12 Tutte's Spring Embedding Theorem$^{\beta}$

In 1963, William Tutte published a paper ambitiously entitled "How to Draw a Graph". Let Σ be any planar embedding of any simple planar graph G.

- Nail the vertices of the outer face of Σ to the vertices of an arbitrary strictly convex polygon P in the plane, in cyclic order.
- Build the edges of G out of springs or rubber bands.
- Let go!

Tutte proved that if the input graph G is sufficiently well-connected, then this physical system converges to a *strictly convex* planar embedding of G!

Let me state the parameters of the theorem more precisely, in slightly more generality than Tutte did.[1] A *Tutte drawing* of a planar graph G is described by a *position* function $p\colon V \to \mathbb{R}^2$ mapping the vertices to points in the plane, subject to the two conditions:

(1) The vertices of the outer face f_∞ of some planar embedding of G are mapped to the vertices of a strictly convex polygon in cyclic order. In particular, the boundary of F_∞ must be a simple cycle.

(2) Each vertex v that is not in f_∞ maps to a point in the interior of the convex hull of its neighbors; that is, we have

$$\sum_{u \to v} \lambda_{u\to v}(p_v - p_u) = 0$$

for some positive real coefficients $\lambda_{u\to v}$ on the darts into v.

(I will use subscript notation p_v instead of function notation $p(v)$ throughout this chapter.) The edges of a Tutte drawing are line segments connecting their endpoints. Let me emphasize that that the *definition* of a Tutte drawing does not require mapping edges to *disjoint* segments, or even mapping vertices to *distinct* points. Moreover, the dart coefficients are not required to be symmetric; it is possible that $\lambda_{u\to v} \neq \lambda_{v\to u}$.

A graph G is *3-connected* if we can delete any two vertices without disconnecting the graph, or equivalently (by Menger's theorem) if every pair of vertices is connected by at least three vertex-disjoint paths.

Finally, a planar embedding is *strictly convex* if the boundary of every face of the embedding is a convex polygon, and no two edges on any face boundary are collinear.

Tutte's spring-embedding theorem: *Every Tutte drawing of a simple 3-connected planar graph G is a strictly convex straight-line embedding.*

It is not hard to see that 3-connectivity is required. If G has an articulation vertex v, that is, a vertex whose deleting leaves a disconnected subgraph, then a Tutte drawing of G can map an entire component of $G \setminus v$ to the point p_v. Similarly, if G has two vertices v and v such that $G \setminus \{u, v\}$ is disconnected, a Tutte drawing of G can map an entire component of $G \setminus \{u, v\}$ to the line segment $p_u p_v$. In both cases, the Tutte drawing is not even an embedding, much less a strictly convex embedding.

[1] The formulation and proof of Tutte's theorem that I'm presenting here follows a lecture note by Dan Spielman (2018), which is based on papers by Michael Floater (1997); László Lovász (1999); Steven Gortler, Craig Gotsman, and Dylan Thurston (2006); and Jim Geelen (2012).

12.1 Outer Face is Outer

Whitney (1932) proved that every simple 3-connected graph G has a unique embedding on the sphere (up to homeomorphism), or equivalently, a unique planar rotation system. I will describe Whitney's proof later in this note. Thus, in every planar embedding of G, the faces are bounded by the same set of cycles; we can reasonably call these cycles the *faces of G*.

The definition of a Tutte drawing requires choosing one of the faces of G to be the outer face f_∞. We call the vertices of f_∞ *boundary* vertices, and the remaining vertices of G *interior* vertices. Similarly, we call the edges of f_∞ *boundary* edges, and the remaining edges of G *interior* edges. This terminology is justified by the following observation:

Outer face lemma: *In every Tutte drawing of a simple 3-connected planar graph G, every interior vertex maps to a point in the interior of the outer face. In particular, no interior vertex maps to the same point as a boundary vertex.*

Proof: We say that an interior vertex w *directly reaches* a boundary vertex z, or symmetrically that z is *directly reachable* from w, if there is a path from w to z using only interior edges. 3-connectivity implies that every interior vertex of G can directly reach at least three boundary vertices of G.

We prove the lemma by applying Gaussian elimination to the system of linear equations defined by condition (2). Linear system (2) expresses the position p_v of any vertex v as a strict convex combination of the positions of its neighbors in G, that is, a weighted average where every neighbor of v has positive weight. By pivoting on that row, we can remove the variables p_v from the system.

