Random Cantor Set Decoupling

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Our goal is to

1 Toy Problem: Gaussians Supported on Fractal Intervals

Let's consider a toy problem, which is easier than the Cantor set by virtue of the fact that it has less arithmetic structure. Fix an exponent p, fix a large integer N, a quantity $0 < s < 1$, and then set M to be the closest integer to N^s. Choose M points $\xi_1, \dots, \xi_M$ on $\mathbb{T}$, uniformly at random. For $1 \leqslant k \leqslant M$, let

$$f_k(x) = N^{-1/p} e^{2\pi i \xi_k \cdot x} \phi(x/N).$$

Then f_k is L^p normalized, and roughly speaking, has phase space support on the set

$$\{(x, \xi) : |x| \leqslant N \text{ and } |\xi - \xi_k| \leqslant 1/N\}.$$

If $s < 1/2$, the intervals I_j are disjoint from one another with high probability.

Lemma 1.1. *If $s < 1/2$, then for any $\varepsilon > 0$, if N is sufficiently large, the intervals $I_1, \dots, I_M$ will be disjoint from one another with probability at least $1 - O(N^{2s-1})$. We can also (TODO: Prove this) get this property with 90% probability for $s = 1/2$ if $M = N^{1/2}/100$.*

Proof. Let P_l denote the probability that $I_1, \dots, I_l$ are $1/N$ separated from one another, i.e.

$$d(I_j, I_k) \geqslant 1/N$$

for $j \neq k$. Then $P_1 = 1$, and we can obtain an inductive lower bound for the other l. Namely, if $I_1, \dots, I_l$ are $1/N$ separated from one another, then $I_1, \dots, I_{l+1}$ will be $1/N$ separated from one another if ξ_{l+1} lies away from a $3/N$ neighborhood of $\xi_1, \dots, \xi_l$, which is a set of measure at least $1 - (3/N)l$. Thus we find that

$$P_{l+1} \geqslant P_l(1 - 3l/N).$$

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Thus we find that

$$P_l \geqslant \prod_{j=1}^{l-1}(1-3l/N) = (-3/N)^{l-1}\frac{\Gamma(l-N/3)}{\Gamma(1-N/3)}.$$

Using asymptotics for the ratio of a Gamma function (Tracomi and Erdelyi, 1951, though TODO: Assumptions of that paper might not hold uniformly in the l we need)

$$P_l \geqslant 1 - 1.5\left(\frac{l(l-1)}{N}\right) + O(1/N^2).$$

In particular, for $l = M$, we find that

$$P_M \geqslant 1 - 1.5M^2/N + O(1/N^2) \geqslant 1 - 1.5N^{2s-1} + O(1/N^2) = 1 - O(N^{2s-1}). \qquad \square$$

Thus in this case, the functions $\{f_k\}$ are all orthogonal to one another. For larger s, we cannot expect the functions to be orthogonal with high probability, but we can expect them to be almost orthogonal to one another.

Lemma 1.2. *With probability exceeding $1 - O(1/N^{10})$, all of the sets*

$$A_i = \{j : |\xi_i - \xi_j| \geqslant 10/N\}$$

have cardinality $O_s(1)$.

Proof. Let I be an interval of length L. Then the random variable

$$Z_I = \#\{k : \xi_k \in I\}$$

is a $\mathrm{Bin}(M, L)$ random variable. In particular, a Chernoff bound implies that for $t \geqslant ML$,

$$\mathbb{P}(Z_I \geqslant t) \leqslant e^{-ML}\left(\frac{e\mu}{t}\right)^t.$$

Now let $I_1, \dots, I_N \subset \mathbb{T}$ be the family of all sidelength $3/N$ intervals whose endpoints lie on integer multiples of $1/N$. The bound above implies that for any $1 \leqslant j \leqslant N$, and any $t \geqslant 3/N^{1-s}$,

$$\mathbb{P}(Z_{I_j} \geqslant t) \leqslant e^{-3/N^{1-s}}\left(\frac{3e}{N^{1-s}}\right)^t \leqslant \left(\frac{3e}{N^{1-s}t}\right)^t.$$

In particular,

$$\mathbb{P}\left(Z_{I_j} \geqslant \frac{20}{1-s}\right) \leqslant \left(\frac{3e(1-s)}{20N^{1-s}}\right)^{\frac{20}{1-s}} \leqslant 1/N^{20}.$$

Taking a union bound, we conclude that

$$\mathbb{P}\left(\max_j Z_{I_j} \geqslant \frac{10}{1-s}\right) \leqslant 1/N^{19} \leqslant 1/N^{10}.$$

But the fact that no interval Z_{I_j} contains more than $O_s(1)$ points of $\{\xi_1, \dots, \xi_M\}$ implies what was needed to be proved. $\square$

This condition shows that with high probability, the functions $\{f_k\}$, roughly speaking, have close to disjoint Fourier support. In particular, they are almost orthogonal, which almost immediately implies that for any constants $a_1, \dots, a_M$,

$$\left\| \sum_{k=1}^M a_k f_k \right\|_{L^2(\mathbb{R}^d)} \lesssim N^{(d/2)(1/2-1/p)} \left(\sum_{k=1}^M |a_k|^2 \right)^{1/2}.$$

