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The harmonic oscillator propagator

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The Feynman propagator for the harmonic oscillator is evaluated by a variety of path-integral-based means.

I. INTRODUCTION

Recently, a number of papers in this journal have focused on the propagator for the linear potential.[1] In particular, they have demonstrated how the result may be obtained by a variety of different means. Since the linear potential is an important one—corresponding to motion in a constant gravitational or electric field—it is useful to have at one's disposal such a range of derivations, not necessarily to use at any one time, but rather to appropriately exploit when the propagator is required in different contexts. Perhaps even more important physically, however, is the case of the harmonic oscillator propagator, since harmonic motion is associated with motion in a uniform magnetic field or is omnipresent whenever the amplitude of oscillation becomes small. The form of the propagator associated with the Lagrangian

$$L=\tfrac{1}{2}m\dot{z}^2-\tfrac{1}{2}m\omega^2z^2 \tag{1}$$

describing such motion is well known:[2]

$$D_F(z_f,t_f;z_i,t_i)=\sqrt{\frac{m\omega}{2\pi i\ \sin\ \omega(t_f-t_i)}}\times\exp\ i\ \frac{m\omega}{2}\left[(z_f^2+z_i^2)\cot\ \omega(t_f-t_i)-2z_fz_i\ \csc\ \omega(t_f-t_i)\right]. \tag{2}$$

However, the path integral derivations of this result given in typical texts are somewhat standard and limited. It is the purpose of this note to show how this form can be derived via a variety of path-integral-based procedures, each of which yields the desired form Eq. (2) but some of which may also be of utility in another context wherein standard methods may not suffice. Although the methods described below all utilize the Feynman path integral, it should be noted for completeness that alternative techniques also exist. For example, Nardone has obtained the propagator via an innovative use of the Heisenberg representation.[3] Saxon obtains the result by use of a somewhat formal procedure involving the use of oscillator creation and annihilation operators.[4] Merzbacher, in the forthcoming third edition of his quantum mechanics text, will show how the propagator can be formed from exponentiation of a solution of the Hamilton–Jacobi equation.[5] However, for reasons of conciseness and self-containment we shall confine our discussion to procedures based on path integration.

In the next section then we review the standard path integral derivations of Eq. (2) which utilize either the propagator completeness relation or the Euler identity[6]

$$\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right)=\frac{\sin x}{x}. \tag{3}$$

Following this, we show how the harmonic propagator may be derived via alternative techniques, some of which have interesting connections with methods used in contemporary field theory. Then we summarize our findings in a concluding Sec. III.

II. DERIVING THE HARMONIC OSCILLATOR PROPAGATOR

The derivation of the harmonic oscillator propagator begins with the representation of the propagator in terms of the Feynman path integral (note we use $\hbar=1$)

$$D_F=\int[dz]\exp iS[z], \tag{4}$$

where $\int[dz]$ represents the sum over all paths connecting the initial and final spacetime points and $S[z]$ is the action associated with each path $z(t)$. One then writes an arbitrary path $z(t)$ in terms of its deviation $\delta z(t)$ from the classical trajectory $z_{\rm cl}(t)$, which satisfies

$$\ddot{z}_{\rm cl}(t)=-\omega^2 z_{\rm cl}(t),\quad z_{\rm cl}(t_i)=z_i,\quad z_{\rm cl}(t_f)=z_f. \tag{5}$$

Then,

$$\begin{aligned}S[z_{\rm cl}]&=\int_{t_i}^{t_f}dt\,\frac{m}{2}\,[(\dot{z}_{\rm cl}+\delta\dot{z})^2-\omega^2(z_{\rm cl}+\delta z)^2]\\&=S[z_{\rm cl}]+\int_{t_i}^{t_f}dt\,\frac{m}{2}\,((\delta\dot{z})^2-\omega^2(\delta z)^2),\end{aligned} \tag{6}$$

since the term linear in δz vanishes by Hamilton's principle, so that the propagator can be written as

$$D_F(z_f,t_f;z_i,t_i)=F(t_f-t_i)\exp iS[z_{\rm cl}], \tag{7}$$

where

$$F(t_f-t_i)=\int[d\,\delta z]\exp i\int_{t_i}^{t_f}dt\,\frac{m}{2}\,[(\delta\dot{z})^2-\omega^2(\delta z)^2] \tag{8}$$

is a harmonic oscillator path integral subject to the end-point conditions $\delta z(t_f)=\delta z(t_i)=0$.

