Radial Multipliers

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Chapter 1

Notation

- We use the normalization of the Fourier transform

$$\widehat{f}(\xi) = \int f(x) e^{-2\pi i \xi \cdot x} \, dx$$

which is standard in classical analysis.

- We let Dil_t be the dilation operator on functions, i.e. the operator such that

$$\mathrm{Dil}_t f(x) \coloneqq f(x/t).$$

Overloading notation, we consider dyadic dilations

$$\mathrm{Dil}_j f(x) \coloneqq f(x/2^j).$$

There is no confusion here since the dyadic dilation operator will only be used along with symbols that stand for integers, like n, m, j, or k, whereas the other dilation operator will be used in all other cases, so which dilation we use should be clear from the context.

- We use ∂_j to denote the usual partial derivative operators, and D_j to denote the self-adjoint normalization $D_j f \coloneqq (2\pi i)^{-1} \partial_j$. This notation has the convenience that for a polynomial P,

$$P(D_j)\{f\} = \int P(\xi) \widehat{f}(\xi) e^{2\pi i \xi \cdot x} \, d\xi,$$

which simplifies some of the studies of pseudodifferential and Fourier integral operators.

- We will often use the Japanese bracket $\langle x \rangle := (1 + |x|^2)^{1/2}$ for $x \in \mathbb{R}^d$.
- For integers n and m, we let $[\![n, m]\!] := \{n, n+1, \ldots, m\}$.

Part I

Background

Chapter 2

Fourier Multipliers

The question of the regularity of translation-invariant operators on $\mathbb{R}^d$ has proved central to the development of modern harmonic analysis and the theory of linear partial differential operators. This is because for essentially any translation invariant operator T, we can find a tempered distribution $m : \mathbb{R}^d \to \mathbb{C}$, the *symbol* of T, such that for any Schwartz function $f \in \mathcal{S}(\mathbb{R}^d)$,

$$Tf(x) = \int_{\mathbb{R}^d} m(\xi)\widehat{f}(\xi)e^{2\pi i \xi \cdot x}\, d\xi.$$

In other words, we find that

$$\widehat{Tf}(\xi) = m(\xi)\widehat{f}(\xi),$$

which is why such operators are also called *Fourier multipliers*. Applying the notation of spectral calculus, one might also write this operator as $m(D)$, where $D = (2\pi i)^{-1}\nabla$ is a self-adjoint normalization of the gradient operator. Thus the study of the boundedness of translation invariant operators is closely connected to the study of the interactions of the operators

$$E_\xi f(x) \coloneqq \widehat{f}(\xi)e^{2\pi i \xi \cdot x},$$

which act as projections onto the common eigenspaces of the components of D, since we can write $m(D)$ as a vector-valued integral of the form

$$m(D) = \int_{\mathbb{R}^d} m(\xi) E_\xi\, d\xi.$$

Thus $m(D)$ is represented as a weighted average of the operators $\{E_\xi\}$.

The study of translation invariant operators emerges from classical questions in analysis, like those related to the convergence of Fourier series, or problems in mathematical physics via the study of the heat, wave, and Schrödinger equation. These physical equations also often have *rotational* symmetry, so it is natural to restrict our attention to translation-invariant operators which are also rotation-invariant. Such operators are precisely those operators associated with *radial* symbols $m : \mathbb{R}^d \to \mathbb{C}$, i.e. symbols for which there exists a function $h : [0, \infty) \to \mathbb{R}$ such that

$$m(\xi) = h(|\xi|)$$

for some function $h : [0, \infty) \to \mathbb{C}$. This is the class of *radial Fourier multipliers*. The notation of spectral calculus leads us to write

$$m(D) = h\left(\sqrt{-\Delta}\right),$$

where $\Delta = \sum D_i^2 = (2\pi)^{-2} \sum \partial_i^2$ is the Laplacian on $\mathbb{R}^d$. Thus the study of radial multipliers is closely connected to interactions between the spherical restriction operators

$$E_\lambda f(x) \coloneqq \int_{|\xi|=\lambda} \widehat{f}(\xi) e^{2\pi i \xi \cdot x}\, d\xi,$$

for $0 < \lambda < \infty$, which are the projections onto the eigenspaces of Δ. Similar to the study of $m(D)$, we then have

$$h\left(\sqrt{-\Delta}\right) = \int_0^\infty h(\lambda) E_\lambda\, d\lambda.$$

Thus studying the regularity of radial Fourier multipliers allows us to understand the interactions between the operators $\{E_\lambda\}$.

Suggested by the discussion above, the theory of radial multipliers can be extended from $\mathbb{R}^d$ to the much more general setting of Riemannian manifolds. On any such manifold X, we can define a Laplace-Beltrami operator Δ as an unbounded operator on $L^2(X)$, and provided X is geodesically complete, this operator will be essentially self-adjoint. Thus we can consider a spectral calculus. In particular, we can study the unbounded operators $h(\sqrt{-\Delta})$ on $L^2(X)$, for bounded, Borel measurable functions $h : [0, \infty) \to \mathbb{C}$. The study of such operators should tell us how different eigenfunctions of Δ interact with one another. Some techniques of analyzing radial multipliers on $\mathbb{R}^d$ extend to the Riemannian case. In other cases, new tools are required.

The main focus of this research project is the study of necessary and sufficient conditions to guarantee the L^p boundedness of radial multiplier operators, both in the Euclidean setting, and also in the setting of Riemannian manifolds.

2.1 Convolution Kernels of Fourier Multipliers

It is often useful to study spatial representations of these operators, since one can often exploit certain geometry to obtain useful results. Given any translation invariant operator T, we can associate a tempered distribution $k : \mathbb{R}^d \to \mathbb{C}$, the *convolution kernel* of T, such that for any Schwartz function $f \in \mathcal{S}(\mathbb{R}^d)$,

$$Tf(y) = \int_{\mathbb{R}^d} k(x) f(y - x)\, dx,$$

If T is radial, we can write $k(x) = a(|x|)$ for some function $a : [0, \infty) \to \mathbb{C}$, and then we have a representation

$$T = \int_0^\infty a(r) S_r\, dr,$$

where

$$S_r f(x) := \int_{|y|=r} f(x + y)\, dy,$$

are the *spherical averaging operators*. Thus problems about radial translation-invariant operators are also connected to spherical averaging problems.

With the notation as above, the function k is the *Fourier transform* of m, and the function a is a *Bessel transform* of h, i.e. $a = \mathcal{B}_d h$, where

$$\mathcal{B}_d h(r) := (2\pi) r^{1-d/2} \int_0^\infty h(\lambda) J_{d/2-1}(2\pi\lambda r) \lambda^{d/2}\, d\lambda.$$

Here $J_{d/2-1}$ is the Bessel function of order $d/2 - 1$, given by the formula

$$J_\alpha(\lambda) := \frac{(\lambda/2)^\alpha}{\Gamma(\alpha + 1/2)} \int_{-1}^1 e^{i\lambda s} (1 - s^2)^{\alpha - 1/2}\, ds.$$

For later reference, we note that, using the theory of stationary phase, for each d, we can find symbols s_1 and s_2 of order $-1/2$ such that

$$J_d(\lambda) = e^{2\pi i(\lambda - \omega)} s_1(\lambda) + e^{-2\pi i(\lambda - \omega)} s_2(\lambda),$$

which makes it easier to understand the function for large values of λ.

2.2 Multipliers on Euclidean Space

The general study of the boundedness properties of Fourier multipliers in multiple variables was initiated in the 1950s, as connections of the theory to partial differential equations became more fully realized[1]. It was quickly realized that the most fundamental estimates were L^p bounds of the form

$$\|Tf\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

It is therefore natural to introduce the spaces $M^{p,q}(\mathbb{R}^d)$, consisting of all symbols m which induce a Fourier multiplier operator T bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$. The space $M^{p,q}(\mathbb{R}^d)$ is then a Banach space under the operator norm

$$\|m\|_{M^{p,q}(\mathbb{R}^d)} := \sup \left\{ \frac{\|Tf\|_{L^q(\mathbb{R}^d)}}{\|f\|_{L^p(\mathbb{R}^d)}} : f \in \mathcal{S}(\mathbb{R}^d) \right\}.$$

For notational convenience, $M^{p,p}(\mathbb{R}^d)$ is denoted by $M^p(\mathbb{R}^d)$. Duality implies that $M^{p,q}(\mathbb{R}^d)$ is isometric to $M^{q^*,p^*}(\mathbb{R}^d)$, where p^* and q^* are the conjugates to p and q. And the fact these operators are translation invariant, together with Littlewood's Principle, implies that $M^{p,q}(\mathbb{R}^d) = \{0\}$, unless $q \geqslant p$. Combining these reductions, we see that it suffices to study the spaces $M^{p,q}(\mathbb{R}^d)$ where $1 \leqslant p \leqslant 2$ and where $q \geqslant p$, or alternatively, the spaces $M^{p,q}(\mathbb{R}^d)$, where $2 \leqslant p \leqslant \infty$ and $q \geqslant p$.

The spaces $M^{p,q}(\mathbb{R}^d)$ are difficult to characterize in general, but do have ismple characterizations for very particular parameters of p and q:

- The spaces $M^{1,q}(\mathbb{R}^d) = M^{q^*,\infty}(\mathbb{R}^d)$ are easily characterized by virtue of the fact that the study of the boundedness of operators with domain $L^1(\mathbb{R}^d)$, or range $L^\infty(\mathbb{R}^d)$ is often trivial; we have

  $$M^{1,q}(\mathbb{R}^d) = \widehat{L^q(\mathbb{R}^d)} \quad \text{for } q > 1, \quad \text{and} \quad M^{1,1}(\mathbb{R}^d) = \widehat{M(\mathbb{R}^d)},$$

  Here $M(\mathbb{R}^d)$ is the space of all finite signed Borel measures, equipped with the total variation norm. Stated more quantitatively, this equality asserts that for any symbol m, if $k = \widehat{m}$ is the associated convolution kernel, then

  $$\|m\|_{M^{1,q}(\mathbb{R}^d)} = \|k\|_{L^q(\mathbb{R}^d)} \quad \text{for } q > 1, \quad \text{and} \quad \|m\|_{M^{1,1}(\mathbb{R}^d)} = \|k\|_{M(\mathbb{R}^d)}.$$

  The proof follows from Schur's Lemma, which often gives tight estimates when obtaining bounds with domain $L^1(\mathbb{R}^d)$.

[1] See [10] for a discussion of what was discovered at this time, and for a more detailed exposition of the content of this section.

- The unitary nature of the Fourier transform implies $M^2(\mathbb{R}^d) = L^\infty(\mathbb{R}^d)$, i.e. for any symbol m,
$$\|m\|_{M^2(\mathbb{R}^d)} = \|m\|_{L^\infty(\mathbb{R}^d)}.$$
The proof follows from Parseval's identity.

It is surprising that these are the *only* currently known necessary and sufficient characterizations of the spaces $M^{p,q}(\mathbb{R}^d)$. No simple characterizations of these spaces are known for any other values of p and q, and perhaps no simple characterizations exist. Nonetheless, several tools for analyzing Fourier multipliers in this range have been developed, and we end this section by briefly summarizing some relevant results.

We begin with the major tool of *Littlewood-Paley* theory, which makes it natural to restrict to the study of Fourier multipliers compactly suported on dyadic annuli. Fix a smooth function $\beta \in C_c^\infty(\mathbb{R}^d)$ compactly supported away from the origin on a neighborhood of the unit annulus, such that for $x \neq 0$,

$$\sum_{j \in \mathbb{Z}} (\text{Dil}_j \beta)(x) = 1.$$

For a symbol m, for $t > 0$, and an integer j, we define

$$m_t := (\text{Dil}_{1/t} m) \cdot \beta \quad \text{and} \quad m_j := (\text{Dil}_{1/2^j} m) \cdot \beta.$$

Then m_t describes the behaviour of the multiplier m restricted to the annulus of frequencies $|\xi| \sim t$, but rescaled so that this behaviour is now lying on the annulus $|\xi| \sim 1$. Similarily, m_j describes the rescaled behaviour of m on the annulus of frequencies $|\xi| \sim 2^j$, appropriately rescaled. We have

$$m(D) = \sum_j m_j(D/2^j).$$

Littlewood-Paley theory tells us that if we define P_j to be the Fourier multiplier with symbol $\text{Dil}_j \beta$, then for any $g : \mathbb{R}^d \to \mathbb{C}$, and any $1 < p < \infty$,

$$\|g\|_{L^p(\mathbb{R}^d)} \sim_p \left\| \left(\sum_{j \in \mathbb{Z}} |P_j g|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)}.$$

If we introduce a function $\tilde{\beta}$, which is smooth, compactly supported, with the same properties as β, but equal to one on the support of β, and we introduce the analogous projection operators $\tilde{P}_j$, then we find that

$$[P_j \circ m(D)]\{f\} = [m_j(D/2^j)]\{f\} = [m_j(D/2^j)]\{\tilde{P}_j f\}.$$

A similar Littlewood-Paley inequality holds for the projections $\{\tilde{P}_j\}$, and so we conclude that, for an input function $f : \mathbb{R}^d \to \mathbb{C}$, if we set $f_j = \tilde{P}_j f$, then to show $m \in M^{p,q}(\mathbb{R}^d)$ for $1 < p, q < \infty$, the Littlewood-Paley inequalities for $\{P_j\}$ and $\{\tilde{P}_j\}$ imply that a vector-valued inequality of the form

$$\left\| \left(\sum_j |m_j(D/2^j) f_j|^2 \right)^{1/2} \right\|_{L^q(\mathbb{R}^d)} \lesssim \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \tag{2.1}$$

is equivalent to the bound $\|m(D)f\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$, i.e. equivalent to the inequality

$$\left\| \sum m_j(D/2^j) f_j \right\|_{L^q(\mathbb{R}^d)} \lesssim \| \sum f_j \|_{L^p(\mathbb{R}^d)}. \tag{2.2}$$

The advantage of using (2.1) over (2.2) is that the l^2 sum gives a square root cancellation which makes it easier to combine bounds on the multipliers $m_j(D/2^j)$ for separate j. It is therefore natural to *first* restrict the analysis of multipliers to the individual multipliers $m_j(D/2^j)$, which have the perk that they are compactly supported in frequency space on dyadic annuli, and so their Fourier transforms are all smooth. The rescaling symmetries of $\mathbb{R}^d$ imply that we can actually restrict our analysis to the study to the multipliers $m_j(D)$, which are supported on the annulus $\{1/2 \leqslant |\xi| \leqslant 2\}$. In the sequel, we call multipliers supported on this annulus *unit scale multipliers*. Once bounds for compactly supported multipliers are obtained, one can then rely on various techniques, such as the technology of Hardy spaces, together with the square root cancellation obtained by Littlewood-Paley theory, to show that in the expansion

$$m(D)f = \sum_j m_j(D/2^j) f,$$

the terms on the right hand side do not interact significantly with one another, and so the estimates obtained for unit scale multipliers generalize to a more general family of multipliers. We will therefore find that most conditions guaranteeing the boundedness of Fourier multiplier operators are given in terms of individual control on each of the symbols $\{m_j\}$, since these individual controls will be sufficient to guarantee these symbols do not interact with one another.

Let us state some sufficient conditions currently known which allow one to determine whether a multiplier is bounded:

- Young's convolution inequality, a very crude bound, not taking into account any oscillation in the convolution kernel, implies that if $p < q$ and

$$1/p - 1/q = 1 - 1/r,$$

then

$$\|m\|_{M^{p,q}(\mathbb{R}^d)} \leqslant \|k\|_{L^r(\mathbb{R}^d)},$$

and if $p = q$, then

$$\|m\|_{M^{p,q}(\mathbb{R}^d)} \leqslant \|k\|_{M^1(\mathbb{R}^d)}.$$

- The *Hörmander-Mikhlin multiplier theorem* states that for $1 < p < \infty$ and $\varepsilon > 0$, if we let k_t be the convolution kernel corresponding to the unit scale multiplier $m_t(D)$, then

$$\|m\|_{M^p(\mathbb{R}^d)} \lesssim_{p,\varepsilon} \sup_{t>0} \int |k_t(x)| \langle x \rangle^\varepsilon \, dx.$$

  In particular, we have

$$\|m\|_{M^p(\mathbb{R}^d)} \lesssim_{p,d} \sup_{0 \leqslant k \leqslant d/2+1} \sup_{\xi \in \mathbb{R}^d} |\xi|^k |\nabla^k m(\xi)|.$$

  Notice that the assumption of the Hörmander-Mikhlin multiplier theorem is an assumption on each of the multipliers $\{m_t\}$ separately.

- The *Marcinkiewicz multiplier theorem* states that for $1 < p < \infty$, if $m \in L^\infty(\mathbb{R}^d)$, and for any integer $0 < l \leqslant d$, and any subset $\alpha \subset [\![1, d]\!]$, if we let

$$\Xi_\alpha = \{\xi \in \mathbb{R}^d : \xi_i \neq 0 \text{ for all } i \in \alpha^c\},$$

  and for $k \subset \mathbb{Z}^d$, let $R_{\alpha,k}(\xi)$ denote the $|\alpha|$ dimensional set

$$\{\xi + \eta : 2^{k_i - 1} \leqslant |\eta_i| \leqslant 2^{k_i} \text{ for all } i \in \alpha \text{ and } \eta_i = 0 \text{ for all } i \notin \alpha\}.$$

  Then for $1 < p < \infty$,

$$\|m\|_{M^p(\mathbb{R}^d)} \lesssim_{p,d} \|m\|_{L^\infty(\mathbb{R}^d)} + \sup_{\alpha \subset [\![1,d]\!]} \sup_{\xi \in \Xi_\alpha} \sup_{k \in \mathbb{Z}^l} \int_{R_{\alpha,k}(\xi)} |\partial^\alpha m|.$$

  In particular, we have

$$\|m\|_{M^p(\mathbb{R}^d)} \lesssim_{p,d} \sup_{\alpha \subset [\![1,d]\!]} \sup_{\xi \in \mathbb{R}} |\partial^\alpha m(\xi)| |\xi_1|^{\alpha_1} \cdots |\xi_d|^{\alpha_d}.$$

Conversely, we know some necessary conditions, that show some control over the mass and oscillatory properties of the multiplier m is necessary in order to have $m \in M^{p,q}(\mathbb{R}^d)$ for some exponents p and q. For instance,

$$\sup_{t>0} t^{d(1/p-1/q)} \|m_t\|_{L^{q^*}(\mathbb{R}^d)} \lesssim \|m\|_{M^{p,q}(\mathbb{R}^d)} \quad \text{for } 1 \leqslant p \leqslant q \leqslant 2.$$

This implies, in particular, that $M^{p,q}(\mathbb{R}^d) \subset L^{q^*}_{\mathrm{loc}}(\mathbb{R}^d - \{0\})$, and in particular, any unit scale multiplier in $M^{p,q}(\mathbb{R}^d)$ lies in $L^{q^*}(\mathbb{R}^d)$. However, if $p < 2 < q$, there are unit scale multipliers in $M^{p,q}(\mathbb{R}^d)$ which are distributions of positive order, i.e. that are not even measures. On the convolution kernel side, for any $p \leqslant q$, if k_t is the convolution kernel of the multiplier $m_t(D)$, if $\phi \in C_c^\infty(\mathbb{R}^d)$ has Fourier transform equal to one on the annulus $1/4 \leqslant |\xi| \leqslant 4$, then

$$\|k_t\|_{L^q(\mathbb{R}^d)} = \|k_t * \phi\|_{L^q(\mathbb{R}^d)} = \|m_t(D)\phi\|_{L^q(\mathbb{R}^d)} \lesssim \|m\|_{M^{p,q}(\mathbb{R}^d)}.$$

Noting that $k_t = t^{-d} \mathrm{Dil}_t\{P_t k\}$, we can rewrite this inequality as saying that

$$\|t^{d/q^*} P_t k\|_{L_t^\infty L^q(\mathbb{R}^d)} \lesssim \|m\|_{M^{p,q}(\mathbb{R}^d)}.$$

Heuristically, the convolution kernel k must have d/q^* derivatives in L^q. More precisely, if we introduce the homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^d)$ to be the family of all distributions f for which the Besov norm

$$\|f\|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} \sim \|2^{sj} P_j f\|_{l^r(\mathbb{Z}) L^p(\mathbb{R}^d)}$$

is finite, then the identity above says precisely that k lies in $\dot{B}^{d/q^*}_{q,\infty}(\mathbb{R}^d)$. If we also define the homogeneous spaces $\dot{A}^t_{p,r}(\mathbb{R}^d)$ to be the space of all distributions for which

$$\|f\|_{\dot{A}^t_{p,r}(\mathbb{R}^d)} \sim \|2^{sj} L_j m\|_{l^r(\mathbb{Z}) L^p(\mathbb{R}^d)},$$

where $L_j m = m \cdot \mathrm{Dil}_j \phi$ are the spatial localizations to dyadic annuli, then the discussion in this paragraph entails that

$$M^{p,q}(\mathbb{R}^d) \subset \widehat{\dot{B}^{d/q^*}_{q,\infty}}(\mathbb{R}^d) \cap \dot{A}^{-d/q}_{q^*,\infty}.$$

for $1 \leqslant p \leqslant q \leqslant 2$. Thus regularity on the spatial and frequency side is necessary to ensure a Fourier multiplier is bounded. We note that Hausdorff-Young ensures that

$$\widehat{\dot{B}^{d/q^*}_{q,\infty}}(\mathbb{R}^d) \subset \dot{A}^{-d/p}_{q^*,\infty},$$

which gives weaker control over the behaviour of the multiplier m for large ξ when compared to the condition of being an element of $\dot{A}^{-d/q}_{q^*,\infty}$, but stronger control for small ξ. But for *unit scale multipliers*, the two conditions are equivalent, i.e. because only $O(1)$ Littlewood-Paley projections are non-zero.

2.3 The Radial Multiplier Conjecture

Despite the continuing lack of a complete characterization of the classes $M^{p,q}(\mathbb{R}^d)$, it is surprising that we *can* conjecture a characterization of the subspace of $M^{p,q}(\mathbb{R}^d)$ consisting of *radial symbols*, for an appropriate range of exponents. The conjectured range of estimates was first suggested by the result of [5], which concerned radial multipliers m whose associated operator T is bounded from the L^p norm to the L^q norm *restricted to radial functions*, i.e. such that the norm

$$\|m\|_{M^{p,q}_{\text{rad}}(\mathbb{R}^d)} = \sup\left\{ \frac{\|Tf\|_{L^q(\mathbb{R}^d)}}{\|f\|_{L^p(\mathbb{R}^d)}} : f \in \mathcal{S}(\mathbb{R}^d) \text{ and } f \text{ is radial} \right\}$$

is finite. The main result of [5] is that if $d > 1$, if $1 < p < 2d/(d+1)$, and if $1 \leqslant p \leqslant q < 2$, then $M^{p,q}_{\text{rad}}(\mathbb{R}^d)$ is a subset of $L^1_{\text{loc}}(\mathbb{R}^d)$, and for any radial multiplier m, if k it's convolution kernel, then

$$M^{p,q}_{\text{rad}}(\mathbb{R}^d) = \widehat{B^{d/q^*}_{q,\infty}}(\mathbb{R}^d),$$

Moreover, this range of p and q gives give *precisely the range* under which this characterization holds. It is natural to conjecture that the same constraint continues to hold when we remove the constraint that our inputs f are radial, i.e. that for unit scale, integrable, radial symbols m, for $d > 1$, $1 < p < 2d/(d+1)$, and for $p \leqslant q < 2$,

$$\|m\|_{M^{p,q}(\mathbb{R}^d)} \sim_{p,q,d} \|k\|_{L^q(\mathbb{R}^d)}$$

In the sequel, we call this the *radial multiplier conjecture* in $\mathbb{R}^d$.

2.4 Range of Exponents in the Conjecture

In this section, we show why the range of the radial multiplier cannot be improved, i.e. that the conjecture cannot hold for $p \geqslant 2d/(d+1)$. One can see quite easily that the conjecture cannot hold for $p > 2d/(d+1)$ by using the *ball multiplier operator* analysis due to Fefferman [4], as a black box. To see why the result cannot hold when $p = 2d/(d+1)$, we apply an analogous argument to Fefferman's analysis (a reduction to Kakeya-type phenomena) applied to a family of *Bochner-Riesz bump functions*, which shows a uniform bound for $p = 2d/(d+1)$ as suggested by the radial multiplier conjecture is impossible.

Let's begin with the ball-multiplier argument, which is less technical. Consider the multiplier

$$m(\xi) = \beta(|\xi|)\mathbf{I}(|\xi| \leqslant 1),$$

where β is smooth, equal to one on $\{1/2 \leqslant |\xi| \leqslant 2\}$, and has support on $\{1/4 \leqslant |\xi| \leqslant 4\}$. Then the function m differs from the ball multiplier $\mathbf{I}(|\xi| \leqslant 1)$ by a compactly supported, smooth symbol, and thus the L^p mapping properties of the multiplier $m(D)$ are identically to the mapping properties of the ball multiplier. In particular, a result of [4] guarantees the ball multiplier does not lie in $M^{p,q}(\mathbb{R}^d)$ when $d > 1$ for any values of p and q except when $p = q = 2$, so the same is true of the multiplier m given above. Now if k is the convolution kernel of m, then integrating in polar coordinates gives that

$$k(x) = \int_{0 \leqslant r \leqslant 5} r^{d-1}\beta(r)\widehat{\sigma}(rx)\, dr,$$

where σ is the surface measure of the unit sphere. Standard stationary phase asymptotics show that

$$\widehat{\sigma}(rx) = s_+(r|x|)e^{2\pi i r|x|} + s_-(r|x|)e^{-2\pi i r|x|},$$

where s_+ and s_- are symbols of order $-(d-1)/2$. Integrating by parts gives that

$$\begin{aligned}\int_0^1 r^{d-1}\beta(r)s_+(r|x|)e^{2\pi i r|x|}\, dr = \frac{s_+(|x|)e^{2\pi i|x|}}{2\pi i|x|}& \\ - \frac{1}{2\pi i|x|}\int_0^1 \frac{d}{dr}\left\{r^{d-1}\beta(r)s_+(r|x|)\right\} e^{2\pi i r|x|}\, dr.&\end{aligned}$$

Each of these terms is $O(|x|^{-\frac{d+1}{2}})$. A similar result holds for s_-. Thus

$$|k(x)| \lesssim \langle x\rangle^{-\frac{d+1}{2}}.$$

This allows us to conclude that $k \in L^q(\mathbb{R}^d)$ for $q > 2d/(d+1)$. Because of the L^p mapping properties of m, we conclude that the radial multiplier conjecture cannot hold for $2d/(d+1) < p < 2$, and for any $q \geqslant p$.

To show the radial multiplier conjecture cannot hold at the endpoint $p = 2d/(d+1)$, we must be slightly more careful in our analysis. To do this, we will find a family of smooth, unit scale multipliers $\{m_\delta\}$ such that for $2d/(d+1) \leqslant p \leqslant q \leqslant 2$,

$$\lim_{\delta\to 0}\frac{\|m_\delta\|_{M^{p,q}(\mathbb{R}^d)}}{\|k_\delta\|_{L^q(\mathbb{R}^d)}} = \infty.$$

We will choose $\{m_\delta\}$ to be more and more singular near the boundary of the unit ball as $\delta \to 0$, proceeding as in Fefferman's analysis. Consider the multiplier

$$m_\delta(\xi) = h\left(\frac{|\xi| - 1}{\delta}\right),$$

where $h \in C_c^\infty(\mathbb{R})$ is supported on $|\lambda| \leqslant 1/2$. Thus m_δ is a Bochner-Riesz bump function, i.e. a smooth function supported on a δ neighborhood of the unit sphere. If k_δ is the convolution kernel corresponding to the multiplier m_δ, then we calculate, as above, that

$$k_\delta(x) = \int r^{d-1} h\left(\frac{r-1}{\delta}\right) \widehat{\sigma}(rx)\, dr,$$

can be written as a sum of terms of the form

$$\int r^{d-1} h\left(\frac{r-1}{\delta}\right) s_\pm(r|x|) e^{\pm 2\pi i r|x|}\, dr,$$

where $s_\pm$ is a symbol of order $-(d-1)/2$. We focus on the s_+ term, with the s_- term handled in an analogous fashion. The support of the integrand lies away from the origin, we can integrate by parts arbitrarily many times to write this integral as

$$\left(\frac{-1}{2\pi i|x|}\right)^N \int \left(\frac{d}{dr}\right)^N \left\{ r^{d-1} h\left(\frac{r-1}{\delta}\right) s_+(r|x|) \right\} e^{\pm 2\pi i r|x|}\, dr.$$

Applying the product rule to the derivative, and then integrating over the width δ interval upon which the oscillatory integral is supported, we conclude that this term is $O_N(\delta^{1-N}|x|^{-\frac{d-1}{2}-N})$ for all $N \geqslant 0$. For $|x| \geqslant 1/\delta$, it's better to take the error estimates for large N, whereas for $|x| \leqslant 1/\delta$, it's better to take the error term with $N = 0$. In particular, when calculating the L^q norm, using the trivial bound $\|k_\delta\|_{L^\infty} \leqslant \delta$ for $|x| \leqslant 1$, using the bound with $N = 0$ for $1 \leqslant |x| \leqslant \delta^{-1}$, and using the bound for arbitrarily large N for $|x| \geqslant \delta^{-1}$, we conclude that

$$\|k_\delta\|_{L^q(\mathbb{R}^d)} \lesssim_q \delta^{\frac{d+1}{2} - \frac{d}{q}}.$$

In particular, we note that the convolution kernels $\{k_\delta\}$ are uniformly bounded in δ provided that $q \geqslant 2d/(d+1)$. If the radial multiplier conjecture held at this exponent, we would have for any Schwartz function $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\|k_\delta * f\|_{L^q(\mathbb{R}^d)} \lesssim \delta^{\frac{d+1}{2} - \frac{d}{q}} \|f\|_{L^p(\mathbb{R}^d)},$$

where the implicit constant is independent of δ. We now use the existence of Kakeya sets, ala Fefferman's analysis, to justify that such a bound cannot be possible when $2d/(d+1) \leqslant p \leqslant 2$ and where $q \geqslant p$. It will be more convenient to dualize, and show that for $2 \leqslant p \leqslant 2d/(d-1)$ and $q \geqslant p$, a bound of the form

$$\|k_\delta * f\|_{L^q(\mathbb{R}^d)} \lesssim \delta^{\frac{d}{p}-\frac{d-1}{2}} \|f\|_{L^p(\mathbb{R}^d)}$$

is not possible, which is equivalent to the original bound.

For each $\delta > 0$, we will choose a function f_δ so that the inequality above is not possible uniformly in δ as $\delta \to 0$. The function f_δ will be chosen to have Fourier support on the annulus $1-2\delta \leqslant |\xi| \leqslant 1+2\delta$. Cover the unit sphere of $\mathbb{R}^d$ by a maximal collection Θ_δ of δ-separated points. Then $\#(\Theta_\delta) \sim \delta^{1-d}$. For each $\theta \in \Theta_\delta$, consider a cap centered at $(1-1.4\delta)\theta$, with length δ in the θ direction, and with length $\delta^{1/2}$ in the $d-1$ directions tangential to θ. The choice is made so that κ_θ has essentially 10% of it's mass on the support of m_δ. We then consider a family of bump functions $\{\phi_\theta\}$ supported on the family $\{\kappa_\theta\}$, with magnitude roughly $\delta^{-(d+1)/2}$ on Θ_δ. For any points $\{x_\theta : \theta \in \Theta\}$, consider the family of functions $\{\chi_\theta\}$, with

$$\widehat{\chi_\theta}(\xi) = e^{-2\pi i x_\theta \cdot \xi}\phi_\theta(\xi).$$

For each θ, χ_θ then has mass concentrated on the dual rectangle T_θ of κ_θ, which is centered at x_θ, and has magnitude roughly $\delta^{-(d+1)/2}$ on T_θ. The action of the Fourier multiplier $m_\delta(\sqrt{-\Delta})$, roughly speaking, is to cut the Fourier support of χ_θ by a tenth. We lose 90% of the Fourier mass of θ, but our support is also made ten times thinner in the tangential direction. As a result, $k_\delta * \chi_\theta$ will now have mass concentrated on a tube T_θ^*, with the same center, but which is ten times longer than T_θ in the tangential direction. Let's let T_θ^+ be the portion of T_θ^* which lies at the opposite end of T_θ^* to T_θ, but with the same dimensions as T_θ.

Our construction now relies on *Kakeya* like phenomena. For any $\varepsilon > 0$, there exists a large δ_0 such that for $\delta \leqslant \delta_0$, we can pick $\{x_\theta\}$ such that tubes $\{T_\theta\}$ are disjoint from one another, but such that the tubes $\{T_\theta^+\}$ have large overlap, in the sense that

$$|\bigcup_\theta T_\theta^+| \leqslant \varepsilon |\bigcup_\theta T_\theta|.$$

The supports of $k_\delta * \chi_\theta$ thus have lots of overlap on this set. However, it is difficult to determine the sum $\sum_\theta k_\delta * \chi_\theta$, since these functions might have different signs where they overlap. To fix this, we define

$$f_\delta = \sum_\theta \varepsilon_\theta \chi_\theta$$

where $\{\varepsilon_\theta\}$ are independent $\{-1,+1\}$ valued Bernoulli random variables, because Khintchine's inequality implies that we have the pointwise inequality, for any $1 < r < \infty$, of the form

$$\left(\mathbb{E}\left|\sum_\theta \varepsilon_\theta \chi_\theta\right|^r\right)^{1/r} \sim \left(\sum_\Theta |\chi_\theta|^2\right)^{1/2}.$$

It follows that, since the tubes T_θ are disjoint,

$$\begin{aligned}\mathbb{E}\,\|f_\delta\|_{L^p(\mathbb{R}^d)}^p &\sim \int\left(\sum_\theta |\chi_\theta(x)|^2\right)^{p/2} dx\\ &\sim \left(\delta^{-\frac{d+1}{2}}\right)^p \left|\bigcup_\theta T_\theta\right|.\end{aligned}$$

Similarily, Khintchine's inequality can again be applied to

$$k_\delta * f_\delta = \sum_\theta \varepsilon_\theta (k_\delta * \chi_\theta).$$

to conclude that

$$\begin{aligned}\mathbb{E}\,\|k_\delta * f_\delta\|_{L^q(\mathbb{R}^d)}^q &\sim \int\left(\sum_\theta |k_\delta * \chi_\theta(x)|^2\right)^{q/2} dx\\ &\gtrsim \left(\delta^{-\frac{d+1}{2}}\right)^q \int\left(\sum_\theta \mathbf{I}_{T_\theta^+}(x)\right)^{q/2} dx.\end{aligned}$$

If the radial multiplier conjecture held for the exponents p and q considered, we would therefore conclude that by Jensen's inequality,

$$\begin{aligned}
\left(\delta^{-\frac{d+1}{2}}\right)^q \int \left(\sum_\theta \mathbf{I}_{T_\theta^+}(x)\right)^{q/2} dx &\lesssim \mathbb{E}\,\|k_\delta * f_\delta\|^q_{L^q(\mathbb{R}^d)} \\
&\lesssim \delta^{q\left(d/p - \frac{d-1}{2}\right)}\,\mathbb{E}\,\|f_\delta\|^q_{L^p(\mathbb{R}^d)} \\
&\lesssim \delta^{q\left(d/p - \frac{d-1}{2}\right)} \left(\mathbb{E}\,\|f_\delta\|^p_{L^p(\mathbb{R}^d)}\right)^{q/p} \\
&\lesssim \delta^{q\left(d/p - \frac{d-1}{2}\right)} \delta^{-q\left(\frac{d+1}{2}\right)} \left|\bigcup_\theta T_\theta\right|^{q/p} \\
&\lesssim \left(\delta^{-d(1-1/p)}\right)^q \left|\bigcup_\theta T_\theta\right|^{q/p}.
\end{aligned}$$

Rearranging, and taking the qth root on both sides, we conclude that

$$\left(\int \left(\sum_\theta \mathbf{I}_{T_\theta^+}(x)\right)^{q/2} dx\right)^{1/q} \lesssim \delta^{\frac{d}{p} - \frac{d-1}{2}} \left|\bigcup_\theta T_\theta\right|^{1/p}$$

Interpolating the trivial bound

$$\int \left(\sum_\theta \mathbf{I}_{T_\theta^+}(x)\right) dx \sim \left|\bigcup_\theta T_\theta\right|$$

with the fact that the sum is supported on the finite measure set $\bigcup_\theta T_\theta^+$ yields that

$$\left(\int \left(\sum_\theta \mathbf{I}_{T_\theta^+}(x)\right)^{q/2} dx\right)^{1/q} \gtrsim \left|\bigcup_\theta T_\theta^+\right|^{-(1/2-1/q)} \left|\bigcup_\theta T_\theta\right|^{1/2},$$

Substituting and rearranging, we obtain that

$$\left|\bigcup_\theta T_\theta\right|^{1/2-1/p} \lesssim \delta^{d/p - \frac{d-1}{2}} \left|\bigcup_\theta T_\theta^+\right|^{1/2-1/q}.$$

Now $p \leqslant 2d/(d-1)$, which implies the exponent of δ is positive, which allows us to conclude that for $\delta \leqslant 1$,

$$\left|\bigcup_\theta T_\theta\right| \lesssim \left|\bigcup_\theta T_\theta^+\right|^{\frac{1/2-1/q}{1/2-1/p}}.$$

Note that because $q \geqslant p \geqslant 2$,

$$\frac{1/2-1/p}{1/2-1/q} \geqslant 1.$$

Thus if $|\bigcup_\theta T_\theta^+| \leqslant 1$,

$$\left|\bigcup_\theta T_\theta\right| \lessapprox \left|\bigcup_\theta T_\theta^+\right|^{\frac{1/2-1/q}{1/2-1/p}} \lessapprox \left|\bigcup_\theta T_\theta^+\right|.$$

This is impossible, because the existence of Kakeya sets implies we can choose these tubes so that as $\delta \to 0$,

$$\left|\bigcup_\theta T_\theta^+\right| = o\left(\left|\bigcup_\theta T_\theta\right|\right),$$

which would give a contradiction to this bound.

Remark. The argument above can still be applied for general values $2 \leqslant p \leqslant q \leqslant \infty$, and shows that if the radial multiplier conjecture holds for certain estimates, then certainly Minkowski dimension bounds hold for all Kakeya sets. We should expect such an analysis, since the multiplier conjecture for a particular value of p would imply the Bochner-Riesz conjecture for that same p, and [20] shows this implies results for the restriction conjecture, and thus the Kakeya maximal conjecture.

If we use the disjointness of the $\{T_\theta\}$, we have that

$$\left|\bigcup_\theta T_\theta\right| \sim \delta^{-\frac{3d-1}{2}}.$$

Thus we can rearrange the inequality to read that

$$\left|\bigcup_\theta T_\theta^+\right| \gtrapprox \delta^{\frac{-1}{1-2/q}\left(\frac{d+1}{2}-\frac{d-1}{p}\right)}.$$

This bounds tells us more information about Kakeya sets. Namely, if K is a Kakeya set, then we can choose a family of tubes $\{T_\theta^+\}$ as above, whose union is contained in a $\delta^{-1/2}$ neighborhood of $\delta^{-1}K$. It follows that if $\varepsilon = \delta^{1/2}$, then the volume of the ε neighborhood K_ε of K satisfies the bound

$$|K_\varepsilon| \gtrapprox \varepsilon^{2d-\frac{1}{1/2-1/q}\left(\frac{d+1}{2}-\frac{d-1}{p}\right)}.$$

Taking $\varepsilon \to 0$, we conclude that K has Minkowski dimension at least

$$\frac{1}{1/2 - 1/q}\left(\frac{d+1}{2} - \frac{d-1}{p}\right) - d,$$

and in fact for

$$q < 4 \cdot \frac{p}{p+2} \cdot \frac{d}{d-1},$$

we would conclude that K must have positive Lebesgue measure, which would give a contradiction give the existence of Kakeya sets, and thus the conjecture certainly cannot hold for such exponents. For $p \geqslant 2(d+1)/(d-1)$, no such choice of q exists with $q \geqslant p$ TODO: Why isn't this $p \geqslant 2d/(d-1)$?

2.5 Summary of Prior Results

We now know, by results of Heo, Nazarov, and Seeger [9] that the radial multiplier conjecture is true when $d \geqslant 4$ and when $1 < p < (2d-2)/(d+1)$. A summary of the proof strategies of this argument is provided in Chapter 9. When $d = 4$, this was improved by Cladek [3], who showed that the conjecture is true here when $1 < p < 36/29$, where $36/29 \approx (2d - 1.79)/(d+1)$. In the same paper, Cladek also established the first results in the $d = 3$ case, obtaining a *restricted weak type* bound

$$\|Tf\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^{p,1}(\mathbb{R}^n)},$$

where $1 < p < 13/12$, where $13/12 \approx (2d - 1.66)/(d+1)$. Summaries of the improvements Cladek obtained are described in Chapter 10. But the radial multiplier conjecture has not yet been completely resolved in any dimension n, we do not have any strong type L^p bounds when $d = 3$, and we don't have any bounds whatsoever when $d = 2$. One goal of this research project is to investigate whether one can use modern research techniques to improve upon these bounds.

The full proof of the radial multiplier is likely far beyond current research techniques. Indeed, it remains a major open problem in harmonic analysis to determine the range of exponents for which *specific* radial Fourier multipliers are bounded in the range where the conjecture would apply, such as the Fourier multiplier on $\mathbb{R}^d$ with symbol $m_\lambda(\xi) = (1 - |\xi|)_+^\lambda$, the family of *Bochner-Riesz multipliers*. The radial multiplier conjecture characterizes the range of the Bochner-Riesz multipliers, and thus the conjecture would also imply the Kakeya and restriction conjectures. All three of these results are major unsolved problems in harmonic analysis. On the other hand, the Bochner Riesz conjecture is completely resolved when $d = 2$, while in contrast, no results related to the radial multiplier conjecture are known in this dimension at all. And in any dimension $d > 2$, the range under which the Bochner-Riesz multiplier is known to hold [7] is strictly larger than the range under which the radial multiplier conjecture is known to hold, even for the restricted weak-type bounds obtained in [3]. Thus it still seems within hope that the techniques recently applied to improve results for Bochner-Riesz problem, such as broad-narrow analysis [2], the polynomial Wolff axioms [11], and methods of incidence geometry and polynomial partitioning [22] can be applied to give improvements to current results characterizing the boundedness of general radial Fourier multipliers.

Our hopes are further emboldened when we consult the proofs in [9] and [3], which reduce the radial multiplier conjecture to the study of upper bounds of quantities of the form

$$\left\| \sum_{(y,r)\in\mathcal{E}} F_{y,r} \right\|_{L^p(\mathbb{R}^n)},$$

where $\mathcal{E} \subset \mathbb{R}^n \times (0,\infty)$ is a finite collection of pairs, and $F_{y,r}$ is an oscillating function supported on a $O(1)$ neighborhood of a sphere of radius r centered at a point y. The L^p norm of this sum is closely related to the study of the tangential intersections of these spheres, a problem successfully studied in more combinatorial settings using incidence geometry and polynomial partitioning methods [23], which provides further estimates that these methods might yield further estimates on the radial multiplier conjecture.

When $d = 3$, the results of [3] are only able to obtain bounds on the L^p sums in the last paragraph when $\mathcal{E}$ is a Cartesian product of two subsets of $(0,\infty)$ and $\mathbb{R}^d$. This is why only restricted weak-type bounds have been obtained in this dimension. It is therefore an interesting question whether different techniques enable one to extend the L^p bounds of these sums when the set $\mathcal{E}$ is *not* a Cartesian product, which would allow us to upgrade the result of [3] in $d = 3$ to give strong L^p bounds. This question also has independent interest, because it would imply new results for the 'sharp' local smoothing conjecture, which concerns the regularity of solutions to the wave equation in $\mathbb{R}^d$. Incidence geometry has been recently applied to yield results on the 'non-sharp' local smoothing conjecture [8], which again suggests these techniques might be applied to yield the estimates needed to upgrade the result of [3] to give strong L^p-type bounds.

2.6 Bessel Transforms and Radial Multipliers

Given a function h on $[0, \infty)$, we define the *d-dimensional Fourier-Bessel transform* of h as

$$\mathcal{B}_d h(r) = r^{-\frac{d-2}{2}} \int_0^\infty \rho^{d/2} h(\rho) J_{d/2-1}(\rho r)\, d\rho,$$

where $J_{d/2-1}$ is the Bessel function of order $d/2 - 1$. If m is the function on $\mathbb{R}^d$ defined by setting $m(\xi) = h(|\xi|)$, then

$$\widehat{m}(x) = \mathcal{B}_d h(|x|).$$

The condition in the radial multiplier conjecture for unit scale multipliers can thus be restated that

$$\|m\|_{M^{p,q}(\mathbb{R}^d)} \sim \left(\int_0^\infty r^{d-1} |\mathcal{B}_d h(r)|^q \, dr \right)^{1/q}.$$

One can use Bessel function asymptotics to relate this quantity to a condition on the one-dimensional Fourier transform of h (extended to an even function on $\mathbb{R}$), if h is a unit scale multiplier. Indeed, in [5], Garrigós and Seeger show that for $1 < q < 2$, for such multipliers, we have

$$\left(\int_0^\infty r^{d-1} |\mathcal{B}_d h(r)|^q \, dr \right)^{1/q} \sim_{d,q} \left(\int_0^\infty \langle \lambda \rangle^{(d-1)(1-q/2)} |\widehat{h}(\lambda)|^q \right)^{1/q}.$$

Let us briefly prove this equivalence.

Theorem 2.1. *TODO*

Proof. Write the left hand side as A, and the right hand side as B. We will begin by proving that $A \lesssim B$. To begin with, using the fact that h is even, we can write

$$\mathcal{B}_d h(r) = r^{-\frac{d-2}{2}} \sum_{\pm} \int_0^\infty \rho^{d/2} h(\rho) s_\pm(\rho r) e^{\pm 2\pi i \rho r} \, d\rho,$$

where s_+ and s_- are symbols of order $-1/2$. In particular, expanding out the asymptotics of the symbols, we see that for any $N > 0$, we can write $\mathcal{B}_d h(r)$ as a linear combination of terms of the form

$$r^{-\frac{d-1}{2}-\alpha} \int_0^\infty \rho^{\frac{d-1}{2}-\alpha} h(\rho) e^{\pm 2\pi i \rho r} \, d\rho$$

for $|\alpha| < N$, plus a remainder term given by

$$\sum_{\pm} r^{-\frac{d-2}{2}} \int_0^\infty \rho^{d/2} h(\rho) a_\pm(\rho r) e^{\pm 2\pi i \rho r},$$

where a_+ and a_- are symbols of order $-1/2 - N$. If $k_{\pm,N}$ is the inverse Fourier transform of $\rho^{N-1/2} a_\pm(\rho)$, then we can write the remainder above as a linear combination of terms of the form

$$r^{-N-\frac{d-1}{2}} \int k_{\pm,N}(t/r) [D^{d/2-N+1/2} \widehat{h}](\pm r - t)\, dt.$$

The fact that $\rho^{N-1/2} a_\pm(\rho)$ is a symbol of order -1 implies that we have

$$|k_{\pm,N}(t)| \lesssim_N 1.$$

Set

$$Rf(r) = \int k_{\pm,N}(t/r) f(\pm r - t)\, dt.$$

A weighted version of Schur's Lemma, together with this bound, implies that if N is suitably large, then

$$\left(r^{(d-1)-q\left(N+\frac{d-1}{2}\right)} |Rf(r)|^q \right)^{1/q} \lesssim \left(\langle \lambda \rangle^{(d-1)(1-q/2)} |f(r)|^q . \right)^{1/q}$$

TODO

where $|k(t)| \lesssim_N \langle t \rangle^{-N}$ for all $N > 0$.

given by replacing $s_\pm$ with it's order $\leqslant$

$$r^{-\frac{d-1}{2}-N} \left((\partial_i^{d/2} \widehat{h}) * \right) (\pm r)$$

$$O_N \left(\langle r \rangle^{-\frac{d-1}{2}-N} \right).$$

The terms above for a particular α can be written as a constant multiple of

$$r^{-\frac{d-2}{2}-\alpha} \left(\partial^\alpha \widehat{h} \right) (\pm r).$$

Applying the triangle inequality to each of these terms, we see that $\square$

This characterization of the conclusion of the radial multiplier conjecture will come in handy later on when we discuss the extension of the radial multiplier conjecture to Riemannian manifolds. In the sequel, we let

$$C_q(h) = \left(\int_0^\infty \langle \lambda \rangle^{(d-1)(1-q/2)} |\widehat{h}(\lambda)|^q \right)^{1/q}.$$

Thus the radial multiplier conjecture, restricted to unit scale radial multipliers $m(\xi) = h(|\xi|)$, is to determine whether, for certain p and q, $\|m\|_{M^{p,q}(\mathbb{R}^d)} \sim C_q(h)$.

2.7 Multipliers on Riemannian Manifolds

Fix a geodesically complete Riemannian manifold X. The operator $\sqrt{-\Delta}$, defined initially on $C_c^\infty(X)$, is then essentially self-adjoint on $L^2(X)$. The spectral calculus of unbounded operators can then be used to define operators of the form $h(\sqrt{-\Delta})$ for each locally bounded, Borel measurable function $h : [0,\infty) \to \mathbb{C}$. These operators are analogous to the radial multipliers studied in the Euclidean setting, and we will also call these operators *radial multipliers on X*. Just like multiplier operators on $\mathbb{R}^d$ are crucial to an understanding of the interactions between the functions $e_\xi(x) = e^{2\pi i \xi \cdot x}$ on $\mathbb{R}^n$, understanding the operators $h(\sqrt{-\Delta})$ is crucial to understanding the interactions of eigenfunctions of the Laplace-Beltrami operator on the manifold X.

We let $M^{p,q}(X, \sqrt{-\Delta})$ denote the family of all locally bounded, Borel measurable functions h such that the operator $T_h = h(\sqrt{-\Delta})$ extends to a bounded operator from $L^p(X)$ to $L^q(X)$, with the analogous operator norm, though, when there is no ambiguity, we will overload notation and write this space as $M^{p,q}(X)$.

To avoid technicalities, we will mainly focus on the study of radial multipliers on *compact* Riemannian manifolds. Such manifolds are automatically geodesically complete. Moreover, on such a manifold, the spectrum of Δ forms a discrete set $\Lambda \subset (0,\infty)$, and for each $\lambda \in \Lambda$, there is a finite dimensional space $H_\lambda \subset C^\infty(X)$, such that we have an orthogonal decomposition

$$L^2(X) = \bigoplus_{\lambda \in \Lambda} H_\lambda,$$

and for $f \in H_\lambda$, we have $\Delta f = -\lambda^2 f$. Such a function is called a *Laplace-Beltrami eigenfunction*. For any function $h : [0,\infty) \to \mathbb{C}$, we then have that for $f \in C^\infty(X)$,

$$h(\sqrt{-\Delta}) f = \sum_{\lambda \in \Lambda} h(\lambda) P_\lambda f,$$

where $P_\lambda : L^2(X) \to H_\lambda$ is orthogonal projection onto H_λ. If, for each H_λ, we fix an orthonormal basis $\{e_{\lambda,n}\}$, then we can write

$$h(\sqrt{-\Delta})f = \sum_{\lambda \in \Lambda} \sum_n h(\lambda)\langle f, e_{\lambda,n}\rangle e_{\lambda,n},$$

Thus $h(\sqrt{-\Delta})$ is diagonalized by the basis $\{e_{\lambda,n}\}$.

There are several model cases of study, each of which having constant curvature and a sufficient amount of symmetry to be able to describe the class of Laplace-Beltrami eigenfunctions:

- The torii $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, for which

  $$\Lambda = \left\{ \sqrt{n} : n > 0 \right\},$$

  and for which H_λ is, for each $\lambda \in \Lambda$, spanned by the functions of the form

  $$e_\xi(x) = e^{i\xi \cdot x}$$

  where $\xi \in \mathbb{Z}^d$, and $|\xi| = \lambda$. The study of radial multipliers in this setting is thus naturally connected to the theory of Fourier series, and is very similar to the theory of multipliers on $\mathbb{R}^d$.

- The sphere S^d in $\mathbb{R}^{d+1}$. Here

  $$\Lambda = \left\{ \sqrt{n(n+d-1)} : n > 0 \right\}$$

  If $\lambda_n = \sqrt{n(n+d-1)}$, then H_{λ_n} is the space of all *spherical harmonics* of degree n. Then

  $$\dim(H_{\lambda_n}) = \binom{d+n-1}{n} - \binom{d+n-3}{n-2} \lesssim_d n^{d-2}.$$

  The theory of the Laplacian on S^d is closely connected to the representation theory of the non-commutative Lie group $SO(d+1)$.

- For a discrete, cocompact subgroup $\Gamma \subset PSL(d, \mathbb{R})$, we can consider the Riemannian manifold $\mathbb{H}^d/\Gamma$, obtained by quotienting hyperbolic space by the family of isometries corresponding to Γ. It is not quite as easy to explicitly write the eigenfunctions of this manifold, but the spectral theory of Δ is closely tied to the study of the representation theory of the non-commutative group Lie group G/Γ.

The Killing-Hopf theorem says that every manifold of constant curvature has either $\mathbb{R}^d$, S^d, or $\mathbb{H}^d$ as a universal cover, so these provide a good family of simple manifolds, with a fairly clear family of eigenfunctions, with which we can begin our analysis.

Setting up the regularity study of multipliers on a compact Riemannian manifold has a certain technical problem not present in the Euclidean problem. If h has compact support, then the operator

$$h\left(\sqrt{-\Delta}\right) f = \sum h(\lambda_n)\langle f, e_n\rangle \cdot e_n$$

has a finite sum on the right hand side, and each term of the sum is individually well behaved, since e_n is smooth. And so by the triangle inequality, any such operator will be trivially bounded from $L^p(X)$ to $L^q(X)$ for any exponents p and q. Thus $M^{p,q}(X)$ contains all compactly supported radial multipliers, trivializing the study of compactly supported radial multipliers. This is the complete opposite of the Euclidean case, where dyadic decomposition type technology allowed us to reduce the study of general multipliers to compactly supported radial multipliers. The key step to fixing this problem is to recognize that Euclidean multipliers automatically have rescaling symmetries which prevent the case of compact support from being trivial, whereas this is not present in the case of compact Riemannian manifolds. To get around this we add a rescaling symmetry into our problem, i.e. we study conditions that ensure we have a bound of the form

$$\sup_R R^{d(1/q-1/p)} \|\mathrm{Dil}_R h\|_{M^{p,q}(X)} < \infty.$$

The exponent of R here emerges from the fact that

$$\|\mathrm{Dil}_R h\|_{M^{p,q}(\mathbb{R}^d)} = R^{-d(1/q-1/p)} \|h\|_{M^{p,q}(\mathbb{R}^d)}.$$

We let $M^{p,q}_{\mathrm{Dil}}(X)$ denote the family of all multipliers for which the inequality above holds, and give it the norm induced by the quantity on the left hand side, and set $M^p_{\mathrm{Dil}}(X) = M^{p,p}_{\mathrm{Dil}}(X)$. A transference principle of Mitjagin [17] (See [12] for a similar result, written in English and available online) shows that if X is a compact Riemannian manifold, and $h : (0,\infty) \to \mathbb{C}$ is a bounded, Borel measurable function, then

$$\|h\|_{M^{p,q}(\mathbb{R}^d, \sqrt{-\Delta})} \lesssim_{X,p,q} \|h\|_{M^{p,q}_{\mathrm{Dil}}(X, \sqrt{-\Delta})}.$$

Thus, in some sense, the dilation-invariant Fourier multiplier problem on a compact manifold X is at least as hard as it is on $\mathbb{R}^d$. Another goal of this research project is to extend the radial multiplier conjecture to the setting of dilation invariant bounds for multipliers of the Laplacian on Riemannian manifolds.