Such a pivot is equivalent to deleting vertex v from the graph and adding new edges between the neighbors of v, with appropriate positive coefficients on their darts.[2] (Of course the resulting graph may not be planar.) Pivoting out one interior vertex does not change which boundary vertices are directly reachable from any other interior vertex. Thus, if we eliminate all but one interior vertex w, the remaining constraint expresses w as a strict convex combination of at least three boundary vertices.

The same elimination argument implies that *every* assignment of positive dart coefficients $\lambda_{u\to v} > 0$ defines a *unique* Tutte drawing; the linear system containing the equation

$$\sum_{u\to v} \lambda_{u\to v}(p_v - p_u) = 0$$

for every interior vertex v always has full rank.

12.2 Laplacian linear systems and energy minimization

Tutte's original formulation required that every interior vertex lie at the center of mass of its neighbors; this is equivalent to requiring $\lambda_{u\to v} = 1$ for every dart $u\to v$.[3] More generally, the

[2] This modification is called a *star-mesh transformation*; the special case of removing a vertex of degree 3 is called a *Y-Δ transformation*.

[3] It is sometimes more convenient to formalize Tutte's description as $\lambda_{u\to v} = 1/\deg(v)$, so that the weights of all darts into each vertex v sum to 1. This formalization is inconsistent with the physical spring analogy, but instead treats weights as transition probabilities of a (backward) random walk. Both formalizations lead to the same Tutte drawing.

physical interpretation in terms of springs corresponds to the special case where dart coefficients are symmetric.

Suppose each edge uv is to a (first-order linear) spring with spring constant $\omega_{uv} = \lambda_{u\to v} = \lambda_{v\to u}$. For any vertex placement $p \in (\mathbb{R}^2)^V$, the total potential energy in the network of springs is

$$\Phi(p) := \frac{1}{2}\sum_{u,v}\omega_{uv}\|p_u - p_v\|^2.$$

If we fix the positions of the outer vertices, Φ becomes a strictly convex[4] function of the interior vertex coordinates. If we let the interior vertex positions vary, the network of springs will come to rest at a configuration with locally minimal potential energy. The unique minimum of Φ can be computed by setting the gradient of Φ to the zero vector and solving for the interior coordinates; thus we recover the original linear constraints

$$\sum_{v}\omega_{uv}(p_u - p_v) = 0$$

for every interior vertex u. The underlying matrix of this linear system is called a weighted *Laplacian* matrix of G. This matrix is positive definite[5] and therefore non-singular, so a unique equilibrium configuration always exists.

When the dart coefficients are not symmetric, this physical intuition goes out the window; the linear system of balance equations is no longer the gradient of a convex function. Nevertheless, as we've already argued, any choice of positive coefficients $\lambda_{u\to v}$ corresponds to a unique straight-line drawing of G. None of the actual proof of Tutte's theorem relies on any special properties of the coefficients $\lambda_{u\to v}$ other than positivity.

Given the graph G, the outer convex polygon, and the dart coefficients, we can compute the corresponding vertex positions in $O(n^3)$ time via Gaussian elimination. (There are faster algorithms to solve this linear system. In particular, a numerically approximate solution can be computed in $O(n \log n)$ time in theory, or in $O(n \operatorname{polylog} n)$ time in practice.)

12.3 Slicing with Lines

For the rest of this note, fix a simple 3-connected planar graph G and a Tutte drawing p. At the risk of confusing the reader, I will generally not distinguish between features of the abstract graph G (vertices, edges, faces, cycles, paths, and so on) and their images under the Tutte drawing (points, line segments, polygons, polygonal chains, and so on). For example, an *edge* of the Tutte drawing p is the (possibly degenerate) line segment between the images of the endpoints of an edge of H, and a *face* of the Tutte drawing p is the (not necessarily simple) polygon whose vertices are the images of the vertices of a face of G in cyclic order.

Both sides lemma: *For any interior vertex v and any line ℓ through p_v, either all neighbors of v lie on ℓ, or v has neighbors on both sides of ℓ.*

Proof: Suppose all of v's neighbors lie in one closed halfplane bounded by ℓ. Then the convex hull of v's neighbors also lies in that halfspace, which implies that v does not lie in the interior of that convex hull, contradicting the definition of a Tutte drawing. □

[4]The Hessian of Φ is positive definite, meaning all of its eigenvalues are positive.
[5]The Laplacian matrix is just the Hessian of Φ.