The classical trajectory satisfying the boundary conditions Eq. (5) is easily found to be

$$\begin{aligned}z_{\rm cl}(t)=&\frac{z_f-z_i\cos\omega(t_f-t_i)}{\sin\omega(t_f-t_i)}\sin\omega(t-t_i)\\&+z_i\cos\omega(t-t_i)\end{aligned} \tag{9}$$

and substitution into the action integral yields, after a bit of algebra,

$$\begin{aligned}S[z_{\rm cl}]=&\frac{m\omega}{2}\,[(z_f^2+z_i^2)\cot\omega(t_f-t_i)\\&-2z_f z_i\csc\omega(t_f-t_i)],\end{aligned} \tag{10}$$

which agrees with the phase in Eq. (2). The problem then reduces to finding the prefactor $F(t_f-t_i)$. One common method,[7] requiring nothing but algebra, uses the completeness relation

$$\begin{aligned}F(t_f-t_i)&=D_F(0,t_f;0,t_i)\\&=\int_{-\infty}^{\infty}dz\;D_F(0,t_f;z,t)D_F(z,t;0,t_i),\end{aligned}$$

where

$$t_i<t<t_f. \tag{11}$$

Using the representation given in Eqs. (6) and (7) this yields

$$\begin{aligned}F(t_f-t_i)=&F(t_f-t)F(t-t_i)\\&\times\int_{-\infty}^{\infty}dz\,\exp i\,\frac{m\omega}{2}\,z^2(\cot\omega(t_f-t)\\&+\cot\omega(t-t_i)),\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{F(t_2-t_1)}{F(t_f-t)F(t-t_i)}&=\sqrt{\frac{2\pi i}{m\omega}\,\frac{1}{\cot\omega(t_f-t)+\cot\omega(t-t_i)}}\\&=\sqrt{\frac{2\pi i}{m\omega}\,\frac{\sin\omega(t_f-t)\sin\omega(t-t_i)}{\sin\omega(t_f-t_i)}},\end{aligned} \tag{12}$$

whose solution is

$$F(t_f-t_i)=\sqrt{\frac{m\omega}{2\pi i\sin\omega(t_f-t_i)}}, \tag{13}$$

in agreement with the known result Eq. (2). However, this procedure must in all honesty be categorized as a "trick" and certainly does not represent a systematic approach which can be applied in other circumstances.

An alternative and more general procedure involves performing the sum over paths in Eq. (8) using[2]

$$\begin{aligned}S[\delta z(t)]&=\int_{t_i}^{t_f}dt\,\frac{m}{2}\,[(\delta\dot{z}(t))^2-\omega^2(\delta z(t))^2]\\&\equiv-\frac{m}{2}\int_{t_i}^{t_f}dt\;\delta z(t)\mathcal{O}\;\delta z(t),\end{aligned}$$

with

$$\mathcal{O}=\frac{d^2}{dt^2}+\omega^2. \tag{14}$$

Now expand $\delta z(t)$ in terms of eigenfunctions $z_n(t)$ of the operator $\mathcal{O}$:

$$\delta z(t)=\sum_n a_n z_n(t), \tag{15}$$

where

$$\mathcal{O}z_n(t)=\lambda_n z_n(t). \tag{16}$$

Here, the eigenfunctions satisfy Dirichlet boundary conditions $z_n(t_i)=z_n(t_f)=0$ and are subject to the orthogonality condition

$$\int_{t_i}^{t_f} dt\ z_n(t) z_m(t)=\delta_{nm}. \tag{17}$$

The sum over all paths can then be performed by integration over all expansion coefficients a_n,

$$\begin{aligned}D_F(0,t_f;0,t_i) &= N\prod_{j=1}^{\infty}\left(\int_{-\infty}^{\infty} da_j\right)\\ &\quad\times \exp i\int_{t_i}^{t_f} dt\ \delta z(t)\mathscr{O}\ \delta z(t)\\ &= N\prod_{j=1}^{\infty}\left(\int_{-\infty}^{\infty} da_j\right)\\ &\quad\times\exp i\int_{t_i}^{t_f} dt\sum_{k=1}^{\infty} a_k z_k(t)\sum_{l=1}^{\infty}\lambda_l a_l z_l(t)\\ &= N\prod_{j=1}^{\infty}\left(\int_{-\infty}^{\infty} da_j\ \exp i\lambda_j a_j^2\right)\\ &= N'(\det\mathscr{O})^{-1/2},\end{aligned} \tag{18}$$

