Fourier Analysis and patterns in sets

Geometric Measure theory

$\left\{ \begin{matrix} \text{compact sets} \\ E \subseteq \mathbb{T}^d \end{matrix} \right\}$ $\left\{ \begin{matrix} \text{Finite measures } \mu \\ \text{with } \operatorname{supp}(\mu) \subseteq E \end{matrix} \right\}$

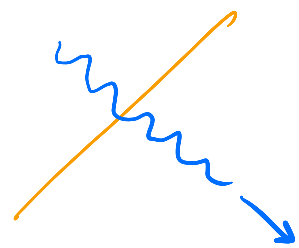
Minkowski / Hausdorff dimension s $\qquad \displaystyle\sum_{k \in \mathbb{Z}^d - \{0\}} |\hat{\mu}(k)|^2 |k|^{s-d} < \infty.$

$|N_\delta(E)| \lesssim \delta^{d-s}$

Fourier dimension s $\qquad |\hat{\mu}(k)| \leq |k|^{-s/2}$ for <u>all</u> n.

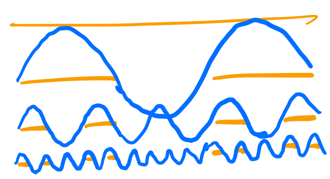
$$\dim_{\mathbb{F}}(E) \leq \dim_{\mathbb{H}}(E) \leq \dim_{\mathbb{M}}(E)$$

Example: $S \subseteq \mathbb{T}^d$ curved hypersurface.
Then if $\mu = \psi \, d\sigma$, $|\hat{\mu}(\xi)| \lesssim |\xi|^{-(d-1)/2}$.
$\dim_{\mathbb{F}}(S) = \dim_{\mathbb{H}}(S) = \dim_{\mathbb{M}}(S) = d-1$.



Example: $H \subseteq \mathbb{T}^d$ is a hyperplane.
Then $\dim_{\mathbb{H}}(H) = \dim_{\mathbb{M}}(H) = d-1$.
BUT $\dim_{\mathbb{F}}(H) = 0$.

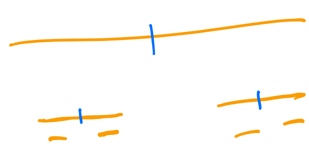
Example: $C \subseteq \mathbb{T}$ the Cantor set



For $\delta = 1/3^n$, $|C_\delta| \lesssim 2^n/3^n = \delta^{1-\log_3 2}$
so $\dim_{\mathbb{M}}(C) = \dim_{\mathbb{H}}(C) = \log_3 2$.
BUT $\dim_{\mathbb{F}}(C) = 0$.

Heuristic: Randomness helps create Salem sets.
Nonrandomness often destroys Fourier dimension.

Define a random Cantor set C:



Then <u>almost surely</u>, $\dim_{\mathbb{F}}(C) = \log_3 2$.

If $\dim(E)$ is large, does E 'contain patterns'

Structure in Large Sets

Like Ramsey Theory but more analytical

- Does E contain solns. to $m_1x_1 + \dots + m_nx_n = 0$ for some $n > 0$.
- (Keleti, 1999): There is $E \subseteq \mathbb{T}$ with $\dim_{\mathbb{H}}(E) = 1$ s.t. for any $m \in \mathbb{Z}^n - \{0\}$ and distinct $x_1, \dots, x_n \in E$,

$$m_1x_1 + \dots + m_nx_n \neq 0.$$

- If $\dim_{\mathbb{F}}(E) > 0$, there is $m \in \mathbb{Z}^n - \{0\}$ and distinct $x_1, \dots, x_n \in E$ s.t. $m_1x_1 + \dots + m_nx_n = 0$.

proof: If $|\hat{\mu}(k)| \lesssim |k|^{-\varepsilon}$, pick $n > 1/\varepsilon$.

Let $\nu = \mu * \dots * \mu$.

Then $|\hat{\nu}(k)| = |\hat{\mu}(k)|^n \lesssim |k|^{-n\varepsilon}$

So $\hat{\nu} \in L^1(\mathbb{Z})$, so $\nu \in C(\mathbb{T})$.

Thus $\mathrm{supp}(\nu) \subseteq \mathrm{supp}(\mu) + \dots + \mathrm{supp}(\mu)$ contains an interval.

- If $\dim_{\mathbb{F}}(E) > 2/n$, E contains solutions to an n-term equation.