Halfplane lemma: *Let H be any halfplane that contains at least one vertex of G. The subgraph of G induced by all vertices in H is connected.*

Proof: Without loss of generality, assume that H is the halfplane above the x-axis. Let t be any vertex with maximum y-coordinate; the outer face lemma implies that t is a boundary vertex. I claim that for any vertex $u \in H$, there is a directed path in G from u to t, where the y-coordinates never decrease. There are two cases to consider:

- If t and u have the same y-coordinate, the outer-face lemma implies that either $t = u$ or tu is an edge of the outer face. In either case the claim is trivial.
- Otherwise, u must lie below t. Let U be the set of all vertices reachable from u along horizontal edges of G. Because G is connected, some vertex $v \in U$ has a neighbor that is not in U. The both-sides lemma implies that v has a neighbor w that has larger y-coordinate than v. The induction hypothesis implies that there is a y-monotone path in G from $w \rightsquigarrow t$. Thus, $u \rightsquigarrow v \rightarrow w \rightsquigarrow t$ is a y-monotone path, which proves the claim. $\square$

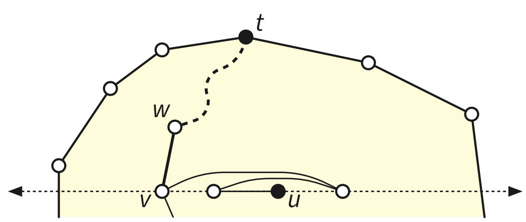

Figure 1: The halfplane lemma.

12.4 No Degenerate Vertex Neighborhoods

None of the previous lemmas actually require the planar graph G to be 3-connected. The main technical challenge in proving Tutte's theorem is showing that if G is 3-connected, then every Tutte drawing of G is non-degenerate. The assumption of 3-connectivity is necessary[6]—if G is 2-connected but not 3-connected, then some subgraphs of G can degenerate to line segments in the Tutte drawing, and if G is connected but not 2-connected, some subgraphs of G *will* degenerate to single points.

Utility lemma: *The complete bipartite graph $K_{3,3}$ is not planar.*

Proof: $K_{3,3}$ has $n = 6$ vertices and $m = 9$ edges, so by Euler's formula, any planar embedding would have exactly $2 + m - n = 5$ faces. On the other hand, because $K_{3,3}$ is simple and bipartite, every face in any planar embedding would have degree at least 4. Thus, a planar embedding of $K_{3,3}$ would imply $20 = 4f \leq 2m = 18$, which is obviously impossible. $\square$

Nondegeneracy lemma: *No vertex of G is collinear with all of its neighbors.*

Proof: By definition, no three boundary vertices are collinear, and thus no boundary vertex is collinear with all of its neighbors.

For the sake of argument, suppose some vertex u and all of its neighbors lies on a common line ℓ, which without loss of generality is horizontal. Let V^+ and V^- be the subsets of

[6] In fact, we only need the weaker assumption that G is *internally* 3-connected, meaning each interior vertex has three vertex-disjoint paths to the outer face.

vertices above and below ℓ, respectively. Let U be the set of all vertices that are reachable from u and whose neighbors all lie on ℓ. The halfplane lemma implies that the induced subgraphs $G[V^+]$ and $G[V^-]$ are connected, and the induced subgraph $G[U]$ is connected by definition. Fix arbitrary vertices $v^+ \in V^+$ and $v^- \in V^-$.

Finally, let W denote the set of all vertices that lie on line ℓ and are adjacent to vertices in U, but are not in U themselves. Every vertex in W has at least one neighbor not in ℓ, so by the both-sides lemma, every vertex in W has neighbors in both V^+ and V^-. Deleting the vertices in W disconnects U from the rest of the graph. Thus, **because G is 3-connected**, W contains at least three vertices w_1, w_2, w_3.

Now suppose we contract the induced subgraphs $G[V^+]$, $G[V^-]$, and $G[U]$ to the vertices v^+, v^-, and u, respectively. The resulting minor of G contains the complete bipartite graph $\{v^+, v^-, u\} \times \{w_1, w_2, w_3\} = K_{3,3}$. But this is impossible, **because G is planar** and therefore every minor of G is planar. □

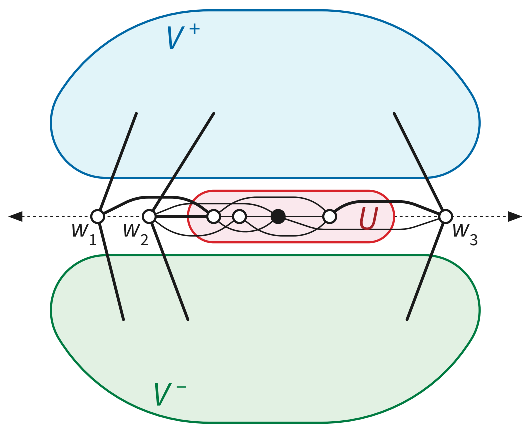

Figure 2: A collinear vertex neighborhood implies a $K_{3,3}$ minor.