where $N,\ N'$ are normalization constants and

$$\det\mathscr{O}\equiv\prod_{j=1}^{\infty}\lambda_j \tag{19}$$

is the product of operator eigenvalues. The determinant itself is quite singular and difficult to handle. Thus one generally deals instead with the better defined *ratio* of determinants. For example, for the well-known case of the free propagator[2]

$$D_F^{(0)}(z_f,t_f;z_i,t_i)=\sqrt{\frac{m}{2\pi i(t_f-t_i)}}\ \exp\left[i\ \frac{m(z_f-z_i)^2}{2(t_f-t_i)}\right], \tag{20}$$

we identify

$$S^{(0)}[z_{\rm cl}]=\frac{m(z_f-z_i)^2}{2(t_f-t_i)} \tag{21}$$

as the classical action and

$$F^{(0)}(t_f-t_i)=\sqrt{\frac{m}{2\pi i(t_f-t_i)}}=N'(\det\mathscr{O}^{(0)})^{-1/2} \tag{22}$$

as the prefactor, where $\mathscr{O}^{(0)}=d^2/dt^2$. Using this result we have then

$$F(t_f-t_i)=\left(\frac{\det\mathscr{O}}{\det\mathscr{O}^{(0)}}\right)^{-1/2}\sqrt{\frac{m}{2\pi i(t_f-t_i)}}. \tag{23}$$

Note that this result, while formal, is completely general and is valid no matter what the form of $\mathscr{O}$. In the case of the harmonic oscillator, however, one can obtain a simple closed form for the result by various means, as we shall demonstrate.

(i) Reference 8—one procedure makes use of the Euler identity Eq. (3), which arises since the functions which satisfy the (Dirichlet) boundary conditions at $t=t_i, t_f$ are of the form

$$z_n(t)=\sin\omega_n(t-t_i)$$

with

$$\omega_n=\frac{n\pi}{t_f-t_i},\quad n=1,2,3,\ldots. \tag{24}$$

The corresponding eigenvalues are

$$\lambda_n=\omega^2-\left(\frac{n\pi}{t_f-t_i}\right)^2. \tag{25}$$

Then we have

$$\begin{aligned}\frac{\det\mathscr{O}}{\det\mathscr{O}^{(0)}} &= \prod_{n=1}^{\infty}\frac{\omega^2-\dfrac{n^2\pi^2}{(t_f-t_i)^2}}{-\dfrac{n^2\pi^2}{(t_f-t_i)^2}}=\prod_{n=1}^{\infty}\left(1-\frac{\omega^2(t_f-t_i)^2}{n^2\pi^2}\right)\\ &= \frac{\sin\omega(t_f-t_i)}{\omega(t_f-t_i)}\end{aligned} \tag{26}$$

and

$$\begin{aligned}F(t_f-t_i) &= \sqrt{\frac{\omega(t_f-t_i)}{\sin\omega(t_f-t_i)}}\ \sqrt{\frac{m}{2\pi i(t_f-t_i)}}\\ &= \sqrt{\frac{m\omega}{\sin\omega(t_f-t_i)}},\end{aligned} \tag{27}$$

as desired.

(ii) Reference 9—one can avoid use of the Euler identity and perform the calculation directly by taking the logarithm and making use of the integral relation[10]

$$\ln\mathscr{D}=-\int_0^{\infty}\frac{ds}{s}\exp(-s\mathscr{D})+C, \tag{28}$$

where C is a divergent constant having no physical consequences. For convergence reasons it is useful to make the analytic continuation to imaginary time $t\to -iT$. Then, using the identity

$$\ln\det\mathscr{O}=\ln\prod_n\lambda_n=\sum_n\ \ln\lambda_n=\mathrm{tr}\ \ln\mathscr{O}, \tag{29}$$

we can write

$$\begin{aligned}\ln\frac{\det\mathscr{O}}{\det\mathscr{O}^{(0)}} &= -(1-\lim_{\omega\to 0})\int_0^{\infty} ds\ s^{\epsilon-1}\exp(-s\ \det\mathscr{O})\\ &= -(1-\lim_{\omega\to 0})\sum_{n=1}^{\infty}\int_0^{\infty} ds\ s^{\epsilon-1}\\ &\quad\times\exp\left[-s\left(\omega^2+\frac{n^2\pi^2}{(T_f-T_i)^2}\right)\right],\end{aligned} \tag{30}$$

where the ϵ is inserted in order to regulate the singularity as $s\to 0$. The sum may be performed using the identity[11]

$$\sum_{n=-\infty}^{\infty}\exp(-n^2\pi x)=\sqrt{\frac{1}{x}}\sum_{n=-\infty}^{\infty}\exp(-n^2\pi/x). \tag{31}$$

