

## 20a Systems of Cycles and Homology<sup>a</sup>

*Homology* is a natural equivalence relation between cycles, similar to but both simpler and coarser than homotopy; where homotopy treats cycles as *sequences* of darts, homology treats cycles as *sets of edges* (or more generally, *linear combinations* of darts). Homology can be defined with respect to any “coefficient ring”, but to keep the presentation simple, I’ll describe only the simplest special case ( $\mathbb{Z}_2$ -homology) in this section, and return to a slightly more complicated special case ( $\mathbb{R}$ -homology) in a later note.

### 20a.1 Cycles and Boundaries

Fix a surface map  $\Sigma = (V, E, F)$  with Euler genus  $\bar{g}$ .

$\mathbb{Z}_2$ -homology is an equivalence relation between certain *subgraphs* of  $\Sigma$ , formally represented as subsets of  $E$ .

An *even subgraph* of  $\Sigma$  is a subgraph  $H$  such that  $\deg_H(v)$  is even for every vertex  $v \in V(\Sigma)$ . The empty subgraph is an even subgraph, as is every simple cycle. Every even subgraph is the union (or symmetric difference) of simple edge-disjoint cycles.

For every edge  $e \notin T$ , let  $\text{cycle}_T(e)$  denote the unique simple undirected cycle in the graph  $T + e$ ; we call  $\text{cycle}_T(e)$  a *fundamental cycle* with respect to  $T$ . Let  $\mathcal{C} = \{\text{cycle}_T(e) \mid e \in L\}$ . The set  $\mathcal{C}$  is called a *system of cycles* for the map  $\Sigma$ .

**Fundamental Cycle Lemma:** *Let  $T$  be an arbitrary spanning tree of an arbitrary graph  $G$  (sic). For every even subgraph  $H$  of  $G$ , we have*

$$H = \bigoplus_{e \in H \setminus T} \text{cycle}_T(e).$$

*Thus, even subgraphs are symmetric differences of fundamental cycles.*

**Proof:** Let  $H$  be an arbitrary even subgraph of  $G$ , and let  $H' = \bigoplus_{e \in H \setminus T} \text{cycle}_T(e)$ . The symmetric difference of any two even subgraphs is even, so  $H \oplus H'$  is an even subgraph and therefore the union of edge-disjoint cycles. On the other hand,  $H \oplus H'$  is a subgraph of  $T$  and therefore acyclic. We conclude that  $H \oplus H'$  is empty, or equivalently,  $H = H'$ .  $\square$

Mnemonically, any even subgraph can be *named* by listing its edges in  $C \cup L$ .

Let  $Z$  be a subset of the faces of  $\Sigma$ . The *boundary* of  $Z$ , denoted  $\partial Z$ , is the subgraph of  $\Sigma$  containing every edge that is incident to both a face in  $Z$  and a face in  $F \setminus Z$ . A *boundary subgraph* is any subgraph that is the boundary of a subset of faces. Every boundary subgraph is an even subgraph. Conversely, if  $\Sigma$  is a planar map, the Jordan curve theorem implies that every even subgraph is a boundary subgraph, but this equivalence does not extend to more complex surfaces.

**Fundamental Boundary Lemma:** *Let  $(T, L, C)$  be an arbitrary tree-cotree decomposition of a surface map  $\Sigma$ . For every boundary subgraph  $B$  of  $\Sigma$ , we have*

$$B = \bigoplus_{e \in B \cap C} \text{bdry}_C(e).$$

*Thus, boundary subgraphs are symmetric differences of fundamental boundaries.*

**Proof:** We mirror the proof of the Fundamental Cycle lemma. Let  $B$  be any boundary subgraph, and let  $B' = \bigoplus_{e \in B \cap C} \text{bdry}_C(e)$ . The boundary space is closed under symmetric difference, so  $B' \oplus B$  is a boundary subgraph. On the other hand,  $B \oplus B'$  has no edges in  $C$ , so  $B \oplus B'$  is a subgraph of the cut graph  $T \cup L$ . We conclude that  $B \oplus B'$  is empty, or equivalently,  $B = B'$ .  $\square$ .