Both sides redux: *Every interior vertex v has neighbors on both sides of any line through p_v.*

12.5 No Degenerate Faces

It remains to prove that the faces of the Tutte drawing are nondegenerate. First we need a combinatorial lemma, similar to Fáry's lemma that any simple planar map can be refined into a simple triangulation.

Geelen's Lemma: *Let uv be any edge of G, let f and f' be the faces incident to uv, and let S and S' be the vertices of these two faces other than u and v. Let P be any path that starts at a vertex in S and ends at a vertex of S'. Then every path from u to v in G either consists of the edge uv or contains a vertex of P.*

Proof: Fix any planar embedding of G (not necessarily the Tutte drawing!) where uv is an interior edge. The faces incident to uv are disjoint disks on either side of uv. Let s and t be the endpoints of P. Let P' be a path from l to r through the union of the faces incident to uv, crossing the edge uv once. The closed curve $C = P + P'$ separates u from v. Thus, by the Jordan curve theorem, every path Q from u to v crosses C, which implies that either $Q = uv$ or Q contains a vertex of P. □

Split Faces Lemma: *Let uv be any interior edge of G, let f and f' be the faces incident to uv, and*

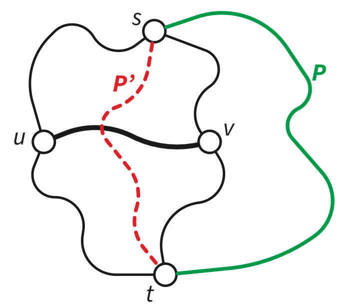

Figure 3: Geelen's lemma.

let S and S' be the vertices of these two faces other than u and v. Finally, let ℓ be any line through p_u and p_v. Then S and S' lie on opposite sides of ℓ; in particular, no vertex in $S \cup S'$ lies on ℓ.

Proof: Without loss of generality, assume ℓ is horizontal. For the sake of argument, suppose both S and S' contain vertices s and t that lie on or below ℓ. If s lies on ℓ, the nondegeneracy lemma implies that s has a neighbor s' strictly below ℓ; otherwise, let $s' = s$. Similarly, if t lies on ℓ, the nondegeneracy lemma implies that t has a neighbor t' strictly below ℓ; otherwise, let $t' = t$. The halfspace lemma implies that there is a path P' in G from s' to t' that lies entirely below ℓ. Let P be the path from s to t consisting of the edge ss' (if $s \neq s'$), the path P', and the edge $t't$ (if $t \neq t'$).

The nondegeneracy lemma also implies that u and v have respective neighbors u' and v' strictly above ℓ, and the halfspace lemma implies that there is a path Q' from u' to v' that lies strictly above ℓ. Let Q be the path from u to v consisting of the edge uu', the path Q', and the edge $v'v$.

The edge uv and the path P satisfy the conditions of Geelen's lemma. The path Q clearly avoids the edge uv, so Q must cross P. But P and Q lie on opposite sides of ℓ. We have reached a contradiction, completing the proof. □

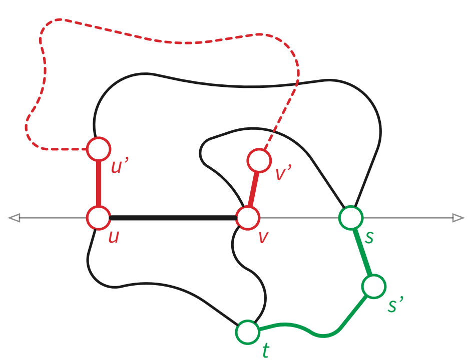

Figure 4: The split faces lemma.