Rewriting this as

$$\sum_{n=1}^{\infty} \exp(-n^2 \pi x) = \frac{1}{2}\left(\sqrt{\frac{1}{x}} - 1\right) + \sqrt{\frac{1}{x}} \sum_{n=1}^{\infty} \exp(-n^2 \pi / x), \quad (32)$$

we find

$$\ln \frac{\det \mathscr{O}}{\det \mathscr{O}^{(0)}} = -(1 - \lim_{\omega \to 0}) \int_0^{\infty} ds\; e^{-\omega^2 s}\left[\frac{T_f - T_i}{2\sqrt{\pi}}\, s^{-3/2} - \frac{1}{2}\, s^{\epsilon - 1} + \frac{T_f - T_i}{\sqrt{\pi}} \sum_{n=1}^{\infty} s^{-3/2} \times \exp\left(-\frac{n^2 (T_f - T_i)^2}{s}\right)\right]. \quad (33)$$

Then, using the integrals[12]

$$\int_0^{\infty} dx\; x^{\alpha - 1} \exp-\left(\gamma x + \frac{\beta}{x}\right) = 2\left(\frac{\beta}{\gamma}\right)^{\alpha/2} K_{\alpha}(2\sqrt{\beta\gamma}), \quad (34)$$

$$\int_0^{\infty} dx\; x^{\alpha - 1} \exp - \gamma x = \gamma^{-\alpha} \Gamma(\alpha),$$

we have

$$\begin{aligned}\ln \frac{\det \mathscr{O}}{\det \mathscr{O}^{(0)}} &= -(1 - \lim_{\omega \to 0})\left[\omega\, \frac{T_f - T_i}{2\sqrt{\pi}}\, \Gamma\left(-\frac{1}{2}\right) - \omega^{-2\epsilon} \Gamma(\epsilon) + \sum_{n=1}^{\infty} \frac{2(T_f - T_i)}{\sqrt{\pi}} \left(\frac{\omega}{n(T_f - T_i)}\right)^{-1/2} \times K_{-1/2}(2n\omega(T_f - T_i))\right] \\ &= -(1 - \lim_{\omega \to 0})\left[\sum_{n=1}^{\infty} \frac{1}{n} \exp - 2n\omega(T_f - T_i) - \omega(T_f - T_i) + \ln\, \omega - \frac{1}{2}\, \Gamma(\epsilon)\right], \end{aligned} \quad (35)$$

where we have used the results

$$K_{-1/2}(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x}, \quad \omega^{-2\epsilon} = 1 - 2\epsilon \ln\, \omega + \cdots. \quad (36)$$

The series may be summed via

$$\sum_{n=1}^{\infty} \frac{1}{n}\, x^n = -\ln(1 - x) \quad (37)$$

and yields

$$\begin{aligned}\ln \frac{\det \mathscr{O}}{\det \mathscr{O}^{(0)}} &= -(1 - \lim_{\omega \to 0})\left[-\ln(1 - e^{-2\omega(T_f - T_i)}) - \omega(T_f - T_i) + \ln\, \omega - \frac{1}{2}\, \Gamma(\epsilon)\right] \\ &= -(1 - \lim_{\omega \to 0})[-\ln 2 \sinh\, \omega(T_f - T_i) - \tfrac{1}{2}\Gamma(\epsilon) + \ln\, \omega] \\ &= \ln\left(\frac{\sinh\, \omega(T_f - T_i)}{\omega(T_f - T_i)}\right), \end{aligned} \quad (38)$$

where we have used the identity $1 - e^{-2x} = 2 \sinh x / e^x$. At this point we make the continuation $T \to it$ back to real time and the form Eq. (13) again obtains.

