20a Systems of Cycles and Homology$^\alpha$

Homology is a natural equivalence relation between cycles, similar to but both simpler and coarser than homotopy; where homotopy treats cycles as *sequences* of darts, homology treats cycles as *sets of edges* (or more generally, *linear combinations* of darts). Homology can be defined with respect to any "coefficient ring", but to keep the presentation simple, I'll describe only the simplest special case ($\mathbb{Z}_2$-homology) in this section, and return to a slightly more complicated special case ($\mathbb{R}$-homology) in a later note.

20a.1 Cycles and Boundaries

Fix a surface map $\Sigma = (V, E, F)$ with Euler genus $\bar{g}$.

$\mathbb{Z}_2$-homology is an equivalence relation between certain *subgraphs* of Σ, formally represented as subsets of E.

An *even subgraph* of Σ is a subgraph H such that $\deg_H(v)$ is even for every vertex $v \in V(\Sigma)$. The empty subgraph is an even subgraph, as is every simple cycle. Every even subgraph is the union (or symmetric difference) of simple edge-disjoint cycles.

For every edge $e \notin T$, let $\mathsf{cycle}_T(e)$ denote the unique simple undirected cycle in the graph $T + e$; we call $\mathsf{cycle}_T(e)$ a *fundamental cycle* with respect to T. Let $\mathcal{C} = \{\mathsf{cycle}_T(e) \mid e \in L\}$. The set $\mathcal{C}$ is called a *system of cycles* for the map Σ.

Fundamental Cycle Lemma: *Let T be an arbitrary spanning tree of an arbitrary graph G (sic). For every even subgraph H of G, we have*

$$H = \bigoplus_{e \in H \setminus T} \mathit{cycle}_T(e).$$

Thus, even subgraphs are symmetric differences of fundamental cycles.

Proof: Let H be an arbitrary even subgraph of G, and let $H' = \bigoplus_{e \in H \setminus T} \mathsf{cycle}_T(e)$. The symmetric difference of any two even subgraphs is even, so $H \oplus H'$ is an even subgraph and therefore the union of edge-disjoint cycles. On the other hand, $H \oplus H'$ is a subgraph of T and therefore acyclic. We conclude that $H \oplus H'$ is empty, or equivalently, $H = H'$. $\square$

Mnemonically, any even subgraph can be *named* by listing its edges in $C \cup L$.

Let Z be a subset of the faces of Σ. The *boundary* of Z, denoted ∂Z, is the subgraph of Σ containing every edge that is incident to both a face in Z and a face in $F \setminus Z$. A *boundary subgraph* is any subgraph that is the boundary of a subset of faces. Every boundary subgraph is an even subgraph. Conversely, if Σ is a planar map, the Jordan curve theorem implies that every even subgraph is a boundary subgraph, but this equivalence does not extend to more complex surfaces.

Fundamental Boundary Lemma: *Let (T, L, C) be an arbitrary tree-cotree decomposition of a surface map Σ. For every boundary subgraph B of Σ, we have*

$$B = \bigoplus_{e \in B \cap C} \mathit{bdry}_C(e).$$

Thus, boundary subgraphs are symmetric differences of fundamental boundaries.

Proof: We mirror the proof of the Fundamental Cycle lemma. Let B be any boundary subgraph, and let $B' = \bigoplus_{e \in B \cap C} \mathsf{bdry}_C(e)$. The boundary space is closed under symmetric difference, so $B' \oplus B$ is a boundary subgraph. On the other hand, $B \oplus B'$ has no edges in C, so $B \oplus B'$ is a subgraph of the cut graph $T \cup L$. We conclude that $B \oplus B'$ is empty, or equivalently, $B = B'$. $\square$.

Mnemonically, any boundary subgraph can be *named* by listing its edges in C.

20a.2 Homology

Finally, two subgraphs A and B of Σ are *($\mathbb{Z}_2$)-homologous* if their symmetric difference $A \oplus B$ is a boundary subgraph of Σ. For example, every boundary subgraph is homologous with the empty subgraph, which is why boundary subgraphs are also called *null-homologous* subgraphs. Straightforward definition-chasing implies that ($\mathbb{Z}_2$)-homology is an equivalence relation, whose equivalence classes are obviously called *($\mathbb{Z}_2$)-homology classes*. We usually omit "$\mathbb{Z}_2$" if the type of homology is clear from context.

