UPPSALA
UNIVERSITET

Exact Solutions of the Ising Model

Bachelor Degree Project 15 c

Author: Ludwig Ridderstolpe
Supervisor: Giuseppe Dibitetto
Subject reader: Ulf Danielsson

Department of Physics and Astronomy
Division of Theoretical Physics
Uppsala University

September 7, 2017

Abstract

This report presents the general Ising model and its basic assumptions. This study aims to, from diagonalization of the Transfer Matrix, obtain the Helmholtz free energy and the exclusion of a phase transition for the one-dimensional Ising model under an external magnetic field. Furthermore from establishing the commutation relations of the Transfer matrices and using the Kramers–Wannier duality one finds the free energy and the presence of a phase transition for the square-lattice Ising model.

Sammanfattning

Den här rapporten presenterar den generella Ising modellen och dess antaganden. Studien syftar till att från diagonalisering av överföringsmatrisen erhålla Helmholtz fria energi och utesluta fasövergångar för den endimensionella Ising modellen under ett externt magnetiskt fält. Ytterligare från att etablera kommutationsrelationer av överföringsmatriserna och av Kramers–Wannier dualitet finner vi den fria energin och en fasövergång för den tvådimensionella Ising modellen.

Contents

1 Introduction

The Ising model was introduced by Wilhelm Lenz (1920) and assumes particles in a crystal structure that may without restriction rotate freely around a fixed lattice point [1]. The model in its entirety arises from expressing such a system's Hamiltonian in two parts: one as the energy contribution from particle to particle interaction and one as the energy contribution from constraints on the system. Developing Lenz proposal with Statistical mechanics, the Ising model is completely characterized by Helmholtz free energy and by exactly calculating the free energy is regarded as the solution of the model. The field of Statistical mechanics is the study of summarizing microscopic dynamics of a system to a macroscopic description of the system's thermodynamic equilibrium properties [2]. Additionally statistical mechanics offers a formalism such that through the partition function one finds the free energy and from the free energy one can derive the magnetization and phase transitions.

Lenz (1922) gave the proposed model to his student Ernst Ising. Later Ising (1925) published a summarization of a exact calculation of the free energy for the one-dimensional case with an external field and further proved for that case it demonstrates no phase transitions at any temperature. Firstly the model was regarded as an oversimplification with no practical applications, Ising followed with making the claim that there is no phase transitions in higher dimensional models and later abandon the field of physics altogether [1]. Despite this due to a growing demand of a deeper understanding of ferromagnetism the Ising model attracted attention and Rudolf Peierls (1936) published a paper where he claimed in contrary to Ising's statement that the Ising model in two and three dimensions demonstrates phase-transitions [3]. Hendrik Kramers and Gregory Wannier (1941) located the critical temperature for the two-dimensional Ising model by relating the low and high temperature expansion with a transformation (*Kramers-Wannier Duality*). The big breakthrough came when Lars Onsager (1942) announced his analytical solutions to the Square-lattice Ising model. There has yet been presented any exact solution to Square-lattice model with an external field or for higher-dimensional models.

It is of great interest to understand ferromagnetic system's. The Ising model being the simplest system in we can observe phase transitions analytically [4], it is a powerful model which explains dynamics behind ferromagnets spontaneous magnetization and offers method of determining phase transitions and critical temperatures.

The aim of this thesis is to study the Ising model to obtain the Helmholtz free energy and magnetic phase transitions for ferromagnetic systems. By firstly outline the properties and assumptions of the general Ising model and establish its validity as a description of ferromagnets. Then with the *transfer matrix method* calculate the free energy of the one-dimensional Ising model with an external magnetic field and confirm the exclusion of a phase transition. Lastly from the *commuting transfer matrix method* and operators calculate the free energy of the Square-lattice Ising model and confirm the existence of phase transitions and the models critical temperature with Kramer-Wannier duality.

The content of this report are as follows. In chapter 1 a brief review of necessary concept of statistical mechanics such as: the partition function, the free energy and phase transitions. In chapter 2 the Hamiltonian of the General Ising model is formulated. In chapter 3 the solution of the One-dimensional Ising model and review of the transfer matrix method. In chapter 4 solution of the Square-lattice Ising model, review of the commuting transfer matrix method. Appendices explain Kramers–Wannier duality and necessary complex analysis for the solution of the Square-lattice Ising model.

2 Background Statistical Mechanics

In this chapter we will review basic statistical mechanical concepts necessary to understand the development of the solutions for the Ising model. We start with firstly covering the partition function and the Helmholtz free energy, together with some comments about their significance. Lastly we explain the concept of magnetization and phase transitions in a system.

2.1 The Partition Function

Consider a system made up of microstates labeled r with the energy of each microstate E_r, then the probability of the system being in a state r is given by p_r such that

$$p_r = \frac{1}{Z} e^{-\beta E_r} \tag{2.1}$$

where $\beta = k_B T$, k_B is the Boltzmann constant, T is the system's temperature and $e^{-\beta E_r}$ is the *Boltzmann weight*. This probability distribution is called the *Boltzman-distribution*.

Where the *partition function* Z normalizes the Boltzmann distribution with:

$$Z = \sum_r e^{-\beta E_r} \tag{2.2}$$

The partition function expresses all accessible states of the system [5], but its significance is how it can relate the system microscopic properties that constitute the energy of each microstate to macroscopic thermodynamic quantities.

2.2 Helmholtz Free Energy

For a system held at constant temperature and volume, from the Legendre transform of the first law of thermodynamic one get the thermodynamic potential F called the *Helmholtz free energy*:

$$F = -k_B T \ln Z \tag{2.3}$$

The Helmholtz free energy can be compared to the entropy S which for a closed adiabatic system attains it maximum when the system is at equilibrium. For a system in contact to a heath bath the Helmholtz free energy attains its minimum at equilibrium. This is a powerful property of the free energy and it is regarded as state function [6].

2.3 Magnetization

The magnetization M is a macroscopic quantity which describes the alignment of particles magnetic moment $\vec{\mu}$ in a certain direction. The mean magnetization for a system of N

particles labeled i becomes then

$$\langle M \rangle = \mu \sum_{i}^{N} \vec{s}_i \tag{2.4}$$

where $\vec{s}_i$ is the direction of the magnetic moment for a particle i. When a system's magnetic moment align such that the mean magnetization is of macroscopic size the systems is said to be *ferromagnetic* [7].

One can introduce a constraint to the system in the form of a magnetic field $\vec{H}$, then the interaction energy between the magnetic moment of particle and the magnetic field is:

$$E = -\vec{\mu} \cdot \vec{H} \tag{2.5}$$

Moreover the magnetization M can be calculated from the Helmholtz free energy by:

$$M = -\frac{\partial F}{\partial H} \tag{2.6}$$

This demonstrates the need for the free energy when considering mainly solids and the study of their magnetization.

2.4 Phase transitions and critical points

A system is said to be in a thermodynamic phase when considered physical properties change continuously, from that a phase transition is then associated with a discontinuous change of the physical property [6].

Consider a ferromagnet magnetized by an magnetic field H, for $0 \ll H$ a large magnetization M will be induced in the directions of the external field. But if we reverse the field $0 \gg H$ a large but opposite magnetization M will be induced. Further we note if we let the magnetic field decrease to 0^+ or 0^- we will still measure a non-zero *spontaneous magnetization* M_0 with directions dependenent on the previous field..

This implies a discontinuity around $H = 0$ as shown in fig. (1.a). As we discussed above this can be regarded as a phase transition.

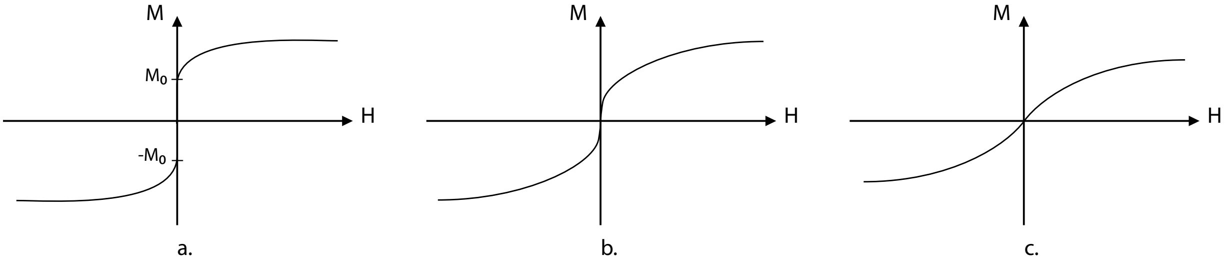


Figure 1: *Graphs of M(H, T) for (a)* $T < T_c$*, (b)* $T = T_c$*, (c)* $T > T_c$*. The pictures is reconstruction from* [8, P. 2].

Moreover M is found to be dependent on the temperature, increases in T will result in a lower M_0. For a critical temperature T_c we have that M_0 vanishes fig. (1.b) and $M(H, T)$ is continuous function with an infinite slope at $H = 0$. For temprature $T > T_c$ we find that $M(H, T)$ becomes analytical everywhere fig. (1.c) [8].

3 The General Ising Model

In this chapter we will cover an introduction to the General Ising model and its main assumptions. Will determine the form of the Hamiltonian, establish a connection to ferromagnetism and discuss the concept of exactly solvable models.

3.1 Hamiltonian of the Ising model

The Ising model is a discrete mathematical description of particles, where the particle's magnetic moment is independent and fixed to lattice configuration of a finite number of sites. We restrict the magnetic moment for all particles to the same direction and allow they be parallel or antiparallel. Let the total number sites to be N, labeled $i = \{1, 2, 3, ...\}$ and to each site assign a spin variable $\sigma_i = \pm 1$ indicating "upwards" (+1) or "downwards" (-1) magnetic moment [8].

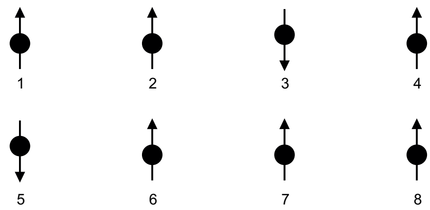


Figure 2: *Example of a 2D lattice configuration, with arrows indicating directions of magnetic moment*

Each spin configuration corresponds to one microstate where then the Hamiltonian of the system depends on the total spin configuration. The spin configuration σ is the set of the direction for the spins for all sites

$$\sigma = \{\sigma_1, ..., \sigma_N\} \tag{3.1}$$

then we have in total 2^N number of possible spin configurations. The Hamiltonian is assumed to be

$$E(\sigma) = E_0(\sigma) + E_1(\sigma) \tag{3.2}$$

with E_0 the energy contribution from intermolecular interactions between sites and if the system is under the presence of an external magnetic field then E_1 the energy contribution from spin interaction with the external field. Lastly we note that flipping the direction of all magnetic moments i.e. negating the spin configuration must clearly not affect intermolecular energy, giving us:

$$E_0(\sigma) = E_0(-\sigma) \tag{3.3}$$

For the intermolecular interaction we assume that only near-neighbouring sites interacts. Meaning of two sites with spin variables σ_i and σ_j is aligned ($\pm$) and ($\pm$) we have a energy contribution and if they are antiparallel ($\pm$) and ($\mp$) they cancell each other and give no energy contribution. Let then

$$E_0(\sigma) = -J \sum_{(i,j)} \sigma_i \sigma_j \tag{3.4}$$

where J is the interactions constant with unit of energy, this is called the *nearest-Neighbour Ising Model*. From eq. (3.4) we note that the lowest energy state occurs when all spins are aligned, meaning the model "favours" alignment. Meaning it can demonstrate a spontaneous magnetization i.e. it is a ferromagnetic model.

Lastly from eq. (2.4) we get the energy from particle interaction with the field to

$$E_1(\sigma) = -H \sum_{i} \sigma_i \tag{3.5}$$

where H is the relation between the direction of field and the magnetic moment.

3.2 Exactly solvable models

Exactly solvable models is often but not necessary models describing completely *integrable systems*, informally meaning that quantities in our case the free energy can be expressed in a integral form. In the field of statistical mechanics the concept of exactly solved models is more or less synonyms to quantum integrable models where it is possible by using *Yang baxter equation* to establish *Transfer matrices* from which one can express the Hamiltonian of the system [9].

4 The One-dimensional Ising Model

In this chapter we will go through the transfer matrix method of solving the one-dimensional Ising model. The method of finding the partition function will later be repurposed when calculating the partition function for the Square-lattice Ising model. Lastly the mean magnetization can easily and directly be determined from the free energy and by examining its analyticity we will find that there can not exist a phase transition for this model.

4.1 The Partition Function

Consider a one-dimensional lattice configuration of N sites, we label each site with a integer j and we assume periodic boundary conditions such that $\sigma_1 = \sigma_{N+1}$. This can be thought of as connecting the end point to form a circle as following: This can be thought of as connecting the end point to form a circle as following:

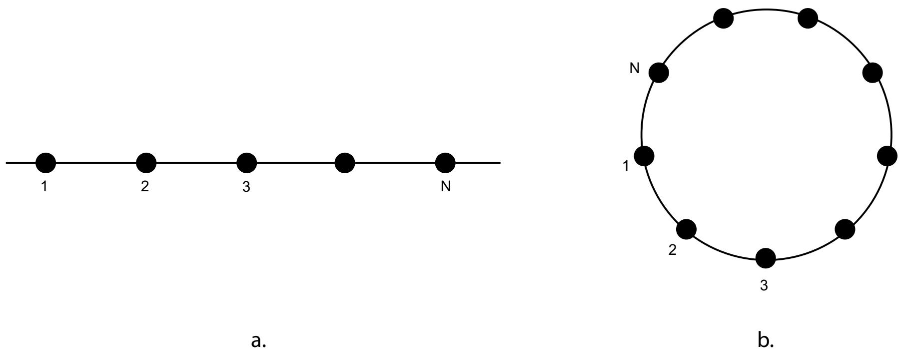


Figure 3: *Solid lines denotes interaction. (a) One dimensional lattice configuration, (b) one dimensional lattice configuration with periodic boundary condition*

The periodic boundary conditions except from simplifying the solutions of the model, is the only valid finite version of the system we can consider as closed i.e. no source terms in the end-points.

Furthermore we assume a nearest-neighbour interaction, we assign a spin variable σ_j to each site from eq. (3.4) the interaction energy is then

$$E_0 = -J \sum_{j}^{N} \sigma_j \sigma_{j+1} \tag{4.1}$$

and we allow the system to be subject to an external field. From eq. (3.4) the field interaction energy becomes then:

$$E_1 = -H \sum_{j}^{N} \sigma_j \tag{4.2}$$

Lastly from eq. (3.2) the energy for each spin-configuration becomes

$$E(\sigma) \;=\; -J\sum_{j}^{N}\sigma_j\sigma_{j+1} - H\sum_{j}^{N}\sigma_j \tag{4.3}$$

and the partition function Z_N follows from eq. (2.2) as

$$Z_N \;=\; \sum_{\sigma}\exp\Big[k\sum_{j}^{N}\sigma_j\sigma_{j+1} + h\sum_{j}^{N}\sigma_j\Big] \tag{4.4}$$

with $k = \beta j$ and $h = \beta H$ and where $\sum_\sigma$ is the summation over all possible spin-configurations.