The study of dilation invariant radial multipliers on $\mathbb{T}^d$ is essentially the same as on $\mathbb{R}^d$, as we might guess from the crude observation that the family of eigenfunctions to the Laplacian are similar in both domains. More precisely, we can show that for any locally bounded, Borel measurable function $h : (0, \infty) \to \mathbb{R}$, such that every point in $(0, \infty)$ is a Lebesgue point of h,

$$\|h\|_{M^{p,q}_{\text{Dil}}(\mathbb{T}^d)} \sim_{p,q,d} \|h\|_{M^{p,q}(\mathbb{R}^d)}.$$

A proof of this result is given by Theorem 3.6.7 of [6]. Whether an analogous result remains true for more general Riemannian manifolds remains unclear, since the family of eigenfunctions to the Laplacian can take on various different forms on these manifolds, that can look quite different to the Euclidean case (TODO: Does the existence of low dimension Kakeya sets on certain manifolds show that the radial multiplier conjecture cannot be true in general).

What *is* easy to establish is that the theory of multipliers in $M^2_{\text{Dil}}(X)$ is relatively the same. Indeed, applying orthogonality, we calculate that for any function $h : (0, \infty) \to \mathbb{C}$, we have

$$\begin{aligned} \|h(\sqrt{-\Delta})f\|_{L^2(X)} &= \left\| \sum_\lambda h(\lambda) E_\lambda f \right\|_{L^2(X)} \\ &= \left(\sum_\lambda |h(\lambda)|^2 \|E_\lambda f\|_{L^2(X)}^2 \right)^{1/2} \\ &\leqslant \left(\sup_{\lambda \in \sigma(\sqrt{-\Delta})} |h(\lambda)| \right) \left(\sum_\lambda \|E_\lambda f\|_{L^2(X)}^2 \right)^{1/2} \\ &= \left(\sup_{\lambda \in \sigma(\sqrt{-\Delta})} |h(\lambda)| \right) \|f\|_{L^2(X)}. \end{aligned}$$

Taking f to be an eigenfunction with eigenvalue λ which maximizes the value of $|h(\lambda)|$ shows this inequality is tight, i.e. we have

$$\|h\|_{M^{2,2}(X)} = \sup_{\lambda \in \sigma(\sqrt{-\Delta})} |h(\lambda)|.$$

Now applying an arbitrary dilation to h, we conclude that

$$\|h\|_{M^{2,2}_{\text{Dil}}(X)} = \sup_{\lambda > 0} |h(\lambda)|.$$

Thus $M^2_{\mathrm{Dil}}(X)$ consists precisely of the bounded functions on $(0,\infty)$.

How about the spaces $M^{p,2}(X)$, for $1 \leqslant p < 2$? We know by the transference principle that any multiplier h in $M^{p,2}_{\mathrm{Dil}}(X)$ must satisfy a bound of the form

$$\left(\int_0^\infty r^{d-1} |h(r)|^q \right)^{1/q} < \infty,$$

where $q = 2p/(2-p)$. Is this sufficient? The analogous characterization on $\mathbb{R}^d$ is proved using the Hausdorff-Young inequality, plus orthogonality. One way to state the Hausdorff-Young inequality on $\mathbb{T}^d$ is that for $1 \leqslant p \leqslant 2$, and $f \in C^\infty(\mathbb{T}^d)$,

$$\left(\sum_\lambda \sum_n |\langle f, e_{\lambda,n} \rangle|^{p^*} \right)^{1/p^*} \leqslant \|f\|_{L^p(\mathbb{T}^d)}.$$

To obtain an analogous result on a general compact manifold X, we can interpolate Parseval's inequality

$$\left(\sum_\lambda \sum_n |\langle f, e_{\lambda,n} \rangle|^2 \right)^2 \leqslant \|f\|_{L^2(X)}.$$

with

$$\|P_\lambda f\|_{L^2(M)}$$

$$\left(\sum_\lambda \|P_\lambda f\| \right)$$

with the results of Sogge (TODO: FIND PAPER) that say that

$$\sup_{\lambda,n} |\langle f, e_{\lambda,n} \rangle| \lambda^{-\frac{d-1}{2}} \lesssim \|f\|_{L^1(X)},$$

which yields that for $1 \leqslant p < 2$, that

$$\left(\sum_{\lambda,n} |\langle f, e_{\lambda,n} \rangle|^{p^*} \lambda^{-(d-1)(1/p-1/2)} \right)^{1/p^*} \lesssim \|f\|_{L^p(X)}.$$

We therefore conclude that for $f \in C^\infty(X)$,

$$
\begin{aligned}
&\|h(\sqrt{-\Delta})f\|_{L^2(X)} \\
&\quad = \left(\sum_\lambda \sum_n |h(\lambda)|^2 |\langle f, e_{\lambda,n}\rangle|^2 \right)^{1/2} \\
&\quad = \Bigg(\sum_\lambda \sum_n \left[|h(\lambda)|^2 \lambda^{(d-1)(1/p-1/2)(2/p^*)} \right] \\
&\qquad\qquad\qquad \left[|\langle f, e_{\lambda,n}\rangle|^2 \lambda^{-(d-1)(1/p-1/2)(2/p^*)} \right] \Bigg)^{1/2} \\
&\quad \leqslant \left(\sum_\lambda \sum_n |h(\lambda)|^{\frac{2p}{2-p}} \lambda^{(d-1)(1-1/p)} \right)^{\frac{2-p}{2p}} \\
&\qquad\qquad \left(\sum |\langle f, e_{\lambda,n}\rangle|^{p^*} \lambda^{-(d-1)(1/p-1/2)} \right)^{1/p^*} \\
&\quad \lesssim \left(\sum_\lambda \sum_n |h(\lambda)|^q \lambda^{(d-1)(1-1/p)} \right)^{1/q} \|f\|_{L^p(X)}.
\end{aligned}
$$

In the model case of S^d, we have that

$$
\begin{aligned}
\|h(\sqrt{-\Delta})f\|_{L^2(X)} &\lesssim \left(\sum_n |h(\lambda_n)|^q n^{2(d-1)(1-1/p)+(d-2)} \right)^{1/q} \|f\|_{L^p(X)} \\
&\lesssim \left(\sum_n |h(\lambda_n)|^q n^{(d-1)(2(1-1/p)+1)-1} \right)^{1/q} \|f\|_{L^p(X)}.
\end{aligned}
$$

In particular, we have that

$$
\|h\|_{M^{p,2}_{\mathrm{Dil}}(X)} \lesssim \sup_R \left(\sum_\lambda \sum_n |h(\lambda_n/R)|^q n^{(d-1)(2(1-1/p)+1)-1} \right)^{1/q}.
$$

TODO: Determine if we can do something more optimal here. TODO: In light of Sogge's bounds, there's probably a counterexample here.

The spaces $M^{1,q}_{\mathrm{Dil}}(X)$ are a little more tricky, since we do not have a precise theory of the Fourier transform in the general setting of Riemannian manifolds.

To take a look at these bounds, we recall that $L^1 \to L^q$ bounds of an operator are characterized by Schur's test. If $\{e_n\}$ is a $C^\infty(X)$ basis of eigenfunctions on X, with $\Delta e_n = -\lambda_n^2 e_n$, then

$$h(\sqrt{-\Delta})f(x) = \sum h(\lambda_n)\langle f, e_n\rangle e_n(x) = \int \left(\sum_n h(\lambda_n) e_n(x)\overline{e_n(y)} \right) f(y)\, dy.$$

Thus the kernel of $h(\sqrt{-\Delta})$ is precisely $K(x,y) = \sum_n h(\lambda_n)e_n(x)\overline{e_n(y)}$, and we conclude by Schur's test that

$$\|h\|_{M^{1,q}(X)} = \left\| \sum_n h(\lambda_n) e_n(x)\overline{e_n(y)} \right\|_{L^\infty_y L^q_x}.$$

In the case $X = \mathbb{R}^n$, the analogous kernel is

$$K(x,y) = \int_{\mathbb{R}^d} h(|\xi|) e^{2\pi i \xi \cdot x}\overline{e^{2\pi i \xi \cdot y}},$$

which can be explicitly reduced to $K(x,y) = \mathcal{B}_d h(|x-y|)$, and the condition of being contained in $M^{1,q}(\mathbb{R}^d)$ then becomes that

$$\left(\int r^{d-1} |\mathcal{B}_d h(r)|^q \, dr \right)^{1/q} < \infty.$$

If h is compactly supported, and $1 < q < 2$, then we have seen that this condition is equivalent to the condition that

$$\left(\int \langle t \rangle^{(d-1)(1-q/2)} |\widehat{h}(t)|^q \, dt \right)^{1/q} < \infty.$$

In the general setting we do not have quite as nice a formula, but we can still *force* the Fourier transform into the equation to see if it can be used to understand these quantities (which will be necessary for studying the radial multiplier conjecture).

There are two approaches here, the first approach is to write

$$\begin{aligned}
K(x,y) &= \sum_n h(\lambda_n) e_n(x)\overline{e_n(y)} \\
&= \sum_n \left(\int \widehat{h}(t) e^{2\pi i t\lambda_n} e_n(x)\overline{e_n(y)}\ dt \right) \\
&= \int \widehat{h}(t) \left(\sum_n e^{2\pi i t\lambda_n} e_n(x)\overline{e_n(y)} \right)\ dt \\
&= \int \widehat{h}(t) HW_t(x,y)\ dt,
\end{aligned}$$

where $HW_t(x,y) = \sum_n e^{2\pi i t\lambda_n} e_n(x)\overline{e_n(y)}$ is the kernel of the *half-wave propogator* $e^{2\pi i t\sqrt{-\Delta}}$ on X. The connection between radial multipliers on X and the Fourier transform of their symbol is therefore closely related to the study of the solutions to the half-wave equation $\partial_t = \sqrt{-\Delta}$ on X. Alternatively, we can employ the cosine transform, since h is assumed to be extended to an even function, and write

$$\begin{aligned}
K(x,y) &= \sum_n h(\lambda_n) e_n(x)\overline{e_n(y)} \\
&= \sum_n \left(2\int_0^\infty \widehat{h}(t)\cos(2\pi t\lambda_n) e_n(x)\overline{e_n(y)}\ dt \right) \\
&= 2\int_0^\infty \widehat{h}(t) \left(\sum_n \cos(2\pi t\lambda_n) e_n(x)\overline{e_n(y)} \right)\ dt \\
&= 2\int_0^\infty \widehat{h}(t) W_t(x,y)\ dt,
\end{aligned}$$

where $W_t(x,y)$ is the kernel of the *wave propogator* $\cos(2\pi t\sqrt{-\Delta})$ on X. Thus the connenction of radial multipliers on X and their Fourier transform is related to the study of the solutions to the wave equation $\partial_t^2 = \Delta$. The half-wave equation and the wave equation are certainly connected, but the latter has the advantage of finite speed of propogation.

An analogous characterization of the form

$$\|h\|_{M^{1,q}(X)} \lesssim \left(\int \langle t\rangle^{(d-1)(1-q/2)} |\widehat{h}(t)|^q\ dt \right)^{1/q}$$

would follow if and only if we could prove a general inequality of the form

$$\left\| \int a(t) W_t(x,y)\, dt \right\|_{L^\infty_y L^q_x} \lesssim \left(\int \langle t \rangle^{(d-1)(1-q/2)} |a(t)|^q\, dt \right)^{1/q}$$

to hold. By interpolation, it would suffice to prove that

$$\left\| \int a(t) W_t(x,y)\, dt \right\|_{L^\infty_y L^{1,\infty}_x} \lesssim \int \langle t \rangle^{\frac{d-1}{2}} |a(t)|\, dt$$

and

$$\left\| \int a(t) W_t(x,y)\, dt \right\|_{L^\infty_y L^{2,\infty}_x} \lesssim \left(\int |a(t)|^2\, dt \right)^{1/2}$$

By finite speed of propogation, we may replace this over an integral of x over a ball of radius t about y. This integral does become singular as we approach x near the boundary of this ball, i.e. it blows up like $(t^2 - d(x,y)^2)^{-\frac{d-1}{2}}$. This is non-integral, unless $d = 2$, in which case the integral scales like $\theta(t)$. Thus the L^1 integral becomes

$$\begin{aligned} &\int_0^t r^{d-1} (t^2 - r^2)^{-\frac{d-1}{2}}\, dr \\ &\quad = t \int_0^{\pi/2} \sin(\theta)^{d-1} \cos(\theta)^{2-d}\, d\theta. \end{aligned}$$

One way to interpret $\int a(t) W_t(x,y)\, dt\, dx$ is as th

TODO: What techniques can we use to obtain this bound? TODO: Can we come up with a proof of this bound in the model case $X = \mathbb{R}^d$, i.e. an alternate proof of the characterization of $L^1 \to L^q$ boundedness? TODO: Can we apply the theory of fourier integral operators here?

Directly translating the assumptions of the radial multiplier conjecture to this setting yields the following statement: If $h : [0,\infty) \to \mathbb{R}$ is a function supported at the unit scale, and we define

$$C_q(h) = \left(\int |\widehat{h}(s)|^q \langle s \rangle^{(d-1)(1-q/2)}\, ds \right)^{1/q},$$

then for what values of p and q is is true that the inequality

$$\|h\|_{M^{p,q}_{\mathrm{Dil}}(X)} \lesssim C_q(h)$$

holds. Mitjagin's result implies that we require $1 < p < 2d/(d+1)$ and $p \leqslant q < 2$, and we conjecture that, perhaps under appropriate assumptions on X, we can achieve similar ranges of exponents as have been obtained for the Euclidean radial multiplier conjecture.

On general compact manifolds, there are difficulties arising from a generalization of the radial multiplier conjecture, connected to the fact that analogues of the Kakeya / Nikodym conjecture are false in this general setting [16]. But these problems do not arise for constant curvature manifolds, like the sphere. The sphere also has over special properties which make it especially amenable to analysis, such as the fact that solutions to the wave equation on spheres are periodic. Best of all, there are already results which achieve the analogue of [5] on the sphere. Thus it seems reasonable that current research techniques can obtain interesting results for radial multipliers on the sphere, at least in the ranges established in [9] or even those results in [3].

2.8 Summary

In conclusion, the results of [9] and [3] indicate three lines of questioning about radial Fourier multiplier operators, which current research techniques place us in reach of resolving. The first question is whether we can extend the range of exponents upon which the conjecture of [5] is true, at least in the case $d = 2$ where Bochner-Riesz has been solved. The second is whether we can use more sophisticated arguments to prove the L^p sum bounds obtained in [3] when $d = 3$ when the sums are no longer Cartesian products, thus obtaining strong L^p characterizations in this settiong, as well as new results about the sharp local smoothing conjecture. The third question is whether we can generalize these bounds obtained in these two papers to study radial Fourier multipliers on the sphere.

Chapter 3

Local Symplectic Geometry

There is an inherent symplectic structure to the study of oscillatory integrals. Indeed, given any distribution expressed by an oscillatory integral of the form

$$u(x) = \int a(x,\theta) e^{2\pi i \phi(x,\theta)},$$

and under the assumption that ϕ is *non-degenerate*, i.e. it is homogeneous of degree one in the θ variable and satisfies the property that

$$D_{x,\theta}(\nabla_\theta \phi) \text{ has full rank whenever } \nabla_\theta \phi = 0,$$

then the set

$$\{(x, \nabla_x \phi) : \nabla_\theta \phi = 0\}$$

is a *Lagrangian submanifold* of the symplectic manifold $\mathbb{R}^n \times \mathbb{R}^n$. Since we deal with oscillatory integrals localized in space, we will profit from a more in depth understanding of the *local* theory of symplectic geometry.

A *symplectic vector space* V is a finite dimensional vector space equipped with a non-degenerate, skew-symmetric bilinear form ω, which we call a *symplectic form*. Here are two useful examples:

- For any vector space W, the vector space $V = W \oplus W^*$ is naturally a symplectic vector space, equipped with the symplectic form

  $$\omega((x_1,\xi_1),(x_2,\xi_2)) = \langle \xi_2, x_1 \rangle - \langle \xi_1, x_2 \rangle.$$

  In particular, for any n, $\mathbb{R}^n \times \mathbb{R}^n$ is naturally a symplectic manifold.

- Let $V = \mathbb{C}^n$, viewed as a real-vector space of dimension $2n$. We let

$$\omega(z, w) = -\text{Im}\,(z \cdot \overline{w})\,.$$

  Then V is a symplectic vector space.

Remark. Symplectic geometry original arose in the study of physical problems, and so it is helpful to use physical intuition in the study of symplectic manifolds. Consider a force field $F : \mathbb{R}^3 \to \mathbb{R}^3$ in space, with $F(x) = v$ for all $x \in \mathbb{R}^3$, and some fixed vector $v \in \mathbb{R}^3$. Then F is conservative, and so each particle with unit mass, at a point $x \in \mathbb{R}^3$ and having momentum $\xi \in \mathbb{R}^3$, has a well defined energy given by the function

$$E(x, \xi) = \frac{\xi^2}{2} - v \cdot x,$$

the sum of kinetic and potential energies. If we consider a small pertubation $(dx, d\xi)$ of the position and momentum of a particle, then, to first order, the energy will change by

$$\frac{\partial E}{\partial x}\, dx + \frac{\partial E}{\partial \xi}\, d\xi = -v\, dx + \xi\, d\xi.$$

Notice this quantity is $\omega((\xi, v), (dx, d\xi))$. Thus the bilinear function ω measures the amount of energy required to change the location of a particle in position-momentum space by $(dx, d\xi)$, such that the physical system wants to travel in the direction (ξ, v). Under this interpretation, it is completely natural to see that ω is skew-adjoint, because of the principle of conservation of energy. This intuition continues to hold if F is a general conservative vector field. We will return to this discussion using a more differential geometric language later on in this chapter.

Spectral arguments can be used to show every symplectic vector space V is even dimensional, and in any such space, there is a pair of linearly independent sets $\{e_i\}$ and $\{f^i\}$, together forming a basis of V, such that $\omega(e_i, e_j) = \omega(f^i, f^j) = 0$, and $\omega(e_i, f^i) = 1$. A basis of the form above is called a *Darboux basis*, and we conclude from it's existence that all symplectic vector spaces of the same dimension are isomorphic.

Let W be a subspace of a symplectic vector space (V, ω). The non-degeneracy of ω implies that $\dim(W) + \dim(W^\perp) = \dim(V)$ and $W^{\perp\perp} = W$. We say that:

- W is *isotropic* if $W \subset W^\perp$.

- W is *coisotropic*, or *involutive* if $W^\perp \subset W$.

- W is *Lagrangian* if $W = W^\perp$.
- W is *symplectic* if $W \cap W^\perp = \{0\}$.

Note that if $\dim(V) = 2n$, then the equation above implies that isotropic subspaces have dimension at most n, coisotropic subspaces have dimension at least n, and Lagrangian subspaces must have dimension n.

Example. *Let $A : W \to W^*$ be a linear map. Then the graph $\Gamma(A)$ of A forms a Lagrangian subspace of $W \oplus W^*$ if and only if A is self-adjoint. Indeed,* $\dim(\Gamma(A)) = \dim(W)$*, and we have $W \subset W^\perp$ if and only if for all $w_1, w_2 \in W$,*

$$\langle Aw_2, w_1 \rangle - \langle w_2, Aw_1 \rangle = 0,$$

which implies A is self-adjoint.

Example. *For any $\theta \in \mathbb{R}$, the space $e^{i\theta}\,\mathbb{R}^n$ is a Lagrangian subspace of $\mathbb{C}^n$. Indeed, we calculate that for any $x_1, x_2 \in \mathbb{R}^n$,*

$$\omega(e^{i\theta}x_1, e^{i\theta}x_2) = -Im(x_1 \cdot x_2) = 0.$$

The fact that $\dim(e^{i\theta}\,\mathbb{R}^d) = n$ *implies the subspace is Lagrangian.*

Lemma 3.1. *If W_1 and W_2 are isotropic subspaces of a symplectic vector space V, and $W_1 \cap W_2 = \{0\}$, then there exists two Lagrangian subspaces L_1 and L_2 of V, containing W_1 and W_2 respectively, such that $L_1 \cap L_2 = \{0\}$.*

Proof. Let V be $2n$ dimensional. Consider a *maximal* pair of isotropic subspaces W_1^* and W_2^*, which are disjoint from one another, and contain W_1 and W_2 respectively. If one of these spaces is not Lagrangian, swapping the order of spaces if necessary, we may assume that $\dim(W_1^*) < n$. Find $x_0 \in [(W_1)^*]^{\perp\perp} - W_1^*$. Then $W_1 \oplus \mathbb{R}\,x_0$ is also isotropic, which gives a contradiction. ◻

The following Lemma is a coordinatized version of the previous Lemma.

Lemma 3.2. *Let V be a symplectic vector space, and let*

$$\{e_1, \dots, e_{n_1}\} \cup \{f^1, \dots, f^{n_2}\}$$

be linearly independent vectors such that $\omega(e_i, e_j) = \omega(f^i, f^j) = 0$, and $\omega(e_i, f^j) = \delta(i, j)$. Then we can extend these sets to a full Darboux basis for V.

Proof. Let W_1 denote the linear space of $\{e_1, \ldots, e_{n_1}\}$, and let W_2 denote the space of $\{f^1, \ldots, f^{n_2}\}$. Then W_1 and W_2 are isotropic, and $W_1 \cap W_2 = \{0\}$. We can therefore find transverse Lagrangian subspaces L_1 and L_2 with $W_1 \subset L_1$ and $W_2 \subset L_2$. By identifying L_1 with the dual of L_2, we can find $\{e_{n_1+1}, \ldots, e_n\}$, and $\{f^{n_2}, \ldots, f^n\}$, such that the two bases are dual to one another. This is the required Darboux basis. $\square$

Lemma 3.3. *If V_0 and V_1 are Lagrangian subspaces of a symplectic vector space V, then we can find a third Lagrangian subspace V_2 which is transverse to both V_0 and V_1.*

Proof. Define $W_1 = L_1$, and choose W_2 such that $L_2 = (L_1 \cap L_2) \oplus W_2$. Then W_1 and W_2 are disjoint, and isotropic, and so the previous Lemma implies we can find a Lagrangian subspace L_2' containing W_2, and transverse to L_1. If we now identify V with $\mathbb{C}^n$, in such a way that $L_1 = \mathbb{R}^n$, and $L_2' = i\,\mathbb{R}^n$, then defining $L_3 = e^{i\pi/4} L_2$ gives the required Lagrangian subspace. $\square$

3.1 Symplectic Maps

A linear map $T : V \to W$ between two symplectic vector spaces is called *symplectic* if it preserves the symplectic form on V. We will denote the set of all symplectic isomorphisms (symplectomorphisms) by $\mathrm{Sp}(V)$, and we call this set the *symplectic group*. If $\dim(V) = 2n$, then the group is a connected, non-compact Lie group of dimension $2n^2 + n$. The symplectic group $\mathrm{Sp}(\mathbb{R}^{2n})$ is denoted by $\mathrm{Sp}(n)$, and can be identified with the family of $2n \times 2n$ matrices which satisfy $M^t \Omega M = \Omega$, where Ω is the $2n \times 2n$ matrix

$$\begin{pmatrix} 0 & +I \\ -I & 0 \end{pmatrix}.$$

We note that $\Omega^2 = -I$.

Given any linear transformation $T : V \to W$ between two symplectic vector spaces, we can define a symplectic-adjoint $T^\dagger : W \to V$ by the equation

$$\omega(Tv, w) = \omega(v, T^\dagger w).$$

A symplectomorphism is then precisely a map T such that $T^\dagger = T^{-1}$. On $\mathbb{R}^{2n}$, we calculate that

$$\omega(Tv, w) = v^t T^t \Omega w = v^t \Omega(-\Omega T^t \Omega w) = \omega(v, -\Omega T^t \Omega w),$$

and thus $T^\dagger = -\Omega T^t \Omega$. Since $\det(\Omega) = 1$, we conclude that $\det(T^\dagger) = \det(T)$ for any linear endomorphism $T : V \to V$. In particular, if T is a symplectomorphism, then $T^\dagger = T^{-1}$, and thus $\det(T) = \pm 1$. In particular, if we write

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for $n \times n$ square matrices A, B, C, and D, then

$$T^\dagger = \begin{pmatrix} +D^t & -B^t \\ -C^t & +A^t \end{pmatrix}.$$

Thus T is symplectic precisely when $A^t D - B^t C = I$, and that $A^t C$, $D^t B$, $B^t A$, and $C^t D$ are all symmetric matrices. If T is within a neighborhood of the identity, then we can write $B = S^t A^{-1}$ and $C = (A^t)^{-1} L$ for two symmetric matrices S and T, and then $D = A^{-1}(I - (A^t)^{-1} S T A^{-1})$. Thus symplectic matrices in a neighborhood of the identity can be expressed uniquely as a matrix $A \in GL(n)$, and two symmetric matrices S and L, each having $n(n+1)/2$ independent coordinates. This immediately tells us the dimension of $\mathrm{Sp}(n)$, i.e. $2n^2 + n$.

Now consider a symplectic matrix M. Noting that

$$\Omega M = (M^t)^{-1} \Omega \quad \text{and} \quad \Omega M^t = M^{-1} \Omega,$$

we calculate that

$$(M^t M) \Omega (M^t M) = M^t M M^{-1} (M^t)^{-1} \Omega = \Omega.$$

If we set $P = (M^t M)^{1/2}$, then we conclude that for all k,

$$P^{2k} \Omega P^{2k} = \Omega.$$

But, taking power series, this means that for *any* analytic function f defined in a neighborhood of $\sigma(P) \subset (0, \infty)$,

$$f(P^2) \Omega f(P^2) = \Omega.$$

But taking $f(t) = t^{\tau/2}$ gives that

$$P^\tau \Omega P^\tau = \Omega.$$

Since P is positive definite, we can write $P = e^L$ for some symmetric matrix L. But then plugging this into the equation above, and differentiating in τ as $\tau \to 0$ yields that $L\Omega + \Omega L = 0$. Applying the polar decomposition theorem, there exists a unique unitary matrix U such that $M = PU$. The formula above for $\tau = 1$ implies that P is symplectic, and since $\mathrm{Sp}(n)$ is a group, U is also symplectic and unitary. Now since $L\Omega + \Omega L = 0$, we can write

$$L = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix},$$

where A and B are symmetric. But

$$\begin{pmatrix} +A & B \\ B & -A \end{pmatrix} = \begin{pmatrix} A & -B \\ +B & A \end{pmatrix} \cdot \Omega.$$

The matrix S on the right hand side is a *complex* linear symmetric matrix, if we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$. Thus we conclude that we can write *any* element of $\mathrm{Sp}(n)$ uniquely as $e^{S\Omega}U$, where U is a unitary transformation on $\mathbb{C}^n$, and S is a complex linear symmetric map on $\mathbb{C}^n$. In particular, we conclude that $\mathrm{SP}(n)$ is diffeomorphic to $\mathbb{C}^{n(n+1)/2} \times U(n)$.

3.2 Linear Canonical Relations

If V_X and V_Y are symplectic vector spaces, then $V_X \oplus V_Y$ can be made into a symplectic vector space, if we equip it either with with the symplectic form $\omega_X - \omega_Y$, or with $\omega_Y - \omega_X$. In either case, we call a Lagrangian subspace C of $V_X \oplus V_Y$ a *linear canonical relation*. We recall that a *symplectic linear map* $A : V_1 \to V_2$ between symplectic vector spaces is a map preserving the symplectic form.

Lemma 3.4. *Let $C \subset V_X \oplus V_Y$ be a linear canonical relation. Then we can find orthogonal decompositions $V_X = V_{X,1} \oplus V_{X,2}$ and $V_Y = V_{Y,1} \oplus V_{Y,2}$ such that*

$$C = C_X \oplus \Gamma \oplus C_Y,$$

where C_X is a Lagrangian submanifold of $V_{X,1}$, C_Y is a Lagrangian submanifold of $V_{Y,1}$, and Γ is the graph of a symplectic isomorphism $A : V_{X,2} \to V_{Y,2}$.

Proof. Let

$$C_X = \{x \in V_X : (x,0) \in C\} \quad \text{and} \quad C_Y = \{y \in V_Y : (0,y) \in C\}.$$

Because C is Lagrangian, C is contained in $C_X^\perp \oplus C_Y^\perp$. In other words, if $x_1 \in C_X$, $y_1 \in C_Y$, and $(x_2, y_2) \in C$, then

$$\omega(x_1, x_2) = \omega(y_1, y_2) = 0.$$

In particular, $C_X \subset C_X^\perp$ and $C_Y \subset C_Y^\perp$. Find orthogonal $V_{X,2}$ and $V_{Y,2}$ such that

$$C_X^\perp = C_X \oplus V_{X,2} \quad \text{and} \quad C_Y^\perp = C_Y \oplus V_{Y,2}.$$

Then $C \subset C_X \oplus C_Y \oplus (V_{X,2} \oplus V_{Y,2})$. We can thus find $\Gamma \subset V_{X,2} \oplus V_{Y,2}$ such that $C = C_X \oplus C_Y \oplus \Gamma$. We claim Γ projects bijectively onto $V_{X,2}$ and $V_{Y,2}$. For instance, suppose $x \in V_{X,2}$ and $y_1, y_2 \in V_{Y,2}$ are such that (x, y_1) and (x, y_2) lie in Γ. Then $(0, y_1 - y_2)$ lies on Γ and on C_Y, so by orthogonality, $y_1 = y_2$. Thus Γ is the graph of a symplectic isomorphism $A : V_{X,2} \to V_{Y,2}$. To define $V_{X,1}$ and $V_{Y,1}$, let

$$\dim(V_X) = 2n \quad \dim(V_Y) = 2m \quad \dim(C_X) = a \quad \dim(C_Y) = b.$$

We can then consider Darboux bases

$$\{e_{X,1}, \ldots, e_{X,n}\} \cup \{f_{X,1}, \ldots, f_{X,m}\}$$

and

$$\{e_{Y,1}, \ldots, e_{Y,m}\} \cup \{f_{Y,1}, \ldots, f_{Y,m}\}$$

for V_X and V_Y, such that

$$C_X = \operatorname{span}(\{e_{X,1}, \ldots, e_{X,a}\})$$

and

$$C_Y = \operatorname{span}(\{e_{Y,1}, \ldots, e_{Y,b}\}).$$

But then we immediately see that

$$V_{X,2} = \operatorname{span}(\{e_{X,a+1}, \ldots, e_{X,n}\} \cup \{f_{X,a+1}, \ldots, f_{X,n}\})$$

and

$$V_{Y,2} = \operatorname{span}(\{e_{Y,b+1}, \ldots, e_{Y,m}\} \cup \{f_{Y,a+1}, \ldots, f_{Y,m}\}).$$

We then simply define

$$V_{X,1} = \operatorname{span}(\{f_{X,1}, \ldots, f_{X,a}\})$$

and

$$V_{Y,1} = \operatorname{span}(\{f_{Y,1}, \ldots, f_{Y,b}\}),$$

and the remaining parts of the proof follow immediately. $\square$

3.3 Symplectic Manifolds

A *symplectic manifold* M is a manifold equipped with a symplectic form ω, i.e. a closed two form which gives each of the tangent spaces of M a symplectic structure. The basic example here is $M = T^*X$, where X is any smooth manifold; the natural two form here is

$$\omega = dx \wedge d\xi = d\theta,$$

where $\theta = \sum \xi_i dx^i$ is the *tautological one form* on T^*M. It has the property that for any section $s : M \to T^*M$, we have $s^*\theta = s$.

Example. *For a manifold M, the main example of a symplectic manifold, for our purposes, will be the manifold T^*M, which is equipped with a two form ω given by identifying $T(T^*M)$ with $TM \oplus T^*M$, i.e. in coordinates (x, ξ) for T^*M, induced from a coordinate system x on M, we have*

$$\omega = \sum d\xi_i \wedge dx^i.$$

The form is closed, because it is exact, i.e. $\omega = d\theta$, where $\theta = \sum \xi_i dx^i$ is the fundamental symplectic one form.

Like with the Riemannian form on a Riemannian manifold, the symplectic form on a symplectic manifold gives a natural bundle isomorphism $J : T^*M \to TM$. In particular, given a function $H : M \to \mathbb{R}$, we can define the *symplectic gradient* $\nabla_\Xi H$ as the vector field which is identified under the bundle isomorphism with the covector field dH. As an example, if $M = T^*X$, then in local coordinates (x, ξ) on T^*X, the identification is given by

$$J(d\xi^i) = \frac{\partial}{\partial x^i} \quad \text{and} \quad J\left(dx^i\right) = -\frac{\partial}{\partial \xi^i}.$$

Thus for a function $H : T^*X \to \mathbb{R}$, we have

$$\nabla_\Xi H = \sum \frac{\partial H}{\partial \xi^i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial \xi^i}.$$

Often this gradient is often just denoted by X_H. This gradient is closely related to the theory of Hamiltonian vector fields, i.e. since, given a Hamiltonian H on some phase space representing a physical system, X_H gives the Hamiltonian flow of that physical system. We can define the *Poisson bracket* of two functions $H : M \to \mathbb{R}$ and $I : M \to \mathbb{R}$ by setting

$$\{H, I\} = \nabla_{\Xi} H(I) = \omega(dH, dI) = [X_H, X_I].$$

For $M = T^*X$, we can write this in coordinates as

$$\{H, I\} = \sum \frac{\partial H}{\partial \xi_i} \frac{\partial I}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial I}{\partial \xi_i},$$

which agrees with the classical Poisson bracket. Equipped with the Poisson bracket, the space $C^\infty(M)$ becomes a Lie algebra. We have an exact sequence

$$0 \to H^0(M) \to C^\infty(M) \to \mathbb{H}(M) \to 0,$$

where $\mathbb{H}(M)$ is the space of all *Hamiltonian vector fields* on M, i.e. because $\nabla_{\Xi} f = 0$ if and only if $df = 0$, which implies that f lies in $H^0(M)$.

A *symplectic* map $F : M_1 \to M_2$ between two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is one which preserves the symplectic form on tangent spaces, i.e. such that $F^*\omega_2 = \omega_1$. A *Darboux chart* on an open set U is a symplectic diffeomorphism $U \to \mathbb{R}^{2n}$. The coordinates $\{e_i\} \cup \{f^j\}$ are called *Darboux coordinates* on the manifold.We shall prove that Darboux charts cover any symplectic manifold. Thus local symplectic geometry differs drastically from local Riemannian geometry; any two symplectic manifolds of the same dimension are locally symplectomorphic, whereas that is certainly not true of two Riemannian manifolds. We will actually now prove a stronger extension result, due to A. Weinstein.

Theorem 3.5. *Let X be a C^∞ manifold, Y a closed submanifold, and let ω_1 and ω_2 be two symplectic forms on X, which agree on the tangent spaces of Y. Then there exists two open neighborhoods U_1 and U_2 of Y and a symplectomorphism $\Phi : (U_1, \omega_1) \to (U_2, \omega_2)$.*

Proof. The form $\omega_2 - \omega_1$ vanishes on Y. Thus we can find a tubular neighborhood U of Y, and a one-form α on U_1, such that $d\alpha = \omega_2 - \omega_1$, and such that both α and it's first derivatives vanish on Y. Set

$$\omega_t = \omega_0 + t(\omega_1 - \omega_0).$$

Then ω_t is skew-symmetric, but not necessarily non-degenerate for all t. But we can make it so by shrinking the tubular neighborhood U_1, since $\omega_1 - \omega_0$ vanishes on U_1. Now let X_t be the vector field identified under ω_t with α. Let us integrate this vector field, constructing maps $\{\phi_t(s)\}$ such that

$$\frac{d\phi_t}{ds} = X_t \circ \phi_t.$$

Set $\Phi_t(x) = \phi_t(x, t)$. Note that X_t, and it's first derivatives, all vanish on Y. It follows that, thinning U_1 again, we may assume that U_1 is mapped diffeomorphically into itself by Φ_t for all $t \in [0, 1]$. Let U_2 be the image of U_1 under Φ_1. We claim that

$$\Phi_t^*(\omega_t) = \omega_0.$$

Indeed, the product rule implies that

$$\frac{\partial}{\partial t}\{\Phi_t^*(\omega_t)\} = \Phi_t^* \left\{ \frac{d\omega_t}{dt} + L_{X_t}(\omega_t) \right\}.$$

But Cartan's formula implies that, since $d\omega_t = 0$,

$$L_{X_t}(\omega_t) = i_{X_t} d\omega_t + d(i_{X_t}\omega_t) = d(i_{X_t}\omega_t),$$

and $i_{X_t}\omega_t(Y) = \omega_t(X_t, Y) = \alpha(Y)$, so $i_{X_t}\omega_t = \alpha$. But this means that

$$L_{X_t}\omega_t = d\alpha,$$

and so since $d\omega_t/dt = \omega_1 - \omega_0$,

$$\Phi_t^* \left\{ \frac{d\omega_t}{dt} + L_{X_t}(\omega_t) \right\} = \Phi_t^* \{(\omega_1 - \omega_0) + d\alpha\} = \Phi_t^*(0) = 0.$$

Thus we conclude that $\Phi_t^*(\omega_t)$ is constant, and thus equal to ω_0 since Φ_0 is the identity. But this means that $\Phi_1^*(\omega_1) = \omega_0$, which verifies the claim. □

We also have an extension theorem for Darboux coordinates.

Theorem 3.6. *Let X be a manifold, fix $x \in X$, and a neighborhood U containing x, and let $\{e_1, \dots, e_r\}$ and $\{f^1, \dots, f^s\}$ be C^∞ functions defined in a neighborhood of x, such that x lies on the common zero set of all these functions, the differentials $\{de_i\} \cup \{df^j\}$ are linearly independent at x, and we have the Poisson bracket relations*

$$\{e_i, f^j\} = 0, \quad \{e_i, e_j\} = \delta_{ij}, \quad \textit{and} \quad \{f^i, f^j\} = \delta_{ij}$$

on a neighborhood of x. Then these functions extend to a full Darboux coordinate system on a neighborhood of x.

Proof. Shrink U small enough that it is contained in a coordinate system extending the functions $\{e_i\} \cup \{f^j\}$. Let $\{x_{s+1}, \dots, x_n\} \cup \{y_{r+1}, \dots, y_n\}$ denote the extra coordinates needed to form this coordinate system. Suppose first that $r < s$. For any manifold M which is transverse to the vector fields $\{\nabla_\Xi e_i\} \cup \{\nabla_\Xi f_j\}$, by the Frobenius integrability theorem, we can foliate the manifold locally by manifolds upon which the functions $\{e_i\}$ and $\{f^j\}$ are constant. In particular, this implies we can guarantee the existence of a smooth function u such that on M, u agrees with x_{r+1} on M, but satisfies $X_{e_i} u = \{e_i, u\} = 0$ for $1 \leqslant i \leqslant r$, and $X_{f_j} u = \{f_j, u\} = 0$ for $1 \leqslant j \leqslant s$, *except* for $j = r + 1$, in which case $X_{f_{r+1}} u = \{f_{r+1}, u\} = -1$. Set $e_{r+1} = u$. Then the Darboux basis equations continue to hold on this extension. Iterating this procedure if necessary, swapping $\{e_i\}$ and $\{f^j\}$, we may assume that $r = s$. But then the common zero set of these functions forms a symplectic manifold, and so we may take a Darboux coordinate system on this symplectic manifold, which together form a Darboux coordinate system for the whole manifold. □

The $2n$-form on X given by

$$V = \frac{(-1)^{[n/2]}}{n!} \omega \wedge \cdots \wedge \omega$$

is called the *canonical volume*, or *Pfaffian* on X. In symplectic coordinates, we have

$$V = de^1 \wedge \cdots \wedge de^n \wedge df_1 \wedge \cdots \wedge df_n,$$

The canonical volume is certainly invariant under symplectomorphisms, including the flow of Hamiltonian vector fields. We can pullback this form to get a volume form on any submanifold of X. In particular, if Y is symplectic, this is just the Pfaffian on Y. Since V is non-vanishing, we conclude that *any symplectic manifold is orientable*.

The concepts of *isotropic*, *coisotropic*, *Lagrangian*, and *symplectic* submanifolds are analogous to the linear concepts introduced at the beginning of the chapter (i.e. a submanifold such that the tangent space at each point fits these properties).

Given a symplectic manifold (X^{2n}, ω), we often specify a submanifold Y of X by fixing m functions $H_1, \dots, H_m$, such that the differentials $dH_1, \dots, dH_m$ are linearly independent on their common zero set $Z(H_1, \dots, H_m)$. Then

$$Y = \{x : H_1(x) = \cdots = H_m(x) = 0\}.$$

is a submanifold of X with codimension m. Moreover, every submanifold of codimension m can be given locally in this form. We are interested on conditions on the functions $\{H_1, \ldots, H_m\}$ which guarantee that Y fits the properties above.

Theorem 3.7. *The manifold Y is a symplectic submanifold if and only if the $m \times m$ matrix with coefficients given by the Poisson bracket $\{H_i, H_j\}$ is nonsingular.*

Proof. The manifold Y is a symplectic submanifold if and only if ω, restricted to Y, is nondegenerate. For each $y_0 \in Y$, the set of values orthogonal to the space $T_{y_0}Y$, viewed as a subspace of $T_{y_0}X$, and with respect to the bilinear form ω, is *precisely* the family of Hamiltonian vector fields $\{\nabla_\Xi H_j\}$. The space $T_{y_0}Y$ is symplectic if and only if the vector space spanned by $\{\nabla_\Xi H_j\}$ is symplectic, and this holds precisely when the matrix with coefficients given by the Poisson brackets $\{H_i, H_j\}$ form an invertible matrix (so that the form ω is nondegenerate here). $\square$

Remark. The result above can thus only hold if m is even. Given the assumption that Y is symplectic, let $\dim(Y) = 2r$. The nonsingularity property above implies that if Y is symplectic, then we can find smooth functions $\{a_{ij} : 1 \leqslant i \leqslant r\}$ and $\{b_{ij} : 1 \leqslant i \leqslant r\}$ such that if we set

$$e_i = \sum a_{ij} H_j \quad \text{and} \quad f^i = \sum b_{ij} H_j,$$

then $\{e_i\}$ and $\{f^i\}$ form a partial Darboux coordinate system. If we extend this to a full Darboux coordinate system, the remaining coordinates would form a Darboux coordinate system on Y.

Example. *If $H, I \in C^\infty(X)$ and $\{H, I\} \neq 0$ on the zero set $Z(H, I)$, then $Z(H, I)$ is a symplectic manifold of codimension two.*

Next, we move on to the problem of determining when Y is coisotropic.

Theorem 3.8. *The manifold Y is coisotropic if and only if $\{H_j, H_k\} = 0$ on Y, for all j and k. It then follows that we can find functions $\{c_{jkl}\}$ in $C^\infty(Y)$ such that, on Y,*

$$[\nabla_\Xi H_j, \nabla_\Xi H_k] = \sum c_{jkl} \nabla_\Xi H_l,$$

i.e. so the distribution defined locally by the functions $\{H_j\}$ and $\{H_k\}$ is integrable.

Proof. Fix $y_0 \in Y$. If $T_{y_0}Y$ is an coisotropic subspace of $T_{y_0}X$, then it's orthogonal complement V_{y_0} with respect to ω is an isotropic subspace of $T_{y_0}X$. This orthogonal complement is spanned by the vectors $(\nabla_\Xi H_j)(y_0)$, and thus this space is isotropic if and only if

$$\omega(\nabla_\Xi H_j(y_0), \nabla_\Xi H_k(y_0)) = \{H_j, H_k\}(y_0) = 0.$$

This proves the first part of the claim. The second part follows immediately from a calculation. □

Suppose Y is coisotropic, and $2n - m$ dimensional for some $m \leqslant n$. The theorem above implies that the m dimensional bundle $(TY)^\perp$ is an integrable distribution on Y, so we can use the Frobenius theorem to foliate Y by m dimensional submanifolds with $(TY)^\perp$ as their tangent spaces. These manifolds are called the *characteristics* of Y, and each is a isotropic submanifold of X, because $(TY)^\perp$ is contained in $(TY)^{\perp\perp} = TY$.

Example. *If $H \in C^\infty(X)$, and dH is non-vanishing on $Z(H)$, then $Z(H)$ is coisotropic, and it's characteristics are precisely the integral curves of the Hamiltonian vector field X_H.*

It is often the case that the family Y_0 of characteristics of Y can be given the structure of a smooth $2(n - m)$ dimensional manifold, such that the projection map $Y \to Y_0$ is a submersion. This is at least the case locally. It then follows that this projection induces a bijection of $TY/(TY)^\perp$ with TY_0, and (since $TY/(TY)^\perp$ is symplectic), this means that Y_0 is naturally a symplectic manifold.

Theorem 3.9. *A submanifold Y of X is coisotropic if and only if every point $y_0 \in Y$ is contained in a Lagrangian submanifold of X, which is also a submanifold of Y.*

Proof. Suppose Y is coisotropic. Consider the class Y_0 of characteristics of Y. Locally at least, we can consider Y_0 as a symplectic manifold. Given any $y_0 \in Y$, and let L_0 be a Lagrangian submanifold of Y_0 containing the characteristic of Y passing through y_0. Because the projection $Y \to Y_0$ is symplectic, the collection of points in L_0 forms an isotropic submanifold of X. But this collection of points is n dimensional, and thus a Lagrangian submanifold of X. Conversely, if this property holds, then Y is a union of Lagrangian submanifolds of X. But if L is a Lagrangian submanifold of Y, then

$$(TY)^\perp \subset (TL)^\perp = TL,$$

and so $(TY)^\perp \subset TY$. □

Theorem 3.10. *Suppose Y is an m dimensional isotropic submanifold of X. Then at every point $y_0 \in Y$, we can write Y locally as the intersection of a $2m$ dimensional symplectic submanifold of X, and a Lagrangian submanifold of X.*

Proof. Fix $y_0 \in Y$. Let S be any smooth submanifold, containing Y in a neighborhood of y_0, and such that $T_{y_0}S$ is symplectic. We then know that S is symplectic in a neighborhood of y_0. Then Y is a Lagrangian submanifold of S. By working in a Darboux coordinate system for S, extended to a Darboux coordinate system for X, we can find another symplectic manifold S' transverse to S. Taking a Lagrangian submanifold of S' containing Y yields the claim. □

A Lagrangian submanifold Y of X^{2n} is clearly just an n dimensional isotropic (or coisotropic) manifold. One simple way to check that the submanifold is Lagrangian is to check that $i^*\omega = 0$, where $i : Y \to X$ is the inclusion map.

Example. *Let M' be a smooth, k dimensional submanifold of an n dimensional manifold M. We define it's* conormal bundle*, N^*M', to be the the set of all cotangent vectors (x, ξ) in T^*M, with $x \in M'$ and such that ξ vanishes when restricted to M'. If we work locally in coordinates in which M' is a plane, then we immediately see that M' is Lagrangian.*

Example. *Most Lagrangian manifolds will arise from our study of oscillatory integrals in the following way. Let $\phi : M^d \times \mathbb{R}^N_\theta \to \mathbb{R}$ be a* non-degenerate phase function*, i.e. a smooth function homogeneous in the θ variable, and such that whenever $\nabla_\theta \phi = 0$, the N covectors $d_{x,\theta}(\partial^1_\theta \phi), \dots, d_{x,\theta}(\partial^N_\theta \phi)$ are linearly independent. It then follows that the set*

$$\Sigma_\phi = \{(x, \theta) : \nabla_\theta \phi = 0\}$$

is a conic d-dimensional submanifold of $M \times \mathbb{R}^N$, and the map $\Sigma_\phi \to T^*M$ given by $F(x, \theta) \mapsto (x, d_x\phi)$ is an immersion. The image is an (immersed) Lagrangian submanifold of T^*M. To see this, we note that $F^*(\theta) = d_x\phi$; since $d_\theta \phi = 0$ when restricted to Σ_ϕ, we conclude that, on Σ_ϕ, $d_x\phi = d_x\phi + d_\theta\phi = d_{x,\theta}\phi$ on Σ_ϕ. But this means that $d(d_x\phi) = 0$, which is sufficient to show that $F^*(d\omega) = d(F^*\theta) = d(d_x\phi) = 0$.

3.4 Existence of Phase Functions

We now specialize to the study of the symplectic manifold T^*M.

Theorem 3.11. *Fix smooth functions $p_1, \dots, p_m$ on T^*M, and suppose that the differentials $d_\xi p_1, \dots, d_\xi p_m$ are linearly independent on the set $Y = Z(p_1, \dots, p_m)$, i.e. so that Y is a submanifold of M with codimension m. Then Y is coisotropic if and only if locally, around each point (x_0, ξ^0) in Y, we can find an open set U of x_0 in M, and a real-valued smooth function ϕ defined on U such that $p_i(d\phi(x)) = 0$ for $1 \leqslant i \leqslant m$, and $d\phi(x_0) = \xi^0$.*

Proof. The condition implies Y is coisotropic. Indeed, the graph $d\phi : M \to T^*M$ is an immersed Lagrangian submanifold of T^*M. It is contained in Y by assumption, and passes through (x_0, ξ^0). But we then know that Y is coisotropic, since it is a union of Lagrangian manifolds.

Conversely, we suppose Y is coisotropic. Then we can foliate Y by it's m dimensional characteristic submanifolds, each an isotropic submanifold of X. Suppose we can find an $n - m$ dimensional isotropic submanifold L_0 of Y, which is transverse to all these characteristics of Y. Then the union of all characteristics passing through L_0 forms an isotropic submanifold L of Y of dimension n, which is thus Lagrangian. We will pick L to be the graph of a function $x \mapsto a(x)$, chosen in such a way that the projection of the fundamental form θ on L onto M induces a closed form which we can locally write as $d\phi$, where ϕ is the function we are required to construct.

Given the assumption that the differentials $d_\xi p_1, \dots, d_\xi p_m$ are linearly independent, we can (possibly after reordering coordinates), locally write

$$Y = \{(x, \xi) : \xi'' = f_j(x, \xi')\},$$

where $\xi = (\xi', \xi'')$, for $\xi' \in \mathbb{R}^{n-m}$ and $\xi'' \in \mathbb{R}^m$, and $f = (f_{n-m+1}, \dots, f_n)$. Consider the codimension m hyperplane

$$\Sigma = \{x : x'' = x_0''\}.$$

Let $H_j = \xi_j - f_j(x, (\xi^0)')$, for $n - m < j \leqslant n$. Because the functions $\{dp_j(y)\}$ span the same subspace of T^*M as $\{dH_j(y)\}$ for $y \in Y$, we conclude that $\{\nabla_\Xi p_j(y)\}$ spans the same subspace of TM as $\{\nabla_\Xi H_j(y)\}$. But

$$\nabla_\Xi H_j = \frac{\partial}{\partial x^j} + \sum_k \frac{\partial f_j}{\partial x^k} \frac{\partial}{\partial \xi^k},$$

and the projection of this vector is transverse to Σ, which is spanned by $\partial/\partial x^j$ for $1 \leqslant j \leqslant n - m$. Thus the projection of the vectors $\nabla_\Xi p_j$ are also transverse to Σ.

Define a function $\psi : \Sigma \to \mathbb{R}$, such that for $1 \leqslant j \leqslant n - m$,

$$\frac{\partial \psi}{\partial x^j} = \xi_j^0.$$

Then define a function $a = (a', a'')$ on Σ, such that $a' = (a_1, \dots, a_{n-m})$ is defined by the equations

$$a_j(x) = \partial \psi / \partial x^j$$

for $1 \leqslant j \leqslant n - m$, and $a'' = (a_{n-m+1}, \dots, a_n)$ is defined by

$$a_j(x) = f_j(x, a'(x)).$$

The map $x \mapsto (x, a(x))$ is then a submanifold L of Y, of dimension $n - m$ containing (x_0, ξ^0). The submanifold L_0 is transverse to the characteristics of Y, precisely because the projections of $\nabla_\Xi p_j(y)$ are transverse to Σ. As discussed above, the union of characteristics passing through L_0 is then a Lagrangian manifold L. Taking the derivative of the projection $Y \to M$ induces a bijection of TL_0 with $T\Sigma$, and maps $TY^\perp$ onto a subspace transverse to $T\Sigma$. But then counting dimensions implies that the projection map $L \to M$ is a local diffeomorphism in a neighborhood of (x_0, ξ^0). The inverse map $M \to L$ allows us to extend the function a to be defined away from Σ.

If $\pi : L \to M$ is a projection map, and $\alpha = \sum a_j(x) dx^j$, then

$$\pi^* \alpha = \sum \xi_j dx^j = \theta$$

is the fundamental form. Since L is Lagrangian, θ is closed on L, and thus α is closed in M. The restriction of α to Σ is exact, and thus we conclude that there is a neighborhood of Σ upon which α is exact, i.e. we can find a function ϕ on M such that $\alpha = d\phi$. But this means precisely that we have found a function ϕ such that for $1 \leqslant j \leqslant n$,

$$\frac{\partial \phi}{\partial x^j} = a_j(x).$$

In particular, this means that $p_i(d\phi(x)) = 0$. And ϕ equals ψ on Σ, so in particular, this means that $d\phi(x_0) = \xi^0$. □

We will later see *all* conic Lagrangian manifolds are locally of the form above.

Example. *If $\phi : M \to \mathbb{R}$ is a smooth function, then $d\phi : M \to T^*M$ is an immersed Lagrangian submanifold of T^*M, since*

$$(d\phi)^*(\theta) = \sum \frac{\partial \phi}{\partial x^i} dx^i = d\phi,$$

is closed, and thus

$$(d\phi^*)(\omega) = (d\phi)^*(d\theta) = d(d(\phi)^*\theta) = d^2\phi = 0.$$

The next lemma shows that all Lagrangian sections are locally of this form.

Lemma 3.12. *The image of a section $s : X \to T^*X$ is an immersed Lagrangian submanifold if and only if locally we can write*

$$s = df$$

for some function $f : X \to \mathbb{R}$.

Proof. If $\theta = \sum \xi^i dx^i$, then $d\theta$ is the symplectic form ω, and $s^*\theta = s$. Thus s gives a Lagrangian manifold if and only if $s^*\omega = s^*(d\theta) = d(s^*\theta) = ds = 0$. The result above now follows by Poincaré's Lemma. □

A similar result holds if Λ is a conic Lagrangian submanifold of T^*X.

Lemma 3.13. *Suppose that $\Lambda \subset T^*X - \{0_X\}$ is a conic Lagrangian manifold containing some covector $(x_0, \xi_0) \in T^*X$. Then local coordinates (x, U) can be chosen, centered at x_0, inducing coordinates (x, ξ) on T^*U, such that the map $(x, \xi) \mapsto \xi$ is a diffeomorphism in an open neighborhood of (x_0, ξ_0) in Λ, and there exists a smooth, homogeneous function H such that, in a small neighborhood of (x_0, ξ_0) in T^*X, Λ locally agrees with set of points $\{(\nabla H(\xi), \xi)\}$ in the coordinate system.*

Proof. We first choose coordinates (x, U) such that $(x, \xi) \mapsto \xi$ is a diffeomorphism. Begin by choosing coordinates (y, V) centered at x_0 such that $\nu = dy_1$. The tangent plane V_0 to Λ at ν must be Lagrangian. If V_0 is transverse to the Lagrangian plane given in (y, η) coordinates by

$$V_1 = \{(0, a) : a \in \mathbb{R}^d\},$$

then we can set x to be y. Otherwise, we find V_2 Lagrangian and transverse to both V_0 and V_1. Since V_2 is transverse to V_1, it can be identified with a linear section, and the last result thus implies that we can find a quadratic form $Q : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$V_2 = \{(a, dQ(a)) : a \in \mathbb{R}^d\}.$$

If we set $x_1 = y_1 + Q(y)$, and $x_i = y_i$ for $2 \leqslant i \leqslant n$, then these are the required coordinates.

