1 Nodal Domains via Diffusion Processes

After Georgiev and Muckherjee [2]

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Abstract

We describe the theory of Itô diffusions on compact manifolds, and it's application by Georgiev and Muckherjee to the study of the geometry of nodal sets to the Laplace-Beltrami operator on manifolds.

Let M^d be a compact Riemannian manifold, and let $e_\lambda \in C^\infty(M)$ be an eigenfunction for the Laplacian, such that $\Delta e_\lambda = -\lambda^2 e_\lambda$. Our goal is to use the theory of stochastic diffusions to study asymptotic properties of *nodal domains* D_λ, open connected components of $\{x \in M : e_\lambda(x) \neq 0\}$, as $\lambda \to \infty$. In particular, we show it is impossible to fit D_λ in a $O(\lambda^{-1})$ tubular neighborhood of a 'flat' embedded surface $\Sigma_\lambda \subset M$. Steinerberger [6] gave the first version of this result using diffusion. Here we discuss a more robust result, using similar techniques, due to Georgiev and Muckherjee [2].

Theorem 1. *Let M be a Riemannian manifold. Then there exists $c > 0$ such that for any $\lambda > 0$, if Σ_λ is a smooth surface in M of dimension k such that for any $x \in M$ with $d(x, \Sigma_\lambda) \leq \lambda^{-1}$, there exists a unique point on Σ_λ closest to x, then no nodal domain D_λ is contained in a $c \cdot \lambda^{-1}$ neighborhood of Σ_λ for all $\lambda > 0$.*

Why should we expect the theory of diffusions to give us information about nodal sets? A major reason is that eigenfunctions to the Laplace-Beltrami operator behave well under the heat equation, i.e. if $e^{t\Delta}$ are the propogators for the heat equation $\partial_t = \Delta$ on M, then $(e^{t\Delta} e_\lambda)(x) = e^{-\lambda^2 t} e_\lambda(x)$. The heat equation mathematically describes the distribution of particles diffusing through a medium in which they are subject to random molecular bombardments. The theory of *diffusions* in probability gives an alternate mathematical model of this situation, so it is reasonable that applying the theory will bring light upon the theory of eigenfunctions. In particular, we will see that it gives us a theory of *exit times*, that give us a way to study the rate of propogation of a diffusion process.

2 Probabilistic Tools

Let us begin by introducing the probabilistic machinery required to describe Itô diffusions. We work over a fixed probability space S, and study a *continuous stochastic process* on S, valued in a space M. There are three useful ways to think of such a process. The first is as a Borel-measurable function X from S to $C([0,\infty), M)$, intuitively, a random continuous path on M. The second is as a family of variables $\{X_t : t \in [0,\infty)\}$. The third is as a *law* which describes how random variables in the 'future' depend on random variables in the past, which leads to the study of the conditional operators $\mathbb{E}^x[f(X)]$, defined for $x \in M$ and a random statistic $f(X)$ associated with the process X, which give the average value of the statistic given that we start the process at the state x, i.e. we let $X_0 = x$, and then let the process evolve according to the law defining the process.

The most basic Itô diffusion is *Brownian motion*. A one-dimensional Brownian motion B is a continuous stochastic process on $\mathbb{R}$ such that for any interval $I = [t, s]$, the increments $\Delta_I B = B_s - B_t$ are mean zero, variance $s - t$ Gaussian random variables, and for any almost disjoint family of intervals $\{I_1, \ldots, I_n\}$ in $[0,\infty)$, the random variables $\{\Delta_{I_1} B, \ldots, \Delta_{I_n} B\}$ are independent. A Brownian motion on $\mathbb{R}^d$ is precisely a continuous process whose coordinates are independent one-dimensional Brownian motions.

By tweaking Brownian motion locally, we end up with a more general *Itô diffusion*. Suppose that for each $x \in \mathbb{R}^d$, we are given a $d \times d$ positive semidefinite symmetric matrix $A(x)$. Then we obtain a continuous process X defined by the law given by the *stochastic differential equation*

$$dX = A(X)dB.$$

The formal definition of this differential equation is quite technical, but for our purposes, the equation means that there a Brownian motion B such that

$$X_{t+\delta} = X_t + A(X_t) \cdot [B_{t+\delta} - B_t] + o(\delta),$$

where the $o(\delta)$ term is a random variable with mean $o(\delta)$, and with L^3 norm $O(\delta)$. As one might expect, one can analogously define Itô diffusions on a compact manifold M given a section A of $\mathrm{Hom}(TM)$, which will satisfy analogous formulae. Thus the diffusion acts like Brownian motion, except that instead of acting radially, it spreads out unevenly from a point x in the directions dictated by the extent of the matrix $A(x)$.