4.2 The Transfer Matrix V

We want now to rewrite the partition function in a way so we can drop all the summations from eq. (4.4) and write the parition functions on a form that the free energy can be calculated. We do this by firstly expanding the sum in the Boltzmann weight as

$$Z_N = \sum_{\sigma}\exp\big[k(\sigma_1\sigma_2 + \sigma_2\sigma_3 + ...) \;+\; h(\sigma_1 + \sigma_2 + ...)\big] \;=$$

and we split the h-terms

$$= \sum_{\sigma}\exp\Big[k\sigma_1\sigma_2 + k\sigma_2\sigma_3 + ... + \frac{h}{2}\sigma_1 + \frac{h}{2}\sigma_1 + \frac{h}{2}\sigma_2 + \frac{h}{2}\sigma_2 + ...)\Big] \;=$$

and express the partition function as a product of Boltzmann weight for just the near-neighbouring interactions:

$$= \sum_{\sigma}\exp\Big[k\sigma_1\sigma_2 + \frac{h}{2}(\sigma_1 + \sigma_2)\Big] \;\cdot\; ... \;\cdot\; \exp\Big[k\sigma_N\sigma_1 + \frac{h}{2}(\sigma_N + \sigma_1)\Big] \tag{4.5}$$

In the last factor we used the periodic boundary condition. If we define the function $V(\sigma, \sigma')$ such as

$$V(\sigma, \sigma') \;=\; \exp\big[k\sigma\sigma' + \frac{h}{2}(\sigma + \sigma')\big] \tag{4.6}$$

using $V(\sigma, \sigma')$ we get the partition function to

$$Z_N \;=\; \sum_{\sigma_1}\sum_{\sigma_2}...\sum_{\sigma_N}V(\sigma_1, \sigma_2)V(\sigma_2, \sigma_3)\cdot ... \cdot V(\sigma_N, \sigma_1) \tag{4.7}$$

where each summation is over the possible spin state of each spin variable [8].

Consider a 2 by 2 matrix V such that all possible spin configurations of $V(\sigma, \sigma')$ is represented in the elements, such that V is given by

$$V \;= \begin{pmatrix} V(+, +) & V(+, -) \\ V(-, +) & V(-, -) \end{pmatrix} = \begin{pmatrix} e^{k+h} & e^{-k} \\ e^{-k} & e^{k-h} \end{pmatrix} \tag{4.8}$$

and if we then return to our partition function. Each factor $V(\sigma_j,\, \sigma_{j+1})V(\sigma_{j+2},\, \sigma_{j+3})$ when summed over respective spin variable can be regarded as the matrix product of V. We consider

$$Z_N = \sum_{\sigma_1} \sum_{\sigma_2} ... \sum_{\sigma_N} V(\sigma_1,\, \sigma_2) \,\cdot ... \cdot\, V(\sigma_N,\, \sigma_1) \,=$$

and from the definition of the matrix product we can regard this as the matrix V^N up the summation of σ_1

$$= \sum_{\sigma_1} V(\sigma_1,\, \sigma_1)^N \tag{4.9}$$

and from the definition of the trace of matrix, the eq. (4.9) must becomes $\mathrm{Tr}(V^N)$ giving us the partition function as:

$$Z_N \,=\, \mathrm{Tr}(V^N) \tag{4.10}$$

This is a important result, leaving the partition dependent only on the interactions constants.

With V being a real and symmetric matrix it can be diagonalized as

$$V \,=\, PDP^{-1} \tag{4.11}$$

where matrix D is a diagonal matrix with the eigenvalues λ_1 and λ_2 of V in its diagonal

$$D \,= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{4.12}$$

and P a matrix containing the eigenvectors x_1 and x_2 of V

$$P \,=(\vec{x}_1,\, \vec{x}_2) \tag{4.13}$$

We insert eq. (4.11) into eq. (5.67), we get the power of V to

$$V^N \,=\, PDP^{-1}PDP^{-1} \cdot ... \cdot PDP^{-1} \,=\, PD^N P^{-1} \tag{4.14}$$

and use the property of the matrix trace:

$$\mathrm{Tr}(PD^N P^{-1}) \,=\, \mathrm{Tr}(D^N) \tag{4.15}$$

Then clearly we have that the partition function can be written as:

$$Z_N \,=\, \mathrm{Tr}(D^N) \,=\, \lambda_1^N + \lambda_2^N \tag{4.16}$$

4.3 Free Energy and Phase Transition

The last step is to calculate the the Helmholtz free energy. From eq. (2.3) we have

$$-(k_BT)^{-1}F \;=\; \ln(Z_N) \;=\; \ln(\lambda_1^N + \lambda_2^N) \;=\; \ln\left[\lambda_1^N\left(1 + \frac{\lambda_2^N}{\lambda_1^N}\right)\right]$$

and we introduce the *free energy per lattice site* $f = F/N$. With the free energy F related to the size of the system we want a quantity in the thermodynamic limit that is not proportional to N [8]. We have

$$-(k_BT)^{-1}N^{-1}F \;=\; \ln(\lambda_1) + N^{-1}\ln\left[1 + \left(\frac{\lambda_2}{\lambda_1}\right)^N\right] \tag{4.17}$$

we assume $\lambda_1 > \lambda_2$. In the thermodynamic limit we expect $N \to \infty$ then the last term in the RHS vanish

$$\lim_{N\to\infty} N^{-1}\ln\left[1 + \left(\frac{\lambda_2}{\lambda_1}\right)^N\right] \;=\; 0$$

giving us the free energy per lattice site to:

$$f \;=\; -k_BT\ln(\lambda_1) \tag{4.18}$$

The last step is to calculate the eigenvalues which can be obtained with the characteristic equation

$$0 \;=\; \det(V-\lambda) \;=\; \begin{vmatrix} e^{k+h}-\lambda & e^{-k} \\ e^{-k} & e^{k-h}-\lambda \end{vmatrix} \;=\; (e^{k+h}-\lambda)(e^{k-h}-\lambda) - e^{-2k}$$

the eigenvalues becomes then

$$\lambda_\pm \;=\; \frac{1}{2}e^{-2k}(e^{3k+h} + e^{3k-h} \pm (e^{2(3k+h)} + e^{2(3k-h)} - 2e^{6k} + 4e^{2k})^{1/2}) \tag{4.19}$$

using hyperbolic functions gives us the final form of the eigenvalues:

$$\lambda_\pm \;=\; e^k\cosh h \pm (e^{2k}\sinh^2 h + e^{-2k})^{1/2} \tag{4.20}$$

For given interactions constants we get the free energy per lattice site to a functions of the the temperature T and h.

$$f(h,T) \;=\; -k_BT\left[e^k\cosh h + (e^{2k}\sinh^2 h + e^{-2k})^{1/2}\right] \tag{4.21}$$

With equation eq. (2.6) we get the magnetization

$$M(H,\,T) \;=\; \frac{e^k\sinh\beta H}{(e^{2k}\sinh^2\beta H + e^{-2k})^{1/2}} \tag{4.22}$$

we have that $M(H,\,T)$ is an analytical function for for real H and positive T. Meaning no phase transitions for positive temperatures.

5 The Square-lattice Ising Model

In this chapter we will cover the solutions of the Square-lattice model, the partition function is determined in terms of the transfer matrices V and W. In the following chapters will interpret Baxter Rodney's solutions method of the *commuting transfer matrices* [8].

With the *Star-triangle transformation* a commutation relations of the transfer matrices is found. From commutation of the transfer matrices and the operators R, C and together with Kramers-Wannier duality a eigenvalues relation is determined. Which lastly is parametrized with elliptic function and the free energy calculated. The existence of a phase transition and the critical temperature is found with Kramers-Wannier duality and analysing singularities in the free energy.

5.1 Partition function Square-lattice Ising model

Consider a square-lattice configuration of N sites under the presence of no external field. We impose periodic boundary conditions such that end-site of each row and column is connected, the lattice configuration can be thought of as the surface of a toroidal.

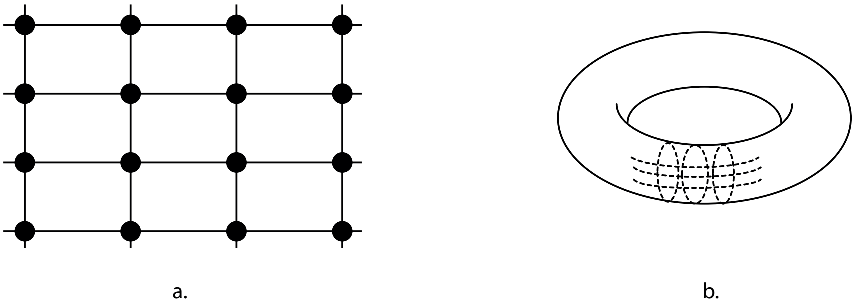


Figure 4: *(a) Square lattice configuration, (b) square lattice configuration with periodic boundary condition*

Furthermore we assume a nearest-neighbour interaction between the sites, where we allow anisotropy in the interactions i.e. the interaction constant between horizontal oriented sites is J and between vertical oriented sites it is J'. This results in four interactions for one site as following:

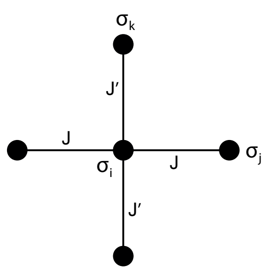


Figure 5: *Nearest-neighbour interaction in one site*

We formulate the systems energy as a pairwise summation of the interactions terms for all sites in horizontal and vertical orientation

$$E(\sigma) = -J \sum_{(i,j)} \sigma_i \sigma_j - J' \sum_{(i,k)} \sigma_i \sigma_k \tag{5.1}$$

and the partition function from eq. (2.2).

$$Z_N = \sum_{\sigma} \exp\left[K \sum_{(i,j)} \sigma_i \sigma_j + L \sum_{(i,k)} \sigma_i \sigma_k \right] \tag{5.2}$$

with $K = J\beta$ and $L = J'\beta$.

5.2 The Transfer Matrices V and W

The objective is to formulate a transfer matrix description of the Square-lattice model similar to the one-dimensional case. We do this by rotating our lattices as seen in fig. (6).

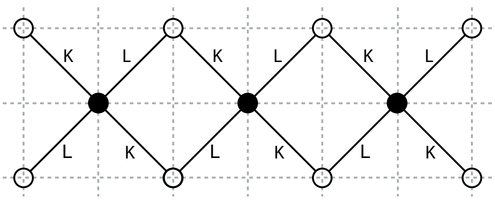


Figure 6: *Square lattice configuration with rotated orientation and different colors of the sites to easier distinguish different rows.*

Let the total number of rows be m, number if sites in each row n and denote the spin-configuration of each row ϕ_r were then $1 \leq r \leq m$ and $\phi_r = \{\sigma_1, ..., \sigma_n\}$.

We want to describe how rows interact with near-neighbouring rows, ahead when considering two rows let ϕ denote the lower row and ϕ' denote the upper row. We then return to fig. (6) and consider the lowest and the middle row. From fig. (7)

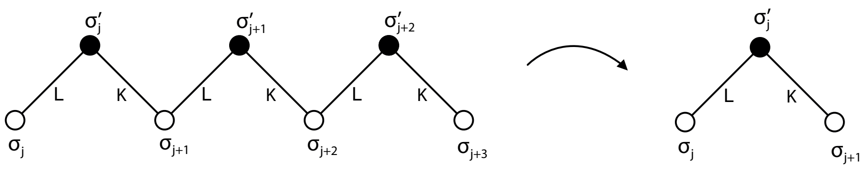


Figure 7: *The interaction between two rows is described as a repetition of the right side interactions*

we identify left-side pattern to be a repetition of the right side interaction, mainly that σ_j interacts with σ'_j and σ_{j+1} interacts with σ'_j. Introduce the Boltzmann weight describing the row to row interaction as the function $V(\phi,\,\phi')$ where:

$$V(\phi,\,\phi')\;=\;\exp\Big[\sum_{j=1}^{n}(K\sigma_{j+1}\sigma'_j+L\sigma_j\sigma'_j)\Big] \tag{5.3}$$

Further we consider the middle and highest row in fig.(6). We have from fig. (8)

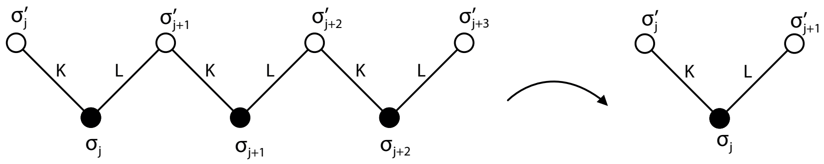


Figure 8: *The interaction between two rows is described as a repetition of the right side interactions, we note the the interaction share the middle row with fig. (7)*

that we make the a similar conclusion and introduce the Boltzmann weight in this case $W(\phi,\,\phi')$ where:

$$W(\phi,\,\phi')\;=\;\exp\Big[\sum_{j=1}^{n}(K\sigma_j\sigma'_j+L\sigma_j\sigma'_{j+1})\Big] \tag{5.4}$$

The strength of this formalism is that the partition function becomes a consecutive product of the functions $V(\phi,\,\phi')$ and $W(\phi,\,\phi')$ as

$$Z_N\;=\;\sum_{\phi_1}\sum_{\phi_2}...\sum_{\phi_m}V(\phi_1,\,\phi_2)W(\phi_2,\,\phi_3)\;\cdot ...\cdot\;V(\phi_{m-1},\,\phi_m)W(\phi_m,\,\phi_1) \tag{5.5}$$

where now sum for over the spin configuration of each row and we used the periodic boundary conditions in the last weight.

We introduce the matrices V and W with elements $V_{\phi,\phi'}$ and $W_{\phi,\phi'}$ such that each element is one possible spin configuration of $V(\phi,\,\phi')$ or $W(\phi,\,\phi')$. With 2^n possible spin configuration for a row we get V and W to be 2^n by 2^n matrices. This is our transfer matrices for the Square-lattice model, we can write the partition function as

$$\begin{aligned} Z_N &= \sum_{\phi_1}\sum_{\phi_2}...\sum_{\phi_m}V_{\phi_1,\phi_2}W_{\phi_2,\phi_3}\cdot ...\cdot W_{\phi_m,\phi_1}\;= \\ &= \sum_{\phi_1}(VWV\cdot ...\cdot W)_{\phi_1\phi_1}\;=\;\sum_{\phi_1}((VW)^{m/2})_{\phi_1,\phi_1}\;= \\ &= \;\mathrm{Tr}\,(VW)^{m/2} \end{aligned} \tag{5.6}$$

where we used the definition of the matrix product to rewrite the first summation and the definition of the trace of a matrix to rewrite the last summation.

