25 Maximum Flows in Surface Graphs$^{\varnothing}$

25.1 Real Homology

25.2 Homologous Feasible Flows

25.3 Shortest Paths with Negative Edges

$O(n\log^2 n/\log\log n)$ time (Real RAM)

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- ***[[[Move to planarization chapter?]]]$$$
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25.4 Ellipsoid Method (Sketch)

25.5 Summary

To simplify notation, assume $C = n^{O(1)}$ and (because off-the-shelf algorithms are faster otherwise) $g = o(n^{1/4})$.

- # iterations $N = O(d\log\Delta)$
- Dimension $d = O(g)$
- Aspect ratio $\Delta = C^{O(g)}$
- So $N = O(g^2\log C)$
- Oracle time $T_s = O(n\log^2 n)$
- Iteration time $O(T_s + d^2)$ arithmetic operations
- kth iteration requires $O(k)$ bits of precision
- So kth iteration takes $O(T_s A(k) + d^2 M(k))$ time
- Total time is $O(N\,A(N)\,T_s + d^2\,N\,M(N))$
- Real RAM *without* square roots: $A(N) = O(1)$ and $M(N) = O(\log\log N)$ (for square roots)

$$\begin{aligned}
O(N\,A(N)\,T_s + d^2\,N\,M(N)) &= O(N\,T_s + d^2\,N\log\log N)\\
&= O(g^2\log C)\cdot O(n\log^2 n) + O(g^2)\cdot O(g^2\log C\log\log(g\log C))\\
&= O(g^2 n\log^2 n\log^2 C) + O(g^4\log C\log\log(g\log C))\\
&= O(g^2 n\log^4 n)
\end{aligned}$$

- Bit RAM, grade school arithmetic: $A(N) = O(N)$ and $M(N) = O(N^2)$

$$\begin{aligned}
O(N\,A(N)\,T_s + d^2\,N\,M(N)) &= O(N^2 T_s + N^3 d^2)\\
&= O(g^4\log^2 C)\cdot O(n\log^2 n) + O(g^6\log^3 C)\cdot O(g^2)\\
&= O(g^4 n\log^2 n\log^2 C) + O(g^8\log^3 C)\\
&= O(g^4 n\log^4 n) + O(g^8\log^3 n)
\end{aligned}$$

First term dominates because $g = O((n\log n)^{1/4})$.

- Fast bit RAM: $A(N) = O(N)$ and $M(N) = O(N \log N)$

$$\begin{aligned}
O(N\,A(N)\,T_s + d^2\,N\,M(N)) &= O(N^2 T_s + d^2 N^2 \log N) \\
&= O(g^4 \log^2 C) \cdot O(n \log^2 n) \\
&\qquad + O(g^2) \cdot O(g^4 \log^2 C \log(g \log C)) \\
&= O(g^4 n \log^2 n \log^2 C) + O(g^6 \log^2 C \log(g \log C)) \\
&= O(g^4 n \log^4 n)
\end{aligned}$$

  First term dominates because $g = o(\sqrt{n \log n})$.

- Fast word RAM: $A(N) = M(N) = O(N)$ — Need to verify square root time

$$\begin{aligned}
O(N\,A(N)\,T_s + d^2\,N\,M(N)) &= O(N^2 T_s + d^2 N^2) \\
&= O(g^4 \log^2 C) \cdot O(n \log^2 n) + O(g^2) \cdot O(g^4 \log^2 C) \\
&= O(g^4 n \log^2 n \log^2 C) + O(g^6 \log^2 C) \\
&= O(g^4 n \log^4 n)
\end{aligned}$$

  ... because $g = o(\sqrt{n})$

Theorem: *Let Σ be a surface map with n vertices, genus $g = o(\sqrt{n \log n})$, positive integer edge capacities less than $n^{O(1)}$, and two vertices s and t. We can compute the maximum (s,t)-flow in Σ in $O(g^4 n \log^4 n)$ time.*

25.6 References

1. Alt JACM 1988
2. Brent JACM 1976
3. Chambers Erickson Nayyeri
4. fast integer multiplication
5. Fürer arXiv:1402.1811

25.7 Aptly Named Sir

- Directed graphs
- Non-orientable surfaces(?)
- Multi-dimensional parametric search
- Min-cost homologous circulations
- Spectral min-cost-flow algorithms, scaling