Now we connect diffusions to semielliptic differential operators. For any diffusion X, we can associate such an operator L, known as the *generator* of the diffusion, and this is a one to one correspondence. As an example, Brownian motion on $\mathbb{R}^d$ has $\Delta/2$ as it's generator. This motivates us to *define* Brownian motion on a manifold M as a process generated by $\Delta/2$, i.e. half the Laplace-Beltrami operator. This correspondence becomes useful in several scenarios. First, for any $f \in C^2(M)$, we have

$$\lim_{t \to 0} \frac{\mathbb{E}^x[f(X_t)] - f(x)}{t} = (Lf)(x).$$

To see how L emerges, use the approximation $X_t = x + A(x)B_t + o(t)$ and expand $f(X_t)$ in a Taylor series about x. On a sample by sample basis, the first order terms in the expansion of $f(X_t)$ will behave badly, on the order of $O(t^{1/2})$, since Brownian motion is non-differentiable everywhere. But thankfully these terms vanish in the expectation since B is highly oscillatory. The second order terms will involve squares of Brownian motion, and the fact that $\mathbb{E}^0[B_t^2] = t$ implies that these terms will have expected value $O(t)$. Higher order terms will be $O(t^{3/2})$, and thus not affect the value of the limit as $t \to 0$. *Dynkin's formula* follows formally from this calculation by time-homogeneity and the fundamental theorem of calculus, implying that for any time T (possibly a *random time*, provided it is an 'integral stopping time'),

$$\mathbb{E}^x[f(X_T)] = f(x) + \mathbb{E}^x\left[\int_0^T (Lf)(X_s)\, ds\right].$$

To test Dynkin's formula, if B is a Brownian motion on $\mathbb{R}^n$, T is the first time that B exits an open ball of radius R about the origin (an *exit time*, which will always be an integrable stopping time for any bounded open set), and if $f(y) = |y|^2$, then $f(X_T) = R^2$ and $(\Delta/2)f = n$, so

$$R^2 = \mathbb{E}^x[f(X_T)] = f(x) + \mathbb{E}^x\left[\int_0^T n\, ds\right] = n \cdot \mathbb{E}^x[T].$$

Thus we see that on average Brownian motion diffuses a distance $O(R)$ in R^2 units of time. It is interesting to note that if R is sufficiently small, one can perform a similar calculation for Brownian motion on a Riemannian manifold with the exit time of a geodesic ball. If x is fixed, and $f(y) = d(x,y)^2$ is the geodesic distance to x, then one can show (see Section 2.4 of [5]) that

$(\Delta/2)f(x) - n$ is proportional to the *rate of volume expansion* in the manifold at the point x. Thus we expect Brownian motion to diffuse more slowly in positively curved regions (with a negative rate of expansion), and faster in regions of negative curvature (with positive rate of expansion).

The correspondence also applies in the reverse manner, giving the *Feynman-Kac formula* and it's variants; if f is a function on M, and we define $u(x,t) = \mathbb{E}^x[f(X_t)]$, then u is the solution to the partial differential equation $\partial_t u = Lu$ on M, with initial conditions f. We can also consider boundary value problems. If D is a bounded region of $\mathbb{R}^d$, and $\tau_D = \inf\{t > 0 : X_t \notin D\}$ is the *exit time* for D, then the Dynkin formula tells us that the unique solution to the Dirichlet problem $Lv = -h$ on D with boundary conditions ϕ on ∂D is given by

$$v(x) = \mathbb{E}^x[\phi(X_{\tau_D})] + \mathbb{E}^x\left[\int_0^{\tau_D} h(X_t)\, dt\right].$$

One can also solve the heat equation $\partial_t u = Lu$ with absorbing boundary conditions $u(x,t) = 0$ for $x \in \partial D$, and initial condition $u(x,0) = f(x)$ by setting $u(x,t) = \mathbb{E}^x[f(X_t)\chi_t]$, where $\chi_t = 1$ if $t < \tau_D$, and $\chi_t = 0$ otherwise (we kill paths that reach the boundary and are 'absorbed'). There is also a way to consider solutions to the heat equation with an insulating boundary, i.e. finding a solution to $\partial_t u = \Delta u$ such that $\partial u/\partial \eta = 0$ on ∂D by considering a *reflecting Brownian motion* which 'bounces off the boundary' instead of being killed.