Mnemonically, any boundary subgraph can be *named* by listing its edges in  $C$ .

## 20a.2 Homology

Finally, two subgraphs  $A$  and  $B$  of  $\Sigma$  are  $(\mathbb{Z}_2)$ -homologous if their symmetric difference  $A \oplus B$  is a boundary subgraph of  $\Sigma$ . For example, every boundary subgraph is homologous with the empty subgraph, which is why boundary subgraphs are also called *null-homologous* subgraphs. Straightforward definition-chasing implies that  $(\mathbb{Z}_2)$ -homology is an equivalence relation, whose equivalence classes are obviously called  $(\mathbb{Z}_2)$ -homology classes. We usually omit “ $\mathbb{Z}_2$ ” if the type of homology is clear from context.

**Lemma:** *Let  $(T, L, C)$  be an arbitrary tree-cotree decomposition of a surface map  $\Sigma$ . The only boundary subgraph of the cut graph  $T \cup L$  is the empty graph.*

**Proof:** Let  $H$  be a non-empty cut graph in  $\Sigma$ ;  $H$  must be the boundary of a non-empty proper subset  $Z$  of the faces in  $\Sigma$ . Consider the fundamental domain  $\Delta = \Sigma \setminus (T \cup L)$ . Because both  $Z$  and its complement are non-empty, some interior edge  $e$  of  $\Delta$  separates a face in  $Z$  from a face not in  $Z$ . But the interior edges of  $\Delta$  are precisely the edges in  $C$ .  $\square$

**Lemma:** *Let  $(T, L, C)$  be an arbitrary tree-cotree decomposition of a surface map  $\Sigma$ . Every even subgraph in  $\Sigma$  is homologous with an even subgraph of the cut graph  $T \cup L$ .*

**Proof:** It suffices to prove that every edge  $e \in C$  is homologous with a subgraph of  $T \cup L$  that has even degree everywhere except the endpoints of  $e$ .

Consider the fundamental domain  $\Delta = \Sigma \setminus (T \cup L)$ . Every edge  $e \in C$  appears in  $\Delta$  as a boundary-to-boundary chord, which partitions the faces of  $\Delta$  into two disjoint subsets  $Y \sqcup Z$ . (Recall that no edge in  $C$  can be an isthmus!) Every face of  $\Delta$  is a face of the original map  $\Sigma$  and vice versa; let  $\beta$  denote the boundary of  $Y$  (or equivalently, the boundary of  $Z$ ) in  $\Sigma$ . Because  $\beta$  is a boundary subgraph in  $\Sigma$ ,  $e$  is homologous with  $\beta \oplus e$ . Finally, every edge in  $\beta \oplus e$  is an edge in the cut graph  $T \cup L$ .  $\square$

**Lemma:** *Let  $(T, L, C)$  be an arbitrary tree-cotree decomposition of a surface map  $\Sigma$ . Every subgraph of  $\Sigma$  is homologous with a symmetric difference of cycles in  $\mathcal{C}$ .*

**Proof:** By the previous lemma, it suffices to consider only even subgraphs of the cut graph  $T \cup L$ . Every even subgraph of  $T \cup L$  is the symmetric difference of simple cycles in  $T \cup L$ . The simple cycles in  $T \cup L$  are precisely the cycles in  $\mathcal{C}$ .  $\square$

**Homology Basis Theorem:** *Let  $(T, L, C)$  be an arbitrary tree-cotree decomposition of a surface map  $\Sigma$ . For every even subgraph  $H$  of  $\Sigma$ , we have*

$$H = \left( \bigoplus_{i \in I(H)} \text{cycle}_T(\ell_i) \right) \oplus \left( \bigoplus_{e \in H \cap C} \text{bdry}_T(e) \right)$$

for some subset  $I(H) \subseteq \{1, 2, \dots, \bar{g}\}$ . Thus, every even subgraph is homologous with the symmetric difference of a **unique** subset of cycles in  $\mathcal{C}$ , which is nonempty if and only if  $H$  is a boundary

subgraph.

