A Characterization of L^p Boundedness For Spectral Multipliers

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Fourier Multipliers on $\mathbb{R}^d$

Given $a : \mathbb{R}^d \to \mathbb{C}$, consider the Fourier Multiplier

$$F_a g(x) = \int_{\mathbb{R}^d} a(\xi)\, \hat{g}(\xi)\, e^{2\pi i \xi \cdot x}\, d\xi$$

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For $f:S^d \to \mathbb{C}$, define $\Delta f = \Delta_{\mathbb{R}^{d+1}} \tilde{f}$

Then every has expansion

s.t. and

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Then every f has expansion $f = \sum_\lambda f_\lambda$

s.t. $\Delta f_\lambda = -\lambda^2 f_\lambda$ and $\|f\|_{L^2}^2 = \sum_\lambda \|f_\lambda\|_{L^2}^2$

For $m:[0,\infty) \to \mathbb{C}$ define $T_m g = \sum m(\lambda) g_\lambda$

Main Question

How does T_m relate to F_m?

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$\Delta(\xi) = -|\xi|^2$

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Correct Formulation

Relation between F_m and uniform boundedness of T_{m_R} where $m_R(\lambda) = m(\lambda/R)$

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Supp m compact subset of $(0,\infty)$

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Result holds for any M with periodic geodesic flow

S^d $\mathbb{RP}^n$ $\mathbb{CP}^n$ $\mathbb{HP}^n$ $\mathbb{OP}^n$

First transference result from $\mathbb{R}^d$ to M for any M and any $p \neq 2$

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"Geometrically $F_m = \lim_{R \to \infty} T_{m_R}$"

There are reasons to believe Main Theorem fails when extended to all M and all p

<u>Main Theorem</u> (D., In Preparation)

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RADIAL MULTIPLIER (SUPER) CONJECTURE

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Radial Multiplier (Super) Conjecture

If a is radial and has compact support, $1 < p < \frac{2d}{d+1}$,

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Proof of **Main Theorem** follows by proving analog of HNS

$$\sup_R \|T_{m_R}\|_{L^p \to L^p} \lesssim \|\widehat{m(|\cdot|)}\|_{L^p(\mathbb{R}^d)}$$

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$$\lesssim \|F_m\| \lesssim \sup_R \|T_{m_R}\| \lesssim \|\widehat{m(|\cdot|)}\|_{L^p(\mathbb{R}^d)}$$

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$$\sup_R \|T_{m_R}\|_{L^p \to L^p} \lesssim \|\widehat{m(|\cdot|)}\|_{L^p(\mathbb{R}^d)}$$

$$\|\widehat{m(|\cdot|)}\|_{L^p(\mathbb{R}^d)} \lesssim \|F_m\| \lesssim \sup_R \|T_{m_R}\| \lesssim \|\widehat{m(|\cdot|)}\|_{L^p(\mathbb{R}^d)}$$

$F_m(\delta_0) = \widehat{m(|\cdot|)}$

Mikisagin

Corollaries Of Main Result

Main Theorem (D., In Preparation)

If $1 < p < \frac{2(d-1)}{d+1}$, $\|F_m\|_{L^p \to L^p} \sim \sup_R \|T_{m_R}\|_{L^p \to L^p}$

Characterization of L^p Boundedness

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If $p < \frac{2(d-1)}{d+1}$ and $\mathrm{supp}(m)$ is compact

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Partial Sums: If $p < \frac{2(d-1)}{d+1}$ and m has compact support

$$\lim_{R \to \infty} \sum m(\lambda/R) f_\lambda = 0 \quad \text{for all } f \in L^p(S^d)$$

iff $\|\widehat{m(|\cdot|)}\|_{L^p(\mathbb{R}^d)} < \infty$

Corollaries Of Main Result

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Partial Sums: If $p < \frac{2(d-1)}{d+1}$ then

$$\lim_{R \to \infty} \sum m(\lambda/R) f_\lambda = f \quad \text{for all} \quad f \in L^p(S^d)$$

iff $m(0) = 1$ and $\sup_{j \in \mathbb{Z}} \|\widehat{m_j(|\cdot|)}\|_{L^p(\mathbb{R}^d)} < \infty$

only once compactness assumption is removed

$$\boxed{\sup_R \|T_{m_R}\| \lesssim \|\widehat{m(|\cdot|)}\|_{L^p}}$$

(1) Fourier Inversion: $m(\lambda/R) = \int R\hat{m}(Rt)e^{2\pi i t\lambda}\,dt$

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(2) Lax-Hörmander Parametrix: For High Frequency inputs

