Higher Linear Algebra
MATH2601 UNSW

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2023T2

Contents

*With some inspiration from Hussain Nawaz's Notes

1 Group and Fields

1.1 Groups

Definition A group G is a non-empty set with a binary operation defined on it. That is

1. **Closure:** for all a, b in G a composition $a * b$ is defined and in G,
2. **Associativity:** $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$,
3. **Identity:** there is an element $e \in G$ such that $a * e = e * a$ for all $a \in G$,
4. **Inverse:** for each $a \in G$ there is an a' in G such that $a * a' = a' * a = e$,

If G is a finite set then the order of G is $|G|$, the number of elements in G.

Groups are defined as $(G, *)$. We say this as "the group G under the operation $*$".

Abelian Groups A group G is abelian if the operation satisfies the commutative law

$$a * b = b * a \qquad \text{for all} \quad a, b \in G$$

Notation

- We use power notation for repeated applications: $a * a \cdots * a = a^n$ and $a^{-n} = (a^{-1})^n$.
- For group operation, $\times$ we use 1 for the identity and a^{-1} for inverse of a.
- For group operation, $+$ we use 0 for the identity and $-a$ for the inverse of a.
- We would then write na for $a + a + \cdots a$ (repeated addition, not multiplying by n).

Trivial Groups The trivial group is the group consisting of exactly one element, $\{e\}$. It is the smallest possible group, since there has to be at least one element in a group.

More Properties of Groups

- There is only one identity element in G.
- Each element of G only has one inverse.
- For each $a \in G, (a^{-1})^{-1} = a$
- For every, $a, b \in G, (a * b)^{-1} = b^{-1} * a^{-1}$.
- Let $a, b, c \in G$. Then if $a * b = a * c, b = c$.

1.1.1 Permutation Groups

Let $\Omega_n = \{1, 2, \ldots, n\}$. As an ordered set $\Omega_n = (1, 2, \ldots, n)$ has $n!$ rearrangements. We may think of these permutations as being functions $f : \Omega_n \to \Omega_n$. These are bijections.

Observe that the set $\mathcal{S}_n$ of all permutations of n objects forms a group under composition of order $n!$.

Small Finite Groups Small groups can be pictured using a multiplication table, where the row element is multiplied on the left of the column element.

In a multiplication table of finite group each row must be a permutation of the elements of the group, because:

- If we had repetition in a row (or column), so that $xa = xb$, then the cancellation rule will give $a = b$. Hence each element occurs no more than once in a row (or column).
- If $a^2 = a$ then multiplying by a^{-1} gives $a = e$, so the identity is the only element that can be fixed.

1.2 Fields

A field $(\mathbb{F}, +, \times)$ is a set $\mathbb{F}$ with two binary operations on it, addition $(+)$ and multiplcation $(\times)$, where

1. $(\mathbb{F}, +)$ is an abelian group,
2. $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$ is an abelian group under multiplication,
3. The distributive laws $a \times (b + c) = a \times b + a \times c$ and $(a + b) \times c = a \times c + b \times c$ hold.

Additional Notes

- Our definition is equivalent to saying $\mathbb{F}$ satisfies the $12 = 5 + 5 + 2$ number laws.
- We use juxtaposition for the multiplication in fields and 1 for the identity under multiplication.
- The smallest possible field has two elements, and is written $\{0, 1\}$ with $1 + 1 = 0$.

Finite Fields The only finite fields are those of size p^k for some prime p (referred to as the characteristic of the field) and positive integer k. These fields are called Galois fields of size p^k, $\mathrm{GF}(p^k)$. Note that $\mathrm{GF}(p^k) \neq \mathbb{Z}_{p^k}$ unless $k = 1$.

Properties of Fields Let $\mathbb{F}$ be a field and $a, b, c \in \mathbb{F}$. Then

- $a0 = 0$
- $a(-b) = -(ab)$
- $a(b - c) = ab - ac$
- if $ab = 0$ then either $a = 0$ or $b = 0$.

1.3 Subgroups and Subfields

Subgroups Let $(G, *)$ be a group and H a non-empty subset of G. If H is a group under the restriction of $*$ to H, we call it a subgroup of G. We write this as $H \leq G$ and say H inherits the group structure from G.

The Subgroup Lemma Let $(G, *)$ be a group and H a non-empty subset of G. Then H is a subgroup of G if and only if

1. for all $a, b \in H, a * b \in H$
2. for all $a \in H, a^{-1} \in H$.

i.e. H is closed under $*$ and $^{-1}$.