(iii) Reference 13—a third procedure involves no summing of series but exploits the feature that

$$\frac{d}{d\omega} \ln \mathscr{O} = \frac{2\omega}{\mathscr{O}}, \quad (39)$$

where $\langle t|1/\mathscr{O}|t'\rangle \equiv G_{\omega}(t,t')$ is the Green's function satisfying

$$\mathscr{O}_t G_{\omega}(t,t') = \left(\frac{d^2}{dt^2} + \omega^2\right) G_{\omega}(t,t') = \delta(t - t'). \quad (40)$$

The form of the Green's function satisfying Dirichlet boundary conditions at $t = t_i, t_f$ is easily seen to be

$$G_{\omega}(t,t') = K \sin\, \omega(t_f - t_>) \sin\, \omega(t_< - t_i), \quad (41)$$

where $t_>, t_<$ is the greater, lesser of t, t'. The constant K can be found from the condition that

$$1 = \int_{t' - \epsilon}^{t' + \epsilon} dt\; \delta(t - t') = \frac{\partial}{\partial t}\, G_{\omega}(t,t')\Bigg|_{t' - \epsilon}^{t' + \epsilon}, \quad \text{i.e.,} \quad K = -\frac{1}{\omega \sin\, \omega(t_f - t_i)}. \quad (42)$$

We may then construct the desired ratio of determinants by integrating with respect to ω,

$$\begin{aligned}\ln \frac{\det \mathscr{O}}{\det \mathscr{O}^{(0)}} &= \mathrm{tr}\, \ln \frac{\mathscr{O}}{\mathscr{O}^{(0)}} = \int_0^{\omega} d\omega' 2\omega' \int_{t_i}^{t_f} dt\; G_{\omega'}(t,t) \\ &= -\int_0^{\omega} d\omega' \frac{2}{\sin\, \omega'(t_f - t_i)} \times \int_{t_i}^{t_f} dt\, \sin\, \omega'(t_f - t) \sin\, \omega'(t - t_i). \end{aligned} \quad (43)$$

Using a trigonometric identity on the product of sines this may be integrated directly,

$$\begin{aligned}\ln \frac{\det \mathscr{O}}{\det \mathscr{O}^{(0)}} &= -\int_0^{\omega} \frac{d\omega'}{\sin\, \omega'(t_f - t_i)} \int_{t_i}^{t_f} dt (\cos\, \omega'(t_f + t_i - 2t) - \cos\, \omega'(t_f - t_i)) \\ &= \int_0^{\omega(t_f - t_i)} ds\left(\cot s - \frac{1}{s}\right) \\ &= \ln\left(\frac{\sin\, \omega(t_f - t_i)}{\omega(t_f - t_i)}\right), \end{aligned} \quad (44)$$

and once more leads to Eq. (8).

(iv) Reference 14—as a fourth example, we note that, although convenient, it is not necessary to go to Fourier component space in order to evaluate the determinant. In fact one can use the more intuitive coordinate representation and the familiar time-slicing procedure in order to achieve the same results. Thus use the path integral representation Eq. (8) and divide the time interval $t_f - t_i$ into n slices of size ϵ, yielding

$$\begin{aligned} F(t_f - t_i) &= D_F(0,t_f;0,t_i) \\ &= \left(\frac{m}{2\pi i\epsilon}\right)^{n/2} \prod_{i=1}^{n-1} \int_{-\infty}^{\infty} dz_i \\ &\quad \times \exp i\left[\sum_{j=1}^{n-1} \frac{m(z_j - z_{j-1})^2}{2\epsilon} - \frac{1}{2} m\omega^2 z_i^2 \epsilon\right], \end{aligned} \tag{45}$$

where $z_n = z_0 = 0$. Notice that the argument of the exponential is a quadratic function of the integration variables z_i. If we define the $(n-1)$-component vector $Z \equiv (z_1, z_2, \ldots, z_{n-1})$, then the exponential given in Eq. (45) can be written as

$$\exp i\,\frac{m}{2\epsilon} \sum_{ij} Z_i \mathscr{K}_{n-1}^{ij} Z_j,$$

where

$$\mathscr{K}_{n-1} = \begin{pmatrix} \gamma & -1 & 0 & \cdots & 0 \\ -1 & \gamma & -1 & \cdots & 0 \\ 0 & -1 & \gamma & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & \gamma \end{pmatrix} \tag{46}$$

and we have defined $\gamma = 2 - \epsilon^2\omega^2$. We can imagine diagonalizing this matrix, whereupon we recognize that the path integral becomes simply

$$F(t_f - t_i) = \left(\frac{m}{2\pi i\epsilon}\right)^{n/2} \prod_{i=1}^{n-1} \int_{-\infty}^{\infty} dy_i \exp i\,\frac{m}{2\epsilon}\lambda_i y_i^2, \tag{47}$$

