Exam 1/P Formula Sheets

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Sets

De Morgan's Law

$(A \cup B)^c = A^c \cap B^c \qquad (A \cap B)^c = A^c \cup B^c$

Inclusion-Exclusion Principle

$|A \cup B| = |A| + |B| - |A \cap B|$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

Derivatives

Function	Derivative
c	0
x^r	rx^{r-1}
$cf(x)$	$cf'(x)$
$f(x) + g(x)$	$f'(x) + g'(x)$
$f(x) \cdot g(x)$	$f'(x) \cdot g(x) + f(x) \cdot g'(x)$
$f(g(x))$	$f'(g(x)) \cdot g'(x)$
e^x	e^x
$\ln x$	$\frac{1}{x}$
a^x	$a^x \cdot \ln a$

Integrals

Properties of Integrals

$$\int_a^b c\,dx = c \cdot (b-a)$$

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Applications of the FTC

$$\int_a^b x^r\,dx = \left.\frac{x^{r+1}}{r+1}\right|_a^b \quad (r \neq -1) \qquad \int_a^b \frac{1}{x}\,dx = \ln|x|\Big|_a^b$$

$$\int_a^b e^x\,dx = e^x\Big|_a^b \qquad \int_a^b c^x\,dx = \left.\frac{c^x}{\ln c}\right|_a^b$$

Integrals Cont.

Substitution

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Integration by Parts

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Other Useful Identities

$$\int_a^b xe^{cx}\,dx = \left[\frac{xe^{cx}}{c} - \frac{e^{cx}}{c^2}\right]\Bigg|_a^b \qquad c \neq 0$$

$$\int_0^\infty x^n e^{-cx}\,dx = \frac{n!}{c^{n+1}} \qquad n \in \mathbb{N}, c > 0$$

Probability Axioms

Probability Function Definition

1. $P(S) = 1$
2. $P(A) \geq 0$ for all A
3. For mutually disjoint events $A_1, A_2, \ldots$

$$P\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty P(A_i)$$

Axiom Consequences

$P(\emptyset) = 0$

$P(A^c) = 1 - P(A)$

$A \subseteq B \implies P(B \cap A^c) = P(B) - P(A)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

If S is finite with equally likely outcomes then $P(A) = \frac{|A|}{|S|}$

Counting Techniques

Multiplication Rule

A compound experiment consisting of 2 sub-experiments with a and b possible outcomes resp. has $a \cdot b$ possible outcomes.

Permutations

An ordered arrangement of k elements from an n element set

Count: ${}_nP_k = n(n-1)\cdots(n-(k-1)) = \frac{n!}{(n-k)!}$

Combinations

A k-element subset of an n element set

Count: ${}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$

Properties of Binomial Coefficients

$$\sum_{k=0}^n \binom{n}{k} = 2^n \qquad \binom{n}{k} = \binom{n}{n-k}$$

$$n\binom{n-1}{k-1} = k\binom{n}{k} \qquad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\binom{m+n}{r} = \sum_{k=0}^r \binom{m}{k}\binom{n}{r-k}$$

Counting Rules

\# of ways to select k elements from n total elements:

	order matters	order doesn't matter
replace	n^k	$\binom{n+k-1}{k}$
don't replace	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$

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Conditional Probability

Definition

$P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ provided $P(B) > 0$

Conditional Probabilities are Probabilities

$P(A^c \mid B) = 1 - P(A \mid B)$

$P(A \cup B \mid C) = P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$

Multiplication Rule $P(A) > 0,\ P(B) > 0$

$P(A \cap B) = P(A \mid B) \cdot P(B) = P(B \mid A) \cdot P(A)$

Bayes' Theorem

$P(A \mid B) = \dfrac{P(B \mid A) \cdot P(A)}{P(B)}$ provided $P(A) > 0,\ P(B) > 0$