The Homology Basis Theorem immediately implies an algorithm to decide if two even subgraphs  $H$  and  $H'$  are homologous: Compute their canonical decompositions into fundamental cycles and boundaries, with respect to the same tree-cotree decomposition, and then compare the index sets  $I(H)$  and  $I'(H)$ . A careful implementation of this algorithm runs in  $O(\bar{g}n)$  time; details are left as an exercise (because we're about to describe simpler algorithms).

### 20a.3 Relax, it's just linear algebra!

Unlike our earlier characterization of homotopy, our characterization of homology is unique; every even subgraph is homologous with the symmetric difference of *exactly one* subset of cycles in  $\mathcal{C}$ . The easiest way to prove this fact is to observe that subgraphs, even subgraphs, boundary subgraphs, and homology classes all define vector spaces over the finite field  $\mathbb{Z}_2 = (\{0, 1\}, \oplus, \cdot)$ . In particular, homology can be viewed as a linear map between vector spaces.

Subgraphs (subsets of  $E$ ) comprise the **edge space** (or *first chain space*)  $C_1(\Sigma) = \mathbb{Z}_2^{|E|}$ . The (indicator vectors of) individual edges in  $\Sigma$  comprise a basis of the edge space.

Even subgraphs of  $\Sigma$  comprise a subspace of  $C_1(\Sigma)$  called the **cycle space**  $Z_1(\Sigma)$ . The Fundamental Cycle Lemma implies that the fundamental cycles  $\text{cycle}_T(e)$ , for all  $e \notin T$ , define a basis for the cycle space. The number of fundamental cycles is equal to the number of edges not in  $T$ , which is  $|E| - (|V| - 1)$ . Thus,  $Z_1(\Sigma) = \mathbb{Z}_2^{|E| - |V| + 1}$ .

Boundary subgraphs of  $\Sigma$  comprise a subspace of  $Z_1(\Sigma)$  called the **boundary space**  $B_1(\Sigma)$ . The Fundamental Boundary lemma implies that the fundamental boundaries  $\text{bdry}_C(e)$ , for all  $e \in C$ , define a basis for the boundary space. The number of fundamental boundaries is equal to the number of edges of  $C$ , which is  $|F| - 1$ . Thus,  $B_1(\Sigma) = \mathbb{Z}_2^{|F| - 1}$ .

Finally, the set of homology classes of even subgraphs of  $\Sigma$  comprise the **(first) homology space**, which is the quotient space

$$\begin{aligned} H_1(\Sigma) &:= Z_1(\Sigma)/B_1(\Sigma) \\ &= \mathbb{Z}_2^{|E| - |V| + 1} / \mathbb{Z}_2^{|F| - 1} \\ &\cong \mathbb{Z}_2^{|E| - (|V| - 1) - (|F| - 1)} \\ &= \mathbb{Z}_2^{|E| - |T| - |C|} = \mathbb{Z}_2^{|L|} = \mathbb{Z}_2^{\bar{g}}. \end{aligned}$$

(Hey look, we proved Euler's formula again!) The Homology Basis Theorem implies that homology classes of fundamental cycles  $\text{cycle}_T(e)$ , for all  $e \in L$ , define a basis for the homology space. In particular, there are exactly  $2^{\bar{g}}$  distinct homology classes.

### 20a.4 Crossing Numbers

Another way to characterize the homology class of an even subgraph  $H$  is to determine which cycles in a system of cycles *cross*  $H$ . The definition of "cross" is rather subtle, but mirrors the intuition of transverse intersection.

Consider two distinct simple cycles  $\alpha$  and  $\beta$ , and let  $\pi$  be one of the components of the intersection  $\alpha \cap \beta$ . (Because  $\alpha \neq \beta$ , the intersection  $\pi$  must be either a single vertex of a common subpath.)

