Technical Glossary

Topology

The Line

- The *line* $\mathbb{R}$ is the set of all real numbers.
- There are several types of *intervals* on the real line:
  - *Bounded open* interval $(a,b) := \{x \mid a < x < b\}$.
  - *Bounded closed* interval $(a,b) := \{x \mid a \leq x \leq b\}$.
  - *Bounded half-open* intervals $(a,b] := \{x \mid a < x \leq b\}$ and $[a,b) := \{x \mid a \leq x < b\}$.
  - *Open half-lines* $(a,\infty) := \{x \mid a < x\}$ and $(-\infty,b) := \{x \mid x < b\}$.
  - *Closed half-lines* $[a,\infty) := \{x \mid a \leq x\}$ and $(-\infty,b] := \{x \mid x \leq b\}$.
  - The empty set $\varnothing = (a,a)$ for any $a \in \mathbb{R}$.
  - The entire real line $\mathbb{R} = (-\infty,\infty)$.
- A subset $X \subseteq \mathbb{R}$ is *open* if, for every point $x \in X$, there are real numbers a and b such that $x \in (a,b) \subseteq X$. More concisely: A subset of $\mathbb{R}$ is open if and only if it is the union of bounded open intervals. The empty set, bounded open intervals, open halflines, and the entire real line are open.
- A subset $X \subseteq \mathbb{R}$ is *closed* if and only if its complement $\mathbb{R} \setminus X$ is open. The empty set, bounded closed intervals, closed halflines, and the entire real line are closed. **Closed does *not* mean "not open"!**
- A subset $X \subseteq \mathbb{R}$ is *bounded* if it is a subset of a bounded interval.
- A subset $X \subseteq \mathbb{R}$ is *compact* if it is both closed and bounded.

The Plane

- The *plane* $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the set of all ordered pairs of real numbers,
- An *open box* is the Cartesian product of two open intervals.
- An *open disk* is the interior of any circle.
- A subset of the plane is *open* if it is the union of open boxes, or equivalently, the union of open disks.

Topological Spaces

- A *topological space* is a set X together with a set $\mathcal{U}$ of subsets of X, called the *open subsets* of X, satisfying two conditions:
  - The union of any subset of $\mathcal{U}$ is an element of $\mathcal{U}$. That is, unions of open sets are open.
  - The intersection of any *finite* subset of $\mathcal{U}$ is an element of $\mathcal{U}$. That is, *finite* intersections of open sets are open.
- A subset Y of a topological space X is *closed* if its complement $X \setminus Y$ is open. **Closed does not mean "not open"!**
  - The intersection of any collection of closed sets is closed.
  - The union of any *finite* collection of closed sets is closed.

- Let Y be any subset of a topological space X.
  - The *interior* Y° of Y is the union of all open subsets of Y.
  - The *closure* $Y^\bullet$ of Y is the intersection of all closed subsets of Y.
  - The *boundary* ∂Y is $Y^\bullet \setminus Y^\circ$. (Beware: This word is overloaded!)
- A *cover* of X is a collection of subsets of X whose union is X.
  - An *open cover* of X is a cover by open subsets of X.
  - A *finite cover* of X is a cover by a finite number of subsets of X.
  - If $\mathcal{C}$ is a cover of X, a *subcover* of $\mathcal{C}$ is any subset of $\mathcal{C}$ that is also a cover of X.
  - **Caveat lector:** The word "cover" is also used as a synonym for "covering space"!
- A topological space X is *compact* if every open cover of X has a finite subcover.
  - Bolzano-Weirstrauß: The two definitions of a compact subset of $\mathbb{R}$ agree.

Building new spaces

Let X and Y be topological spaces.

- A **subspace** of X is a subset $Z \subseteq X$ equipped with the **subset topology**: A subset $U \subseteq Z$ is open if and only if $U = V \cap Z$ for some open subset $V \subseteq X$.

- product space / topology

- Let $\sim$ be any equivalence relation over X. The **quotient space** $X/\sim$ is the set of equivalence classes $\{[x]_\sim \mid x \in X\}$ equipped with the **quotient topology**: A subset $Z \subseteq X/\sim$ is open if and only if $\{x \in X \mid [x] \in U\}$ is an open subset of X.

  - For any subspace $Z \subseteq X$, let $\sim_Z$ be the equivalence relation where $x \sim_Z y$ if and only if $x = y$ or $(x \in Z$ and $y \in Z)$. Then X/Z is another name for the quotient space $X/\sim_Z$.

- metric space / topology

Examples

- Plane
  - Product topology
  - Metric topology
- The *circle* S^1 can be defined in several equivalent ways (up to homeomorphism):
  - Subset: $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.
  - Quotient: $[0, 1]/(0 \sim 1)$ or $\mathbb{R}/\mathbb{Z}$
- Disk
  - Subset:
  - Product:
- Sphere
  - Subset:
  - Quotient:

Functions

Fix arbitrary topological spaces X and Y.

- A function $f : X \to Y$ is **continuous** if the preimage $f^{-1}(Z)$ of any open subset $Z \subseteq Y$ is an open subset of X. Continuous functions are sometimes (unfortunately) called *maps*.
- A function $f : X \to Y$ is a **homeomorphism** if f is a continuous bijection, and its inverse $f^{-1} \colon Y \to X$ is also continuous.
  - Spaces X and Y are *homeomorphic* if there is a homeomorphism from X to Y.

Paths, Cycles, and Connectivity

Fix an arbitrary topological space X.

- A *path* in X is a continuous function from the interval $[0,1]$ to X.
- A *cycle* in X is a continuous function from the circle S^1 to X.
- A path or cycle is *simple* if it is injective.
- A topological space X is *disconnected* if it the union of two disjoint non-empty open sets, and *connected* otherwise.
  - Maximal connected subspaces of X are called the *components* of X.
- Two points x and y in a topological space X are *path-connected* if there is a path in X from x to y.
  - Path-connectivity is an equivalence relation, whose equivalence classes are called the *path-components* of X
  - X is path connected if it has exactly one path-component.
  - Every path-connected space is connected, but not vice versa.
  - Every connected open subset of $\mathbb{R}^2$ is path-connected.

Geometry

Algorithms and Data Structures