We claim now that if (x, U) gives a diffeomorphism, then we can find H. Shrinking U if necessary, there exists a radial $\phi : \mathbb{R}^d \to U$ such that

$$\Lambda \cap U = \{(\phi(\xi), \xi) : \xi \in \mathbb{R}^d\}.$$

Since Λ is Lagrangian, if $\psi(\xi) = (\phi(\xi), \xi)$, then

$$\psi^* \theta = \sum \xi_i d\phi_i = 0.$$

If we set $H(\xi) = \sum \xi_i \phi_i(\xi)$, then $\nabla H = \phi$, giving the required result. □

The structure of linear canonical relations can give us results about 'nonlinear' canonical relations, i.e. a conic Lagrangian submanifold of $T^*X \times T^*Y$ for two manifolds X^n and Y^m.

Lemma 3.14. *Let X and Y be smooth manifolds, and let C be a conic Lagrangian submanifold of $T^*X \times T^*Y$. Fix $(x_0, y_0; \xi_0, \eta_0) \in C$, and assume that the vector*

$$0\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} + \xi_0 \frac{\partial}{\partial \xi} + \eta_0 \frac{\partial}{\partial \eta}$$

is not tangent to C. Then X and Y have coordinate systems $x = (x', x'')$ and $y = (y', y'')$ centered at x_0 and y_0, such that $\xi_0 = (1, \dots, 0)$, $\eta_0 = (1, \dots, 0)$, and the tangent plane to C is given by

$$dx' = dy' \quad \text{and} \quad d\xi' = d\eta' \quad \text{and} \quad d\xi'' = 0 \quad \text{and} \quad d\eta'' = 0.$$

Then (x', x'', η', y'') can be used as local coordinates for C, and we can find a phase function $\phi(x, y'', \eta')$ such that C is parameterized by ϕ, in the sense that C locally agrees with Λ_ϕ on the coordinate system. Alternately, we can also find a phase $\phi(\xi', x'', y)$ such that C locally agrees with Λ_ϕ.

Given an assumption of the existence of the coordinates of the form above, we can easily predict what their dimensions should be. If we define C_{y_0,η_0} to be the projection of $[T^*X \times \{(y_0, \eta_0)\}] \cap C$ onto T^*X, then we can see from the tangent space calculation above that if x' and y' are k-dimensional coordinates, then the dimension of the projection will be equal to $k + n$. The same is true for the projection of C_{x_0,ξ_0} onto T^*Y. *In particular* if the projection is n-dimensional, then we can choose the function ϕ to have zero-dimensional homogeneous part. In this circumstance, we find that C is precisely given locally by the graph of a section $dH : X \times Y \to T^*X \times T^*Y$, i.e. locally C agrees with the set

$$\{(x, y, \nabla_x H, \nabla_y H)\},$$

and moreover, we find that the second derivatives of H vanish at the origin.

3.5 Physical Explanation of Symplectic Geometry

How does symplectic geometry arise in classical mechanics? In classical mechanics, the state of a physical system can be described by a point in some manifold M, together with a momentum vector $\xi \in T_x^* M$; the reason momentum is naturally a covector is because, given some velocity vector $v \in T_x M$, and some momentum vector ξ, the value $\xi(v)$ can naturally be identified with the *kinectic energy* of the system, up to some constant of proportionality. The motion of the system is then induced by a function $H : T^* M \to \mathbb{R}$, which gives, for each position $x \in M$ and $\xi \in T_x^* M$, the Hamiltonian $H(x, \xi)$ of the system (the total energy). If we define a function $L : TM \oplus T^* M \to \mathbb{R}$ by setting, for each $(x, v, \xi) \in TM \oplus T^* M$,

$$L(x, v, \xi) = \xi(v) - H(x, \xi),$$

then motion is then governed by the *principle of least action*, which states that, the motion $(x(t), \xi(t))$ of the system is a *local minimizer* of the *action* S of the physical system, given between two times t_1 and t_2 by

$$S = \int_{t_1}^{t_2} L(x(t), \dot{x}(t), \xi(t)) \; dt.$$

Variational methods lead to Hamilton's equations of motion, expressed in coordinates by the equation

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial \xi_i} \quad \text{and} \quad \frac{d\xi_i}{dt} = -\frac{\partial H}{\partial x_i}.$$

It is not immediately obvious from this equation that these equations are coordinate invariant, but we can see the invariance by seeing the relation to the symplectic geometry of the mechanics. Indeed, since $T^* M$ is a symplectic manifold, the covector field $dH : T^* M \to T^*(T^* M)$ can be identified using the symplectic bilinear form ω on each tangent space with a vector field $X_H : T^* M \to T(T^* M)$, and the equations of motion are given precisely by this vector field. If (x, ξ) are the equations of motion, for any $(v_0, \dot{\xi}_0) \in T(T^* M)$, we have

$$\omega((v_0, \dot{\xi}_0), (\dot{x}, \dot{\xi})) = \omega((v_0, \dot{\xi}_0), X_H) = dH(v_0, \dot{\xi}_0),$$

so, tested against the tangent vector to motion, the symplectic form gives the change of energy from a particular shift in the state of a particle. Since energy is conserved, it is therefore natural that the form emerging from mechanics is skew adjoint. The form ω is of course independent of the function H, but one can use this intuition to interpret what the quantity $\omega((v_0, \dot{\xi}_0), (v, \dot{\xi}))$ is at any point for two elements in $T_{(x,\xi)}(T^*M)$; there exists a choice of energy function H which causes a system starting at (x, ξ) to move in the direction $(v, \dot{\xi})$, and the quantity above then gives the infinitisimal change of energy caused from shifting the system in the direction $(v_0, \dot{\xi}_0)$.

From this point of view, it is natural to see how the concepts of symplectic geometry came into being. For any *canonical transformation*, i.e. any symplectic isomorphism $F : T^*M \to T^*M$ of T^*M, which we can view as a change of coordinates, motion in the new coordinates generated by the Hamiltonian $H : T^*M \to \mathbb{R}$ can be identified, using the isomorphism, with the motion generated by the Hamiltonian $H \circ F$. The concept of a canonical transformation was first introduced to allow one to simplify the form of a given Hamiltonian, i.e. historically, those emerging in the study of the motion of planets. We will see that, in a very similar light, symplectic geometry will allow us to simplify the *phases* that occur in the oscillatory integrals that emerge in harmonic analysis (from the point of view of physics, this can be viewed as a *quantization* of Hamiltonian mechanics).

For any two real-valued functions H and I on T^*M, we can consider their *Poisson bracket*

$$\{H, I\} = \omega(X_H, X_I) = dI(X_H),$$

which is also the *Lie bracket* of the vector fields X_H and X_I. If we consider the motion induced by H, then it follows that $\{H, I\}$ gives the infinitisimal change in I as a result of this motion. In particular, if $\{H, I\} = 0$, then I is preserved by the Hamiltonian dynamics induced by H. We then call I an *conserved quantity* of the system. This is how, for instance, conservation of momentum, conservation of angular momentum, and so on arises in physics, as a particular function I which is conserved by Hamiltonians arising in Newtonian physics.

One way Lagrangian manifolds arise is via the *Liouville-Arnold theorem*. A system of functions $I_1, \dots, I_d : T^*M \to \mathbb{R}$ are called an *integrable system* if they are *generically* linearly independent, i.e. $dI_1 \wedge \cdots \wedge dI_d$ is non-vanishing on a dense subset of T^*M, and the *Poisson bracket* $\{I_j, I_k\} = 0$ for all j and k. Given an integrable system, we can consider the function $I : T^*M \to \mathbb{R}^d$. For each regular value $\lambda \in \mathbb{R}^d$ of this function, the level set $\Lambda = F^{-1}(\lambda)$ is a d dimensional submanifold of T^*M, and is actually a *Lagrangian submanifold*. Indeed, the tangent space of Λ is generated by $X_{I_1}, \dots, X_{I_d}$, and we know by the Poisson bracket assumption that $[X_{I_j}, X_{I_k}] = 0$ for all j and k, so that the tangent space is Lagrangian. The Liouville-Arnold theorem states that for any j, one can choose local coordinates (θ, ω) on $U \subset T^*M$ around each point in Λ such that

$$\Lambda \cap U = \{\omega = 0\},$$

and $\{I_j, \theta_i\} = \omega_i$. If we set $H = I_j$, we therefore see that the flow induced by H is given locally in this coordinate system by linearly shifting θ by ω, and fixing ω. Such coordinates are called *action-angle coordinates*. The Liouville-Arnold theorem also proves the global results that the manifold Λ is compact and connected, then it is isomorphic to $\mathbb{T}^d$.

Example. *A harmonic oscillator on $\mathbb{R}^d$ is given in a Newtonian formulation, in Euclidean coordinates by the equations*

$$\ddot{x}_i = -k_i^2 x_i,$$

The energy function here is

$$H(x, \xi) = \xi \cdot x + (1/2) \sum k_i^2 x_i^2 = \sum \xi_i x_i + (1/2) k_i^2 x_i^2.$$

If we define

$$H_i(x, \xi) = \xi_i x_i + (1/2) k_i^2 x_i^2$$

then

$$X_{H_i} = -x_i \frac{\partial}{\partial x_i} + (\xi_i + k_i^2) \frac{\partial}{\partial \xi^i}.$$

It is simple to check that $[X_{H_j}, X_{H_k}] = 0$, and that $H = \sum H_j$. It follows that under the harmonic oscillation, each of the quantities $\{H_j\}$ is conserved. At any point away from the origin, the Liouville-Arnold theorem tells us the existence of an action-angle coordinate system (θ, ω) around any point (x^0, ξ^0) in Λ which locally linearizes the motion of the system. Since we have

$$(x_j(t), \xi_j(t)) = e^{itk_j}(x_j(t), \xi_j(t)),$$

we see that we should set

$$\theta_j = (x_j^2 + \xi_j^2) \cdot \angle\{(x_j, \xi_j), (x_j^0, \xi_j^0)\}$$

and

$$\omega_j = k_j(x_j^2 + \xi_j^2).$$

Thus the coordinates $\{\theta_j\}$ really are 'angles' in this sense. Note also that for each fixed $\lambda \neq 0$, the manifold $\Lambda = (H_1, \dots, H_d)^{-1}(\lambda)$ is isomorphic to $\mathbb{T}^d$, as guaranteed by the Liouville-Arnold theorem.

TODO: If geodesic flow is integrable, then the Liouville-Arnold theorem implies that the geodesic flow is periodic?

3.6 Semiclassical Approximations

How do we move from classical mechanics to quantum physics? The story begins with the hydrogen atom, which consists of a negatively charged electron orbiting a positively charged nucleus. Classically, the perception was that the electron orbited the nucleus, like a moon orbits planets. But this cannot be possible, since charged particles radiate energy when accelerated, causing them to lose energy. Thus an orbiting electron would radiate energy, and thus spiral into the nucleus of the proton.

In the early 20th century, de Broglie noticed that we could model the state of a particle as a wave

$$\psi = e^{(i/\hbar)(\xi \cdot x - Ht)}.$$

Assume the Hamiltonian H can be given by

$$H = \frac{|\xi|^2}{2m} + V,$$

where V depends only on space. Then the equation can be rewritten as

$$H\psi - \frac{|\xi|^2}{2m}\psi - V(x)\psi = 0.$$

Now in any given evolution of a particle, the Hamiltonian is constant, so we conclude that

$$H\psi = i\hbar\frac{\partial \psi}{\partial t}.$$

Next, we have that

$$\xi\psi = -i\hbar\nabla_x\psi$$

and thus

$$|\xi|^2\psi = -(\hbar^2/m)\Delta_x\psi.$$

But this means that the constancy of energy equation becomes

$$i\hbar\frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m}\Delta_x\psi = V\psi.$$

This is *precisely* the Schrödinger equation.

In this form, Schrödinger's equation is just a restatement of the conservation of energy in Hamiltonian mechanics. De Broglie's relation is also implicitly given by the equation, since the momentum of the particle modelled by ψ is equal to $\hbar$

times the location of the spatial Fourier transform of ψ in the x variable. Quantum mechanics follows when we enforce the *Born rule*, a postulate of quantum mechanics which states that a particle exists in an unknown probabilistic state modelled by a *square integrable* solution ψ to the Schrödinger equation, whose spatial distribution is given by $|\psi|^2$, and whose momentum distribution is given by $|\widehat{\psi}|^2$. The uncertainty principle prevents a particle from being arbitrarily localized in position-momentum space. But at large scales, the princ diple tends not to cause too much trouble, i.e. so that working with a classical mechanical formulation of physics on a large scale does not introduce too much error. *Semiclassical analysis* is the study of how much intuition from classical mechanics holds at an *intermediate scale* between classical and quantum mechanics, as well as applications of similar quantizations of other partial differential equations. In other words, it determines at what scale the *correspondence principle* holds.

Chapter 4

Parametrices

The Fourier transform provides a key way to understand spectral multipliers on a compact manifold M. If $h : (0, \infty) \to \mathbb{C}$ is a function, and P is a first order, elliptic, self-adjoint pseudodifferential operator on M, and $T = h(P)$, and if $\{e_n\} \subset C^\infty(M)$ is an orthonormal basis of eigenfunctions diagonalizing $L^2(M)$, such that $Pe_n = -\lambda_n^2 e_n$, then we conclude that we have a kernel representation

$$\begin{aligned} Tf(x) &= \sum h(\lambda_n) \langle f, e_n \rangle e_n \\ &= \int_M \left(\sum_n h(\lambda_n) e_n(x) \overline{e_n(y)} \right) f(y)\, dy \\ &= \int_M K(x,y) f(y)\, dy, \end{aligned}$$

where $K(x,y) = \sum_n h(\lambda_n) e_n(x) \overline{e_n(y)}$. The fact that this sum is often highly oscillatory prevents us from obtaining too much information directly from this expression aside from L^2 orthogonality. We can instead get more information by applying the Fourier transform. Namely, if H is the cosine transform of h, i.e.

$$H(t) = \int_0^\infty h(\lambda) \cos(2\pi t \lambda)\, d\lambda,$$

then we have an inversion formula

$$h(\lambda) = 4 \int H(t) \cos(2\pi t \lambda)\, dt.$$

We can thus write

$$\begin{aligned}
\sum_n h(\lambda_n) e_n(x) \overline{e_n(y)} &= \sum_n \left[4 \int H(t) \cos(2\pi t \lambda_n) \right] e_n(x) \overline{e_n(y)} \\
&= 4 \int H(t) \left[\sum_n \cos(2\pi t \lambda_n) e_n(x) \overline{e_n(y)} \right] \\
&= 4 \int H(t) \cos(2\pi t P) \; dt.
\end{aligned}$$

The advantage of this approach is that if we know h satisfies certain smoothness conditions, we can guarantee that H has suitable decay. And we can understand the multiplier operators

$$\cos(2\pi t P)$$

by virtue of the fact that these operators are the solution operators to the *wave equation* $\partial_t^2 = 4\pi^2 P$ on the manifold (with initial conditions starting with zero velocity), and so can be interpreted geometrically as wave propogators. If we treat h as an even function, and apply the Fourier inversion formula, we could also write this equation as

$$\sum_n h(\lambda_n) e_n(x) \overline{e_n(y)} = \int_{-\infty}^{\infty} \widehat{h}(t) e^{2\pi i t P} \, dt,$$

where the operators $e^{2\pi i t P}$ are the propogators to the *half-wave equation* $\partial_t = 2\pi i P$, which, being a first order operator, is often less technical to work with, though has the disadvantage of not having a *finite speed of propogation*.

A technicality behind the study of the wave equations above is that, being defined on a general compact manifold, these operators are no longer constant-coefficient partial differential equations, even if $P = \sqrt{-\Delta}$ and we are free to work locally in any coordinate system that we choose, because the manifold has *curvature*. Thus we propogators cannot be treated as Fourier multipliers, as in the Euclidean case. We get around this by using the theory of Fourier integral operators to construct *small-time parametrices* to the wave propogator. By a parametrix for the half-wave equation, we mean a family of operators $\{S(t)\}$ such that for $|t| \lesssim 1$, $S(t)e^{2\pi i t P} - 1$ is a smoothing operator, i.e. it is an operator with a kernel in $C^\infty(M \times M)$. And by a parametrix for the wave equation, we mean a family of operators $\{S(t)\}$ such that for $|t| \lesssim 1$, $S(t)\cos(2\pi i t P) - 1$ is a smoothing operator.

There are two choices of parametrix. The *Lax parametrix* has a less explicit formula, but is a more general and applies to any pseudodifferential operator P.

The *Hadamard parametrix* applies to the wave equation, and when $P = \sqrt{-\Delta}$, but gives more geometric information about the behaviour of the operator, and also has the advantage that it can be constructed for larger times (up to the injectivity radius of the manifold rather than $O(1)$ time).

4.1 The Lax Parametrix

Let P be a first order, classical, self-adjoint, elliptic pseudodifferential operator on a compact manifold M. Here we discuss a method due to Lax, which constructs a *parametrix* for the equation $\partial_t = 2\pi i P$ for $|t| \lesssim 1$, which is an operator given by *Fourier integrals*. A similar parametrix exists for the wave equation $\partial_t^2 = -4\pi^2 P$, obtained by using the fact that

$$\cos\left(2\pi t\sqrt{-\Delta}\right) = (1/2)e^{2\pi i P} + (1/2)e^{-2\pi i P}.$$

The equation $\partial_t = 2\pi i P$ is a pseudodifferential variant of a first order hyperbolic partial differential equation, which is where the method describe below first originated, in the work of Lax and Hörmander (TODO: Find these citations).

To state the main result of the Lax parametrix construction, we introduce the following notation: let $p : T^*M \to (0,\infty)$ denote the principal symbol of the operator P. We can then define a corresponding *Hamiltonian vector field* X_p, obtained by applying duality to the covector field $dp : T^*M \to T^*(T^*M)$, with respect to the symplectic form on T^*M; in coordinates, we can write

$$X_p = \sum_j \frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j}.$$

Let $\{\varphi_t : T^*M \to T^*M\}$ denote the *Hamiltonian flow* corresponding to the Hamiltonian vector field X_p. We can now state the main result of this section.

Theorem 4.1. *There exists an interval $I = \{|t| \lesssim 1\}$ centered at the origin, and a Schwartz operator S mapping functions on M to functions on $I \times M$, such that*

$$(\partial_t + 2\pi i P) \circ S$$

is a smoothing operator*, i.e. with a Schwartz kernel lying in $C^\infty(I \times M \times M)$, such that S is a Fourier integral operator of order* $1/4$ *associated with the canonical relation*

$$C = \{(t, x, y; \tau, \xi, \eta) : (x, \xi) = \varphi_t(y, \eta) \text{ and } \tau = p(y, \eta)\}.$$

For each $t \in I$, the operator $S(t)$ on M given by freezing the t variable is then a Fourier integral operator of order zero, associated with the canonical relation

$$C_t = \{(x,y;\xi,\eta) : (y,\eta) = \varphi_t(x,\xi)\}.$$

If we assume that for each $x \in M$, the cospheres

$$\{\varphi_t(x,\xi) : \xi \in T_x^*(M)\}$$

have everywhere non-vanishing Gaussian curvature, then C_t is the conormal bundle of the $2d-1$ dimensional submanifold

$$\Sigma_t = \{(x,y) : y = \varphi_t(x,\xi) \text{ for some } \xi \in T_x^*(M)\}.$$

of $M \times M$.

We hope to find a parametrix for the half-wave operator, expressed as a Fourier integral operator of the form

$$Sf(x,t) = S(t)f(x) = \int s(t,x,y,\xi)e^{2\pi i\Phi(t,x,y,\xi)}f(y)\,dy\,d\xi,$$

such that:

- $\Phi(t,x,y,\xi) = \phi(x,y,\xi) + tp(y,\xi)$, where ϕ is smooth away from $\xi = 0$, homogeneous of degree one, and $\phi(x,y,\xi) \approx (x-y)\cdot\xi$ in the sense that on the support of s,

  $$\partial_\xi^\beta\{\phi(x,y,\xi) - (x-y)\cdot\xi\} \lesssim_\beta |x-y|^2|\xi|^{1-\beta}.$$

  In particular, this implies that $\phi(x,y,\xi) = 0$ when $(x-y)\cdot\xi = 0$.

- s is a symbol of order zero, supported on $|x-y| \lesssim 1$ and on $|\xi| \geqslant 1$, in such a way that

  $$|\nabla_\xi\phi(x,y,\xi)| \gtrsim |x-y| \quad \text{and} \quad |\nabla_x\phi(x,y,\xi)| \gtrsim |\xi|$$

  for $(x,y) \in \text{supp}_x(s) \times \text{supp}_y(s)$.

Our goal is to choose ϕ and s such that:

- The operator $(\partial_t - 2\pi iP)\circ S - 1$ is a smoothing operator on $(-\varepsilon,\varepsilon)\times M$.

- $S(0)$ differs from the identity operator by a smoothing operator.

It then follows that $e^{2\pi it\sqrt{-\Delta}} \circ S(t)$ is a smoothing operator for $|t| < \varepsilon$. An operator like S, which has a kernel given by an oscillatory integral with a homogeneous phase, is called a *Fourier integral operator*.

To find a choice of ϕ and s which gives us this parametrix, let us start by determining what properties these functions should satisfy. Let us fix a coordinate system (x, U), where $x(U)$ is a precompact subset of $\mathbb{R}^n$. Let us assume that in these coordinates, P has symbol $P(x,\xi)$ and principal symbol $p(x,\xi)$. Then the kernel of $(\partial_t - 2\pi iP) \circ S$ in this coordinate system is

$$\int \{\partial_t - 2\pi iP(x, D_x)\} \left\{ s(t, x, y, \xi) e^{2\pi i\Phi(t,x,y,\xi)} \right\} \; d\xi.$$

This operator can also be expressed as a Fourier integral, using the calculus of Fourier integral operators. Namely, we can write the operator as

$$\int s'(t, x, y, \xi) e^{2\pi i\Phi(t,x,y,\xi)},$$

where s' is a symbol with an asymptotic expansion of the form

$$\begin{aligned} s'(t, x, y, \xi) \sim \partial_t s(t, x, y, \xi) &+ 2\pi ip(y, \xi)s(t, x, y, \xi) \\ &- \sum_\alpha \frac{2\pi i}{\alpha!} (\partial_\xi^\alpha P)(x, \nabla_x \phi(t, x, y, \xi)) \; D_z^\alpha \left\{ s(t, z, y, \xi) e^{2\pi i\Phi'(t,x,y,z,\xi)} \right\} \Big|_{z=x}, \end{aligned}$$

where

$$\Phi'(t, x, y, z, \xi) = \Phi(t, z, y, \xi) - \Phi(t, x, y, \xi) + (x - z) \cdot \nabla_x \Phi(t, x, y, \xi).$$

If we expect this operator to be smoothing, then the order zero part of the operator must vanish, i.e. we must first conclude that

$$(2\pi i)s(t, x, y, \xi) \cdot [p(y, \xi) - p(x, \nabla_x \phi(t, x, y, \xi))]$$

be a symbol of order 0. It is therefore natural to choose ϕ such that

$$p(y, \xi) = p(x, \nabla_x \phi(t, x, y, \xi)),$$

at least when $|x - y|$ is small (i.e. on the support of the symbol s), and $|\xi| \gtrsim 1$ (since the integral over $|\xi| \lesssim 1$ gives a smoothing operator). This is an *Eikonal* *equation*, and we will briefly take an aside to discuss how such an equation using the basic theory of Hamilton-Jacobi equations.

TODO: The manifold

$$\Sigma = \{(x, y, \xi, \eta) : p(y, \eta) = p(x, \xi)\}$$

is coisotropic of codimension one, since the differential of the map $(x, y, \xi, \eta) \mapsto p(y, \eta) - p(x, \xi)$ is no-vanishing given that p is elliptic. BUT MAYBE THE CURRENT PHASE FUNCTION RESULTS WE HAVE HERE DON'T GUARANTEE EXISTENCE YET.

The principal symbol p, a function on the cotangent bundle T^*M, induces a Hamiltonian vector field $H^p \in \Gamma(TM)$, such that for $f \in C^\infty(T^*M)$,

$$H^p(f) = \{p, f\} = \nabla_\xi p \cdot \nabla_x f - \nabla_x p \cdot \nabla_\xi f,$$

where $\{\cdot, \cdot\}$ is the Poisson bracket. Let $\{\psi_t\}$ denote the phase flow on M induced by this vector field. Since M is compact, and the vector field is homogeneous in the ξ variable, one can see that the flow exists for all times.

Lemma 4.2. *For any t, ψ_t is a canonical transformation of T^*M, and*

$$p \circ \psi_t = p,$$

i.e. p is constant on it's bicharacteristics *(the integral curves of the Hamiltonian vector field).*

Proof. For any $(x_0, \xi_0) \in T^*M$, the function

$$f(t) = p(\psi_t(x_0, \xi_0))$$

has derivative

$$f'(t) = H^p(p)(\psi_t(x_0, \xi_0)) = 0,$$

since $H^p(p) = 0$. But this means f is a constant function, yielding the second part of the Lemma.

To show that ψ_t is a canonical transformation, let

$$\sigma = \sum d\xi^j \wedge dx^j.$$

We use a similar argument, i.e. calculating

$$\frac{d}{dt}\psi_t^* \sigma.$$

Since $\{\psi_t\}$ is a semigroup, we have that

$$\frac{d}{dt}\,\psi_t^*\sigma|_{t=t_0} = \psi_{t_0}^*\frac{d}{dt}\,\psi_t^*\sigma|_{t=0}\,.$$

But

$$\begin{aligned}\psi_t^*\sigma &= \sum_j\left(d\xi^j - td\left\{\frac{\partial p}{\partial x^j}\right\}\right)\wedge\left(dx^j + td\left\{\frac{\partial p}{\partial \xi^j}\right\}\right) + O(t^2)\\ &= \sum_j(d\xi^j\wedge dx^j) + t\left(d\xi^j\wedge d\left\{\frac{\partial p}{\partial \xi^j}\right\} - dx^j\wedge\left\{\frac{\partial p}{\partial x^j}\right\}\right) + O(t^2),\end{aligned}$$

and to show the derivative above is zero, it suffices to note that the mixed partial cancel out above. □

Now we can obtain the existence result. Working in coordinates, we may assume without loss of generality that we are working in $\mathbb{R}^d$. We return to finding a solution ϕ to the equation

$$p(y,\xi) = p(x,\nabla_x\phi(x,y,\xi)).$$

To do this, we fix y, and $|\xi_0| = 1$, and try and define $\psi(x) = \phi(x,y,\xi)$ such that

$$p(y,\xi) = p(x,\nabla_x\psi(x)),$$

such that $\psi(x) = 0$ if x lies in the hyperplane

$$H = \{x : x\cdot\xi = y\cdot\xi\},$$

and such that $\nabla_x\psi(y) = \xi$. This latter assumption, that $\xi \neq 0$, together with the implicit function theorem, shows there exists a unique function a defined uniquely on a neighborhood of y in H in such a way that $p(y,\xi) = p(x,a(x))$. Define

$$S_0(x) = (x,a(x)),$$

Then the image of S_0 is a symplectic submanifold of T^*H, and $p(y,\xi) = p\circ S_0$. If we now define a section S on a neighborhood of y by extending S_0 along the bicharacteristics of p, then the fact that p is constant along the bicharacteristics implies that $p(y,\xi) = p\circ S$, and that the bicharacteristics are canonical transformations implies that S is a Lagrangian section. But for *any* Lagrangian section, we can find a function ψ, defined in a neighborhood of y, such that

$$S(x) = (x,\nabla_x\psi(x)).$$

This is the function we were required to find. TODO: Draw a picture of what's going on.

Now

If we set

$$s'(t,x,y,\xi) = e^{-2\pi i\Phi(t,x,y,\xi)}(\partial_t - 2\pi iP(x,D))\left\{s(t,\cdot,y,\xi)e^{2\pi i\Phi(t,\cdot,y,\xi)}\right\}$$

then the kernel is

$$\int s'(t,x,y,\xi)e^{2\pi i\Phi(t,x,y,\xi)}\ d\xi.$$

Provided that s' is a symbol of order $-\infty$ for $0 < |t| \leqslant \varepsilon$, integration by parts shows that $(\partial_t + 2\pi iP) \circ S$ is smoothing, and so we will try to choose ϕ and s so as to obtain such a result.

In our discussion of pseudodifferential operators, we have already discussed an asymptotic formula for s', namely, if

$$r_{x,y}(z) = \nabla_x\phi(x,z,\xi)\cdot(x-z) - \{\phi(x,y,\xi) - \phi(z,y,\xi)\}.$$

then for any $N > 0$, if $a \sim \sum_{k=-\infty}^{1} a_k$, where a_k is homogeneous of degree k, and if $\xi_\phi = \nabla_x\Phi(t,x,y,\xi) = \nabla_x\phi(x,y,\xi)$,

$$\begin{aligned} &s'(t,x,y,\xi) \\ &\quad = \underbrace{(p(y,\xi) - a(x,\xi_\phi))\cdot s(t,x,y,\xi)}_{\text{symbols of order 1}} \\ &\quad + \underbrace{\partial_t s(t,x,y,\xi)}_{\text{symbol of order 0}} \\ &\quad - \sum_{1\leqslant|\beta|<N} \underbrace{\frac{2\pi i}{\beta!\cdot(2\pi i)^\beta}\cdot\partial_\xi^\beta a(x,\xi_\phi)\partial_z^\beta\{e^{2\pi i r_{x,y}(z)}s(t,z,y,\xi)\}|_{z=y}}_{\text{symbols of order } 1-\lceil|\beta|/2\rceil} \\ &\quad + R_N(t,x,y,\xi). \end{aligned}$$

where, because $|\nabla_x\Phi(t,x,y,\xi)| \gtrsim |\xi|$ on the support of s,

$$\langle\xi\rangle^{t-\lceil N/2\rceil}R_N \in L^\infty((-\varepsilon,\varepsilon)\times U\times U\times\mathbb{R}^d).$$

It is simple to establish estimates of the form

$$|\partial_x^\alpha\partial_y^\beta\partial_\xi^\lambda s'(t,x,y,\xi)| \lesssim \langle\xi\rangle^{N_{\alpha\beta\lambda}}.$$

Thus if we can justify that $|s'(t,x,y,\xi)| \lesssim_N \langle \xi \rangle^{-N}$ for all $N > 0$, then it will follow that s' is a symbol of order $-\infty$. We now determine the propoerties of the symbol s and the symbol ϕ which will give us these estimates.

To begin with, let us specify the function ϕ. In order to guarantee that s' is a symbol of order zero, the expansion above shows that $(p(y,\xi) - p(x,\xi_\phi)) \cdot s(t,x,y,\xi)$ must be a symbol of order zero. This will be true if we can pick ϕ such that, on the support of s, and for $|\xi| \gtrsim 1$,

$$p(x, \nabla_x \phi(x,y,\xi)) = p(y,\xi).$$

This is an example of an *Eikonal equation*, e.g. an equation of the form

$$q(z, \nabla_z \psi(z)) = 0$$

for some function $q(z,\zeta)$. In our case, $z = (x,y,\xi)$, so $\zeta = (\zeta_x, \zeta_y, \zeta_\xi)$, and so

$$q(z,\zeta) = p(x,\zeta_x) - p(y,\xi).$$

Let us make some further remarks we desire about our choice of function ϕ:

- We want ϕ to be homogeneous and smooth away from the origin. If we solve the equation for all $|\xi| = 1$, and then extend ϕ such that for $\lambda > 0$ and $|\xi| = 1$,
$$\phi(x,y,\lambda\xi) = \lambda\phi(x,y,\xi)\psi(\lambda),$$
where ψ is smooth, equal to one for $|\lambda| \geqslant 3/4$, and vanishing for $|\lambda| \leqslant 1/2$, then ϕ will satisfy the equation for all $|\xi| \gtrsim 1$. This means that
$$p(x, \nabla_x \phi(x,y,\xi)) - p(y,\xi)$$
is smooth and supported on $|\xi| \lesssim 1$, which implies it is a symbol of order $-\infty$, which suffices for our construction. Thus it suffices to solve the equation for $|\xi| = 1$.

- Since ϕ is smooth away from the origin and homogeneous, the equation
$$|\partial_\xi^\beta \{\phi(x,y,\xi) - (x-y)\cdot\xi\}| \lesssim_\beta |x-y|^2 \langle \xi \rangle^{1-\beta}$$
holds if, for $|\xi| = 1$, we have $\phi(x,y,\xi) = 0$ whenever $(x-y)\cdot\xi = 0$, and $\nabla_x \phi(x,y,\xi) = \xi$ whenever $x = y$. Thus we have some *initial conditions* for our Eikonal equation.

The second condition constitutes a type of initial condition for ϕ, since it specifies it's behaviour on a hypersurface, a kind of Cauchy condition, and thus we should expect these are close to the conditions that give unique solutions to the equation. And the following Lemma indeed shows that there is a unique function ϕ, defined for $|x - y| \lesssim 1$ and $|\xi| = 1$ with these properties.

Lemma 4.3. *Let Z be a smooth manifold, and let $q(z, \zeta)$ be a real-valued, smooth function defined locally around a point $(z_0, \zeta_0) \in T^*Z$. Let S be a smooth hypersurface in Z passing through z_0 with conormal vector ζ_S at z_0, such that*

$$\frac{\partial q}{\partial \zeta_S}(z_0, \zeta_0) = \lim_{t \to 0} \frac{q(z_0, \zeta_0 + t\zeta_S) - q(z_0, \zeta_0)}{t}$$

is nonzero. Suppose that ψ is any smooth function defined on S locally about z_0, such that $d\psi(z_0)$ agrees with the action of ζ_0 on $T_{x_0}S$. Then there exists a unique smooth function ϕ defined in a neighborhood of z_0, which agrees with ψ on S, satisfies the Eikonal equation $q(z, \nabla_z \phi(z)) = 0$, and has $\nabla_z \phi(z_0) = \zeta_0$.

Proof. TODO: See Sogge, Theorem 4.1.1. □

In our case,

$$Z = \{(x, y, \xi) : |\xi| = 1\}.$$

We have $z_0 = (x_0, x_0, \xi_0)$, $\zeta_0 = (\xi, \xi, 0)$, and

$$S = \{(x, y, \xi) : |\xi| = 1 \quad \text{and} \quad (x - y) \cdot \xi = 0\}.$$

The conormal vector ξ_S of S at z_0 is a multiple of $(\xi_0, -\xi_0, 0)$, and so by homogeneity,

$$\frac{\partial q}{\partial \xi_S} = \lim_{t \to 0} \frac{p(x_0, (1 + t)\xi_0) - p(x_0, \xi_0)}{t} = p(x_0, \xi_0),$$

which is nonvanishing because P is elliptic. If we define ψ equal to zero on S, then $d\psi = 0$, which agrees with the action of ζ_0 on S. Thus the theorem applies local uniqueness and existence to solutions to the Eikonal equation, and by compactness of Z we can patch such solutions together to find a solution defined for all $|x - y| \lesssim 1$.

We therefore conclude that there exists a unique choice of ϕ such that, if s has small enough support, $s'(t, x, y, \xi)$ is a symbol of order zero. Next, let us see what constraints are forced on us in order to ensure that $S(0)$ differs from the identity by a smoothing operator. The kernel of U is precisely

$$\int s(0, x, y, \xi) e^{2\pi i \phi(x, y, \xi)} \, d\xi.$$

We now show that this operator is actually a *pseudodifferential operator* of order zero, and determine it's symbol up to first order.

To do this, we write $\phi_\alpha(x, y, \xi) = (1 - \alpha)\phi(x, y, \xi) + \alpha(x - y) \cdot \xi$. Let U_α be the operator with kernel

$$\int s(0, x, y, \xi) e^{2\pi i \phi_\alpha(x,y,\xi)} \, d\xi.$$

Assume the support of s is close enough to the diagonal such that

$$|\nabla_\xi \phi_\alpha(x, y, \xi)| \gtrsim |x - y|$$

on the support of s. Then $\partial_\alpha^n U_t$ has kernel

$$\int (2\pi i)^n (\phi_1 - \phi_0)^n s(0, x, y, \xi) e^{2\pi i \phi_\alpha(x,y,\xi)} \, d\xi.$$

This is an oscillatory integral defined by a symbol of order n. However, when $t = 1$, the fact that $\phi(x, y, \xi) \approx (x - y) \cdot \xi$, together with the formula for converting pseudodifferential operators with compound symbols into standard Kohn-Nirenberg type symbols shows that $\partial_\alpha^n U_1$ is actually a pseudodifferential operator of order $-n$. Integration by parts, similarily, shows that $\partial_\alpha^n U_t$ is defined by an oscillator integral against a symbol of order $-n$. But this means that if we define a pseudodifferential operator by the asymptotic formula

$$V \sim \sum \frac{(-1)^n}{n!} \partial_\alpha^n U_1,$$

then $U - V$ is smoothing. Indeed, for any n, by Taylor's formula we have

$$U = \sum_{k=0}^{n-1} \frac{(-1)^n}{n!} \partial_\alpha^n U_1 + \frac{(-1)^n}{n!} \int_0^1 \alpha^{n-1} \partial_\alpha^n U_\alpha \, d\alpha$$

The integral here is an oscillatory integral defined against a symbol of order $-n$, and thus taking $n \to \infty$ verifies the claim.

It is an important remark that reversing this argument shows that *any* pseudodifferential operator can be written in the form above for the particular choice of ϕ we have given. This is a special case of the *equivalence of phase functions* theorem. This in particular guarantees that we can choose a symbol $I(x, y, \xi)$ of order zero such that $U - 1$ is smoothing if and only if $s(0, x, y, \xi) - I(x, y, \xi)$ is a symbol of order $-\infty$. The symbol I can be chosen to be vanishing for $|x-y| \gtrsim 1$, since the difference will be a smoothing pseudodifferential operator.

Next, the quantity

$$\begin{aligned}
&\partial_t s(t,x,y,\xi) \\
&\quad + \sum_{k=1}^{d} \partial_\xi^k a(x,\xi_\phi) \partial_x^k s(t,x,y,\xi) \\
&\quad + \left(a_0(x,\xi_\phi) + \frac{1}{2\pi} \sum_{|\beta|=2} \partial_\xi^\beta p(x,\xi_\phi) \partial_x^\beta \phi(x,y,\xi) \right) s(t,x,y,\xi).
\end{aligned}$$

must be a symbol of order -1. But because the coefficients of this equation are smooth, and all derivatives are bounded, it follows from the general theory of transport equations that there exists a unique smooth, function s_0 defined for $|t| \leqslant \varepsilon$, which is a symbol of order zero, such that $s_0(0,x,y,\xi) = I(x,y,\xi)$, s_0 vanishes for $|x-y| \gtrsim 1$, and satisfies the transport equation

$$\begin{aligned}
&\partial_t s_0(t,x,y,\xi) \\
&\quad + \sum_{k=1}^{d} \partial_\xi^k a(x,\xi_\phi) \partial_x^k s_0(t,x,y,\xi) \\
&\quad + \left(a_0(x,\xi_\phi) + \frac{1}{2\pi} \sum_{|\beta|=2} \partial_\xi^\beta p(x,\xi_\phi) \partial_x^\beta \phi(x,y,\xi) \right) s_0(t,x,y,\xi) = 0.
\end{aligned}$$

We have thus justified that the quantity

$$R_0(t,x,y,\xi) = e^{-2\pi i \Phi(t,x,y,\xi)} (\partial_t - 2\pi i P)(s_0(t,\cdot,y,\xi) e^{2\pi i \Phi(t,x,y,\xi)})$$

is a symbol of order -1. Now we come to a quirk of this parametrix, which does not occur in the study of hyperbolic partial differential equations. Since the operator $P(x,D)$ is only *pseudolocal* rather than completely local, the remainder term R_0 is *not* necessarily supported on a neighborhood of the origin. To fix this, we now successively define the terms $\{s_k\}$ for $k < 0$, which are symbols of order $-k$, such that $s_k(0,x,y,\xi) = 0$, and

$$TODO : SPECIFYREQUIREDEQUATION.$$

Again, solutions exist for small time periods. And this implies that $e^{-2\pi i \Phi(t,x,y,\xi)}(\partial_t + 2\pi i P)((s_0 + \cdots + s_{-k}) e^{2\pi i \Phi(t,x,y,\xi)})$ is a symbol of order $-k$ (TODO: Is It), and we can continue the calculation to complete the argument.

Chapter 5

Distributions on Riemannian Manifolds

How do we work with distributions on $\mathbb{R}^d$? We first identify a vector space of test functions, say, the space $\mathcal{D}(\mathbb{R}^d)$ of smooth, compactly supported functions, the space $\mathcal{E}(\mathbb{R}^d)$ of all smooth functions, or the space $\mathcal{S}(\mathbb{R}^d)$ of Schwartz functions. The distributions are then formally defined as the dual space of this class of test functions. To actually work with these distributions, we find an explicit way to represent them, via a bilinear pairing; for $\mathcal{D}(\mathbb{R}^d)$, the bilinear pairing $\mathcal{E}(\mathbb{R}^d) \times \mathcal{D}(\mathbb{R}^d) \to \mathbb{C}$ given by

$$(\phi, \psi) \mapsto \int_{\mathbb{R}^d} \phi(x)\psi(x)\, dx.$$

This pairing induces a natural identification of $\mathcal{E}(\mathbb{R}^d)$ with a dense subclass of $\mathcal{D}(\mathbb{R}^d)^*$. Thus we can intuitively study elements of $\mathcal{D}(\mathbb{R}^d)^*$ as if they behaved like elements of $\mathcal{E}(\mathbb{R}^d)$, at least when integrated against elements of $\mathcal{D}(\mathbb{R}^d)$. Reversing this pairing allows us to think of elements of $\mathcal{E}(\mathbb{R}^d)^*$ as elements of $\mathcal{D}(\mathbb{R}^d)$, and the pairing $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ allows us to think of tempered distributions of $\mathcal{S}(\mathbb{R}^d)$ as if they themselves were elements of $\mathcal{S}(\mathbb{R}^d)$.

The bilinear pairings here often behave well with respect to natural operations on the respective spaces. For instance, we have integration by parts identities

$$(\partial^\alpha \phi, \psi) = (-1)^{|\alpha|}(\phi, \partial^\alpha \psi)$$

for all the pairings above. And for the pairing of $\mathcal{S}(\mathbb{R}^d)$, we have the multiplication formula

$$(\mathcal{F}\phi, \psi) = \Big(\phi, \mathcal{F}\psi\Big),$$

where $\mathcal{F}$ is the Fourier transform. By taking weak limits, this allows us to extend the derivative and Fourier transform operations for distributions. In particular, we can take the derivatives of elements of $\mathcal{D}(\mathbb{R}^d)^*$, $\mathcal{E}(\mathbb{R}^d)^*$ and $\mathcal{S}(\mathbb{R}^d)^*$, and we can take the Fourier transform of elements of $\mathcal{S}(\mathbb{R}^d)^*$. By taking pairings of vector fields, i.e. by setting

$$(X, Y) = \int (X \cdot Y)\, dx$$

where X and Y are smooth vector fields, with X having compact support, we can define vector-valued distributions, and the theory above extends analogously.

We can also define other operations on distributions as if they were functions, provided that we have information about their *wavefront sets*. For any distribution u, it's wavefront set $\mathrm{WF}(u)$ to be a subset of $\mathbb{R}^d_x \times \mathbb{R}^d_\xi$, conic in the variable ξ, which gives information about the position and 'direction' of the singularities of u. As an example of how the wavefront set is useful, if $f : \mathbb{R}^n_x \to \mathbb{R}^m_y$ is smooth, and $u \in \mathcal{D}(\mathbb{R}^m)$ is a distribution such that $\mathrm{WF}(u)$ does not contain any point of the form (y, η) for which there exists $x \in f^{-1}(y)$ with $Df(x)^t\eta = 0$, then we can define a distribution f^*u on $\mathbb{R}^n$, which can be interpreted as a weak limit of a sequence $\{f^*u_n\}$, where $u_n \in \mathcal{E}(\mathbb{R}^m)$ converging weakly to u in a way respecting the wavefront set of u; the advantage of this is that f^*u_n is just $u_n \circ f$, and thus easy to work with.

Now let's do the same thing on a Riemannian manifold M^d. Here we also have natural spaces of test functions, i.e. the spaces $\mathcal{E}(M)$ and $\mathcal{D}(M)$ of smooth functions, the latter of which specified to have compact support. Here the natural pairing $\mathcal{E}(M) \times \mathcal{D}(M) \to \mathbb{C}$ is given by integration against the *volume measure* on the manifold M, i.e. the pairing is given by

$$(\phi, \psi)_g \mapsto \int_M \phi(x)\psi(x)\, dV_g(x).$$

This pairing allows us to identify $\mathcal{E}(M)^*$ and $\mathcal{D}(M)^*$ with weak limits of elements of $\mathcal{D}(M)$ and $\mathcal{E}(M)$ respectively. It will also be convenient to consider *vector-valued distributions*, i.e. the dual spaces of the spaces $\mathcal{E}(\Gamma(TM))$ and $\mathcal{D}(\Gamma(TM))$ of smooth vector fields, the latter of which limited to have compact support. These spaces have a pairing given by

$$(X, Y)_g \mapsto \int_M \langle X, Y \rangle_g\, dV_g.$$

Thus we can identify the dual spaces $\mathcal{E}(\Gamma(TM))^*$ and $\mathcal{D}(\Gamma(TM))^*$ with weak limits of vector fields, the latter being uniformly compactly supported.

On a Riemannian manifold, the natural derivative operators to consider are the *gradient operator*, which, for a given function $f \in \mathcal{E}(M)$, gives a smooth vector field $\nabla_g f \in \Gamma(TM)$, which has the property that for any other smooth vector field $X \in \Gamma(TM)$,

$$X(f) = \langle X, \nabla_g f \rangle_g.$$

We also have a *divergence operator*, which associates with any smooth vector field $X \in \Gamma(TM)$ a smooth function $\nabla_g \cdot X$ such that the integration by parts identity

$$\int_M \left\langle \nabla_g f, X \right\rangle_g \, dV_g = -\int_M f \; (\nabla_g \cdot X) \; dV_g$$

holds when either X or f has compact support. This formula gives us a way to interpret the gradient $\nabla_g u \in \mathcal{D}(\Gamma(TM))^*$ of a general distribution $u \in \mathcal{D}(M)^*$; i.e. such that $\nabla_g u$ is the vector-valued distribution such that for any $X \in \mathcal{D}(\Gamma(M))$,

$$(\nabla_g u, X) = -(u, \nabla_g \cdot X).$$

Similarily, we can consider the divergence $\nabla_g \cdot X \in \mathcal{D}(M)^*$ of a distributional vector fields $X \in \mathcal{D}(\Gamma(TM))^*$. Combining these operators gives us the *Laplace-Beltrami operator* $\Delta_g f = \nabla_g \cdot \nabla_g f$, which we can now consider as a map from $\mathcal{D}(M)^*$ to itself, and from $\mathcal{E}(M)^*$ to itself. We note that in coordinates, we have

$$\nabla_g f = \sum_{i,j} \frac{\partial f}{\partial x^i} g^{ij} \frac{\partial}{\partial x^j},$$

and the divergence is given by

$$\nabla_g \cdot X = |g|^{-1/2} \sum_i \frac{\partial}{\partial x^i} \left\{ |g|^{1/2} X^i \right\},$$

where $|g|$ is the determinant of the matrix with coefficients $\{g_{ij}\}$, which we can roughly think of as the volume of the unit ball in the metric.

Chapter 6

The Hadamard Parametrix

The Hadamard parametrix gives an alternate expression as a wave to invert the wave equation on a Riemannian manifold, though one requires much tighter control of the geometry of the underlying Riemannian manifold. We begin with summarizing facts about the fundamental solution of the standard wave equation on $\mathbb{R}^{d+1}$, before moving onto fundamental solutions of the wave equation on $\mathbb{R}^{d+1}$ with an arbitrary *constant coefficient* Riemannian metric, and then we move to constructing a small-time parametrix on an arbitrary Riemannian manifold.

6.1 Euclidean Case

Let's begin with the standard wave equation on $\mathbb{R}^d$, equipped with the standard metric. We are therefore concerned with constructing a fundamental solution for the d'Alembertian operator $\square = \partial_t^2 - \Delta_x$ on $\mathbb{R}^{d+1}$. One choice of fundamental solution for $\square$ is the *forward fundamental solution*, defined for $d \geqslant 2$ by the equation

$$E_+(x,t) = c_d \cdot \frac{H(t)}{\operatorname{Im}\left\{ (|x|^2 - (t+i0)^2)^{\frac{d-1}{2}} \right\}}$$

where H is the heaviside step function, and

$$c_d = \frac{2}{(n+1)A_{n+1}},$$

where A_{n+1} denotes the surface area of S^n. We have

$$\operatorname{Supp}(E_+) = \{(x,t) : t \geqslant 0 \text{ and } |x| \leqslant t\},$$

Furthermore, we have

$$\text{Sing Supp}(E_+) = \{(x,t) : t \geqslant 0 \text{ and } |x| = t\},$$

i.e. E_+ has singular support on the *forward light cone*. When d is odd, one can see from the formula above that E_+ actually has *support* on the forward light cone; this is *Huygen's principle*. For $d = 2$, we have

$$E_+(t,x) = H(t)\frac{H(t^2 - |x|^2)}{2\pi(t^2 - |x|^2)^{1/2}},$$

where the right hand side is locally integrable, and thus defines a distribution. For $d = 3$, we have

$$E_+(t,x) = H(t)\frac{\delta(t^2 - |x|^2)}{2\pi} = H(t)\frac{\delta(t - |x|)}{4\pi t}.$$

When $d \geqslant 4$, the forward fundamental solution becomes a distribution of higher order, i.e. becoming more singular on the forward light cone. For $d = 2$ the equation above no longer applies, but we have the simpler formula

$$E_+(t,x) = \frac{H(t)H(t^2 - |x|^2)}{2}.$$

It is interesting to note that E_+ is the *unique* fundamental solution which has the finite speed of propogation property, i.e. it is supported on the interior of the forward light cone.

Lemma 6.1. *If v is a fundamental solution of the D'Alembertian, supported on the interior of the forward light cone, then $v = E_+$.*

Proof. If $u = v - E_+$, then u is supported on the interior of the forward light cone and $\Box u = 0$. But this means that

$$u = \delta * u = \Box E_+ * u = E_+ * \Box u = E_+ * 0 = 0,$$

where these convolutions are well defined precisely because of the support of all the quantities involved. ∎

The reflection of the forward fundamental solution about the origin $t = 0$ is another fundamental solution to the wave equation, which we denote by E_-. It is supported on the interior of the backward light cone, and called the *backward fundamental solution*. Taking convex combinations of these fundamental solutions gives a plethora of other fundamental solutions, like the solution

$$E_{\text{FW}} = \frac{E_+ + E_-}{2},$$

the *Feynman-Wheeler* fundamental solution.

Using the Fourier transform, we can write

$$E_+(x,t) = \frac{H(t)}{2\pi} \int \frac{\sin(2\pi t|\xi|)}{|\xi|} e^{2\pi i \xi \cdot x}\, d\xi.$$

Modifying this formula gives Fourier expressions for the backward fundamental solution, and the Feynman-Wheeler fundamental solution. We also have a solution of the form

$$E_F(t,x) = \frac{1}{4\pi i} \int e^{2\pi i(\xi \cdot x + |t\xi|)}\, \frac{d\xi}{|\xi|}\, d\xi,$$

the *Feynman fundamental solution*. This fundamental solution has the quirk that

$$\text{supp}(E_F) = \mathbb{R}^d_x \times \mathbb{R}_t,$$

which is counterintuitive given the finite propogation speed of the wave equation.

Let us use these fundamental solutions to solve the Cauchy problem for the wave equation.

Theorem 6.2. *Suppose $f, g \in C^\infty(\mathbb{R}^d)$, and $F \in C^\infty(\mathbb{R}^d \times [0,\infty))$. Then the Cauchy problem*

$$\Box u(x,t) = F(x,t)$$

with $u(0,t) = f(x)$ and $\partial_t u(x,0) = g(x)$ has a unique solution in $C^\infty(\mathbb{R}^d \times [0,\infty))$, and we can write this solution as

$$u(t) = \partial_t E_+(t) * f + E_+(t) * g + \int_0^t E_+(t-s) * F(s)\, ds.$$

If we assume $f, g \in \mathcal{S}(\mathbb{R}^d)$, and $F \in C^\infty(\mathbb{R}_t, \mathcal{S}(\mathbb{R}^d_x))$, then we can also write

$$u(x,t) = \cos(2\pi t\sqrt{-\Delta}) f + \frac{\sin(2\pi t\sqrt{-\Delta})}{\sqrt{-\Delta}} g + \int_0^t \frac{\sin(2\pi (t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s)\, ds.$$

Proof. The Fourier multiplier formula follows from the Fourier expression of E_+. The expression above is well defined since $\partial_t E_+(t)$ and $E_+(t)$ are smooth functions of t valued in $\mathcal{E}^*(\mathbb{R}^d)$. Now

$$\Box E_+(t) = \Box \partial_t E_+(t) = 0$$

for $t > 0$, so it is clear that $v(t) = \partial_t E_+(t) * f + E_+(t) * g$ solves the wave equation for $t > 0$. Since $(E_+)(0+) = 0$, and $(\partial_t E_+)(0+) = \delta_0$, $v(t)$ has the required initial conditions, so would solve the equation provided there were no forcing term. Thus it suffices to show that

$$w(t) = \int_0^t E_+(t-s) * F(s)\, ds$$

solves the equation $\Box w = F$, with vanishing initial conditions. Now since $E_+(0+) = 0$,

$$\partial_t w(t) = \int_0^t (\partial_t E_+)(t-s) * F(s)\, ds$$

and since $\partial_t E_+(0+) = \delta_0$,

$$\begin{aligned}
\partial_t^2 w(t) &= F(t) + \int_0^t (\partial_t^2 E_+)(t-s) * F(s)\, ds. \\
&= F(t) + \int_0^t (\Delta E_+)(t-s) * F(s)\, ds \\
&= F(t) + \Delta w(t).
\end{aligned}$$

Thus $\Box w = F$. It is clear from the above formulas that $w(0+) = \partial_t w(0+) = 0$. Thus we proved the *existence* of solutions to the wave equation. Uniqueness follows from our uniqueness argument that E_+ is the unique fundamental solution, since it suffices to show that there is no nonzero $u \in C^\infty(\mathbb{R}^d \times [0,\infty))$ with $\Box u = 0$ and with vanishing initial conditions. $\square$

In order to construct fundamental solutions to the wave equation on a Riemannian manifold, it is helpful to note that we can find constants $\{a_\nu : \nu \geqslant 1\}$ such that if

$$E_\nu(x,t) = a_\nu \cdot H(t) \cdot \operatorname{Im}\left\{|x|^2 - (t+i0)^2\right\}^{\nu - \frac{d-1}{2}}$$

then $\Box E_\nu = \nu E_{\nu-1}$, where $E_0 = E_+$. The precise quantities $\{a_\nu\}$ are difficult to calculate; the most convenient precise description of the distribution is that given by the equation

$$E_\nu(x,t) = \lim_{\varepsilon\to 0} \nu! \int_{\mathbb{R}^{d+1}} e^{2\pi i(x\cdot\xi + t\tau)}(|\xi|^2 - (\tau - i\varepsilon)^2)^{-\nu-1}\, d\xi\, d\tau,$$

i.e. the spacetime Fourier transform of the function $(\xi,\tau) \mapsto (|\xi|^2 - (\tau - i0)^2)^{-\nu-1}$. This function is holomorphic on the upper half plane, so Paley-Wiener implies that $\text{supp}_t(E_\nu) \subset [0,\infty)$, and it is fairly simple to check that E_0, as we have defined it here, is a fundamental solution to the wave equation, so that the uniqueness result we proved above for the fundamental solution implies that $E_+ = E_0$. One then verifies that $\Box E_\nu = \nu E_{\nu-1}$ quite simply. We note also the two formulas

$$\nabla E_\nu(x,t) = (-1/2)xE_{\nu-1} \quad \text{and} \quad \partial_t E_\nu(x,t) = (1/2)tE_{\nu-1}.$$

As $\nu \to \infty$, the distributions $\{E_\nu\}$ become less and less singular on the forward light cone. By induction, using the recurrence formula

$$E_\nu(x,t) = \nu \int_0^t \frac{\sin(2\pi(t-s)|\xi|)}{|\xi|} \widehat{E}_{\nu-1}(\xi,s)\, d\xi\, ds,$$

we can write E_ν is a finite linear combination of terms of the form

$$t^j H(t) \int \frac{e^{2\pi i(x\cdot\xi + t|\xi|)}}{|\xi|^{\nu+k+1}}\, d\xi$$

where j and k range over non-negative integers satisfying $j + k = \nu$. In particular, we see from this formula that E_ν is a Lagrangian distribution of order at most $d/4 - \nu - 1$ on $\mathbb{R}^{d+1}$.