where λ_i are the eigenvalues of the matrix $\mathscr{K}_{n-1}$. Performing the integration, we find that

$$\begin{aligned} F(t_f - t_i) &= \left(\frac{m}{2\pi i\epsilon}\right)^{(n/2)} \prod_{i=1}^{n-1} \sqrt{\frac{2\pi i\epsilon}{m\lambda_i}} \\ &= \sqrt{\frac{m}{2\pi i(t_f - t_i)}} \lim_{n\to\infty} \sqrt{\frac{n}{\det \mathscr{K}_{n-1}}}. \end{aligned} \tag{48}$$

We may evaluate the determinant by using the top row and noting that if D_{n-1} represents the determinant of this $(n-1)\times(n-1)$ component matrix, then

$$D_{n-1} = \gamma D_{n-2} - D_{n-3}. \tag{49}$$

Defining $x = \omega\epsilon = \omega(t_f - t_i)/n$ and writing out a few terms

$$\begin{aligned} D_1 &= 2 - x^2, \\ D_2 &= 3 - 4x^2 + x^4, \\ D_3 &= 4 - 10x^2 + 6x^4 - x^6, \\ D_4 &= \cdots, \end{aligned} \tag{50}$$

we soon recognize that the solution of this recursion relation can be written as

$$\begin{aligned} D_n &= n + 1 - \frac{(n+2)(n+1)n}{3!} x^2 \\ &\quad + \frac{(n+3)(n+2)(n+1)n(n-1)}{5!} x^4 - \cdots \\ &= \sum_{k=0}^{n-1} (-)^k \frac{(n+k+1)!}{(n-k)!(2k+1)!} x^{2k}. \end{aligned} \tag{51}$$

Then

$$\lim_{n\to\infty} \frac{\det \mathscr{K}_{n-1}}{n} = 1 - \frac{n^2x^2}{3!} + \frac{n^4x^4}{5!} - \cdots = \frac{\sin \omega(t_f - t_i)}{\omega(t_f - t_i)}, \tag{52}$$

so that, using Eq. (48), we reproduce the familiar answer—Eq. (8)—as required.

We have seen then how the prefactor $F(t_f - t_i)$ can be determined via a variety of techniques using differing means to evaluate the operator determinant. As a final example, we show how, using a representation in terms of a continued fraction, the *full result* for the propagator may be constructed via the time-slicing procedure.[15] In this case we begin with the standard time-sliced representation—Eq. (45)—but now with $z_0 = z_i$, $z_n = z_f \neq 0$. The procedure then involves successive (quadratic) integrations over $z_1, z_2 \ldots, z_{n-1}$. Thus the integration over z_1 involves

$$\begin{aligned} I_1 &= \left(\frac{m}{2\pi i\epsilon}\right)^{1/2} \int_{-\infty}^{\infty} dz_1 \exp i\,\frac{m}{2\epsilon}\left((z_2 - z_1)^2 + (z_1 - z_0)^2 \right. \\ &\quad \left. - \omega^2\epsilon^2 z_1^2\right) \\ &= C_1 \exp i\,\frac{m}{2\epsilon}\left((1 - C_1^2)(z_2^2 + z_0^2) - 2z_2 z_0 C_1^2\right), \end{aligned} \tag{53}$$

with

$$C_1^2 = \frac{1}{2 - \omega^2\epsilon^2} \equiv \frac{1}{\gamma}. \tag{54}$$

Likewise that over z_2 gives

$$\begin{aligned} I_2 &= \left(\frac{m}{2\pi i\epsilon}\right)^{1/2} \int_{-\infty}^{\infty} dz_2 \exp i\,\frac{m}{2\epsilon}\left((z_3 - z_2)^2 - \omega^2\epsilon^2 z_2^2\right) I_1 \\ &= C_1 C_2 \exp i\,\frac{m}{2\epsilon}\left((1 - C_2^2)(z_0^2 + z_3^2) - 2z_3 z_0 C_2^2 C_1^2\right), \end{aligned} \tag{55}$$

where

$$C_2^2 = \frac{1}{\gamma - C_1^2} = \frac{1}{\gamma - \dfrac{1}{\gamma}}. \tag{56}$$