Now the product VW is not symmetric, but in general we have that the trace of a matrix is the sum of its eigenvalues and from eq (2.2) we get the partition function

$$Z_N\;=\;\Lambda_1^m+\Lambda_2^m+...+\Lambda_{2^n}^m \tag{5.7}$$

where $\Lambda_1^2,\ ,\Lambda_2^2, ...$ is the eigenvalues of VW.

We want now to calculate the eigenvalues of the matrix VW, to do so we need to determine a set of properties of the product.

5.3 Star-Triangle Transformation and Commutation Relation

First let V and W instead be functions of the interactions terms as $V(K,\, L)$ and $W(K,\, L)$, this choice will become apparent in next chapters. Furthermore we want to allow a more general interaction, consider the product

$$V(K,\, L)\, W(K',\, L') \tag{5.8}$$

where $K,\, L,\, K'$ and L' is now any complex constants and denote the rows as following:

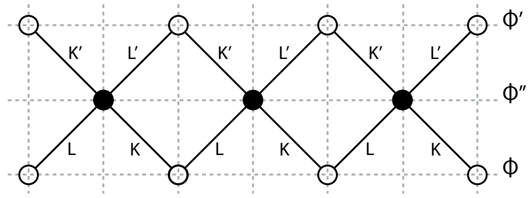


Figure 9: *Square lattice configuration with four interaction constants, with rows labeled ϕ, ϕ' and ϕ''*

The elements of the matrix product from eq.s (5.3) and (5.4) is given by

$$\begin{aligned}
(VW)_{\phi,\,\phi'} &= \sum_{\phi''} V_{\phi,\,\phi''} W_{\phi'',\,\phi'} = \\
&= \sum_{\phi''} \exp\Big[\sum_{j=1}^{n} (K\sigma_{j+1}\sigma_j'' + L\sigma_j\sigma_j'')\Big] \exp\Big[\sum_{j=1}^{n} (K'\sigma_j''\sigma_j' + L'\sigma_j''\sigma_{j+1}')\Big] = \\
&= \sum_{\phi''} \prod_{j=1}^{N} \exp\Big[\sigma_j''(K\sigma_{j+1} + L\sigma_j + K'\sigma_j' + L'\sigma_{j+1}')\Big]
\end{aligned} \tag{5.9}$$

Where the spin variable $\sigma_j'' = \pm 1$ will only for every factor change the overall sign of the exponents, it is useful to define

$$(VW)_{\phi,\,\phi'} = \prod_{j=1}^{N} X(\sigma_j,\, \sigma_{j+1};\, \sigma_j',\, \sigma_{j+1}') \tag{5.10}$$

with the function

$$\begin{aligned}
X(\sigma_j,\, \sigma_{j+1};\, \sigma_j',\, \sigma_{j+1}') &= \sum_{\sigma''} \exp\Big[(\sigma_j''(K\sigma_{j+1} + L\sigma_j + K'\sigma_j' + L'\sigma_{j+1}')\Big] \\
&= 2\cosh(K\sigma_{j+1} + L\sigma_j + K'\sigma_j' + L'\sigma_{j+1}')
\end{aligned} \tag{5.11}$$

where we expanded the sum and used the definition of hyperbolic function cosh x. More important we have that $X(\sigma_j,\, \sigma_{j+1};\, \sigma'_j,\, \sigma'_{j+1})$ describes a single star-interaction as seen in fig. (10).

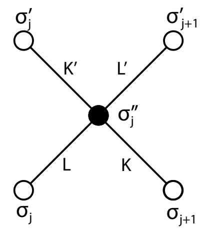


Figure 10: *Graphic representation of the interaction described by* $X(\sigma_j,\, \sigma_{j+1};\, \sigma'_j,\, \sigma'_{j+1})$

The main idea of this solution of the square-lattice Ising model is to establish that V and W both commute i.e. we ask whether V and W satisfy the equation:

$$V(K,\, L)\, W(K',\, L') \;=\; V(K',\, L')\, W(K,\, L) \tag{5.12}$$

To anwer this we need to examine a different lattice configuration, consider a three-vertex interaction as seen in fig. (11).

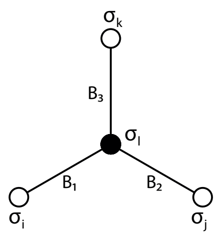


Figure 11: *Two dimensional lattice configuration with three-vertx*

We want to find a function that similar to $X(\sigma_j,\, \sigma_{j+1};\, \sigma'_j,\, \sigma'_{j+1})$ that describes the interaction, introduce the function w such as:

$$\begin{aligned} w(\sigma_i,\, \sigma_j,\, \sigma_k) = \sum_{\sigma_l} \exp\left[\sigma_l(B_1\sigma_i + B_2\sigma_j + B_3\sigma_k)\right] \;= \\ = 2\cosh(B_1\sigma_i + B_2\sigma_j + B_3\sigma_k) \end{aligned} \tag{5.13}$$

We note that $w(\sigma_i,\, \sigma_j,\, \sigma_k)$ firstly is unchanged if we negate all the spins. Also for eq. (5.13) there must exist constants R, C_1, C_2, C_3 such that

$$w(\sigma_i,\, \sigma_j,\, \sigma_k) \;=\; R\exp\left[C_1\sigma_i\sigma_j + C_2\sigma_j\sigma_k + C_3\sigma_k\sigma_i\right] \tag{5.14}$$

which is describing the system as a interaction in a triangle.

This is called the *star-triangle transformation.* It is graphically represented in fig. (12).

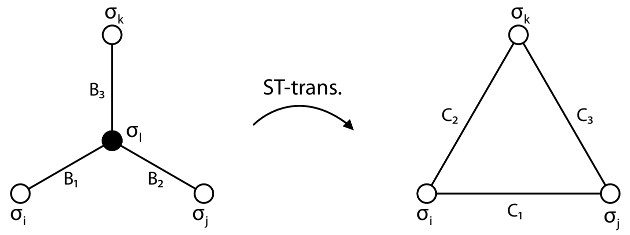


Figure 12: *Performing ST-transformation on a vertex.*

Now if we set eq. (5.13) equal to eq. (5.14) and evaluate for all possible spin configurations

$$w(+,+,+) = 2\cosh(B_1+B_2+B_3) = R\exp[C_1+C_2+C_3] \tag{5.15}$$

$$w(+,+,-) = 2\cosh(B_1+B_2-B_3) = R\exp[C_1+C_2-C_3] \tag{5.16}$$

$$w(-,+,+) = 2\cosh(-B_1+B_2+B_3) = R\exp[-C_1+C_2-C_3] \tag{5.17}$$

$$w(+,-,+) = 2\cosh(B_1-B_2+B_3) = R\exp[-C_1-C_2+C_3] \tag{5.18}$$

this the *Star-Triangle relations* [8]. They relate the interactions terms of different lattice structures, but for our case they is key to determine if V and W commute.

We return to the interaction in fig. (10) and we add a particle to particle interaction M such that we form the triangle which result in a interaction as in fig. (13),

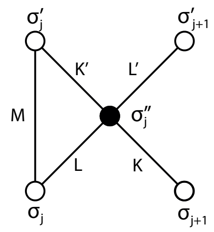


Figure 13: *Square lattice interaction with added interaction, we will use the triangle in the ST-relations*

which can be described by the function w_1 where:

$$\begin{aligned} w_1(\sigma_j,\,\sigma_{j+1};\,\sigma'_j,\,\sigma'_{j+1}) &= \sum_{\sigma''}\exp\left[M\sigma_j\sigma'_j + L\sigma_j\sigma''_j + K\sigma_{j+1}\sigma''_j + L'\sigma'_{j+1}\sigma''_j + K'\sigma'_j\sigma''_j\right] = \\ &= e^{M\sigma_j\sigma'_j}\sum_{\sigma''}\exp\left[\sigma''_j(L\sigma_j + K\sigma_{j+1} + L'\sigma'_{j+1} + K'\sigma'_j)\right] \end{aligned} \tag{5.19}$$

We note that the added interaction M is isolated in the term $e^{M\sigma_j\sigma'_j}$.

We want to investigate eq. (5.12), we interchange L and K with L' and K' and lastly also add the interaction M which result in a interaction as in fig. (14)

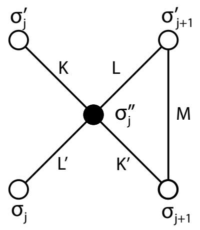


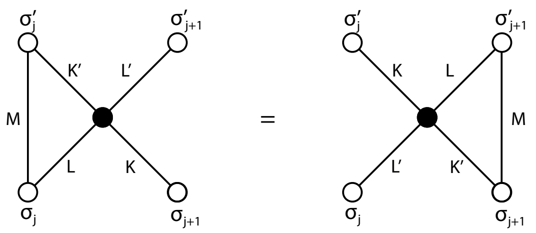
Figure 14: *Square lattice interaction with added interaction and interchanged interaction constants.*

which can be described by the function w_2 where:

$$w_2(\sigma_j,\, \sigma_{j+1};\, \sigma'_j,\, \sigma'_{j+1}) = \sum_{\sigma''_j} \exp\left[M\sigma_{j+1}\sigma'_{j+1} + L\sigma'_{j+1}\sigma''_j + K\sigma'_j 1\sigma''_j + L'\sigma_j\sigma''_j + K'\sigma_{j+1}\sigma''_j\right] =$$

$$= e^{M\sigma_{j+1}\sigma'_{j+1}} \sum_{\sigma''_j} \exp\left[\sigma''_j(L\sigma'_{j+1} + K\sigma_{j'} + L'\sigma_j + K'\sigma_{j+1})\right] \quad (5.20)$$

Now we ask whether if the exist M such that $w_1(\sigma_j,\, \sigma_{j+1};\, \sigma'_j,\, \sigma'_{j+1}) \;=\; w_2(\sigma_j,\, \sigma_{j+1};\, \sigma'_j,\, \sigma'_{j+1})$ mainly if we have



then we identify the interaction constants in each of the two triangles in fig. (5.3) and relate them to corresponding interactions constants in the ST-relations. This gives us that:

$$C_1 = L \quad C_2 = K' \quad C_3 = M \quad (5.21)$$

We can then perform the ST-transformation which will result in:

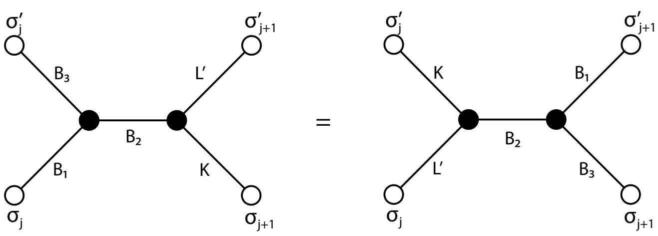


Figure 15: *ST-transformed triangles*

If we then compapare this to the ST-relations we identify that

$$B_1 = L' \quad B_3 = K \tag{5.22}$$

We insert (5.21) and (5.22) in the star-triangle relations and get the following:

$$
\begin{aligned}
2\cosh(L' + B_2 + K) &= R\exp[L + K' + M] && (5.23)\\
2\cosh(L' + B_2 - K) &= R\exp[L - K' - M] && (5.24)\\
2\cosh(-L' + B_2 + K) &= R\exp[-L + K' - M] && (5.25)\\
2\cosh(L' - B_2 + K) &= R\exp[-L - K' + M] && (5.26)
\end{aligned}
$$

We then add/subtract eq. (5.23) with eq. (5.26) and we get

$$
\begin{aligned}
4\cosh(B_2)\cosh(K + L') &= 2Re^{M}\cosh(L + K') && (5.27)\\
4\sinh(B_2)\sinh(K + L') &= 2Re^{M}\sinh(L + K') && (5.28)
\end{aligned}
$$

furthermore we add/subtract eq. (5.24) with eq. (5.25) giving us

$$
\begin{aligned}
4\cosh(B_2)\cosh(K - L') &= 2Re^{-M}\cosh(L - K') && (5.29)\\
4\sinh(B_2)\sinh(K - L') &= 2Re^{-M}\sinh(L - K') && (5.30)
\end{aligned}
$$

Lastly we divide eq. (5.27) with eq. (5.28) and eq. (5.30) with eq. (5.29) and multiply the result as

$$\frac{\cosh(K + L')\sinh(K - L')}{\cosh(K - L')\sinh(K + L')} = \frac{\cosh(L + K')\sinh(L - K')}{\sinh(L + K')\cosh(L - K')} \tag{5.31}$$

this simplifies to the relation

$$\sinh(2K)\sinh(2L) = \sinh(2K')\sinh(2L') \tag{5.32}$$

which do not depend on M, we can then set M to something more preferable like zero. Mainly if K, L, K' and L' satisfies eq. (5.32) we can interchange the interaction constants; giving us that the commutations relation (5.12) is true.

5.4 Inversion of the Transfer Matrices

Another important property is to express the matrix product in a spin variable free form, this is done by asking for the inverse of the matrix product. To be more specific we ask for a given K and L what choices of K' and L' leaves the product VW diagonal.

To ensure diagonality we firstly note that for elements not on the diagonal i.e. $\phi \neq \phi'$ we have

$$\sigma_j = \sigma'_j, \quad \sigma_{j+1} \neq \sigma'_{j+1} \qquad \text{or} \qquad \sigma_j \neq \sigma'_j, \quad \sigma_{j+1} = \sigma'_{j+1} \tag{5.33}$$

and then we must require $X = 0$. And for elements on the diagonal we have $\phi = \phi'$ and $\phi = -\phi'$ because negating elements in V and W before the matrix product leaves the product unchanged. We have

$$\sigma_j = \sigma_j', \quad \sigma_{j+1} = \sigma_{j+1}' \qquad \text{or} \qquad \sigma_j \neq \sigma_j', \quad \sigma_{j+1} \neq \sigma_{j+1}' \tag{5.34}$$

then we require $X \neq 0$.

Using requirement (5.33) with eq. (5.11) we have

$$\cosh(L + K - K' + L') = 0 \tag{5.35}$$

$$\cosh(L - K - K' - L') = 0 \tag{5.36}$$

where $\cosh(z) = 0$ have the solutions $z = \frac{1}{2}i\pi(2n+1)$, we get for $n = 0$ that the interactions constants satisfy

$$L + K - K' + L' = \frac{1}{2}i\pi$$
$$L - K - K' - L' = \frac{1}{2}i\pi$$

solving for K' and L' gives

$$K' = L + \frac{1}{2}i\pi, \quad L' = -K \tag{5.37}$$

where this is the choice of K' and L' that leaves the product diagonal.