6.2 Constant Coefficient Metric

Now let's move on to the case of the wave equation on $\mathbb{R}^{d+1}$, where $\mathbb{R}^d$ is equipped with a different, constant-coefficient metric $g = \sum g_{ij} dx^i dx^j$. The Laplace-Beltrami operator then becomes

$$\Delta_g f = \sum_{i,j} g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} = \nabla \cdot \nabla_g f.$$

Let G denote the positive definite $d \times d$ matrix with coefficients g_{ij}. Then

$$\nabla_g f = \sum_{i,j} \frac{\partial f}{\partial x^i} g^{ij} \frac{\partial}{\partial x^j} = G^{-1} \cdot \nabla f.$$

Consider a new coordinate system $y = Tx$ for some invertible linear operator $T : \mathbb{R}^d \to \mathbb{R}^d$. The chain rule implies that

$$\nabla_g(T^* f) = G^{-1} \cdot \nabla(T^* f) = (G^{-1}T^t)(\nabla f \circ T).$$

Thus

$$\begin{aligned}
\Delta_g(T^* f) &= \nabla \cdot \nabla_g f \\
&= \sum_{i,j} \frac{\partial}{\partial x^i} \left\{ (G^{-1}T^t)_{ij} \left(\frac{\partial f}{\partial y^j} \circ T \right) \right\} \\
&= \sum_{k,i,j} T_{ki} (G^{-1}T^t)_{ij} \frac{\partial x^k}{\partial y^i} \frac{\partial^2 f}{\partial y^k \partial y^j} \circ T \\
&= \sum_{k,j} (TG^{-1}T^t)_{kj} \frac{\partial^2 f}{\partial y^k \partial y^j}.
\end{aligned}$$

If we choose T such that $TG^{-1}T^t = I$, which happens precisely when $G = T^tT$, then we obtain that

$$\Delta_g(T^* f) = T^*(\Delta f),$$

where Δf is the Euclidean Laplacian of h. Now let E denote the forward fundamental solution to $\square$ on $\mathbb{R}^{d+1}$. Let $F : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ given by $F(x,t) = F(Tx,t)$. Noting that $\det(T) = \det(G)^{1/2}$, if we set

$$E_0 = \det(G)^{1/2} F^* E,$$

i.e. so that

$$E_0(x,t) = \det(G)^{1/2} E(Tx,t) = H(t) \frac{c_d \det(G)^{1/2}}{\mathrm{Im}(|x|_g^2 - (t + i0)^2)^{\frac{d-1}{2}}},$$

then $\Box_g E_0 = \delta$. Indeed, for $\phi \in \mathcal{D}(\mathbb{R}^{d+1})$, a change of variables tells us that

$$\begin{aligned}
(\Box_g E_0, \phi) &= \det(G)^{1/2} \lim_{\varepsilon \to 0} \left(\Box_g F^* E_\varepsilon, \phi \right) \\
&= \det(G)^{1/2} \lim_{\varepsilon \to 0} \iint \Box_g (F^* E_\varepsilon)(x,t)\ \phi(x,t)\ dx\ dt \\
&= \det(G)^{1/2} \lim_{\varepsilon \to 0} \iint (\Box E_\varepsilon)(Tx,t)\ \phi(x,t)\ dx\ dt \\
&= \det(G)^{1/2} \lim_{\varepsilon \to 0} \det(T)^{-1} \iint (\Box E_\varepsilon)(y,t)\ \phi(T^{-1}y,t)\ dy\ dt \\
&= \det(G)^{1/2} \det(T)^{-1} \phi(T^{-1}0,0) \\
&= \phi(0,0).
\end{aligned}$$

Thus we've found a fundamental solution.

Analogous to the behaviour of the forward fundamental solution with respect to the standard metric, the fundamental solution here has support on the interior of the light cone, i.e. the set

$$\{(x,t) \in \mathbb{R}^{d+1} : t \geqslant 0 \text{ and } |x|_g \leqslant t\}.$$

We also have a Fourier transform representation of this fundamental solution, namely, E_0 is given by an oscillatory integral distribution of the form

$$\begin{aligned}
E_0(x,t) &= H(t) \int \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|} e^{2\pi i T^t \xi \cdot x}\ d\xi \\
&= H(t) \int \det(G)^{-1/2} \frac{\sin(2\pi t|\xi|_g)}{2\pi|\xi|_g} e^{2\pi i \xi \cdot x}\ d\xi.
\end{aligned}$$

Thus in particular, for $\phi \in \mathcal{D}(\mathbb{R}^d \times \mathbb{R})$, we have

$$\int E_0(x,t)\phi(x,t)\ dV_g(x)\ dt = \int_0^\infty \iint \phi(x,t) \frac{\sin(2\pi t|\xi|_g)}{2\pi|\xi|_g} e^{2\pi i \xi \cdot x}\ dx\ d\xi\ dt.$$

This is somewhat 'coordinate invariant', because the left hand side is given by integration with respect to the Riemannian volume measure, which changes nicely in coordinates, and the right hand side is given by integration against the natural Louiville volume measure $dx\ d\xi$ defined on $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$.

6.3 General Riemannian Metric

Let us now address the wave equation on a general Riemannian manifold M. Because we have a non-constant coefficient equation, we cannot expect to obtain a fundamental solution, instead only a *parametrix*, which gives a solution *modulo smoothing terms*. More precisely, let r_0 be the *injectivity radius* of M, defined in subsequent paragraphs. If δ and t_0 are positive numbers with $t_0 + \delta < r_0$, and if we set $I_0 = \{t : |t| < t_0\}$, then we will construct a Schwartz operator T_H from M to $I_0 \times M$ such that the operator $\square_g \circ T_H$ has a kernel lying in $C^\infty(I_0 \times M \times M)$. The operator T_H is called the *Hadamard parametrix*.

Let us start by introducing *normal coordinates*, which is where the injectivity radius enters the picture. Namely, r_0 is the largest quantity such that for all $r < r_0$, for a point $q \in M$, and if B is the radius r ball centered at the origin in T_qM (defined with respect to the metric on T_qM induced by g), the geodesic map $\exp_q : B \to M$ is well-defined and a diffeomorphism. The image is then the metric ball

$$B_r(q) = \{p \in M : d_g(p,q) < r\}.$$

Let us fix $q \in M$. If we fix an arbitrary orthonormal basis on T_qM, then we obtain an isomorphism $T_pM \cong \mathbb{R}^d$, and we call the induced diffeomorphism x from $B_\delta(q)$ to the ball B of radius δ in $\mathbb{R}^d$ a system of *normal coordinates* at q.

Let's review some useful properties of this correspondence. In the coordinate system x, we can write the Riemannian metric as

$$\sum g_{ij}(x) dx^i dx^j$$

for smooth functions $\{g_{ij}(x)\}$ giving th coefficients of a symmetric matrix $G(x)$. The fact that we have used geodesic normal coordinates means the matrices $\{G(x)\}$ have two useful properties:

- $G(0)$ is the identity map.
- (The Gauss Lemma) For each $x \in B$, $G(x)x = G(0)x$.

These equations are key to the construction of the Hadamard parametrix, which we might predict given that the wave equation on Euclidean space propogates radially, and the Gauss Lemma tells us that in normal coordinates, the Riemannian metric behaves like the Euclidean metric in the radial direction.

In coordinates, if we write

$$\nabla_g f = \sum_{i,j} g^{ij} (\partial_j f) \frac{\partial}{\partial x^j},$$

then the Laplace-Beltrami operator can be written as

$$\Delta_g f = |g|^{-1/2}\nabla \cdot (|g|^{1/2}\nabla_g f).$$

Using the product rule, we can write $\Delta_g f = L_g f + R_g f$, where

$$L_g f = \mathrm{Div}(\nabla_g f) \quad \text{and} \quad R_g f = X \cdot \nabla f,$$

and

$$X = |g|^{-1/2} \sum\nolimits_{i,j} \partial_i \{|g|^{1/2} g^{ij}\} \frac{\partial}{\partial x^j}.$$

We first claim that if we consider

$$E_0(x,t) = H(t)\frac{c_d}{\mathrm{Im}(|x|^2 - (t+i0)^2)^{\frac{d-1}{2}}},$$

then, restricted to $B \times I_0$, $(\partial_t^2 - L_g)E_0 = \delta$.

We begin with a computation involving smooth functions. If $E \in C^\infty(B)$ is smooth and *radial*, with $E(x) = F(|x|^2)$, then we claim that

$$(\nabla_g E)(x) = 2xF'(|x|^2) = (\nabla E)(x).$$

The nontrivial part is proving the first inequality, which follows by the Gauss Lemma because

$$\begin{aligned}
(\nabla_g E)(x) &= G(x)^{-1}\{(\nabla E)(x)\} \\
&= G(x)^{-1}\left(2xF'(|x|^2)\right) \\
&= 2F'(|x|^2)G(x)^{-1}x \\
&= 2F'(|x|^2)x.
\end{aligned}$$

By taking weak limits, we conclude that $\nabla_g E = \nabla E$ for any distribution E that is a weak limit of radial functions. We thus obtain the following result.

Lemma 6.3. *Consider the distribution $E_{0,q}$ on $M \times I_0$, such that supp$(E_{0,q}) \subset B_\delta(q)$, and such that, with respect to the normal coordinate system $x : B_\delta(q) \to B$ constructed above, we have*

$$E_{0,q}(x,t) = H(t) \cdot \frac{c_d}{Im\left(|x|^2 - (t+i0)^2\right)^{\frac{d-1}{2}}}.$$

Then $(\partial_t^2 - L_g)E_{0,q} = \delta_q$.

Proof. In the normal coordinate system x, for each fixed t, $E_{0,q}(\cdot, t)$ is a radial function. Thus

$$L_g E_{0,q} = \operatorname{Div}(\nabla_g E_{0,q}) = \operatorname{Div}(\nabla E_{0,q}) = \Delta E_{0,q} = \delta_q. \qquad \square$$

We have constructed a fundamental solution for L_g, which gives the 'second order terms' for the Laplace-Beltrami operator. We must thus adjust our solution so that the 'lower order terms' R_g are also negligible, which will give us our parametrix. We note that, as in the Euclidean case, we can use similar ideas to the Lemma above to define distributions $E_{\nu,q}$ for all $\nu > 0$, such that

$$(\partial_t^2 - L_g)\{E_{\nu,q}\} = \nu E_{\nu-1,q}.$$

These distributions are just the Euclidean distributions E_ν when studied in normal coordinates about q. Our goal is now to recursively find a sequence of smooth functions $\{\beta_\nu : \nu \geqslant 0\}$ such that, for any $N \geqslant 0$, if we define

$$S_{q,N}(x,t) = \sum_{\nu=0}^{N} \beta_\nu(x) E_{\nu,q}(x,t),$$

then

$$(\Box_g S_{q,N})(x,t) = \delta_q - (\Delta_g \beta_N)(x) E_{N,q}(x,t).$$

The distributions $E_{N,q}$ becomes increasingly smooth as $N \to \infty$, so we therefore construct an arbitrarily smooth approximation to a fundamental solution to the wave equation at q.

TODO: FIX THE COMMENTED OUT CALCULATIONS WHICH SHOW HOW TO FIND

If we choose a orthogonal frame on some open neighborhood of the closure of an open set $U \subset M$, then this induces a smooth family of normal coordinate systems x_q. Since the objects constructed above depend smoothly on the Riemannian metric and coordinate system given, we can therefore find a family of distributions E_ν such that for $d(p,q) < \delta$, and if $\omega = f\ dx_q$ is an order one scalar density, then

$$\begin{aligned}\int_M & E_\nu(p,q,t)\omega(p) \\ &= c_{d,\nu} H(t) \int_{\mathbb{R}^d} \frac{f(x_q^{-1}(x))}{\operatorname{Im}\left(|x|^2 - (t+i0)^2\right)^{\frac{d-1}{2}+\nu}}\ dx.\end{aligned}$$

We can write this integral as a linear combination of terms of the form

$$\begin{aligned} t^j H(t) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(x_q^{-1}(x))}{|\xi|^{\nu+k+1}} e^{2\pi i x \cdot \xi \pm t|\xi|} \; dx \; d\xi \\ = t^j H(t) \int_0^\infty \int_{\mathbb{R}^d} \rho^{d-\nu-k-2} f(x_q^{-1}(x)) B_d(\rho|x|) e^{\pm 2\pi i \rho t} \; dx \; d\rho, \end{aligned}$$

which can in term be written as a linear combination of terms of the form

$$t^j H(t) \int_0^\infty \int_{\mathbb{R}^d} a(\rho|x|) \rho^{d-\nu-k-2} f(x_q^{-1}(x)) e^{2\pi i \rho(\pm|x| \pm t)} \; dx \; d\rho,$$

where a is a symbol of order $-(d-1)/2$. We can write this integral in a coordinate independent way as

$$t^j H(t) \int_M \left(\int_0^\infty a(\rho d_g(p,q)) \rho^{d-\nu-k-2} e^{2\pi i \rho(\pm d_g(p,q) \pm t)} \; d\rho \right) \omega(p) \; dp.$$

Thus the distribution E_ν can be written as a sum of terms given by oscillatory integrals of the form

$$t^j \int_0^\infty a(\rho d_g(p,q)) \rho^s e^{2\pi i \rho(\pm d_g(p,q) \pm t)} \; d\rho.$$

Let us rescale this integral as $t^j d_g(p,q)^{s-1}$ times

$$\int_0^\infty a(\rho) \rho^s e^{2\pi i \rho(\pm d_g(p,q) \pm t)} \; d\rho.$$

Let us consider the integral over $\rho \leqslant 1$, i.e. an integral of the form

$$t^j d_g(p,q)^{s-1} \int_0^\infty \chi(\rho) \rho^s e^{2\pi i \rho(\pm d_g(p,q) \pm t)} \; d\rho.$$

If $s \geqslant 1$, this integral is automatically a smoothing kernel by virtue of the fact that $\chi(\rho)\rho^s$ is a symbol of order $-\infty$. If $\rho \leqslant 1$,

This is the Fourier transform of the function $\chi(\rho) \cdot \rho^s$, evaluated at the point $\mp d_g(p,q) \mp t$.

Modulo a smooth kernel, we may assume $a(\rho) = 0$ for $|\rho| \leqslant 1$.

Let us consider the composition of these integrals at two different times t_0 and t_1. This composition can be expressed as a sum of terms of the form

$$t_0^{j_1} t_1^{j_2} \int_0^\infty a(\rho_1 d_g(p,a)) a(\rho_2 d_g(a,q)) \rho_1^{s_1} \rho_2^{s_2} e^{2\pi i [\rho_1(\pm d_g(p,a) \pm t_1) + \rho_2(\pm d_g(a,q) \pm t_2)]} \; da \; d\rho_0 \; d\rho_1.$$

By finite speed of propogation, we need only integrate over values a such that $d_g(p,a) \leqslant t_0$ and $d_g(a,q) \leqslant t_1$.

$$\chi(P/R) = \int R\widehat{\chi}(Rt)e^{2\pi itP}$$

the kernel of this operator is thus a linear combination of terms of the form

$$R^{-j_0}\int\int_0^\infty \widehat{\chi}(t_0)t_0^{j_0}\beta(d_g(p,q))a(\rho_0 d_g(p,q))\rho_0^{s_0}e^{2\pi i\rho_0(\pm d_g(p,q)\pm t_0/R)}\,dt_0\,d\rho_0$$

Write

$$\alpha_0(t_0,p,q,\rho_0) = \widehat{\chi}(t_0)t_0^{j_0}\beta(d_g(p,q))a(\rho_0 d_g(p,q))\rho_0^{s_0}$$

and

$$\alpha_1(t,p,q,\rho_1) = a(\rho_1 d_g(p,q))\rho_1^{s}.$$

Then the composition of $\chi(P/R)$ with our parametrix at time t is a linear combination of terms of the form

$$\int \alpha_1(t,p,a,\rho_1)\alpha_0(t_0,a,q,\rho_0)e^{2\pi i[\rho_1(\pm d_g(p,a)\pm t)+\rho_0(\pm d_g(a,q)\pm t_0/R)}\,dt_0\,d\rho_0\,d\rho_1\,da.$$

Suppose we localize ρ_1 to values $|\rho_1| \lesssim R$, i.e. we multiply by some function $\eta(\rho_1/R)$. This is equivalent to consider a kernel of the form

$$R\int \chi(\rho_1)\alpha_1(t,p,a,R\rho_1)\alpha_0(t_0,a,q,\rho_0)e^{2\pi i[R\rho_1(\pm d_g(p,a)\pm t)+\rho_0(\pm d_g(a,q)\pm t_0/R)}\,dt_0\,d\rho_0\,d\rho_1\,da.$$

If we now compose it with our parametrix, we find the kernel is a linear combination of terms of the form

$$\int R\widehat{\chi}(Rt_0)t_0^{j'}H(t)\beta(d_g(a,q))a(\rho_0 d_g(a,q))\rho_0^{s'}e^{2\pi i\rho_0(\pm d_g(a,q)\pm t_0)}\,dt_0\,d\rho_0\,da$$

we get terms of the form

$$t^{j+j'}H(t)\int_0^\infty R\widehat{\chi}(Rt')a(\rho_1 d_g(p,q))a(\rho_0 d_g(p,q))\rho_1^{s}\rho_0^{s'}e^{2\pi i[\rho_1(\pm d_g(p,q)\pm t)+\rho_0(\pm d_g(p,q)\pm t')]}d\rho_1 d\rho_0 dt'.$$

If we have $|\rho_1| \lesssim R$ on the support of this integral,

If we assume that on the support of this integral, we either have $|\rho_1| \lesssim R$ or $|\rho_1| \gtrsim R$, We can then write S_N as a sum of terms of the form

$$t^j H(t)\beta(p,q)\int_0^\infty a(\rho d_g(p,q))\rho^s e^{2\pi i\rho(\pm d_g(p,q)\pm t)}\,d\rho.$$

At least on the sphere, by rotational symmetry of the geodesic flow, β is a function of $d_g(p,q)$.

If we now define an operator $T_{H,N}$ on $\{|t| < r_0 - r\} \times M$ to have Schwartz kernel $K_{H,N}$ on $M \times \{|t| < r_0 - r\} \times M$, supported in

$$\{(p,t,q) : |t| < r_0 - r \text{ and } d_g(p,q) < t\},$$

and in the coordinates B about q, agreeing with S_N, then we find that for smooth, compactly supported functions f on M,

$$\Box_g T_{H,N} f - \delta \otimes f$$

is a Lagrangian distribution of order $d/4 - N - 2$ on $\mathbb{R}^{d+1}$, supported with the Lagrangian manifold given on the light cone. Applying asymptotics (TODO: ASK ANDREAS), modulo a smoothing operator, we can then find an operator T_H, whose Schwartz kernel is a Lagrangian distribution of order $d/4 - 1$, such that $\Box_g T_H f - \delta \otimes f$ is a smoothing operator. TODO: Applying this parametrix, we can construct a parametrix of $\cos(2\pi t\sqrt{-\Delta_g})$, which is a Lagrangian distribution of order $d/4$.

TODO: USE HADAMARD PARAMETRIX TO SHOW WAVE EQUATION HAS FINITE PROPOGATION SPEED. SOGGE Chapter 2.

6.4 Explicit Hadamard Parametrix For The Sphere

TODO

Consider geodesic normal coordinates for the sphere in $\mathbb{R}^{d+1}_x$, centered at the south pole. Without loss of generality, to compute this metric we may assume our sphere is the locus given by the equation

$$(y_0 - 1)^2 + y_1^2 + \cdots + y_d^2 = 1$$

and that the south pole is the origin. The metric in these coordinates is precisely the restriction of the metric $\sum dy_i^2$ on $T\mathbb{R}^{d+1}$ to S^d. To work out the geodesic normal coordinates $x = (x_1, \ldots, x_d)$ centered at the south pole. Then

$$y_0 = 1 - \cos(r) \quad \text{and} \quad (y_1, \ldots, y_d) = \sin(r) \cdot \frac{x}{|x|}$$

It will be simpler to work in polar coordinates r and θ, where θ is a unit vector in $\mathbb{R}^d$. Then

$$y_0 = 1 - \cos(r) \quad \text{and} \quad (y_1, \dots, y_d) = \sin(r)\theta.$$

Thus

$$dy_0 = \sin(r)\ dr$$

and for $1 \leqslant i \leqslant d$,

$$dy_i = \cos(r)\theta_i\ dr + \sin(r)d\theta_i.$$

Thus $g^2 = dr^2 + \sin(r)^2 \sum_i (d\theta_i)^2$. The Laplace-Beltrami operator in these coordinates is thus

$$\Delta_g = \frac{1}{\sin(r)^{d-2}} \frac{\partial}{\partial r} \left\{ \sin(r)^{d-2} \frac{\partial}{\partial r} \right\} + \frac{1}{\sin r)^2} \Delta_\theta.$$

And we have

$$|g| = \sin(r)^{2(d-1)}$$

Thus, using the terminology of Sogge, Chapter 2,

$$a^r(x) = -\frac{(d-1)}{\sin(r)}$$

$$\rho = -(d-1)\frac{r}{\sin(r)}$$

$$\alpha_0(r) = \frac{c}{\sin(x)^{\frac{d-1}{2}}}$$

$$-\frac{(d-1)}{2}\cot(r)\alpha_0 = \partial_r \alpha_0$$

The highest order term in the Hadamard Parametrix for the Laplace-Beltrami operator is

$$E_0(t,x) = E_+(t,r)$$

$$g_{rr} = 1$$
$$g_{\theta_i\theta_i} = \sin(r)^2$$
$$g^{rr} = 1$$
$$g^{\theta_i\theta_i} = 1/\sin(r)^2$$

$$g^2 = \left(\cos^2(r)(4d(\sin(r) - 1)^2 + 1)\right) dr^2 + \dots$$

$$g^2 = \left(2\cos(r)\cos\left(\frac{r}{2} + \frac{\pi}{4}\right)\right)^2 dr^2 + \dots$$

and for $1 \leqslant i \leqslant d$,

$$dy_i = (2|x|\cos|x|(1 - \sin|x|) - \sin|x|(2 - \sin|x|))\,\frac{x_i(x \cdot dx)}{|x|^3} 2\cos|x|(1{-}\sin|x|)\frac{x_i(x \cdot dx)}{|x|^2} - \frac{\sin|x|(2 - \text{s}}{}$$

it will help to work with polar normal coordinates $x = r \cdot \theta$, where $r > 0$ and $\theta \in S^{d-1}$. Then $y_0 = \sin(r)$ and $(y_1, \dots, y_d) = \sin(r)(2 - \sin(r))\theta$

point at a height $y_0 = a$ is given by the curve $\gamma(t) = (a,)$

$$\int_0^t$$

Consider the geodesic normal coordinates $y = (y_1, \dots, y_d)$ centered at the south pole.

What is the metric $G = \{g\}$ in these coordinates. In the normal (x, y, z) coordinates, the metric

Chapter 7

Geometrical Optics

Geometrical optics are useful not only in the classical analysis of light, but also as a first order approximation to the quantum behaviour of waves, and statistical optics (the study of randomness in waves – This website might be good to look into this in more detail).

Chapter 8

Notes on Bochner-Riesz

The goal of this section is to compare and contrast approaches to understanding the Bochner-Riesz conjecture on Euclidean space and on compact Riemannian manifolds, in order to reflect on the differences in understanding multipliers on $\mathbb{R}^d$ vs on a compact manifold X before we attack the more general multiplier problem in this setting. We define the Riesz multipliers via symbols $r_\rho^\delta : [0,\infty] \to [0,\infty)$, defined for $\rho > 0$ and a real number δ by setting, for $\tau > 0$,

$$r_\rho^\delta(\tau) = (1 - \tau/\rho)_+^\delta.$$

Here $s_+ = \max(s,0)$. The resulting radial multipliers on $\mathbb{R}^n$, and on a compact Riemannian manifold X, will be denoted by

$$R_\rho^\delta = r_\rho^\delta\left(\sqrt{-\Delta}\right).$$

The goal of the Bochner-Riesz conjecture is to determine bounds on the operators $\{R_\rho^\delta\}$ invariant under dilation of the symbol.

8.1 Euclidean Case

Let's review a reduction of Bochner-Riesz to Tomas Stein:

- First, we can *rescale the problem*. If $r^\delta = r_1^\delta$, then

  $$r_\rho^\delta(\lambda) = r^\delta(\lambda/\rho).$$

  Thus if $R^\delta = R_1^\delta$, then $R_\rho^\delta = R^\delta \circ \mathrm{Dil}_{1/\rho}$, and so the operators $\{R_\rho^\delta\}$ are uniformly bounded from L^p to L^p for all ρ if and only if R^δ is bounded from L^p to L^p.

- We now perform a *spatial decomposition*. Let k^δ be the convolution kernel corresponding to the operator R^δ. We break up the effects of the operator spatially into dyadic annuli, i.e. writing

$$k^\delta(x) = \sum_{j=0}^{\infty} k_j^\delta(2^j x),$$

where k_0^δ is supported on $|x| \leqslant 2$, and all of the other kernels k_j^δ are supported on the annuli $\{1/2 \leqslant |x| \leqslant 1\}$, and can be written as

$$k_j^\delta(x) = \phi \cdot \mathrm{Dil}_{1/2^j} k^\delta$$

for some $\phi \in C_c^\infty$ supported on the annulus $\{1/2 \leqslant |x| \leqslant 2\}$ and equal to one on the annulus $\{3/4 \leqslant |x| \leqslant 3/2\}$. We analyze each of the convolution kernels separately and then collect up each of the bounds we obtain by applying the triangle inequality. Thus we let R_j^δ be the operator with convolution kernel k_j^δ. Provided we can obtain a bound of the form

$$\|R_j^\delta f\|_{L^p(\mathbb{R}^d)} \lesssim 2^{-\varepsilon j} \|f\|_{L^p(\mathbb{R}^d)}$$

for some $\varepsilon > 0$, and some implicit constant uniform in j, we can sum up the bounds using the triangle inequality to bound R^δ.

- Spatial localization means that the operators $\{R_j^\delta\}$ are *local*, i.e. for any function f, the support of $R_j^\delta f$ is contained in a $O(1)$ neighborhood of the support of f. A decomposition argument, thus implies that it suffices to obtain a bound of the form

$$\|R_j^\delta f\|_{L^p(\mathbb{R}^d)} \lesssim 2^{-\varepsilon j} \|f\|_{L^p(\mathbb{R}^d)}$$

for functions f *supported on balls of radius* 1, since the general bound will follow from this.

- We *reduce to L^2 bounds*: Now that f is suported on a ball of radius 1, R_j^δ is supported on a ball of radius $O(1)$, and so for $p \leqslant 2$ we have

$$\|R_j^\delta f\|_{L^p(\mathbb{R}^d)} \lesssim \|R_j^\delta f\|_{L^2(\mathbb{R}^d)}.$$

Thus it suffices to obtain a bound of the form $\|R_j^\delta f\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$. Switching from the L^p norm to the L^2 norm is the most inefficient part of the proof, but it enables us to apply more powerful tools which we only have in $L^2(\mathbb{R}^d)$. Getting around this reduction is key to improving the currently known Bochner-Riesz bounds.

- We reduce the problem to Tomas-Stein. Since we are now in $L^2(\mathbb{R}^d)$, we can apply Plancherel. If ψ_j^δ is the Fourier transform of k_j^δ, then we obtain that
$$\|R_j^\delta f\|_{L^2(\mathbb{R}^d)} = \|\psi_j^\delta \cdot \widehat{f}\|_{L^2(\mathbb{R}^d)}.$$
A stationary phase calculation shows that ψ_j^δ has the majority of its mass on an annulus of radius 2^j and width $O(1)$, and has magnitude $O(2^{-j\delta})$ there, i.e.
$$|\psi_j^\delta(\xi)| \lesssim_N 2^{-\delta j} \langle 2^j - |\xi| \rangle^{-N}.$$
Thus by Tomas-Stein, if R_S denotes the restriction operator to the unit sphere S, we find that
$$\begin{aligned}
\|\psi_j^\delta \cdot \widehat{f}\|_{L^2(\mathbb{R}^d)} &\lesssim_N 2^{-\delta j} \left(\int_0^\infty \langle 2^j |1-r| \rangle^{-2N} \int_{|\xi|=1} |\widehat{f}(r\xi)|^2 \, d\sigma \, r^{d-1} dr \right)^{1/2} \\
&\lesssim 2^{-\delta j} \left(\int_0^\infty \langle 2^j |1-r| \rangle^{-2N} \|R_S \circ \mathrm{Dil}_r f\|_{L^2(S^{n-1})}^2 \frac{dr}{r} \right)^{1/2} \\
&\lesssim 2^{-\delta j} \|f\|_{L^p(\mathbb{R}^d)} \left(\int_0^\infty \langle 2^j |1-r| \rangle^{-2N} r^{2d/p-1} dr \right)^{1/2} \\
&\lesssim 2^{-\delta j} \|f\|_{L^p(\mathbb{R}^d)}.
\end{aligned}$$
This bound is summable in j, which yields the required result.

Let us end our discussion of the Euclidean case by expanding on the computation of the inequality
$$|\psi_j^\delta(\xi)| \lesssim_N 2^{-j\delta} \langle 2^j - |\xi| \rangle^{-N}.$$
Before this, let's see why the result is *intuitive*. The function ψ_j^δ is obtained by localizing the frequency multiplier m^δ on the spatial side and then rescaling. Thus our result is intuitively saying that the phase-portrait of the multiplier is concentrated on a neighborhood of the set
$$\{(x,\xi) : |\xi| \leqslant 1 \text{ and } ||\xi| - 1| = 1/|x|\}.$$
This makes sense, since the 'high frequency' components of m^δ should be distributed near the boundary of the unit ball, since this is where the symbol becomes singular; that the spatial part should be inversely proportional to the distance to the boundary can be detected by taking derivatives of m in the frequency variable, i.e. noting that if $||\xi| - 1| \sim 1/2^j$, then
$$|\nabla^N m^\delta(\xi)| \lesssim_{N,\delta} (1 - |\xi|)^{\delta - N} \sim 2^{-j\delta} 2^{jN}.$$

And we see the derivative grows in N as a power of 2^j, which is inversely propoertional to $||\xi| - 1|$. Working more precisely, we have

$$\psi_j^\delta = 2^{-jd}\left[\widehat{\phi} * \mathrm{Dil}_{2^j} m^\delta\right].$$

The function is the average of $\widehat{\phi}$ over a ball of radius $O(2^j)$ so we immediately obtain a bound by using the rapid decay of $\widehat{\phi}$, thus obtaining that

$$|\psi_j^\delta(\xi)| \lesssim_N \langle 2^j - |\xi| \rangle^{-N}.$$

Thus we see that ψ_j^δ has the majority of it's support on the ball of radius 2^j. But we can do much better than this using the fact that $\widehat{\phi}$ is *oscillatory*, since ϕ is supported away from the origin, and m^δ is *mostly* smooth. More precisely, $\widehat{\phi}$ oscillates at frequencies ~ 1, so we should expect integration by parts to yield useful decay on a quantity $\widehat{\phi} * \mathrm{Dil}_{2^j} f$ if we had a bound $|\nabla^N f| \ll 2^{Nj}$ for large $N > 0$. This is true of m^δ away from a thickness $O(2^{-j})$ annulus containing the unit ball. Thus we are motivated to define $m^\delta = a_j^\delta + b_j^\delta$, where

$$a_j^\delta(\xi) = m^\delta(\xi)\chi(2^j(1 - |\xi|)) \quad \text{and} \quad b_j^\delta(\xi) = m_j^\delta(\xi)(1 - \chi(2^j(1 - |\xi|)))$$

where $\chi(t)$ is supported on $|t| \leqslant 1$ and equal to one for $|t| \leqslant 1/2$. The function b_j^δ is therefore supported on $|\xi| \leqslant 1 - 1/2^{j+1}$. For $N > 0$, we have

$$|\nabla^N m_j^\delta(\xi)| \lesssim_{N,\delta} (1 - |\xi|)^{\delta - N}.$$

By the product rule, $\nabla^N b_j^\delta$ is a sum of derivatives of m_j^δ and of derivatives of $1 - \chi(2^j(|\xi| - 1))$. The support of any derivative of the latter is supported on $|\xi| \geqslant 1 - 1/2^j$. Thus we have

$$|\nabla^N b_j^\delta(\xi)| \lesssim_N (1 - |\xi|)^{\delta - N}\mathbf{I}(|\xi| \leqslant 1 - 1/2^{j+1}) + 2^{j(N-\delta)}\mathbf{I}(1 - 1/2^j \leqslant |\xi| \leqslant 1).$$

Since ϕ is supported away from the origin, we may antidifferentiate $\widehat{\phi}$ arbitrarily many times without any singular behaviour emerging. But now averaging the Nth antiderivative of $\widehat{\phi}$, which is rapidly decaying, with the Nth derivative of $\mathrm{Dil}_{2^j} b_j^\delta$, which is rapidly decaying outside of an annulus of width 1 and radius 2^j, we find that

$$|(\widehat{\phi} * \mathrm{Dil}_{2^j} b_j^\delta)(\xi)| \lesssim 2^{j(d-\delta)}\langle 2^j - \xi \rangle^{-N}.$$

The multiplier a_j^δ is not so smooth, but it is supported on a very thin annulus of radius 1 and thickness $O(2^{-j})$, and m^δ has magnitude at most $2^{-\delta j}$ on this annulus, which gives that

$$|(\widehat{\phi} * \mathrm{Dil}_{2^j} a_j^\delta)(\xi)| \lesssim 2^{-j\delta} \int_{||\eta|-2^j| \leqslant 1} |\widehat{\phi}(\xi - \eta)| \, d\eta \lesssim_N 2^{j(d-\delta)} \langle 2^j - \xi \rangle^{-N}.$$

Putting these results together gives the required bound.

8.2 Manifold Case

The analogue of the Tomas Stein theorem on a compact Riemannian manifold X is a result due to Sogge, so let's see if we can obtain a result for compact manifolds using similar techniques:

- The first problem is that on a compact Riemannian manifold we do not have a rescaling symmetry which we can use to reduce the study of the Bochner-Riesz multipliers R_ρ^δ to the case $\rho = 1$. Thus we must analyze a general multiplier of the form R_ρ^δ for all $\rho > 0$. The case of small ρ is easily dealt with using the triangle inequality, so we may assume that $\rho \gtrsim 1$ in what follows.

- Now we try and reduce to Sogge's spectral cluster bounds, which are analogous to the Tomas-Stein bounds in $\mathbb{R}^d$. If we are able to justify that $K_{\rho,j}^\delta$ behaves like a spectral band projection operator, as in the Euclidean setting, we'd be able to apply this bound. Plancherel does not quite have an analogy to the L^2 setting on a manifold. But we can instead use the wave operator and it's parametrices, i.e. that

$$\begin{aligned} R_\rho^\delta &= \sum_\lambda r^\delta(\lambda/\rho) E_\lambda \\ &= \rho \int_0^\infty \widehat{r^\delta}(\rho t) e^{2\pi i t \sqrt{-\Delta}} \, dt \\ &= c_\delta \cdot \rho^{-\delta} \int_0^\infty e^{2\pi i \rho t} (t + i0)^{-\delta - 1} e^{2\pi i t \sqrt{-\Delta}} \, dt. \end{aligned}$$

  The singularity in the definition of this integral occurs at $t = 0$, so the operator should, for large t, be relatively well behaved.

- Since we expect the function is well behaved for large t, let's bound these terms so we may reduce to controlling the integral over $t \lesssim 1$. Fix $\alpha \in C_c^\infty(\mathbb{R})$ equal to one in a neighborhood of zero, and consider the behaviour of R_ρ^δ for large t, i.e. the operator

$$R_\rho^\delta = c_\delta \cdot \rho^{-\delta} \int_0^\infty (1 - \alpha(t)) \cdot e^{-2\pi i \rho t} t^{-\delta - 1} e^{2\pi i t \sqrt{-\Delta}}\, dt.$$

If ψ is the inverse Fourier transform of $c_\delta t^{-\delta-1}(1-\alpha(t))$, then ψ is bounded and rapidly decreasing because all of the derivatives of it's Fourier transform are smooth and integrable. We thus can revert back to the multiplier setting and write

$$R_\rho^\delta = \rho^{-\delta} \sum_\lambda \psi(\lambda - \rho) E_\lambda.$$

The rapid decay here means we can be fairly lazy in controlling this operator, for instance, employing the Sobolev embedding bound

$$\|E_\lambda f\|_{L^2(X)} \lesssim \langle \lambda \rangle^{d(1/p-1/2)-1/2} \|f\|_{L^p(X)}$$

and the triangle inequality, using the rapid decay to obtain that

$$\|R_\rho^\delta f\|_{L^p(X)} \lesssim \langle \rho \rangle^{-[\delta - d(1/p-1/2)+1/2]} \|f\|_{L^p(X)},$$

which is better than what we need. Thus we now need only bound the operator

$$\tilde{R}_\rho^\delta = c_\delta \cdot \rho^{-\delta} \int_0^\infty \alpha(t) e^{2\pi i \rho t} (t + i0)^{-\delta - 1} e^{2\pi i t \sqrt{-\Delta}}\, dt.$$

The advantage of this localization is that Fourier integral operator methods tend to only give understand of the half-wave operator for times $|t| \lesssim 1$.

- We now 'spatially localize' as in the Euclidean case, though things look different here since we are dealing with the wave equation. We choose β such that

$$1 = \chi + \sum_{j=1}^\infty \mathrm{Dil}_{2^j} \beta.$$

We then write

$$R_\rho^\delta = \sum_{j=0}^{O(\log \rho)} R_{\rho,j}^\delta$$

where for $j > 0$

$$R^\delta_{\rho,j} = c_\delta \cdot \rho^{-\delta} \int_0^\infty \alpha(t)(\mathrm{Dil}_{2^j}\beta)(\rho t) e^{-2\pi i \rho t} t^{-\delta-1} e^{2\pi i \sqrt{-\Delta}}\, dt,$$

and

$$\begin{aligned} R^\delta_{\rho,0} &= c_\delta \cdot \rho^{-\delta} \int_0^\infty \alpha(t)\beta(\rho t) e^{-2\pi i \rho t} t^{-\delta-1} e^{2\pi i \sqrt{-\Delta}}\, dt \\ &= c_\delta \cdot \rho^{-\delta} \int_0^\infty \beta(\rho t) e^{-2\pi i \rho t} t^{-\delta-1} e^{2\pi i \sqrt{-\Delta}}\, dt, \end{aligned}$$

where the last identity follows because the result the support of the integral is on $t \lesssim 1/\rho$, and we are assuming ρ is large so that α may be assumed equal to one on the support of the integral. Thus R^δ_ρ is an integral over times $|t| \sim 2^j/\rho$. This is analogous to the spatial decomposition we performed in the Euclidean setting, except now we have the wave equation involved, and the 'pseudolocal' finite speed of propogation for the wave equation now must substitute for the explicit spatial localization we obtained in the Euclidean decomposition.

- Despite the singularity that occurs at the origin, the case $j = 0$ is simplest to deal with. If we define

$$m(\lambda) = c_\delta(\widehat{\beta} * r_\delta)$$

then $R^\delta_{\rho,0}$ is a multiplier operator with symbol

$$m_\rho(\lambda) = \rho^{-\delta}\mathrm{Dil}_\rho m.$$

We have estimates of the form

$$|\nabla^N m(\lambda)| \lesssim_N \langle \lambda \rangle^{-M}.$$

Thus

$$|\nabla^N m_\rho(\lambda)| \lesssim_N \rho^{-\delta-N}\langle \lambda/\rho \rangle^{-M}.$$

In particular, taking $M = N$ and $M = 0$ yields that

$$|\nabla^N m_\rho(\lambda)| \lesssim_N \rho^{-\delta}\langle \lambda \rangle^{-N}.$$

Thus $\{\rho^\delta m_\rho\}$ are a uniformly bounded family of symbols of order zero. Thus we obtain that

$$\|m_\rho(\sqrt{-\Delta})f\|_{L^p(X)} \lesssim \rho^{-\delta}\|f\|_{L^p(X)} \lesssim \|f\|_{L^p(X)}.$$

TODO: Check there isn't an error here since the $\rho^{-\delta}$ terms helps us out, but shouldn't our bounds be scale invariant?

- Now we deal with the $j > 0$ terms, and we must use the pseudolocal finite speed of propogation of the wave equation as a substitute for explicit localization. Since we have localized to times $t \lesssim 1$. We deal with this by using the Lax parametrix for the wave equation, but first we must ensure the remainder terms from employing the parametrix are well behaved. For $t \lesssim 1$, we can write $e^{2\pi it\sqrt{-\Delta}} = Q(t) + R(t)$, where $Q(t)$ is a Fourier integral operator supported on a $O(1)$ neighborhood of the diagonal $\Delta = \{(x,x) : x \in X\}$, and with kernel given in coordinates by

$$(x,y) \mapsto \int e^{2\pi i[\phi(x,y,\xi)+t|\xi|]} q(t,x,y,\xi)\, d\xi$$

where q is a symbol of order zero, and ϕ is homogeneous of order one in ξ, with $\phi(x,y,\xi) \approx (x-y)\cdot\xi$, in the sense that

$$|\nabla_\xi^N[\phi(x,y,\xi) - (x-y)\cdot\xi]| \lesssim_N |x-y|^2|\xi|^{1-N}$$

for all $N > 0$. The operators $\{R(t)\}$ are smoothing, i.e. with a joint kernel A uniformly in $C^\infty([-1,1]\times X\times X)$. Thus we write

$$\begin{aligned} R^\delta_{\rho,j} &= c_\delta \cdot \rho^{-\delta}\int_0^\infty \alpha(t)(\mathrm{Dil}_{2^j}\beta)(\rho t)e^{-2\pi i\rho t}t^{-\delta-1}(Q(t)+R(t))\, dt \\ &= R^\delta_{\rho,j,Q} + R^\delta_{\rho,j,R}. \end{aligned}$$

Let's control the $R(t)$ term. Computing the integral of the kernel defining $R^\delta_{\rho,j,R}$ leads to a term of the form

$$c_\delta 2^j\rho^{-1-\delta}(\widehat{\alpha A} * \mathrm{Dil}_{\rho/2^j}\widehat{\beta} * r^\delta)(\rho).$$

The function αA is smooth and compactly supported in the t variable, so it's Fourier transform is rapidly decaying. The same is true of $\widehat{\beta}$, except it is rescaled so we can imagine the majority of it's mass occurs on $|\lambda| \lesssim \rho/2^j$.

Finally, r^δ is concentrated on $|\lambda| \lesssim 1$. Thus the kernel is pointwise bound from above by a constant times

$$2^j \rho^{-1-\delta} \int_{\rho - O(1)}^{\rho + O(1)} (\widehat{\alpha A} * \mathrm{Dil}_{\rho/2^j} \widehat{\beta})(\lambda)\, d\lambda.$$

Taking advantage of the oscillation of $\widehat{\beta}$, and the smoothness of $\widehat{\alpha A}$, i.e. integrating by parts, one can shows that for $|\lambda - \rho| \lesssim 1$

$$|(\widehat{\alpha A} * \mathrm{Dil}_{\rho/2^j} \widehat{\beta})(\lambda)| \lesssim_{N,M} (\rho/2^j)^N \cdot \rho^{-M} \cdot (\rho/2^j) = \rho^{1+N-M} 2^{-(N+1)j},$$

Taking $N = M$ gives that the kernel is bounded above by

$$2^j \rho^{-1-\delta} \left(\rho 2^{-(N+1)j}\right) = 2^{-Nj}.$$

But now trivial estimates, e.g. using Schur's lemma implies that

$$\|R^\delta_{\rho,j,R} f\|_{L^p(X)} \lesssim_N \rho^{-\delta} 2^{-Nj} \|f\|_{L^p(X)},$$

a bound that can be summed in j by taking, e.g. $N = 1$. Thus we are now reduced to the study of the oscillatory integral operators $R^\delta_{\rho,j,Q}$.

- Now let's localize. First off, the condition that $K_{\rho,j}$ is supported on the diagonal, and the compactness of X, means we need only prove the result restricted to a single coordinate chart. Let $K_{\rho,j,t}$ be the kernel of the operator $R^\delta_{\rho,j,Q}$. Intuitively, the wave equation travels at unit speed, so, since $R^\delta_{\rho,j,Q}$ involves the wave equation localized to times $t \sim 2^j/\rho$, we should expect this kernel to be localized to $|x - y| \lesssim 2^j/\rho$. In fact, we will show that the restricted kernel

$$K'_{\rho,j,t}(x,y) = K_{\rho,j,t}(x,y) \cdot \mathbf{I}(|x-y| \geqslant 2^{j(1+\varepsilon)}/\rho)$$

has $L^\infty_y L^1_x$ and $L^\infty_x L^1_y$ bounds of the form $O_{\varepsilon,N}(2^{-jN})$, so that Schur's lemma implies that if we write $(R^\delta_{\rho,j,Q})'$ as the operator with kernel $K'_{\rho,j,t}$, then

$$\|(R^\delta_{\rho,j,Q})' f\|_{L^p(X)} \lesssim_N 2^{-jN} \|f\|_{L^p(X)}.$$

This reduces us to proving localized estimates of the following form: for some $\varepsilon > 0$, and for any function f supported on a ball of radius $2^j/\rho$, we have a bound

$$\|R^\delta_{\rho,j,Q} f\|_{L^p(O(2^j/\rho))} \lesssim 2^{-j\varepsilon} \|f\|_{L^p(X)}.$$

Notice the localization we get here is slightly weaker than in the Euclidean setting (the operators are localized to balls of radius $O(2^{j(1+\varepsilon)}/\rho)$ for any $\varepsilon > 0$ rather than localized to balls of radius $O(2^j/\rho)$) which means our bounds here need the slightly greater decay in j (the $O(2^{-j\varepsilon})$ bound above) rather than a bound independent of j.

To prove the bounds for the restricted kernel $K'_{\rho,j,t}$ above, we just apply the principle of nonstationary phase to the integral representation, which says that for $|x-y| \gtrsim 2^{j(1+\varepsilon)}/\rho$ we have, taking the Fourier inversion formula in the t variable,

$$\begin{aligned} K'_{\rho,j,t} &= c_\delta \rho^{-\delta} \int_0^\infty \int \alpha(t)(\mathrm{Dil}_{2^j}\beta)(\rho t)(t+i0)^{-\delta-1} q(t,x,y,\xi) e^{2\pi i[\phi(x,y,\xi)+t|\xi|-\rho t]}\, d\xi\, dt \\ &= \int a^\delta_{\rho,j}(x,y,\xi,|\xi|-\rho) e^{2\pi i \phi(x,y,\xi)}\, d\xi, \end{aligned}$$

where

$$a^\delta_{\rho,j}(x,y,\xi,\cdot) = c_\delta 2^j \rho^{-1-\delta}(\alpha \widehat{q(\cdot,x,y},\xi) * \mathrm{Dil}_{\rho/2^j}\beta * r^\delta * q.(x,y,\xi))$$

and therefore TODO satisfies estimates of the form

$$|\nabla_t^n \nabla_\xi^m a^\delta_{\rho,j}| \lesssim_{n,m,N} 2^{-j\delta}(2^j/\rho)^n \langle 2^j \tau/\rho\rangle^{-N} \langle \xi \rangle^{-m}$$

Nonstationary phase TODO thus gives the required bounds.

- It now suffices to show that for some $\varepsilon > 0$, and for any function f supported on a ball B of radius $2^j/\rho$, we have a bound

  $$\|R^\delta_{\rho,j,Q} f\|_{L^p(O(1)\cdot B)} \lesssim 2^{-j\varepsilon}\|f\|_{L^p(X)}.$$

  Since we are localized, we can now, like in the Euclidean case, reduce to an L^2 bound, i.e. writing

  $$\|R^\delta_{\rho,j,Q} f\|_{L^p(O(1)\cdot B)} \lesssim (2^j/\rho)^{d(1/p-1/2)}\|R^\delta_{\rho,j,Q} f\|_{L^2(O(1)\cdot B)}.$$

  It now suffices to note TODO that $R^\delta_{\rho,j,Q}$ is a Fourier multiplier operator with symbol which is pointwise bouned by $O_N(2^{-j\delta}\langle 2^j\tau/\rho\rangle^{-N})$, so we can now TODO apply Sogge's version of Tomas Stein on manifolds summed over geometric intervals to yield the required bounds.

Part II

Review of Relevent Literature

Chapter 9

Heo, Nazarov, and Seeger: Initial Radial-Multiplier Conjecture Results

In this chapter we give a description of the techniques of Heo, Nazarov, and Seeger's 2011 paper *Radial Fourier Multipliers in High Dimensions* [9]. One of the main goals of this paper is to verify the radial multiplier conjecture in $\mathbb{R}^d$ for $d \geqslant 4$, and $1 < p < p_d$, where $p_d = 2(d-1)/(d+1)$, i.e. that if $m \in L^\infty(\mathbb{R}^d)$ is a radial function, $d \geqslant 4$, and $\beta \in \mathcal{S}(\mathbb{R}^d)$ is nonzero, then

$$\|m\|_{M^p(\mathbb{R}^d)} \sim \sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \beta)\|_{L^p(\mathbb{R}^d)} \quad \text{for } p \in \left(1, \frac{2(d-1)}{d+1}\right),$$

where the implicit constant depends on p and β. We have

$$\sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \beta)\|_{L^p(\mathbb{R}^d)} \sim \sup_{t>0} \frac{\|T_m(\mathrm{Dil}_t \beta)\|_{L^p(\mathbb{R}^d)}}{\|\mathrm{Dil}_t \beta\|_{L^p(\mathbb{R}^d)}}.$$

Thus we find that the boundedness of T_m on $\mathcal{S}(\mathbb{R}^d)$ is equivalent to it's boundedness on the family of inputs $\{\mathrm{Dil}_t \beta\}$. If we make the assumption that m is compactly supported, then the assumption is equivalent to the fact that the convolution kernel k associated with m is in $L^p(\mathbb{R}^n)$.

Another consequence of the techniques of this paper is that an 'sharp' result for local smoothing. Namely, the techniques of the paper imply that if $d \geqslant 4$, and $q > 2 + 4/(d-3)$, then

$$\frac{1}{2L} \int_{-L}^{L} \|e^{it\sqrt{-\Delta}} f\|_{L^q(\mathbb{R}^d)}^q \, dt \lesssim_{q,d} \|(I - L^2 \Delta)^{\alpha/2} f\|_{L^q(\mathbb{R}^d)}^q,$$

where $\alpha = d(1/2 - 1/q) - 1/2$. This is an sharp result because α can be set *equal* to the tight local smoothing exponent, rather than just arbitrarily close as in results that use decoupling type techniques.

9.1 Discretized Reduction

It is obvious that

$$\|m\|_{M^p(\mathbb{R}^d)} \gtrsim_\beta \sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \beta)\|_{L^p(\mathbb{R}^d)},$$

so it suffices to show that

$$\|m\|_{M^p(\mathbb{R}^d)} \lesssim_\beta \sup_{t>0} t^{d/p} \|T_m(\mathrm{Dil}_t \beta)\|_{L^p(\mathbb{R}^d)},$$

We will show this via a discrete convolution inequality, which can also be used to prove local smoothing results for the wave equation.

Let σ_r be the surface measure for the sphere of radius r centered at the origin in $\mathbb{R}^d$. Also fix a nonzero, radial, compactly supported function $\psi \in \mathcal{S}(\mathbb{R}^d)$ whose Fourier transform is non-negative, and vanishes to high order at the origin. Given $x \in \mathbb{R}^d$ and $r \geqslant 1$, define $\chi_{xr} = \mathrm{Trans}_x(\sigma_r * \psi)$, which we view as a smooth function oscillation on a thickness ≈ 1 annulus of radius r centered at x. Our goal is to prove the following inequality.

Lemma 9.1. *For any $a : \mathbb{R}^d \times [1, \infty) \to \mathbb{C}$, and $1 \leqslant p < p_d$,*

$$\left\| \int_{\mathbb{R}^d} \int_1^\infty a(x,r) \chi_{x,r} \, dx \, dr \right\|_{L^p(\mathbb{R}^d)} \lesssim \left(\int_{\mathbb{R}^d} \int_1^\infty |a(x,r)|^p r^{d-1} dr dx \right)^{1/p}.$$

The implicit constant here depends on p, d, and ψ.

How does Lemma 9.1 prove the required result? Suppose $m : \mathbb{R}^d \to \mathbb{C}$ is a radial multiplier, so we can consider it's convolution kernel $k : \mathbb{R}^d \to \mathbb{C}$, which is also radial, with $k(\cdot) = b(|\cdot|)$ for some function $b : [0, \infty) \to \mathbb{C}$. If we set $a(x,r) = g(x) b(r)$ for any function $g : \mathbb{R}^d \to \mathbb{C}$, then the function

$$F = \int_{\mathbb{R}^d} \int_1^\infty a(x,r) \chi_{x,r} \, dx \, dr,$$

is equal to $k * \psi * g$. In this setting, Lemma 9.1 says that

$$\|k * \psi * g\|_{L^p(\mathbb{R}^d)} \lesssim \|k\|_{L^p(\mathbb{R}^d)} \|g\|_{L^p(\mathbb{R}^d)}.$$

The left hand side is a Fourier multiplier operator applied to g, with symbol equal to $\widehat{\psi} \cdot m$, which is clearly related to the bounds we want to show. In particular, if m is compactly supported away from the origin, let's say, on the annulus $1/2 \leqslant |\xi| \leqslant 2$. If we chose ψ so that $\widehat{\psi}$ is nonvanishing on the annulus $1/4 \leqslant |\xi| \leqslant 2$, then the multiplier $1/\widehat{\psi}$ is smooth on the support of m, and so satisfies $L^p \to L^p$ bounds for all $1 < p < \infty$ restricted to functions with Fourier support on m. In particular, we conclude that m is bounded from L^p to L^p if it's Fourier transform lies in $L^p(\mathbb{R}^d)$. We can then use other tools (Hardy space technology and the like) to study more general multipliers that aren't compactly supported.

To prove Lemma 9.1, it suffices to prove the following discretized estimate where we replace integrals with sums.

