \lesssim M(\sum_{i=0}^{\infty} |m_i(D)f|^2)^{1/2}$, which implies

$$\begin{aligned}
\|S_1 f\|_{L^2(\mathbf{R}^d)} &\lesssim \|M(\sum_{i=0}^{\infty} |m_i(D)f|^2)^{1/2}\|_{L^2(\mathbf{R}^d)} \\
&\lesssim \left\| (\sum_{i=0}^{\infty} |m_i(D)f|^2)^{1/2} \right\|_{L^2(\mathbf{R}^d)} \\
&= \left(\sum_{i=0}^{\infty} \|m_i(D)f\|_{L^2(\mathbf{R}^d)}^2 \right)^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
\|S_1 f\|_{L^\infty(\mathbf{R}^d)} &\leqslant \|M(\sum_{|i+l(x)| \leqslant \tilde{\log}(B/A)}^{\infty} |m_i(D)f|^2)^{1/2}\|_{L^\infty(\mathbf{R}^d)} \\
&\leqslant \left\| \left(\sum_{|i+l(x)| \leqslant \tilde{\log}(B/A)} |m_i(D)f|^2 \right)^{1/2} \right\|_{L^\infty(\mathbf{R}^d)} \\
&\lesssim \tilde{\log}(B/A)^{1/2} \sup_i \|m_i(D)f\|_{L^\infty(\mathbf{R}^d)}
\end{aligned}$$

Interpolation gives $\|S_1 f\|_{L^p(\mathbf{R}^d)} \lesssim \tilde{\log}(B/A)^{1/2-1/p} \|m_i(D)f\|_{L^p_x(l^p_i)}$. But now Littlewood-Paley theory shows that

$$\|m_i(D)f\|_{L^p_x(l^p_i)} \leqslant A \left(\sum_{i=0}^{\infty} \|P_i f\|_{L^p(\mathbf{R}^d)} \right)^{1/p} \leqslant A \left(\sum_{i=0}^{\infty} \|P_i f\|_{L^p(\mathbf{R}^d)}^2 \right)^{1/2} \lesssim A\|f\|_{L^p}.$$

Thus $\|S_1 f\|_{L^p(\mathbf{R}^d)} \lesssim A\tilde{\log}(B/A)^{1/2-1/p} \|f\|_{L^p(\mathbf{R}^d)}$.

On the other hand, if i is much smaller than $l(x)$, we should expect the error between $m_i(D)f(y)$ and $⨍_{Q_x} m_i(D)f(z)\, dz$ to be even smaller, and if i is much bigger, then $m_i(D)f$ is no longer constant at this scale, and so the averages should be small, so $m_i(D)f(x)$ should dominate $⨍_{Q_x} m_i(D)f(z)$. Now since our assumption implues that $\|m(D)f\|_{L^2(\mathbf{R}^d)} \lesssim \|f\|_{L^2(\mathbf{R}^d)}$, it is not so difficult to prove that

$$\|S_2 f\|_{L^2(\mathbf{R}^d)} \lesssim A\|f\|_{L^2(\mathbf{R}^d)} \sim A \left\| \left(\sum |P_i f|^2 \right)^{1/2} \right\|_{L^2(\mathbf{R}^d)}.$$

The difficulty is proving $\|S_2 f\|_{L^\infty(\mathbf{R}^d)} \lesssim A\| \left(\sum |P_i f|^{1/2}\right) \|_{L^\infty(\mathbf{R}^d)}$, which we can interpolate into an inequality like above where we can apply Littlewood-Paley theory. To do this we perform another decomposition, writing

$$S_2 f = If + IIf$$

where

$$If(x) = \fint_{Q_x} \sum_{|i+l(x)| \geqslant \widetilde{\log}(B/A)} \left(m_i(D)(\mathbf{I}_{2Q_x} f)(y) - \fint_{Q_x} m_i(D)(\mathbf{I}_{2Q_x} f)(z)\, dz \right) \chi_i(x,y)\, dy.$$

and

$$If(x) = \fint_{Q_x} \sum_{|i+l(x)| \geqslant \widetilde{\log}(B/A)} \left(m_i(D)(\mathbf{I}_{(2Q_x)^c} f)(y) - \fint_{Q_x} m_i(D)(\mathbf{I}_{(2Q_x)^c} f)(z)\, dz \right) \chi_i(x,y)\, dy.$$

Now

$$\|If\|_{L^\infty} \leqslant \sup_x \fint_{Q_x} \left(\sum |m_i(D)(\mathbf{I}_{2Q_x} f)|^2 \right)^{1/2} dy \leqslant \sup_x |Q_x|^{-1/2} \left(\sum \|m_i(D)(\mathbf{I}_{2Q_x} f)\|_{L^2(\mathbf{R}^d)}^2 \right)^{1/2} \lesssim A|Q$$

Part IV

Stuff to Read in More Detail

- Sogge, L^p Estimates For the Wave Equation and Applications (1993).