We call  $\pi$  a *crossing* between  $\alpha$  and  $\beta$  (or we say that  $\alpha$  and  $\beta$  *cross* at  $\pi$ ) if, after contracting the path  $\pi$  to a point  $p$ , the contracted curves  $\alpha/\pi$  and  $\beta/\pi$  intersect transversely at  $p$ .

Equivalently,  $\alpha$  and  $\beta$  cross at  $\pi$  if, no matter how we perturb the two curves within a small neighborhood of  $\pi$ , the two perturbed curves  $\tilde{\alpha}$  and  $\tilde{\beta}$  intersect. By convention, no two-sided cycle crosses itself (because we can perturb two copies of a two-sided cycle so that they are disjoint), but every one-sided cycle crosses itself once (because we cannot).

For any simple cycles  $\alpha$  and  $\beta$ , the *crossing number*  $cr(\alpha, \beta)$  is the number of crossings between  $\alpha$  and  $\beta$ , modulo 2. In particular,  $cr(\alpha, \alpha) = 0$  if for every two-sided cycle  $\alpha$ , and  $cr(\beta, \beta) = 1$  for every one-sided cycle  $\beta$ .

We can extend this definition of crossing number to even subgraphs by linearity:  $cr(\alpha \oplus \beta, \gamma) = cr(\alpha, \gamma) \oplus cr(\beta, \gamma)$ . Although one can express any even subgraph as a symmetric difference of cycles in many different ways, crossing numbers are the same for every such decomposition.

For any face  $f$  and any cycle  $\gamma$ , we have  $cr(\partial f, \gamma) = 0$ . It follows by linearity that if either  $\gamma$  or  $\delta$  is a boundary subgraph, then  $cr(\delta, \gamma) = 0$ . More generally, it follows that crossing numbers are a *homology invariant*: if  $\alpha$  and  $\beta$  are homologous even subgraphs, then  $cr(\alpha, \gamma) = cr(\beta, \gamma)$  for every cycle  $\gamma$ , because  $\alpha \oplus \beta$  is the symmetric difference of face boundaries.

**Lemma:** For any even subgraphs  $H$  and  $H'$ , if  $cr(H, H') = 1$ , then neither  $H$  nor  $H'$  is a boundary subgraph.

**Proof:** If (say)  $H$  is a boundary subgraph, then  $H$  is the symmetric difference of face boundaries, and therefore  $cr(H, H') = 0$  by linearity.  $\square$

**Lemma:** Let  $\sigma$  be a simple cycle and let  $\mathcal{C} = \{\gamma_1, \gamma_2, \dots, \gamma_{\bar{g}}\}$  be a system of cycles in a surface map  $\Sigma$ . Then  $\sigma$  is boundary cycle if and only if  $cr(\sigma, \gamma_i) = 0$  for every cycle  $\gamma_i \in \mathcal{C}$ .

**Proof:** If  $\sigma$  is a boundary cycle, homology invariance immediately implies  $cr(\sigma, \gamma_i) = cr(\emptyset, \gamma_i) = 0$ .

Suppose on the other hand that  $\sigma$  is not a boundary cycle. Then by definition the sliced surface  $\Sigma \setminus \sigma$  is connected. Let  $v$  be a vertex of  $\sigma$ , and let  $\pi$  be any path from  $v^+$  to  $v^-$  in  $\Sigma \setminus \sigma$ . This path  $\pi$  appears in  $\Sigma$  as a closed walk that crosses  $\sigma$  exactly once, so  $cr(\pi, \sigma) = 1$ . It follows from the previous lemma that  $\pi$  is not a boundary cycle. Thus, by the Homology Basis theorem,  $\pi$  is homologous with  $\bigoplus_{i \in I} \gamma_i$  for some non-empty subset  $I \subseteq \{1, 2, \dots, \bar{g}\}$ . Finally, homology invariance implies  $cr(\pi, \sigma) = \bigoplus_{i \in I} cr(\gamma_i, \sigma) = 1$ , so we must have  $cr(\gamma_i, \sigma) = 1$  for an odd number of indices  $i \in I$ , and therefore for at least one such index.  $\square$

**Corollary:** Let  $\mathcal{C}$  be a system of cycles in a surface map  $\Sigma$ . An even subgraph  $H$  of  $\Sigma$  is a boundary subgraph if and only if  $cr(H, \gamma_i) = 0$  for every cycle  $\gamma_i \in \mathcal{C}$ . Two even subgraphs  $H$  and  $H'$  of  $\Sigma$  are homologous if and only if  $cr(H, \gamma_i) = cr(H', \gamma_i)$  for every cycle  $\gamma_i \in \mathcal{C}$ .