Theorem 9.2. *Fix a finite family of pairs $\mathcal{E} \subset \mathbb{R}^d \times [1, \infty)$, which is* discretized *in the sense that for any (x_1, r_1) and (x_2, r_2) in $\mathcal{E}$, one either has $x_1 = x_2$, or $|x_1 - x_2| \geqslant 1$, and one either has $r_1 = r_2$, or $|r_1 - r_2| \geqslant 1$. Then for any $a : \mathcal{E} \to \mathbb{C}$ and $1 \leqslant p < 2(d-1)/(d+1)$,*

$$\left\| \sum_{(x,r) \in \mathcal{E}} a(x,r) \chi_{x,r} \right\|_{L^p(\mathbb{R}^d)} \lesssim \left(\sum_{(x,r) \in \mathcal{E}} |a(x,r)|^p r^{d-1} \right)^{1/p},$$

where the implicit constant depends on p, d, and ψ, but most importantly, is independent of $\mathcal{E}$.

Proof of Lemma 9.1 from Lemma 9.2. For any $a : \mathbb{R}^d \times [1, \infty) \to \mathbb{C}$, if we consider the vector-valued function $\mathbf{a}(x,r) = a(x,r)\chi_{x,r}$, then

$$\int_{\mathbb{R}^d} \int_1^\infty \mathbf{a}(x,r)\, dr\, dx = \int_{[0,1)^d} \int_0^1 \sum\nolimits_{n \in \mathbb{Z}^d} \sum\nolimits_{m > 0} \mathrm{Trans}_{n,m} \mathbf{a}(x,r)\, dr\, dx$$

The triangle inequality, and then the increasing property of norms on $[0,1)^d \times [0,1]$

implies that

$$
\begin{aligned}
&\left\| \int_{\mathbb{R}^d} \int_1^\infty \mathbf{a}(x,r)\, dr\, dx \right\|_{L^p(\mathbb{R}^d)} \\
&\quad \leqslant \int_{[0,1)^d} \int_0^1 \left\| \sum\nolimits_{n \in \mathbb{Z}^d} \sum\nolimits_{m>0} \mathrm{Trans}_{n,m} \mathbf{a}(x,r) \right\|_{L^p(\mathbb{R}^d)}\, dr\, dx \\
&\quad \lesssim \int_{[0,1)^d} \int_0^1 \left(\sum\nolimits_{n \in \mathbb{Z}^d} \sum\nolimits_{m>0} |a(x-n, r+m)|^p r^{d-1} \right)^{1/p}\, dr\, dx \\
&\quad \leqslant \left(\int_{[0,1)^d} \int_0^1 \sum\nolimits_{n \in \mathbb{Z}^d} \sum\nolimits_{m>0} |a(x-n, r+m)|^p r^{d-1}\, dr\, dx \right)^{1/p} \\
&\quad = \left(\int_{\mathbb{R}^d} \int_1^\infty |a(x,r)|^p r^{d-1} dr dx \right)^{1/p}. \qquad \square
\end{aligned}
$$

Lemma 9.2 is further reduced by considering it as a bound on the operator $a \mapsto \sum_{(x,r) \in \mathcal{E}} a(x,r) \chi_{x,r}$. In particular, applying real interpolation, it suffices for us to prove a restricted strong type bound. Given any discretized set $\mathcal{E}$, let $\mathcal{E}_k$ be the set of $(x,r) \in \mathcal{E}$ with $2^k \leqslant r < 2^{k+1}$. Then Lemma 9.2 is implied by the following Lemma.

Lemma 9.3. *For any* $1 \leqslant p < 2(d-1)/(d+1)$ *and* $k \geqslant 1$,

$$
\left\| \sum_{(x,r) \in \mathcal{E}} \chi_{x,r} \right\|_{L^p(\mathbb{R}^d)} \lesssim_p \left(\sum_{k \geqslant 1} 2^{k(d-1)} \#(\mathcal{E}_k) \right)^{1/p}.
$$

Remark. Note that if $2^k \leqslant r \leqslant 2^{k+1}$, then because $\|\chi_{x,r}\|_{L^p(\mathbb{R}^d)} \sim 2^{k(d-1)/p}$, we can write this as

$$
\left\| \sum_{(x,r) \in \mathcal{E}} \chi_{x,r} \right\|_{L^p(\mathbb{R}^d)} \lesssim_p \left(\sum_{(x,r) \in \mathcal{E}} \|\chi_{x,r}\|_{L^p(\mathbb{R}^d)}^p \right)^{1/p}.
$$

Thus we are proving a kind of $l^p L^p$ decoupling for the functions $\chi_{x,r}$. This is strictly weaker than an $l^2 L^p$ decoupling bound. TODO: Could we possibly get an $l^2 L^p$ decoupling bound here?

9.2 Density Decomposition

To control these sums, we apply a 'density decomposition', somewhat analogous to a Calderon Zygmund decomposition, which will enable us to obtain L^2 bounds.

We say a 1-separated set $\mathcal{E}$ in $\mathbb{R}^d \times [R, 2R)$ is of *density type* (u, R) if

$$\#(B \cap \mathcal{E}) \leqslant u \cdot \mathrm{diam}(B)$$

for each ball B in $\mathbb{R}^{d+1}$ with diameter $\leqslant R$.

Theorem 9.4. *For any 1-separated set $\mathcal{E}_k \subset \mathbb{R}^d \times [2^k, 2^{k+1})$, we can consider a disjoint union $\mathcal{E}_k = \bigcup_{m=1}^{\infty} \mathcal{E}_k(2^m)$ with the following properties:*

- *For each m, $\mathcal{E}_k(2^m)$ has density type $(2^m, 2^k)$.*
- *If B is a ball of radius $r \leqslant 2^k$ containing at least $2^m \cdot r$ points of $\mathcal{E}_k$, then*

$$B \cap \mathcal{E}_k \subset \bigcup_{m' \geqslant m} \mathcal{E}_k(2^{m'}).$$

- *For each m, there are disjoint balls $\{B_i\}$, with radii $\{r_i\}$, each at most 2^k, such that*

$$\sum_i r_i \leqslant \frac{\#(\mathcal{E}_k)}{2^m}$$

such that $\bigcup B_i^*$ covers $\bigcup_{m' \geqslant m} \mathcal{E}_k(2^{m'})$, where B_i^* denotes the ball with the same center as B_i but 5 times the radius.

Proof. Define a function $M : \mathcal{E}_k \to [0, \infty)$ by setting

$$M(x, r) = \sup \left\{ \frac{\#(\mathcal{E}_k \cap B)}{\mathrm{rad}(B)} : (x, r) \in B \text{ and } \mathrm{rad}(B) \leqslant 2^k \right\}.$$

We can establish a kind of weak L^1 estimate for M using a Vitali type argument. Let

$$\widehat{\mathcal{E}}_k(2^m) = \{(x, r) \in \mathcal{E}_k : M(x, r) \geqslant 2^m\}.$$

We can therefore cover $\widehat{\mathcal{E}}_k(2^m)$ by a family of balls $\{B\}$ such that $\#(\mathcal{E}_k \cap B) \geqslant 2^m \mathrm{rad}(B)$. The Vitali covering lemma allows us to find a disjoint subcollection of balls $B_1, \dots, B_N$ such that $B_1^*, \dots, B_N^*$ covers $\widehat{\mathcal{E}}_k(2^m)$. We find that

$$\#(\mathcal{E}_k) \geqslant \sum_i \#(B_i \cap \mathcal{E}_k) \geqslant 2^m \sum_i \mathrm{rad}(B_i),$$

Setting $\mathcal{E}_k = \widehat{\mathcal{E}}_k(2^m) - \bigcup_{k' > k} \widehat{\mathcal{E}}_{k'}(2^m)$ thus gives the required result. $\square$

To prove Lemma 9.3, we perform a decomposition of $\mathcal{E}_k$ for each k, into the sets $\mathcal{E}_k(2^m)$, and then define $\mathcal{E}^m = \bigcup_{k\geqslant 1} \mathcal{E}_k^m$. For appropriate exponents, we will prove L^p bounds on the functions

$$F^m = \sum_{(x,r)\in\mathcal{E}^m} \chi_{x,r}$$

which are exponentially decaying in m, i.e. that

$$\|F^m\|_{L^p(\mathbb{R}^d)} \lesssim m \cdot 2^{-m(1/p-1/p_d)} \left(\sum_k 2^{k(d-1)} \#(\mathcal{E}_k) \right)^{1/p}.$$

Thus summing in m using the triangle inequality gives a bound on $F = \sum_m F^m$, in the range $1 < p < p_d$, i.e. that

$$\|F\|_{L^p(\mathbb{R}^d)} \lesssim \left(\sum_k 2^{k(d-1)} \#(\mathcal{E}_k) \right)^{1/p},$$

proving Lemma 9.3. To get the bound on F^m, we interpolate between an L^2 bound for F^m, and an L^0 bound (i.e. a bound on the measure of the support of F^m). First, we calculate the support of F^m.

Lemma 9.5. *For each k,*

$$|supp(F_k^m)| \lesssim 2^{-m} 2^{k(d-1)} \#\mathcal{E}_k.$$

Thus we have

$$|supp(F^m)| \leqslant \sum_k |supp(F_k^m)| \lesssim \sum_k 2^{-m} 2^{k(d-1)} \#\mathcal{E}_k.$$

Proof. We recall that for each k and m, we can find disjoint balls $B_1, \dots, B_N$ with radii $r_1, \dots, r_N \leqslant 2^k$ such that

$$\sum_{i=1}^N r_i \leqslant 2^{-m} \#\mathcal{E}_k,$$

where $\mathcal{E}_k(2^m)$ is covered by the expanded balls $B_1^* \cup \cdots \cup B_N^*$. If we write

$$F_{k,i}^m = \sum_{(x,r)\in\mathcal{E}_k(2^m)\cap B_i^*} \chi_{x,r},$$

then $\text{supp}(F_k^m) \subset \bigcup_i \text{supp}(F_{k,i}^m)$. For each $(x,r) \in B_i^* \cap \mathcal{E}_k(2^m)$, the support of $\chi_{x,r}$, an annulus of thickness $O(1)$ and radius r, is contained in an annulus of thickness $O(r_i)$ and radius $O(2^k)$ with the same centre as B_i. Thus we conclude that

$$|\text{supp}(F_{k,i}^m)| \lesssim r_i 2^{k(d-1)},$$

and it follows that

$$|\text{supp}(F_k^m)| \leqslant \sum_i r_i 2^{k(d-1)} \leqslant 2^{-m} 2^{k(d-1)} \#\mathcal{E}_k. \qquad \square$$

From interpolation, it therefore suffices to prove the following L^2 estimate on the function F^m.

Lemma 9.6. *Suppose $\mathcal{E} = \bigcup_k \mathcal{E}_k$ is a one-separated set, where $\mathcal{E}_k \subset \mathbb{R}^d \times [2^k, 2^{k+1})$ is a set of density type $(2^m, 2^k)$. Then*

$$\left\| \sum_{(x,r)\in\mathcal{E}} \chi_{x,r} \right\|_{L^2(\mathbb{R}^d)} \lesssim \sqrt{m} \cdot 2^{\frac{m}{d-1}} \left(\sum_k 2^{k(d-1)} \#(\mathcal{E}_k) \right)^{1/2}.$$

Note that this bound gets worse and worse as m grows, whereas the support bound gets better and better, since annuli are concentrating in a small set, which is bad from the perspective of constructive interference, but absolutely fine from the perspective of a support bound. Interpolation gives a bound exponentially decaying in m for $1 < p < p_d$.

9.3 L^2 Bounds

Proving 9.6 is where the weak-orthogonality bounds from Lemma 9.7 come into play. Indeed, we can write the inequality as

$$\left\| \sum_{(x,r)\in\mathcal{E}} \chi_{x,r} \right\|_{L^2(\mathbb{R}^d)} \lesssim \sqrt{m} \cdot 2^{\frac{m}{d-1}} \left(\sum_{(x,r)\in\mathcal{E}} \|\chi_{x,r}\|_{L^2(\mathbb{R}^d)}^2 \right)^{1/2},$$

and if we had perfect orthogonality, or even almost orthogonality, then we could replace the $\sqrt{m} \cdot 2^{\frac{m}{d-1}}$ term with a constant.

To prove this L^2 bound, we require an analysis of the interference patterns of the functions $\chi_{x,r}$, which are supported on various annuli, but oscillate on these annuli. We will use almost orthogonality principles to understand these interference patterns which work the best now we have reduced our analysis to L^2 bounds.

Lemma 9.7. *For any $N > 0$, $x_1, x_2 \in \mathbb{R}^d$ and $r_1, r_2 \geqslant 1$,*

$$|\langle \chi_{x_1,r_1}, \chi_{x_2,r_2} \rangle| \lesssim_N (r_1 r_2)^{(d-1)/2} (1 + |r_1 - r_2| + |x_1 - x_2|)^{-(d-1)/2} \sum_{\pm,\pm} (1 + ||x_1 - x_2| \pm r_1 \pm r_2|)^{-N}.$$

In particular,

$$|\langle \chi_{x_1,r_1}, \chi_{x_2,r_2} \rangle| \lesssim \left(\frac{r_1 r_2}{|(x_1, r_1) - (x_2, r_2)|} \right)^{(d-1)/2}$$

Remark. Suppose $r_1 \leqslant r_2$. Then Lemma 9.7 implies that χ_{x_1,r_1} and χ_{x_2,r_2} are roughly uncorrelated, except when $|x_1 - x_2|$ and $|r_1 - r_2|$ is small, and in addition, one of the following two properties hold:

- $r_1 + r_2 \approx |x_1 - x_2|$, which holds when the two annuli are 'approximately' externally tangent to one another.
- $r_2 - r_1 \approx |x_1 - x_2|$, which holds when the two annuli are 'approximately' internally tangent to one another.

Heo, Nazarov, and Seeger do not exploit the tangency information, though utilizing the tangencies seems important to improve the results they obtain. Laura Cladek's paper exploits this tangency information, to some extent, to obtain the improved result in her paper.

Proof. We write

$$\begin{aligned}
\langle \chi_{x_1 r_1}, \chi_{x_2 r_2} \rangle &= \langle \widehat{\chi}_{x_1 r_1}, \widehat{\chi}_{x_2 r_2} \rangle \\
&= \int_{\mathbb{R}^d} \widehat{\sigma_{r_1} * \psi}(\xi) \cdot \overline{\widehat{\sigma_{r_2} * \psi}(\xi)} e^{2\pi i (x_2 - x_1) \cdot \xi} \, d\xi \\
&= (r_1 r_2)^{d-1} \int_{\mathbb{R}^d} \widehat{\sigma}(r_1 \xi) \overline{\widehat{\sigma}(r_2 \xi)} |\widehat{\psi}(\xi)|^2 e^{2\pi i (x_2 - x_1) \cdot \xi} \, d\xi.
\end{aligned}$$

Define functions A and B such that $B(|\xi|) = \widehat{\sigma}(\xi)$, and $A(|\xi|) = |\widehat{\psi}(\xi)|^2$. Then

$$\langle \chi_{x_1,r_1}, \chi_{x_2,r_2} \rangle = C_d (r_1 r_2)^{d-1} \int_0^\infty s^{d-1} A(s) B(r_1 s) B(r_2 s) B(|x_2 - x_1| s) \, ds.$$

Using well known asymptotics for the Fourier transform for the spherical measure, we have

$$B(s) = s^{-(d-1)/2} \sum_{n=0}^{N-1} (c_{n,+} e^{2\pi i s} + c_{n,-} e^{-2\pi i s}) s^{-n} + O_N(s^{-N}).$$

But now substituting in, assuming $A(s)$ vanishes to order $100N$ at the origin, we conclude that

$$\begin{aligned}
\langle \chi_{x_1 r_1}, \chi_{x_2 r_2} \rangle &= C_d \left(\frac{r_1 r_2}{|x_1 - x_2|} \right)^{(d-1)/2} \sum_{n,\tau} c_{n,\tau} r_1^{-n_1} r_2^{-n_2} |x_2 - x_1|^{-n_3} \\
&\qquad \left\{ \int_0^\infty A(s) s^{-(d-1)/2} s^{-n_1-n_2-n_3} e^{2\pi i (\tau_1 r_1 + \tau_2 r_2 + \tau_3 |x_2 - x_1|) s}\, ds \right\} \\
&\lesssim_N \left(\frac{r_1 r_2}{|x_1 - x_2|} \right)^{\frac{d-1}{2}} \left(1 + \frac{1}{|x_1 - x_2|^N} \right) \sum_\tau \left(1 + |\tau_1 r_1 + \tau_2 r_2 + \tau_3 |x_2 - x_1|| \right)^{-5N} \\
&\lesssim_N \left(\frac{r_1 r_2}{|x_1 - x_2|} \right)^{\frac{d-1}{2}} \left(1 + \frac{1}{|x_1 - x_2|^N} \right) \sum_\tau \left(1 + |\tau_1 \tau_3 r_1 + \tau_2 \tau_3 r_2 + |x_2 - x_1|| \right)^{-5N}.
\end{aligned}$$

This gives the result provided that $1 + |x_1 - x_2| \geqslant |r_1 - r_2|/10$ and $|x_1 - x_2| \geqslant 1$. If $1 + |x_1 - x_2| \leqslant |r_1 - r_2|/10$, then the supports of χ_{x_1,r_1} and χ_{x_2,r_2} are disjoint, so the inequality is trivial. On the other hand, if $|x_1 - x_2| \leqslant 1$, then the bound is trivial by the last sentence unless $|r_1 - r_2| \leqslant 10$, and in this case the inequality reduces to the simple inequality

$$\langle \chi_{x_1,r_1}, \chi_{x_2,r_2} \rangle \lesssim_N (r_1 r_2)^{(d-1)/2}.$$

But this follows immediately from the Cauchy-Schwartz inequality. □

The exponent $(d-1)/2$ in Lemma 9.7 is too weak to apply almost orthogonality directly to obtain L^2 bounds on $\sum_{(x,r)\in\mathcal{E}_k} \chi_{xr}$ on it's own, but together with the density decomposition assumption we will be able to obtain Lemma 9.6.

Proof of Lemma 9.6. Without loss of generality, we may assume that the k such that $\mathcal{E}_k \neq \varnothing$ is 10-separated. Write

$$F = \sum_{(x,r)\in\mathcal{E}} \chi_{x,r}$$

and $F_k = \sum_{(x,r)\in\mathcal{E}_k} \chi_{x,r}$. First, we deal with $F_{\lesssim m} = \sum_{k\leqslant 10m} F_k$ trivially, i.e. writing

$$\|F\|_{L^2(\mathbb{R}^d)} \lesssim m^{1/2} \left(\sum_{k\leqslant 10m} \|F_k\|^2_{L^2(\mathbb{R}^d)} + \| \sum_{k>10m} F_k \|_{L^2(\mathbb{R}^d)} \right)^{1/2}.$$

We then decompose

$$\| \sum_{k>10m} F_k\|^2_{L^2(\mathbb{R}^d)} \leqslant \sum_{k>10m} \|F_k\|^2_{L^2(\mathbb{R}^d)} + 2 \sum_{k'>k>10m} |\langle F_k, F_{k'}\rangle|.$$

Let us analyze $\langle F_k, F_{k'}\rangle$. The term will become a sum of the form $\langle \chi_{x,r}, \chi_{y,s}\rangle$, where $r \sim 2^k$ and $s \sim 2^{k'}$. Because of our assumption of being 10-separated, we have $r \leqslant s/2^{10}$. If $\langle \chi_{x,r}, \chi_{y,s}\rangle \neq 0$, then since the support of $\chi_{y,s}$ is an annulus of radius s centered at y, with thickness $O(1)$, and $\chi_{x,r}$ has support on an annulus of radius r centered at x, with thickness $O(1)$, the fact that r is comparatively smaller than s implies that (x,r) must be contained in the annulus of radius s centered at y, with thickness $O(2^k)$. Such an annulus is covered by $O(2^{(k'-k)(d-1)})$ balls of radius 2^k. Each ball can only contain 2^{k+m} points (x,r), and so there can be at most

$$O(2^{k'(d-1)}2^{-k(d-1)}2^{k+m}) = O(2^{k'(d-1)-k(d-2)+m}).$$

pairs $(x,r) \in \mathcal{E}_k$ for which $\langle \chi_{x,r}, \chi_{y,s}\rangle \neq 0$. For such pairs we have

$$|\langle \chi_{x,r}, \chi_{y,s}\rangle| \lesssim \left(\frac{2^k 2^{k'}}{2^{k'}} \right)^{\frac{d-1}{2}} = 2^{\frac{k(d-1)}{2}}.$$

Thus we conclude that

$$|\langle F_k, \chi_{y,s}\rangle| \lesssim 2^{-k(\frac{d-3}{2})+k'(d-1)+m}.$$

Summing over $10m < k < k'$, we conclude that since $d \geqslant 4$,

$$\sum_{10m<k<k'} |\langle F_k, \chi_{y,s}\rangle| \lesssim 2^{k'(d-1)+m} \sum_{10m<k<k'} 2^{-k\frac{d-3}{2}} \lesssim 2^{k'(d-1)+m} 2^{-5m} \lesssim 2^{k'(d-1)}.$$

But this means that

$$\sum_{10m<k<k'} |\langle F_k, F_{k'}\rangle| \lesssim 2^{k'(d-1)} \cdot \#(\mathcal{E}_{k'}).$$

This means that

$$\| \sum_{k>10m} F_k\|^2_{L^2(\mathbb{R}^d)} \lesssim \sum_{k>10m} \|F_k\|^2_{L^2(\mathbb{R}^d)} + \sum_{k'} 2^{k'(d-1)}\#(\mathcal{E}_{k'}),$$

and it now suffices to deal with estimates the $\|F_k\|_{L^2(\mathbb{R}^d)}$, i.e. the interactions of functions supported on radii of comparable magnitude. To deal with these, we further decompose the radii, writing $[2^k, 2^{k+1})$ as the disjoint union of intervals $I_{k,\mu} = [2^k + (\mu-1)2^{am}, 2^k + \mu 2^{am}]$, for some a to be chosen later. These interval induces a decomposition $\mathcal{E}_k = \bigcup_\mu \mathcal{E}_{k,\mu}$. Again, incurring a constant loss at most, we may assume that the μ such that $\mathcal{E}_{k,\mu} \neq \emptyset$ are 10 separated. We write $F_k = \sum F_{k,\mu}$, and we have

$$\|F_k\|^2_{L^2(\mathbb{R}^d)} = \sum_\mu \|F_{k,\mu}\|^2_{L^2(\mathbb{R}^d)} + \sum_{\mu<\mu'} |\langle F_{k,\mu}, F_{k,\mu'}\rangle|.$$

We now consider $\chi_{x,r}$ and $\chi_{y,s}$ with $r \in I_{k,\mu}$ and $s \in I_{k',\mu}$. Then we must have $|x-y| \lesssim 2^k$ and $2^{am} \leqslant |r-s| \lesssim 2^k$, and so we have

$$\begin{aligned} |\sum_{\mu<\mu'}\langle F_{k,\mu}, \chi_{y,s}\rangle| &\lesssim 2^{k(d-1)} \sum_{\substack{(x,r)\in\mathcal{E}_k \\ 2^{am}\leqslant|(x,r)-(y,s)|\lesssim 2^k}} |(x,r)-(y,s)|^{-\frac{d-1}{2}} \\ &\lesssim 2^{k(d-1)} \sum_{am\leqslant l\leqslant k} 2^{-l(d-1)/2}\#\{(x,r)\in\mathcal{E}_k : |(x,r)-(y,s)| \sim 2^l\}. \end{aligned}$$

Using the density assumption,

$$\#\{(x,r)\in\mathcal{E}_k : |(x,r)-(y,s)| \sim 2^l\} \lesssim 2^{l+m}$$

and so we obtain that, again using the assumption that $d \geqslant 4$,

$$|\sum_{\mu<\mu'}\langle F_{k,\mu}, \chi_{y,s}\rangle| \lesssim 2^{k(d-1)} 2^{m(1-a(d-3)/2)}.$$

Now summing over all (y,s), we obtain that

$$|\sum_{\mu<\mu'}\langle F_{k,\mu}, F_{k,\mu'}\rangle| \lesssim 2^{k(d-1)} 2^{m(1-a(d-3)/2)}\#(\mathcal{E}_{k,\mu'}).$$

and now summing over μ' gives that

$$\|F_k\|^2_{L^2(\mathbb{R}^d)} \lesssim \sum_\mu \|F_{k,\mu}\|^2_{L^2(\mathbb{R}^d)} + 2^{k(d-1)} 2^{m(1-a(d-3)/2)}\#\mathcal{E}_k,$$

which is a good enough bound if we pick a to be large enough. Now we are left to analyze $\|F_{k,\mu}\|_{L^2(\mathbb{R}^d)}$, i.e. analyzing interactions between annuli which have radii differing from one another by at most $O(2^{am})$. Since the family of all possible radii are discrete, the set $\mathcal{R}_{k,\mu}$ of all possible radii has cardinality $O(2^{am})$. We do not really have any orthogonality to play with here, so we just apply Cauchy-Schwartz, writing $F_{k,\mu} = \sum_{r \in \mathcal{R}_{k,\mu}} F_{k,\mu,r}$, to write

$$\|F_{k,\mu}\|_{L^2(\mathbb{R}^d)}^2 \lesssim 2^{am} \sum_r \|F_{k,\mu,r}\|_{L^2(\mathbb{R}^d)}^2.$$

Recall that $\chi_{x,r} = \text{Trans}_x(\sigma_r * \psi)$, where ψ is a compactly supported function whose Fourier transform is non-negative and vanishes to high order at the origin. In particular, we now make the additional assumption that $\psi = \psi_\circ * \psi_\circ$ for some other compactly function $\psi_\circ$ whose Fourier transform is non-negative and vanishes to high order at the origin. Then we find that $F_{k,\mu,r}$ is equal to the convolution of the function

$$A_r = \sum_{(x,r) \in \mathcal{E}} \text{Trans}_x \psi_\circ$$

with the function $\sigma_r * \psi_\circ$. Using the standard asymptotics for the Fourier transform of σ_r, i.e. that for $|\xi| \geqslant 1$,

$$|\widehat{\sigma}_r(\xi)| \lesssim r^{d-1}(1 + r|\xi|)^{-\frac{d-1}{2}},$$

and since $|\widehat{\psi_\circ}(\xi)| \lesssim_N |\xi|^N$, we get that if $r \geqslant 1$, then for $|\xi| \leqslant 1/r$,

$$|\widehat{\sigma}_r(\xi)\widehat{\psi_\circ}(\xi)| \lesssim_N r^{d-1-N}$$

and for $|\xi| \geqslant 1/r$,

$$|\widehat{\sigma}_r(\xi)\widehat{\psi_\circ}(\xi)| \lesssim_N r^{\frac{d-1}{2}} |\xi|^{-N}.$$

Thus in particular,the L^∞ norm of the Fourier transform of $\sigma_r * \psi_\circ$ is $O(r^{(d-1)/2})$. Now the functions $\psi_\circ$ are compactly supported, so since the set of x such that $(x, r) \in \mathcal{E}$ is one-separated, we find that

$$\|A_r\|_{L^2(\mathbb{R}^d)} \lesssim \#\{x : (x, r) \in \mathcal{E}\}^{1/2}.$$

But this means that

$$\|F_{k,\mu,r}\|_{L^2(\mathbb{R}^d)} = \|A_r * (\sigma_r * \psi_\circ)\|_{L^2(\mathbb{R}^d)} \lesssim r^{\frac{d-1}{2}} \#\{x : (x, r) \in \mathcal{E}\}^{1/2}.$$

Thus we have that
$$\|F_{k,\mu}\|_{L^2(\mathbb{R}^d)}^2 = 2^{am} \cdot \#\mathcal{E}_{k,\mu} \cdot 2^{k(d-1)}.$$
Summing over μ gives that
$$\|F_k\|_{L^2(\mathbb{R}^d)}^2 = 2^{k(d-1)}\#\mathcal{E}_k(2^{am} + 2^{m(1-a(d-3)/2)}).$$
Picking $a = 2/(d-1)$ optimizes this bound, giving
$$\|F_k\|_{L^2(\mathbb{R}^d)} \lesssim 2^{m/(d-1)}2^{k(d-1)/2}(\#\mathcal{E}_k)^{1/2}.$$
Plugging this into the estimates we got for F gives the required bound. □

9.4 Reformulating Heo, Nazarov, and Seeger Using Half-Wave Equations

In variable coefficient settings, the method of breaking down an operator into it's behaviours on spheres, as was done in this argument, breaks down. Indeed, in the study of multipliers of the Laplacian on Riemannian manifolds, there is no analogue of the property that 'the Fourier transform of a radial function is radial'. A more robust alternative is to use the *half-wave operator*, i.e. writing
$$h\big(\sqrt{-\Delta}\big) = \int \widehat{h}(t) W(t)\; dt,$$
where $W(t)$ is a smoothed out version of the wave propogator. In the Euclidean case, $W(t)$ is just the Fourier multiplier with symbol $\chi(\xi)e^{2\pi i t|\xi|}$, for some smooth, compactly supported function χ supported on $|\xi| \sim 1$. In Section 2.6 of these notes, we showed that the condition used in 9 is equivalent to the assumption that
$$\left(\int \left(|\widehat{h}(t)| \langle t \rangle^{s_p} \right)^p \right)^{1/p} < \infty.$$
In this section, we try and reformulate the argument of Heo Nazarov and Seeger using the half-wave equation.

We begin with the discretization step. The desired inequality to use which replaces that of Lemma (TODO) is the inequality
$$\left\| \int_{\mathbb{R}^d} \int_1^\infty a(x,t) W(t,x,\cdot)\; dt\; dx \right\|_{L^p(\mathbb{R}^d)} \lesssim \left(\int |a(x,t)|^p \langle t \rangle^{(d-1)(1-p/2)}\; dt\; dx \right)^{1/p}.$$

The same trick allows us to discretize (TO ASK ANDREAS: DO THE SINGULARITIES MATTER), and then real interpolation shows it suffices to consider the bound

$$\left\| \sum_{(x,t)\in\mathcal{E}} W(t,x,\cdot) \right\|_{L^p(\mathbb{R}^d)} \lesssim \left(\sum_{k=1}^{\infty} \left(\#(\mathcal{E}_k) 2^{ks_p} \right)^p \right)^{1/p}.$$

To get L^2 orthogonality properties of the functions $\{W(t,x,\cdot)\}$, we use Parseval's inequality to calculate that

$$\begin{aligned}
&\langle W(t,x,\cdot), W(t',x',\cdot)\rangle \\
&\quad = \int \chi(\xi)^2 e^{2\pi i[(t-t')|\xi| + (x-x')\cdot\xi]}\, d\xi \\
&\quad = |x-x'|^{-\frac{d-2}{2}} \int_0^\infty \rho^{d/2} \chi(\rho)^2 e^{2\pi i(t-t')\rho} J_{\frac{d-2}{2}}(|x-x'|\rho)\, d\rho \\
&\quad = |x-x'|^{-\frac{d-1}{2}} \int_0^\infty \rho^{\frac{d-1}{2}} \chi(\rho)^2 a_\pm(|x-x'|\rho) e^{2\pi i\rho[(t-t')\pm|x-x'|]}.
\end{aligned}$$

Integrating by parts when the oscillatory integral isn't stationary, we obtain that for $|x-x'| \gtrsim 1$, that

$$|\langle W(t,x,\cdot), W(t',x',\cdot)\rangle| \lesssim_N |x-x'|^{-\frac{d-1}{2}} \left\langle |(t-t') \pm |x-x'|| \right\rangle^{-N}.$$

Conversely, for $|x-x'| \lesssim 1$, but provided $|t-t'| \gtrsim 1$, we can simply integrate by parts in the formula

$$\int \chi(\xi)^2 e^{2\pi i[(t-t')|\xi| + (x-x')\cdot\xi]}$$

to get that

$$|\langle W(t,x,\cdot), W(t',x',\cdot)\rangle| \lesssim_N |t-t'|^{-N}.$$

Finally, for $|x-x'| \lesssim 1$ and $|t-t'| \lesssim 1$,

$$|\langle W(t,x,\cdot), W(t',x',\cdot)\rangle| \lesssim 1,$$

which can be calculated from the oscillatory integral equation, or just simply by Cauchy-Schwartz, i.e. because the L^2 norm of each term is $O(1)$, since this is the L^2 norm of $\chi(\cdot)e^{2\pi i t|\cdot|}$.

We now use this, together with a density decomposition argument, to obtain bounds on the required quantities. TODO

Chapter 10

Cladek: Improvements Using Incidence Geometry

The results of Heo, Nazarov, and Seeger only apply when $d \geqslant 4$. Cladek found a method to get an initial radial multiplier conjecture result in $\mathbb{R}^3$, and an improvmeent of the bounds obtained by Heo, Nazarov, and Seeger when $d = 3$. The idea is to exploit the fact that one need only prove a version of 9.2 for a set $\mathcal{E} = \mathcal{E}_X \times \mathcal{E}_R$, where $\mathcal{E}_X$ is a one-separated family of points, and $\mathcal{E}_R$ are a family of radii. One can then exploit this Cartesian product structure when analyzing functions of the form

$$F = \sum_{(x,r)\in\mathcal{E}} \chi_{x,r},$$

in particular, improving upon the result of [9].

10.1 Result in 3 Dimensions

As in [9], Cladek first performs a density decomposition, i.e. writing

$$F = \sum F_k^m$$

where

$$F_k^m = \sum_{(x,r)\in\mathcal{E}_k(2^m)} \chi_{x,r}.$$

Cladek then interpolates between an L^0 bound and an L^2 bound on the resulting functions. The L^0 bound is exactly the same bound used in [9].

Theorem 10.1. *For the function F, we have*

$$|supp(F_k^m)| \lesssim 2^{-m}4^k\#\mathcal{E}_k$$

and thus

$$|supp(F^m)| \lesssim \sum_k 2^{-m}4^k\#\mathcal{E}_k.$$

The L^2 bound is improved upon, which is what allows us to obtain a new result in three dimensions.

Lemma 10.2. *Suppose $\mathcal{E} = \bigcup_k \mathcal{E}_k$ is a one-separated set, where $\mathcal{E}_k \subset \mathbb{R}^d \times [2^k, 2^{k+1})$ is a set of density type $(2^m, 2^k)$. Then*

$$\left\| \sum_{(x,r)\in\mathcal{E}} \chi_{x,r} \right\|_{L^2(\mathbb{R}^d)} \lesssim_\varepsilon 2^{[(11/13)+\varepsilon]m} \sum_k 4^k\#\mathcal{E}_k.$$

Interpolation thus yields that for a set of density type 2^m as in this Lemma,

$$\| \sum_{(x,r)\in\mathcal{E}} \chi_{x,r}\|_{L^p(\mathbb{R}^d)} \lesssim_\varepsilon 2^{-m(1/p-12/13-\varepsilon)}(\sum_k 4^k\#\mathcal{E}_k)^{1/p}.$$

If $1 < p < 13/12$, this sum is favorable in m, and may be summed without harm to prove the radial multiplier conjecture for unit scale radial multipliers in this range.

Proof of Lemma 10.2. Write

$$F_k = \sum_{(x,r)\in\mathcal{E}_k} \chi_{x,r}.$$

As before, we can throw away terms for $k \leqslant 10m$, i.e. obtaining that

$$\| \sum F_k\|_{L^2(\mathbb{R}^d)} \lesssim m^{1/2} \left(\sum_k \|F_k\|^2_{L^2(\mathbb{R}^d)} + \sum_{10m<k<k'} |\langle F_k, F_{k'}\rangle| \right)^{1/2}.$$

Our proof thus splits into two cases: where the radii are incomparable, and where the radii are comparable.

TODO: □

10.2 Results in 4 Dimensions

TODO

Chapter 11

Mockenhaupt, Seeger, and Sogge: Exploiting Periodicity

The main goal of the paper *Local Smoothing of Fourier Integral Operators and Carleson-Sjölin Estimates* is to prove local regularity theorems for a class of Fourier integral operators in $I^\mu(Z, Y; C)$, where Y is a manifold of dimension $n \geqslant 2$, and Z is a manifold of dimension $n + 1$, which naturally arise from the study of wave equations. A consequence of this result will be a local smoothing result for solutions to the wave equation, i.e. that if $2 < p < \infty$, then there is δ depending on p and n, such that if $T : Y \to Y \times \mathbb{R}$ is the solution operator to the wave equation, and Y is a compact manifold whose geodesics are periodic, then T is continuous from from $L^p_c(Y)$ to $L^p_{\alpha,\text{loc}}(Y \times \mathbb{R})$ for $\alpha \leqslant -(n-1)|1/2 - 1/p| + \delta$. Such a result is called local smoothing, since if we define $Tf(t,x) = T_t f(x)$, then the operator T_t is, for each t, a Fourier integral operator of order zero, with canonical relation

$$C_t = \{(x, y; \xi, \xi) : x = y + t\widehat{\xi}\},$$

where $\widehat{\xi} = \xi/|\xi|$ is the normalization of ξ. Standard results about the regularity of hyperbolic partial differential equations show that each of the operators T_t is continuous from $L^p_c(Y)$ To $L^p_{\alpha,\text{loc}}(Y \times \mathbb{R})$ for $\alpha \leqslant -(n-1)|1/2 - 1/p|$, and that this bound is sharp. Thus T is *smoothing* in the t variable, so that for any $f \in L^p$, the functions $T_t f$ 'on average' gain a regularity of δ over the worst case regularity at each time. The local smoothing conjecture states that this result is true for any $\delta < 1/p$.

The class of Fourier integral operators studied are those satisfying the following condition: as is standard, the canonical relation C is a conic Lagrangian manifold of dimension $2n+1$. The fact that C is Lagrangian implies C is locally parameterized by $(\nabla_\zeta H(\zeta,\eta), \nabla_\eta H(\zeta,\eta), \zeta, \eta)$, where H is a smooth, real homogeneous function of order one. If we assume $C \to T^*Y$ is a submersion, then $D_\xi[\nabla_\eta H(\zeta,\eta)]$ has full rank, which implies $D_\eta[\nabla \xi H(\zeta,\eta)] = (D_\xi[\nabla_\eta H(\zeta,\eta)])^t$ has full rank, and thus the projection $C \to T^*Z$ is an immersion. We make the further assumption that the projection $C \to Z$ is a submersion, from which it follows that for each z in the image of this projection, the projection of points in C onto $T^*_z Z$ is a conic hypersurface Γ_z of dimension n. The final assumption we make is that all principal curvatures of Γ_z are non-vanishing.

Remark. The projection properties of C imply that, in $T^*(Z \times Y)$, there exists a smooth phase ϕ defined on an open subset of $Z \times T^*Y$, homogeneous in T^*Y, such that locally we can write C as $(z, \nabla_z\phi(z,\eta), \nabla_\eta\phi(z,\eta), \eta)$ for $\eta \neq 0$. Then, working locally on conic sets,

$$\Gamma_z = \{(\nabla_z\phi(z,\eta))\},$$

and the curvature condition becomes that the Hessian $H_{\eta\eta}\langle\nabla_z\phi, \nu\rangle$ has constant rank $n-1$, where ν is the normal vector to Γ_z. This is a natural homogeneous analogue of the Carleson-Sjölin condition for non-homogeneous oscillatory integral operators, i.e. the Carleson-Sjölin condition is allowed to assume $H_{\eta\eta}\phi$ has rank n, which cannot be possible in our case, since ϕ is homogeneous here. An approach using the analytic interpolation method of Stein or the Strichartz / Fractional Integral approach generalizes the Carleson-Sjölin theorem to show that for any smooth, non-homogeneous phase function $\Phi : \mathbb{R}^{n+1} \times \mathbb{R}^n \to \mathbb{R}$, and any compactly supported smooth amplitude a on $\mathbb{R}^{n+1} \times \mathbb{R}^n$. Consider the operators

$$T_\lambda f(z) = \int a(z,y) e^{2\pi i \lambda \Phi(z,y)} f(y)\, dy.$$

If the associated canonical relation C, if C projects submersively onto $T^*\mathbb{R}^n$, so that for each $z \in \mathbb{R}^{n+1}$ in the image of the projection map C, the set $S_z \subset \mathbb{R}^{n+1}$ obtained from the inverse image of the projection of $C \to Z$ at z is a n dimensional hypersurface with k non-vanishing curvatures. Then for $1 \leqslant p \leqslant 2$,

$$\|T_\lambda f\|_{L^q(\mathbb{R}^{n+1})} \lesssim \lambda^{-(n+1)/q} \|f\|_{L^p(\mathbb{R}^n)}.$$

where $q = p^*(1 + 2/k)$.

Remark. We can also see these assumptions as analogues in the framework of cinematic curvature, splitting the z coordinates into 'time-like' and 'space-like' parts. Working locally, because $C \to T^*Y$ is a submersion, we can consider coordinates $z = (x,t)$ so that, with the phase ϕ introduced above, $D_x(\nabla_\eta \phi)$ has full rank n, and that $\partial_t \phi(x,t,\eta) \neq 0$. Then for each $z = (x,t)$, we can locally write $\partial_t \phi(x,t,\eta) = q(x,t,\nabla_x \phi(x,t,\eta))$, homogeneous in η, and then

$$C = \{(x,t,y;\xi,\tau,\eta) : (x,\xi) = \chi_t(y,\eta), \tau = q(x,t,\xi)\},$$

where χ_t is a canonical transformation. Our curvature conditions becomes that $H_{\xi\xi} q$ has full rank $n-1$. This is the cinematic curvature condition introduced by Sogge.

Under these assumptions, the paper proves that any Fourier integral operator T in $I^{\mu-1/4}(Z,Y;C)$ maps $L^2_c(Y)$ to $L^q_{\text{loc}}(Z)$ if

$$2\left(\frac{n+1}{n-1}\right) \leqslant q < \infty \quad \text{and} \quad \mu \leqslant -n(1/2 - 1/q) + 1/q.$$

and maps $L^p_c(Y)$ to $L^p_{\text{loc}}(Z)$ if

$$p > 2 \quad \text{and} \quad \mu \leqslant -(n-1)(1/2 - 1/p) + \delta(p,n).$$

If we introduce time and space variables locally as in the remark above, any operator in $I^{\mu-1/4}(Z,Y;C)$ can be written locally as a finite sum of operators of the form

$$Tf(x) = \int_{-\infty}^{\infty} T_t f(x),$$

where

$$T_t f(x) = \int a(t,x,\eta) e^{2\pi i \phi(x,t,y,\eta)} f(y)\, dy\, d\eta.$$

is a Fourier integral operator whose canonical relation is a locally a canonical graph, then the general theory implies that each of the maps T_t maps $L^2_c(Y)$ to $L^q_{\text{loc}}(X)$ if

$$2 \leqslant q \leqslant \infty \quad \text{and} \quad \mu \leqslant -n(1/2 - 1/q)$$

so that here we get local smoothing of order $1/q$, and also maps $L^p_c(Y)$ to $L^p_{\text{loc}}(X)$ if

$$1 < p < \infty \quad \text{and} \quad \mu \leqslant -(n-1)|1/p - 1/2|$$

so we get $\delta(p,n)$ smoothing. A consequence of the smoothing, via Sobolev embedding, is a maximal theorem result for the operator T_t, i.e. that for any finite interval I, the operator

$$Mf = \sup_{t \in I} |T_t f|$$

maps $L^p_c(Y)$ to $L^p_{\text{loc}}(X)$ if $\mu < -(n-1)(1/2 - 1/p) - (1/p - \delta(p,n))$. If the local smoothing conjecture held, we would conclude that, except at the endpoint T^* has the same $L^p_c(Y)$ to $L^p_{\text{loc}}(X)$ mapping properties as each of the operators T_t. We also get square function estimates, such that for any finite interval I, if we consider

$$S f(x) = \left(\int_I |T_t f(x)|^2 \, dt \right)^{1/2},$$

then for

$$2\frac{n+1}{n-1} \leqslant q < \infty \quad \text{and} \quad \mu \leqslant -n(1/2 - 1/q) + 1/2,$$

the operator S is bounded from $L^2_c(Y)$ to $L^q_{\text{loc}}(X)$.

Our main reason to focus on this paper is the results of the latter half of the paper applying these techniques to radial multipliers on compact manifolds with periodic geodesics. Thus we consider a compact Riemannian manifold M, such that the geodesic flow is periodic with minimal period $2\pi \cdot \Pi$. We consider $m \in L^\infty(\mathbb{R})$, such that $\sup_{s>0} \|\beta \cdot \text{Dil}_s m\|_{L^2_\alpha(\mathbb{R})} = A_\alpha$ is finite for some $\alpha > 1/2$ and some $\beta \in C^\infty_c(\mathbb{R})$. We define a 'radial multiplier' operator

$$T f = \sum_\lambda m(\lambda) E_\lambda f$$

where E_λ is the projection of f onto the space of eigenfunctions for the operator $\sqrt{-\Delta}$ on M with eigenvalue λ. We can also write this operator as $m(\sqrt{-\Delta})$. Then the wave propogation operator $e^{2\pi i t \sqrt{-\Delta}}$ is periodic of period Π. The Weyl formula tells us that the number of eigenvalues of $\sqrt{-\Delta}$ which are smaller than λ is equal to $V(M) \cdot \lambda^n + O(\lambda^{n-1})$.

Theorem 11.1. *Let $m \in L^2_\alpha(\mathbb{R})$ be supported on $(1,2)$, and assume $\alpha > 1/2$, then for $2 \leqslant p \leqslant 4$, $f \in L^p(M)$, and for any integer k,*

$$\left\| \sup_{2^k \leqslant \tau \leqslant 2^{k+1}} |Dil_\tau m(\sqrt{-\Delta}) f| \right\|_{L^p(M)} \lesssim_\alpha \|m\|_{L^2_\alpha(M)} \|f\|_{L^p(M)}.$$

Proof. To understand the radial multipliers we apply the Fourier transform, writing

$$T_\tau f = (\text{Dil}_\tau m)(\sqrt{-\Delta}) f = m(\sqrt{-\Delta}/\tau) f = \int_{-\infty}^{\infty} \tau \widehat{m}(t\tau) e^{2\pi i t \sqrt{-\Delta}} f \, dt.$$

If we define $\beta \in C_c^\infty((1/2, 8))$ such that $\beta(s) = 1$ for $1 \leqslant s \leqslant 4$, and set $L_k f = \mathrm{Dil}_{2^k} \beta(\sqrt{-\Delta}) f$, then for $2^k \leqslant \tau \leqslant 2^{k+1}$

$$T_\tau f = (\mathrm{Dil}_\tau m)(\sqrt{-\Delta}) f = (\mathrm{Dil}_\tau m \cdot \mathrm{Dil}_{2^k} \beta)(\sqrt{-\Delta}) = T_\tau L_k f.$$

so Cauchy-Schwartz implies that

$$\begin{aligned}
|T_\tau f(x)| &= \left| \int_{-\infty}^{\infty} \tau \widehat{m}(\tau) e^{2\pi i t \sqrt{-\Delta}} L_k f(x)\, dt \right| \\
&\leqslant \|m\|_{L_\alpha^2(M)} \left(\int_{-\infty}^{\infty} \frac{\tau}{(1 + |t\tau|^2)^\alpha} |e^{2\pi i t \sqrt{-\Delta}} L_k f(x)|^2 \right)^{1/2} \\
&\leqslant \|m\|_{L_\alpha^2(M)} \left(\int_{-\infty}^{\infty} \frac{2^k}{(1 + |2^k t|^2)^\alpha} |e^{2\pi i t \sqrt{-\Delta}} L_k f(x)|^2 \right)^{1/2}
\end{aligned}$$

Because of periodicity, if we set $w_k(t) = 2^k/(1 + |2^k t|^2)^\alpha$, it suffices to prove that for $\alpha > 1/2$,

$$\left\| \left(\int_0^\Pi w_k(t) |e^{2\pi i t \sqrt{-\Delta}} L_k f(x)|^2\, dt \right)^{1/2} \right\|_{L^p(M)} \lesssim_{\alpha,p} \|f\|_{L^p(M)}.$$

This is a weighted combination of the wave propogators, roughly speaking, assigning weight 2^k for $t \lesssim 1/2^k$, and assigning weight $1/t$ to values $t \gtrsim 1/2^k$.

For a fixed $0 < \delta$, we can split this using a partition of unity into a region where $t \gtrsim \delta$ and a region where $t \lesssim \delta$, where δ is independent of k. For each t, the wave propogation $e^{2\pi i t \sqrt{-\Delta}}$ is a Fourier integral operator of order zero (we have an explicit formula for small t, and the composition calculus for Fourier integral operators can then be used to give a representation of the propogation operators for all times t, such that the symbols of these operators are locally uniformly bounded in S^0). Thus the square function estimate above can be applied in the region where $t \gtrsim \delta$, because the weighted square integral above has weight $O_\delta(1)$ uniformly in k.

Next, we move onto the region $t \lesssim 1/2^k$. The symbol of the operator $e^{2\pi i t \sqrt{-\Delta}}$

Finally we move onto the region $1/2^k \lesssim t \lesssim \delta$. On this region we have $w_k(t) \sim 1/t$, which hints we should try using dyadic estimates. In particular, suppose that for $\gamma \leqslant \delta$, we have a family of dyadic estimates of the form

$$\left\| \left(\int_\gamma^{2\gamma} |e^{2\pi i t \sqrt{-\Delta}} L_k f|^2\, dt \right)^{1/2} \right\|_{L^p(M)} \lesssim \gamma^{1/2} (1 + \gamma 2^k)^\varepsilon \cdot \|f\|_{L^p(M)}.$$

Summing over the $O(k)$ dyadic numbers between $1/2^k$ and δ gives

$$\left\| \left(\int_{1/2^k \lesssim t \lesssim \delta} |e^{2\pi it\sqrt{-\Delta}} L_k f|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(M)} \lesssim 2^{\varepsilon k} \|f\|_{L^p(M)}$$

If we were able to obtain this inequality for some $\varepsilon > 0$, then we could bound that for all $0 < \gamma < \Pi/2$

If we localize near $t \lesssim 1/2^k$ by multiplying by $\phi(2^k t)$ for some compactly supported smooth ϕ supported on $|t| \lesssim 1$, then for t on the support of $\phi(2^k t)$ we have a weight proportional to 2^k, and rescaling shows that it suffices to bound the quantities

$$\left\| \left(\int \phi(t) |e^{2\pi i(t/2^k)\sqrt{-\Delta}} L_k f(x)|^2 \, dt \right)^{1/2} \right\|$$

the family of functions

$$\left\| \left(\int |\phi(t) e^{2\pi i(t/2^k)\sqrt{-\Delta}} L_k f(x)|^2 \, Dt \right)^{1/2} \right\|_{L^p_x} \lesssim \sup \|e^{2\pi i(t/2^k)\sqrt{-\Delta}} L_k f\|_{L^p_x}$$

$a_k(t) = 2^{-k/2} \widehat{\phi}(t/2^k) \beta(\tau/2^k)$

it suffices to uniformly bound quantities of the form

$$\left\| \left(\int 2^k \phi(2^k t) |e^{2\pi i\sqrt{-\Delta}} L_k f(x)|^2 \, dt \right)^{1/2} \right\|_{L^p(M)} \lesssim_{\alpha,p} \|f\|_{L^p(M)}$$

We now apply a dyadic decomposition to deal with the smaller values of t. Let us assume for simplicity of notation that $\delta < 1$, and then consider a partition of unity $1 = \sum_{j=1}^\infty \phi(2^j t)$ for $0 \leqslant t \leqslant 1$, and such that ϕ is localized near $1/4 \leqslant t \leqslant 2$, then our goal is to bound the quantities

$$\left\| \left(\int_{-\infty}^{\infty} \phi(2^j t) \frac{2^k}{(1 + |2^k t|^2)^\alpha} |A_t L_k f(x)|^2 \, dt \right)^{1/2} \right\|_{L^p(M)},$$

which are each proportional to

$$s$$

$\square$

Chapter 12

Lee and Seeger: Decomposition Arguments For FIOs with Cinematic Curvature

Let's now discuss a paper [15] entitled *Lebesgue Space Estimates For a Class of Fourier Integral Operators Associated With Wave Propogation*. In this paper, Lee and Seeger prove a variable coefficient version of the result of Heo, Nazarov, and Seeger, i.e. generalizing that result as it applies to sharp local smoothing on $\mathbb{R}^d$ to the local smoothing of Fourier integral operators satisfying the cinematic curvature condition.

We consider a Fourier integral operator $T : \mathcal{D}(Y) \to \mathcal{D}^*(Z)$ of order $\mu - 1/4$, where $\dim(Y) = d$ and $\dim(Z) = d + 1$, with a canonical relation C satisfying the following properties:

- The projection map $\pi_R : C \to T^*Y$ is a submersion. It follows that around any point $p = (z_0, y_0; \zeta_0, \eta_0)$ we can choose coordinate systems y and (x, t) on Y and Z respectively, centered at z_0 and y_0, such that $\zeta_0 = dx_1$, $\eta_0 = dy_1$, and the tangent plane to C at this point at p is given by

  $$dx = dy \quad \text{and} \quad d\xi = d\eta \quad \text{and} \quad d\tau = 0.$$

  In particular, it follows that $\pi_Z : C \to Z$ is a submersion, and we can locally find a function $\phi(z, \eta)$, homogeneous in η, such that, locally,

  $$C = \Big\{(z, \nabla_z \phi(z, \eta)) \times (\nabla_\eta \phi(z, \eta), \eta) \in T^*Z \times T^*Y\Big\}.$$

By assumption on the tangent space of C,

$$(D_\eta \nabla_x)\phi(0, e_1) = I \quad (D_\eta \partial_t)\phi(0, e_1) = 0.$$

$$\nabla_\eta \phi(0, e_1) = 0 \quad \nabla_x \phi(0, e_1) = e_1 \quad \partial_t \phi(0, e_1) = 0.$$

The equivalence of phase theorem implies we can find a symbol $a(x, t, y, \eta)$ of order μ such that, after appropriately microlocalizing the inputs and outputs of the operator T, we have

$$Tf(x,t) = \int a(x,t,y,\eta) e^{2\pi i[\phi(x,t,\eta) - y\cdot\eta]} f(y)\, d\eta\, dy.$$

- The last assumption implies that for each z_0, $\Sigma_{z_0}\pi_Z^{-1}(z_0)$ is a d dimensional submanifold of C. Moreover, our choice of coordinates makes it easy to see that the natural map $\Sigma_{z_0} \to T^*_{z_0}Z$ is an immersion, whose image is the immersed hypersurface Γ_{z_0} of $T^*_{z_0}$. Indeed, the tangent plane to Σ_{z_0} at the point above is given in coordinates by

  $$dx = dy = dt = d\tau = 0 \quad \text{and} \quad d\xi = d\eta.$$

  And this is projected injectively to the plane defined by $d\tau = 0$ in $T^*_{z_0}Z$. Our other assumption we make about C is an assumption on *cinematic curvature*. We assume that for each z_0, the hypersurface Σ_{z_0} is a cone with l nonvanishing principal curvatures, for some $1 \leqslant l \leqslant d-1$. Since

  $$\Sigma_{z_0} = \Big\{(z_0; \nabla_z \phi(z_0, \eta_0)\Big\}.$$

  Then the curvature assumptions amount to the fact that the Hessian matrix

  $$H_\eta\{\partial_t \phi\}$$

  has rank at least l on a neighborhood on the support of a.

Given these assumptions, the following result is obtained.

Theorem 12.1. *If*

$$l \geqslant 3 \quad \textit{and} \quad \frac{2l}{l-2} < q < \infty \quad \textit{and} \quad \mu \leqslant \frac{d}{q} - \frac{d-1}{2}$$

then T maps $L^q_c(Y)$ into $L^q_{loc}(Z)$.