Lemma: *Let (T, L, C) be an arbitrary tree-cotree decomposition of a surface map Σ. The only boundary subgraph of the cut graph $T \cup L$ is the empty graph.*

Proof: Let H be a non-empty cut graph in Σ; H must be the boundary of a non-empty proper subset Z of the faces in Σ. Consider the fundamental domain $\Delta = \Sigma \mathbin{\backslash\!\!\backslash} (T \cup L)$. Because both Z and its complement are non-empty, some interior edge e of Δ separates a face in Z from a face not in Z. But the interior edges of Δ are precisely the edges in C. $\square$

Lemma: *Let (T, L, C) be an arbitrary tree-cotree decomposition of a surface map Σ. Every even subgraph in Σ is homologous with an even subgraph of the cut graph $T \cup L$.*

Proof: It suffices to prove that every edge $e \in C$ is homologous with a subgraph of $T \cup L$ that has even degree everywhere except the endpoints of e.

Consider the fundamental domain $\Delta = \Sigma \mathbin{\backslash\!\!\backslash} (T \cup L)$. Every edge $e \in C$ appears in Δ as a boundary-to-boundary chord, which partitions the faces of Δ into two disjoint subsets $Y \sqcup Z$. (Recall that no edge in C can be an isthmus!) Every face of Δ is a face of the original map Σ and vice versa; let β denote the boundary of Y (or equivalently, the boundary of Z) in Σ. Because β is a boundary subgraph in Σ, e is homologous with $\beta \oplus e$. Finally, every edge in $\beta \oplus e$ is an edge in the cut graph $T \cup L$. $\square$

Lemma: *Let (T, L, C) be an arbitrary tree-cotree decomposition of a surface map Σ. Every subgraph of Σ is homologous with a symmetric difference of cycles in $\mathcal{C}$.*

Proof: By the previous lemma, it suffices to consider only even subgraphs of the cur graph $T \cup L$. Every even subgraph of $T \cup L$ is the symmetric difference of simple cycles in $T \cup L$. The simple cycles in $T \cup L$ are precisely the cycles in $\mathcal{C}$. $\square$

Homology Basis Theorem: *Let (T, L, C) be an arbitrary tree-cotree decomposition of a surface map Σ. For every even subgraph H of Σ, we have*

$$H = \left(\bigoplus_{i \in I(H)} \mathit{cycle}_T(\ell_i) \right) \oplus \left(\bigoplus_{e \in H \cap C} \mathit{bdry}_T(e) \right)$$

*for some subset $I(H) \subseteq \{1, 2, \dots, \bar{g}\}$. Thus, every even subgraph is homologous with the symmetric difference of a **unique** subset of cycles in $\mathcal{C}$, which is nonempty if and only if H is a boundary* *subgraph.*

The Homology Basis Theorem immediately implies an algorithm to decide if two even subgraphs H and H' are homologous: Compute their canonical decompositions into fundamental cycles and boundaries, with respect to the same tree-cotree decomposition, and then compare the index sets $I(H)$ and $I'(H)$. A careful implementation of this algorithm runs in $O(\bar{g}n)$ time; details are left as an exercise (because we're about to describe simpler algorithms).

20a.3 Relax, it's just linear algebra!

Unlike our earlier characterization of homotopy, our characterization of homology is unique; every even subgraph is homologous with the symmetric difference of *exactly one* subset of cycles in $\mathcal{C}$. The easiest way to prove this fact is to observe that subgraphs, even subgraphs, boundary subgraphs, and homology classes all define vector spaces over the finite field $\mathbb{Z}_2 = (\{0,1\}, \oplus, \cdot)$. In particular, homology can be viewed as a linear map between vector spaces.

Subgraphs (subsets of E) comprise the ***edge space*** (or *first chain space*) $C_1(\Sigma) = \mathbb{Z}_2^{|E|}$. The (indicator vectors of) individual edges in Σ comprise a basis of the edge space.