Law of Total Probability

If $A_1, A_2, \ldots, A_n$ partition S with $P(A_i) > 0$, then

$P(B) = P(B \mid A_1) \cdot P(A_1) + \cdots + P(B \mid A_n) \cdot P(A_n)$

Independent Events

Definition: $P(A \cap B) = P(A) \cdot P(B)$

(A, B) indep. pair $\implies$ (A, B^c), (A^c, B), (A^c, B^c) also indep. pairs

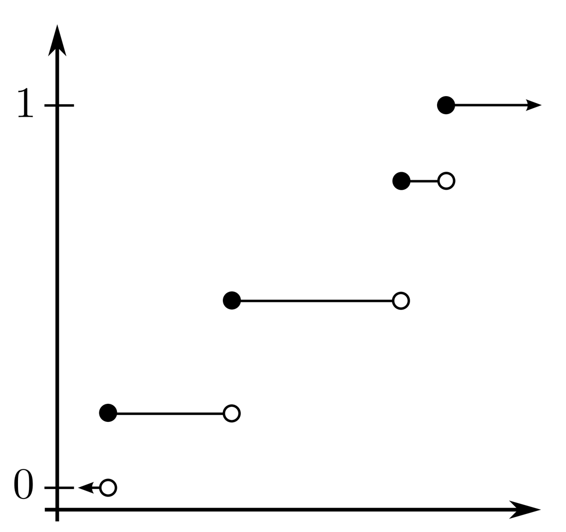
Random Variables

A random variable, X, is a function from the sample space S to $\mathbb{R}$

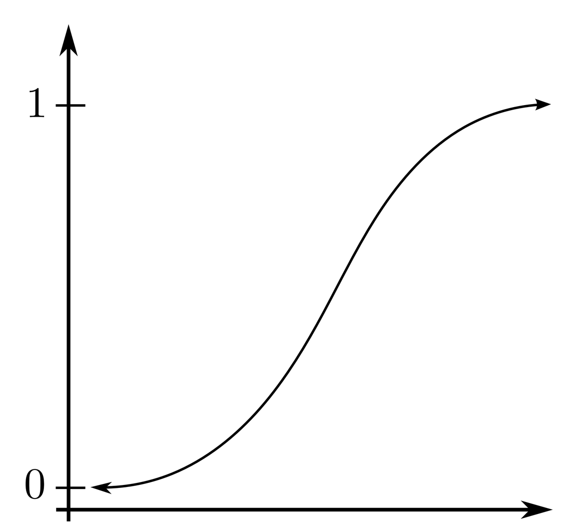
Cumulative Distribution Function

$F(x) = P(X \le x)$

Discrete CDF



Continuous CDF



A valid CDF is nondecreasing, right-continuous and

$\lim_{x \to -\infty} F(x) = 0,\ \lim_{x \to \infty} F(x) = 1$

Random Variables Cont.

Discrete Random Variable

X has finite or countably infinite (listable) support.

Probability Mass Function: $p(x) = P(X = x)$

A valid PMF has $p(x) \ge 0$ and $\sum_x p(x) = 1$

Jumps in the CDF are the probabilities (values of the PMF).

Continuous Random Variable

X has continuous CDF differentiable except at finitely many points.

Probability Density Function: $f(x) = F'(x)$

A valid PDF has $f(x) \ge 0$ and $\int_{-\infty}^{\infty} f(x) = 1$

$P(a \le X \le b) = \int_a^b f(x)\, dx = F(b) - F(a)$

Mixed Type Random Variable

X's CDF is a weighted average of a continuous and discrete CDF:

$F(x) = \alpha \cdot F_C(x) + (1 - \alpha) \cdot F_D(x) \qquad 0 < \alpha < 1$

Summary Statistics

Expected Value

$E[X] = \sum_x x \cdot p(x) \qquad E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$

Law of the Unconscious Statistician (LOTUS)

$E[g(X)] = \sum_x g(x) \cdot p(x) \qquad E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx$

Expected Value Linearity

$E[aX + b] = a \cdot E[X] + b \qquad E[X + Y] = E[X] + E[Y]$

Survival Shortcut

If X is nonnegative integer-valued, then $E[X] = \sum_{k=0}^{\infty} P(X > k)$

If X is nonnegative continuous, then $E[X] = \int_0^{\infty} [1 - F(x)]\, dx$

Summary Statistics Cont.