Using requirement (5.34) we have the two cases, firstly when the pairs are alike given by X_{like} as

$$\begin{aligned} X_{like} &= X_{like}(\sigma_j = \sigma_j'; \sigma_{j+1} = \sigma_{j+1}') \\ &= 2\cosh\left(\sigma_j(L+K') + \sigma_{j+1}(K+L')\right) = \\ &= \cosh(\sigma_j(L+K'))\cosh(\sigma_{j+1}(K+L')) + \sinh(\sigma_j(L+K'))\sinh(\sigma_{j+1}(K+L')) = \end{aligned}$$

inserting (5.37) we get

$$= 2\cosh\left(\sigma_j(2L + \frac{1}{2}i\pi)\right)$$

and dropping the σ_j because $\cosh Z$ is a even function, we get

$$X_{like} = 2i\sinh 2L \tag{5.38}$$

Secondly when the pairs are unlike given by X_{unlike}

$$\begin{aligned} X_{unlike} &= X_{unlike}(\sigma_j \neq \sigma_j'; \sigma_{j+1} \neq \sigma_{j+1}') \\ &= 2\cosh\left(\sigma_j(L-K') + \sigma_{j+1}(K-L')\right) = \\ &= \cosh(\sigma_j(L-K'))\cosh(\sigma_{j+1}(K+L')) + \sinh(\sigma_j(L-K'))\sinh(\sigma_{j+1}(K-L')) = \end{aligned}$$

inserting (5.37) we get

$$= -\sinh(\sigma_j \frac{1}{2} i\pi) \sinh\left(\sigma_{j+1} 2K\right)$$

we have sinh Z is an uneven function and breaking out σ_j and σ_{j+1} gives us the change in sign, we get

$$X_{unlike} = -2i\sigma_j\sigma_{j+1} \sinh 2K \tag{5.39}$$

Using these results the matrix elements becomes of VW becomes

$$(2i \sinh 2L)^n \delta(\sigma_1, \sigma_1') \ldots \delta(\sigma_n, \sigma_n') + (-2i \sinh 2K)^n \delta(\sigma_1, -\sigma_1') \ldots \delta(\sigma_n, -\sigma_n') \tag{5.40}$$

where the factors $\delta(\sigma_1, \sigma_1') \ldots \delta(\sigma_n, \sigma_n')$ ensures that the term vanish if all the pairs is not alike and the factors $\delta(\sigma_1, -\sigma_1') \ldots \delta(\sigma_n, -\sigma_n')$ ensures that the term vanish if all the pairs is not unlike. We can write this in matrix form

$$V(K, L)W(L + \frac{1}{2} i\pi, -K) = (2i \sinh 2L)^n I + (-2i \sinh 2K)^n R \tag{5.41}$$

where we used eq. (5.37) and I is the 2^n by 2^n identity matrix and R is the 2^n by 2^n matrix with elements

$$R_{\phi\,\phi'} = \delta(\sigma_1, -\sigma_1') \ldots \delta(\sigma_n, -\sigma_n') \tag{5.42}$$

Eq. (5.41) is very useful relations and the RHS of eq. (5.41) as desired ensures a inverse to the matrix product.

5.5 The Negation Operator R

We want to further discuss the matrix R. This is a useful operator which shifts the sign of the interaction terms, we can see this if we consider $V(K, L)R$ where the matrix elements is given by:

$$\begin{aligned}\big(V(K, L)R\big)_{\phi, \phi'} &= \sum_{\phi''} V_{\phi, \phi''} R_{\phi'', \phi} = \\ &= \sum_{\phi''} \exp\Big[\sum_{j=1}^{n} (K\sigma_{j+1}\sigma_j'' + L\sigma_j\sigma_j'')\Big] \delta(\sigma_1'', -\sigma_1') \ldots \delta(\sigma_n'' - \sigma_n') =\end{aligned}$$

The only non-vanishing terms is the when $\sigma_j'' = -\sigma_j'$, we get

$$= \exp\Big[\sum_{j=1}^{n} (-K\sigma_{j+1}\sigma_j' - L\sigma_j\sigma_j')\Big]$$

this gives us that

$$V(-K, -L) = RV(K, L) = V(K, L)R \tag{5.43}$$

and similarly can be shown for $W(K, L)$. Also we have the elements of the inverse operator R^{-1}

$$R_{\phi, \phi'} = \delta(-\sigma_1, \sigma_1') \ldots \delta(-\sigma_n, \sigma_n') \tag{5.44}$$

clearly the relation

$$V(K,\, L) \;=\; R^{-1} V(K,\, L) R \tag{5.45}$$

holds. Again similarly can be shown for $W(K,\, L)$. The significance of eq. (5.45) is that $V(K,\, L)$ and $W(K,\, L)$ then both commutes with R.

5.6 The Shift Operator C

We want to develop further relations for how V and W changes under alteration of the interactions terms. Consider that for $V(K,\, L)$ we interchange σ_j, σ_j' and K, L.

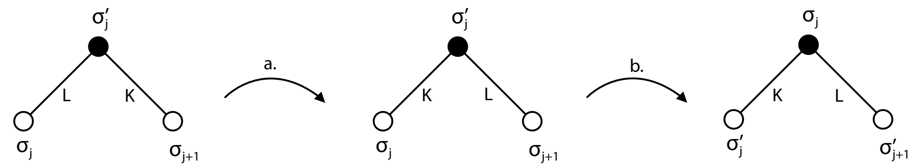


Figure 16: *Interaction described by $V(K,\, L)$, (a) interchange K and L, (b) interchange σ_j and σ_j'. This result in a interaction described by $W(K, L)$*

By interchanging σ_j, σ_j' is the same as switching the indices $V_{\phi'\phi}$ i.e. transpose the matrix V and as we can se in fig. (16) by also switching K and L we result in $W(K,\, L)$. This represent:

$$W(K,\, L) \;=\; V^T(L,\, K) \tag{5.46}$$

And by performing the same interchange for $V(K,\, L)W(K,\, L)$.

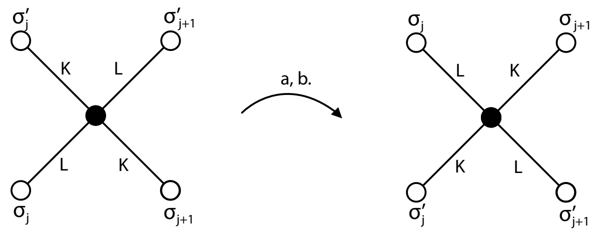


Figure 17: *Interaction described by $V(K,\, L)W(K,\, L)$, (a) interchange K and L, (b) interchange σ_j and σ_j'. Leaves the interaction unchanged.*

From fig. (17) together with previous reasoning we have

$$V(K,\, L)W(K,\, L) \;=\; \big[V(L,\, K)W(L,\, K)\big]^T \tag{5.47}$$

or to be more clear this transformation leaves the interaction unchanged.

We want to represent these results as a operator by introducing the 2^n by 2^n matrix operator C with elements:

$$C_{\phi,\, \phi'} \;=\; \delta(\sigma_1,\, \sigma_2') \ldots \delta(\sigma_n,\, \sigma_1') \tag{5.48}$$

The operator C shift the spin label one column [8]. We consider $V(K,\, L)C$ where the elements is given by:

$$\big(V(K,\, L)C\big)_{\phi,\,\phi'} \,=\, \sum_{\phi''} V_{\phi,\,\phi''}C_{\phi'',\,\phi'} \,= \tag{5.49}$$

$$= \sum_{\phi''} \exp\Big[\sum_{j=1}^{n}(K\sigma_{j+1}\sigma''_j + L\sigma_j\sigma''_j)\Big]\delta(\sigma''_1,\, \sigma'_2)\,...\,\delta(\sigma''_n,\, \sigma'_1) \,=$$

we have that the only non-vanishing terms is when $\sigma''_j = \sigma'_{j+1}$, we get

$$= \exp\Big[\sum_{j=1}^{n}(K\sigma_{j+1}\sigma'_{j+1} + L\sigma_j\sigma'_{j+1})\Big] \tag{5.50}$$

This can be graphically represented as seen in fig. (18)

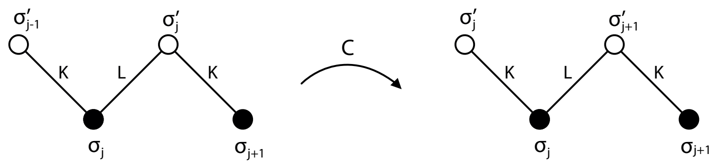


Figure 18: *Transformation with the operator C, which results in the interaction $W(K,\, L)$.*

meaning shifting the spin variables gives us the relation:

$$W(K,\, L) \,=\, V(K,\, L)C \tag{5.51}$$

Furthermore we also have the inverse operator C^{-1} with elements given by

$$C^{-1}_{\phi,\,\phi'} \,=\, \delta(\sigma_2,\, \sigma'_1)\,...\,\delta(\sigma_1,\, \sigma'_n) \tag{5.52}$$

and if we consider $C^{-1}V(K,\, L)C$ with elements is given by

$$\big(C^{-1}V(K,\, L)C\big)_{\phi,\,\phi'} \,=\, \sum_{\phi''}\sum_{\phi'''} C^{-1}_{\phi,\,\phi''}V_{\phi'',\,\phi'''}C_{\phi''',\,\phi'} \,=$$

$$= \sum_{\phi''}\sum_{\phi'''} \delta(\sigma_2,\, \sigma''_1)\,...\,\delta(\sigma_1,\, \sigma''_n)\exp\Big[\sum_{j=1}^{n}(K\sigma''_{j+1}\sigma'''_j + L\sigma''_j\sigma'''_j)\Big]\cdot$$

$$\cdot\,\delta(\sigma'''_1,\, \sigma'_2)\,...\,\delta(\sigma'''_n,\, \sigma'_1)$$

we have that the only surviving terms is when $\sigma''_j = \sigma_{j+1}$ and $\sigma'''_j = \sigma'_{j+1}$ we get

$$= \exp\Big[\sum_{j=1}^{n}(K\sigma_{j+2}\sigma'_{j+1} + L\sigma_{j+1}\sigma'_{j+1})\Big] \,=\, \exp\Big[\sum_{j=1}^{n}(K\sigma_{j+1}\sigma'_j + L\sigma_j\sigma'_j)\Big]$$

not surprisingly $V(K,\, L)$ remains unchanged. In other words we have established

$$V(K,\, L) \,=\, C^{-1}V(K,\, L)C \tag{5.53}$$

and similar can be said for $W(K,\, L)$. We have the V and W also commute with C.

From this result we can simplify the matrix product VW, we have as long as K, L and K' and L' satisfies relation (5.32)

$$\sinh 2K \sinh 2K \;=\; \sinh 2K' \sinh 2L$$

we have then that eq. (5.12):

$$V(K,\, L)\, W(K',\, L') = \; V(K',\, L')\, W(K,\, L)$$

With eq. (5.51) this can be written as

$$V(K,\, L)\, V(K',\, L') = \; V(K',\, L')\, V(K,\, L) \tag{5.54}$$

The same can be said for $W(K,\, L)$. Eq. (5.54) states that V and W commute with them self. Furthermore from eq. (5.41) we have then

$$V(K,\, L)V(L + \frac{1}{2}i\pi,\, -K)C \;=\; (2i\sinh 2L)^n I + (-2i\sinh 2K)^n R \tag{5.55}$$

and again similarly for $W(K,\, L)$.

5.7 Alternate Form of the Transfer Matrices

At this point it is unnecessary to continue consider the sum over the spin variables in $V(K,\, L)$. We are only interested in the overall number of changes in sign and not the specific sign in each site. Let

$$r = \text{ number of unlike pairs of } \;(\sigma_{j+1}, \sigma'_j) \tag{5.56}$$

$$s = \text{ number of unlike pairs of } \;(\sigma_j, \sigma'_j) \tag{5.57}$$

Now return to $V(K,\, L)$ and expand the sum

$$\exp\Big[\sum_{j=1}^{n}(K\sigma_{j+1}\sigma'_j + L\sigma_j\sigma'_j)\Big] \;=\; \exp\big[K(\sigma_2\sigma'_1 + \sigma_3\sigma'_2 + ...) + L(\sigma_1\sigma'_1 + \sigma_2\sigma_2 + ...)\big] \tag{5.58}$$

for every unlike pair $(\sigma_{j+1}, \sigma'_j)$ we will have contribution of a (-1) in the parenthesis canceling out a like pair, with n pair in total and r changes in sign we get $(n-2r)$ surviving K terms. Similarly for the unlike pairs (σ_j, σ'_j), given us $(n-2s)$ surviving L terms.

We have now the elements for the transfer matrix V can be expressed without the spin variables as

$$V(K,\, L) = \; \exp[(n-2r)K + (n-2s)L] \;= \tag{5.59}$$

in the thermodynamic limit we can assume without loss of generality n to be even. Let $n = 2p$, then

$$= \exp[2(p-r)K + 2(p-s)L]$$

this simplifies the result if we introduce r' and s' such that

$$V(K,\, L) \;=\; \exp[\pm 2r'K \pm 2s'L] \tag{5.60}$$

where r' and s' are positive integers. The (± 1) arises from that $\exp[2r'K+2s'L]$ represent two spin configurations where the other one is obtained by negating all spin variable. As discussed in Appendix A. both r and s must be even, this gives us that s' and r' is both be either even or odd. Then if we consider

$$\exp[2K]^{r'} \exp[2L]^{s'}$$

and negating

$$\big(-\exp[2K]\big)^{r'} \big(-\exp[2L]\big)^{s'}$$

will then leave the RHS of eq. (5.60) unchanged. This condition can be expressed as allowing $K = K \pm \frac{1}{2}\pi i$ which will yield $\exp[2K \pm \pi i] = -\exp[2K]$ and same for $\exp[2L]$. This gives us

$$V(K \pm \frac{1}{2}i\pi,\, L \pm \frac{1}{2}i\pi) \;=\; V(K,\, L) \tag{5.61}$$

and similarly for $W(K,\, L)$.

5.8 Functional Relation of the Eigenvalues

We have now all the necessary properties of the transfer matrices to calculate the eigenvalues of the matrix product VW. Firstly we define

$$k \;=\; \big(\sinh 2K \,\sinh 2L\big)^{-1} \tag{5.62}$$

where k is a real number. We have from Appendix A. that for $T = T_c$ then $k = 1$. We regard now the complex variables K and L constrained to a fixed k given by eq. (5.62). By variation of K and L we can generate transfer matrices $V(K,\, L)$.