Even subgraphs of Σ comprise a subspace of $C_1(\Sigma)$ called the ***cycle space*** $Z_1(\Sigma)$. The Fundamental Cycle Lemma implies that the fundamental cycles $\mathsf{cycle}_T(e)$, for all $e \notin T$, define a basis for the cycle space. The number of fundamental cycles is equal to the number of edges not in T, which is $|E| - (|V| - 1)$. Thus, $Z_1(\Sigma) = \mathbb{Z}_2^{|E|-|V|+1}$.

Boundary subgraphs of Σ comprise a subspace of $Z_1(\Sigma)$ called the ***boundary space*** $B_1(\Sigma)$. The Fundamental Boundary lemma implies that the fundamental boundaries $\mathsf{bdry}_C(e)$, for all $e \in C$, define a basis for the boundary space. The number of fundamental boundaries is equal to the number of edges of C, which is $|F| - 1$. Thus, $B_1(\Sigma) = \mathbb{Z}_2^{|F|-1}$.

Finally, the set of homology classes of even subgraphs of Σ comprise the ***(first) homology space***, which is the quotient space

$$\begin{aligned} H_1(\Sigma) &:= Z_1(\Sigma)/B_1(\Sigma) \\ &= \mathbb{Z}_2^{|E|-|V|+1} \big/ \mathbb{Z}_2^{|F|-1} \\ &\cong \mathbb{Z}_2^{|E|-(|V|-1)-(|F|-1)} \\ &= \mathbb{Z}_2^{|E|-|T|-|C|} = \mathbb{Z}_2^{|L|} = \mathbb{Z}_2^{\bar{g}}. \end{aligned}$$

(Hey look, we proved Euler's formula again!) The Homology Basis Theorem implies that homology classes of fundamental cycles $\mathsf{cycle}_T(e)$, for all $e \in L$, define a basis for the homology space. In particular, there are exactly $2^{\bar{g}}$ distinct homology classes.

20a.4 Crossing Numbers

Another way to characterize the homology class of an even subgraph H is to determine which cycles in a system of cycles *cross* H. The definition of "cross" is rather subtle, but mirrors the intuition of transverse intersection.

Consider two distinct simple cycles α and β, and let π be one of the components of the intersection $\alpha \cap \beta$. (Because $\alpha \neq \beta$, the intersection π must be either a single vertex of a common subpath.)

We call π a *crossing* between α and β (or we say that α and β *cross* at π) if, after contracting the path π to a point p, the contracted curves α/π and β/π intersect transversely at p.

Equivalently, α and β cross at π if, no matter how we perturb the two curves within a small neighborhood of π, the two perturbed curves $\tilde{\alpha}$ and $\tilde{\beta}$ intersect. By convention, no two-sided cycle crosses itself (because we can perturb two copies of a two-sided cycle so that they are disjoint), but every one-sided cycle crosses itself once (because we cannot).

For any simple cycles α and β, the *crossing number* $\mathsf{cr}(\alpha,\beta)$ is the number of crossings between α and β, modulo 2. In particular, $\mathsf{cr}(\alpha,\alpha)=0$ if for every two-sided cycle α, and $\mathsf{cr}(\beta,\beta)=1$ for every one-sided cycle β.

We can extend this definition of crossing number to even subgraphs by linearity: $\mathsf{cr}(\alpha\oplus\beta,\gamma)=\mathsf{cr}(\alpha,\gamma)\oplus\mathsf{cr}(\beta,\gamma)$. Although one can express any even subgraph as a symmetric difference of cycles in many different ways, crossing numbers are the same for every such decomposition.

For any face f and any cycle γ, we have $\mathsf{cr}(\partial f,\gamma)=0$. It follows by linearity that if either γ or δ is a boundary subgraph, then $\mathsf{cr}(\delta,\gamma)=0$. More generally, it follows that crossing numbers are a *homology invariant*: if α and β are homologous even subgraphs, then $\mathsf{cr}(\alpha,\gamma)=\mathsf{cr}(\beta,\gamma)$ for every cycle γ, because $\alpha\oplus\beta$ is the symmetric difference of face boundaries.