Continuing, we eventually find the result

$$D_F(z_f,t_f;z_i,t_i)=\lim_{n\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{1/2}\prod_{i=1}^{n-1}C_i\ \exp\ i\ \frac{m}{2\epsilon}$$
$$\times\left((1-C_{n-1}^2)(z_n^2+z_0^2)-2z_nz_0\prod_{i=1}^{n-1}C_i^2\right), \tag{57}$$

where

$$C_{n-1}^2=\frac{1}{\gamma-C_{n-2}^2}=\frac{1}{\gamma-\cfrac{1}{\gamma-\cfrac{1}{\gamma-\cfrac{1}{\cdots n-1}}}}. \tag{58}$$

In order to evaluate the products of the C_i which arise here we note that C_i^2 obeys the recursion relation

$$C_i^2=\frac{1}{\gamma-C_{i-1}^2}. \tag{59}$$

Defining $C_i^2=A_i/B_i$ we see that this implies that

$$\frac{A_i}{B_i}=\frac{B_{i-1}}{\gamma B_{i-1}-A_{i-1}}, \tag{60}$$

whose solution is

$$A_i=B_{i-1},\quad B_i=\gamma B_{i-1}-B_{i-2}. \tag{61}$$

We observe that the recursion relation for B_i is identical to that for D_i [cf. Eq. (49)] and thus has the same solution

$$\lim_{n\to\infty}\frac{B_{n-1}}{n}=\frac{\sin\ \omega(t_f-t_i)}{\omega(t_f-t_i)}. \tag{62}$$

Then for the prefactor we have from Eq. (57)

$$F(t_f-t_i)=\left(\frac{m}{2\pi i(t_f-t_i)}\right)^{1/2}\lim_{n\to\infty}\left(n\prod_{i=1}^{n-1}C_i^2\right)^{1/2}. \tag{63}$$

But we note that

$$\prod_{i=1}^{n-1}C_i^2=\frac{A_1}{B_1}\times\frac{A_2}{B_2}\times\cdots\times\frac{A_{n-1}}{B_{n-1}}=\frac{1}{B_{n-1}}, \tag{64}$$

where we have used the result that $A_{i+1}=B_i$ and $A_1=1$. Thus

$$F(t_f-t_i)=\left(\frac{m}{2\pi i(t_f-t_i)}\right)^{1/2}\left(\frac{\omega(t_f-t_i}{\sin\ \omega(t_f-t_i)}\right)^{1/2}$$
$$=\left(\frac{m\omega}{\sin\ \omega(t_f-t_i)}\right)^{1/2}, \tag{65}$$

as expected.

We can correspondingly evaluate the phase factor, which we write as

$$\phi=\lim_{n\to\infty}\left(\frac{m}{2(t_f-t_i)}\right)\left((z_n^2+z_0^2)n(1-C_{n-1}^2)\right.$$
$$\left.-2z_nz_0n\prod_{i=1}^{n-1}C_i^2\right). \tag{66}$$

Note that we can write

$$1-C_{n-1}^2=1-\frac{A_{n-1}}{B_{n-1}}=\frac{B_{n-1}-B_{n-2}}{B_{n-1}}, \tag{67}$$

where

$$B_{n-1}-B_{n-2}=[n-(n-1)]-\frac{1}{3!}\,n(n-1)[(n+1)-(n$$
$$-2)]\omega^2\epsilon^2+\frac{1}{5!}\,(n-2)(n-1)n(n+1)$$
$$\times[(n+2)-(n-3)]\omega^4\epsilon^4+\cdots$$
$$=1-\frac{1}{2!}\,n(n-1)\omega^2\epsilon^2+\frac{1}{4!}\,(n-2)$$
$$\times(n-1)n(n+1)\omega^4\epsilon^4$$
$$+\cdots\xrightarrow{n\to\infty}\ \cos\ \omega(t_f-t_i). \tag{68}$$

Then

$$\phi=\left(\frac{m}{2(t_f-t_i)}\right)[(z_f^2+z_i^2)\cos\ \omega(t_f-t_i)$$
$$-2z_fz_i)]\lim_{n\to\infty}\frac{n}{B_{n-1}}$$
$$=\left(\frac{m\omega}{2\sin\ \omega(t_f-t_i)}\right)[(z_f^2+z_i^2)\cos\ \omega(t_f-t_i)-2z_fz_i], \tag{69}$$

in agreement with the known result Eq. (2). Thus we have derived the full propagator via a straightforward time-slicing method.