Corollary: *No edge of G maps to a single point.*

Proof: Suppose $p_u = p_v$ for some edge uv. Let ℓ be any line through $p_u = p_v$ and some other vertex on a face incident to uv. We immediately have a contradiction to the previous lemma. □

Convexity lemma: *Every face of G maps to a strictly convex polygon.*

Proof: Let f be any face of G, let uv be any edge of f, and let ℓ be the unique line containing p_u and p_v. If uv is a boundary edge, the outer face lemma implies that every vertex of f except u and v lies strictly on one side of ℓ. Similarly, if uv is an interior edge, the split faces lemma implies that every vertex of f except u and v lies strictly on one side of ℓ. In particular, no other vertex of f lies on the line ℓ. It follows that uv is an edge of the convex hull of f. We conclude that f coincides with its convex hull. $\square$

Now we are finally ready to prove the main theorem.

Proof: Call a point *generic* if it does not lie in the image of the Tutte drawing. Consider any path from a generic point p out to infinity that does not pass through any vertex in the drawing. The split faces lemma implies that whenever the moving point crosses an edge e, it leaves one face and enters another. When the moving point reaches infinity, it is only in the outer face. Thus, every generic point lies in exactly one face.

For the sake of argument, suppose two edges uv and xy intersect in the Tutte drawing. Then any generic point near the intersection $uv \cap xy$ must lie in two different faces, which we just showed is impossible. We conclude that the Tutte drawing is an embedding; in particular, every face is a simple polygon. We already proved that every face in this embedding is strictly convex. $\square$

12.6 Whitney's Uniqueness Theorem

Tutte's theorem is as strong as possible in the following sense: Every planar graph with a strictly convex embedding is 3-connected. This observation follows from an earlier study of 3-connected planar graphs by Hassler Whitney.

A planar map is *polyhedral* if (1) the boundary of every face is a simple cycle, and (2) the intersection of any two facial cycles is either empty, a single vertex, or a single edge. Every strictly convex planar map is polyhedral.

Lemma: *Every planar embedding of a 3-connected planar graph is polyhedral.*

Proof: Fix a planar embedding Σ of some graph G.

Suppose the boundary of some face f is not a simple cycle. The boundary of f must have a repeated vertex v. So the radial map $\Sigma^\diamond$ contains a cycle of length 2 through v and f, which has at least one other vertex of Σ on either side. It follows that $G \setminus v$ must be disconnected.

Suppose two faces f and g have two vertices u and v in common, but not the edge uv. Then the radial map $\Sigma^\diamond$ contains a simple cycle with vertices f, u, g, v, which has at least one other vertex of G on either side. It follows that $G \setminus \{u, v\}$ is disconnected.

We conclude that if Σ is not polyhedral, then G is not 3-connected. $\square$

Lemma: *If a graph G has a polyhedral planar embedding, then G is 3-connected.*

Proof: Let G be any graph that is not 3-connected, and let Σ be any planar embedding of G. Again, there are two cases to consider:

- Suppose $G \setminus v$ is disconnected, for some vertex v. Then the same face of Σ must be incident to v twice.

- Suppose $G \setminus \{u, v\}$ is disconnected, for some vertices u and v. We can assume without loss of generality that u and v are not adjacent, since otherwise $G \setminus u$ is already disconnected. Then some pair of faces f and g must have both u and v on their boundaries, but not the edge uv.

In both cases, we conclude that Σ is not polyhedral. □

Lemma: *The dual Σ^* of any polyhedral planar map Σ is polyhedral.*

Proof: Suppose Σ has a face f whose boundary is not a simple cycle. Then the boundary walk of f encounters some vertex v more than once; in other words, v and f are incident more than once. Thus, in the dual map Σ^*, the dual vertex f^* and the dual face v^* are incident more than once, so the boundary of v^* is not a simple cycle.

On the other hand, suppose Σ has two faces f and g that share two vertices v and w, but there is no dart with endpoints v and w and shores f and g. It follows that the dual faces v^* and w^* in Σ^* share the dual vertices f^* and g^*, but there is no dart with endpoints f^* and g^* and shores v^* and w^*.

We conclude that if Σ is not polyhedral, then neither is Σ^*. □

Lemma (Whitney): *Every planar graph has at most one polyhedral embedding.*

Proof: Let Σ be a polyhedral planar embedding of some graph G (which must be planar and 3-connected by previous lemmas), and let Σ' be any embedding of G that is not equivalent to Σ. Let succ and succ$'$ be the successor permutations of Σ and Σ', respectively. Because Σ and Σ' are not equivalent, succ$'$ is not equal to either succ or succ^{-1}.