Lemma: *For any even subgraphs H and H', if $cr(H,H')=1$, then neither H nor H' is a boundary subgraph.*

Proof: If (say) H is a boundary subgraph, then H is the symmetric difference of face boundaries, and therefore $\mathsf{cr}(H,H')=0$ by linearity. □

Lemma: *Let σ be a simple cycle and let $\mathcal{C}=\{\gamma_1,\gamma_2,\dots,\gamma_{\bar{g}}\}$ be a system of cycles in a surface map Σ. Then σ is boundary cycle if and only if $\mathsf{cr}(\sigma,\gamma_i)=0$ for every cycle $\gamma_i\in\mathcal{C}$.*

Proof: If σ is a boundary cycle, homology invariance immediately implies $\mathsf{cr}(\sigma,\gamma_i)=\mathsf{cr}(\varnothing,\gamma_i)=0$.

Suppose on the other hand that σ is not a boundary cycle. Then by definition the sliced surface $\Sigma \mathbin{\backslash\!\!\backslash} \sigma$ is connected. Let v be a vertex of σ, and let π be any path from v^+ to v^- in $\Sigma \mathbin{\backslash\!\!\backslash} \sigma$. This path π appears in Σ as a closed walk that crosses σ exactly once, so $\mathsf{cr}(\pi,\sigma)=1$. It follows from the previous lemma that π is not a boundary cycle. Thus, by the Homology Basis theorem, π is homologous with $\bigoplus_{i\in I}\gamma_i$ for some non-empty subset $I\subseteq\{1,2,\dots,\bar{g}\}$. Finally, homology invariance implies $\mathsf{cr}(\pi,\sigma)=\bigoplus_{i\in I}\mathsf{cr}(\gamma_i,\sigma)=1$, so we must have $\mathsf{cr}(\gamma_i,\sigma)=1$ for an odd number of indices $i\in I$, and therefore for at least one such index. □

Corollary: *Let $\mathcal{C}$ be a system of cycles in a surface map Σ. An even subgraph H of Σ is a boundary subgraph if and only if $\mathsf{cr}(H,\gamma_i)=0$ for every cycle $\gamma_i\in\mathcal{C}$. Two even subgraphs H and H' of Σ are homologous if and only if $\mathsf{cr}(H,\gamma_i)=\mathsf{cr}(H',\gamma_i)$ for every cycle $\gamma_i\in\mathcal{C}$.*

20a.5 Systems of Cocycles and Cohomology

Cohomology is the dual of homology. While homology is an equivalence relation between subgraphs of maps, cohomology is an equivalence relation between subgraphs of *dual* maps. In fact, it's the *dual* equivalence relation between subgraphs of dual maps. Two subgraphs A and B of Σ are *cohomologous* if and only if the corresponding dual subgraphs A^* and B^* of Σ^* are homologous.

I'll adopt the convenient convention of adding the prefix "co" to indicate the dual of a structure in the dual map. Mnemonically, a cosnarfle in Σ is the dual of snarfle in Σ^*.

- We've already defined a *spanning co-tree* of Σ is a subset of edges whose corresponding dual edges comprise a spanning tree of Σ^*. Less formally, a spanning cotree of Σ is the dual of a spanning tree of Σ^*.
- A *cocycle* in Σ is the dual of a cycle in Σ^*. (In planar graphs, every cocycle is a minimal edge cut, but that equivalence does not extend to more complex surfaces.)
- A *co-even subgraph* of Σ is the dual of an even subgraph of Σ^*. That is, a subgraph H of Σ is co-even if every face of Σ has an even number of incidences with H. No edge in a co-even subgraph is a loop, because loops are co-isthmuses.
- The *coboundary* if a subset X of vertices of Σ, denoted δX, is the dual of the boundary of the corresponding subset X^* of faces of Σ^* . That is, δX is the subset of edges with one endpoint in X and one endpoint not in X. A *coboundary* subgraph is the coboundary of some subset of vertices. Every coboundary subgraph is co-even.
- Finally, two co-even subgraphs are cohomologous if their symmetric difference is a coboundary subgraph.

As usual, fix a tree-cotree decomposition (T, L, C) of a surface map Σ. For every edge $e \in T \cup L$, let $\mathsf{cocycle}_C(e)$ denote the subgraph of Σ dual to the fundamental cycle $\mathsf{cycle}_{c^*}(e^*)$ in the dual map Σ^*. Finally, let $\mathcal{K} = \{\mathsf{cocycle}_C(e) \mid e \in T\}$. The following lemmas follow immediately from our earlier characterization of homology.