If we take $l = d - 1$, we get the full assumption of 'cinematic curvature' and we can use this to get results about local smoothing of the wave equation on compact Riemannian manifolds, which recovers the local smoothing result of Heo, Nazarov, and Seeger obtained in their paper on radial Fourier multipliers.

Theorem 12.2. *If M^d is a compact Riemannian manifold,*

$$d \geqslant 4 \quad \text{and} \quad \frac{2(d-1)}{d-3} < q < \infty,$$

and we set $\alpha = (d-1)/2 - d/q$, then

$$\|e^{it\sqrt{-\Delta}} f\|_{L^q_{t,loc} L^q_x(M)} \lesssim \|f\|_{L^q_\alpha(M)}.$$

Proof. The solution operator for the half-wave equation is a Fourier integral operator of order $-1/4$, where $\mu = 0$, associated with the canonical relation

$$C = \{(x, t, y; \eta, \omega, \eta) : x = \exp_y(t\xi/|\xi|) \text{ and } \omega = |\xi|_g\}.$$

It is immediate that the projection maps $C \to T^*Y$ and $C \to Z$ are submersions. For each $z_0 = (x_0, t_0)$,

$$\Gamma_{z_0} = \{(\xi, \omega) : \omega = |\xi|_g\}$$

is a spherical cone, and thus has $d-1$ nonvanishing principal curvatures. Consider the operator

$$T = L \circ (I - \Delta)^{-\alpha/2}.$$

Then T is a Fourier integral operator of order $-1/4 + \alpha$, associated with the same canonical relation C. Applying the Theorem above, we conclude that for the α specified, T maps $L^q(M)$ into $L^q_{t,\text{loc}}(\mathbb{R})L^q_x(M)$. But this implies that if $g = (I - \sqrt{\Delta})^{\alpha/2} f$, then

$$\|e^{it\sqrt{-\Delta}} f\|_{L^q_{t,\text{loc}} L^q_x(M)} = \|Tg\|_{L^q_{t,\text{loc}} L^q_x(M)} \lesssim \|g\|_{L^q(M)} = \|f\|_{L^q_\alpha(M)}. \qquad \square$$

In Seeger, Sogge, and Stein, independent of the curvature in this problem, it is proved that

$$\|T_R f\|_{L^\infty} \lesssim R^{\frac{d-1}{2} - \frac{d}{q}} \|f\|_{L^\infty}.$$

By interpolating this bound, to obtain the result above, it is sufficient to obtain a restricted weak-type bound at the endpoint value of q.

12.1 Frequency Localization and Discretization

Let us describe the idea of the proof. Let $K(z,y)$ denote the kernel of T, i.e.

$$K(z,y) = \int a(z,y,\eta) e^{2\pi i[\phi(z,\eta) - y\cdot\eta]}\, d\eta.$$

Without loss of generality, we may assume that a is supported on $|\eta| \geqslant 1$, since integrals over small frequencies give a smoothing operator. We localize in frequency, writing, working modulo smoothing operators by ignoring small frequencies,

$$K(x,t,y) = \sum_{j\geqslant 100} 2^{j\mu} K_{2^j}(z,y),$$

where, for $R \geqslant 1$,

$$\begin{aligned} K_R(z,y) &= R^{-\mu} \int a(z,y,\eta)\chi(\eta/R) e^{2\pi i[\phi(z,\eta) - y\cdot\eta]}\, d\eta \\ &= \int a_R(z,y,\eta/R) e^{2\pi i[\phi(z,\eta) - y\cdot\eta]}\, d\eta \end{aligned}$$

where $a_R(z,y,\eta) = R^{-\mu} a(z,y,R\eta)\chi(\eta)$ is chosen to have uniform compact support, and such that

$$|D^{\alpha}_{z,y,\eta} a_R(z,y,\eta)| \lesssim_\alpha 1$$

holds uniformly in R. Let T_R be the operator with kernel K_R.

12.2 Discretizing the Problem

It is more natural to establish estimates for the adjoint operator T_R^*. We will establish restricted estimates, i.e. bounding the behaviour of $T_R^*\{\chi_E\}$ for a measurable set $E \subset \mathbb{R}^{d+1}$. We have

$$T_R^*\{\chi_E\}(y) = \int \chi_E(x,t)\overline{a_R(z,y,\eta/R)} e^{2\pi i[y\cdot\eta - \phi(z,\eta)]}\, d\eta\, dx\, dt.$$

The majority of the behaviour of T_R^* depends on the behaviour of χ_E at frequencies with magnitude $\Theta(R)$, so we perform a discretization at a scale $1/R$. Taking a Fourier series in the y variable, assuming $\mathrm{supp}_y(a_R)$ is contained in $[-1/2,1/2]^d$, we write

$$a_R(z,y,\eta) = \sum_{\nu\in\mathbb{Z}^d} a_{R,\nu}(z,\eta/R)\chi(y) e^{2\pi i \nu\cdot y},$$

where χ is some smooth, compactly supported function equal to one on $\text{supp}_y(K)$. We may then write

$$T_R^*\{\chi_E\}(y) = \sum_{\nu \in \mathbb{Z}^d} T_{R,\nu}^*\{\chi_E\}(y) e^{2\pi i \nu \cdot y},$$

where $T_{R,\nu}$ is the operator with kernel

$$K_{R,\nu}(z, y) = \int a_{R,\nu}(z, \eta/R)\chi(y) e^{2\pi i[\phi(z,\eta) - y \cdot \eta]}\, d\eta.$$

The smoothness and support properties of the symbols $\{a_R\}$ imply that

$$|D^\alpha_{x,t,\eta} a_{R,\nu}(z, \eta)| \lesssim_{\alpha,N} |\nu|^{-N} \quad \text{for all } N > 0,$$

and so we can likely ignore the interactions between the operators $T_{R,\nu}^*$ as we vary ν, i.e. by using the triangle inequality.

Let us now study the behaviour of the function $T_{R,\nu}^*\{\chi_E\}$ on the set $\text{supp}_y(K)$. To do this, we discretize our operator at a scale $1/R$. Let $\mathcal{Z}_R$ be the set of points on the lattice $R^{-1}\,\mathbb{Z}^{d+1}$ which lies in some neighborhood of $\text{supp}_z(K)$. For $\zeta \in \mathcal{Z}_R$, we let Q_ζ be the sidelength R^{-1} cube centred at ζ. Then $\{Q_\zeta\}$ is an almost disjoint family of cubes covering $\text{supp}_z(K)$. Define

$$a_{R,\nu,\zeta}(\eta) = \int_{Q_\zeta \cap E} a_{R,\nu}(z, \eta/R)\chi(y) e^{2\pi i[\phi(\zeta,\eta) - \phi(z,\eta)]}\, dz.$$

Then if we define

$$S_{R,\nu,\zeta}(y) = \int a_{R,\nu,\zeta}(\eta) e^{2\pi i[y \cdot \eta - \phi(\zeta,\eta)]}\, d\eta,$$

then for $y \in \text{supp}_y(K)$,

$$T_{R,\nu}^*\{\chi_E\}(y) = \sum_\zeta S_{R,\nu,\zeta}(y).$$

Now for $z \in Q_\zeta$ and $|\eta| \sim R$,

$$|\partial_\eta^\alpha\{\phi(\zeta, \eta) - \phi(z, \eta)\}| \lesssim_\alpha R^{-\alpha} \quad \text{for all } \alpha,$$

which allows us to conclude that

$$|D_\eta^\alpha a_{R,\nu,\zeta}(\eta)| \lesssim_{\alpha,N} |Q_\zeta \cap E| \cdot R^{-\alpha} \langle \nu \rangle^{-N} \quad \text{for all } \alpha \text{ and } \nu > 0.$$

We are thus reduced to the analysis of the quantities

$$\sum_{\zeta \in \mathcal{Z}_R} S_{R,\nu,\zeta}.$$

If, for $m \geqslant 0$, we write

$$Z_{R,m} = \{\zeta : |Q_\zeta \cap E| \sim 2^{-m}(1/R)^{d+1}\},$$

then in the next section, we will obtain estimates of the form

$$\left\| \sum_{\zeta \in Z_{R,m}} S_{R,\nu,\zeta} \right\|_{L^{p,\infty}} \lesssim_N \langle \nu \rangle^{-N} [2^{-m}(1/R)^{d+1}] \cdot R^{\frac{d+1}{2} - \frac{1}{p}} \#(Z_{R,m})^{1/p}.$$

If we set $E_{R,m} = \bigcup_{\zeta \in Z_{R,m}} Q_\zeta \cap E$, then $E = \bigcup_m E_{R,m}$. For each m,

$$\#(Z_{R,m}) \sim 2^m R^{d+1} |E_{R,m}|,$$

and so the estimate above implies that

$$\left\| \sum_{\zeta \in Z_{R,m}} S_{R,\nu,\zeta} \right\|_{L^{p,\infty}} \lesssim_N \langle \nu \rangle^{-N} 2^{-m/q} R^{d(1/p-1/2)-1/2} |E_{R,m}|^{1/p}.$$

Summing in $m \geqslant 0$, and using Hölder's inequality to prove that for any non-negative numbers $\{a_m\}$, we have

$$\sum_m 2^{-m/q} a_m^{1/p} \lesssim_q \left(\sum a_m\right)^{1/p},$$

we conclude that

$$\begin{aligned}
\|T^*_{R,\nu}\{\chi_E\}\|_{L^{p,\infty}} &= \left\| \sum_{\zeta \in \mathcal{Z}_R} S_{R,\nu,\zeta} \right\|_{L^{p,\infty}} \\
&\lesssim \sum_m \left\| \sum_{\zeta \in Z_{R,m}} S_{R,\nu,\zeta} \right\|_{L^{p,\infty}} \\
&\lesssim_N \langle \nu \rangle^{-N} R^{d(1/p-1/2)-1/2} |E|^{1/p}.
\end{aligned}$$

Summing over ν, we conclude that

$$\|T^*_R\{\chi_E\}\|_{L^{p,\infty}} \lesssim R^{d(1/p-1/2)-1/2} |E|^{1/p}.$$

Thus we have proved a restricted weak-type bound for the operators T^*_R.

12.3 Interactions of Discretized Operators

Our proof now rests on the following problem. We fix a large constant $M > 0$, to be specified later. Our goal is to prove the following Lemma.

Lemma 12.3. *For each $Z \subset \mathcal{Z}_R$, and each $C > 0$, suppose there exists a symbol $a_\zeta(\eta)$ supported on $|\eta| \sim R$, and satisfying derivative bounds of the form*

$$|(\partial_\xi^\alpha a_\zeta)(\eta)| \leqslant CR^{-\alpha} \quad \text{for } |\alpha| \leqslant M.$$

If we define

$$S_\zeta(y) = \int a_\zeta(\eta) e^{2\pi i[y\cdot\eta - \phi(\zeta,\eta)]}\, d\eta,$$

then

$$\left\| \sum_{\zeta\in Z} S_\zeta \right\|_{L^{p,\infty}} \lesssim CR^{\frac{d+1}{2}-\frac{1}{p}} \#(Z)^{1/p},$$

where the implicit constant is independent of C, R, and Z.

Lemma 12.3 is clearly sufficient to prove the estimate mentioned in the last section. So let's now prove it. By linearity, we may assume without loss of generality that $C = 1$.

We will obtain the Lemma by interpolating a combination of more elementary estimates, namely, a pair of L^1 and L^∞ bounds for the sum, which do not take advantage of the curvature in the problem, and an L^2 bound which does take into account this curvature.

To understand the individual behaviour of the functions $\{S_\zeta\}$, we perform a 'double dyadic decomposition', i.e. taking a $R^{-1/2}$ discretized subset Θ_R of unit vectors in $\mathbb{R}^d$, consider some smooth partition of unity adapted to the $O(1/R^{1/2})$ neighborhoods of these unit vectors, and thus consider the associated decomposition $a_\zeta = \sum_\theta a_{\zeta,\theta}$. Applyig non-stationary phase, we get that, for the resulting decomposition $S_\zeta = \sum_\theta S_{\zeta,\theta}$, the function $S_{\zeta,\theta}$ has the majority of it's support on a cap about the point $\nabla_\xi \varphi(\zeta,\theta)$, with thickness R^{-1} in the θ-direction, and thickness $R^{-1/2}$ in the directions orthogonal to θ. Moreover, on this cap, $S_{\zeta,\theta}$ has magnitude $O(R^{(d+1)/2})$. Completely ignoring the interactions between these caps, which do not matter anyhow given we are taking the L^1 norm of the functions, the triangle inequality implies that

$$\|S_\zeta\|_{L^1} \leqslant \sum_\theta \|S_{\zeta,\theta}\|_{L^1} \lesssim R^{\frac{d-1}{2}} \cdot R^{\frac{d+1}{2}} \cdot R^{-1} R^{-\frac{d-1}{2}} = R^{\frac{d-1}{2}}.$$

We interpolate this bound with some L^2 orthogonality bounds for the family $\{S_\zeta\}$ to prove the required result.

We will use Fourier analysis to obtain these L^2 bounds, which is more simple given that the symbols we now have in our Fourier integral operators are independent of y. Namely, we have

$$\widehat{S}_\zeta(\eta) = a_\zeta(\eta) e^{-2\pi i \phi(\zeta,\eta)}.$$

Fix $\zeta = (x,t)$ and $\zeta' = (x',t')$. By the multiplication formula,

$$\begin{aligned}
\langle S_\zeta, S_{\zeta'} \rangle &= \langle \widehat{S}_\zeta, \widehat{S}_{\zeta'} \rangle \\
&= \int a_\zeta(\eta) a_{\zeta'}(\eta) e^{2\pi i [\phi(\zeta',\eta) - \phi(\zeta,\eta)]}\, d\eta \\
&= R^d \int a_\zeta(R\eta) a_{\zeta'}(R\eta) e^{2\pi i R[\phi(\zeta',\eta) - \phi(\zeta,\eta)]}.
\end{aligned}$$

Let us write $\phi_{\zeta,\zeta'}(\eta)$ for the phase in this oscillatory integral. Then

$$\nabla_\eta \phi_{\zeta,\zeta'}(\eta) = \nabla_\eta \phi(\zeta',\eta) - \nabla_\eta \phi(\zeta,\eta).$$

Provided we have localized our analysis to a suitably small neighborhood of $(0,e_1)$, our assumptions that

$$(D_\eta \nabla_x \phi)(0,e_1) = I \quad \text{and} \quad (D_\eta \partial_t \phi)(0,e_1) = 0$$

imply that

$$|\nabla_\eta \phi_{\zeta,\zeta'}(\eta)| \geqslant 0.5|x - x'| - 0.1|t - t'|.$$

Thus we conclude using this formula that if $|x - x'| \geqslant 0.5|t - t'|$, then we have a non-stationary phase, and we can integrate by parts to conclude that for all $N \leqslant M$,

$$\langle S_\zeta, S_{\zeta'} \rangle \lesssim_N R^{-N}.$$

On the other hand, suppose that $|x - x'| \leqslant 0.5|t - t'|$. Set $\zeta(s) = \zeta + s(\zeta' - \zeta)$, and write

$$\frac{\phi(\zeta',\eta) - \phi(\zeta,\eta)}{t' - t} = \int_0^1 \left[\partial_t \phi(\zeta_s,\eta) + \left(\frac{x' - x}{t' - t} \right) \cdot \nabla_x \phi(\zeta_s,\eta) \right]\, ds.$$

It follows that the left hand side is a pertubation of $\partial_t \phi(0,\eta)$. The phase

$$(t,t') \mapsto (t' - t) \cdot \partial_t \phi(0,\eta)$$

has a stationary point at $(0, e_1)$ by assumption that $(D_\eta \partial_t)\phi(0, e_1) = 0$. But the Hessian $H_\eta(\partial_t \phi)$ has rank l at $(0, e_1)$, so the principle of stationary phase 'with parameters' (to account for the pertubation in phase, see Hörmander's FIO I paper for details) implies that

$$\langle S_\zeta, S_{\zeta'} \rangle \lesssim R^d \langle R|t - t'| \rangle^{-l/2} \quad \text{if } |x - x'| \leqslant 0.5|t - t'|.$$

Putting this bound together with the previous bound, we conclude that for any two ζ and ζ', we have

$$\langle S_\zeta, S_{\zeta'} \rangle \lesssim \frac{R^d}{\langle R|\zeta - \zeta'| \rangle^{l/2}}.$$

Together with a form of the Calderón-Zygmund decomposition introduced by Heo, Nasarov and Seeger, this bound will be sufficient to prove we get the bound required. TODO.

Chapter 13

Beltran, Hickman, and Sogge: Decoupling for Fourier Integral Operators

The paper we now discuss extends the theory of decoupling, which was originally used to establish local smoothing for the wave equation on Euclidean space, to the setting of more general FIOs. Here we attempt to study L^p to L^p estimates for Fourier integral operators given by

$$Tf(x,t) = \int_{\mathbb{R}^d} e^{2\pi i \phi(x,t;\xi)} b(x,t;\xi)(1+|\xi|^2)^{\mu/2} \widehat{f}(\xi)\, d\xi$$

where b is a compactly supported symbol of order zero, compactly supported in x and t, and ϕ is a phase function, homogeneous of degree one in the ξ variable. We let

$$K(x,t;y) = \int e^{2\pi i [\phi(x,t;\xi) - y\cdot\xi]} b(x,t;\xi)(1+|\xi|^2)^{\mu/2}\, d\xi$$

denote the kernel. Since $\nabla_\xi \phi(x,t;\xi)$ is homogeneous of degree zero in ϕ, the sets

$$\Sigma_{(x,t)} = \{\nabla_\xi \phi(x,t,\xi) : \xi \in \mathbb{R}^n\} \subset \mathbb{R}^n_y$$

are usually manifolds of dimension $n-1$. They are related to the singular support of K.

To study the L^p behaviour of T, we break up the behaviour of the operator dyadically in the ξ variable, thus setting

$$T = T_{\leqslant 1} + \sum_{n=1}^{\infty} T_n,$$

where, for a given $\lambda > 0$, we let T^λ be an operator with kernel K^λ given by

$$K^\lambda(x,t;\xi) = \int e^{2\pi i[\phi(x,t;\xi)-y\cdot\xi]} b(x,t;\xi)(1+|\xi|^2)^{\mu/2}\beta(\xi/\lambda)\, d\xi.$$

It can be verified that $T_{\leqslant 1}$ is a pseudo-differential operator of order 0, and is therefore bound on L^p for all $1 < p < \infty$. It therefore suffices to show that as $\lambda \to \infty$,

$$\|T^\lambda f\|_{L^p(\mathbb{R}^d)} \lesssim \lambda^{-\varepsilon}\|f\|_{L^p(\mathbb{R}^d)}$$

so that we may sum in n in the expansion of T via the triangle inequality to obtain an L^p bound for the original operator.

For large λ, the principle of stationary phase tells us we should expect K^λ to be concentrated in the set

$$\{(x,t;y) : |\nabla_\xi \phi(x,t;\xi) - y| \leqslant 1/\lambda \text{ for some } \xi\},$$

since the phase oscillates to a significant degree for $|\nabla\phi(x,t;\xi)-y| \gtrsim 1/\lambda$, roughly a $1/\lambda$ neighborhood of the singular support of K. Also we have $\|K^\lambda\|_{L^\infty_x L^\infty_y} \lesssim \lambda^{\mu+d}$ trivially by taking in absolute values. This gives the crude estimate that $\|K_n\|_{L^\infty_x L^1_y} \lesssim \lambda^{\mu+d-1}$. Thus we obtain by Schur's Lemma that

$$\|T^\lambda f\|_{L^1(\mathbb{R}^{d+1})} \lesssim \lambda^{\mu+d-1}\|f\|_{L^1(\mathbb{R}^d)}.$$

We will get a much better bound by a more sophisticated decomposition of the kernels $\{K^\lambda\}$.

For a given λ, let $\{\xi^\lambda_\nu\}$ be a maximal, $\lambda^{-1/2}$ separated subset of the unit sphere in $\mathbb{R}^n$, where ν ranges over some set Θ^λ with $\#(\Theta^\lambda) \sim \lambda^{(d-1)/2}$. Let

$$\Gamma^\lambda_\nu = \{\xi \in \mathbb{R}^d_\xi : |\xi \cdot \xi^\lambda_\nu| \geqslant (1 - c\lambda^{-1/2}) \cdot |\xi|\}$$

for some suitably small constant $c > 0$. Let $\{\chi^\lambda_\nu\}$ be a smooth partition of unity, homogeneous of degree zero, adapted to the Γ^λ_ν. We thus have

$$|D^\alpha \chi^\lambda_\nu(\xi)| \lesssim_\alpha \lambda^{|\alpha|/2}|\xi|^{1-\alpha}.$$

We thus consider operators T^λ_ν with kernels K^λ_ν given by

$$K^\lambda_\nu(x,t;y) = \int e^{2\pi i(\phi(x,t;\xi)-y\cdot\xi)} b^\lambda_\nu(x,t;\xi)(1+|\xi|^2)^{\mu/2}$$

where

$$b_\nu^\lambda(x,t;\xi) = b(x,t;\xi)\beta(\xi/\lambda)\chi_\nu^\lambda(\xi).$$

Stationary phase again tell us that $K_\nu^\lambda(x,t;y)$ satisfies the bounds

$$|K_\nu^\lambda(x,t;y)| \lesssim_N \frac{\lambda^{\mu+(d+1)/2}}{\langle \lambda|\pi_{\xi_\nu^\lambda}(y - \nabla_\xi \phi(x,t,\xi_\nu^\lambda)| + \lambda^{1/2}|\pi_{\xi_\nu^\lambda}^\perp(y - \nabla_\xi \phi(x,t,\xi_\nu^\lambda))|\rangle^N}.$$

This bound immediately yields via Schur's Lemma that for all $1 \leqslant p \leqslant \infty$,

$$\|K_\nu^\lambda\|_{L^\infty_{x,t} L^1_y} \lesssim \lambda^\mu,$$

and thus that

$$\|T_\nu^\lambda f\|_{L^\infty(\mathbb{R}^{d+1})} \lesssim \lambda^\mu \|f\|_{L^\infty(\mathbb{R}^d)},$$

a much better bound than was obtained trivially than from the global sum.

We might hope to then combine this still fairly trivial bound with a square function estimate of the form

$$\|T_\nu^\lambda f\|_{L^p(\mathbb{R}^{d+1})} \lesssim_\varepsilon \lambda^\varepsilon \|S^\lambda f\|_{L^p(\mathbb{R}^{d+1})}$$

where

$$S^\lambda f = \left(\sum_\nu |T_\nu^\lambda f|^2 \right)^{1/2},$$

which in some sense, captures the orthogonality of the operators $\{T_\nu^\lambda\}$. This then yields that for $p \geqslant 2$, that

$$\begin{aligned}
\|T_\nu^\lambda f\|_{L^p_{x,t}} &\lesssim_\varepsilon \lambda^\varepsilon \|T_\nu^\lambda f\|_{L^p_{x,t} l^2_\nu} \\
&\leqslant \lambda^\varepsilon \|T_\nu^\lambda f\|_{L^p_{x,t} l^p_\nu} \\
&= \lambda^\varepsilon \|T_\nu^\lambda f\|_{l^p_\nu L^p_{x,t})} \\
&\lesssim \lambda^\varepsilon \lambda^{\mu+(d-1)/2} \#(\Theta^\lambda)^{1/p} \\
&= \lambda^{\varepsilon+\mu+(d-1)/p},
\end{aligned}$$

thus giving bounds for $\mu > (d-1)/2$, i.e., the non-endpoint local smoothing.

Wolff noticed that the non-endpoint local smoothing results could be obtained with a weaker bound than a square function estimate, namely, an l^p *decoupling inequality* of the form

$$\|T^\lambda f\|_{L^p(\mathbb{R}^{d+1})} \lesssim \lambda^{\alpha(p)+\varepsilon} \|T_\nu^\lambda f\|_{l^p_\nu L^p_{x,t}},$$

where if $2 \leqslant p \leqslant 2(d+1)/(d-1)$, then

$$\alpha(p) = (d-1)|1/p - 1/2|,$$

and for $2(d+1)/(d-1) \leqslant p < \infty$,

$$\alpha(p) = (d-1)|1/p - 1/2| - 1/p.$$

The L^p norm of the localized pieces is much easier to estimate. For instance, we have

$$\|T_\nu^\lambda f\|_{L^\infty_{x,t}} \lesssim \lambda^\mu \|f\|_{L^\infty},$$

and thus

$$\|T_\nu^\lambda f\|_{l_\nu^\infty L^\infty_{x,t}} \lesssim \lambda^\mu \|f\|_{L^\infty}.$$

On the other hand, we have an L^2 energy conservation estimate of the form

$$\|T_\nu^\lambda f\|_{L^2_{x,t}} \lesssim \|T_\nu^\lambda f\|_{L_t^\infty L_x^2} \lesssim \lambda^\mu \|f_\nu^\lambda\|_{L^2}$$

where f_ν^λ is the localization of f_ν^λ on the Fourier side to the support of χ_ν^λ. This immediately yields via Parseval's inequality and orthogonality that

$$\|T_\nu^\lambda f\|_{l_\nu^2 L^2_{x,t}} \lesssim \lambda^\mu \|f_\nu^\lambda\|_{l_\nu^2 L_x^2} \lesssim \lambda^\mu \|f\|_{L_x^2}.$$

Interpolation thus yields that for $2 \leqslant p \leqslant \infty$,

$$\|T_\nu^\lambda f\|_{l_\nu^p L^p_{x,t}} \lesssim \lambda^\mu \|f\|_{L^p},$$

and thus that, together with Wolff's decoupling inequality,

$$\|T^\lambda f\|_{L^p(\mathbb{R}^{d+1})} \lesssim_\varepsilon \lambda^{\alpha(p)+\mu+\varepsilon} \|f\|_{L^p(\mathbb{R}^d)},$$

and thus we get boundedness of T for $\mu < \alpha(p)$, which gives $1/p$ degrees of local smoothing.

Chapter 14

Kim: Variable-Coefficient Generalizations of the Radial Multiplier Conjecture

This chapter discusses Jongchon Kim's 2017 paper *Endpoint Bounds for Quasiradial Fourier Multipliers* [13], and his 2018 paper *Endpoint Bounds for a Class of Spectral Multipliers on Compact Manifolds* [14]. These two papers introduce some useful techniques which can be used to generalize some of the results of Heo-Nasarov-Seeger to variable-coefficient settings.

Lets begin with the results of [13]. Given a homogeneous function

$$a : \mathbb{R}^d \to (0, \infty)$$

of degree one, smooth away from the origin, the paper discusses the problem of bounding 'quasiradial' Fourier multipliers with symbols of the form

$$m(\xi) = h(a(\xi)).$$

The homogeneity and non-negativity condition implies that the 'cosphere'

$$\Sigma = \{\xi : a(\xi) = 1\}$$

is a smooth hypersurface in $\mathbb{R}^d$. Under the additional assumption that this surface has everywhere non-vanishing Gaussian curvature, Kim proves that for $d \geqslant 4$, and $1 < p < 2(d-1)/(d+1)$, if h is a unit scale multiplier, then

$$\|h \circ a\|_{M^p(\mathbb{R}^d)} \lesssim C_p(h).$$

This is analogous to the result of Heo, Nasarov, and Seeger, but applied to multipliers that are now only *quasi-radial* rather than radial.

Similar results are obtained in [14], but in the setting of compact manifolds. Given a smooth, compact manifold M, and a first-order classical, elliptic, formally positive, self-adjoint pseudodifferential operator P, such that the cospheres

$$\Sigma_x = \{\xi \in T_x M : p(x,\xi) = 1\}$$

have non-vanishing Gaussian curvature, Kim proves that, for a unit-scale function h and for $1 < p < 2(d+1)/(d+3)$,

$$\|h\|_{M^{p,q}_{\text{Dil}}(M,P)} \lesssim \|h\|_{B^2_{d(1/p-1/2),q}(\mathbb{R})},$$

This improves results of Seeger and Sogge (1989), who proved the result with $B^{2,q}_{\alpha_p}$ replaced with L^2_α for $\alpha > \alpha_p$, and a result of Seeger (1991), who replaced $B^{2,q}_{\alpha_p}$ with a subspace $R^{2,q}_{\alpha_p}$ consisting of functions which have decompositions similar to the Bochner-Riesz multipliers. TODO: Understand the relation between these two results.

14.1 Quasi-Radial Multipliers

Let's begin by describing the new ideas introduced in [13]. Let h and a be as above. If η is supported on $\{1/4 \leqslant |\xi| \leqslant 4\}$, and equal to one on $\{1/2 \leqslant |\xi| \leqslant 2\}$, then

$$\begin{aligned} m(D)f(x) &= \int h(a(\xi)) e^{2\pi i \xi \cdot (x-y)} f(y)\; dy\; d\xi \\ &= \int h(a(\xi))\eta(a(\xi)) e^{2\pi i \xi \cdot (x-y)} f(y)\; dy\; d\xi \\ &= \int \widehat{h}(t)\eta(a(\xi)) e^{2\pi i [\xi \cdot (x-y) + t a(\xi)]} f(y)\; dy\; d\xi\; dt \\ &= \int \widehat{h}(t)(K_t * f)(x)\; dx\; dt, \end{aligned}$$

where K_t is the kernel with

$$\widehat{K}_t(\xi) = \eta(a(\xi)) e^{2\pi i t a(\xi)}.$$

Let's start with some L^1 estimates for the kernels $\{K_t\}$, which do not even use the curvature properties of Σ.

Theorem 14.1.
$$\|K_t\|_{L^1(\mathbb{R}^d)} \lesssim \langle t \rangle^{\frac{d-1}{2}}.$$

Proof. Let ψ be the inverse Fourier transform of the function $\eta(a(\cdot))$. Then the Fourier transform of $\mathrm{Dil}_{1/t}\psi$ is equal to $t^{-d}\mathrm{Dil}_t(\eta \circ a)$. We calculate that

$$\begin{aligned} K_t(tx) &= \int (\eta \circ a)(\xi) e^{2\pi i [ta(\xi) + \xi \cdot tx]}\, d\xi \\ &= \int t^{-d}\mathrm{Dil}_t(\eta \circ a)(\xi) e^{2\pi i [a(\xi) + \xi \cdot x]}\, d\xi \\ &= e^{2\pi i a(D)}(\mathrm{Dil}_{1/t}\psi)(x). \end{aligned}$$

Thus
$$\|K_t\|_{L^1(\mathbb{R}^d)} = t^{-d}\|e^{2\pi i a(D)}(\mathrm{Dil}_{1/t}\psi)\|_{L^1(\mathbb{R}^d)}.$$

The operator $e^{2\pi i a(D)}$ is a Fourier integral operator on $\mathbb{R}^d$, whose canonical relation is the conormal bundle to the surface Σ, which is, in particular, a canonical graph. Thus Theorem 2.2 of [18] implies that

$$\|e^{2\pi i a(D)}(\mathrm{Dil}_{1/t}f)\|_{L^1(\mathbb{R}^d)} \lesssim \|(1+\Delta)^{\frac{d-1}{2}}\{\mathrm{Dil}_{1/t}f\}\|_{H^1(\mathbb{R}^d)} \lesssim t^{\frac{d-1}{2}}.$$

TODO: Finish off. □

By the curvature hypothesis, for each $z \in \mathbb{R}^d - \{0\}$, there are exactly two points ξ_+ and ξ_- on Σ such that z is normal to Σ at these points[1]. Moreover, we can globally parameterize these normal points, such that $z \mapsto \xi_+(z)$ and $z \mapsto \xi_-(z)$ are smooth functions for $z \in \mathbb{R}^d - \{0\}$. We let

$$\psi_+(z) = \xi_+ \cdot z \quad \text{and} \quad \psi_-(z) = \xi_- \cdot z.$$

Now consider the coordinate system $(0,\infty) \times \Sigma \to \mathbb{R}^d$ given by $(\rho,\omega) \mapsto \rho\omega$. In coordinates, we have

$$d\xi = \rho^{d-1}(\omega \cdot n(\omega)) d\rho d\sigma(\omega),$$

[1] To prove at least two such points exist, simply take the maxima and minima of the function $f(\xi) = z \cdot \xi$. The curvature condition on Σ implies that any extremal point ξ^* on Σ is either a global maxima or a global minima, since Σ must, by the curvature condition, always curve away from the hyperplane normal to z at each point.

We can thus write

$$
\begin{aligned}
K_t(z) &= \int \eta(a(\xi)) e^{2\pi i [t a(\xi) + \xi \cdot z]} \, d\xi \\
&= \int_0^\infty \int_\Sigma \rho^{d-1} \eta(\rho) \left(\langle \omega, n(\omega) \rangle e^{2\pi i \rho [t + \omega \cdot z]} \right) \, d\sigma(\omega) \, d\rho \\
&= \int_0^\infty \rho^{d-1} \eta(\rho) \Big(b_+(\rho z) e^{2\pi i \rho (t + \psi_+(z))} + b_-(\rho z) e^{-2\pi i \rho (t + \psi_-(z))} \Big) \, d\rho,
\end{aligned}
$$

where b_+ and b_- are symbols of order $-(d-1)/2$, and where b_+ and b_- have principal symbols which are scalar multiples of

$$
z \mapsto |z|^{-\frac{d-1}{2}} \langle \xi_+, n(\xi_+) \rangle \quad \text{and} \quad z \mapsto |z|^{-\frac{d-1}{2}} \langle \xi_-, n(\xi_-) \rangle
$$

respectively. Exploiting the oscillation of $e^{2\pi i \rho (t + \psi_+(z))}$ and $e^{-2\pi i \rho (t + \psi_-(z))}$, integrating by parts, we conclude that for all $N \geqslant 0$,

$$
|K_t(z)| \lesssim_N (1 + |t| + |x|)^{-\frac{d-1}{2}} \sum_{\pm} \langle t + \psi_\pm(x) \rangle^{-N}. \tag{14.1}
$$

We note that $\psi_+(x) \geqslant 0 \geqslant \psi_-(x)$, that both functions are homogeneous of degree one, and that in the basic situation where Σ is the unit sphere, i.e. when $a(\xi) = |\xi|$,

$$
\psi_+(x) = x \quad \text{and} \quad \psi_-(x) = -x.
$$

For $t > 0$, we thus see that (14.1) reads

$$
|K_t(z)| \lesssim_N (1 + |t| + |x|)^{-\frac{d-1}{2}} \langle t + \psi_-(x) \rangle^{-N}.
$$

Similarily, we can apply Parseval's identity, together with the fact that

$$
\widehat{K}_t(\zeta) = \eta(a(\zeta)) e^{2\pi i t a(\zeta)},
$$

to conclude that

$$
\begin{aligned}
&\int K_{t_0}(x - x_0) \overline{K_{t_1}(x - x_1)} \, dx \\
&\qquad = \int |\eta(a(\zeta))|^2 e^{2\pi i [(x_0 - x_1) \cdot \zeta + (t_0 - t_1) a(\zeta)]},
\end{aligned}
$$

and similar stationary phase estimates to above show that for $N \geqslant 0$,

$$
\begin{aligned}
&\left| \int K_{t_0}(x - x_0) \overline{K_{t_1}(x - x_1)} \, dx \right| \\
&\qquad \lesssim_N (1 + |x_0 - x_1| + |t_0 - t_1|)^{-\frac{d-1}{2}} \sum_{\pm} \langle (t_0 - t_1) + \phi_\pm(x_0 - x_1) \rangle^{-N}.
\end{aligned}
$$

Remark. Using (14.1), one sees the kernels K_t decay rapidly away from

$$\{x : |t + \psi_-(x)| \leqslant 1\},$$

a set well approximated by an annulus of thickness $O(1)$ and radius $\sim t$, in particular having measure $O(t^{d-1})$. On this set, K_t has magnitude $O(t^{-\frac{d-1}{2}})$. This gives an alternate method to obtain the L^1 bound

$$\|K_t\|_{L^1(\mathbb{R}^d)} \lesssim t^{d-1} t^{-\frac{d-1}{2}} = t^{\frac{d-1}{2}},$$

that was obtained without the use of the curvature hypothesis.

Together with a discretization argument, and the density decomposition arguments introduced in Heo-Nazarov-Seeger, the result follows. One slight difference is that, since we are using the half-wave equation formalism, our discretized estimates have slightly different weights, corresponding to the definition of $C_p(h)$. Indeed, it is proved that

$$\left\| \sum_{n\geqslant 2} K_{n+u} * f(n,\cdot) \right\|_{L^p(\mathbb{R}^d)} \lesssim \left\| (1+n+u)^{s_p} f(n+u,y) \right\|_{l_n^p(\mathbb{N}) L_y^p(\mathbb{R}^d)},$$

uniformly for $0 \leqslant u \leqslant 1$. Applying Minkowski's inequality, and the fact that $L^1[0,1] \leqslant L^p[0,1]$,

$$\begin{aligned}
\left\| \int_2^\infty (K_t * f)(t,\cdot)\, dt \right\|_{L^p(\mathbb{R}^d)} &\lesssim \Big\| \sum_{n\geqslant 2} K_{n+u} * f(n+u,\cdot) \Big\|_{L_u^1[0,1] L^p(\mathbb{R}^d)} \\
&\leqslant \Big\| \sum_{n\geqslant 2} K_{n+u} * f(n+u,\cdot) \Big\|_{L_u^p[0,1] L^p(\mathbb{R}^d)} \\
&\lesssim \Big\| (1+n+u)^{s_p} f(n+u,y) \|_{L_u^p[0,1] l_n^p(\mathbb{N}) L_y^p(\mathbb{R}^d)} \\
&= \left(\iint_{t\geqslant 1} (1+t)^{(d-1)(1-p/2)} |f(t,y)|^p \, dt\, dy \right)^{1/p}.
\end{aligned}$$

The right hand side of this inequality is comparable to $C_p(h)$ times $\|g\|_{L^p(\mathbb{R}^d)}$ if $f(t,y) = \widehat{m}(t) g(y)$, in which case the computed quantity is essentially the L^p norm of $m(a(D))\{g\}$. This inequality is equivalent to the boundedness of the convolution kernel in terms of $C_p(h)$.

Next, discretization is done in the y-variable. To prove the inequality above, it suffices to show that for functions $b_{n,z}$ concentrated on neighborhoods of $z \in \mathbb{Z}^d$,

$$\left\| \sum_{n \geqslant 2} \sum_{z \in \mathbb{Z}^d} c(n,z)(K_n * b_{n,z}) \right\|_{L^p(\mathbb{R}^d)} \lesssim \left\| c(n,z)(1+n)^{s_p} \right\|_{l_n^p l_z^p}.$$

For interpolation purposes, it is better to reweight this inequality as

$$\left\| \sum_{n \geqslant 2} \sum_{z \in \mathbb{Z}^d} (1+n)^{\frac{d-1}{2}} c(n,z)(K_n * b_{n,z}) \right\|_{L^p(\mathbb{R}^d)} \lesssim \left(\sum_n \sum_n |c(n,z)|^p (1+n)^{d-1} \right)^{1/p}.$$

14.2 Spectral Multipliers

The result of [14] uses similar techniques, combined with the Lax parametrix to substitute for the convolution kernel decomposition possible in [13] because of the translation invariance of the operators studied.

In the proof, by a partition of unity argument, it suffices to prove the result assuming that our inputs functions f lie in $L^p(\Omega_0)$, where Ω_0 is a compact subset of M contained in a single coordinate chart Ω of M. It will help to fix a compact set Ω_1 whose interior contains Ω_0.

To prove the result, we write

$$m(P/R)f = \int [R\widehat{m}(R\xi)] e^{2\pi i t P} f.$$

Write $m = \sum_{j \geqslant 0} m_j$, where for $j > 0$, m_j is a Littlewood-Paley cutoff on an annulus at a scale 2^j, and m_0 is a Littlewood-Paley cutoff on the unit ball.

For $2^j \lesssim 1$, we can reduce the study of the operators $m_j(P/R)$ to symbol estimates. For $2^j \gtrsim R$, one can use the local constancy property of the multipliers, together with $L^p(M)$ to $L^2(M)$ by using the fact that M is compact, and thus has finite volume, and then apply orthogonality, to obtain the boundedness of $m_j(P/R)$. We mainly concentrate on the new techniques for controlling the interactions between the multipliers $\{m_j : 1 \lesssim j \lesssim \log R\}$.

We're going to need some even, smooth cutoff functions $\eta \prec \eta' \prec \eta''$:

- η is supported on $\{|\lambda| \in [1/4, 4]\}$, and equal to one on $\{|\lambda| \in [1/2, 2]\}$.
- η' is supported on $\{|\lambda| \in [1/6, 6]\}$, and equal to one on $\{|\lambda| \in [1/4, 4]\}$.

- η'' is supported on $\{|\lambda| \in [1/16, 16]\}$, and equal to one on $\{|\lambda| \in [1/8, 8]\}$.

We let η_j, η'_j, and η''_j denote the dilations of these cutoffs by 2^j.

It will help us to localize the function f to the eigenband $\lambda \sim R$. For $f \in L^p(\Omega_0)$, we can write

$$\eta''(P/R)f = S_R f + A_R f,$$

where $\|A_R f\|_{L^p(M)} \lesssim_N R^{-N}\|f\|_{L^p(\Omega_0)}$ for all $f \in L^p(\Omega_0)$, and where $S_R f$ is supported on Ω_1 for $f \in L^p(\Omega_0)$.

We could conclude our argument if we could show that

$$\left\| \sum_{1 \lesssim j \lesssim \log R} m_j(P/R) f \right\|_{L^{p,q}(M)} \lesssim \|m\|_{B^{2,q}_{\alpha_p}(\mathbb{R})} \|f\|_{L^p(\Omega_0)},$$

for all functions $f \in L^p(\Omega_0)$. We claim this result follows from the following proposition.

Lemma 14.2. *Fix a family of functions $\{b_j : 1 \lesssim j \lesssim \log R\}$ such that:*

- $\|b_j\|_{L^2(\mathbb{R})} \lesssim 1$.
- *$\widehat{b}_j$ was supported on $\{|t| \sim 2^j\}$.*
- *For any n and M, and $|\lambda| \notin [1/8, 8]$,*

$$|\partial^n b_j(\lambda)| \lesssim_{n,M} 2^{-jM} \langle 2^j \lambda \rangle^{-M}.$$

Then for functions $\{f_j\}$ in $L^p(\Omega_0)$,

$$\left\| \sum_j 2^{jd/2} b_j(P/R)\{S_R f_j\} \right\|_{L^p(\Omega)} \lesssim \left(\sum_j 2^{jd} \|f_j\|^p_{L^p(M)} \right)^{1/p}.$$

It follows simply from this that

$$\left\| \sum_j 2^{jd/2} b_j(P/R) f_j \right\|_{L^p(M)} \lesssim \left(\sum_j 2^{jd} \|f_j\|^p_{L^p(\Omega_0)} \right)^{1/p}.$$

An interpolation lemma (Lemma 2.4 of Lee, Rogers, and Seeger) yields a square function estimate of the form.

$$\left\| \sum_j 2^{-jd(1/p-1/2)} b_j(P/R) f_j \right\|_{L^{p,q}(M)} \lesssim \left\| \left(|f_j|^q \right)^{1/q} \right\|_{L^p(\Omega_0)}$$

How does this result imply the required inequality? We can write

$$m_j = \eta_j(D)m_j = \eta_j(D)\{m_j\eta'\} + \eta_j(D)\{m_j(1-\eta')\}.$$

The multipliers $\eta_j(D)\{m_j(1-\eta')\}$ are rapidly decaying (TODO: I Can't see intuitively why this should be the case), so it's easy to bound the corresponding multiplier operators. We therefore reduce our required estimate to

$$\left\| \sum_j [\eta_j(D)\{m_j\eta'\}](P/R)f \right\|_{L^{p,q}(M)} \lesssim \|m\|_{B^{2,q}_{\alpha_p}(\mathbb{R})} \|f\|_{L^p(\Omega)},$$

for f supported on Ω. If $b_j = \|m_j\|^{-1}_{L^2(\mathbb{R})}\eta_j(D)\{m_j\eta'\}$, then $\{b_j\}$ satisfies the assumptions of the lemma above, and the interpolated result, with $f_j = 2^{j\alpha_p}\|m_j\|_{L^2(\mathbb{R})}f$.

The Lemma will be proved by a reduction to a restricted weak-type inequality, namely, that for any family of finite subsets $\mathcal{E}_j$ of $(\mathbb{Z}/R)^d$,

$$\left\| \sum_j 2^{jd/2} \sum_{n\in\mathcal{E}_j} b_j(P/R)\chi_{j,n} \right\|_{L^{p,\infty}(\Omega)} \lesssim R^{-d/p} \left(\sum_j 2^{jd}\#\mathcal{E}_j \right)^{1/p}.$$

We will mainly focus now on the L^2 estimates that imply this restricted weak-type inequality.

For a fixed $\alpha > 0$, we cover $\mathbb{R}^d$ by essentially disjoint cubes with sidelength $2^j/R$, and denote the collection of cubes in $\mathcal{Q}_j$. Let $\mathcal{Q}_j(\lambda)$ be the collection of all $Q \in \mathcal{Q}_j$ such that $Q \cap \#\mathcal{E}_j > \lambda^p$. This allows us to write

$$\mathcal{E}_j = \mathcal{E}_j^{\text{High}}(\lambda) \cup \mathcal{E}_j^{\text{Low}}(\lambda),$$

where $\mathcal{E}_j^{\text{High}}(\lambda)$ is the collection of points in $\mathcal{E}_j$ that lie in a cube Q lying in $\mathcal{Q}_j(\lambda)$, i.e. the set of 'high density points'. Fix $C > 0$ suitably large, and for each cube Q, let Q^* denote the cube with the same center as Q, but C times the sidelength. Then

$$\sum_j \sum_{Q\in\mathcal{Q}_j(\lambda)} |Q| \lesssim \sum_j \sum_{Q\in\mathcal{Q}_j} (2^{jd}/R)\alpha^{-p}\#(\mathcal{E}_j \cap Q) \leqslant \alpha^{-p} \sum_j 2^{jd}t^{-d}\#(\mathcal{E}_j).$$

This inequality shows that it is fair to throw out the union of the cubes in $\mathcal{Q}_j(\lambda)$ in the analysis of the measure of the set

$$\left\{ x \in \Omega : \left| \sum_j 2^{jd/2} \sum_{Q\in\mathcal{Q}_j(\lambda)} b_j(P/R)\chi_{j,n} \right| > \alpha \right\}.$$

The required bound would follow if we could show that

$$\lambda^{-1} \sum_j 2^{jd/2} \left\| \sum_{Q \in \mathcal{Q}_j(\lambda)} [b_j(P/R)\chi_{j,Q}] \mathbf{I}_{(Q^*)^c} \right\|_{L^1(\Omega)} \lesssim \lambda^{-p} \sum_j 2^{jd} R^{-d} \# \mathcal{E}_j.$$

But (see end of Section 5.1 for more details), this follows from the fact that the Lax parametrix is supported on a small neighborhood of the diagonal.

We are now left with the analysis of $\mathcal{E}_j^{\text{Low}}(\lambda)$. To simplify notation, we will assume that $\mathcal{E}_j = \mathcal{E}_j^{\text{Low}}(\lambda)$ for all j. We will try and control

$$\left\| \sum_j 2^{jd/2} \sum_{n \in \mathcal{E}_j} b_j(P/R)\chi_{j,n} \right\|_{L^2(\Omega)}^2 \lesssim \alpha^{2-p} \log \alpha \sum_j 2^{jd} R^{-d} \# \mathcal{E}_j.$$

Define $G_j = \sum_{n \in \mathcal{E}_j} b_j(P/R) f_{j,n}$. Then, applying the triangle inequality for $j \lesssim \log \alpha$, we conclude that

$$\| \sum_j 2^{jd/2} G_j \|_{L^2(\Omega)}^2 \lesssim \log \alpha \left(\sum_j 2^{jd} \|G_j\|_{L^2(\Omega)}^2 + \sum_j \sum_{\log \alpha \lesssim k \lesssim j} 2^{\frac{k+j}{2}} |\langle G_j, G_k \rangle| \right).$$

We will bound each of these quantities separately.

First, let's bounded $\|G_j\|_{L^2(\Omega)}$.

Let us suppose the Lax parametrix applies on times $|t| \leqslant \varepsilon$. Times larger than this are dealt with fairly simply

We can apply the Lax parametrix for times $|t| \leqslant \varepsilon$. The large times are dealt with easily, using compactness to reduce $L^p(M)$ estimates to $L^2(M)$ estimates, and then applying orthogonality. Similarily, small times can also be dealt with by reducing to the study of pseudodifferential operators.

Part III

Review of Square Function Techniques

Chapter 15

Heo-Nasarov-Seeger: Extending Radial Multiplier Bounds to $\mathbb{R}^d$

Let's review how the method of atomic decompositions is used in [9] to extend results for unit scale multipliers to a general result for multipliers. Let us consider a general multiplier m, and write

$$m_j(\xi) = \eta(\xi) m(2^j \xi),$$

Let $H_j = \widehat{m}_j$, and let $K_j = 2^{jd} H_j(2^j \xi)$. Then our assumptions are that

$$\|K_j\|_{L^p(\mathbb{R}^d)} \leqslant A 2^{-jd(1-1/p)}.$$

Given a general input f, let us define $f_j = \eta(D/2^j) f$. We utilize a non-tangential variant of Peetre's maximal square function

$$Sf(x) = \left(\sum_j \left(2^{jd} \sup_{|y| \lesssim 2^{-j}} |f_j(x+y)| \right)^2 \right)^{1/2}.$$

Then for $1 < p < \infty$,

$$\|Sf\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

We will try and show that

$$\left\| \sum_j K_j * f_j \right\|_{L^p(\mathbb{R}^d)} \lesssim A \|Sf\|_{L^p(\mathbb{R}^d)},$$

from which the result would follow.

An initial estimate is obtained from the cancellation of the forms $\{K_j\}$, i.e.

$$\| \sum_j \eta(D/2^j) f_j \|_{L^p(\mathbb{R}^d)} \lesssim \| f_j \|_{l^p(\mathbb{N}) L^p(\mathbb{R}^d)}.$$

The inequality is easily obtained using Plancherel for $p = 2$, and the triangle inequality for $p = 1$. We complement this inequality with an estimate obtained from an 'atomic decomposition' for f_j. We let

$$\Omega_s = \{x : Sf(x) > 2^s\}$$

and then set

$$\Omega_s^* = \{x : M\chi_{\Omega_s} > 100^{-d}\}.$$

The weak L^1 boundedness for the maximal function implies $|\Omega_s^*| \lesssim |\Omega_s|$. We also let $\mathcal{Q}_s^j$ be the set of all dyadic cubes of sidelength 2^{-j}, such that

$$|Q \cap \Omega_s| \geqslant \frac{|Q|}{2} \quad \text{and} \quad |Q \cap \Omega_{s+1}| < \frac{|Q|}{2}.$$

We then perform a Whitney decomposition of Ω_s^* into dyadic cubes $\mathcal{W}_s$, i.e. a maximal set of cubes W such that the 20-fold dilate W^* is contained in Ω_s^*. Each cube in $\mathcal{Q}_s^j$ is contained in a unique cube in $\mathcal{W}_s$. This is because if $Q \in \mathcal{Q}_s^j$, then

$$\frac{|Q^* \cap \Omega_s|}{|Q^*|} \geqslant 20^{-d} \frac{|Q \cap \Omega_s|}{|Q|} \geqslant \frac{20^{-d}}{2} \geqslant 100^{-d}.$$

Thus $Q^* \subset \Omega_s^*$, implying Q is contained in some cube in $\mathcal{W}_s$.

We now break our functions down into 'atoms'. For $W \in \mathcal{W}_s$, set

$$a_{j,s,W} = \sum_{\substack{Q \in \mathcal{Q}_s^j \\ Q \subset W}} f_j \chi_Q.$$

We then consider 'cumulative atoms'

$$a_{j,W} = \sum_{W \in \mathcal{W}_s} A_{j,s,W}.$$

Then we have an atomic decomposition

$$f_j = \sum_W a_{j,W} = \sum_s \sum_{W \in \mathcal{W}_s} a_{j,s,W}.$$

We have two important bounds for the atoms. First, we have that

$$\left(\sum_{W \in \mathcal{W}_s} \sum_j \|a_{j,s,W}\|_{L^2}^2 \right) \lesssim 2^j |\Omega_j|^{1/2}.$$

To see this, we calculate that

$$\begin{aligned}
\sum_{W \in \mathcal{W}_s} \sum_j \|a_{j,s,W}\|_{L^2(\mathbb{R}^d)}^2 &= \sum_s \sum_{Q \in Q_s^j} \|f_j \chi_Q\|_{L^2(\mathbb{R}^d)}^2 \\
&\leqslant \sum_s \sum_{Q \in Q_s^j} |Q| \|f_j \chi_Q\|_{L^\infty(\mathbb{R}^d)}^2 \\
&\lesssim \sum_s \sum_{Q \in Q_s^j} |Q - \Omega_{j+1}| \|f_j \chi_Q\|_{L^\infty(\mathbb{R}^d)}^2 \\
&\leqslant \int_{\Omega_s^* - \Omega_{s+1}} \sum_j \sup_{|y| \leqslant 2^{-j}} |f_j(x+y)|^2 \, dx \\
&\lesssim 2^{2s} |\Omega_s^*| \lesssim 2^{2s} |\Omega_s|.
\end{aligned}$$

Another useful bound is that, for any assignment $j(W)$ for each $W \in \mathcal{W}_s$, and for each $0 \leqslant p \leqslant 2$,

$$\left(\sum_{W \in \mathcal{W}_s} |W| \; \|A_{j(W),s,W}\|_{L^\infty}^p \right)^{1/p} \lesssim 2^s |\Omega_j|^{1/p}.$$

To see this result, we simply note that for each $W \in \mathcal{W}_s$, and each $Q \in Q_s^j$, $Q \subset \Omega_s^* - \Omega_{s+1}$,

$$\|A_{j,s,W}\|_{L^\infty(\mathbb{R}^d)} \lesssim \sup_{\substack{Q \in Q_s^j \\ Q \subset W}} \|f_j \chi_Q\|_{L^\infty(\mathbb{R}^d)} \leqslant \sup_{x \in \Omega_s^* - \Omega_{s+1}} |Sf(x)| \lesssim 2^s,$$

and $\sum_{W \in \mathcal{W}_s} |W| \leqslant |\Omega_j^*| \lesssim |\Omega_j|$.

