1. On Trilinear Oscillatory Integral Inequalities and Related Topics

After M. Christ [1]

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Abstract. We discuss decay estimates for trilinear oscillatory integrals on $L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$ which give power decay in the frequency parameter with exponent having magnitude larger than $1/2$, under non-degeneracy conditions on the phase. The main consequence of these estimates, for the purpose of this summer school, are smoothing bounds for singular Brascamp-Lieb type multilinear forms, which in particular, include bounds on the $L^1(\mathbb{R}^2)$ norm of $f(x)g(x+y)h(x+y^2)$, in terms of negative index Sobolev estimates on the functions f, g, and h.

1.1. **Introduction.**

Consider an oscillatory integral of the form

$$T_\lambda^\phi(f) = \int_{\mathbb{R}^J} e^{i\lambda\phi(x)} f(x)\, dx, \tag{1}$$

for a function $\phi : \mathbb{R}^J \to \mathbb{R}$ and $f : \mathbb{R}^J \to \mathbb{C}$. If $f : \mathbb{R}^J \to \mathbb{C}$ is appropriately smooth and ϕ is appropriately 'non-stationary' then, as $\lambda \to \infty$, the integrand in (1) begins to oscillate faster and faster, and so we expect greater and greater cancellation to occur in the integral. In particular, if $\operatorname{supp}(f)$ is contained in a fixed compact set $K \subset \mathbb{R}^J$, the principle of stationary phase can then guarantee a bound $|T_\lambda^\phi(f)| \lesssim_K \lambda^{-\gamma} \max_{|\alpha| \leq n} \|\partial^\alpha f\|_{L^\infty(\mathbb{R}^J)}$ with power decay in λ, for appropriate exponents γ and n.

If we remove the smoothness assumptions on f, then for an arbitrary input it is impossible to obtain any bound on (1) with a power decay in λ. If we set $f_\lambda(x) = \chi_K(x) e^{-i\lambda\phi(x)}$, then $\|f_\lambda\|_{L^p(\mathbb{R}^J)} \lesssim_K 1$ and $|T_\lambda^\phi(f_\lambda)| = |K|$. Thus the only bound of the form

$$|T_\lambda^\phi(f)| \lesssim_K \lambda^{-\gamma} \|f\|_{L^p(\mathbb{R}^J)} \tag{2}$$

which can hold uniformly over all $\lambda > 0$ and all $f \in L^p(\mathbb{R}^J)$ supported on a compact set K is the trivial bound with $\gamma = 0$. The main result about oscillatory inegrals we wish to discuss from [1] establishes non-trivial decay estimates of the form (2) for $p = 2$, $J = 3$, and $K = [0,1]^3$, under a *structural assumption* on the functions f. Namely, we assume that f can be written as $f_1 \otimes \cdots \otimes f_J$ for some functions $f_1, \dots, f_J : \mathbb{R} \to \mathbb{C}$, i.e. so that $f(x) = f_1(x_1) \cdots f_J(x_J)$. Equation (2) can then be written as a multilinear inequality of the form

$$|T_\lambda^\phi(f_1 \otimes \cdots \otimes f_J)| \lesssim \lambda^{-\gamma} \|f_1\|_{L^2(\mathbb{R})} \cdots \|f_J\|_{L^2(\mathbb{R})}. \tag{3}$$

Unlike the generality of (2), the multilinearity in (3) allows for cancellation, since the functions f_λ are no longer counterexamples to (3) with $\gamma > 0$ unless the phase ϕ can be written as

$$\phi(x) = \phi_1(x_1) + \cdots + \phi_J(x_J) \tag{4}$$

for some functions $\phi_1, \dots, \phi_J : \mathbb{R} \to \mathbb{R}$. One might hope that the best possible decay in (3) is closely related to how 'far' a phase function ϕ is from being decomposed as in (4). Current research is far from an optimal quantitative understanding of this relation, but [1] obtains improved decay estimates by making assumptions naturally related to preventing a decomposition of the form (4), along with non-linear analogues, called *rank one degeneracies.*

1.2. **Degeneracies.** One can prevent a decomposition of the form (4) by assuming a mixed derivative, such as $\partial_1 \partial_2 \phi$, is non-vanishing on $[0,1]^3$. It is then a result of Hörmander [2] that the operator

$$S_z f(x) = \int_{[0,1]} e^{i\lambda\phi(x,y,z)} f(y)\, dy \tag{5}$$

satisfies, uniformly for $z \in [0,1]$,

$$\|S_z f\|_{L^2[0,1] \to L^2[0,1]} \lesssim \lambda^{-1/2}. \tag{6}$$

Using (6), Cauchy-Schwartz and Hölder's inequality then implies

$$|T_\lambda^\phi(f)| \lesssim \lambda^{-1/2} \|f_1\|_{L^2[0,1]} \|f_2\|_{L^2[0,1]} \|f_3\|_{L^1[0,1]} \leq \lambda^{-1/2} \|f\|_{L^2[0,1]}. \tag{7}$$