3 Nodal Sets Via Brownian Motion

Let us use the theory we have introduced to prove Theorem 1. Let e_λ be an eigenfunction, and D_λ a nodal domain of e_λ. We may assume without loss of generality that e_λ is positive on D_λ. Consider two solutions $p(x,t)$ and $u(x,t)$ to the heat equation on D_λ, with initial conditions $p(x,0) = 0$, $u(x,0) = e_\lambda(x)$, and with boundary conditions $p(x,t) = 1$ and $u(x,t) = 0$ for $x \in \partial D_\lambda$. The Feynman-Kac formula tells us that

$$p(x,t) = 1 - \mathbb{E}^x[\chi_t] = \mathbb{P}^x[t > \tau_\lambda] \quad \text{and} \quad u(x,t) = \mathbb{E}^x[e_\lambda(B_t)\chi_t].$$

where $\chi_t = \mathbb{I}(t \leq \tau_\lambda)$, and $\tau_\lambda = \inf\{t > 0 : B_t \in D_\lambda^c\}$ is the exit time of D_λ. If $x_0 \in D_\lambda$ maximizes e_λ on D_λ, then

$$e^{-\lambda^2 t} e_\lambda(x) = u(x,t) = \mathbb{E}^x[e_\lambda(B_t)\chi_t] \leq e_\lambda(x_0)\mathbb{E}^x[\chi_t] = e_\lambda(x_0)(1 - p(x,t)).$$

In particular, $p(x_0, t) \leq 1 - e^{-\lambda^2 t}$, so $\mathbb{P}^{x_0}[\tau_\lambda \geq t\lambda^{-2}] \geq e^{-t}$. Thus $\mathbb{E}[\tau_\lambda] \gtrsim \lambda^{-2}$, and the heuristic diffusion rate of Brownian motion therefore leads us to believe that x_0 lies roughly $\gtrsim \lambda^{-1}$ from ∂D_λ. If we had $M = \mathbb{R}^d$, this would immediately yield a contradiction if D_λ was contained in a $c \cdot \lambda^{-1}$ neighborhood of a k dimensional plane Σ, because we have a strong quantification of this heuristic; if B' is the projection of Brownian motion onto the $d-k$ dimensional plane normal to Σ, then a result due to Kent [3] implies that for any $\varepsilon > 0$, there is $c > 0$ such that

$$\mathbb{P}^0 \left(\sup_{0 \leq t \leq \lambda^{-2}} |B'_t| \leq c \cdot \lambda^{-1} \right) \leq \varepsilon.$$

Letting c corresponding to $\varepsilon = (1/2)e^{-1}$ gives a contradiction, since then

$$e^{-1} \leq \mathbb{P}^{x_0}[\tau_\lambda \geq \lambda^{-2}] \leq \mathbb{P}^0 \left(\sup_{0 \leq t \leq \lambda^{-2}} |B'_t| \leq c \cdot \lambda^{-1} \right) \leq (1/2)e^{-1}.$$

Extending this to the non-Euclidean setting is not too difficult. Given a general k dimensional hypersurface Σ on a manifold M satisfying the assumptions of the result we are trying to prove, we suppose that D_λ is contained in a $c \cdot \lambda^{-1}$ neighborhood U_λ of Σ_λ. The flatness assumptions imply that we can find a normal coordinate system ϕ on a $2c \cdot \lambda^{-1}$ neighborhood of Σ_λ using geodesics. Then $\phi(\Sigma_\lambda)$ is a k dimensional plane, and $\phi(U_\lambda)$ is a $c \cdot \lambda^{-1}$ neighborhood of this plane. Because the Euclidean and Riemannian metrics are comparable, Brownian motion on M should behave in coordinates analogously to Brownian motion on $\mathbb{R}^d$. The rate of diffusion of both processes leads us to believe the behaviour should be similar up to times $c \cdot \lambda^{-2}$, before which both Brownian motions are highly unlikely to leave the neighborhoods U_λ and $\phi(U_\lambda)$. Thus our proof can be completed by an application of the following 'comparison result' for hitting times, which is Theorem 2.2 of [2].

Theorem 2. *Let M^d be a compact Riemannian manifold, and consider an open geodesic ball $B \subset M$ around a point x_0 with radius r smaller than the injectivity radius of M. Let (U, ϕ) be a chart on M with $B \subset U$, and suppose that the metric of M is comparable to the Euclidean metric in the coordinates ϕ. Fix a compact set $K \subset B$. Let B^1 be a Brownian motion on M, and B^2 be a Brownian motion on $\mathbb{R}^d$. If τ_1 denotes the time that B^1 exits K, and τ_2 denotes the time that B^2 exits $\phi(K)$, then for any $c > 0$, there exists $C > 0$ such that*

$$(1/C) \cdot \mathbb{P}(\tau_2 \leq cr^2) \leq \mathbb{P}(\tau_1 \leq cr^2) \leq C \cdot \mathbb{P}(\tau_2 \leq cr^2).$$

References

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