We must argue that

$$\left\| \sum_{s,j} \sum_{l \geqslant 0} \sum_{\substack{W \in \mathcal{W}_s \\ l(W) = 2^{l-j}}} K_j * A_{j,s,W} \right\|_{L^p(\mathbb{R}^d)} \lesssim A \|Sf\|_{L^p(\mathbb{R}^d)}.$$

Let us split the convolution operator K_j into a 'short range piece' and a 'long range piece', writing $K_j = K_{j,l}^{\text{short}} + K_{j,l}^{\text{long}}$, where

$$K_{j,l}^{\text{short}}(x) = 2^{jd} H_{j,l}^{\text{short}}(2^j x) \quad \text{and} \quad K_{j,l}^{\text{long}}(x) = 2^{jd} H_{j,l}^{\text{long}}(2^j x),$$

where

$$H_{j,l}^{\text{short}}(x) = H_{j,l}(x)\mathbf{I}(|x| \leqslant 2^l) \quad \text{and} \quad H_{j,l}^{\text{long}}(x) = H_{j,l}(x)\mathbf{I}(|x| > 2^l).$$

It suffices to show that for $1 \leqslant q \leqslant 2$, and each fixed s,

$$\left\| \sum_j \sum_l \sum_{\substack{W \in \mathcal{W}_s \\ l(W) = 2^{l-s}}} K_{j,l}^{\text{short}} * A_{j,s,W} \right\|_{L^q(\mathbb{R}^d)}^q \lesssim B 2^{sq} |\Omega_s|.$$

This is because, setting $q = p$,

$$\sum_s 2^{sp} |\Omega_s| \lesssim \|Sf\|_{L^p(\mathbb{R}^d)}^p.$$

It suffices to prove the result for $q = 2$, since the result for $q < 2$ follows by Hölder's inequality. To do this, we note that for a radial kernel k whose Fourier transform decays rapidly away from the unit annulus, and $p < 2d/(d+1)$,

$$\|\widehat{k}\|_{L^\infty} \lesssim \|k\|_{L^p(\mathbb{R}^d)}.$$

The result follows from Bessel asymptotics. Since the sets $\{W^* : W \in \mathcal{W}_s\}$ have bounded overlap, and with Parseval,

$$\begin{aligned}
&\left\| \sum_{W \in \mathcal{W}_s} \sum_j K_{j,l(W)+s}^{\text{short}} * A_{j,s,W} \right\|_{L^2(\mathbb{R}^d)}^2 \\
&\qquad \lesssim \sum_{W \in \mathcal{W}_s} \left\| \sum_s K_{j,l(W)+s}^{\text{short}} * A_{j,s,W} \right\|_{L^2(\mathbb{R}^d)}^2 \\
&\qquad \leqslant \sum_{W \in \mathcal{W}_s} \|\widehat{K}_{j,l(W)+s}^{\text{short}}\|_{L^\infty(\mathbb{R}^d)}^2 \sum_{W \in \mathcal{W}_s} \sum_j \|A_{j,s,W}\|_{L^2(\mathbb{R}^d)}^2.
\end{aligned}$$

But we know this last quantity is bounded by $B 2^{2j} |\Omega_j|$.

Now we look at the long range estimates. We are helped here by an exponential bound on the L^p norm of smoothed convolution kernels supported on large radii. TODO: Finish Off last part of argument on Page 23, and Proposition 6.1.

Chapter 16

Seeger: Singular Convolution Operators in L^p Spaces

Let $m : \mathbb{R}^d \to \mathbb{C}$ be the symbol for a Fourier multiplier operator $m(D)$. If the operator was bounded from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$, then it would also be bounded 'at all frequency scales'. That is, if we consider a Littlewood-Paley decomposition $1 = \sum \eta(\xi/2^j)$, and define $m_j(\xi) = m(2^j\xi)\eta_j(\xi)$, i.e. so that

$$m(\xi) = \sum_{j \geqslant 0} m_j(\xi/2^j),$$

then we would have estimates of the form

$$\|m_j(D)f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}, \tag{16.1}$$

where the implicit constant is uniform in j. The main focus of the paper in question is to determine whether a uniform bound of the form (16.1) implies $m(D)$ is bounded. More precisely, for which values of p is it true that

$$\|m\|_{M^p(\mathbb{R}^d)} \lesssim \sup\nolimits_{j \geqslant 0} \|m_j\|_{M^p(\mathbb{R}^d)}. \tag{16.2}$$

It is clear that (16.2) is true for $p = 2$, since in this case (16.2) is equivalent to an inequality of the form

$$\|m\|_{L^\infty(\mathbb{R}^d)} \lesssim \sup_{j \geqslant 0} \|m_j\|_{L^\infty(\mathbb{R}^d)},$$

which holds because the supports of the symbols $\{m_j\}$ have the bounded finite intersection property.

In $M^1(\mathbb{R})$, the result fails. The Hilbert transform H is a Fourier multiplier with symbol $m(\xi) = \mathrm{sgn}(\xi)$. For each $j > 0$, $m_j = \eta(\xi)\mathrm{sgn}(\xi)$ is invariant of j, with inverse fourier transform given by $k = H\eta$. In particular,

$$\|m_j\|_{M^1(\mathbb{R})} = \|k\|_{L^1(\mathbb{R})} = \|H\widehat{\eta}\|_{L^1(\mathbb{R})} \lesssim 1.$$

The multipliers $\{m_j(D)\}$ are therefore uniformly bounded on $L^1(\mathbb{R})$. It follows that (16.2) cannot hold for $p = 1$, since $H = m(D)$ is not bounded on $L^1(\mathbb{R})$. A substitute in this setting is the Hörmander-Mikhlin multiplier criterion discussed in the introduction, which gives boundedness of m in $L^{1,\infty}(\mathbb{R})$ given a slightly stronger assumption on the individual multipliers $\{m_j\}$, i.e. given that for some $\varepsilon > 0$, the convolution kernels $\{k_j\}$ associated with the multipliers $\{m_j\}$ satisfy

$$\sup_j \int \langle x\rangle^\varepsilon |k_j(x)|\, dx \leqslant B,$$

the theorem then guarantees that $\|m(D)f\|_{L^{1,\infty}(\mathbb{R})} \lesssim_\varepsilon B\|f\|_{L^1(\mathbb{R})}$.

For $1 < p < \infty$, with $p \neq 2$, (16.2) *also fails*, due to a modification of an example of Littman, McCarthy, and Riviére, due to Triebel. For simplicity, let's work in $\mathbb{R}$, though generalizations exist in higher dimensions. If we fix an even bump function $\phi \in C_c^\infty(\mathbb{R})$ supported on a small enough neighborhood of the origin, and set

$$m_N(\xi) = \sum_{j=N}^{2N} e^{2\pi i(2^j\xi)}\phi(\xi - 2^j),$$

then

$$m_{N,j}(\xi) = m_N(2^j\xi)\eta(\xi) = e^{2\pi i(4^j\xi)}\phi(2^j(\xi - 1)),$$

and thus the multiplier operator corresponding to $m_{N,j}$ has convolution kernel

$$k_{N,j}(x) = 2^{-j}e^{2\pi i(4^j+x)}\widehat{\phi}(2^j + x/2^j)$$

Modulation and translation invariance of $L^1(\mathbb{R})$ imply that

$$\|m_{N,j}\|_{M^p(\mathbb{R})} \leqslant \|m_{N,j}\|_{M^1(\mathbb{R})} = \|k_{N,j}\|_{L^1(\mathbb{R})} \lesssim 1 \quad \text{uniformly in } N \text{ and } j.$$

On the other hand, we have

$$\|m_N\|_{M^p(\mathbb{R})} \gtrsim N^{|1/p-1/2|}.$$

To see this, define a function f_N such that

$$\widehat{f}_N(\xi) = \sum_{j=N}^{2N} e^{-2\pi i 2^j \xi} \phi(\xi - 2^j).$$

Then by the compactness of ϕ,

$$m_N(\xi)\widehat{f}_N(\xi) = \sum_{j=N}^{2N} \phi(\xi - 2^j)^2.$$

This is a sum of functions supported on different dyadic frequency scales, and so Littlewood-Paley theory implies that square-root cancellation occurs, so that for $1 < p < \infty$,

$$\|m_N(D)\{f\}\|_{L^p(\mathbb{R})} \sim_p N^{1/2}.$$

On the other hand, we have, by orthogonality,

$$\|f_N\|_{L^2(\mathbb{R})} = \|\widehat{f}_N\|_{L^2(\mathbb{R})} = N^{1/2},$$

and by the triangle inequality,

$$\|f_N\|_{L^1(\mathbb{R})} \lesssim N \quad \text{and} \quad \|f_N\|_{L^\infty(\mathbb{R})} \lesssim N.$$

Interpolation implies that

$$\|f_N\|_{L^p(\mathbb{R})} \lesssim N^{1/2+|1/p-1/2|}.$$

But putting this together with the lower bound on the L^p norm of $m_N(D)\{f_N\}$ we conclude that

$$\|m_N\|_{M^p(\mathbb{R})} \gtrsim N^{|1/p-1/2|}.$$

The Baire category theorem can then produce a counterexample to (16.2).

In this paper, it is shown one can combine bounds at each scale in L^p, by assuming the Hörmander-Mikhlin condition, but with a logarithmic improvement to that obtained using the standard Calderon-Zygmund type argument.

Theorem 16.1. *Suppose*

$$\sup_{j>0} \|m_j\|_{M^p(\mathbb{R}^d)} \leqslant A$$

and

$$\sup_{j>0} \int \langle x \rangle^\varepsilon |k_j(x)| \leqslant B,$$

then

$$\|m\|_{M^p(\mathbb{R}^d)} \lesssim_{\varepsilon,p} A\widetilde{\log}(B/A)^{|1/p-1/2|}.$$

Remark. Any multiplier m to which the theorem above applies satisfies the assumptions of the Hörmander-Mikhlin multiplier theorem, and so that theorem already guarantees that $m \in M^p(\mathbb{R}^d)$. The gain in this theorem is the logarithmic dependance on the constant in the Hörmander-Mikhlin criterion.

Triebel's example shows this result is tight. Indeed, since

$$|D^\alpha m_N| \lesssim_\alpha (2^{2N})^{|\alpha|} \quad \text{for } \alpha \geqslant 0,$$

we can apply the theorem above with $B = 2^{2N}$, which gives

$$\|m\|_{M^p(\mathbb{R})} \lesssim N^{|1/p-1/2|},$$

and we have already seen that $\|m\|_{M^p(\mathbb{R})} \gtrsim N^{|1/p-1/2|}$. The usual Hörmander-Mikhlin multiplier argument would only be able to obtain bounds which were exponential in N.

Let us prove this result. Without loss of generality, we may assume $2 \leqslant p \leqslant \infty$. To prove the result, we rely on vector-valued variants of Littlewood-Paley theory and the Fefferman-Stein sharp maximal function. Given a Banach space X, and an X-valued function $f : \mathbb{R}^d \to X$, we define the sharp maximal function to be

$$f^\#(x) = \sup_Q \fint_Q \|f(y) - f_Q\|_X \, dy \quad \text{where} \quad f_Q = \fint_Q f(y)\, dy.$$

For $p_0 \in [1,\infty]$ and $p \in (1,\infty]$ with $p \geqslant p_0$, the key fact about this function is that if $f \in L^{p_0}(\mathbb{R}^d, X)$, and $f^\# \in L^p(\mathbb{R}^d)$, then $Mf \in L^p(\mathbb{R}^d)$, and

$$\|Mf\|_{L^p(\mathbb{R}^d)} \lesssim \|f^\#\|_{L^p(\mathbb{R}^d)},$$

where M is the Hardy-Littlewood Maximal Function. To exploit this, we set $X = l^2(\mathbb{N})$, and define a vector valued operator $Sf = \{m_j(D/2^j)f\}$. If we can show that

$$\|(Sf)^\#\|_{L^p(\mathbb{R}^d)} \lesssim A\widetilde{\log}(B/A)^{1/2-1/p}\|f\|_{L^p(\mathbb{R}^d)},$$

then our proof would be completed, because it follows by Littlewood-Paley theory that, since $m(D) = \sum_j m_j(D/2^j)$,

$$\begin{aligned}
\|m(D)f\|_{L^p(\mathbb{R}^d)} &\sim \|Sf\|_{L^p(\mathbb{R}^d)l^2(\mathbb{N})} \\
&\lesssim \|M(Sf)\|_{L^p(\mathbb{R}^d)} \\
&\lesssim \|S^\# f\|_{L^p(\mathbb{R}^d)} \\
&\lesssim A\widetilde{\log}(B/A)^{1/2-1/p}\|f\|_{L^p(\mathbb{R}^d)}.
\end{aligned}$$

Thus our goal is to prove the estimate for

$$S^{\#} f(x) = \sup_{x \in Q} \fint_Q \left(\sum_{i=0}^{\infty} \left| m_j(D/2^j) f(y) - \fint_Q m_j(D/2^j) f(z)\, dz \right|^2 \right)^{1/2} dy.$$

For the purposes of interpolation, we will help to linearize $S^{\#}$ using duality, i.e. picking a dyadic cube Q_x for each x, and a family of functions $\{\chi_j(x,y)\}$ such that

$$\left(\sum |\chi_j(x,y)|^2 \right)^{1/2} \leqslant 1,$$

chosen such that

$$S^{\#} f(x) \approx \fint_{Q_x} \sum_{j=0}^{\infty} \left(m_j(D/2^j) f(y) - \fint_{Q_x} m_j(D/2^j) f(z)\, dz \right) \chi_j(x,y)\, dy.$$

Also, let $N = \widetilde{\log}(B/A)$. Then we can consider a decomposition $S^{\#} f = S_1 f + S_2 f$, where if $l : \mathbb{R}^d \to \mathbb{Z}$ is the function such that Q_x has sidelength $2^{l(x)}$,

$$S_1 f(x) = \sum_{|j+l(x)| \leqslant N} \fint_{Q_x} \left(m_j(D/2^j) f(y) - [m_j(D/2^j) f]_{Q_x} \right) \chi_j(x,y)\, dy$$

and

$$S_2 f(x) = \sum_{|j+l(x)| > N} \fint_{Q_x} \left(m_j(D/2^j) f(y) - [m_j(D/2^j) f]_{Q_x} \right) \chi_j(x,y)\, dy.$$

If $|j + l(x)|$ is small, then the uncertainty principle should tell us that $m_j(D/2^j) f$ is roughly constant on squares of radius Q_x, up to some small error, so not much cancellation can be expected to be obtained in the analysis of the function $S_1 f$. We will prove the more general fact that for any functions $\{f_j\}$, if we define

$$\tilde{S}_1\{f_j\}(x) = \sum_{|j+l(x)| \leqslant N} \fint_{Q_x} \left(\eta(D/2^j) f_j(y) - [\eta(D/2^j) f_j]_{Q_x} \right) \chi_j(x,y)\, dy,$$

then $\|\tilde{S}_1\{f_j\}\|_{L^p(\mathbb{R}^d)} \lesssim N^{1/2-1/p} \|f_j\|_{L^p(\mathbb{R}^d) l^p(\mathbb{N})}$, and in our particular case, where $f_j = m_j(D/2^j) f$, we can use the uniform boundedness of the operators $\{m_j(D/2^j)\}$ to argue that, since $L^p(\mathbb{R}^d) l^p(\mathbb{N}) = l^p(\mathbb{N}) L^p(\mathbb{R}^d)$,

$$\begin{aligned}
\|S_1 f\|_{L^p(\mathbb{R}^d)} &\lesssim N^{1/2-1/p} \|m_j(D/2^j) f\|_{L^p(\mathbb{R}^d) l^p(\mathbb{R}^d)} \\
&= N^{1/2-1/p} \|m_j(D/2^j) \eta(D/2^j) f\|_{L^p(\mathbb{R}^d) l^p(\mathbb{N})} \\
&\leqslant A N^{1/2-1/p} \|\eta(D/2^j) f\|_{l^p(\mathbb{N}) L^p(\mathbb{R}^d)} \\
&\lesssim A N^{1/2-1/p} \|f\|_{L^p(\mathbb{R}^d)}.
\end{aligned}$$

To obtain the bound for $\tilde{S}_1$, we interpolate. We start by applying Cauchy-Schwartz, and then the result of Stein and Fefferman, to obtain a pointwise bound

$$|\tilde{S}_1\{f_j\}| \lesssim M\left(\left(\sum_j |\eta(D/2^j)f_j|^2\right)^{1/2}\right).$$

The operator M is L^2 bounded, so we obtain

$$\begin{aligned}\|\tilde{S}_1\{f_j\}\|_{L^2(\mathbb{R}^d)} &\lesssim \left\|\Big(\sum_j |\eta(D/2^j)f_j|^2\Big)^{1/2}\right\|_{L^2(\mathbb{R}^d)} \\ &= \|\eta_j(D/2^j)f_j\|_{L^2(\mathbb{R}^d)l^2(\mathbb{N})} \\ &\leqslant \|f_j\|_{L^2(\mathbb{R}^d)l^2(\mathbb{N})}.\end{aligned}$$

For $p = \infty$, we apply the slightly more precise pointwise bound

$$|\tilde{S}_1\{f_j\}| \lesssim \sup_l M\left(\left(\sum_{|j-l|\leqslant N} |\eta(D/2^j)f_j|^2\right)^{1/2}\right),$$

and since M is bounded in L^∞,

$$\begin{aligned}\|\tilde{S}_1\{f_j\}\|_{L^\infty} &\lesssim \sup_l \left\|\left(\sum_{|j-l|\leqslant N} |\eta(D/2^j)f_j|^2\right)^{1/2}\right\|_{L^\infty(\mathbb{R}^d)} \\ &\lesssim N^{1/2}\sup_j \|\eta(D/2^j)f_j\|_{L^\infty(\mathbb{R}^d)} \lesssim N^{1/2}\sup_j \|f_j\|_{L^\infty(\mathbb{R}^d)}.\end{aligned}$$

Interpolating between the $L^2(\mathbb{R}^d)l^2(\mathbb{N}) \to L^2(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)l^\infty(\mathbb{N}) \to L^\infty(\mathbb{R}^d)$ bound gives the required bound.

The function $S_2 f$ is obtained by averaging the functions $m_j(D/2^j)f$ over cubes either much smaller than the frequency scale, or much larger. For small cubes, we should therefore be able to show the function is close to it's average, and for large cubes, we should be able to exploit some cancelation. We will again consider a operator of the form

$$\tilde{S}_2\{f_j\}(x) = \sum_{|k+l(x)|>N} ⨍_{Q_x} \big[m_j(D/2^j)f_j(y) - [m_j(D/2^j)f_j]_{Q_x}\big]\chi_j(x,y)\, dy.$$

We will prove that

$$\|\tilde{S}_2\{f_j\}\|_{L^p(\mathbb{R}^d)} \lesssim A\|f_j\|_{L^p(\mathbb{R}^d)l^2(\mathbb{N})},$$

and then, with $f_j = \eta(D/2^j)f$, we conclude from the inequality above, and Littlewood-Paley, that

$$\|S_2 f\|_{L^p(\mathbb{R}^d)} \lesssim A\|\eta(D/2^j)f\|_{L^p(\mathbb{R}^d)l^2(\mathbb{N})} \lesssim A\|f\|_{L^p(\mathbb{R}^d)}.$$

This would then complete the proof of the argument.

Again, we obtain this result by interpolation between an L^2 bound, and an L^∞ bound. To get L^2 boundedness, we use Cauchy-Schwartz to get rid of the functions $\{\chi_j\}$, apply the L^2 boundedness of the maximal function, and Plancherel to exploit the disjointness of the supports of the symbols $\{m_j(\cdot/2^j)\}$, to conclude that

$$\begin{aligned}
\|\tilde{S}_2\{f_j\}\|_{L^2(\mathbb{R}^d)} &\lesssim \left\| M\left(\sum |m_j(D/2^j)f_j|^2\right)^{1/2}\right\|_{L^2(\mathbb{R}^d)} \\
&\lesssim \|m_j(D/2^j)f_j\|_{L^2(\mathbb{R}^d)l^2(\mathbb{N})} \\
&= \left\| m\cdot\left(\sum_j |\widehat{f_j}|^2\right)^{1/2}\right\|_{L^2(\mathbb{R}^d)} \\
&= \left\| m(D)\left\{\mathcal{F}^{-1}\left(\sum_j |\widehat{f_j}|^2\right)^{1/2}\right\}\right\|_{L^2(\mathbb{R}^d)} \\
&\leqslant A\left\|\mathcal{F}^{-1}\left(\sum_j |\widehat{f_j}|^2\right)^{1/2}\right\|_{L^2(\mathbb{R}^d)} \\
&= A\|f_j\|_{L^2(\mathbb{R}^d)l^2(\mathbb{N})}.
\end{aligned}$$

It now suffices to prove a bound of the form $\|\tilde{S}_2\{f_j\}\| \lesssim A\|f_j\|_{L^\infty(\mathbb{R}^d)l^2(\mathbb{N})}$, and we can then interpolate to get the required L^p bound.

To obtain this bound, let's consider an upper bound $|\tilde{S}_2| \lesssim I + II$, where

$$I(x) = ⨍_{Q_x}\left(\sum_j |m_j(D/2^j)\{Q_x^* f_j\}|^2|\right)^{1/2}$$

and

$$II(x) = ⨍_{Q_x}\left(\sum_{|j+l(x)|>N}\left| m_j(D/2^j)\{(Q_x^*)^c f_j\}(y) - [m_j(D/2^j)\{(Q_x^*)^c f_j\}]_{Q_x}\right|^2\right)^{1/2} dy.$$

Indeed, both bounds can be obtained by applying Cauchy-Schwartz to get rid of the functions $\{\chi_j\}$, and for I, we can discard the averaging terms $[m_j(D/2^j)\{Q_x^* f_j\}]_{Q_x}$ using the triangle inequality.

Applying Hölder's inequality and then Plancherel as before to exploit the disjointness of the $\{m_j(D/2^j)\}$, we get that

$$\begin{aligned}
|I(x)| &\leqslant |Q_x|^{-1/2}\|m_j(D/2^j)\{Q_x^* f_j\}\|_{l^2(\mathbb{N})L^2(\mathbb{R}^d)} \\
&\leqslant A|Q_x|^{-1/2}\|Q_x^* f_j\|_{l^2(\mathbb{N})L^2(\mathbb{R}^d)} \\
&\lesssim A\left(\fint_{Q_x}\sum |f_j|^2\right)^{1/2} \\
&\leqslant A\|f_j\|_{L^\infty(\mathbb{R}^d)l^2(\mathbb{N})}.
\end{aligned}$$

On the other hand, to bound II, we now exploit cancellation. Wr write

$$\begin{aligned}
&\left|m_j(D/2^j)\{(Q_x^*)^c f_j\}(y) - [m_j(D/2^j)\{(Q_x^*)^c f_j\}]_{Q_x}\right|^2 \\
&\quad\leqslant \sup_{y'\in Q_x}\left|m_j(D/2^j)\{(Q_x^*)^c f_j\}(y) - m_j(D/2^j)\{(Q_x^*)^c f_j\}(y')\right|.
\end{aligned}$$

If we write k_j for the convolution kernel of m_j, then we can write

$$\begin{aligned}
&\left|m_j(D/2^j)\{(Q_x^*)^c f_j\}(y) - m_j(D/2^j)\{(Q_x^*)^c f_j\}(y')\right| \\
&\quad= \left|2^{jd}\int_{(Q_x^*)^c}[k_j(2^j(y-z)) - k_j(2^j(y'-z))]f_j(z)\,dz\right| \\
&\quad\leqslant 2^{jd}\|f_j\|_{L^\infty(\mathbb{R}^d)}\int_{(Q_x^*)^c}|k_j(2^j(y-z)) - k_j(2^j(y'-z))|.
\end{aligned}$$

Let us write

$$E_j(x,y,y') = 2^{jd}\int_{(Q_x^*)^c}|k_j(2^j(y-z)) - k_j(2^j(y'-z))|\,dz.$$

Then we have

$$|II(x)| \lesssim \sup_{y,y'\in Q_x}\left(\sum_{|k+l(x)|>N}|E_j(x,y,y')|^2\right)^{1/2}\|f_j\|_{L^\infty(\mathbb{R}^d)l^\infty(\mathbb{N})}.$$

We claim that $|E_j(x,y,y')| \lesssim B\min(2^{-\varepsilon(j+l(x))}, 2^{j+l(x)})$, to which we can geometrically sum up, using the fact that $|j+l(x)| > N$, to annihilate the B term, and conclude that

$$|II(x)| \lesssim A\|f_j\|_{L^\infty(\mathbb{R}^d)l^\infty(\mathbb{N})}.$$

The bounds on $\{E_j\}$ follow from our assumptions about the multipliers $\{m_j\}$. Indeed, rescaling, we have

$$\begin{aligned}
E_j(x,y,y') &= \int_{2^j(Q_x^*)^c} |k_j(2^j y - z) - k_j(2^j y' - z)|\, dz \\
&\leqslant 2 \int_{|z|>2^{j+l(x)}} |k_j(z)| \\
&\lesssim B 2^{-\varepsilon(j+l(x))}.
\end{aligned}$$

On the other hand, if $j \leqslant -l(x)$, we can utilize the smoothness of m_j, which are uniformly supported on unit frequencies. Using Taylor's formula, we can write

$$\begin{aligned}
|k_j(2^j y - z) - k_j(2^j y' - z)| &= \left| \int_0^1 (\nabla k_j)(2^j(y + t(y-y')) - z) \cdot [2^j(y-y')]\, dt \right| \\
&\lesssim 2^{j+l(x)} \int_0^1 |(\nabla k_j)(2^j(y + t(y-y')) - z)|
\end{aligned}$$

and then for each t, we can write

$$\begin{aligned}
&\int_{2^j(Q_x^*)^c} |k_j(2^j(y + t(y-y')) - z)|\, dz \\
&\quad \leqslant \|\nabla k_j\|_{L^1(\mathbb{R}^d)} = \|\nabla\{\eta * k_j\}\|_{L^1(\mathbb{R}^d)} = \|(\nabla\eta) * k_j\|_{L^1(\mathbb{R}^d)} \leqslant \|k_j\|_{L^1(\mathbb{R}^d)} = B,
\end{aligned}$$

which completes the argument for the required bound, and thus the proof of the entire theorem.

Chapter 17

Seeger: Square Function Techniques for Multipliers on Compact Manifolds

In [**?**], some square function estimates are used to extend bounds on unit scale multipliers, to more general multipliers.

Part IV

Attempts To Solve Problems

Chapter 18

Radial Multipliers On The Sphere

Let us now try and prove certain special cases of the radial multiplier conjectures on the sphere S^d, i.e. multipliers of the operator $P = \sqrt{-\Delta}$. We fix a unit scale function h, and study operators of the form

$$h\,(P/R) = \sum_{\lambda} h(\lambda/R)\mathcal{P}_{\lambda},$$

where $\mathcal{P}_{\lambda}$, for $\lambda > 0$, is the orthogonal projection operator onto the eigenspace of P corresponding to the eigenvalue λ. In particular, we wish to characterize the boundedness properties of the operators $h(P/R)$, in terms of appropriate control of the Fourier transform of the function h. For a given exponent $1 \leqslant p \leqslant \infty$, let $\alpha_p = (d-1)|1/p - 1/2|$. For $1 < p < 2d/(d+1)$, let us assume that

$$C_p(h) = \left(\int_0^{\infty} \left[\langle t \rangle^{\alpha_p} |\widehat{h}(t)| \right]^p \, dt \right)^{1/p}$$

is finite. By Mitjagin's transference principle [17], and because the finiteness of $C_p(h)$ is necessary for $h(|\cdot|)$ to be a bounded Fourier multiplier on $L^p(\mathbb{R}^d)$, we have $C_p(h) \lesssim \|h\|_{M^p(\mathbb{R}^d)} \lesssim \|h\|_{M^p_{\mathrm{Dil}}(S^d)}$. The boundedness of $C_p(h)$ has already been proved sufficient provided the inputs are restricted to *zonal* functions [1]. The main result of this paper is to extend these bounds to general inputs.

Theorem 18.1. *Suppose* $1 \leqslant p < 2d/(d+1)$. *If h is a unit scale multiplier, then*

$$\|h\|_{M^p_{Dil}(S^d)} \lesssim C_p(h),$$

i.e. for $f \in L^p(S^d)$,

$$\|h(P/R)f\|_{L^p(S^d)} \lesssim C_p(h)\|f\|_{L^p(S^d)},$$

where the implicit constant is uniform in the input f, the symbol h, and the scaling parameter R.

Remarks.

1. If $1 \leqslant p < 2d/(d+1)$ and $C_p(h) < \infty$, then [9] implies that the operator $h(|\cdot|)$ is bounded as a Fourier multiplier operator on $L^p(\mathbb{R}^d)$ with

$$\|h(|\cdot|)\|_{M^p(\mathbb{R}^d)} \lesssim C_p(h).$$

Interpolation and duality (see Section 2.5.5 of [6] for more details) implies that the operator is also a Fourier multiplier operator on $L^2(\mathbb{R}^d)$, and so we conclude that

$$\|h\|_{L^\infty(\mathbb{R})} = \|h(|\cdot|)\|_{M^2(\mathbb{R}^d)} \lesssim C_p(h).$$

2. The projection operators $\{\mathcal{P}_\lambda\}$ are each individually smoothing, though not uniformly as $\lambda \to \infty$. They thus individually satisfy bounds of the form $\|\mathcal{P}_\lambda f\|_{L^p(S^d)} \lesssim_\lambda \|f\|_{L^p(M)}$ for all $1 \leqslant p \leqslant \infty$. It thus follows trivially from the triangle inequality, and that there are finitely many eigenvalues for P in $[0,100]$, that for any $R \leqslant 100$,

$$\begin{aligned}\|M_R f\|_{L^p(S^d)} &\leqslant \sum_\lambda |h(\lambda/R)| \|\mathcal{P}_\lambda f\|_{L^p(S^d)} \\ &\leqslant \|h\|_{L^\infty(0,200)} \textstyle\sum_{\lambda \in [0,200]} \|\mathcal{P}_\lambda f\|_{L^p(S^d)} \\ &\lesssim \|h\|_{L^\infty(0,\infty)} \|f\|_{L^p(S^d)}.\end{aligned}$$

Thus, in the analysis that follows, we will always assume that $R \geqslant 100$.

3. Because h is a unit scale multiplier, if we fix a smooth bump function β supported on $\{1/4 \leqslant |t| \leqslant 4\}$, and equal to one on $\{1/2 \leqslant |t| \leqslant 2\}$, set $\beta_R(\lambda) = \beta(\lambda/R)$, and set $Q_R = \beta(P/R)$, then

$$h(P/R) = Q_R \circ h(P/R) \circ Q_R.$$

By including the operators $\{Q_R\}$ in our analysis, we essentially reduce our analysis to the study to inputs and outputs lying in the range of the operators Q_R, which is equal to the finite dimensional subspace V_R of $C^\infty(S^d)$ spanned by eigenfunctions of P with eigenvalue in $R/4 \leqslant \lambda \leqslant 4R$. Since P is positive-semidefinite and self-adjoint, it is often useful to use the heuristic that an element of V_R should behave like a function on $\mathbb{R}^d$ with Fourier support on $\{R/4 \leqslant |\xi| \leqslant 4R\}$.

A particular application of this heuristic is an analogue of Bernstein's inequality on $\mathbb{R}^d$ (see [21], Proposition 5.1), but for functions on a Riemannian manifold lying in V_R. This analogue states that for $1 < r < \infty$, uniformly for $R \geqslant 100$ and $f \in V_R$ we have

$$\|f\|_{L^r_s(S^d)} \lesssim_{r,s} R^s \|f\|_{L^r(S^d)}. \tag{18.1}$$

See Section 3.3 of [19] for a proof.

Another useful inequality follows from the fact that the family of functions $\{\beta_R\}$ form a uniformly bounded subset of the Fréchet space $\mathcal{S}^0$, i.e. satisfying estimates of the form

$$|\partial_\lambda^n \{\beta_R\}(\lambda)| \lesssim_n \langle \lambda \rangle^{-n} \quad \text{uniformly in } R > 0.$$

It follows that the operators $\{Q_R\}$ are pseudodifferential operators of order zero, uniformly bounded as operators on $L^r_s(M)$ for all $1 < r < \infty$, i.e. satisfying

$$\|Q_R f\|_{L^r_s(S^d)} \lesssim_{r,s} \|f\|_{L^r_s(S^d)}. \tag{18.2}$$

See Corollary 4.3.2 of [19] for more details.

To exploit the fact that $C_p(h)$ is finite in the proof, we must employ the Fourier transform of h in some way. A standard method is to apply the Fourier inversion formula to write

$$h(P/R) = \int_{-\infty}^{\infty} R\widehat{h}(Rt) e^{2\pi i t P}\, dt,$$

where

$$e^{2\pi i t P} = \sum_\lambda e^{2\pi i t \lambda} \mathcal{P}_\lambda$$

is the multiplier operator on S^d which, as t varies, gives solutions to the half-wave equation

$$\partial_t - iP = 0.$$

Our goal is thus to study the regularity properties of averages of the half-wave operator.

Consider a cover

$$\{|t| < 100/R\} \cup \{50/R < |t| < 1/100\} \cup \{1/200 < |t| < \infty\}$$

of $\mathbb{R}$, and find a smooth partition of unity $\chi_{I,R}$, $\chi_{II,R}$, and $\chi_{III,R}$ adapted to these sets. Without loss of generality, we may assume all three functions are even, that $\chi_{I,R}(t) = \chi_I(Rt)$, for some smooth, compactly supported function χ_I adapted to the open set $\{|t| < 1\}$, and also assume that $\chi_{III,R} = \chi_{III}$ is independent of R. Given this partition of unity, we now write

$$h(P/R) = I_R + II_R + III_R,$$

where, for $\Pi \in \{I, II, III\}$, the operators

$$\Pi_R = \int \chi_{\Pi,R}(t) R\widehat{h}(Rt) e^{2\pi i t P}\, dt$$

isolate the study of $h(P/R)$ to the behaviour of the half-wave propogators on three different time intervals. The remainder of the argument will consist of separately obtaining L^p boundedness for each of the three operators $Q_R \circ \Pi_R \circ Q_R$, since then the triangle inequality gives the L^p boundedness of

$$(Q_R \circ I_R \circ Q_R) + (Q_R \circ II_R \circ Q_R) + (Q_R \circ II_R \circ Q_R) = h(P/R).$$

The study of the operators $\{I_R\}$ will reduce to a study of pseudodifferential operators, we will be able to apply the endpoint local smoothing inequality of [15] to control the operators $\{III_R\}$, and the study of the operators $\{II_R\}$ will be given by generalizations of the methods of [9] to a variable coefficient setting.

18.1 Analysis of I_R

Let us analyze

$$I_R = \int \chi_I(Rt) R\widehat{h}(Rt) e^{2\pi i t P}\, dt.$$

We are analyzing inputs to I_R coming from the composition of a general element of $C^\infty(S^d)$ with Q_R, which heuristically localizes the 'frequency support' of this function to a band of frequencies with magnitude $\sim R$. Thus, by uncertainty principle heuristics, such functions are locally constant at a scale $1/R$. The half-wave equation propogates a majority of the mass of it's input at a unit speed, and since the operators $\{I_R\}$ are obtained by averaging the half-wave equation over times $\lesssim 1/R$, we should expect that the behaviour of the operators $\{I_R\}$ to behave in a localized manner. In fact, the following analysis will show that the operators $\{I_R\}$ are pseudodifferential operators, which will allow us to conclude these operators are uniformly bounded in $L^p(S^d)$.

Lemma 18.2. *For all $f \in C^\infty(S^d)$,*

$$\|I_R f\|_{L^p(S^d)} \lesssim \|h\|_{L^\infty(\mathbb{R})} \|f\|_{L^p(S^d)} \lesssim C_p(h) \|f\|_{L^p(S^d)},$$

where the implicit constant is uniformly bounded in $R \geqslant 1$ and h, for $1 < p < \infty$. Thus in particular,

$$\|(Q_R \circ I_R \circ Q_R) f\|_{L^p(S^d)} \lesssim C_p(h) \|f\|_{L^p(S^d)}.$$

Proof. Let a be the inverse Fourier transform of the function $t \mapsto \chi_I(t)\widehat{h}(t)$. Then $I_R = a(P/R)$. If ψ denotes the inverse Fourier transform of χ_I, then we can write

$$a(\lambda) = \int h(\alpha)\psi(\lambda - \alpha)\, d\alpha.$$

The fact that h is a unit scale multiplier, and ψ is Schwartz, implies that

$$|\partial^\alpha a(\lambda)| \lesssim_{\alpha,N} \|h\|_{L^\infty(\mathbb{R})} \langle \lambda \rangle^{-N}.$$

If we set $a_R(\lambda) = a(\lambda/R)$, then

$$|\partial^\alpha a_R(\lambda)| \lesssim_\alpha \|h\|_{L^\infty(\mathbb{R})} \langle \lambda \rangle^{-|\alpha|},$$

with an implicit constant independent of R for $R \geqslant 1$. Thus the family of symbols $\{a_R : R \geqslant 1\}$ form a uniformly bounded subset of the Frećhet space $\mathcal{S}^0(\mathbb{R})$ of order zero symbols, and so the operators I_R are pseudodifferential operators of order zero, and uniformly bounded in the $L^p(S^d)$ norm for all $1 < p < \infty$, which yields the required claim. □

18.2 Analysis of III_R

We now show the uniform boundedness of the operators $\{III_R\}$ on $L^p(S^d)$ in the range of p we are considering in this problem, by a reduction to an endpoint local smoothing inequality. This might seem unintuitive, since the operators III_R are obtained by averaging the wave equation over large times $|t| \gtrsim 1$, whereas local smoothing gives bounds for averages of the wave equation over times $|t| \lesssim 1$. We are able to reduce large times to small times by exploiting the *periodicity* of the half-wave equation on the sphere.

Lemma 18.3. *Fix $1 < p < 2d/(d+1)$, let q be the Hölder conjugate to p, and let $I = [-1/2, 1/2]$. Suppose that the sharp local smoothing inequality*

$$\left\| e^{2\pi i t P} f \right\|_{L^q(S^d) L^q_t(I)} \lesssim \|f\|_{L^q_{\alpha_q - 1/q}(S^d)}$$

holds for all $f \in C^\infty(S^d)$. Then the operators $\{III_R\}$ satisfy a bound

$$\|(III_R \circ Q_R) f\|_{L^p(S^d)} \lesssim C_p(h) \|f\|_{L^p(S^d)},$$

with the implicit constant uniformly bounded in R. In particular,

$$\|(Q_R \circ III_R \circ Q_R) f\|_{L^p(S^d)} \lesssim C_p(h) \|f\|_{L^p(S^d)},$$

Proof. For each R, the *class* of operators of the form $\{III_R\}$ formed from a multiplier h satisfying the hypothesis of Theorem 18.1 is closed under taking adjoints. Indeed, if III_R is obtained from h, then III_R^* is obtained from the multiplier $\overline{h}$. Because of this self-adjointness, if we can prove that for any multiplier h satisfying the assumptions of the theorem, the operators $\{III_R\}$ are uniformly bounded in $L^q(S^d)$, where q is the Hölder conjugate to p, then it follows by duality that for any such h, it is also true that the operators $\{III_R\}$ are uniformly bounded back in the original $L^p(S^d)$ norm. In this argument we will prove such L^q estimates, because we will exploit *local smoothing* inequalities, which tend to work better with large Lebesgue exponents, precisely because Lebesgue norms with large exponents are more sensitive to functions with sharp peaks, something explicitly prevented by obtaining control over the smoothness of a function.

We begin by noting that for a pair of Hölder conjugates p and q, $\alpha_q = \alpha_p$. Using the periodicity of the wave equation on S^d, i.e. that

$$e^{2\pi i (t+n) P} = e^{2\pi i t P} \qquad \text{for } n \in \mathbb{Z} \text{ and } t \in \mathbb{R},$$

we can write

$$III_R = \int_{-1/2}^{1/2} H_R(t) e^{2\pi i t P} \, dt,$$

where

$$H_R(t) = \sum_l \chi_{III}(t) R \widehat{h}(R(t+l)) = \sum_l H_{R,l}(t).$$

Now

$$
\begin{aligned}
&\left(\sum_{l\neq 0}\left(|Rl|^{\alpha_q}\|H_{R,l}\|_{L^p[-1/2,1/2]}\right)^p\right)^{1/p} \\
&\qquad \sim R\left(\int_{-1/2}^{1/2}\sum_l\left(|R(t+l)|^{\alpha_q}|\widehat{h}(R(t+l))|\right)^p\ dt\right)^{1/p} \\
&\qquad \sim R\left(\int_{|t|\geqslant 1/2}\left(|Rt|^{\alpha_q}|\widehat{h}(Rt)|\right)^p\ dt\right)^{1/p} \\
&\qquad \lesssim R^{1/q}C_p(h).
\end{aligned}
$$

and

$$
\begin{aligned}
\|H_{R,0}\|_{L^p[-1/2,1/2]} &= \left(\int_{-1/2}^{1/2}|\chi_{III}(t)R\widehat{h}(Rt)|^p\right)^{1/p} \\
&\leqslant \left(\int_{1/200\leqslant|t|\leqslant 1/2}|R\widehat{h}(Rt)|^p\right)^{1/p} \\
&= R^{1/q}\left(\int_{R/3}^{R/2}|\widehat{h}(t)|^p\right)^{1/p} \\
&\lesssim R^{1/q-\alpha_q}C_p(h).
\end{aligned}
$$

Since the family of functions $\{H_{R,l}\}$ could in general be chosen arbitrarily, they can be quite correlated, and so we should expect Hölder's inequality should be efficient, in the worst case. Thus we conclude that, provided $p < 2d/(d+1)$, so that $q > 2d/(d-1)$, and thus

$$
q\alpha_q = (d-1)(q/2-1) > 1,
$$

so we can use Hölder's inequality to conclude that

$$
\begin{aligned}
\|H_R\|_{L^p[-1/2,1/2]} &\leqslant \sum_l \|H_{R,l}\|_{L^p[-1/2,1/2]} \\
&= \|H_{R,0}\|_{L^p[-1/2,1/2]} + \sum_{l\neq 0} \left(|Rl|^{\alpha_q}\|H_{R,l}\|_{L^p[-1/2,1/2]}\right)|Rl|^{-\alpha_q} \\
&\lesssim R^{-\alpha_q+1/q}C_p(h) + (R^{1/q}C_p(h))\left(\sum_{l\neq 0}|Rl|^{-\alpha_q q}\right)^{1/q} \\
&= R^{-\alpha_q+1/q}C_p(h)\left(1+\left(\sum_{l\neq 0}|l|^{-\alpha_q q}\right)^{1/q}\right) \\
&= R^{-\alpha_q+1/q}C_p(h)\left(1+\left(\sum_{l\neq 0}|l|^{-\alpha_q q}\right)^{1/q}\right) \\
&\lesssim_p R^{-\alpha_q+1/q}C_p(h).
\end{aligned}
$$

A further application of Hölder's inequality shows that

$$
\begin{aligned}
|III_R| &= \left|\int_{-1/2}^{1/2} H_R(t)e^{2\pi itP}\,dt\right| \\
&\lesssim C_p(h)R^{-\alpha_q+1/q}\left(\int_{-1/2}^{1/2}|e^{2\pi iP}|^q\,dt\right)^{1/q}.
\end{aligned}
$$

Applying the endpoint local smoothing inequality, we conclude that

$$
\begin{aligned}
\|(III_R\circ Q_R)f\|_{L^q(M)} &\lesssim C_p(h)R^{-\alpha_q+1/q}\|e^{2\pi iP}(Q_Rf)\|_{L^q_tL^q_x} \\
&\lesssim C_p(h)R^{-\alpha_q+1/q}\|Q_Rf\|_{L^q_{\alpha_q-1/q}(M)},
\end{aligned}
$$

Applying Bernstein's inequality gives

$$
\|Q_Rf\|_{L^q_{\alpha_q-1/q}(M)} \lesssim R^{\alpha_q-1/q}\|f\|_{L^q(M)}.
$$

Thus we conclude that

$$
\|(III_R\circ Q_R)f\|_{L^q(M)} \lesssim C_p(h)\|f\|_{L^q(M)}.
$$

We have therefore bounded III_R uniformly in R. □

Corollary 1.2 of [15] establishes that the sharp local smoothing inequality holds for $p < 2(d-1)/(d+1)$, which covers the range of parameters studied in this paper. Thus we have obtained uniform bounds on the operators $\{III_R\}$.

18.3 Analysis of II_R: Density Decompositions

It finally remains to analyze the operator $Q_R \circ II_R \circ Q_R$, where

$$II_R = \int \chi_{II}(t) R\widehat{h}(Rt) e^{2\pi itP}\, dt$$

is obtained by integrating the wave propogators over times $100/R \leqslant |t| \leqslant 0.01$ respectively. To prevent notation from growing too cumbersome later on, let us eschew uses of the subscript R in our operators in this section, e.g. writing II_R as

$$II = \int b(t) e^{2\pi itP}\, dt,$$

where $b(t) = \chi_{II}(t) R\widehat{h}(Rt)$. We then have

$$\|b(t)\langle t\rangle^{s_p}\|_{L^p(\mathbb{R})} \lesssim R^{1-1/p-s_p} C_p(h).$$

Bounding II requires a more subtle analysis of the geometric behaviour of the wave-propogator operators, and we will begin by converting our problem in coordinates on S^d, where the kernels have more explicit representations in oscillatory integrals.

We will employ some restricted weak type bounds, together with interpolation, to obtain L^p estimates on the operators $Q \circ II \circ Q$. We thus introduce a set $E \subset S^d$ and try to obtain $L^{p,\infty}$ bounds on the function $S = (Q \circ II_W \circ Q)\{E\}$. Given that Q already acts, heuristically, by localizing the behaviour of it's inputs to the frequency R, despite the choice of the set E, the uncertainty principle implies $Q\{E\}$ should be locally constant at a scale $1/R$, and so it is natural to discretize at this scale. Consider a maximal $1/2R$ separated subset $\mathcal{X}$ of S^d. Then break E down into a disjoint union of sets $\{E_{x_0} : x_0 \in \mathcal{X}\}$, where for $x_0 \in \mathcal{X}$, the set E_{x_0} is supported on the geodesic ball of radius $1/R$ about x_0. Similarly, let $\mathcal{T}$ be all points in the lattice $\mathbb{Z}/10R$ lying in the set $\{100/R \leqslant |t| \leqslant 1\}$, and write

$$b = \sum_{t \in \mathcal{T}} u(t) b_t,$$

where for each $t \in \mathcal{T}$, $u(t) = \|b\|_{L^\infty[t-10/R, t+10/R]}$, and b_t is a smooth function, compactly supported on the sidelength $1/R$ interval centered at t, satisfying

$$|\partial^\alpha b_t| \lesssim_\alpha R^{|\alpha|},$$

with implicit constants uniform in b and t. By the Plancherel-Polya theorem,

$$\|u(t)\langle t\rangle^{s_p}\|_{l^p(\mathcal{T})} \lesssim R^{1-s_p}.$$

We can then write

$$S = \sum |E_{x_0}| S_{x_0,t_0} \quad \text{where} \quad S_{x_0,t_0} = \int |E_{x_0}|^{-1} b_{t_0}(t)(Q \circ e^{2\pi i t P} \circ Q)\{E_{x_0}\}\, dt.$$

Our computation would be complete if we could show that for any coefficients $\{c(x_0,t_0) : x_0 \in \mathcal{X}, t_0 \in \mathcal{T}\}$,

$$\left\| \sum_{x_0,t_0} c(x_0,t_0) t_0^{\frac{d-1}{2}} S_{x_0,t_0} \right\|_{L^p(S^d)} \lesssim R^{s_p - 1 + d(1-1/p)} \left(\sum_{x_0,t_0} |c(x_0,t_0)|^p t_0^{d-1} \right)^{1/p}.$$

Indeed, we set $c(x_0,t_0) = |E_{x_0}| u(t_0) t_0^{-\frac{d-1}{2}}$ and apply Hölder's inequality, then the inequality above gives exactly that

$$\|S\|_{L^p(S^d)} \lesssim C_p(h)|E|^{1/p},$$

For $p = 1$, this follows from applying the triangle inequality, and using the pointwise estimates

$$|S_{x_0,t_0}(x)| \lesssim_M \frac{R^{d-1}}{(R d_g(x,x_0))^{\frac{d-1}{2}}} \Big\langle R \big| t_0 - d_g(x,x_0) \big| \Big\rangle^{-M}.$$

Applying interpolation, for $p > 1$ we need only prove a restricted weak type version of this inequality. In other words, we can restrict c to be the indicator function of a set $\mathcal{E}$, and take $L^{p,\infty}$ norms on the left hand side. If we write $\mathcal{E} = \bigcup_k \mathcal{E}_k$, where $\mathcal{E}_k$ is the set of $(x,t) \in \mathcal{E}$ with $|t| \sim 2^k/R$, then the inequality reads that

$$\left\| \sum_{k=1}^{\infty} 2^{k\frac{d-1}{2}} \sum_{(x_0,t_0)\in\mathcal{E}_k} S_{x_0,t_0} \right\|_{L^{p,\infty}(S^d)}^p \lesssim R^{(d-1)p-d} \left(\sum_{k=1}^{\infty} 2^{k(d-1)} \#\mathcal{E}_k \right).$$

This is equivalent to showing that for any $\lambda > 0$,

$$\left| \left\{ x : \left| \sum_k 2^{k\frac{d-1}{2}} S_{x_0,t_0}(x) \right| \geqslant \lambda \right\} \right| \lesssim \lambda^{-p} R^{(d-1)p-d} \sum_k 2^{k(d-1)} \#\mathcal{E}_k.$$

The case $\lambda \lesssim R^{d-1}$ follows from the L^1 boundedness we've already proved, so we may assume $\lambda \gtrsim R^{d-1}$ in the sequel.

To obtain this bound, we employ the method of density decompositions, introduced in [9]. Let

$$A = \left(\frac{\lambda}{R^{d-1}}\right)^{(d-1)(1-p/2)} \log\left(\frac{\lambda}{R^{d-1}}\right)^{O(1)}.$$

Then for each k, consider the collection $\mathcal{B}_k(\lambda)$ of all balls B with radius at most $2^k/R$ such that $\#\mathcal{E}_k \cap B \geqslant RA\text{rad}(B)$. Applying the Vitali covering lemma, we can find a disjoint family of balls $\{B_1, \dots, B_N\}$ in $\mathcal{B}_k$ such that the balls $\{B_1^*, \dots, B_N^*\}$ obtained by dilating the balls by 5 cover $\bigcup \mathcal{B}_k(\lambda)$. Then

$$\sum \text{rad}(B_j) \leqslant R^{-1}A^{-1}\#\mathcal{E}_k,$$

and the set $\widehat{\mathcal{E}}_k = \mathcal{E}_k - \bigcup \mathcal{B}_k(\lambda)$ has density type $(RA, 2^k/R)$. Then we conclude that, using the quasi-orthogonality estimates below,

$$\left\| \sum_k \sum_{(x_0,t_0)\in\widehat{\mathcal{E}}_k} 2^{k\frac{d-1}{2}} S_{x_0,t_0} \right\|_{L^2(S^d)}^2 \lesssim_p R^{d-2}\log(A)A^{\frac{2}{d-1}} \sum_k 2^{k(d-1)}\#\mathcal{E}_k.$$

Appling Chebyshev's inequality, and utilizing the choice of A above, we conclude that

$$\begin{aligned}\left| \left\{ x : \left| \sum_k \sum_{(x_0,t_0)\in\widehat{\mathcal{E}}_k} 2^{k\frac{d-1}{2}} S_{x_0,t_0}(x) \right| \geqslant \lambda/2 \right\} \right| &\lesssim R^{d-2}\log(A)A^{\frac{2}{d-1}} \sum_k 2^{k(d-1)}\#\mathcal{E}_k \\ &\lesssim \lambda^{-p}R^{(d-1)p-d} \sum_k 2^{k(d-1)}\#\mathcal{E}_k.\end{aligned}$$

Conversely, we exploit the clustering of the sets $\mathcal{E}_k - \widehat{\mathcal{E}}_k$ to bound

$$\left| \left\{ x : \left| \sum_k \sum_{(x_0,t_0)\in\mathcal{E}_k-\widehat{\mathcal{E}}_k} 2^{k\frac{d-1}{2}} S_{x_0,t_0}(x) \right| \geqslant \lambda/2 \right\} \right|$$

That is, we have found balls $B_1^* < \dots, B_N^*$, each with radius $O(2^k/R)$, such that

$$\sum \text{rad}(B_j) \leqslant R^{-1}A^{-1}\#\mathcal{E}_k.$$

Let (x_j, t_j) denote the center of the ball B_j. Then the function

$$\sum_{(x_0,t_0)\in B_j} S_{x_0,t_0}$$

has mass concentrated on the geodesic annulus $\text{Ann}_j \subset S^d$ with radius t_j and thickness $O(\text{rad}(B_j))$, a set with measure $(2^k/R)^{d-1}\text{rad}(B_j)$. For $(x_0, t_0) \in B_j$, we calculate using the pointwise bounds that

$$\begin{aligned}
\int_{\text{Ann}_j^c} |S_{x_0,t_0}(x)|\, dx &\lesssim R^{d-1} \int_{\text{rad}(B_j)\lesssim|t_j - d_g(x,x_0)|\leqslant 1} \langle R|t_0 - d_g(x, x_0)|\rangle^{-M} \\
&\lesssim R^{d-1} \int_{\text{rad}(B_j)\leqslant|t_j-s|\leqslant 1} s^{d-1}\langle R|t_0 - s|\rangle^{-M}\, ds \\
&\lesssim 2^{k(d-1)} R^{d-1} (R\text{rad}(B_j))^{-M}.
\end{aligned}$$

Because the set of points in $\mathcal{E}_k$ is $1/R$ separated, there can only be at most $O(R\text{rad}(B_j))^{d+1}$ values of (x_0, t_0), and so applying the triangle inequality gives that the sum of the L^1 norm outside of Ann_j is

$$\lesssim 2^{k(d-1)} R^{d-1} (R\ \text{rad}(B_j))^{d+1-M}$$

Note that since $\#\mathcal{E}_k \cap B_j \geqslant RA\text{rad}(B_j)$, and because $\mathcal{E}_k$ is $1/R$ discretized,

$$\text{rad}(B_j) \geqslant (A/R)^{\frac{1}{d-1}},$$

and this, together with Markov's inequality, is enough to justify the required bound. Conversely, since $1 < p < 2(d-1)/(d+1)$, we have

$$\begin{aligned}
\sum |\text{Ann}_j| \lesssim (2^k/R)^{d-1} \sum_j \text{rad}(B_j)& \\
&\lesssim (2^k/R)^{d-1} R^{-1} (L/R^{d-1})^{-(d-1)(1-p/2)} log(L/R^{d-1})^{O(1)} \\
&\lesssim \lambda^{-p} R^{(d-1)p-d} 2^{k(d-1)} \#\mathcal{E}_k,
\end{aligned}$$

Summing over k completes the analysis.

18.4 Analysis of II_R: Quasi-Orthogonality

Our first goal will be to understand how orthogonal the functions $\{S_{x_0,t_0}\}$ are to one another, which will give L^2 estimates for S, that can be interpolated with

L^1 estimates to obtain the required L^p estimates. The rest of this section will be devoted to proving the following inner product estimate, which, together with a density decomposition argument, introduced in [9], can be used to obtain L^2 estimates, which we can then interpolate to obtain L^p estimates for the function S.

Lemma 18.4.

$$|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim_M \frac{R^{d-2}}{(Rd_g(x_0,x_1))^{\frac{d-1}{2}}} \sum_{\pm} \left\langle R\left|(t_0-t_1) \pm d_g(x_1,x_0)\right|\right\rangle^{-M}.$$

Let us proceed with the proof. To begin with, we can use the self-adjointness of the operators Q, the semigroup structure of $\{e^{2\pi itP}\}$, and the fact that multipliers commute, to write

$$\begin{aligned}\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle &= \int \frac{b_{t_0}(t)\overline{b_{t_1}(s)}}{|E_{x_0}||E_{x_1}|} \left\langle (Q\circ e^{2\pi itP}\circ Q)\{E_{x_0}\}, (Q\circ e^{2\pi isP}\circ Q)\{E_{x_1}\}\right\rangle \, dt\, ds \\ &= \int \frac{b_{t_0}(t)\overline{b_{t_1}(s)}}{|E_{x_0}||E_{x_1}|} \left\langle (Q^2\circ e^{2\pi i(t-s)P}\circ Q^2)\{E_{x_0}\}, E_{x_1}\right\rangle \\ &= \int \frac{c_{t_0,t_1}(t)}{|E_{x_0}||E_{x_1}|} \left\langle (Q^2\circ e^{2\pi itP}\circ Q^2)\{E_{x_0}\}, E_{x_1}\right\rangle,\end{aligned}$$

where

$$c_{t_0,t_1}(t) = \int b_{t_0}(u)\overline{b_{t_1}}(u-t)\, dt,$$

is the convolution of b_{t_0} with the reflection of $\overline{b_{t_1}}$ about the y-axis. Thus c_{t_0,t_1} is supported on the length $2/R$ interval centered at $t_0 - t_1$, and has L^1 norm $O(1/R^2)$ by Young's convolution inequality.