Goal: For which s is it true that a generic $E \subseteq \mathbb{T}^d$ with $\dim_{\mathbb{F}}(E) = s$ will avoid a pattern.

$$X_s = \{(E, \mu) : \mathrm{supp}(\mu) \subseteq E,\ |\hat{\mu}(n)| \lesssim |n|^{-s/2}\}$$

(Schmerkin, 2020): For any $y_1, \dots, y_n \in \mathbb{T}^d$, a random set of dimension $(d(n-1)-1)/n$ will not contain a translated, dilated copy of $\{y_1, \dots, y_n\}$ almost surely.

- A generic set of Fourier dimension $(d(n-1)-1)/(n-1/2)$ will not contain a copy of $\{y_1, \dots, y_n\}$.

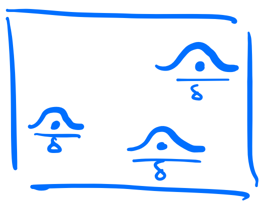
Technical Reductions

- Pattern avoidance will be generic for a fixed s if, for any $\varepsilon > 0$, and any disjoint $I_1, \dots, I_n \subseteq \mathbb{T}^d$, there is a prob. measure μ s.t.
    - for any $x_i \in I_i \cap \mathrm{supp}(\mu)$, $x_n \neq f(x_1, \dots, x_{n-1})$.
    - $|\hat{\mu}(k)| \leq \varepsilon |k|^{-\beta/2}$
- Suppose there are arbitrarily large sets $S = \{x_1, \dots, x_N\}$ s.t.

if $\delta = N^{-1/\beta}$,

- If $x_i \in I_i \cap \mathcal{N}_\delta(S)$, $x_n \neq f(x_1, \dots, x_{n-1})$.
- For any $|\xi| \leq 1/\delta$, $\left| \frac{1}{N} \sum_i e^{2\pi i \xi \cdot x_i} \right| \lessapprox N^{-1/2}$

Then the condition above holds.



If $f(x) = \frac{1}{N\delta^d} \sum_{x_0 \in S} \phi\left(\frac{x - x_0}{\delta}\right)$

$$\hat{f}(\xi) = \underbrace{\hat{\phi}(\delta\xi)}_{\substack{\approx 0 \text{ for} \\ |\xi| \geq 1/\delta}} \times \underbrace{\left(\frac{1}{N} \sum_i e^{2\pi i \xi \cdot x_i}\right)}_{\substack{\lessapprox N^{-1/2} \log N \\ \lessapprox \delta^{\beta/2} \log \delta \\ \lessapprox |\xi|^{-\beta/2}}}$$

- No oscillation
- δ-localized near points x_i

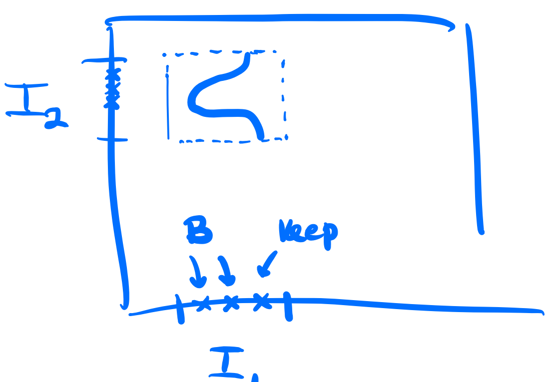
Consider $\{X_{i1}, \dots, X_{iM}\}$ independently selected in I_i, $1 \leq i \leq n$.

Let $B = \left\{ j_n : \begin{array}{l} |X_{nj_n} - f(X_{1j_1}, \dots, X_{n-1, j_{n-1}})| \leq \delta \\ \text{for some } j_1, \dots, j_{n-1} \end{array} \right\}$

Take $S = \{X_{ij}\} - \{X_{nj} : j \in B\}$

Set $N = \#(S)$.

Then $\mathcal{N}_\delta(S)$ avoids solns. to $x_n = f(x_1, \dots, x_{n-1})$.



Goal : Show $|Y_\xi| = \left| \frac{1}{N} \sum_{x \in S} e^{2\pi i \xi \cdot x} \right| \lessapprox N^{-1/2}$

Step 1. $|Y_\xi - \mathbb{E} Y_\xi| \lessapprox N^{-1/2}$.

Step 2. $|\mathbb{E} Y_\xi| \lessapprox N^{-1/2}$.

For each index j,

$$\mathbb{P}(j \in B) = M^{n-1}\, \mathbb{P}(|X_{n j_n} - f(X_{1 j_1}, \dots, X_{(n-1), j_{n-1}})| \le \delta)$$
$$\lesssim M^{n-1} \delta^d$$

If $\delta = M^{-\frac{n-1}{d} - \varepsilon}$, $\mathbb{P}(j \in B) \ll 1$, so Markov's Inequality implies $\#(B) = o(N)$ with high probability (so $N \sim M$).