Variance

$\mathrm{Var}(X) = E[(X - \mu)^2] = E[X^2] - E[X]^2$

Variance Properties

$\mathrm{Var}(X) \ge 0 \qquad \mathrm{Var}(X) = 0 \iff P(X = \mu) = 1$

$\mathrm{Var}(X + c) = \mathrm{Var}(X) \qquad \mathrm{Var}(cX) = c^2 \cdot \mathrm{Var}(X)$

X, Y independent $\implies \mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

Standard Deviation

$\sigma_X = \sqrt{\mathrm{Var}(X)}$

Coefficient of Variation

$\mathrm{CV}(X) = \dfrac{\sigma_X}{\mu_X}$

Skew

$\mathrm{Skew}(X) = E\left[\left(\dfrac{X - \mu}{\sigma}\right)^3\right] = \dfrac{E[X^3] - 3\mu\sigma^2 - \mu^3}{\sigma^3}$

Jensen's Inequality

$g''(x) \ge 0 \implies E[g(X)] \ge g(E[X])$

$g''(x) \le 0 \implies E[g(X)] \le g(E[X])$

$P(g(X) = a + bX) = 1 \iff E[g(X)] = g(E[X])$

Mode

A **mode** is an x value which maximizes the PMF/PDF.

It is possible to have $0, 1, 2, \ldots$ or infinite modes.

Discrete: any values with the largest probability

Continuous: check end points of interval and where $f'(x) = 0$

Percentile

c is a $(100p)^{\text{th}}$ **percentile** of X if $P(X \le c) \ge p$ and $P(X \ge c) \ge 1 - p$

A 50^{th} percentile is called a **median**

Discrete: look for smallest c with $F(c) \ge p$

Continuous: solve for c in $F(c) = p$

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Common Discrete Distributions

Distribution	Description	$P(X=x)$	Expected Value	Variance	MGF	Properties
DUniform($\{a,\dots,b\}$)	Equally likely values $a,\dots,b$	$\frac{1}{b-a+1}$	$\frac{a+b}{2}$	$\frac{(b-a+1)^2-1}{12}$	$\frac{e^{at}-e^{(b+1)t}}{(b-a+1)(1-e^t)}$	
Bernoulli(p)	1 trial w/ success chance p	$P(X=1)=p$ $P(X=0)=1-p$	p	$p(1-p)$	$1-p+pe^t$	
Binomial(n,p)	# of successes in n indep. Bernoulli(p) trials	$\binom{n}{x}p^x(1-p)^{n-x}$	np	$np(1-p)$	$(1-p+pe^t)^n$	$np\in\mathbb{N} \implies np = \text{mode} = \text{median}$
HyperGeom(N,K,n)	# w/ property chosen w/ out replacement from N where K have property	$\frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$	$n\frac{K}{N}$	$n\frac{K}{N}\left(1-\frac{K}{N}\right)\frac{N-n}{N-1}$	ugly	Resembles Binomial($n, \frac{K}{N}$) with large N relative to n
Poisson(λ)	Common frequency dist.	$\frac{e^{-\lambda}\lambda^x}{x!}$	λ	λ	$e^{\lambda(e^t-1)}$	Approximates Binomial(n,p) when $\lambda=np$, n large, p small
Geometric(p)	# of failures before first probability p success	$(1-p)^x p$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$	$\frac{p}{1-(1-p)e^t}$	Only **memoryless** discrete dist.
NegBin(r,p)	# of failures before r^{th} probability p success	$\binom{x+r-1}{r-1}p^r(1-p)^x$	$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$	$\left(\frac{p}{1-(1-p)e^t}\right)^r$	Sum of r iid Geometric(p)