  A survey of results on regularity results for the wave equation. In particular, reviews (without proof) the ideas of Mockenhaipt, Seeger, and Sogge which give local smoothing for Fourier integral operators satisfying the cone condition, as well as mixed norm estimates for non-homogeneous results on wave equations.

- In Sogge's Book, he mentions the main developments in harmonic / microlocal analysis he couldn't discuss in the book were the following:

  - Bennett, Carbery, Tao, On the Multilinear Restriction and Kakeya Conjecture (2006).

    Introduction to multilinear methods in harmonic analysis.

  - Bourgain, Guth, Bounds on Oscillatory Integral Operators Based on Multilinear Estimates (2010).

    Application of multilinear methods to bounding oscillatory integrals.

  - Bourgain, Demeter, The Proof of the l2 Decoupling Conjecture (2014).

    Introduction to Decoupling.

  - Peetre, New Thoughts on Besov-Spaces.

    Characterizes boundedness of Fourier multipliers on homogeneous Besov spaces.

  - Johnson, Maximal Subspaces of Besov-Spaces Invariant Under Multiplication By Characters.

    Shows a Fourier multiplier operator is bounded in the L^p norm if and only if it's translates are all localizably bounded as in Seeger.

- For more background reading in microlocal analysis:

  - Hörmander, The Analysis of Linear Partial Differential Operators, Volumes I-IV.
  - Treves, Introduction to Pseudodifferential and Fourier Integral Operators, Volumes I-II.
  - Taylor.

Bibliography

[1] Jean Bourgain and Larry Guth. Bounds on oscillatory integral operators based on multilinear estimates. *Geom. Funct. Anal.*, 21(6):1239–1295, 2011.

[2] Laura Cladek. Radial Fourier multipliers in $\mathbb{R}^3$ and $\mathbb{R}^4$. *Anal. PDE*, 11(2):467–498, 2018.

[3] Gustavo Garrigós and Andreas Seeger. Characterizations of Hankel multipliers. *Math. Ann.*, 342(1):31–68, 2008.

[4] Shaoming Guo, Changkeun Oh, Wu Shukun Wang, Hong, and Ruixiang Zhang. The Bochner-Riesz problem: An old approach revisited. 2021.

[5] Larry Guth, Hong Wang, and Ruixiang Zhang. A sharp square function estimate for the cone in $\mathbb{R}^3$. *Ann. of Math.*, 192(2):551–581, 2020.

[6] Yaryong Heo, Fdor Nazarov, and Andreas Seeger. Radial Fourier multipliers in high dimensions. *Acta Math.*, 206(1):55–92, 2011.

[7] Nets Hawk Katz and Keith M. Rogers. On the polynomial Wolff axioms. *Geom. Funct. Anal.*, 28(6):1706–1716, 2018.

[8] Sanghyuk Lee and Andreas Seeger. Lebesgue space estimates for a class of Fourier integral operators associated with wave propogation. *Math. Nachr.*, 286(7):745–755, 2013.

[9] William P. Minicozzi and Christopher D. Sogge. Negative results for Nikodym maximal functions and related oscillatory integrals in curved spaces. *Math. Res. Lett.*, 4(2):221–237, 2000.

[10] Boris S. Mitjagin. Divergenz von spektralentwicklungen in L_p-räumen. *Internat. Ser. Numer. Math.*, 25:521–530, 1974.

[11] Andreas Seeger Yaryong Heo, Fëdor Nazrov. Radial Fourier multipliers in high dimensions. 2011.

[12] Joshua Zahl. On the Wolff circular maximal function. *Illinois J. Math.*, 56(4):1281–1295, 2012.

[13] Joshua Zahl. Sphere tangencies, line incidences, and Lie's line-sphere correspondence. *Math. Proc. Cambridge Philos. Soc.*, pages 1–21, 2021.