III. CONCLUSION

In the previous section we have reproduced the harmonic oscillator propagator via a variety of path integral techniques—some based on conventional and familiar time-slicing methods and others using alternative techniques, many of which have close analogs in modern quantum field theoretical applications. As in the corresponding case of the linear potential, only one such derivation is required, of course, in order to generate the form of the propagator. However, seeing the result derived by different means can be of great utility both pedagogically as well as giving an introduction to advanced techniques while still on familiar ground.

[1]See, e.g., R. W. Robinett, ‘‘Quantum Mechanical Time-Development Operator for the Uniformly Accelerated Particle,’’ Am. J. Phys. **64**, 803–808 (1996); L. S. Brown and Y. Zhang, ‘‘Path Integral for the Motion of a Particle in Linear Potential,’’ *ibid*. **62**, 806–808 (1994); G. P. Arrighini, N. L. Durante, and C. Guidotti, ‘‘More on the Quantum Propagator of a Particle in a Linear Potential,’’ *ibid*. **64**, 1036–1041 (1996); B. R. Holstein, ‘‘The Linear Potential Propagator,’’ *ibid*. **65**, 414–418 (1997); P. Nardone, ‘‘Heisenberg Picture in Quantum Mechanics and Linear Evolutionary Systems,’’ *ibid*. **61**, 232–237 (1993).

[2]See, e.g., R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw–Hill, New York, 1965).

[3]P. Nardone, Ref. 1.

[4]D. S. Saxon, *Elementary Quantum Mechanics* (Holden–Day, San Francisco, 1968).

[5]E. Merzbacher, private communication.

[6]M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*

(USGPO, Washington, DC, 1968, #4.3.89; reprinted by Dover Publications, New York).

[7] B. R. Holstein, *Topics in Advanced Quantum Mechanics* (Addison–Wesley, Reading, MA, 1992); B. Felsager, *Geometry, Particles and Fields* (Odense U.P., Odense, 1981); J. V. Narlikar and T. Padmanabhan, *Gravity, Gauge Theories and Quantum Cosmology* (Reidel, New York, 1986).

[8] R. P. Feynman and A. R. Hibbs, Ref. 2; M. Swanson, *Path Integrals and Quantum Processes* (Academic, New York, 1992); B. Felsager, Ref. 7. See also S. Coleman, *Aspects of Symmetry* (Cambridge U.P., New York, 1985), Chap. 7, for an ingenious derivation of the ratio of determinants.

[9] This procedure is similar to that used by L. C. Albuquerque, C. Farina, and S. Rabello, ''Schwinger's Method and the Computation of Determinants,'' Sao Paulo, preprint (1997).

[10] J. F. Donoghue, E. Golowich, and B. R. Holstein, *Dynamics of the Standard Model* (Cambridge U.P., New York, 1992), Appendix B.

[11] E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis* (Cambridge U.P., New York, 1973).

[12] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals Series and Products* (Academic, New York, 1965), #3.471.9 and 3.381.4.

[13] C. Farina, H. Boschi-Filho, and A. de Souza Dutra, ''Green Function Approach for Computing Non-Relativistic Determinants,'' Rio de Janeiro, preprint (1994).

[14] Note that this procedure is equivalent to but simpler than the method of Gelfand and Yaglom, which involves turning the recursion relation into a differential equation via

$$\frac{D_{n-1}-2D_{n-2}+D_{n-3}}{\epsilon^n}=\frac{\gamma-2}{\epsilon^2}D_{n-2}=\omega^2 D_{n-2}, \tag{70}$$

cf. H. Kleinert, *Path Integrals in Quantum Mechanics, Statistical and Polymer Physics* (World Scientific, Singapore, 1995); see also L. S. Schulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1981); and C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw–Hill, New York, 1980), Chap. 9.

[15] The continued fraction form of the prefactor which results herein was first noted by L. Q. English and R. R. Winters, ''Continued Fractions and the Harmonic Oscillator using Feynman's Path Integral,'' Am. J. Phys. **65**, 390–393 (1997). Alternative means of evaluation were noted by K. Unnikrishnan, ''Comment on 'Continued Fractions and the Harmonic Oscillator using Feynman's Path Integral,'' by L. Q. English and R. R. Winters [Am. J. Phys. **65**, 390–393 (1997)],'' *ibid.* **65**, 1212 (1997) and by B. R. Holstein, ''Answer to Question #4. Is there a physics application that is best analyzed in terms of Continued Fractions?'' *ibid.* **65**, 1133–1135 (1997).