Furthermore we have established the relations

$$\begin{array}{rl} V(K,\, L) & \text{commutes with itself} \\ V(K,\, L),\, C & \text{commute} \\ V(K,\, L),\, R & \text{commute} \end{array}$$

the same can be demonstrated for $W(K,\, L)$. For commuting matrices there must be a common set of eigenvectors [10]. Then all matrices $V(K,\, L)$ and $W(K,\, L)$ for which K and L satisfies (5.62) must share at least a common eigenvector x with C and R.

We have that x is a function of K and L but for every K and L we can assign a k, so we regard the eigenvector instead as a function $x(k)$. Further let $v(K, L)$, c, r be the corresponding eigenvalues of $V(K, L)$, C, R. We have then

$$\begin{aligned} V(K, L)x(k) &= v(K, L)x(K) \\ Cx(k) &= cx(k) \\ Rx(k) &= rx(k) \end{aligned} \tag{5.63}$$

If we consider C^n which corresponds to shifting a spin label σ_j with σ_{j+n} which due to the periodic boundary condition is $\sigma_j = \sigma_{j+n}$, we get $C^n = I$. Also we have R^2 which corresponds to negating a spin label twice which gives us $R^2 = I$. The eigenvalues of C^n and R^2 must therefore satisfy:

$$c^n = r^2 = 1 \tag{5.64}$$

We return the to eq. (5.55), multiplying both side with $x(k)$ then

$$V(K, L)V(L + \frac{1}{2}i\pi, -K)Cx(k) = (2i \sinh 2L)^n Ix(k) + (-2i \sinh 2K)^n rx(k)$$

we get then from the commutation relations

$$v(K, L)v(L + \frac{1}{2}i\pi, -K)c = (2i \sinh 2L)^n + (-2i \sinh 2K)^n r \tag{5.65}$$

We recall $V(K, L)W(K, L)$ have the eigenvalues Λ_1^2, $\Lambda_2^2, ..., \Lambda_{2^n}^2$. Using eq. (5.51), we write the matrix product as

$$V(K, L)W(K, L) = V(K, L)V(K, L)C = V^2(K, L)C \tag{5.66}$$

where

$$V^2(K, L)Cx(k) = v^2(K, L)cx(k) \tag{5.67}$$

the LHS of eq. (5.67) is our desired eigenvalues

$$\begin{aligned} \Lambda^2(K, L) &= v^2(K, L)c \\ \Lambda(K, L) &= v(K, L)c^{1/2} \end{aligned} \tag{5.68}$$

inserting eq. (5.68) to eq. (5.65) gives us the eigenvalues relation:

$$\Lambda(K, L)\Lambda(L + \frac{1}{2}i\pi, -K) = (2i \sinh 2L)^n + (-2i \sinh 2K)^n r \tag{5.69}$$

Now with k given by eq.s (5.62) and (5.69) we can not intuitively but through a straightforward parametrization calculate the eigenvalues of VW.

5.9 Parametrization of K and L

Firstly we consider the case $k < 1$ for reasons that soon will be apparent and return to eq. (5.60) the elements of V can be written in a exponential form

$$\exp[\pm 2r'K \pm 2s'L] \;=\; \exp[\pm 2K]^{r'} \exp[\pm 2L]^{s'} \tag{5.70}$$

For a given k we want to parametrize eq. (5.62) such that $\exp(\pm 2K)$ and $\exp(\pm 2L)$ is single-valued *meromorphic* function i.e. analytic over a given domain with exception for isolated singularities each of which is a pole. Introduce the parametrization

$$\sinh 2K = x \tag{5.71}$$

$$\sinh 2L = (kx)^{-1} \tag{5.72}$$

which satisfies eq. (5.62). We then solve for $\exp(2K)$ and $\exp(2L)$

$$\begin{aligned} 2K &= \operatorname{arcsinh} x \;=\; \ln(x + (x^2+1)^{1/2}) \Leftrightarrow \\ \Leftrightarrow \exp(2K) &= x + (x^2+1)^{1/2} \end{aligned} \tag{5.73}$$

and

$$\begin{aligned} 2L &= \operatorname{arcsinh}\,(kx)^{-1} \;=\; \ln((kx)^{-1} + ((kx)^{-1}+1)^{1/2}) \Leftrightarrow \\ \Leftrightarrow \exp(2L) &= (kx)^{-1}[1 + (1+(kx)^2)^{1/2}] \end{aligned} \tag{5.74}$$

but the terms $(x^2+1)^{1/2}$ and $(1+(kx)^2)^{1/2}$ make $\exp(2K)$ and $\exp(2L)$ not single-valued and for a complex number z then $w = \sqrt{z}$ is not meromorphic over the complex plane. There is no parametrization with elementary functions of a general k that makes $(x^2+1)^{1/2}$ and $(1+(kx)^2)^{1/2}$ perfect squares. However, such parametrization can be made with elliptic functions [8].

5.10 Parametrization with Elliptic Functions

A elliptic function is a *meromorphic function* i.e. a functions which is analytic on its domain except for isolated singularities which are poles. Moreover a elliptic functions defined for two variables and is periodic in two directions. Compared to a real function which is defined by the values in a intervall, the elliptic functions is defined by the values lying in a parallelogram. In our case a rectangle further called a *period-rectangle* [11].

Firstly we introduce the *Jacobi elliptic functions*

$$\begin{aligned} \mathrm{sn}\; u &= k^{-1/2} H(u)/\Theta(u) \\ \mathrm{cn}\; u &= (k'/k)^{1/2} H_1(u)/\Theta(u) \\ \mathrm{dn}\; u &= k'^{1/2} \Theta_1(u)/\Theta(u) \end{aligned} \tag{5.75}$$

where $\Theta(u)$, $\Theta_1(u)$, $H(u)$ and $H_1(u)$ is the theta functions [8, P. 456]. The Jacobi elliptic function have two important relations:

$$\mathrm{cn}^2\ u = 1 - \mathrm{sn}^2\ u \tag{5.76}$$
$$\mathrm{dn}^2\ u = 1 - k^2\,\mathrm{sn}\ u \tag{5.77}$$

From which if set the parametrization to

$$x = -i\,\mathrm{sn}(iu) \tag{5.78}$$

we can resolve the square roots such that the result is meromorphic. With eq.s (5.78), (5.76) and (5.77) we get $\exp(\pm 2K)$ and $\exp(\pm 2L)$

$$\begin{aligned} \exp(\pm 2K) &= \mp i\,\mathrm{sn}(iu) + (1-\mathrm{sn}^2(iu))^{1/2} && = \mp\,\mathrm{sn}(iu) + \mathrm{cn}(iu) \\ \exp(\pm 2L) &= (\mp ik)^{-1}(1 + (1-k^2\,\mathrm{sn}^(\,iu))^{1/2}) && = ik^{-1}(\mathrm{dn}(iu)\pm 1)/\,\mathrm{sn}(iu) \end{aligned} \tag{5.79}$$

Using eqs. (5.75) we can express $\exp(\pm 2K)$ and $\exp(\pm 2L)$ in terms H, H_1, Θ and Θ_1 which are function entire functions i.e. analytical everywhere [8]. Resulting in

$$\exp(\pm 2K) = [k'^{1/2}H_1(iu) \mp iH(iu)]/[k^{1/2}\Theta(iu)] \tag{5.80}$$
$$\exp(\pm 2L) = i[k'^{1/2}\Theta_1(iu) \pm \Theta(iu)]/[k^{1/2}H(iu)] \tag{5.81}$$

The fraction together with eq. (B.7) from Appendix B. gives us that $\exp(\pm 2K)$ and $\exp(\pm 2L)$ is meromorphic functions

The reason of only considering $k < 1$ is because as from Appendix B. we find the elliptic functions only to be defined for $0 < k < 1$. We will later return to case when $k = 1$ and $k > 1$.

5.11 The form of $\Lambda(u)$

Next step is to determine the form of eigenvalues in terms of the elliptic functions. We have from earlier the form of the matrix element of V given by

$$V(K,\,L) = \exp[\pm 2K]^{r'}\exp[\pm 2L]^{s'} =$$

using eq. (5.80), (5.81) and $r' = p - r$, $s' = p - s$ we get

$$= \frac{\cdots}{[k^{1/2}\Theta(iu)]^{p-r}[k^{1/2}H(u)]^{p-s}} = \tag{5.82}$$

currently we are only interested in the structure of the poles, let the numerator be ... which is just a entire function

$$= \frac{\ldots}{[\Theta(iu)H(iu)]^p\Theta^{-r}(iu)H^{-s}(iu)}$$

where we let the k factors be a part of $\ldots$, final step gives us

$$V(K,\, L) = \frac{\cdots}{[h(iu)]^p} \tag{5.83}$$

where $h(u) = H(u)\Theta(u)$ and now using the relations (5.63) and eq. (5.68) we get:

$$\Lambda(u) \;=\; \frac{\cdots}{[h(iu)]^p} \tag{5.84}$$

One advantage of the elliptic functions is their periodicity under their half amplitudes I and I'. We want to determine how eq.s (5.79) change if we set $u \to u + 2I'$. Using the peridocicity relations

$$\operatorname{sn}(u + 2iI') \;=\; \operatorname{sn}(u), \quad \operatorname{cn}(u + 2iI') \;=\; -\operatorname{cn}(u), \quad \operatorname{dn}(u + 2iI') \;=\; -\operatorname{dn}(u) \tag{5.85}$$

We get the RHS. of eq.s (5.79) to

$$\mp \operatorname{sn}(iu + 2iI') + \operatorname{cn}(iu + 2iI') \;=\; \mp \operatorname{sn}(iu) - \operatorname{cn}(iu)$$
$$ik^{-1}(\operatorname{dn}(iu + 2iI') \pm 1)/\operatorname{sn}(iu + 2iI') \;=\; ik^{-1}(\operatorname{dn}(iu) \pm 1)/\operatorname{sn}(iu)$$

Eq.s (5.79) is satisfied if $K \to -K \pm \frac{1}{2}i\pi$ and $L \to -L \pm \frac{1}{2}i\pi$, we have

$$\exp(\mp 2K + i\pi) \;=\; \mp \operatorname{sn}(iu) - \operatorname{cn}(iu)$$
$$\exp(\mp 2L + i\pi) \;=\; ik^{-1}(-\operatorname{dn}(iu) \pm 1)/\operatorname{sn}(iu)$$

meaning:

$$u \to u + 2I' \Leftrightarrow \begin{cases} K \to -K \pm \frac{1}{2}i\pi \\ L \to -L \pm \frac{1}{2}i\pi \end{cases} \tag{5.86}$$

We have then using eq. (5.61) and then (5.43)

$$V^2(-K \pm \frac{1}{2}i\pi,\, -L \pm \frac{1}{2}i\pi) \;=\; V^2(-K,\, -L) \;=\; R^2 V^2(K,\, L)$$

and taking the square-root and multiplying both sides with $x(k)$ give us:

$$\Lambda(-K \pm \frac{1}{2}i\pi,\, -L \pm \frac{1}{2}i\pi) \;=\; r\Lambda(K,\, L) \tag{5.87}$$

or expressed as a function of u

$$\Lambda(u + 2I') \;=\; r\Lambda(u) \tag{5.88}$$

where the eigenvalue r is then $r = \pm 1$. Furthermore we want to determine how eq.s (5.79) change if we set $u \to u - 2iI$. Using the relations

$$\operatorname{sn}(u + 2I) \;=\; \operatorname{sn}(u), \quad \operatorname{cn}(u + 2I) \;=\; -\operatorname{cn}(u), \quad \operatorname{dn}(u + 2I) \;=\; \operatorname{dn}(u) \tag{5.89}$$

We get the RHS. of eq.s (5.79) to

$$\begin{aligned} \mp \operatorname{sn}(iu + 2I') + \operatorname{cn}(iu + 2I') &= \pm \operatorname{sn}(iu) - \operatorname{cn}(iu) \\ ik^{-1}(\operatorname{dn}(iu + 2I') \pm 1)/\operatorname{sn}(iu + 2I') &= -ik^{-1}(\operatorname{dn}(iu) \pm 1)/\operatorname{sn}(iu) \end{aligned}$$

Eq.s (5.79) is satisfied if $K \to K \pm \frac{1}{2}i\pi$ and $L \to L \pm \frac{1}{2}i\pi$, we have

$$\begin{aligned} \exp(\pm 2K + i\pi) &= \pm \operatorname{sn}(iu) - \operatorname{cn}(iu)) \\ \exp(\pm 2L + i\pi) &= -ik^{-1}(\operatorname{dn}(iu) \pm 1)/\operatorname{sn}(iu) \end{aligned}$$

meaning:

$$u \to u - 2iI \Leftrightarrow \begin{cases} K \to K \pm \frac{1}{2}i\pi \\ L \to L \pm \frac{1}{2}i\pi \end{cases} \tag{5.90}$$

by a similar discussion as earlier, we get the $\Lambda(u)$ satisfying

$$\Lambda(u - 2iI) = \Lambda(u) \tag{5.91}$$

From eq.s (B.1) we have eq. (5.88) and (5.91) ensures that $\Lambda(u)$ is a *doubly periodic function* see Appendix B. together with theorem (C.3) the function $\Lambda(u)$ must be on the form

$$\Lambda(u) = \rho e^{-\lambda u} \prod_{j=1}^{n} [H(u - u_j)]/H(u - v_j)] \tag{5.92}$$

from eq. (5.84) we know the structure of the poles. We have

$$\prod_{j=1}^{2p} 1/H(u - v_j) = [h(iu)]^{-p} \tag{5.93}$$

letting the minus sign be a part of λ gives us.

$$\Lambda(u) = \rho e^{\lambda u}[h(iu)]^{-p} \prod_{j=1}^{2p} H(iu - iu_j) \tag{5.94}$$

where $u_1, ..., u_{2p}$ are the zeros of $\Lambda(u)$ within a period rectangle, ρ and λ ensure periodicity.