## 20a.5 Systems of Cocycles and Cohomology

Cohomology is the dual of homology. While homology is an equivalence relation between subgraphs of maps, cohomology is an equivalence relation between subgraphs of *dual* maps. In fact, it's the *dual* equivalence relation between subgraphs of dual maps. Two subgraphs  $A$  and  $B$  of  $\Sigma$  are *cohomologous* if and only if the corresponding dual subgraphs  $A^*$  and  $B^*$  of  $\Sigma^*$  are homologous.

I'll adopt the convenient convention of adding the prefix “co” to indicate the dual of a structure in the dual map. Mnemonically, a cosnarfle in  $\Sigma$  is the dual of snarfle in  $\Sigma^*$ .

- We've already defined a *spanning co-tree* of  $\Sigma$  is a subset of edges whose corresponding dual edges comprise a spanning tree of  $\Sigma^*$ . Less formally, a spanning cotree of  $\Sigma$  is the dual of a spanning tree of  $\Sigma^*$ .
- A *cocycle* in  $\Sigma$  is the dual of a cycle in  $\Sigma^*$ . (In planar graphs, every cocycle is a minimal edge cut, but that equivalence does not extend to more complex surfaces.)
- A *co-even subgraph* of  $\Sigma$  is the dual of an even subgraph of  $\Sigma^*$ . That is, a subgraph  $H$  of  $\Sigma$  is co-even if every face of  $\Sigma$  has an even number of incidences with  $H$ . No edge in a co-even subgraph is a loop, because loops are co-isthmuses.
- The *coboundary* of a subset  $X$  of vertices of  $\Sigma$ , denoted  $\delta X$ , is the dual of the boundary of the corresponding subset  $X^*$  of faces of  $\Sigma^*$ . That is,  $\delta X$  is the subset of edges with one endpoint in  $X$  and one endpoint not in  $X$ . A *coboundary subgraph* is the coboundary of some subset of vertices. Every coboundary subgraph is co-even.
- Finally, two co-even subgraphs are cohomologous if their symmetric difference is a coboundary subgraph.

As usual, fix a tree-cotree decomposition  $(T, L, C)$  of a surface map  $\Sigma$ . For every edge  $e \in T \cup L$ , let  $\text{cocycle}_C(e)$  denote the subgraph of  $\Sigma$  dual to the fundamental cycle  $\text{cycle}_{C^*}(e^*)$  in the dual map  $\Sigma^*$ . Finally, let  $\mathcal{K} = \{\text{cocycle}_C(e) \mid e \in T\}$ . The following lemmas follow immediately from our earlier characterization of homology.

**Lemma:** *Let  $(T, L, C)$  be an arbitrary tree-cotree decomposition of a surface map  $\Sigma$ . Every co-even subgraph of  $\Sigma$  is a symmetric difference of fundamental cocycles  $\text{cocycle}_C(e)$  where  $e \notin C$ .*

**Lemma:** *Let  $(T, L, C)$  be an arbitrary tree-cotree decomposition of a surface map  $\Sigma$ . Every co-even subgraph of  $\Sigma$  is cohomologous with a co-even subgraph of the cocut graph  $C \cup L$ .*

**Lemma:** *Let  $(T, L, C)$  be an arbitrary tree-cotree decomposition of a surface map  $\Sigma$ . Every co-even subgraph of  $\Sigma$  is cohomologous with a symmetric difference of cocycles in  $\mathcal{K}$ .*