Common Continuous Distributions

Distribution	$f(x)$	$F(x)$	Expected Value	Variance	MGF	Properties
Uniform(a,b)	$\frac{1}{b-a}$	$\frac{x-a}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{t(b-a)}$	Probabilities are proportional to length
$\mathcal{N}(\mu,\sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\Phi\left(\frac{x-\mu}{\sigma}\right)$	μ	σ^2	$e^{\mu t+\sigma^2t^2/2}$	Approximates sum of n iid rv's w/ mean $\frac{\mu}{n}$ and variance $\frac{\sigma^2}{n}$
Exp(λ)	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$	Only **memoryless** continuous distribution
Gamma(α,λ)	$\frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}$	ugly	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^\alpha$	Sum of α independent Exp(λ) for integer $\alpha>0$

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Discrete Distributions Cont.

Bernoulli

A Bernoulli r.v. is also an **indicator** of the occurrence of $A \subseteq S$

$X \sim \text{Bernoulli}(p) \implies Y = (a-b)X + b$ takes values a and b with probabilities p and $1-p$, respectively.

$$E[Y] = p \cdot a + (1-p) \cdot b$$

$$\text{Var}(Y) = (a-b)^2 \cdot p \cdot (1-p)$$

Binomial

If $X \sim \text{Binomial}(n,p)$, $Y \sim \text{Binomial}(m,p)$ then

$$n - X \sim \text{Binomial}(n, 1-p)$$

$$X, Y \text{ independent} \implies X + Y \sim \text{Binomial}(n+m, p)$$

$$Z \mid X \sim \text{Binomial}(X, q) \implies Z \sim \text{Binomial}(n, pq)$$

Poisson

If $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\kappa)$ then

$$P(X = x+1) = P(X = x) \cdot \frac{\lambda}{x+1}$$

$$\lambda \in \mathbb{N} \implies \lambda,\ \lambda + 1 \text{ are modes}$$

$$X, Y \text{ independent} \implies X + Y \sim \text{Poisson}(\lambda + \kappa)$$

$$Z \mid X \sim \text{Binomial}(X, p) \implies Z \sim \text{Poisson}(\lambda \cdot p)$$

Geometric

If $X \sim \text{Geometric}(p)$ then $X+1$ counts trials until 1^{st} success

$$E[X+1] = \frac{1}{p} \qquad \text{Var}(X+1) = \frac{1-p}{p^2}$$

$P(X \geq n + m \mid X \geq m) = P(X \geq n)$ [memoryless]

Continuous Distributions Cont.

Uniform

$U_s \sim \text{Uniform}(0,1) \implies U = (b-a) \cdot U_s + a \sim \text{Uniform}(a,b)$

$(c,d) \subseteq (a,b) \implies P(c \leq U \leq d) = \dfrac{d-c}{b-a}$,

$$U \mid U \in (c,d) \sim \text{Uniform}(c,d)$$

Normal

$Z \sim \mathcal{N}(0,1) \implies X = \sigma Z + \mu \sim \mathcal{N}(\mu, \sigma^2)$

$\Phi(-z) = 1 - \Phi(z)$

$X \sim \mathcal{N}(\mu_X, \sigma_X^2), Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ independent, then

$$X + Y \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$$

Central Limit Theorem

$X_1, \ldots, X_n$ iid each with mean μ and variance σ^2, then

$X_1 + \cdots + X_n \mathrel{\dot\sim} \mathcal{N}(n\mu, n\sigma^2)$

Approximating consecutive integer-valued X with CLT:

$$P(X \leq x) \approx \Phi\left(\frac{x + \frac{1}{2} - \mu}{\sigma}\right), \quad P(X < x) \approx \Phi\left(\frac{x - \frac{1}{2} - \mu}{\sigma}\right)$$

Exponential

$X \sim \text{Exp}(\lambda) \implies P(a \leq X \leq b) = e^{-\lambda a} - e^{-\lambda b}$