5.12 Zeros of $\Lambda(u)$

The last step is to determine the zeros of $\Lambda(u)$. We return to eq. (5.69):

$$\Lambda(K, L)\Lambda(L + \frac{1}{2}i\pi, -K) = (2i \sinh 2L)^n + (-2i \sinh 2K)^n r$$

We want to express the relation as a function of u, to do this we ask how eq.s (5.79) change for $u \to u + I$. Using the relations

$$\begin{aligned} \operatorname{sn}(u + iI') &= (k \operatorname{sn}(u))^{-1} \\ \operatorname{cn}(u + iI') &= -i \operatorname{dn}(u)/(k \operatorname{sn}(u)) \\ \operatorname{dn}(u + iI') &= -i \operatorname{cn}(u)/\operatorname{sn}(u) \end{aligned} \tag{5.95}$$

we get the RHS of eq.s (5.79) to

$$\begin{aligned} \mp \operatorname{sn}(iu+iI') + \operatorname{cn}(iu+iI') &= ik^{-1s}\big(-\operatorname{dn}(iu) \mp 1\big)/\operatorname{sn}(iu) \\ ik^{-1}\big(\operatorname{dn}(iu+iI' \pm 1)/\operatorname{sn}(iu+iI')\big) &= \pm i\operatorname{sn}(iu) + \operatorname{cn}(iu) \end{aligned} \tag{5.96}$$

and this is satisfied if $K \to L + \frac{1}{2}i\pi$ and $L \to -K$ such that

$$\begin{aligned} \exp(\pm L \pm i\pi) &= ik^{-1s}\big(-\operatorname{dn}(iu) \mp 1\big)/\operatorname{sn}(iu) \\ \exp(\mp K) &= \pm i\operatorname{sn}(iu) + \operatorname{cn}(iu) \end{aligned}$$

meaning that

$$u \to u + I' \Leftrightarrow \begin{cases} K \to L + \frac{1}{2}i\pi \\ L \to -K \end{cases} \tag{5.97}$$

the RHS of (5.69) can be written as $\Lambda(u)\Lambda(u+I')$. Moreover we have from eq.s (5.76), (5.77) and (5.78) we get the relation between K, L and u from:

$$\sinh 2K = -i\operatorname{sn}(iu) \tag{5.98}$$

$$\sinh 2L = \frac{i}{k\operatorname{sn}(iu)} \tag{5.99}$$

Inserting this in the eigenvalue relation we get:

$$\Lambda(u)\Lambda(u+I') = \Big(-\frac{2}{k\operatorname{sn}(iu)}\Big)^n + (-2\operatorname{sn}(iu))^n r \tag{5.100}$$

We now have the relation on a u dependent form, we can also rewrite $\Lambda(u)\Lambda(u+I')$.

Using $n = 2p$ and eq. (5.94) the LHS of eq. (5.100) becomes

$$\begin{aligned} \Lambda(u)\Lambda(u+I') &= \rho e^{\lambda u}[h(iu)]^{-p}\prod_{j=1}^{2p} H(iu-iu_j) \cdot \rho e^{\lambda(u+I')}[h(i(u+I'))]^{-p}\prod_{j=1}^{2p} H(iu-iu_j+iI') = \\ &= \rho^2 e^{\lambda(2u+I')}[h(iu)]^{-2p}\prod_{j=1}^{2p} H(iu-iu_j)H(iu-iu_j+iI') \end{aligned} \tag{5.101}$$

where we used from Appendix B. eq. (B.10) which gives us $h(i(u+I')) = h(iu)$. Furthermore the RHS of eq.(5.100) combined with eq (5.75) gives us:

$$\begin{aligned} \Big(-\frac{2}{k\operatorname{sn}(iu)}\Big)^n + (-2\operatorname{sn}(iu))^n r &= (2/k)^p\Big[\Big(\frac{\Theta(iu)}{H(iu)}\Big)^2 p + \Big(\frac{H(iu)r}{\Theta(iu)}\Big)^2 p\Big] = \\ &= (2/k)^p\Big(\Theta^{4p}(iu) + H^{4p}(iu)r\Big)(h(iu))^{-2p} \end{aligned} \tag{5.102}$$

These results give us the relation

$$\rho^2 e^{\lambda(2u+I')}\prod_{j=1}^{2p} H(iu-iu_j)H(iu-iu_j+iI') = (2/k)^p\Big(\Theta^{4p}(iu) + H^{4p}(iu)r\Big) \tag{5.103}$$

this gives us an overall restriction to the zeros of $\Lambda(u)$. We ask for what u_j and u the LHS and RHS vanish.

5.13 Zeros of the RHS

Firstly we investigate when the RHS of eq. (5.103)

$$(2/k)^p\Big(\Theta^{4p}(iu)+H^{4p}(iu)r\Big)$$

vanish. This occur when

$$\Big(-\frac{2}{k\,\mathrm{sn}(iu)}\Big)^n + (-2\,\mathrm{sn}(iu))^n r = 0$$

this can be simplified by

$$\begin{aligned} 1+(k\,\mathrm{sn}^2(iu))^n r &= 0 \\ r+(k\,\mathrm{sn}^2(iu))^n r^2 &= 0 \end{aligned}$$

where we used $r^2=1$ and $n=2p$ giving us

$$k\,\mathrm{sn}^2(iu))^{2p}+r = 0 \tag{5.104}$$

If we consider $\mathrm{sn}(u)$ we have from the relations (5.85) and (5.89) that it is doubly periodic with $2I$ and $2iI'$. This gives us that $\mathrm{sn}(u)$ have a period rectangle with width $2I$ and hieght $2iI'$. Moreover from eq. (5.75) $\mathrm{sn}(u)$ have a pole when $\Theta(u)=0$, from Appendix B. eq. (B.7) this occur when $u=2mI+i(2n-1)I'$ where n and m is any integers. We return to $(k\,\mathrm{sn}^2(iu))^{2p}+r$ there is then only one pole of order $4p$ per period rectangle. From theorem (C.2) we have that $(k\,\mathrm{sn}^2(iu))^{2p}+r$ have one zero per period rectangle. What is then the structure of u?

It is to our advantages to set

$$u = -\frac{1}{2}I'-i\phi \tag{5.105}$$

we get then that the zeros occur when:

$$(k\,\mathrm{sn}^2(\phi-\frac{1}{2}iI'))^n+r = 0 \tag{5.106}$$

From this we can use the *elliptic amplitude function*

$$\mathrm{Am}(\phi) = -\ln\big[ik^{1/2}\,\mathrm{sn}(\phi-\frac{1}{2}iI')\big] \tag{5.107}$$

with the above mentioned gives us that the zeros occur instead when:

$$\exp[4ip\,\mathrm{Am}(\phi)]+r = 0 \tag{5.108}$$

Eq. (5.108) have the solutions

$$\mathrm{Am}(\phi) = \begin{cases} \frac{1}{2p}\pi n+\frac{1}{4p}\pi & r=1 \\ \frac{1}{2p}\pi n & r=-1 \end{cases} \tag{5.109}$$

where n is an integer. We define

$$\theta_j = \begin{cases} \frac{1}{2p}\pi(j-\frac{1}{2}) & r=1 \\ \frac{1}{2p}\pi j & r=-1 \end{cases} \tag{5.110}$$

where $j = 1, ..., 2p$ we have that eq. (5.108) is satisfied by $\phi = \phi_j$ if

$$\mathrm{Am}(\phi) = \theta_j - \frac{1}{2}\pi \tag{5.111}$$

where we add $-\frac{1}{2}\pi$ for reason that soon will be clear.

Now the advantages of expressing the solutions in terms of the amplitudes function comes from its behaviour between $-I$ and I. We have from the taylor expansion of $\mathrm{Am}(\phi)$ [8]:

$$\mathrm{Am}(\phi) = \frac{\pi}{2I}\phi + 2\sum_{m=1}^{\infty}\frac{q^m/2}{m(1+q^m)}\sin\left(\frac{m\pi}{I}\phi\right) \tag{5.112}$$

Where q is from Appendix B. Most important must $\mathrm{Am}(\phi)$ increases monotonically from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$ as ϕ increases from $-I$ to I and we have:

$$\begin{cases} \frac{\pi}{4p} \leq \theta_j \leq \pi(1-\frac{1}{4p}) & r=1 \\ \frac{\pi}{4p} \leq \theta_j \leq \pi & r=-1 \end{cases} \tag{5.113}$$

Giving us $0 < \theta_j \leq \pi$, with the shift $-\frac{1}{2}\pi$ to θ_j we ensures a distinct and real solutions for every j with $-I < \phi \leq I$.

Lastly from eq. (5.95) we have by setting $u \to u + I'$

$$\begin{aligned} (k\,\mathrm{sn}^2(iu+iI'))^{2p} + r &= 0 \\ 1/(k\,\mathrm{sn}^2(iu))^{2p} + r &= 0 \end{aligned} \tag{5.114}$$

so from eq. (5.114) if u is a solutions so must then $u + I'$ also be a solution, we get

$$u = \mp\frac{1}{2}I' - i\phi_j \tag{5.115}$$

for $j = 1, \ldots, 2p$.

5.14 Zeros of the LHS

Lastly we investigate when the LHS of eq. (5.103)

$$\rho^2 e^{\lambda(2u+I')}\prod_{j=1}^{2p}H(iu-iu_j)H(iu-iu_j+iI')$$

vanish. From eq. (5.115) we have the zeros of $H(iu - iu_j)$ when u_j satisfy

$$u_j = \frac{1}{2}\gamma_j I' - i\phi_j$$

where $\gamma_j = \pm 1$ and $j = 1, ..., 2p$. However $H(iu - iu_j + iI')$ guarantees a solution when $u_j - I'$ giving us the general solutions to

$$u_j \; = \; -\frac{1}{2}\gamma_j I' - i\phi_j \tag{5.116}$$

but not all solutions of eq. (5.116) are allowed.

From the doubly periodicity of eq. (5.94) and restriction of the zeros in eq. (C.2) which can be written as

$$u_1 + ... + u_{2p} \; = \; (p + 2l')I' \; + \; i\big[\frac{1}{2}(1 - r) + 2l\big] \tag{5.117}$$

where l and l' are integers [8]. Then to ensure a double periodicity λ from eq. (C.3) is determined to:

$$\lambda \; = \; -\pi\gamma_j u/4I \tag{5.118}$$

Inserting to eq. (5.94) and we get

$$\Lambda(u) = \; \rho[h(iu)]^{-p} \prod_{j=1}^{2p} e^{-\pi\gamma_j u/4I} H(iu - \phi_j + \frac{1}{2}i\gamma_j I') \tag{5.119}$$

and if we square both sides

$$\Lambda^2(u) = \; \rho^2[h(iu)]^{-2p} \prod_{j=1}^{2p} e^{-\pi\gamma_j u/2I} H^2(iu - \phi_j + \frac{1}{2}i\gamma_j I')$$

this is the eigenvalues of VW. There exist still room for major simplifications.

From the relations (B.10) we have

$$iq^{-1/2}\Theta(iu - \phi_j - \frac{1}{2}i\gamma_j I')/H(iu - \phi_j + \frac{1}{2}i\gamma_j I') \; = \; \exp(-\pi\gamma_j u/2I)$$

resulting in

$$\Lambda^2(u) \; = \; iq^{-1/2} \prod_{j=1}^{2p} \frac{\Theta(iu - \phi_j - \frac{1}{2}i\gamma_j I')H(iu - \phi_j + \frac{1}{2}i\gamma_j I')}{H(iu)\Theta(iu)} \tag{5.120}$$

introduce the constant $\rho' = iq^{-1/2}$ and with eq (5.75) we have

$$k^{1/2}\,\mathrm{sn}(iu - \phi_j + \frac{1}{2}i\gamma_j I')\Theta(iu - \phi_j + \frac{1}{2}i\gamma_j I') \; = \; H(iu - \phi_j + \frac{1}{2}i\gamma_j I') \tag{5.121}$$

we can write the eigenvalues in a γ_j free form

$$\Lambda^2(u) \; = \; \rho' \prod_{j=1}^{2p} \frac{\Theta(iu - \phi_j + \frac{1}{2}i\gamma_j I')\Theta(iu - \phi_j - \frac{1}{2}i\gamma_j I')}{H(iu)\Theta(iu)} k^{1/2}\,\mathrm{sn}(iu - \phi_j + \frac{1}{2}i\gamma_j I') \tag{5.122}$$

because the numerator $\Theta(iu - \phi_j + \frac{1}{2}i\gamma_j I')\Theta(iu - \phi_j - \frac{1}{2}i\gamma_j I')$ does not depend on the choice of γ_j. Introduce D such that

$$\Lambda^2(u) \;=\; D\prod_{j=1}^{2p} k^{1/2}\,\mathrm{sn}(iu - \phi_j + \frac{1}{2}i\gamma_j I') \tag{5.123}$$

where

$$D \;=\; \rho' \prod_{j=1}^{2p} \frac{\Theta(iu - \phi_j + \frac{1}{2}iI')\Theta(iu - \phi_j - \frac{1}{2}iI')}{H(iu)\Theta(iu)} \tag{5.124}$$

This can be simplified by noting D is a doubly periodic function of iu. With

$$\text{poles of order } 2p \text{ when } iu = 0,\, I'$$

and

$$\text{zeros of order } 4p \text{ when } iu = \phi_j \pm iI'$$

As discussed in previous section we have that another function satisfying these requirements is:

$$\frac{(k\,\mathrm{sn}^2\; iu)^{2p} + r}{(k^{1/2}\,\mathrm{sn}\; iu)^{2p}} \tag{5.125}$$

Divide D with eq. (5.125) to form the expression

$$\rho' \prod_{j=1}^{2p} \frac{\Theta(iu - \phi_j + \frac{1}{2}iI')\Theta(iu - \phi_j - \frac{1}{2}iI')}{H(iu)\Theta(iu)} \times \frac{(k^{1/2}\,\mathrm{sn}\; iu)^{2p}}{(k\,\mathrm{sn}^2\; iu)^{2p} + r} \tag{5.126}$$

due to that they share the same zeros and poles this expression will be a entire peridodic function. From theorem $C.1$ this function must be a constant, meaning D must satisfy

$$D \;\propto\; \frac{(k\,\mathrm{sn}^2\; iu)^{2p} + r}{(k^{1/2}\,\mathrm{sn}\; iu)^{2p}} \tag{5.127}$$

We have then

$$\Lambda^2(u) = \; A\frac{(k\,\mathrm{sn}^2\; iu)^{2p} + r}{(k^{1/2}\,\mathrm{sn}\; iu)^{2p}} \prod_{j=1}^{2p} k^{1/2}\,\mathrm{sn}(iu - \phi_j + \frac{1}{2}i\gamma_j I') \tag{5.128}$$

where A is a constant. From eq. 5.114 if we set $u \to u + I'$ we have

$$\Lambda^2(u + I') = \; A\frac{(1 + k\,\mathrm{sn}^2\; iu)^{2p} r}{(k^{1/2}\,\mathrm{sn}\; iu)^{2p}} \prod_{j=1}^{2p} k^{-1/2}[\mathrm{sn}(iu - \phi_j + \frac{1}{2}i\gamma_j I')]^{-1} \tag{5.129}$$

Giving us:

$$\Lambda^2(u)\Lambda^2(u + I') \;=\; A^2\Big[\frac{r}{(k^2\,\mathrm{sn}\; iu)^{2p}} \;+\; (k\,\mathrm{sn}^2\; iu)^{2p} \;+\; 2\Big] \tag{5.130}$$

Then squaring both sides of eq. (5.100) gives us

$$\Lambda^2(u)\Lambda^2(u + I') \;=\; \Big(\frac{4}{k}\Big)^{2p} r\Big[\frac{r}{(k^2\,\mathrm{sn}\; iu)^{2p}} \;+\; (k\,\mathrm{sn}^2\; iu)^{2p} \;+\; 2\Big] \tag{5.131}$$

we compare eq.s (5.130) and (5.131) and identify A as:

$$A = \sqrt{r}\Big(\frac{2}{k^{1/2}}\Big)^{2p} \tag{5.132}$$

Inserting this back to eq. (5.128)

$$\Lambda^2(u) = \tau\Big[\Big(\frac{2}{k\,\mathrm{sn}\,iu}\Big)^{2p} + (2\,\mathrm{sn}\,iu)^{2p}r\Big] \times \prod_{j=1}^{2p} k^{1/2}\,\mathrm{sn}(iu - \phi_j + \frac{1}{2}i\gamma_j I') \tag{5.133}$$

where τ is:

$$\tau = \begin{cases} +1 & r = 1 \\ -i & r = -1 \end{cases} \tag{5.134}$$

We have now fully determined the eigenvalues of the matrix product for $k < 1$, this also concludes our need of elliptic functions.