Note that every non-trivial group G has at least two subgroups: $\{e\}$ and G.

General Linear Groups Let $n \geq 1$ be an integer. The set of invertible $n \times n$ matrices over field $\mathbb{F}$ is a group under matrix multiplication. This is a special case of a bijection function $f : S \to S$ with $S = \mathbb{F}^n$ and is non-abelian if $n > 1$.

It is called the general linear group, $\mathrm{GL}(n, f)$.

The groups $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ are especially important in this course. They have many important subgroups, such as

- the special linear groups $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{SL}(n, \mathbb{C})$ of matrices with determinant 1.
- $O(n) \leq \mathrm{GL}(n, \mathbb{R})$ the group of orthogonal matrices.
- $\mathrm{SO}(n) = O(n) \cap \mathrm{SL}(n, \mathbb{R})$ of special orthogonal matrices.

Subfields If $(\mathbb{F}, +, \times)$ is a field and $\mathbb{E} \subseteq \mathbb{F}$ is also a field under the same operations (restricted to $\mathbb{E}$), then $(\mathbb{E}, +, \times)$ is a subfield of $(\mathbb{F}, +, \times)$, usually written $\mathbb{E} \leq \mathbb{F}$.

The Subfield Lemma Let $\mathbb{E} \neq \{0\}$ be a non-empty subset of field $\mathbb{F}$. Then $\mathbb{E}$ is a subfield of $\mathbb{F}$ if and only iff for all $a, b \in \mathbb{E}$:

$$a + b \in \mathbb{E}, \qquad -b \in \mathbb{E}, \qquad a \times b \in \mathbb{E}, \qquad b^{-1} \in \mathbb{E} \quad \textbf{if} \quad b \neq 0.$$

Rational + Irrational Field Let α be any (non-rational) real or complex number. We defined $\mathbb{Q}(a)$ to be the smallest field containing both $\mathbb{Q}$ and α. Such fields are important in number theory and can clearly be generalised to e.g. $\mathbb{Q}(\alpha, \beta)$. For example, it can be shown

$$\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$$

1.4 Morphisms

A morphism is a category of "nice" maps between the members.

Homomorphism Let $(G, *)$ and $(H, \circ)$ be two groups. A (group) homomorphism from G to H is a map $\phi : G \to H$ that respects the two operations, that is where

$$\phi(a * b) = \phi(a) \circ \phi(b) \quad \text{for all } \ a, b \in G.$$

Isomorphism A bijective homomorphism $\phi : G \to H$ is called an isomorphism: the groups are then said to be isomorphic. That is, $G \cong H$.

Isomorphism Lemmas Let $(G, *)$ and $(H, \circ)$ be two groups and ϕ a homomorphism between them. Then

- ϕ maps the identity of G to the identity of H.
- ϕ maps inverses to inverse, i.e. $\phi(a^{-1} = (\phi(a))^{-1}$ for all $a \in G$.
- if ϕ is an isomorphism from G to H then ϕ^{-1} is an isomorphism from H to G.

Images and Kernel Let $\phi : G \to H$ be a group homorphism, with e' the identity of H. The kernal of ϕ is the set

$$\ker(\phi) = \{g \in G : \phi(g) = e'\}$$

The image of ϕ is the set

$$\mathrm{im}(\phi) = \{h \in H : h = \phi(g),\ \text{some } g \in G\}.$$

Note that $\ker \phi \leq G$ and $\mathrm{im}\, \phi \leq H$.

One-to-One Homomorphism A homomorphism ϕ is one-one if and only if $\ker \phi = \{e\}$, with e the identity of G. If ϕ is one-one then $\mathrm{im}(\phi)$ is isomorphic to G.

Linear Groups A common use of group homomorphisms is to look for a homomorphism $\phi : G \to \mathrm{GL}(n, \mathbb{F})$ for some n and some field $\mathbb{F}$. The group $\mathrm{im}(\phi)$ is called a (linear) representation of G on $\mathbb{F}^n$. If ϕ is one-one (so every element maps to a distinct matrix), we call the representation faithful.

2 Vector Spaces

2.1 Vector Spaces

Motivation for Vector Spaces The concept of a vector space is a natural and important generalisation of $\mathbb{R}^n$. It is natural to consider them whenever possible to add objects and multiply them by scalars.

It may be convenient to consider a field $\mathbb{F}$ as a vector space over one of its subfields.

