22 Distributive Flow Lattices$^{\varnothing}$

Status: Unwritten

22.1 Pseudoflows and Circulations

Let Σ be a planar map. A *pseudoflow* in Σ is an antisymmetric function $\phi : D(\Sigma) \to \mathbb{R}$ from the darts of Σ to the reals; here antisymmetry means $\phi(d) = -\phi(\mathsf{rev}(d))$ for every dart d.[1] Pseudoflows in Σ define a real vector space $C_1(\Sigma)$, unimaginatively called the *chain space* of Σ, whose dimension is equal to the number of *edges* of Σ. If we arbitrarily fix a *reference dart* e^+ for every edge e, then we can specify a unique 1-chain by assigning arbitrary values to the reference darts.

A *circulation* is a pseudoflow that obeys *Kirchhoff's current law*: For every vertex v, the values assigned to darts into v sum to zero. (We previously called circulations *closed discrete 1-forms*) More generally, the *boundary* $\partial\phi$ of any pseudoflow is the function $\partial\phi : V(\Sigma) \to \mathbb{R}$ defined by

$$\partial\phi(v) = \sum_{u \to v} \phi(u \to v) = \sum_{d\,:\,\mathsf{head}(d)=v} \phi(v).$$

(At the risk of confusing the reader, I will use the first summation notation even when Σ has parallel edges.)

Fix a tree-cotree decomposition (T, C) of Σ. For any non-tree edge $e \in C$, the *fundamental cycle* $\mathsf{cycle}_T(e^+)$ is the directed cycle consisting of the reference dart e^+ and the directed path in T from $\mathsf{head}(e^+)$ to $\mathsf{tail}(e^+)$.

Lemma: *Fix an arbitrary planar map Σ and an arbitrary tree-cotree decomposition (T, C) of Σ. Every circulation ϕ in Σ satisfies the identity*

$$\phi = \sum_{e \in C} \phi(e^+) \cdot \mathit{cycle}_T(e^+)$$

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22.2 Aptly Named Sir Not

[1] In the lecture on the Maxwell-Cremona correspondence, we used exactly the same definition for *discrete 1-forms* or *1-cochains*, but topologists should really think of pseudoflows as *1-chains*.