5.15 General Expression for the Eigenvalues

We want now to write the eigenvalues in a more simpler and preferably a form free of elliptic functions. From eq.s (5.107) and (5.108) we get

$$k^{1/2}\,\mathrm{sn}(\phi_j - \frac{1}{2}iI') = -\exp(i\theta_j) \tag{5.135}$$

and from eq.s (5.76) and (5.77)

$$\begin{aligned} (\mathrm{cn}^2\, u - 1)(\mathrm{dn}^2\, u - 1) &= k^2\,\mathrm{sn}^4\, u \\ \mathrm{cn}^2\, u\,\mathrm{sn}^2\, u &= k^2\,\mathrm{sn}^4\, u - \mathrm{sn}^2\, u\,(k^2+1) + 1 \\ &= \mathrm{sn}^2\, u\Big[k^2\,\mathrm{sn}^2\, u + \frac{1}{\mathrm{sn}^2\, u} - (k^2+1)\Big] \end{aligned} \tag{5.136}$$

using eq. (5.135), this can be written as

$$\mathrm{cn}^2(\phi_j - \frac{1}{2}iI')\,\mathrm{dn}^2(\phi_j - \frac{1}{2}iI') = k^{-1}\exp(i\theta_j)^2\Big[k\exp(2i\theta_j) + k\exp(-2i\theta_j) - (k^2+1)\Big] \tag{5.137}$$

and simplifies to:

$$\mathrm{cn}(\phi_j - \frac{1}{2}iI')\,\mathrm{dn}(\phi_j - \frac{1}{2}iI') = -ik^{1/2}\exp(i\theta_j)c_j \tag{5.138}$$

where

$$c_j = \pm k^{-1}(1 + k^2 - 2k\cos(2\theta_j))^{1/2} \tag{5.139}$$

We have for sn u the addition relation [8, P. 463].

$$\mathrm{sn}(u - v) = \frac{\mathrm{sn}\; u\;\mathrm{cn}\; v\;\mathrm{dn}\; v - \mathrm{cn}\; u\;\mathrm{dn}\; u\;\mathrm{sn}\; v}{1 - k^2\,\mathrm{sn}^2\; u\;\mathrm{sn}^2\; v} \tag{5.140}$$

and if the return to $k^{1/2}\,\mathrm{sn}(iu - \phi_j + \frac{1}{2}i\gamma_j I')$ from eq. (5.133), we set $u = iu$ and $v = \phi_j - \frac{1}{2}iI'$ and using the previous eq.s (5.135) and (5.138) we get:

$$k^{1/2}\,\mathrm{sn}(iu - \phi_j + \frac{1}{2}i\gamma_j I') \;=\; \frac{\mathrm{cn}\; iu\; \mathrm{dn}\; iu \;-\; ikc_j\,\mathrm{sn}\; iu}{\exp\left(-i\theta_j\right) - k\exp(i\theta_j)\,\mathrm{sn}^2(iu)} \tag{5.141}$$

We return now to our initial parametrization from eq.s (5.79)

$$\exp(\pm 2K) = \mp\,\mathrm{sn}(iu) + \mathrm{cn}(iu) \tag{5.142}$$

$$\exp(\pm 2L) = ik^{-1}(\mathrm{dn}(iu) \pm 1)/\,\mathrm{sn}(iu) \tag{5.143}$$

we can then easily find from the definitions of hyperbolic functions that:

$$\begin{aligned}
\sinh\, 2K &= -i\,\mathrm{sn}\; iu \\
\cosh\, 2K &= \mathrm{cn}\; iu \\
\sinh\, 2L &= i/(k\,\mathrm{sn}\; iu\,) \\
\cosh\, 2L &= i\,\mathrm{dn}\; iu\,/(k\,\mathrm{sn}\; iu)
\end{aligned} \tag{5.144}$$

Inserting eq.s (5.144) to eq. (5.141) and dropping the the dependence on u we introduce u_j such that

$$u_j \;=\; \frac{\cosh\, 2K\cosh\, 2L + c_j}{\exp(i\theta_j)\sinh\, 2K + \exp(-i\theta_j)\sinh\, 2L} \tag{5.145}$$

and inserting eq. (5.144) to:

$$\tau\Big[\Big(-\frac{2}{k\,\mathrm{sn}(iu)}\Big)^{2p} \;+\; (-2\,\mathrm{sn}(iu))^{2p}r\Big] \;=\; \tau(-4)[(\sinh\, 2L)^{2p} \;+\; r(\sinh\, 2K)^{2p}] \tag{5.146}$$

Lastly we note from eq. (5.95) that we have the $u_j = u_j^{-1}$. Inserting these results into eq. (5.133) gives us

$$\Lambda^2 \;=\; \tau(-4)^p[(\sinh\, 2L)^{2p} \;+\; r(\sinh\, 2K)^{2p}]\prod_{j=1}^{2p}(u_j)^{\gamma_j} \tag{5.147}$$

We have successfully found the eigenvalues of the of the matrix product.

5.16 Analyticity and validity for $k = 1$ and $k > 1$

As previous mentioned when finding the eigenvalues we restricted our solution to $k < 1$ due to our elliptic parametrization only being defined for $k < 1$. What remains now after our parametrization extend our eigenvalues to $k = 1$ and $k > 1$, we investigate whether this is possible for eq. (5.147).

The only term that depends on k is c_j in u_j from eq. (5.145). We have that $(u_j)^{\gamma_j}$ in eq. (5.147) with the numerator

$$\cosh\, 2K\cosh\, 2L + c_j$$

if $\gamma_j = -1$ and

$$\cosh 2K \cosh 2L + c_j = 0$$

our eigenvalues become unanalytical. We want to require c_j to be positive to avoid this.

For $k < 1$ we have c_j to be positive for every θ_j which as found lies in the interval $0 < \theta_j \leq \pi$.

For $k = 1$, we have for $0 < \theta_j < \pi$ then c_j remains positive. If $\theta_j = \pi$ we have shown earlier $r = -1$ and $j = 2p$, this will yield $c_{2p} = 0$

For $k > 1$, we have the two cases when $0 < \theta_j < \pi$ again c_j is positive, but for $r = -1$ and $j = 2p$ we have $\theta_{2p} = \pi$ giving us

$$c_{2p} = (1-k)/k \tag{5.148}$$

which is negative.

To keep Λ^2 analytical we set

$$c_j = k^{-1}(1 + k^2 - 2k\cos(2\theta_j))^{1/2} \tag{5.149}$$

except when $r = -1$, $j = 2p$ and $k > 1$. Then we set

$$c_j = -k^{-1}(1 + k^2 - 2k\cos(2\theta_j))^{1/2} \tag{5.150}$$

5.17 Maximum Eigenvalue and the Free Energy

With our partition function we can finally calculate the free energy. If we keep the number of particles in a row n fixed and let number of rows m in thermodynamic limit satisfy $1 \ll m$. From eq. (5.147) we have our partition function

$$Z_N = \Lambda_1^m + \Lambda_2^m + ... + \Lambda_{2^n}^m$$

where Λ_1^2, , Λ_2^2, ... is the eigenvalues of VW, then in the thermodynamic limit the largest eigenvalue Λ_{max}^2 will grow making the remaining eigenvalues negligible. We have

$$Z_N \approx (\Lambda_{max})^m \tag{5.151}$$

and also together with eq. (2.3) we get the free energy to

$$F = -k_B T \ln(\Lambda_{max})^m \tag{5.152}$$

but we are interested in the free energy per lattice site f. The total number of particles is given by rows times columns i.e. $N = m \times 2p$, we get:

$$f = -\frac{k_b T}{2mp}\ln(\Lambda_{max})^m = -\frac{k_b T}{2p}\ln\, \Lambda_{max} \tag{5.153}$$

We determined the form of the eigenvalues, now we need to maximize Λ^2 from eq. (5.147). With K and L to be given the only parts of Λ^2 we can vary is choice of $r = \pm 1$ and γ_j.

We ask for what r and γ_j eq. (5.147) attains its maximum. From eq.s (5.62), (5.139) and (5.145) we get for real K and L

$$u_j u_j^* = \frac{\cosh 2K \cosh 2L + c_j}{\exp(i\theta_j)\sinh 2K + \exp(-i\theta_j)\sinh 2L} \tag{5.154}$$

and for positive K and L

$$0 \leq c_j \leq \cosh 2L \cosh 2L \tag{5.155}$$

from this we can establish $|u_j| \geq 1$ then eq. (5.147) is maximized when $\gamma_1 = ... = \gamma_{2p} = +1$. With this requirement we can simplify eq. (5.147).

We have then for $r = 1$ and eq.(5.147) that $\theta_j = \frac{\pi}{2p}\big(j - \frac{1}{2}\big)$

$$\prod_{j=1}^{2p} \exp\Big[\,i\frac{\pi}{2p}\big(j - \frac{1}{2}\big)\,\Big] \sinh 2K \,+\, \exp\Big[\,-i\frac{\pi}{2p}\big(j - \frac{1}{2}\big)\Big] \sinh 2L \,=$$

$$= -i(-1)^p \Big[(\sinh 2L)^{2p} \,-\, (\sinh 2K)^2 p\Big]^{-1} \tag{5.156}$$

and for $r = -1$ and eq. (5.147) that $\theta_j = \frac{\pi}{2p} j$

$$\prod_{j=1}^{2p} \exp\Big[\,i\frac{\pi}{2p} j\,\Big] \sinh 2K \,+\, \exp\Big[\,-i\frac{\pi}{2p} j\,\Big] \sinh 2L \,=$$

$$= (-1)^p \Big[(\sinh 2L)^{2p} \,+\, (\sinh 2K)^2\, p\Big]^{-1} \tag{5.157}$$

Inserting this to (5.147) gives us

$$\Lambda_{max}^2 = \prod_{j=1}^{2p} 2(\cosh 2L \cosh 2L + c_j) \tag{5.158}$$

where this is the form of the maximum eigenvalues. We have to consider the values of r before we calculate the free energy.

We define

$$F(\theta) \,=\, \ln 2[\cosh 2K \cosh 2L \,+\, k^{-1}(1 + k^2 - 2K \cos 2\theta)^{1/2}] \tag{5.159}$$

From Appendix C. we have *Perron–Frobenius theorem* which gives us that for a real square matrix with positive entries we can choose the corresponding eigenvector to have strictly positive components. This can only happen when $r = 1$, then inserting $\theta = \frac{\pi}{2p}\big(j - \frac{1}{2}\big)$ we get

$$\Lambda_{max}^2 = \prod_{j=1}^{2p} e^{F[\frac{\pi}{2p}\left(j - \frac{1}{2}\right)]} \tag{5.160}$$

solving for $\ln \Lambda_{max}$ give us:

$$\ln \Lambda_{max} = \sum_{j=1}^{2p} F\Big[\frac{\pi}{2p}\big(j - \frac{1}{2}\big)\Big] \tag{5.161}$$

Lastly inserting to eq. (5.153). For large p the sum can be replaced by the integral with the sub-interval length $\frac{\pi}{2p}$ du to that $0 < \theta_j < \pi$ giving us

$$-f/k_BT \;=\; (2\pi)^{-1}\int_0^{\pi} d\theta F(\theta) \tag{5.162}$$

This the complete expression of the free energy.

5.18 The critical temperature and phase transition

We have from the Kramers-Wannier duality the existence of a phase transition in the square-lattice model. We want to establish if this phase transition have only one corresponding critical temperature, we return to eq. (5.62) and use $K = J/k_BT$, $L = J'/k_BT$. We get

$$k \;=\; \Big(\sinh\, 2J/k_BT\, \sinh\, 2J'/k_BT\Big)^{-1} \tag{5.163}$$

The critical temperature being a discontinuity in the magnetization M is can also be regarded as singularities in f. We ask now when is f singular?

We can split up eq. (5.159) in two parts as:

$$\begin{aligned} F(\theta) &= \ln 2[\cosh\, 2K \cosh\, 2L \;+\; k^{-1}(1+k^2-2K\cos(\theta))^{1/2}] \\ &= \ln\Big(2\cosh\, 2K\cosh\, 2L\Big) \;+\; \ln\Big(1 \;+\; \frac{k^{-1}(1+k^2-2K\cos\, 2\theta)^{1/2}}{\cosh\, 2K\cosh\, 2L}\Big) \end{aligned}$$

Due to that we only searching for singularities we expand such $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + ...$ and get

$$F(\theta) \;\approx\; \ln\Big(2\cosh\, 2K\cosh\, 2L\Big) \;+\; \frac{k^{-1}(1+k^2-2K\cos\, 2\theta)^{1/2}}{\cosh\, 2K\cosh\, 2L} \tag{5.164}$$

inserting to free energy per lattice eq. (5.162) we have

$$-f/k_BT \;=\; (2\pi)^{-1}\int_0^{\pi} d\theta\Big[\ln\Big(2\cosh\, 2K\cosh\, 2L\Big) \;+\; \frac{k^{-1}(1+k^2-2K\cos\, 2\theta)^{1/2}}{\cosh\, 2K\cosh\, 2L}\Big] \tag{5.165}$$

where we can drop the analytical part

$$(2\pi)^{-1}\int_0^{\pi} \ln\Big(2\cosh\, 2K\cosh\, 2L\Big)$$

The singularities must then occur in f_s where

$$-f_s/k_BT \;=\; (2\pi)^{-1}\int_0^{\pi} d\theta \frac{k^{-1}(1+k^2-2K\cos\, 2\theta)^{1/2}}{\cosh\, 2K\cosh\, 2L} \tag{5.166}$$

This can be through rewrites, expansions and dropping analytical parts be written as [8, P. 121]:

$$-f_s/k_BT \;=\; \frac{(1+k)(1-k)^2}{2\pi k\cosh\, 2K\cosh\, 2L}\ln\left|\frac{1+k}{1-k}\right| \tag{5.167}$$

which gives us that f is singular at $k=1$ giving us one critical temperature at

$$\sinh(2J/k_BT_c)\sinh(2J'/k_BT_c) \;=\; 1 \tag{5.168}$$

6 Discussion

This thesis describes the general Ising model and relates it to ferromagnetism. It covers through the transfer matrix method the solution of the One-dimensional Ising model by finding the Helmholtz free energy as an analytical function. From the free energy it is concluded that the One-dimensional Ising model demonstrates no phase transition, due that the magnetization being a analytical function.