Vector Spaces Let $\mathbb{F}$ be a field. A vector space over the field $\mathbb{F}$ consists of an abelian group $(V, +)$ plus a function from $\mathbb{F} \times V$ to V called scalar multiplication and written $\alpha\mathbf{v}$ where

1. $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$ for all $\alpha, \beta \in \mathbb{F}$ for all $\mathbf{v} \in V$.
2. $1\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$.
3. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{u}$ for all $\alpha \in \mathbb{F}$ for all $\mathbf{u}, \mathbf{v} \in V$.
4. $(\alpha + \beta)\mathbf{u} + \alpha\mathbf{u} + \beta\mathbf{u}$ for all $\alpha, \beta \in \mathbb{F}$ for all $\mathbf{u} \in V$.

Properties and Notation for Vector Spaces

1. There are ten axioms here: 5 from the abelian group, closure of scalar multiplication and the four explicit ones.
2. Addition in V is called vector addition to distinguish it from the addition in $\mathbb{F}$.
3. Being a group, V cannot be empty.
4. Bold face letters are used to distinguish elements of V from elements of $\mathbb{F}$.

Vector Space Lemma Let V be a vector space over a field $\mathbb{F}$. For all $\mathbf{v}, \mathbf{w}$ in V and $\lambda \in \mathbb{F}$:

1. $0\mathbf{v} = \mathbf{0}$ and $\lambda\mathbf{0} = \mathbf{0}$.
2. $(-1)\mathbf{v} = -\mathbf{v}$.
3. $\lambda\mathbf{v} = \mathbf{0}$ implies either $\lambda = 0$ or $\mathbf{v} = \mathbf{0}$.
4. if $\lambda\mathbf{v} = \lambda\mathbf{w}$ and $\lambda \neq 0$ then $\mathbf{v} = \mathbf{w}$.

2.2 Standard Examples of Vector Spaces

The Space $\mathbb{F}^n$ over $\mathbb{F}$ The set $\mathbb{F}^n$ consists of all n-tuples of elements of $\mathbb{F}$:

$$\mathbb{F}^n = \left\{ \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} : \alpha_i \in \mathbb{F} \right\}.$$

If $\mathbf{x} = (\alpha_i)_{1 \le i \le n}, \mathbf{y} = (\beta_i)_{1 \le i \le n}$ are elements of $\mathbb{F}^n$, then vector addition on $\mathbb{F}^n$ is defined as

$$\mathbf{x} + \mathbf{y} = (\alpha_i + \beta_i)_{1 \le i \le n}.$$

Scalar multiplication on $\mathbb{F}^n$ is $\lambda\mathbf{x} = (\lambda\alpha_i)_{1 \le i \le n}$.

With these operations, $\mathbb{F}^n$ is a vector space over $\mathbb{F}$.

Geometric Vectors Geometric vectors are ordered pairs of points in $\mathbb{R}^n$, joined by labelled arrows. We add these objects by placing them head to tail and scalar multiplying is just stretching the vector's length while preserving the direction.

The set of all geometric vectors does not form a vector space. However, if you define 2 geometric vectors to be equivalent if one is a translation of the other then the set of equivalence classes of geometric vectors is a vector space.

Matrices For any positive integers p and q the set $M_{p,q}(\mathbb{F})$ is the set of $p \times q$ matrices with element from $\mathbb{F}$. Then $M_{p,q}(\mathbb{F})$ is a vector space over $\mathbb{F}$ with vector addition the usual addition of matrices and scalar multiplication multiplying each element of the matrix.

Polynomials The set of all polynomials with coefficients in $\mathbb{F}, \mathcal{P}(\mathbb{F})$, is a vector space over $\mathbb{F}$ with

$$(f+g)(x) = f(x) + g(x) \quad \text{for all } x \in \mathbb{F}$$
$$(\lambda f)(x) = \lambda f(x) \quad \text{for all } \lambda, x \in \mathbb{F}$$

Similarly, $\mathcal{P}_n(\mathbb{F})$ (polynomials of degree n or less) is a vector space over $\mathbb{F}$.