First, suppose there is a dart d such that $\mathsf{succ}'(d)$ is not equal to either $\mathsf{succ}(d)$ or $\mathsf{succ}^{-1}(d)$. In other words, suppose there is a vertex $v = \mathsf{head}(d)$ where the cyclic orders of darts into v in the two embeddings are different. The darts d and $\mathsf{succ}'(d)$ split the cycle of darts around v into two non-empty intervals; color the darts in one interval red and the other interval blue. In particular, color $\mathsf{succ}(d)$ red and color $\mathsf{succ}^{-1}(d)$ blue. There must be another dart d' that is red or blue, whose successor $\mathsf{succ}'(d)$ in Σ' is either blue or red, respectively.

Let C be the simple cycle in G that bounds face $f = \mathsf{left}'(d) = \mathsf{right}'(\mathsf{succ}'(d))$ in Σ'. (If the boundary of f is not a simple cycle, then Σ' is not polyhedral and we are done.) Similarly, let Let C' be the cycle in G that bounds $f' = \mathsf{left}'(d') = \mathsf{right}'(\mathsf{succ}'(d'))$ in Σ'. The images of C and C' in the polyhedral embedding Σ cross each other at v, and therefore (by the Jordan curve theorem) share at least one other vertex w. It follows that faces f and f' in Σ' share vertices v and w, but do not share the edge vw (if that edge exists). We conclude that Σ' is not polyhedral.

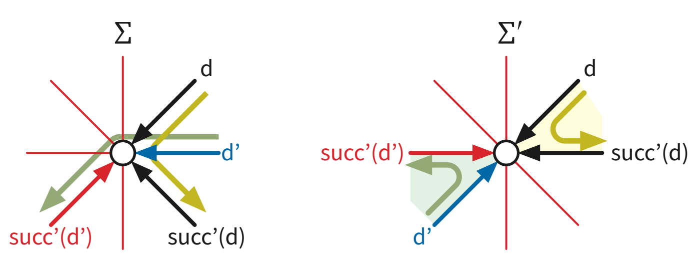

Figure 5: If two embeddings disagree at a vertex, at least one embedding is not polyhedral

On the other hand, suppose are two darts d and d' such that $\mathsf{succ}'(d) = \textsc{succ}(d)$ and $\mathsf{succ}'(d')$ or $\mathsf{succ}^{-1}(d')$. In other words, suppose the dart order around $v = \mathsf{head}(d)$ is the same in both embeddings, but the dart order around $w = \mathsf{head}(d)$ is reversed from one embedding to the other. Without loss of generality, v and w are adjacent, and we can assume $d = v{\to}w$ and $d' = \mathsf{rev}(d) = w{\to}v$.

Let C and C' be the cycles in G that bound faces $f = \mathsf{left}'(d) = \mathsf{right}'(d')$ and $f' = \mathsf{left}'(d') = \mathsf{right}'(d)$ in Σ', respectively. After an arbitrarily small perturbation, the images of C and C' in the polyhedral embedding Σ cross each other at the midpoint vw, and therefore share at least one other vertex x. It follows that the faces f and f' in Σ' have disconnected intersection, and therefore Σ' is not polyhedral. $\square$

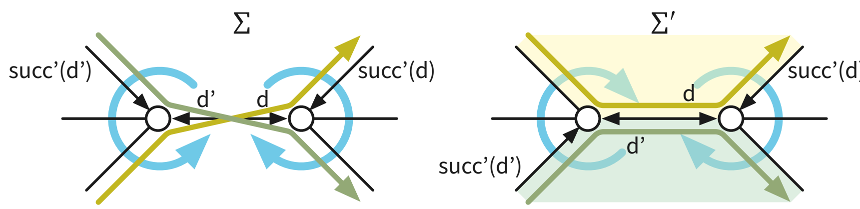

Figure 6: If two embeddings disagree along an edge, at least one embedding is not polyhedral

Together, the previous lemmas now imply Whitney's unique-embedding theorem.

Theorem (Whitney): *Every 3-connected planar graph has a unique planar embedding (up to homeomorphism), which is polyhedral.*

In light of Whitney's observation, Tutte's spring-embedding theorem immediately implies the following corollary:

Convex Embedding Theorem: *For every polyhedral planar embedding, there is an equivalent strictly convex embedding.*

12.7 Not Appearing

- Weakly convex faces and internal 3-connectivity
- Directed version allowing zero dart weights via "strong 3-connectivity"
- Colin de Verdière matrices and spherical spectral embeddings
- More spectral graph algorithms!