Lemma: *Let (T, L, C) be an arbitrary tree-cotree decomposition of a surface map Σ. Every co-even subgraph of Σ a symmetric difference of fundamental cocycles $\mathsf{cocycle}_C(e)$ where $e \notin C$.*

Lemma: *Let (T, L, C) be an arbitrary tree-cotree decomposition of a surface map Σ. Every co-even subgraph of Σ is cohomologous with a co-even subgraph of the cocut graph $C \cup L$.*

Lemma: *Let (T, L, C) be an arbitrary tree-cotree decomposition of a surface map Σ. Every co-even subgraph of Σ is cohomologous with a symmetric difference of cocycles in $\mathcal{K}$.*

20a.6 Homology Signatures

More importantly, however, cohomology offers us a **CO**nvenient method to efficiently **CO**mpute homology classes of even subgraphs of the primal map Σ, by assigning a **CO**ordinate system to the first homology space. Index the leftover edges in L as $\ell_1, \ell_2, \ldots, \ell_{\bar{g}}$.[1] For every edge e in Σ, the *homology signature* $[e]$ is the $\bar{g}$-bit vector indicating which cocycles in $\mathcal{K}$ contain e. Specifically:

$$[e]_i = 1 \iff e \in \mathsf{cocycle}_C(\ell_i).$$

Finally, the homology signature $[H]$ of any subgraph H is the bitwise exclusive-or of the homology signatures of its edges.

The function $H \mapsto [H]$ is a *linear* function from the cycle space $Z_1(\Sigma)$ to the vector space $\mathbb{Z}_2^{\bar{g}}$ of homology signatures. In particular:

Linearity Lemma: *For any two even subgraphs H and H' of Σ, we have $[H \oplus H'] = [H] \oplus [H']$.*

Basis Lemma: *For all indices i and j, we have $[\mathsf{cycle}_T(\ell_i)]_j = 1$ if and only if $i = j$.*

[1]Here I'm using ℓ as a mnemonic for "leftover edge" instead of "loop". We have a lot of other e's flying around, so I don't want to use e_i to denote the ith edge in L.

Proof: The only edge in any fundamental $\mathsf{cycle}_T(e)$ that is *not* in T is the determining edge e. Similarly, the only edge in any fundamental $\mathsf{cocycle}_C(e)$ that is *not* in C is the determining edge e. Thus, $\mathsf{cycle}_T(ell_i) \cap \mathsf{cocycle}_C(\ell_j) = \varnothing$ whenever $i \neq j$, and $\mathsf{cycle}_T(\ell_i) \cap \mathsf{cocycle}_C(\ell_i) = \ell_i$ for every index i. □

Theorem: *Two even subgraphs H and H' of Σ are homologous if and only if $[H] = [H']$.*

Proof: By the Linearity Lemma, it suffices to prove that an even subgraph H is a boundary subgraph if and only if $[H] = 0$.

Let f be any face of Σ, and let λ be any cocycle in Σ. The boundary of f either contains no edges of λ or exactly two edges of λ, depending on whether the dual cycle λ^* contains the dual vertex f^*. It follows that $[\partial f] = 0$ for every face f. The Linearity Lemma implies that $[H] = 0$ for every boundary subgraph H.

Conversely, suppose H is not null-homologous. Then we can write

$$H = \left(\bigoplus_{i \in I} \mathsf{cycle}_T(\ell_i)\right) \oplus \left(\bigoplus_{e \in H \cap C} \mathsf{bdry}_T(e)\right)$$

for some nonempty subset $I \subseteq \{1, 2, \ldots, \bar{g}\}$. The Linearity and Basis lemmas imply that

$$[H] = \left(\bigoplus_{i \in I} [\mathsf{cycle}_T(\ell_i)]\right)$$

and therefore $[H]_i = 1$ if and only if $i \in I$. Because I is non-empty, $[H] \neq 0$. □

20a.7 Separating Cycles

Lemma: *Let γ be a simple cycle in a surface map Σ. The sliced map $\Sigma \mathbin{\backslash\!\backslash} \gamma$ is disconnected if and only if $[\gamma] = 0$*

20a.8 References

20a.9 Aptly Named Sir

- Pants decompositions (except possibly in passing)