Function Spaces Let X be a non-empty set and $\mathbb{F}$ be a field. Then define

$$\mathcal{F}[X] = \{f : X \to \mathbb{F}\}.$$

The set $\mathcal{F}[X]$ is a vector space over $\mathbb{F}$ if we define

- the vzero in $\mathcal{F}[X]$ to be the vzero function: $x \to 0$ for all $x \in X$
- $(f+g)(x) = f(x) + g(x)$ for all $x \in X$
- $(\lambda f)(x) = \lambda(f(x))$ for all $x \in X$

Exotic Example Let $V = \mathbb{R}^+$, the set of positive real numbers. Define addition and scalar multiplication on V by

$$\mathbf{v} \oplus \mathbf{w} = \mathbf{vw}, \qquad \alpha \otimes \mathbf{v} = \mathbf{v}^\alpha$$

Then with these operations, V is a vector space over $\mathbb{R}$ whose addition and multiplication and whose scalar multiplication is exponentiation.

2.3 Subspaces

Subspaces If V is a vector space over $\mathbb{F}$ and $U \subseteq V$, then U is a subspace of V, written $U \leq V$, if it is a vector space over $\mathbb{F}$ with the same addition and scalar multiplication as in V.

Every vector space has $\{\mathbf{0}\}$ (the trivial subspace) and itself as subspaces.

Subspace Test Lemma Suppose V is a vector space over the field $\mathbb{F}$ and U is a non-empty subset of V. Then U is a subspace of V if and only if for all $\mathbf{u}, \mathbf{v} \in U$ and $\alpha \in \mathbb{F}, \alpha\mathbf{u} + \mathbf{v} \in U$.

2.4 Linear Combinations, Spans and Independence

Linear Combination Let V be a vector space over $\mathbb{F}$. A (finite) linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in V is any vector which can be expressed

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$$

where the α_k are scalars.

Span If S is a subset of V, then the span of is

$$\text{span}(S) = \{ \text{ all finite linear combinations of vectors in } S\}.$$

We say that S spans V, or is a spanning set for V, if $\text{span}(S) = V$.

If S is a non-empty subset of a vector space V, then $\text{span}(S)$ is a subspace of V.

Linear Independence A subset S of a vector space V is linearly independent if for all vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in S (with $n \geq 1$) the equation

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n = \mathbf{0}$$

with $\alpha_i \in \mathbb{F}$, implies $\alpha_i = 0$ for all $i = 1 \ldots n$.

Linear Dependence Lemma If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a linearly dependent set of non-zero vectors in V then there is an $i, 2 \leq i \leq k$ such that

$$\mathbf{v}_i = \sum_{j=1}^{i-1} \beta_j \mathbf{v}_j.$$

In other words in a ordered linearly dependent set at least one vector is a linear combination of its predecessors.

Properties of Linear Independence, Dependence and Spanning Sets In any vector space

1. Any subset of a linearly independent set is linearly independent.
2. (a) If $\mathbf{v} \in \text{span}(S)$ and $\mathbf{v} \notin S$, then $S \cup \{\mathbf{v}\}$ is linearly dependent.
   (b) If S is linearily independent and $S \cup \{\mathbf{v}\}$ is linearly dependent then $v \in \text{span}(S)$.
3. (a) If $S_1 \subseteq S_2$, then $\text{span}(S_1) \subseteq \text{span}(S_2)$.
   (b) If $S_1 \subseteq \text{span}(S_2)$, then $\text{span}(S_1) \subseteq \text{span}(S_2)$.
4. $\text{span}(S \cup \{\mathbf{v}\}) = \text{span}(S)$ if and only if $\mathbf{v} \in \text{span}(S)$.
5. If S is linearly dependent, then there is a vector $\mathbf{v}$ in S such that $\text{span}(S \setminus \{\mathbf{v}\}) = \text{span}(S)$.
6. In $\mathbb{F}^p$, if $P \in \text{GL}(p, \mathbb{F})$ is an invertible matrix and $\{\mathbf{v}_i\}$ linearly independent, then the set $\{P\mathbf{v}_i\}$ is also linearly independent.

2.5 Bases

Let $S \subseteq V$. The set S is a basis for V over $\mathbb{F}$ if and only if $V = \text{span}(S)$, and S is a linearly independent set.

2.5.1 Examples of Bases

$\mathbb{F}^n$ over $\mathbb{F}$ The standard basis of $\mathbb{F}^n$ as a vector space over $\mathbb{F}$ is $\mathcal{B} = \{\mathbf{e}_i : 1 \le i \le n\}$ where

$$\mathbf{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th place, written } \begin{pmatrix} \\ 1 \\ \\ \end{pmatrix} \leftarrow i$$

We also use $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as the standard basis of $\mathbb{R}^3$.