The Square-lattice Ising model is solved with the commuting transfer matrices method. The solutions is found by establishing with help from the Star-Triangle transformation and the inverse of the matrix product that the transfer matrices commute with them self, the negation operator R and the spin variable shift operator C. Together with Kramer-Wannier duality a eigenvalue relation can be found, which can be parametrized through elliptic function and the free energy exactly calculated. From the free energy the location of only one critical temperature at $k = 1$ is determined.

The significance of an exact solution of the Square-lattice model is that it offers a understanding of ferromagnetism and a simple analytical description of the phenomena phase transitions. The Star-triangle transformation and the resulting relations have become vital part of solving other models notably the *Chiral Potts models* which allows spin variables to take on higher spin values [12].

There exist different programmes for acquiring the the solutions of the Square-lattice Ising model, firstly is Onsager's original solution and also in addition Bruria Kaufman simplification of Onsager's work which focus on on spinorial representation of the rotation group [13]. The cases of the Ising model covered in the thesis concerns solids i.e. fixed particles, the Ising model can be extended to describe fluids or more specific a *lattice gas*. The lattice gas assumes particle to only exists on lattice sites but allows sites to be unoccupied and particles to change occupations, this is a simple model of density fluctuations and liquid-gas transformations. A similar model to the Square-lattice Ising models is *The spherical model* which is a model on a lattice with an external field, in this case the spin variable is allowed to take any real value constrained to the average value of the square of any spin is one [14].

A way to improve the thesis is to increase the understanding on how the Ising model connects to more general aspects of statistical mechanics. A step to do this is to study the correlation between the spin variables and how their *correlation length* connects to the *Scaling hypothesis*.

The Square-lattice model with an external magnetic field and the three-dimensional model remains unsolved. There is hope resolving the three-dimensional Ising model with *Conformal field theory* which offers descriptions of the Ising model near the vicinity of critical points [15]. An method of solving conformal field theory which is a today ongoing field of research is to solve the three-dimensional Ising model with *Conformal bootstrap*

[16].

To conclude one can obtain an analytical free energy for the One-dimensional nearest-neighbour Ising model with an external field and the Square-lattice nearest-neighbour Ising model. For the one-dimensional Ising Model we have no phase transitions, for the Square Lattice Ising Model we have one phase transition and critical temperature located at eq. 5.168.

References

[1] S.G. Brush. History of the lentz-ising model. *Review of Modern Physics*, 39(4): 883–893, 1967.

[2] K. Huang. *Statstical Mechanics*. John Wiley and sons, 2nd ed. edition, 1987.

[3] R. Peierls. On ising's model of ferromagnetism. *Mathematical Proceedings of the Cambridge Philosophical Society*, 32(3):477–481, 1936.

[4] G. Gallavotti. *Statstical Mechanics*. Texts and Monographs in Physics, 1999.

[5] F. Mandl. *Statstical Physics*. John Wiley and sons, 2nd ed. edition, 1993.

[6] E.M. Lifshitz L.D. Landau. *Statstical Physics*. Pergamon Press, 2nd ed. edition, 1969.

[7] T. Miyazaki and H. Jin. *The Physics of Ferromagnetism*, volume 158. Springer, 2012.

[8] R.J. Baxter. *Exactly Solved Models in Statistical Mechanics*. Academic Press, 3rd print edition, 1989.

[9] G. Mussardo. *Statistical Field Theory: An Introduction to Exactly Solved Models in Statistical Physics*. OUP Oxford, 2009.

[10] K. Hoffman. *Linear Algebra*. Prentice-Hall, 2nd ed. edition, 1971.

[11] N.I. Akhiezer and H.H. McFaden. *Elements of the Theory of Elliptic Functions*. Translations of Mathematical Monographs. American Mathematical Soc., 1970.

[12] H. Au-Yang and J.H.H Perk. Onsager's star-triangle equation: Master key to integrability. In *Integrable Systems in Quantum field theory and statistical mechanics*. Academic Press, 1989.

[13] R. J. Baxter. Onsager and kaufman's calculation of the spontaneous magnetization of the ising model. *Journal of Statistical Physics*, 145(3):518–548, 2011.

[14] T. H. Berlin and M. Kac. The spherical model of a ferromagnet. *Phys. Rev.*, 86: 821–835, Jun 1952.

[15] P. Ginsparg. Applied conformal field theory. 1988, arXiv:hep-th/9108028.

[16] D. Poland et al. S. El-Showk, M.F. Paulos. Solving the 3d ising model with the conformal bootstrap. 1988, arXiv:hep-th/1203.6064v3.

[17] H. A. Kramers and G. H. Wannier. Statistics of the two-dimensional ferromagnet. part i. *Phys. Rev.*, 60:252–262, Aug 1941.

[18] H. A. Kramers and G. H. Wannier. Statistics of the two-dimensional ferromagnet. part ii. *Phys. Rev.*, 60:263–276, Aug 1941.

Appendix A

This appedix cover an unformal discussion over how the Karmers-Wanniers Duality can be used to determine the location of the critical temprature. The relations is the fundation of the paramatrization from which in this report the Square-lattice Ising model is solved.

Kramers–Wannier Duality

Consider a the partition function for a square lattice

$$Z_N(K,\, L) \;=\; \sum_{\{\sigma\}} \exp\Big[K \sum_{(i,j)} \sigma_j \sigma_j \;+\; L \sum_{(i,k)} \sigma_i \sigma_k\Big] \tag{A.1}$$

Let r be number of unlike pairs of $(\sigma_i,\, \sigma_j)$ and s be number of unlike pairs of $(\sigma_i,\, \sigma_k)$. The Boltzmann weight becomes :

$$\exp\big[K(N-2r) \,+\, L(N-2s)\big] \tag{A.2}$$

We are now going to use a dual lattice. Which means firstly we have our square lattice Γ, we are now going to add a dual lattice Γ_D such that it is a square lattice with sites in the center of the faces of Γ .

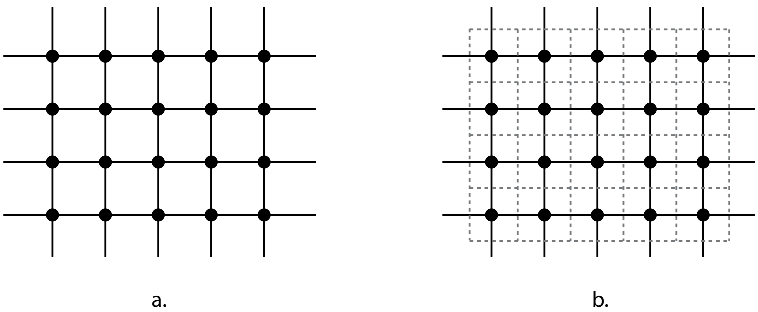


Figure 19: *(1) square lattice Γ, (2) square lattice Γ in solid line, and dual lattice Γ_D in dashed lines*

Instead of regarding the spin labels as sites in Γ we can now think of them as faces in Γ_D. Let now the boundary separating neighbouring faces with different sign be labeled by a solid line and neighbouring faces with the same sign be label with a dashed line.

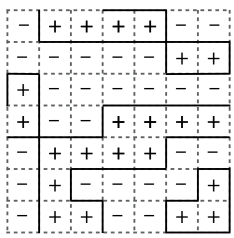


Figure 20: *Example of dual lattice configuration, where boundary between faces indicate if there is a sign change, note periodic boundry conditions is used at the end points.*

We have then r horizontal lines and s vertical lines on Γ_D. Lets consider the sites on Γ, there must be an even number of dashed and solid lines into the vertex of each site. This gives us that must be labeled to link these together and form polygons ex. see fig. (20)

These polygons divide the system in a spin up and spin down domain. We get the partition function to:

$$Z_N(K,\, L) \,=\, 2\sum_{P\subset\Gamma_D} \exp\Big[K(N-2r) \,+\, L(N-2s)\Big] \tag{A.3}$$

where we now sum over all possible polygon configurations on Γ_D, the factor two comes from every set of polygons represent two possible spin configurations one is obtained from the other by negating all the spins.

We evaluate in the low temperature limit. For $T \ll 1$ let $K = K^*$ and $L = L^*$, we get

$$Z_N(K^*,\, L^*) = 2\sum_{P\subset\Gamma_D} \exp\Big[K^*(N-2r) \,+\, L^*(N-2s)\Big] \,= \tag{A.4}$$

$$= 2\exp[N(K^*+L^*)] \sum_{P\subset\Gamma_D} \exp[-2K]^r \exp[-2L]^s \tag{A.5}$$

Further by comparing the low and high temperature expansion one can find the they are related with the transformations:

$$\tanh\, K \,=\, e^{-2L^*} \tag{A.6}$$

$$\tanh\, L = \, e^{-2K^*} \tag{A.7}$$

which guarantees a singularity i.e. a phase transition *Kramers–Wannier duality* [17, 18].

We write eq. (A.6) as

$$\frac{\sinh\, 2K}{(\cosh\, 2K+1)} = \frac{1-e^{-4L^*}}{2\sinh\, L^*}$$

$$\sinh\, 2K \sinh 2L^* = \frac{1}{2}\Big(\cosh\, 2K+1\Big)\Big(1-e^{-4L^*}\Big) \tag{A.8}$$

and by doing the similar for eq. (A.7), we have in the low temperature expansion that L^* and K^* is large and in the high temperature expansion that L and K is small. By using this we get

$$\sinh\, 2K \sinh 2L^* \,=\, 1 \tag{A.9}$$

$$\sinh\, 2K^* \sinh 2L \,=\, 1 \tag{A.10}$$

if the critical point is located at $K = K_c$ and $L = L_c$, assuming there only one line of ciritcal in the (K,L) plane [8]. Then we must require $K = K_c^* = K_c$ and $L = L_c = L_c^*$, we get for the $T = T_c$:

$$\sinh\, 2K \sinh 2L \,=\, 1 \tag{A.11}$$

Appendix B

This appendix lists some properties of the elliptic functions and relating them to the solution of the Square-lattice Ising model.

Properties of Elliptic Functions

When introducing the elliptic functions being periodic in two directions we heavily rely on properties associated functions being doubly periodic.

Definition B.1. Any function f(u) satisfying

$$\begin{aligned} f(u + 2I) &= (-1)^s f(u) \\ f(u + 2iI') &= (-1)^r f(u) \end{aligned} \tag{B.1}$$

where r, s are integers, is said to be *doubly periodic*.

The elliptic functions being depends two variables u and q which oftenly and in our case is a real variable restricted to $0 < q < 1$. We then the half period magnitudes I and I' for the elliptic functions is given by

$$I = \frac{1}{2}\pi \prod_{n=1}^{\infty} \Big(\frac{1+q^{2n-1}}{1-q^{2n-1}} \cdot \frac{1-q^{2n}}{1+q^{2n}}\Big)^2 \tag{B.2}$$

$$I' = \pi^{-1} I \ln(q^{-1}) \tag{B.3}$$

where we can write q simply as:

$$q = \exp(-\pi I'/I) \tag{B.4}$$

Our solutions of the Square-Lattice model is based on the functional relation of K and L where we restrict them to a real constant k. When introducing the elliptic functions for a parametrization of k the modulus k and its conjugate k' takes the form

$$\begin{aligned} k &= 4q^{1/2} \prod_{n=1}^{\infty} \Big(\frac{1+q^{2n}}{1+q^{2n-1}}\Big)^4 \\ k' &= \prod_{n=1}^{\infty} \Big(\frac{1-q^{2n-1}}{1+q^{2n-1}}\Big)^4 \end{aligned} \tag{B.5}$$

but most importantly it is then defined for $0 < k < 1$.

A key property of the theta functions is their analyticity and periodicity mainly

$$H(u) = 0 \qquad \text{when } u = 2mI + 2inI' \tag{B.6}$$

$$\Theta(u) = 0 \qquad \text{when } u = 2mI + i(2n-1)I' \tag{B.7}$$

where n and m are integers.

Lastly the theta functions are related by:

$$H(u+2I) = -H(u) \tag{B.8}$$

$$H(u+2iI') = -q^{-1}\exp(-\pi iu/I)H(u) \tag{B.9}$$

and

$$\Theta(u) = -iq^{1/2}\exp(\frac{1}{2}\pi iu/I)H(u+iI') \tag{B.10}$$

Appendix C

In this Appendix we outline theorems which is vital to solving the Square-lattice Ising model.

I am not going to present any proof to any of these theorems and shamelessly I have taking these from the true hero of the Square-lattice Ising; Baxter Rodney. For more about these theorems and their role in the Ising model read his brilliant book [8].

General Theorems

Liouville's theorem. *Every bounded entire function must be constant*

Corollary C.1. *If a function is doubly periodic (or anti-periodic) and is analytic inside and on a period rectangle, then it is a constant.*

Theorem C.2. *If a function $f(u)$ is doubly periodic (or anti-periodic) and meromorphic, and has n poles per period rectangle,then it also has just n zeros per period rectangle. Multiple poles or zeros of order r being counted r times)*

Theorem C.3. *If a function f(u) is meromorphic and satisfies the (anti-) periodicity conditions and if f(u) has just n poles per period rectangle, at v_1,..., v_n, (counting a pole of order r as r coincident simple poles), then*

$$f(u) = Ce^{i\lambda u} \prod_{j=1}^{n} [H(u - u_j)]/H(u - v_j)] \tag{C.1}$$

where C, λ, v_1, ..., v_n are constants satisfying

$$u_1 + ... + u_n = v_1 + ... + v_n + (r + 2m)I - i(s + 2n)I' \tag{C.2}$$

$$\lambda = \frac{1}{2}\pi(s + 2n)/I \tag{C.3}$$

Theorem C.4. *Perron–Frobenius theorem A real square matrix with positive entries has a unique largest real eigenvalue and that the corresponding eigenvector can be chosen to have strictly positive components.*