We next perform a decomposition of the inner product into various coordinate systems. Cover S^d by a finite family of sets $\{V_\alpha\}$, chosen such that for each V_α, there is a coordinate chart U_α such that the neighbourhood $N(V_\alpha, 0.5)$ is contained in U_α. Let $\{\eta_\alpha\}$ be a partition of unity subordinate to $\{V_\alpha\}$. It will also be convenient to define $V_\alpha^* = N(V_\alpha, 0.1)$. We can then write

$$\langle S_{t_0,x_0}, S_{t_1,x_1}\rangle = \sum_\alpha \int \frac{c_{t_0,t_1}(t)}{|E_{x_0}||E_{x_1}|} \left\langle (Q^2\circ e^{2\pi itP}\circ Q^2)\{\eta_\alpha E_{x_0}\}, E_{x_0}\right\rangle dt.$$

We will bound each of the terms on the right separately from one another, by working with each inner product in the coordinate systems $\{U_\alpha\}$.

The next Lemma allows us to approximate the operator Q, and the propogators $e^{2\pi itP}$, with operators which have more explicit representations in the coordinate system U_α, by an error term which is negligible to our analysis. It utilizes the *Lax-Hörmander parametric* for the half-wave equation over small times, which expresses $e^{2\pi itP}$ in coordinates as a Fourier integral operator.

Lemma 18.5. *For each α, and $|t| \leqslant 1/100$, there exists Schwartz operators Q_α and $W_\alpha(t)$, each with kernel supported on $U_\alpha \times V_\alpha^*$, such that the following properties hold:*

- *For $f \in C^\infty(S^d)$ with $supp(f) \subset V_\alpha^*$,*
$$supp(Q_\alpha f) \subset N(supp(f), 0.1) \quad and \quad supp(W_\alpha(t)f) \subset N(supp(f), 0.1).$$
Moreover,
$$\|(Q^2 - Q_\alpha)f\|_{L^2(S^d)} \lesssim_N R^{-N}\|f\|_{L^2(S^d)}$$
and
$$\left\| \left(Q_\alpha \circ \left(e^{2\pi itP} - W_\alpha(t) \right) \circ Q_\alpha \right)\{f\} \right\|_{L^2(S^d)} \lesssim_N R^{-N}\|f\|_{L^2(S^d)}.$$

- *In the coordinate system of U_α, Q_α is a pseudodifferential operator of order zero given by a symbol $\sigma(x,\xi)$, where*
$$supp(\sigma) \subset \{|\xi| \sim R\},$$
and σ satisfies derivative estimates of the form
$$|\partial_x^\beta \partial_\xi^\kappa \sigma(x,\xi)| \lesssim_{\beta,\kappa} R^{-|\kappa|}.$$

- *In the coordinate system U_α, the operator $W_\alpha(t)$ has a kernel $W_\alpha(t,x,y)$ with an oscillatory integral representation*
$$W_\alpha(t,x,y) = \int s(t,x,y,\xi) e^{2\pi i[\phi(x,y,\xi)+t|\xi|_y]}\, d\xi,$$
where s has compact support in it's x and y coordinates, with
$$supp(s) \subset \{|\xi| \sim R\},$$
where s satisfies derivative estimates of the form
$$|\partial_{t,x,y}^\beta \partial_\xi^\kappa s| \lesssim_{\beta,\kappa} R^{-|\kappa|},$$
and where $|\cdot|_y$ denotes the norm on $\mathbb{R}^n_\xi$ induced by the Riemannian metric on S^d on the contangent space $T_y^* S^d$.

Thus, ignoring errors negligible to our analysis, we need only analyze

$$\left| \int \frac{c_{t_0,t_1}(t)}{|E_{x_0}||E_{x_1}|} \left\langle (Q_\alpha \circ W_\alpha(t) \circ Q_\alpha)\{\phi_\alpha E_{x_0}\}, E_{x_1} \right\rangle du \right|.$$

The behaviour of all operations in this expression are now completely localized to U_α for inputs supported on V_α^*; in particular, this expression is equal to zero unless E_{x_0} and E_{x_1} are both compactly contained in U_α, So we can now naturally work with the kernels of the operators in coordinates to upper bound the inner product, which will complete the required estimate of the inner product.

Lemma 18.6. *Let c be an integrable function supported on the length $10/R$ interval centered at a value t^* with $|t| \leqslant 1/100$. Then*

$$\begin{aligned} \left| \int \frac{c(t)}{|E_{x_0}||E_{x_1}|} \left\langle (Q_\alpha \circ W_\alpha(t) \circ Q_\alpha)\{\phi_\alpha E_{x_0}\}, E_{x_1} \right\rangle dt \right| \\ \lesssim_M R^d \frac{\|c\|_{L^1(\mathbb{R})}}{(Rd_g(x_1, x_0))^{\frac{d-1}{2}}} \sum_{\pm} \left\langle R \left| t^* \pm d_g(x_1, x_0) \right| \right\rangle^{-M} \end{aligned}$$

Proof. We write the integral as

$$\begin{aligned} \int \frac{c(t)}{|E_{x_0}||E_{x_1}|} (\eta_\alpha E_{x_1})(w) \sigma(w, \theta) e^{2\pi i \theta \cdot (w - x)} \\ s(t, x, y, \xi) e^{2\pi i [\phi(x,y,\xi) + t|\xi|_y]} \sigma(y, \eta) e^{2\pi i \eta \cdot (y - z)} E_{x_0}(z) \\ dt\, dx\, dy\, dz\, dw\, d\theta\, d\xi\, d\eta. \end{aligned}$$

The integral looks complicated, but can be simplified considerably by noticing that all the spatial variables are highly localized. To begin with, we use the fact that s is smooth and compactly supported in all it's variables, so s should roughly behave like a linear combination of tensor products; using Fourier series, we can write

$$s(t, x, y, \xi) = \sum_{n \in \mathbb{Z}^d} s_{n,1}(x) s_{n,2}(t, y, \xi),$$

where $s_{n,1}(x) = e^{2\pi i n \cdot x}$, and where

$$|\partial_{t,y}^\alpha \partial_\xi^\kappa \{s_{n,2}\}| \lesssim_{\alpha,\kappa,N} |n|^{-N} R^{-|\kappa|}$$

If we define $a_n(\xi) = a_{n,1}(R\xi) a_{n,2}(R\xi)$, where

$$a_{n,1}(\xi) = |E_{x_1}|^{-1} \int (\eta_\alpha E_{x_1})(w) \sigma(w, \theta) s_{n,1}(x) e^{2\pi i [\theta \cdot (w - x) + (x - x_1) \cdot \xi]}\, d\theta\, dw\, dx$$

and

$$a_{n,2}(\xi) = |E_{x_0}|^{-1} \int c(t) s_{n,2}(t,y,\xi) \sigma(y,\zeta) E_{x_0}(z) e^{2\pi i [(\phi(t^*,x_0,\xi) - \phi(t,y,\xi)) + \zeta \cdot (y-z)]}\, d\zeta\, dt\, dy\, dz,$$

then, rescaling, we can write the required integral as

$$R^d \sum_{n \in \mathbb{Z}^d} \int a_n(\xi) e^{2\pi i R[\phi(x_1,x_0,\xi) + t^* |\xi|_{x_0}]}\, d\xi.$$

Notice that $\operatorname{supp}(a_n) \subset \{|\xi| \sim 1\}$, and

$$|(\nabla_\xi^\kappa a_n)(\xi)| \lesssim_{\kappa,N} |n|^{-N} \|c\|_{L^1(\mathbb{R})}.$$

To obtain an efficient upper bound on this oscillatory integral, it will be convenient to change coordinate systems in a way better respecting the Riemannian metric at x_0, i.e. finding a smooth family of diffeomorphisms $\{F_{x_0} : S^{d-1} \to S^{d-1}\}$ such that $|F_{x_0}|_{x_0} = 1$. We can choose this function such that $F_{x_0}(-x) = -F_{x_0}(x)$. Then if $\tilde{a}_n(\rho,\eta) = a_n(\rho F_{x_0}(\eta)) J F_{x_0}(\eta)$, then a change of variables gives that

$$\int a_n(\xi) e^{2\pi i R[\phi(x_1,x_0,\xi) + t^* |\xi|_{x_0}]} = \int_0^\infty \rho^{d-1} \int_{|\eta|=1} \tilde{a}_n(\rho,\eta) e^{2\pi i R\rho[\phi(x_1,x_0,F_{x_0}(\eta)) + t^*]}\, d\eta\, d\rho.$$

For each fixed ρ, we claim that the phase has exactly two stationary points in the η variable, at the values $\pm\eta_0$, where x_1 lies on the geodesic passing through x_0 tangent to the vector $\eta_0^\sharp$ (here we are using the musical isomorphism to map the cotangent vector η_0 to a tangent vector). Moreover, at these values,

$$\phi(x_1, x_0, F_{x_0}(\pm\eta_0)) = \pm d_g(x_1, x_0),$$

and the Hessian at $\pm\eta_0$ is (positive / negative) definite, with each eigenvalue having magnitude exceeding a constant multiple of $d_g(x_1,x_0)$. It follows from the principle of stationary phase, that the integral above can be written as

$$\frac{1}{(R d_g(x_1,x_0))^{\frac{d-1}{2}}} \sum_{\pm} \int_0^\infty \rho^{\frac{d-1}{2}} f_{n,\pm}(\rho) e^{2\pi i R\rho[t^* \pm d_g(x_1,x_0)]}\, d\rho,$$

where $f_{n,\pm}$ is supported on $|\rho| \sim 1$, and

$$|\partial_\rho^m f_{n,\pm}| \lesssim_{m,N} |n|^{-N} \|c\|_{L^1(\mathbb{R})}.$$

Integrating by parts in the ρ variable if $\pm d_g(x_1, x_0) + t^*$ is large, and then taking in absolute values, we conclude that

$$\left| \int a_n(\xi) e^{2\pi i R[\phi(x_1,x_0,\xi)+t^*|\xi|_{x_0}]} \right| \lesssim_{N,M} |n|^{-N} \frac{\|c\|_{L^1(\mathbb{R})}}{(Rd_g(x_1,x_0))^{\frac{d-1}{2}}} \sum_{\pm} \langle R|t^* \pm d_g(x_1,x_0)|\rangle^{-M}.$$

Taking $N \geqslant d + 1$, and summing in the n variable, we conclude that

$$\left| \sum_n \int a_n(\xi) e^{2\pi i R[\phi(x_1,x_0,\xi)+t'|\xi|_{x_0}]} \right| \lesssim_M \frac{\|c\|_{L^1(\mathbb{R})}}{(Rd_g(x_1,x_0))^{\frac{d-1}{2}}} \sum_{\pm} \langle R|t^* \pm d_g(x_1,x_0)|\rangle^{-M}.$$

But this is precisely an estimate for the quantity we wished to estimate. □

Now applying this Lemma with $c = c_{t_0,t_1}$, and then summing in α, we complete the proof of Lemma 18.4.

18.5 Analysis of II_R: L^2 Estimates

Lemma 18.4 of the last section implies two functions S_{x_0,t_0} and S_{x_1,t_1} can only be correlated in L^2 if $d_g(x_0, x_1) \approx |t_0 - t_1|$. We now exploit this geometry to obtain some L^2 estimates for sums of the functions S_{x_0,t_0}.

Lemma 18.7. *Fix $u \geqslant 1$. Consider a set $\mathcal{E} \subset \mathcal{X} \times \mathcal{T}$. Write*

$$\mathcal{E} = \bigcup_{k=0}^{\infty} \mathcal{E}_k,$$

where $\mathcal{E}_k = \{(x,t) \in \mathcal{E} : |t| \sim 2^k/R\}$. Suppose that each of the sets $\mathcal{E}_k$ has density type $(Ru, 2^k/R)$, i.e. for any set $B \subset \mathcal{X} \times \mathcal{T}$ with $diam(B) \leqslant 2^k/R$,

$$\#(\mathcal{E}_k \cap B) \leqslant Ru\, diam(B).$$

Then

$$\left\| \sum_k \sum_{(x_0,t_0)\in\mathcal{E}_k} 2^{k\frac{d-1}{2}} S_{x_0,t_0} \right\|_{L^2(S^d)}^2 \lesssim R^{d-2} \log_2(u) u^{\frac{2}{d-1}} \sum_k 2^{k(d-1)} \#\mathcal{E}_k.$$

Write $F = \sum F_k$, where

$$F_k = 2^{k\frac{d-1}{2}} \sum_{(x_0,t_0)\in\mathcal{E}} S_{x_0,t_0}.$$

Applying Cauchy-Schwartz, we have

$$\|F\|_{L^2(S^d)}^2 \lesssim \log_2(u) \left(\sum_{k\lesssim \log_2(u)} \|F_k\|_{L^2(S^d)}^2 + \| \sum_{k\gtrsim \log_2(u)} F_k\|_{L^2(S^d)}^2 \right).$$

Without loss of generality by increasing the implicit constant, we can assume that $\{k : \mathcal{E}_k \neq \varnothing\}$ is 10-separated, and that all values of t with $(x,t) \in \mathcal{E}$ are positive (the case where all values of t being negative being treated analogously). Thus if F_k and $F_{k'}$ are both nonzero, then $k = k'$ or $|k - k'| \geqslant 10$. For $k \geqslant k' + 10$, let us estimate $\langle F_k, F_{k'} \rangle$. We can decompose this inner product into a sum of quantities of the form $2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} \langle S_{x_0,t_0}, S_{x_1,t_1} \rangle$, where $t_0 \sim 2^k/R$ and $t_1 \sim 2^{k'}/R$. Now consider the two sets

$$\mathcal{G}_{x_0,t_0,\text{low}} = \{(x_1,t_1) \in \mathcal{E}_{k'} : |d_g(x_0,x_1) - (t_0 - t_1)| \lesssim 2^{k'+10}/R\}$$

and for $l \geqslant k' + 10$, consider the set

$$\mathcal{G}_{x_0,t_0,l} = \{(x_1,t_1) \in \mathcal{E}_{k'} : |d_g(x_0,x_1) - (t_0 - t_1)| \sim 2^l/R\}.$$

Let us use the density properties of $\mathcal{E}$ to control the size of these index sets. First, note that for any $(x_0,t_0) \in \mathcal{E}_k$ and $(x_1,t_1) \in \mathcal{E}_{k'}$, $t_0 - t_1$ lies in a radius $O(2^{k'}/R)$ interval centered at t_0:

- Let us first estimate interactions between the functions S_{x_0,t_0} and S_{x_1,t_1} with $(x_1,t_1) \in \mathcal{G}_{x_0,t_0,\text{low}}$. If $(x_1,t_1) \in \mathcal{G}_{x_0,t_0,\text{low}}$, then x_1 must lie in a width $O(2^{k'}/R)$ and radius $O(2^k/R)$ annulus centered at x_0. Thus $\mathcal{G}_{x_0,t_0,\text{low}}$ is covered by $O(2^{(k-k')(d-1)})$ balls of radius $2^{k'}/R$. The density properties of $\mathcal{E}_{k'}$ implies that
$$\#\mathcal{G}_{x_0,t_0,l} \lesssim Ru\, 2^{(k-k')(d-1)}(2^{k'}/R) = u2^{(k-k')(d-1)+k'}.$$
Together with Lemma 18.4, we conclude that
$$2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} \sum_{(x_1,t_1)\in\mathcal{G}_{x_0,t_0,\text{low}}} |\langle S_{x_0,t_0}, S_{x_1,t_1} \rangle| \lesssim_M R^{d-2} 2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} \left(u2^{(k-k')(d-1)+k'}\right)\left(2^{-k\frac{d-1}{2}}\right).$$
We can now sum over $\log_2(u) \lesssim k' \leqslant k - 10$ and $(x_0,t_0) \in \mathcal{E}_k$ to find
$$2^{k\frac{d-1}{2}} 2^{k'\frac{d-1}{2}} \sum_{(x_0,t_0)\in\mathcal{E}_k} \sum_{k'\leqslant k-10} \sum_{(x_1,t_1)\in\mathcal{G}_{x_0,t_0,\text{low}}} |\langle S_{x_0,t_0}, S_{x_1,t_1} \rangle| \lesssim R^{d-2} 2^{k(d-1)} \#\mathcal{E}_k.$$

- Next, let's estimate interactions between the functions S_{x_0,t_0} and S_{x_1,t_1} with $(x_1, t_1) \in \mathcal{G}_{x_0,t_0,l}$ with $k' + 10 \leqslant l \leqslant k - 5$. If $(x_1, t_1) \in \mathcal{G}_{x_0,t_0,l}$, then x_1 must lie in one of two geodesic annuli centered at x_0, each width $O(2^l/R)$ and radii $O(2^k/R)$. Thus $\mathcal{G}_{x_0,t_0,l}$ is covered by $O(2^{(l-k')}2^{(k-k')(d-1)})$ balls of radius $2^{k'}/R$, and the density of $\mathcal{E}_{k'}$ implies that

$$\#\mathcal{G}_{x_0,t_0,l} \lesssim Ru\, 2^{(l-k')}2^{(k-k')(d-1)}2^{k'}/R = u2^l 2^{(k-k')(d-1)}.$$

  Together with Lemma 18.4, we conclude that

$$2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}} \sum_{(x_1,t_1)\in\mathcal{G}_{x_0,t_0,l}} |\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim_M R^{d-2}2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}}\left(u2^l 2^{(k-k')(d-1)}\right)\left(2^{-k\frac{d-1}{2}}2^{-lM}\right).$$

  Picking $M > 1$, we can sum over $k'+10 \leqslant l \leqslant k-5$, $\log_2(u) \lesssim k' \leqslant k-10$, and $(x_0, t_0) \in \mathcal{E}_k$ to find

$$\sum_{(x_0,t_0)\in\mathcal{E}_k}\ \sum_{k'\leqslant k-10}\ \sum_{k'+10\leqslant l\leqslant k-5}\ \sum_{(x_1,t_1)\in\mathcal{G}_{x_0,t_0,l}} 2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}}|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim R^{d-2}2^{k(d-1)}\#\mathcal{E}_k.$$

- Now let's estimate the interactions between the functions S_{x_0,t_0} and S_{x_1,t_1} with $(x_1, t_1) \in \mathcal{G}_{x_0,t_0,l}$, for $k + 10 \leqslant l \leqslant \log_2 R$, then x_1 must lie in a geodesic ball of radius $O(2^l/R)$ centered at x_0. Such a ball is covered by $O(2^{(l-k')d})$ balls of radius $2^{k'}/R$, and the density of $\mathcal{E}_{k'}$ implies that

$$\#\mathcal{G}_{x_0,t_0,l} \lesssim Ru\, 2^{(l-k')d}(2^{k'}/R) = u2^{(l-k')d}2^{k'}.$$

  Together with Lemma 18.4, we conclude that

$$2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}} \sum_{(x_1,t_1)\in\mathcal{G}_{x_0,t_0,l}} |\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim_M R^{d-2}2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}}\left(u2^{(l-k')d}2^{k'}\right)\left(2^{-lM}\right).$$

  Picking $M > d$, we can sum over $k - 5 \leqslant l \lesssim \log R$, $\log_2(u) \lesssim k' \leqslant k - 10$, and $(x_0, t_0) \in \mathcal{E}_k$ to conclude that

$$2^{k\frac{d-1}{2}}2^{k'\frac{d-1}{2}} \sum_{(x_0,t_0)\in\mathcal{E}_k}\ \sum_{k'\leqslant k-10}\ \sum_{k-5\leqslant l\lesssim \log R}\ \sum_{(x_1,t_1)\in\mathcal{G}_{x_0,t_0,l}} R^{d-2}|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim R^{d-2}.$$

Putting these three bounds together, we conclude that

$$\sum_{\log_2(u)\lesssim k'<k} |\langle F_k, F_{k'}\rangle| \lesssim R^{d-2}\sum_k 2^{k(d-1)}\#\mathcal{E}_k.$$

In particular, we have

$$\|F\|_{L^2(S^d)}^2 \lesssim \log_2(u)\left(\sum_k \|F_k\|_{L^2(S^d)}^2 + R^{d-2}\sum_k 2^{k(d-1)}\#\mathcal{E}_k\right).$$

Next, let us fix some parameter a, and decompose $[2^k/R, 2^{k+1}/R]$ into the disjoint union of length u^a intervals

$$I_{k,\mu} = [2^k/R + (\mu-1)u^a/R, 2^k/R + \mu u^a/R] \quad \text{for } 1 \leqslant \mu \leqslant 2^k/u^a,$$

and thus considering a further decomposition $\mathcal{E}_k = \bigcup \mathcal{E}_{k,\mu}$ and $F_k = \sum F_{k,\mu}$. As before, increasing the implicit constant in the Lemma, we may assume without loss of generality that the set $\{\mu : \mathcal{E}_{k,\mu} \neq \varnothing\}$ is 10-separated. We now estimate

$$\sum_{\mu \geqslant \mu'+10} |\langle F_{k,\mu}, F_{k,\mu'}\rangle|.$$

For $(x_0, t_0) \in \mathcal{E}_{k,\mu}$ and $l \geqslant 1$, define

$$\mathcal{H}_{x_0,t_0,l} = \left\{(x_1,t_1) \in \mathcal{E}_{k,\mu'} : \max(d_g(x_0,x_1), t_0 - t_1) \sim 2^l u^a/R\right\}.$$

Then $\bigcup_{l\geqslant 1} \mathcal{H}_{x_0,t_0,l}$ covers $\bigcup_{\mu \geqslant \mu'+10} \mathcal{E}_{k,\mu'}$. The density properties of $\mathcal{E}_{k,\mu'}$ imply that provided that $l \leqslant k - a\log_2 u + 10$ (so that $2^l u^a/R \leqslant 2^k/R$),

$$\#\mathcal{H}_{x_0,t_0,l} \lesssim (Ru)(2^l u^a/R) = u^{a+1}2^l$$

For $(x_1,t_1) \in \mathcal{H}_{x_0,t_0,l}$, we claim that

$$2^{k(d-1)}|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim R^{d-2}2^{k(d-1)}(2^l u^a)^{-\frac{d-1}{2}}.$$

Indeed, for such tuples we have

$$d_g(x_0,x_1) \gtrsim 2^l u^a/R \quad \text{or} \quad |d_g(x_0,x_1) - (t_0 - t_1)| \gtrsim 2^l u^a/R,$$

and the estimate follows from Lemma 18.4 in either case. Since $d \geqslant 4$,

$$\begin{aligned}\sum_{1\leqslant l\leqslant k-a\log_2 u+10}\ \sum_{(x_1,t_1)\in\mathcal{H}_{x_0,t_0,l}} 2^{k(d-1)}|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| &\lesssim R^{d-2}\sum_{1\leqslant l\leqslant k-a\log_2 u+10}(2^{k(d-1)})(2^l u^a)^{-\frac{d-1}{2}}(u^{a+1}2^l)\\ &\lesssim R^{d-2}\sum_{1\leqslant l\leqslant k-a\log_2 u+10} 2^{k(d-1)}2^{-l\frac{d-3}{2}}u^{1-a\left(\frac{d-3}{2}\right)}\\ &\lesssim R^{d-2}2^{k(d-1)}u^{1-a\left(\frac{d-3}{2}\right)}.\end{aligned}$$

For $l > k - a\log_2 u + 10$, a tuple (x_1, t_1) lies in $\mathcal{H}_{x_0,t_0,l}$ if and only if $d_g(x_0, x_1) \sim 2^l u^a/R$, since we always have

$$|t_0 - t_1| \lesssim 2^k/R \ll 2^l u^a/R.$$

We conclude from Lemma 18.4 that

$$2^{k(d-1)}|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| \lesssim_M R^{d-2}2^{k(d-1)}(2^l u^a)^{-M}.$$

Now $\mathcal{H}_{x_0,t_0,l}$ is covered by $O((2^{l-k}u^a)^d)$ balls of radius $2^k/R$, and the density properties of $\mathcal{E}_k$ imply that

$$\#\mathcal{H}_{x_0,t_0,l} \lesssim (Ru)(2^{l-k}u^a)^d(2^k/R) \lesssim u^{1+ad}2^{ld}2^{-k(d-1)}.$$

Thus, picking $M > \max(d, 1 + ad)$, we conclude that

$$\begin{aligned}\sum_{l\geqslant k-a\log_2 u+10}\ \sum_{(x_1,t_1)\in\mathcal{H}_{x_0,t_0,l}} 2^{k(d-1)}|\langle S_{x_0,t_0}, S_{x_1,t_1}\rangle| &\lesssim R^{d-2}\sum_{l\geqslant k-a\log_2 u+10}(2^{k(d-1)})(2^l u^a)^{-M}u^{1+ad}2^{ld}2^{-k(d-1)}\\ &\lesssim R^{d-2}.\end{aligned}$$

Putting these two bounds together, and then summing over the tuples (x_0, t_0), we conclude that

$$\sum_{\mu\geqslant\mu'+10}|\langle F_{k,\mu}, F_{k,\mu'}\rangle| \lesssim R^{d-2}\left(1 + 2^{k(d-1)}u^{1-a\left(\frac{d-3}{2}\right)}\right)\#\mathcal{E}_{k,\mu}.$$

Now summing in μ, we conclude that

$$\|F_k\|^2_{L^2(S^d)} \lesssim \sum_{\mu}\|F_{k,\mu}\|^2_{L^2(S^d)} + R^{d-2}\left(1 + 2^{k(d-1)}u^{1-a\left(\frac{d-3}{2}\right)}\right)\#\mathcal{E}_k.$$

The functions in the sum defining $F_{k,\mu}$ are highly coupled, and it is difficult to use anything except Cauchy-Schwartz to break them apart. Since $\#(\mathcal{T} \cap I_{k,\mu}) \sim u^a$, if we set $F_{k,\mu} = \sum_{t\in\mathcal{T}\cap I_{k,\mu}} F_{k,\mu,t}$, then we find

$$\|F_{k,\mu}\|^2_{L^2(S^d)} \lesssim u^a\sum_{t\in\mathcal{T}\cap I_{k,\mu}}\|F_{k,\mu,t}\|^2_{L^2(S^d)}.$$

Fortunately, since $\mathcal{X}$ is 1-separated, the functions in $F_{k,\mu,t}$ are quite orthogonal to one another, and so

$$\|F_{k,\mu,t}\|^2_{L^2(S^d)} \lesssim R^{d-2}2^{k(d-1)}\#(\mathcal{E}_k \cap (S^d \times \{t\})).$$

But this means that

$$u^a \sum_t \|F_{k,\mu,t}\|^2_{L^2(S^d)} \lesssim R^{d-2} 2^{k(d-1)} u^a \#\mathcal{E}_{k,\mu}.$$

and so

$$\begin{aligned}
\|F_k\|^2_{L^2(S^d)} &\lesssim \sum_\mu \|F_{k,\mu}\|^2_{L^2(S^d)} + R^{d-2}\left(1 + 2^{k(d-1)} u^{1-a\left(\frac{d-3}{2}\right)}\right) \#\mathcal{E}_k \\
&\lesssim R^{d-2}\left(2^{k(d-1)} u^a + (1 + 2^{k(d-1)} u^{1-a\left(\frac{d-3}{2}\right)}\right) \#\mathcal{E}_k.
\end{aligned}$$

Picking $a = 2/(d-1)$, we conclude that

$$\|F_k\|^2_{L^2(S^d)} \lesssim R^{d-2} 2^{k(d-1)} u^{\frac{2}{d-1}} \#\mathcal{E}_k.$$

Thus, returning to our bound for F, we conclude that

$$\|F\|^2_{L^2(S^d)} \lesssim R^{d-2} \log_2(u) u^{\frac{2}{d-1}} \sum_k 2^{k(d-1)} \#\mathcal{E}_k.$$

This completes the proof of the L^2 density bound.

We now return to a general index set $\mathcal{E} = \bigcup_k \mathcal{E}_k$. Using a density decomposition argument, we can write $\mathcal{E}_k = \bigcup_m \mathcal{E}_{k,m}$, where for each k and m, $\mathcal{E}_{k,m}$ has density type $(R2^m, 2^k/R)$, and there exists a family of balls $\{B_j\}$ covering $\mathcal{E}_{k,m}$, with $\mathrm{rad}(B_j) = r_j$, such that $\sup_j r_j \leqslant 2^k/R$, and $\sum_j r_j \leqslant \#\mathcal{E}_k/R2^m$. Define

$$F^m = \sum_k \sum_{(x_0,t_0)\in\mathcal{E}_{k,m}} 2^{k\frac{d-1}{2}} S_{x_0,t_0}.$$

We have $\#\mathcal{E}_{k,m} \leqslant \#\mathcal{E}_k$ for each m, and so the density bound above implies

$$\|F^m\|^2_{L^2(S^d)} \lesssim R^{d-2} m\, 2^{\frac{2m}{d-1}} \sum_k 2^{k(d-1)} \#\mathcal{E}_k.$$

For $p \ll 1$, we can exploit the fact that the points in $\mathcal{E}_{k,m}$ are highly concentrated for large m to get better bounds, which we will then interpolate.

A similar analysis (TODO: Move to previous section) justifies that

$$S_{x_0,t_0}(x) \leqslant \frac{R^{d-1}}{(Rd_g(x,x_0))^{\frac{d-1}{2}}} \left\langle R\middle|t_0 - d_g(x,x_0)\middle|\right\rangle^{-M}.$$

Thus the function is concentrated on the radius t_0, thickness $1/R$ geodesic annulus centered at x_0, and on this annulus, the function has height $R^{d-1}(Rt_0)^{-\frac{d-1}{2}}$. If we consider a ball B_j with radius r_j, and center x_j, and consider the function

$$F_{k,j}^m = \sum_k \sum_{(x_0,t_0)\in\mathcal{E}_{k,m}\cap B_j} 2^{k\frac{d-1}{2}} S_{x_0,t_0},$$

then the $F_{k,j}^m$ is concentrated on a geodesic annulus with radius $\sim 2^k/R$ and thickness r_j, centered at x_j. By the density properties, $\#(\mathcal{E}_{k,m}\cap B_j) \leqslant 2^m R r_j$, and so $F_{k,j}^m$ has height $O(R^d 2^m r_j)$ on this annulus. We thus conclude that

$$\|F_{k,j}^m\|_{L^\varepsilon(S^d)}^\varepsilon \lesssim \left((2^k/R)^{d-1} r_j\right)(2^m r_j)^\varepsilon \lesssim 2^{k(d-1)} R^{-(d-1-\varepsilon)} 2^{m\varepsilon} r_j.$$

For $\varepsilon < 1$, we can sum in j and k, using the concentrating properties to conclude that

$$\|F_k^m\|_{L^\varepsilon(S^d)}^\varepsilon \lesssim 2^{-m(1-\varepsilon)} 2^{k(d-1)} R^{-(d-\varepsilon)} \#\mathcal{E}_k$$

and thus

$$\|F^m\|_{L^\varepsilon(S^d)}^\varepsilon \lesssim R^{-(d-\varepsilon)} 2^{-m(1-\varepsilon)} \sum_k 2^{k(d-1)} \#\mathcal{E}_k.$$

TODO

So we now have two bounds, namely

$$\|F^m\|_{L^2(S^d)}^2 \lesssim m\, 2^{\frac{2m}{d-1}} \sum_k 2^{k(d-1)} \#\mathcal{E}_k$$

and

$$\|F^m\|_{L^\varepsilon(S^d)}^\varepsilon \lesssim R^{-(d-\varepsilon)} 2^{-m(1-\varepsilon)} \sum_k 2^{k(d-1)} \#\mathcal{E}_k.$$

Can we now interpolate these two bounds? If $1/p = t/\varepsilon + (1-t)/2$, then

$$t = \frac{1/p - 1/2}{1/\varepsilon - 1/2}.$$

and we should expect to get

$$\|F^m\|_{L^p(S^d)}^p \lesssim \left(R^{-(d-\varepsilon)} 2^{-m(1-\varepsilon)}\right)^t \left(m 2^{\frac{2m}{d-1}}\right)^{1-t} \sum_k 2^{k(d-1)} \#\mathcal{E}_k.$$

As long as $d \geqslant 2$, we can choose $\varepsilon = 2^{1/2}(d-1)^{-1/2}(1/p - 1/2)^{-1/2}$, which yields

$$\|F^m\|_{L^p(S^d)}^p \lesssim R^{-(d-\varepsilon)} m 2^{-\delta_{p,d} m} \sum_k 2^{k(d-1)} \#\mathcal{E}_k.$$

Summing in m thus gives that

$$\|F\|_{L^p(S^d)} \lesssim \left(R^{-(d-\varepsilon)} \sum_k 2^{k(d-1)} \# \mathcal{E}_k \right)^{1/p}$$

Chapter 19

Extensions to Non-Periodic Functions

Suppose that M is a manifold on which the geodesic flow is non-periodic. Can one still establish multiplier theorems in this setting? The analysis of small times still carries across perfectly fine. Even the L^2 analysis continues to work, because of the fact that

$$\langle e^{2\pi i t_0 P} f_{x_0}, e^{2\pi i t_1 P} f_{x_1} \rangle = \langle e^{2\pi i (t_0 - t_1)} f_{x_0}, f_{x_1} \rangle,$$

and so provided $|t_0 - t_1| \lesssim 1$, we've reduced our analysis to local times. However, if we're to apply Bourgain's interpolation trick, we also have to get some estimates in L^0 or L^1 so that we can interpolate.

If we write $b(t) = R\widehat{h}(Rt)$, and then consider a decomposition $b(t) = \sum_n b_n(t - n)$, where $\text{supp}(b_n) \subset [-2/3, 2/2]$, then

$$\|b_n\|_{L^p[-1,1]} \lesssim n^{-s_p} R^{1-1/p-s_p} C_p(h).$$

Provided that $p < 2(d-1)/(d+1)$, we have $s_p > 1$, so provided we can show bounds of the form

$$\left\| \int b_n(t-n) e^{2\pi i t P} f \; dt \right\|_{L^1(M)} \lesssim_\varepsilon n^\varepsilon R^{s_p + 1/p - 1} \|f\|_{L^1(M)},$$

then we can interpolate, and then apply the triangle inequality. This is tricky though. It works when M is the sphere, because then the result follows by local smoothing and periodicity. But for other manifolds, other tricks are needed. Composing the Fourier integral operator might work, but requires some technical calculations.

Chapter 20

Are Eigenfunctions of the Laplacian Locally Constant

Suppose that $f \in C^\infty(M)$, and $\Delta f = -\lambda^2 f$. If f was a function on $\mathbb{R}^d$, the uncertainty principle would imply that f was locally constant at a scale $1/\lambda$, i.e. such that for any $|x_0| \leqslant 1/\lambda$,

$$|f(x_0)| \lesssim_N \lambda^d \int |f(x)| w_\lambda(x),$$

where $w_\lambda(x) = (1 + \lambda^2 |x|^2)^{-N}$.

Does an analogous inequality hold for spherical harmonics? Perhaps we can use the Hecke-Funk formula. Namely, if $f \in V_R$, $\phi \in C_c^\infty(\mathbb{R})$ is non-negative, and equal to one on $1/2 \leqslant t \leqslant 2$, and

$$P_R(t) = \sum_k \phi(k/R - 1) G_k(t),$$

then $f = P_R * f$. Experimental evidence (see code in GegenbauerSummationGraph.py) leads us to believe that P_R is concentrated on a $1/R$ neighborhood of $t = 1$, and we should therefore expect to have a bound of the form

$$|f(x)| \lesssim_N R^n \int_{S^n} w_N(R(x \cdot y)) |f(y)| \, dy,$$

where $w_N(t) = \langle t \rangle^{-N}$. Can we formally prove this is the case?

Chapter 21

Uncertainty Principle

Suppose that $u : \mathbb{R}^d \to \mathbb{C}$ is a smooth function, such that

$$\operatorname{supp}(\widehat{u}) \subset \{|\xi| \leqslant R\}.$$

Let B be a ball with radius $1/R$, let χ_B be a smooth bump function adapted to B, and consider the function

$$u_B = \chi_B \cdot u.$$

Let $c_B = R^{d/p} \|u_B\|_{L^p(\mathbb{R}^d)}$ denote the 'magnitude' of the function u_B, and introduce the normalization $u_B = c_B v_B$. Then

$$\|D^\alpha u_B\|_{L^p(\mathbb{R}^d)} \lesssim \sup_{\beta_1+\beta_2=\alpha} \|D^{\beta_1}\chi_B \cdot D^{\beta_2} u\|_{L^p(\mathbb{R}^d)} \lesssim R^{|\alpha|} \|u_B\|_{L^p(\mathbb{R}^d)},$$

and

$$\left(\sum_B R^{-d} c_B^p \right)^{1/p} \lesssim \left(\sum_B \|u_B\|_{L^p}^p \right)^{1/p} \lesssim \|u\|_{L^p(\mathbb{R}^d)}.$$

The Sobolev embedding theorem tell us that

$$\|u_B\|_{L^\infty(\mathbb{R}^d)} \lesssim c_B,$$

and more generally,

$$\|D^\alpha u_B\|_{L^\infty(\mathbb{R}^d)} \lesssim R^{|\alpha|+d/p} \|u_B\|_{L^p(\mathbb{R}^d)}.$$

But this means that

$$\|D^\alpha v_B\|_{L^\infty(\mathbb{R}^d)} \lesssim R^{|\alpha|}.$$

Thus the functions $\{v_B\}$ are smooth, and adapted to the family of balls $\{B\}$.

Chapter 22

Attempt Using Decoupling

Let us try and attack our problem using decoupling. Fix a function $h : (0,\infty) \to \mathbb{R}$, which is compactly supported on $\{1 < \lambda < 2\}$, and use this function to induce a family of radial multiplier operators on the sphere S^d, of the form

$$T_R f = \sum h(\lambda/R)\langle f, e_\lambda\rangle e_\lambda.$$

Our goal is to obtain bounds of the form

$$\sup_{R>0} \|T_R f\|_{L^p(S^d)} \lesssim \|f\|_{L^p(S^d)}.$$

Without loss of generality, as in the last chapter, because of the support of h, for a given R, we may assume that our inputs f are a sum of eigenfunctions on the sphere with eigenvalue between R and $2R$. We rewrite

$$T_R f = \int_{-\infty}^{\infty} R\widehat{h}(Rt)e^{2\pi it\sqrt{-\Delta}} f\, dt.$$

Since we are attempting to obtain bounds in the range that the local smoothing conjecture has been obtained, by the calculations in the last chapter, it suffices to analyze that part of the term above of the form

$$T_R f = \int_{1/R\lesssim|t|\lesssim 1} R\widehat{h}(Rt)e^{2\pi it\sqrt{-\Delta}} f\, dt.$$

In coordinates, modulo a smoothing operator, whose behaviour is negligible, we can write the kernel of $e^{2\pi it\sqrt{-\Delta}}$ as

$$\int a(x,t,y,\xi)e^{2\pi i\Phi(t,x,y,\xi)}\, d\xi,$$

where a is a symbol of order zero with

$$\operatorname{supp}(a) \subset \{(x,t,y,\xi) : |x-y| \lesssim 1 \text{ and } |\xi| \gtrsim 1\},$$

and $\Phi(t,x,y,\xi) \approx (x-y)\cdot\xi + t|\xi|_g$. We can thus write the kernel of $T_R f$ as

$$\int_{-1/2}^{1/2} \int R\widehat{h}(Rt) a(x,t,y,\xi) e^{2\pi i \Phi(t,x,y,\xi)}\, d\xi.$$

We now decompose $T_R f$ in both frequency and time, writing

$$T_R f = T_R^{\leqslant 0} + \sum_{k=1}^{O(\log R)} \sum_{n=1}^{\infty} T_{R,k}^{2^n}$$

where $T_{R,k}^{\lambda}$ has kernel

$$\int_{-1/2}^{1/2} \int \beta(Rt/2^k) R\widehat{h}(Rt) a(x,t,y,\xi) \beta(\xi/\lambda) e^{2\pi i \Phi(t,x,y,\xi)}\, d\xi\, dt,$$

and $T_R^{\leqslant 0}$ is supported on $|\xi| \leqslant 1$, and thus has the right L^p bounds simply by applying the triangle inequality. By the choice of input f, we expect the majority of the contribution of the sum the operators $T_{R,k}^n$ should come from $n \approx \log R$. In the last chapter, we were able to obtain uniform bounds in R by summing up $k \lesssim 1$ and $k \gtrsim \log R$, so it suffices to study the operators $T_R^{n,k}$ in the range $1 \lesssim k \lesssim \log R$.

We now try and apply the decoupling result of Beltran, Hickman, and Sogge; if we cover the unit sphere in $\mathbb{R}^d_\xi$ by $O(\lambda^{-(d-1)/2})$ points Θ_λ, consider an appropriate partition of unity $\{\chi_\lambda^\nu : \nu \in \Theta_\lambda\}$, and thus write

$$T_{R,k}^{\lambda} = \sum T_{R,k}^{\lambda,\nu},$$

where $T_{R,k}^{\lambda,\nu}$ has kernel

$$\begin{aligned}
&\int\int \beta(Rt/2^k) R\widehat{h}(Rt) a(x,t,y,\xi) \beta(\xi/\lambda) \chi_\lambda^\nu(\xi) e^{2\pi i \Phi(t,x,y,\xi)}\, d\xi\, dt \\
&\quad = \int R\widehat{h}(Rt) a_{\lambda,\nu,R,k}(x,t,y,\xi) e^{2\pi i \Phi(t,x,y,\xi)}.
\end{aligned}$$

Let us suppose that, using the techniques of their paper, we can show that

$$\|T_{R,k}^{\lambda} f\|_{L^{p^*}(\mathbb{R}^d)} \lesssim_{p,\varepsilon} \lambda^{\alpha_{p^*}+\varepsilon} \left(\sum_\nu \|T_{R,k}^{\lambda,\nu} f\|_{L^{p^*}(\mathbb{R}^d)}^{p^*} \right)^{1/p^*}.$$

Thus it suffices to analyze the behaviour of each of the operators $T^{\lambda,\nu}_{R,k}$ separately. For each fixed t, energy conservation implies that

$$\begin{aligned}
\|T^{\lambda,\nu}_{R,k} f\|_{L^2(\mathbb{R}^d)} &\lesssim \left(\int_{|t|\sim 2^k/R} R\widehat{h}(Rt)\, dt \right) \|f^\lambda_\nu\|_{L^2(\mathbb{R}^d)} \\
&\lesssim \int_{|t|\sim 2^k/R} R\langle Rs\rangle^{-(d-1)(1/2-1/p)} \|f^\lambda_\nu\|_{L^2(\mathbb{R}^d)} \\
&\lesssim (2^k)^{1-(d-1)(1/2-1/p)} \|f^\lambda_\nu\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Thus L^2 orthogonality implies that

$$\left(\sum_\nu \|T^{\lambda,\nu}_{R,k} f\|^2_{L^2(\mathbb{R}^d)} \right)^{1/2} \lesssim (2^k)^{1-(d-1)(1/2-1/p)} \|f\|_{L^2(\mathbb{R}^d)}.$$

On the other hand, to obtain an interpolation at L^∞, we must understand the operator

$$\sup_\nu \|T^{\lambda,\nu}_{R,k} f\|_{L^\infty(\mathbb{R}^d)}$$

A stationary phase argument shows that, if we write

$$T^{\lambda,\nu}_{R,k} f = \int_{|t|\sim 2^k/R} R\widehat{h}(Rt) T^{\lambda,t}_{R,\nu} f,$$

then the kernel $K^{\lambda,\nu,t}_R$ of $T^{\lambda,\nu,t}_R$ satisfies estimates of the form

$$|K^{\lambda,\nu,t}_R(x,t;y)| \lesssim_N \frac{\lambda^{(d+1)/2}}{\left\langle \lambda|\pi_\nu \nabla_\xi \Phi(x,y,t,\nu)| + \lambda^{1/2}|\pi^\perp_\nu \nabla_\xi \Phi(x,y,t,\nu)| \right\rangle^N}.$$

Here we have $\nabla_\xi \Phi(x,y,t,\nu) = (x-y)+t\nu+O(|x-y|)$. Thus we conclude that, for a fixed x, this kernel has the majority of it's support on a cap centered at the point $x+t\nu$, with thickness $1/\lambda$ in the direction ν, and thickness $1/\lambda^{1/2}$ in directions tangential to ν. But this implies that the kernel of $K^{\lambda,\nu}_{R,k}$ is essentially supported on a $1/\lambda^{1/2}$ neighborhood of the line $\{t\nu : |t| \sim 2^k/R\}$, and moreover, on that line we have

$$|K^{\lambda,\nu}_{R,k}(x;t\nu)| \lesssim \int_{t-1/\lambda}^{t+1/\lambda} R\langle Rs\rangle^{-(d-1)(1/2-1/p^*)} \lambda^{(d+1)/2}\, ds$$

For $\lambda \geqslant R$, since $k \gtrsim 1$, and thus $|t| \geqslant 100/R$ we get that

$$|K^\lambda_{R,\nu}(x;t\nu)| \lesssim \lambda^{-1} R^{1-(d-1)(1/2-1/p^*)} t^{-(d-1)(1/2-1/p^*)}.$$

These same estimates hold replacing tv with $tv + v$ for some v perpendicular to v with $|v| \leqslant \lambda^{-1/2}$. Thus we get that

$$\int |K_{R,v}^{\lambda}(x;y)|\, dy \lesssim 2^k/R.$$

For $\lambda \leqslant R$, and $|t| \leqslant 10/\lambda$, we get that

$$|K_{R,v}^{\lambda}(x;tv)| \lesssim \lambda^{(d+1)/2}$$

and for $|t| \geqslant 10/\lambda$, we get that

$$|K_{R,v}^{\lambda}(x;tv)| \lesssim \lambda^{(d-1)/2} R^{1-(d-1)(1/2-1/p^*)} t^{-(d-1)(1/2-1/p^*)}.$$

Integrating these results gives that

$$\int |K_{R,v}^{\lambda}(x;y)|\, dy \lesssim 1,$$

the same bound as was obtained for $\lambda \geqslant R$. Thus Schur's Lemma gives

$$\sup_{v} \|T_{R,v}^{\lambda} f\|_{L^\infty} \lesssim \|f\|_{L^\infty}.$$

Interpolating gives that

$$\left(\sum \|T_{R,v}^{\lambda} f\|_{L^{p^*}}^{p^*}\right)^{1/p^*} \lesssim \|f\|_{L^{p^*}},$$

and thus that

$$\|T_R^{\lambda} f\|_{L^{p^*}} \lesssim_{\varepsilon} \lambda^{\alpha(p^*)+\varepsilon} \|f\|_{L^{p^*}}.$$

Chapter 23

Trying to Use Hadamard Parametrix

Let

$$T_R f = \sum_\lambda m(\lambda/R)\langle f, e_\lambda\rangle e_\lambda.$$

If we write

$$M(t) = \int_0^\infty m(\lambda)\cos(2\pi t\lambda)\, d\lambda,$$

then the inverse formula implies that

$$m(\lambda/R) = \int M(t)\cos\left(\frac{2\pi\lambda t}{R}\right)\, dt.$$

Thus we have

$$T_R f = \int M(t)\cos\left(\frac{2\pi\sqrt{-\Delta}t}{R}\right) f\, dt$$

Local smoothing implies that we need only analyze an integral of the form

$$T_R f = R\int \eta(Rt)M(Rt)\cos\left(2\pi\sqrt{-\Delta}\cdot t\right) f\, dt,$$

where η has support on an arbitrarily small, but fixed portion of the origin. Localizing the operator by a partition of unity, and then applying the Hadamard parametrix, we should expect to control the kernel $T_R f$ by a finite sum of functions of the form

$$K_R(x,y) = R\iint \frac{c(x,y)\eta(Rt)M(Rt)}{|\xi|^\nu} e^{2\pi i(\xi\cdot d_g(x,y)+t|\xi|)}\, d\xi\, dR,$$

where a is smooth and compactly supported, and d_g denotes the geodesic distance on the manifold. Let us perform a frequency decomposition, writing $K_R = K_{R,0} + 2^{k(d-\nu)} \sum_{k=1}^{\infty} K_{R,k}$, where

$$K_{R,0}(x,t;y) = R \int M(Rt) \frac{c(x,y)\eta(Rt)\psi_0(\xi)}{|\xi|^\nu} e^{2\pi i(\xi \cdot d_g(x,y) + t|\xi|)} \, d\xi \, dt,$$

and for $k \geqslant 1$,

$$K_{R,k}(x,t;y) = 2^{k(\nu-d)} R \iint \frac{a(x,y)\eta(Rt)M(Rt)\tilde{\psi}(\xi/2^k)}{|\xi|^\nu} e^{2\pi i(\xi \cdot d_g(x,y) + t|\xi|)} \, d\xi \, dt.$$

Rescaling, and setting $a_\nu(x,t,y,\xi) = a(x,y)\eta(t)\tilde{\psi}(\xi)/|\xi|^\nu$ gives that

$$K_{R,k}(x,t;y) = \int M(t) \int a_\nu(x,t,y,\xi) e^{2\pi i 2^k (\xi \cdot d_g(x,y) + (t/R)|\xi|)} \, d\xi \, dt.$$

Similarily,

$$K_{R,0}(x,t;y) = \int M(t) a_{\nu,0}(x,y,t,\xi) e^{2\pi i(\xi \cdot d_g(x,y) + (t/R)|\xi|)} \, d\xi \, dt.$$

Our goal is to obtain some L^p estimates on this operator that are summable in k, thus obtaining bounds of the form

$$\|T_{R,k} f\|_{L^p} \lesssim 2^{k(\nu-d-\varepsilon)}$$

for some $\varepsilon > 0$, which hold uniformly in R. The operator $T_{R,0}$ should not be an issue since one can just take in absolute values to obtain the required result.

Stationary phase tells us that the majority of the mass of the kernel $K_{R,k}$ should be concentrated on points $(x,t;y)$ where $|d_g(x,y) - t/R| \lesssim 2^{-k}$, a geodesic annulus of radius t/R, and thickness 2^{-k}. If we are to try a decoupling result, let us split this annulus into a family Θ_k of sectors of aperture $2^{-k/2}$ with the finite itnersection property. We should require a set of sectors $\Theta_{R,k}$ with $\#(\Theta_k) \lesssim 2^{k(d-1)/2}$. If we consider a partition of unity $\{\chi_\theta\}$ localizing the operator to these sectors, we can therefore write $T_{R,k} = \sum_\theta T_{R,k,\theta}$. Let us suppose a Wolff-type decoupling bound held for this operators, i.e. that

$$\|T_{R,k} f\|_{L^p} \lesssim_\varepsilon 2^{k(\alpha(p)+\varepsilon)} \left(\sum_\theta \|T_{R,k,\theta} f\|_{L^p}^p \right)^{1/p}.$$

Let us thus analyze a particular one of these operators $T_{R,k,\theta}$, which has kernel

$$K_{R,k,\theta}(x,t;y) = \iint M(t) a_{\nu,\theta}(x,y,t,\xi) e^{2\pi i 2^k (\xi \cdot d_g(x,y) + (t/R)|\xi|)} \, d\xi \, dt,$$

where $a_{\nu,\theta} = a_\nu \cdot \chi_\theta$. For each t, and x, nonstationary phase tells us that the mass of the kernel $K_{R,k,\theta}(x,t;\cdot)$ should be concentrated on a cap of long thickness 2^{-k} and short thicknesses $2^{-k/2}$, containing in the intersection of the sector θ and the 2^{-k} neighborhood of the annulus of radius t/R centered at y. We should expect that on this cap the kernel should have amplitude equal to $O(M(t)2^{-k(d-1)})$. Thus we have

$$\|T_{R,k,\theta} f\|_{L^\infty(\mathbb{R}^d)} \lesssim C_p(m) 2^{-3k(d+1)/2} \|f\|_{L^\infty}$$

Chapter 24

Attempt Using Heo-Nazarov-Seeger Technique

Suppose h is a radial multiplier with support on $\{1 \leqslant \lambda \leqslant 2\}$, and let

$$b(t) = 2\int_0^\infty h(\lambda)\cos(2\pi\lambda).$$

If $B_R(t) = R\sum_l b(R(t+l))$, our goal is to prove uniform L^p bounds on the radial multiplier operator

$$T_R = \int_1^2 B_R(t)e^{2\pi it\sqrt{-\Delta}}.$$

We may assume our input is a linear combination of eigenfunctions with eigenvalue between R and $2R$. If we reduce to local coordinates, we can write

$$e^{2\pi it\sqrt{-\Delta}}f(x) = \int_{\mathbb{R}^d} a(x,t,y,\xi)e^{2\pi i(\phi(x,y,\xi)-t|\xi|_g)}f(y)\,d\xi,$$

where a is a compactly supported symbol of order zero, and where $\phi(x,y,\xi) \approx (x-y)\cdot\xi$ (this is only up to a smoothing operator, whose behaviour is irrelevant for the purposes of our argument since for such operator trivial L^p estimates hold). Thus we have

$$T_Rf(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_1^2 B_R(t)a(x,t,y,\xi)e^{2\pi i(\phi(x,y,\xi)-t)}f(y)\,dt\,dy\,dx.$$

Now let $\eta \in \mathcal{S}(\mathbb{R}^d)$ be a Schwartz function vanishing to high order at the origin, and consider the operator

$$\tilde{T}_Rf(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_1^2 B_R(t)a(x,t,y,\xi)\eta(\xi)e^{2\pi i(\phi(x,y,\xi)-t|\xi|_g)}f(y)\,dt\,dy\,dx.$$

Then $\tilde{T}_R f = T_R \circ$

$$\left\| \int_{\mathbb{R}^d} \int_1^2 B_R(t) a(x,t,y,\xi) e^{2\pi i(\phi(x,y,\xi) - t|\xi|_g)} f(y) \, d\xi \, dy \, dt \right\|.$$

Part V

Papers to Read in More Detail

- Sogge, L^p Estimates For the Wave Equation and Applications (1993).

  A survey of results on regularity results for the wave equation. In particular, reviews (without proof) the ideas of Mockenhaipt, Seeger, and Sogge which give local smoothing for Fourier integral operators satisfying the cone condition, as well as mixed norm estimates for non-homogeneous results on wave equations.

- In Sogge's Book, he mentions the main developments in harmonic / microlocal analysis he couldn't discuss in the book were the following:

  - Bennett, Carbery, Tao, On the Multilinear Restriction and Kakeya Conjecture (2006).

    Introduction to multilinear methods in harmonic analysis.

  - Bourgain, Guth, Bounds on Oscillatory Integral Operators Based on Multilinear Estimates (2010).

    Application of multilinear methods to bounding oscillatory integrals.

  - Bourgain, Demeter, The Proof of the l2 Decoupling Conjecture (2014).

    Introduction to Decoupling.

  - Peetre, New Thoughts on Besov-Spaces.

    Characterizes boundedness of Fourier multipliers on homogeneous Besov spaces.

  - Johnson, Maximal Subspaces of Besov-Spaces Invariant Under Multiplication By Characters.

    Shows a Fourier multiplier operator is bounded in the L^p norm if and only if it's translates are all localizably bounded as in Seeger.

- For more background reading in microlocal analysis:

  - Hörmander, The Analysis of Linear Partial Differential Operators, Volumes I-IV.
  - Treves, Introduction to Pseudodifferential and Fourier Integral Operators, Volumes I-II.
  - Taylor.

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