<u>Concentration Inequalities</u>

If $Y = f(X_1, \dots, X_N)$, where $X_1, \dots, X_N$ are independent and have 'equal influence' on Y, then $|Y - \mathbb{E} Y| \lesssim \sqrt{N}$ with high probability.

<u>Hoeffding's Inequality</u>: If $|X_i| \le A_i$ for each i, and

$$Y = X_1 + \dots + X_N, \quad \mathbb{P}(|Y - \mathbb{E} Y| \ge t) \le 4 e^{-t^2/2(A_1^2 + \dots + A_N^2)}$$

so $|Y - \mathbb{E}Y| \lessapprox (A_1^2 + \dots + A_N^2)^{1/2}$ with high probability.

<u>Mcdiarmid's Inequality</u>: If

$$|f(x_1, \dots, x_i, \dots, x_n) - f(x_1, \dots, x_i', \dots, x_n)| \le A_i$$

then for $Y = f(X_1, \dots, X_N)$,

$$\mathbb{P}(|Y-\mathbb{E}Y|\geq t)\leq 4e^{-t^2/A_1^2+\dots+A_N^2}.$$

Thus $|Y-\mathbb{E}Y|\lesssim (A_1^2+\dots+A_N^2)^{1/2}$ with high probability

Let $Y_\xi=\frac{1}{N}\sum_{x\in S}e^{2\pi i\xi\cdot x}$.

Then $Y_\xi=(1/N)(Z_\xi-W_\xi)$ where

$$Z_\xi=\frac{1}{N}\sum_{i=0}^{n}\sum_{j=1}^{N}e^{2\pi i\xi\cdot x_{ij}}$$

$$W_\xi=\frac{1}{N}\sum_{j\in B}e^{2\pi i\xi\cdot x_{nj}}$$

- Z_ξ is easy: Apply Hoeffding, $A_{ij}=1$ so $|Z_\xi-\mathbb{E}Z_\xi|\lesssim(\sum A_{ij}^2)^{1/2}\lesssim N^{1/2}$ with high probability.

- W_ξ is non-linear combination of random variables.
    - Mcdiarmid directly fails: $A_{nj}=1$, But $A_{ij}=N$ for $1\leq i\leq n-1$, giving $|W_\xi-\mathbb{E}W_\xi|\lesssim N^{3/2}$.
    - Trick: Average over first variable to reduce influence of other variables.

- Apply Hoeffding only in x_{1j}.

$|W_\xi - \mathbb{E}_1 W_\xi| \lesssim N^{1/2}$ with high probability.

$$\mathbb{E}_1 W_\xi = M \int_\Omega e^{2\pi i \xi \cdot x} \, d\mathbb{P}(x_{1j} = x)$$

where $\Omega = \{ f(x_{1j_1}, \dots, x_{n-1, j_{n-1}}) : 1 \le j_1, \dots, j_n \le M \}_\delta$

Changing x_{1j} only adjusts $|\Omega|$ by $\lesssim M^{n-2} \delta^d$

so adjusts $\mathbb{E}_1 W_\xi$ by at most $O(M^{n-1} \delta^d) = O(1)$.

Thus Mcdiarmid now implies $|\mathbb{E}_1 W_\xi - \mathbb{E} W_\xi| \lesssim N^{1/2}$.

Thus $|Y_\xi - \mathbb{E} Y_\xi| \lesssim N^{-1/2}$ with high probability.

Step 2: $|\mathbb{E} Y_\xi| \lesssim N^{-1/2}$.

$$\mathbb{E} Y_\xi = \int e^{2\pi i \xi \cdot x} \, d\mathbb{P}'(x)$$

where $d\mathbb{P}'(x)$ = Probability $j \in B$ if $X_{nj} = x$.

If $\beta \leq \frac{d}{n-3/4}$,

$$d\mathbb{P}'(x) \approx M^{n-1} \left| f^{-1}(B_\delta(x)) \right| \pm N^{-1/2}$$

so

$$\mathbb{E} Y_\xi \approx M^{n-1} \int_{\mathbb{T}^d} e^{2\pi i \xi \cdot x} \left| f^{-1}(B_\delta(x)) \right| dx$$

$$= M^{n-1} \int_{B_r(0)} \int_{\mathbb{T}^d} \int_{f^{-1}(x+v)} e^{2\pi i \xi \cdot (f(y) - v)} \, dy \, dv$$

$$= M^{n-1} \int_{B_r(0)} \int_{\mathbb{T}^{d(n-1)}} e^{2\pi i \xi \cdot (f(y) - v)} \, dy \, dv$$