Thus (3) holds with $\gamma = 1/2$. It will be enlightening to later discussion to briefly explain the proof of (6) via a microlocal decomposition. For any $\lambda > 0$, and $f \in L^2[0,1]$, we can consider a decomposition of f into wave packets, of the form

$$f(y) = \sum\nolimits_{(y_0,\eta_0) \in \Phi_\lambda} f_{y_0,\eta_0}(y), \tag{8}$$

where Φ_λ is the Cartesian product of $\lambda^{-1/2}\mathbb{Z} \cap [0,1]$ with $\lambda^{1/2}\mathbb{Z}$, where the function $f_{y_0,\eta}$ is supported on a sidelength $\lambda^{-1/2}$ interval centered at y_0, and where it's Fourier transform rapidly decays away from a sidelength λ interval centered at η_0. Our assumption on ϕ, roughly speaking, implies the existence of an injective map $(x_z, \xi_z) : \Phi_\lambda \to \Phi_\lambda$ such that the function $S_z f_{y_0,\eta_0}$ rapidly decays away from a sidelength $\lambda^{-1/2}$ interval centered at $x_z(y_0,\eta_0)$, it's Fourier transform rapidly decays away from a sidelength $\lambda^{1/2}$ interval centered at $\xi_z(y_0,\eta_0)$, and $\|S_z f_{y_0,\eta_0}\|_{L^2[0,1]}^2 \lesssim \lambda^{-1} \|f_{y_0,\eta_0}\|_{L^2[0,1]}^2$. Thus the functions $\{S_z f_{y_0,\eta_0}\}$ are also wave packets, which are essentially disjoint from one another in phase space. They are therefore almost orthogonal, implying that

$$\|S_z f\|_{L^2[0,1]}^2 \lesssim \sum_{y_0,\eta_0} \|S_z f_{y_0,\eta_0}\|_{L^2[0,1]}^2 \lesssim \lambda^{-1} \sum_{y_0,\eta_0} \|f_{y_0,\eta_0}\|_{L^2[0,1]}^2 \lesssim \lambda^{-1} \|f\|_{L^2[0,1]}^2.$$

Thus we have proved (6).

The main result of [1] related to oscillatory integrals is that one can improve Hörmander's estimate under a slightly stronger assumption on ϕ. Say ϕ is *rank one degenerate* if there exists $\psi_1, \dots, \psi_J$, and a hypersurface H such that the function $x \mapsto \phi(x) + \psi_1(x_1) + \cdots + \psi_J(x_J)$ has gradient vanishing on H. Being rank one degenerate is a direct obstruction to proving (3) with $\gamma > 1/2$, since if we set $f_\lambda(x) = e^{i\lambda(\psi_1(x_1)+\cdots+\psi_J(x_J))}\chi(x)$ for a bump function $\chi = \chi_1 \otimes \cdots \otimes \chi_J$ supported on a small neighborhood $x_0 \in H$, then by fibering the neighborhood of x_0 into lines transverse to H and applying stationary phase on each line, we can conclude that $|T_\lambda^\phi f_\lambda| \gtrsim \lambda^{-1/2}$. Theorem 4.1 of [1] says that this is essentially the only obstruction.

Theorem 1 (Theorem 4.1 of [1])**.** *Let $\phi : [0,1]^3 \to \mathbb{R}$ be real analytic, not rank one degenerate, and for each $i \neq j$ suppose $\partial_i \partial_j \phi$ is non-vanishing on $[0,1]^3$. Then (3) holds for some $\gamma > 1/2$.*

The proof of Theorem 4.1 also involves a microlocal decomposition like that discussed in the proof of (6), but with an *additional* decomposition into a 'pseudorandom' part. We fix a small exponent $\sigma > 0$, to be optimized later. Then given $f \in L^2[0,1]$, we partition Φ_λ into $\Phi_{\lambda,0} + \Phi_{\lambda,1}$, where $(y_0, \eta_0) \in \Phi_{\lambda,0}$ if and only if $\|f_{y_0,\eta_0}\|_{L^2[0,1]} \gtrsim \lambda^{-\sigma} \|f\|_{L^2[0,1]}$. Almost orthogonality justifies that $\|f\|_{L^2[0,1]} \sim \sum \|f_{y_0,\eta_0}\|_{L^2[0,1]}^2$, which means that $\#(\Phi_{\lambda,0}) \leq \lambda^{2\sigma}$. The wave packets corresponding to the elements of $\Phi_{\lambda,0}$ are then the sparse part, and the wave packets in $\Phi_{\lambda,1}$ are the pseudorandom part.

In the remainder of this summary, we discuss how this implies certain smoothing inequalities for trilinear forms of singular Brascamp-Lieb type.

1.3. **Smoothing Inequalities.** One potentially surprising application of Theorem 4.1 is to results in which no oscillation is immediately apparent. Consider analytic functions $\varphi_j : \mathbb{R}^2 \to \mathbb{R}$ for $1 \leq j \leq 3$ with nowhere vanishing gradient. If the vectors $\nabla \varphi_j(x_0)$ are linearly independent at x_0, then for a bump function η supported near x_0, it is simple to obtain an inequality of the form

$$\left\| \eta \prod_j f_j(\varphi_j) \right\|_{L^1(\mathbb{R}^2)} \lesssim \prod_j \|f_j\|_{L^{3/2}(\mathbb{R})}.$$

Moreover, one cannot replace 3/2 with any smaller exponent. Nonetheless, other interesting inequalities might be obtained by

References

[1] Christ, M., *On Trilinear Oscillatory Integral Inequalities and Related Topics*. Preprint 2020 (Revised 2022), https://arxiv.org/pdf/2007.12753.

[2] Hörmander, L., *Oscillatory Integrals and Multipliers on FL^p*. Ark. Mat. 11 (1973), 1–11.

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