The continuous analog of the Geometric (rounding & limit)

$P(X \geq s + t \mid X \geq s) = P(X \geq t)$ [memoryless]

Time between events in Poisson process with rate λ

$X_i \sim \text{Exp}(\lambda_i)$ indep. $\implies \min\{X_1, \ldots, X_n\} \sim \text{Exp}(\lambda_1 + \cdots + \lambda_n)$

Gamma

With integer α, $X \sim \text{Gamma}(\alpha, \lambda)$ is

a sum of α independent $\text{Exp}(\lambda)$

the time until α^{th} event in a Poisson process with rate λ

The continuous analog of the Negative Binomial

Moment Generating Functions

Definition

$M_X(t) = E[e^{tX}]$

Properties

Uniquely determines a distribution

$E[X^n] = M_X^{(n)}(0)$

$M_{aX+b}(t) = e^{bt} M_X(at)$

$X,\ Y$ independent $\implies M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$

Variance Shortcut (Cumulant)

$$\text{Var}(X) = \frac{d^2}{dt^2} \ln\left(M(t)\right)\Big|_{t=0}$$

Probability Generating Functions

Defined for nonnegative integer-valued random variables

Definition

$G_X(t) = E[t^X]$

Properties

Uniquely determines a distribution

$P(X = k) = G_X^{(k)}(0)/k!$

$G_{aX+b}(t) = t^b G_X(t^a)$

$X,\ Y$ independent $\implies G_{X+Y}(t) = G_X(t) \cdot G_Y(t)$

Moments

$$G_X^{(n)}(1) = E\left[\frac{X!}{(X-n)!}\right] = E[X(X-1)\ldots(X-n+1)]$$

$E[X] = G_X'(1)$

$\text{Var}(X) = G_X''(1) - (G_X'(1))^2 + G_X'(1)$

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Joint Distributions

Cumulative Distribution Function (CDF)

$F(x,y) = P(X \le x, Y \le y)$

Probability Mass Function (PMF)

$$p(x,y) = P(X = x, Y = y)$$

Probability Density Function (PDF)

$$f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$$

Marginal PMF

$$p_X(x) = \sum_y p(x,y)$$

Marginal PDF

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

Conditional PMF

$$p_{X|Y}(x \mid Y = y) = \frac{p(x,y)}{p_Y(y)}$$

Conditional PDF

$$f_{X|Y}(x \mid Y = y) = \frac{f(x,y)}{f_Y(y)}$$

Independence Criteria (X and Y are independent if *any* hold)

$F(x,y) = F_X(x) \cdot F_Y(y) \quad -\infty < x, y < \infty$

$p(x,y) = p_X(x) \cdot p_Y(y) \quad -\infty < x, y < \infty$

$p_{X|Y}(x \mid Y = y) = p_X(x)$ where $p_Y(y) > 0$

$p_{Y|X}(y \mid X = x) = p_Y(y)$ where $p_X(x) > 0$

$p(x,y) = g(x) \cdot h(y)$ for any $g, h \ge 0$

$F(x,y) = F_X(x) \cdot F_Y(y) \quad -\infty < x, y < \infty$

$f(x,y) = f_X(x) \cdot f_Y(y) \quad -\infty < x, y < \infty$

$f_{X|Y}(x \mid Y = y) = f_X(x)$ where $f_Y(y) > 0$

$f_{Y|X}(y \mid X = x) = f_Y(y)$ where $f_X(x) > 0$

$f(x,y) = g(x) \cdot h(y)$ for any $g, h \ge 0$

2D LOTUS

$$E[h(X,Y)] = \sum_x \sum_y h(x,y) \cdot p(x,y)$$

$$E[h(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y) \cdot f(x,y)\,dx\,dy$$

Conditional Expectation

$$E[X \mid Y = y] = \sum_x x \cdot p_{X|Y}(x \mid Y = y)$$

$$E[X \mid Y = y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x \mid Y = y)\,dx$$