$$e^{2\pi i t\sqrt{-\Delta}}(x,y) = \int_{\mathbb{R}^d} e^{2\pi i(\xi\cdot(x-y)+t|\xi|)}d\xi$$

on $\mathbb{R}^d$

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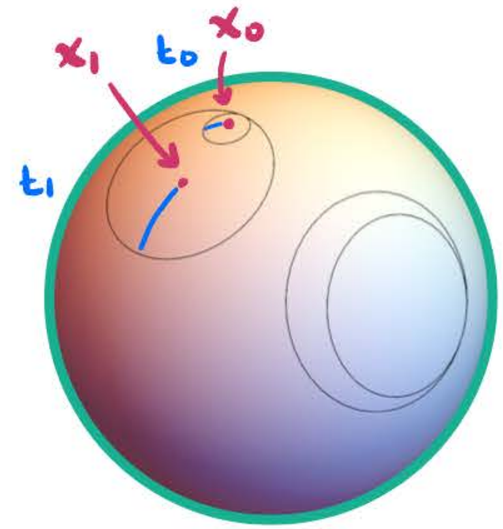

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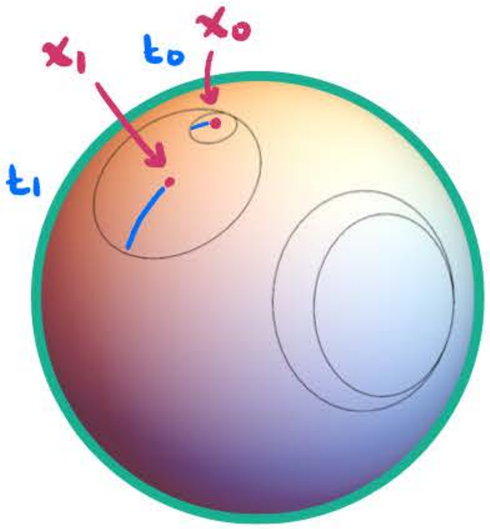

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(3) Density Argument: Suppose $X_K \subseteq M \times [2^K, 2^{K+1}]$ is $1/R$ separated

$\#(X_K \cap B_r) \leq (Ru)r$ for $1/R \leq r \leq 2^K/R$

Then $\left\| \sum_K \sum_{(x,t)\in X_K} e^{2\pi i t\sqrt{-\Delta}} f_x \right\|_{L^2} \lesssim R^d (\log u) u^{\frac{2}{d-1}} \left(\sum_K 2^{K(d-1)} \# X_K \right)^{1/2}$

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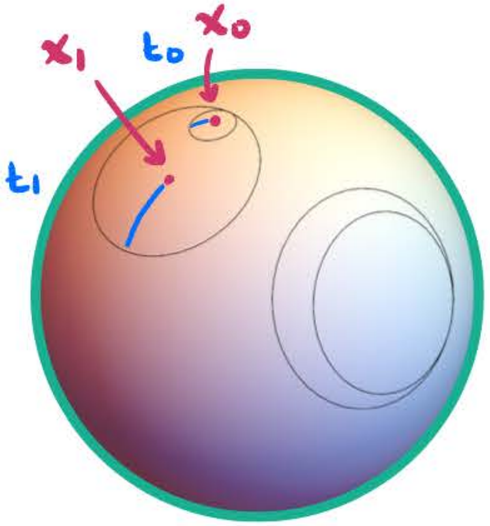

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Thanks For Listening!

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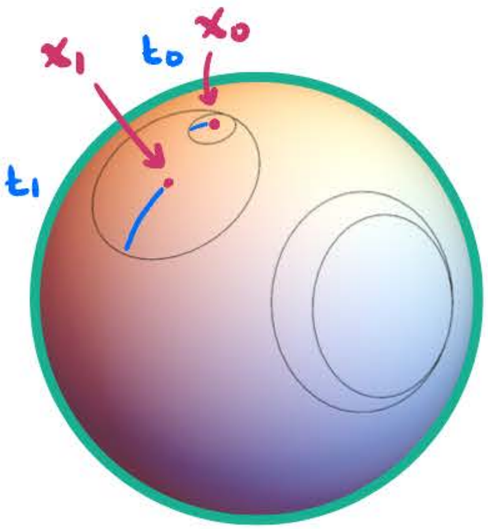

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