## 20a.6 Homology Signatures

More importantly, however, cohomology offers us a **CON**venient method to efficiently **COM**pute homology classes of even subgraphs of the primal map  $\Sigma$ , by assigning a **CO**ordinate system to the first homology space. Index the leftover edges in  $L$  as  $\ell_1, \ell_2, \dots, \ell_{\bar{g}}$ .<sup>1</sup> For every edge  $e$  in  $\Sigma$ , the *homology signature*  $[e]$  is the  $\bar{g}$ -bit vector indicating which cocycles in  $\mathcal{K}$  contain  $e$ . Specifically:

$$[e]_i = 1 \iff e \in \text{cocycle}_C(\ell_i).$$

Finally, the homology signature  $[H]$  of any subgraph  $H$  is the bitwise exclusive-or of the homology signatures of its edges.

The function  $H \mapsto [H]$  is a *linear* function from the cycle space  $Z_1(\Sigma)$  to the vector space  $\mathbb{Z}_2^{\bar{g}}$  of homology signatures. In particular:

**Linearity Lemma:** *For any two even subgraphs  $H$  and  $H'$  of  $\Sigma$ , we have  $[H \oplus H'] = [H] \oplus [H']$ .*

**Basis Lemma:** *For all indices  $i$  and  $j$ , we have  $[\text{cycle}_T(\ell_i)]_j = 1$  if and only if  $i = j$ .*

<sup>1</sup>Here I'm using  $\ell$  as a mnemonic for “leftover edge” instead of “loop”. We have a lot of other  $e$ 's flying around, so I don't want to use  $e_i$  to denote the  $i$ th edge in  $L$ .

**Proof:** The only edge in any fundamental  $\text{cycle}_T(e)$  that is *not* in  $T$  is the determining edge  $e$ . Similarly, the only edge in any fundamental  $\text{cocycle}_C(e)$  that is *not* in  $C$  is the determining edge  $e$ . Thus,  $\text{cycle}_T(\ell_i) \cap \text{cocycle}_C(\ell_j) = \emptyset$  whenever  $i \neq j$ , and  $\text{cycle}_T(\ell_i) \cap \text{cocycle}_C(\ell_i) = \ell_i$  for every index  $i$ .  $\square$

**Theorem:** Two even subgraphs  $H$  and  $H'$  of  $\Sigma$  are homologous if and only if  $[H] = [H']$ .

**Proof:** By the Linearity Lemma, it suffices to prove that an even subgraph  $H$  is a boundary subgraph if and only if  $[H] = 0$ .

Let  $f$  be any face of  $\Sigma$ , and let  $\lambda$  be any cocycle in  $\Sigma$ . The boundary of  $f$  either contains no edges of  $\lambda$  or exactly two edges of  $\lambda$ , depending on whether the dual cycle  $\lambda^*$  contains the dual vertex  $f^*$ . It follows that  $[\partial f] = 0$  for every face  $f$ . The Linearity Lemma implies that  $[H] = 0$  for every boundary subgraph  $H$ .

Conversely, suppose  $H$  is not null-homologous. Then we can write

$$H = \left( \bigoplus_{i \in I} \text{cycle}_T(\ell_i) \right) \oplus \left( \bigoplus_{e \in H \cap C} \text{bdry}_T(e) \right)$$

for some nonempty subset  $I \subseteq \{1, 2, \dots, \bar{g}\}$ . The Linearity and Basis lemmas imply that

$$[H] = \left( \bigoplus_{i \in I} [\text{cycle}_T(\ell_i)] \right)$$

and therefore  $[H]_i = 1$  if and only if  $i \in I$ . Because  $I$  is non-empty,  $[H] \neq 0$ .  $\square$

## 20a.7 Separating Cycles

**Lemma:** Let  $\gamma$  be a simple cycle in a surface map  $\Sigma$ . The sliced map  $\Sigma \setminus \gamma$  is disconnected if and only if  $[\gamma] = 0$

## 20a.8 References

## 20a.9 Aptly Named Sir

- Pants decompositions (except possibly in passing)