Law of Total Expectation/Variance

$E[X] = E[E[X \mid Y]] \qquad \mathrm{Var}(X) = E[\mathrm{Var}(X \mid Y)] + \mathrm{Var}(E[X \mid Y])$

Joint Moment Generating Function

$$M_{X,Y}(s,t) = E[e^{sX+tY}] \qquad E[X^n Y^m] = \frac{\partial^{n+m}}{\partial s^n \partial t^m} M_{X,Y}(s,t)\Big|_{s=t=0} \qquad M_X(s) = M_{X,Y}(s,0)$$

Joint Distributions Cont.

Covariance

Definition: $\mathrm{Cov}(X,Y) = E[(X - \mu_X) \cdot (Y - \mu_Y)] = E[XY] - E[X] \cdot E[Y]$

$\mathrm{Cov}(X,X) = \mathrm{Var}(X) \qquad \mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X) \qquad \mathrm{Cov}(X + c, Y) = \mathrm{Cov}(X,Y)$

$\mathrm{Cov}(X,c) = 0 \qquad \mathrm{Cov}(cX,Y) = c \cdot \mathrm{Cov}(X,Y) \qquad \mathrm{Cov}(X+Y, Z) = \mathrm{Cov}(X,Z) + \mathrm{Cov}(Y,Z)$

$\mathrm{Var}(aX + bY) = a^2 \cdot \mathrm{Var}(X) + b^2 \cdot \mathrm{Var}(Y) + 2ab \cdot \mathrm{Cov}(X,Y)$

Coefficient of Correlation

$$\rho_{X,Y} = \mathrm{Cov}\left(\frac{X - \mu_X}{\sigma_X}, \frac{Y - \mu_Y}{\sigma_Y}\right) = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y} \qquad -1 \le \rho_{X,Y} \le 1$$

Consequences of Independence

$E[g(X) \cdot h(Y)] = E[g(X)] \cdot E[h(Y)] \qquad \mathrm{Cov}(X,Y) = 0$

$M_{X,Y}(s,t) = M_X(s) \cdot M_Y(t) \qquad \rho_{X,Y} = 0$

Bivariate Continuous Uniform

$f(x,y) = \dfrac{1}{\text{Area of support}}$ Probabilities are proportional to areas

Multinomial

$(X_1, \ldots, X_k) \sim \text{Multinomial}(n, p_1, \ldots, p_k)$ if n indep. trials performed, each with k possible outcomes (with respective probabilities $p_1, \ldots, p_k$) and X_i is the number of trials resulting in outcome i.

$$P(X_1 = n_1, \ldots, X_k = n_k) = \frac{n!}{n_1! n_2! \ldots n_k!} \cdot p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$$

$X_i \sim \text{Binomial}(n, p_i) \qquad i \ne j \implies \mathrm{Cov}(X_i, X_j) = -np_i p_j$

Bivariate Normal

$(X,Y) \sim \text{BivNormal}(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho_{X,Y})$ if $aX + bY$ is normal for all $a, b \in \mathbb{R}$.

$Y \mid X = x \sim \mathcal{N}(\mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X), \sigma_Y^2(1 - \rho^2))$

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Transformations

Transformation of Discrete X

$$P(g(X)=y) = \sum_{x|g(x)=y} P(X=x)$$

Transformation of Continuous X

Find sets A_y so $g(X) \le y \iff X \in A_y$

Compute $F_Y(y) = P(g(X) \le y) = P(X \in A_y)$

$f_Y(y) = F_Y'(y)$

Strictly Monotone Transformation of Continuous X

$$f_Y(y) = f_X(g^{-1}(y)) \cdot |(g^{-1})'(y)|$$

1-1 Transformation of Continuous X, Y

If $\begin{cases} g_1(x,y) = u \\ g_2(x,y) = v \end{cases}$ has unique solution $\begin{cases} x = h_1(u,v) \\ y = h_2(u,v) \end{cases}$ and

$$\mathbf{J} = \begin{vmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{vmatrix} = \frac{\partial h_1}{\partial u}\frac{\partial h_2}{\partial v} - \frac{\partial h_1}{\partial v}\frac{\partial h_2}{\partial u} \neq 0$$

Then the joint density of $U = g_1(X,Y)$ and $V = g_2(X,Y)$ is

$g(u,v) = f(h_1(u,v), h_2(u,v)) \cdot |\mathbf{J}|$

Transformations Cont.