Our goal is to extend a result like this to the L^p norm, i.e. to guarantee with high probability that

$$\left\| \sum_{k=1}^M a_k f_k \right\|_{L^p(\mathbb{R}^d)} \lesssim \left(\sum_{k=1}^M |a_k|^2 \right)^{1/2}.$$

Normalizing, it will suffice to prove that with high probability, for *any* constants $a_1, \dots, a_M$ with $\sum |a_i|^2 = 1$,

$$\left\| \sum_{k=1}^M a_k f_k \right\|_{L^p(\mathbb{R}^d)} \lesssim 1.$$

We will obtain such a result using a *Chaining argument.*

For each point a on the unit sphere, we write

$$S(a,x) = \sum_{k=1}^M a_k f_k(x)$$

and then setting

$$Z(a) = \left\| \sum_{k=1}^M a_k f_k \right\|_{L^p(\mathbb{R}^d)}.$$

We now establish an upper bound on the average value of $Z(a)$.

Theorem 1.3. *We have*

$$\mathbb{E}[\sup_{|a|=1} Z(a)] \lesssim (\log N)^{1/2} N^{s/2-1/p}.$$

Proof. Hoeffding's inequality guarantees that for each $x \in \mathbb{R}$,

$$\|S(a,x)\|_{\psi_2} \lesssim |a| N^{-1/p} \phi(x/N).$$

A union bound guarantees that, for all x in the integer lattice,

$$\| \sup_{x \in \mathbb{Z}} S(a,x) \|_{\psi_2} \lesssim |a| (\log N)^{1/2} N^{-1/p}.$$

Applying the local constancy policy, this should show that

$$\|S(a)\|_{\psi_2} \lesssim |a| (\log N)^{1/2} N^{-1/p}.$$

But now the triangle inequality implies that

$$|Z(a) - Z(b)| = |\|S(a)\|_{L^p(\mathbb{R}^d)} - \|S(b)\|_{L^p(\mathbb{R}^d)}| \leqslant \|S(a-b)\|_{L^p(\mathbb{R}^d)}.$$

Thus

$$\|Z(a) - Z(b)\|_{\psi_2} \leqslant \|S(a-b)\|_{L^p(\mathbb{R}^d)} \lesssim |a-b|(\log N)^{1/2} N^{-1/p}.$$

Thus Dudley's integral inequality implies that

$$\mathbb{E}[\sup_{|a|=1} Z(a)] \lesssim (\log N)^{1/2} N^{-1/p} \int_0^\infty (\log N(t))^{1/2}\, dt,$$

where $N(t)$ denotes the number of balls of radius t required to cover the unit sphere in $\mathbb{R}^M$. We have $N(t) \lesssim (1/t)^{M-1}$ for $t \lesssim 1$, and $N(t) = 1$ for $t \gtrsim 1$, which leads to

$$\mathbb{E}[\sup_{|a|=1} Z(a)] \lesssim (\log N)^{1/2} N^{-1/p} M^{1/2} = (\log N)^{1/2} N^{s/2-1/p}.$$

Thus we have a good decoupling constant for $s \geqslant 2/p$. This is good because we therefore need p to be bigger than one to get anything interesting. □

TODO: This is a local bound, and I think we can show this leads to a global bound, e.g. by Demeter's book. Unfortunately, we only have a good bound on the expected value of $\sup_{|a|=1} Z(a)$, and not the tails, so iterating this using Markov's inequality to get a bound on decoupling on a random fractal doesn't yield great results. Indeed, suppose we iteratively construct a fractal at a scale $1/L^k$ consisting of L^{ks} intervals. Then Markov's inequality guarantees that if E is the random fractal constructed, then for sequences $\{C_k\}$ with $\sum 1/C_k < \infty$, the Borel-Cantelli lemma implies that almost surely, if $\delta_k = 1/L^k$, the random set E has a decoupling constant

$$\mathrm{Dec}(E(\delta_k), p) \leqslant C_k \log(1/\delta_k)^{1/2} (1/\delta_k)^{s/2-1/p}$$

for all k. For $s < 2/p$, we can select these constants well, leading to

$$\mathrm{Dec}(E(1/L^k), p) \lesssim 1.$$

In fact, this inequality gets *better as a power in* δ_k as $k \to \infty$. For $s = 2/p$, Markov's inequality only leads to bounds of the form

$$\mathrm{Dec}(E(1/L^k), p) \lesssim k(\log k)^2 \log(1/\delta_k)^{1/2} \lessapprox_{L,\varepsilon} \log(1/\delta_k)^{3/2}.$$

I'll have to (TODO) look into tail bounds on suprema of sub-Gaussian processes if we want to improve the implicit constants in k, i.e. if we want to replace k with a power of $\log k$, so we get a decoupling constant $\widetilde{O}(\log(1/\delta_k)^{1/2})$.

2 Toy Problem # 2: Gaussians Supported on the Cantor Set

Now we consider a different model of random Cantor sets which possesses more arithmetic structure, and thus makes the problem harder. TODO: For normal 1/3 Cantor set, I think the same L^∞ analysis will work *away from frequencies which are a power of 3*, but hopefully this is a small set so something trivial should work here.