Convolution Theorem (X and Y independent)

$$p_{X+Y}(t) = \sum_x p_X(x) \cdot p_Y(t-x) \qquad f_{X+Y}(t) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(t-x)\, dx$$

Order Statistics

$X_{(i)}$ is the i^{th} smallest of iid continuous $X_1, \ldots, X_n$ w/ dist. F, f

$$F_{X_{(j)}}(x) = \sum_{k=j}^{n} \binom{n}{k} F(x)^k (1-F(x))^{n-k}$$

$$f_{X_{(j)}}(x) = n\binom{n-1}{j-1} f(x) F(x)^{j-1} (1-F(x))^{n-j}$$

$$F_{X_{(1)}}(x) = 1-(1-F(x))^n \qquad F_{X_{(n)}}(x) = F(x)^n$$

Mixture Distribution

X has PMF/PDF $f(x) = \alpha_1 \cdot f_1(x) + \cdots + \alpha_n \cdot f_n(x)$

(α_i positive and sum to 1, f_i are valid PMF/PDFs)

$$F_X(x) = \alpha_1 \cdot F_1(x) + \cdots + \alpha_n \cdot F_n(x)$$

$$S_X(x) = \alpha_1 \cdot S_1(x) + \cdots + \alpha_n \cdot S_n(x)$$

$$E[X^k] = \alpha_1 \cdot E[X_1^k] + \cdots + \alpha_n \cdot E[X_n^k]$$

$$M_X(t) = \alpha_1 \cdot M_{X_1}(t) + \cdots + \alpha_n \cdot M_{X_n}(t)$$

Risk and Insurance

Ordinary Deductible d

Payment: $(X-d)_+ = \begin{cases} 0, & X \le d \\ X-d, & X > d \end{cases}$

Mean: $\displaystyle\int_d^\infty (x-d) \cdot f_X(x)\, dx = \int_d^\infty S_X(x)\, dx$

Payment w/ inflation r: $\displaystyle(1+r) \cdot \left(X - \frac{d}{1+r}\right)_+$

Policy Limit u

Payment: $X \wedge u = \begin{cases} X, & X \le u \\ u, & X > u \end{cases}$

Mean: $\displaystyle\int_0^u x \cdot f_X(x)\, dx + u \cdot S_X(u) = \int_0^u S_X(x)\, dx$

Payment w/ inflation r: $\displaystyle(1+r) \cdot \left(X \wedge \frac{u}{1+r}\right)$

Ordinary Deductible d and Policy Limit u Simultaneously

Payment: $X_d^u = \begin{cases} 0, & X \le d \\ X-d, & d < X \le u+d \\ u, & X > u+d \end{cases}$

Mean: $\displaystyle\int_d^{u+d} (x-d) \cdot f_X(x)\, dx + u \cdot S_X(u+d) = \int_d^{u+d} S_X(x)\, dx$

Payment w/ inflation r: $(1+r) \cdot X_{\frac{d}{1+r}}^{\frac{u}{1+r}}$

Loss Given Positive Loss

If $X_C = X \mid X > 0$ and $\alpha = P(X > 0)$, then

$E[X] = \alpha \cdot E[X_C] \qquad E[X^2] = \alpha \cdot E[X_C^2]$

$\text{Var}(X) = \alpha \cdot \text{Var}(X_C) + \alpha \cdot (1-\alpha) \cdot E[X_C]^2$

Michael Mastroianni, ASA