Matrix Spaces Define the matrices

$$E_{ij} = (e_{hl}) = \begin{cases} 1 & h = 1 \text{ and } l = j. \\ 0 & \text{otherwise.} \end{cases}$$

The set

$$\mathcal{B} = \{E_{ij} : 1 \le i \le p, 1 \le j \le q\},$$

is the standard basis of $M_{p,q}(\mathbb{F})$ as a vector space over $\mathbb{F}$.

Polynomial Spaces The standard basis of $\mathcal{P}_n(\mathbb{F})$ as a vector space over $\mathbb{F}$ is

$$\mathcal{B} = \{1, t, \ldots, t^n\}.$$

Function Spaces The space $\mathcal{F}(X)$ has no obvious basis unless X is finite.

Let $X = \{a_1, \ldots, a_n\}$, and for each i for $i = 1, \ldots, n$ define $f_i : X \to \mathbb{F}$ by

$$f_i(a_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & i \ne j \end{cases}$$

The set $\mathcal{B} = \{f_1, f_2, \ldots, f_n\}$ is a basis for $\mathcal{F}(X)$.

(We call the δ_{ij} defined here the Kronecker delta symbol.)

Fields The set $\{1, i\}$ is a basis for $\mathbb{C}$ as a vector space over $\mathbb{R}$. Similarly, $\mathbb{Q}(\sqrt{2})$ as a vector space over $\mathbb{Q}$ has a basis $\{1, \sqrt{2}\}$.

2.6 Dimension

Elements of Bases If vector space V admits a finite spanning set, it admits a finite basis and all bases contain the same number of elements.

Basis and Spanning Sets Let V be a vector space over $\mathbb{F}$ and S a finite spanning set. Then S contains a finite basis for V.

The Exchange Lemma Suppose that S is a finite spanning set for V and that T is a (finite) linearly independent subset of V with $|T| \leq |S|$. Then there is a spanning set S' of V such that

$$T \subseteq S' \text{ and } |S'| = |S|.$$

Independent Set Size If S is a finite spanning set for a vector space V and T is a linearly independent subset of V, then T is finite and $|T| \leq |S|$.

In other words, independent sets are no larger than spanning sets.

Linearly Independent Sets to Basis Let V be a vector space over $\mathbb{F}$ with a finite spanning set and T a linearly independent subset of V. Then there is a basis B of V which contains T.

Dimension The dimension of a vector space V is the size of a basis if V has a finite basis or infinity otherwise. The notation is $\dim(V) = n$ or $\dim(V) = \infty$.

Properties Let V be a finite dimensional vector space and suppose $\dim(V) = n$.

1. The number of elements in any spanning set is at least n.
2. The number of elements in any independent set is no more than n.
3. If $\text{span}(S) = V$ and $|S| = n$ then S is a basis.
4. If S is a linearly independent set and $|S| = n$ then S is a basis.

Combinations, Spanning and Independence Let V be a finite dimensional vector space over $\mathbb{F}$. Then $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V if and only if every $\mathbf{x} \in V$ can be written uniquely as $\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i, \alpha_1 \in \mathbb{F}$.

2.7 Coordinates

Coordinate Suppose V is a vector space of dimension n over $\mathbb{F}$ and $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an ordered basis of V over $\mathbb{F}$. If $\mathbf{v} \in V$ then $\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$ with the α_i unique.

We call $\alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ the coordinate vector of $\mathbf{v}$ with respect to $\mathcal{B}$, and refer to the α_i as the coordinates of $\mathbf{v}$. A useful notation is

$$\alpha = [\mathbf{v}]_{\mathcal{B}} \text{ if } \mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i.$$

Properties of Coordinates

1. $\mathbf{u} = \mathbf{v}$ if and only if $[\mathbf{u}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{B}}$ for all bases $\mathcal{B}$.
2. $[\mathbf{u} + \mathbf{v}]_{\mathcal{B}} = [\mathbf{u}]_{\mathcal{B}} + [\mathbf{v}]_{\mathcal{B}}$ for any basis $\mathcal{B}$.
3. $[\lambda \mathbf{u}]_{\mathcal{B}} = \lambda [\mathbf{u}]_{\mathcal{B}}$ for any basis $\mathcal{B}$.

2.8 Sums and Direct Sums

Definitions The sum $S + T$ of two subspaces is defined as

$$S + T = \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in S, \mathbf{b} \in T\}.$$

If $S \cap T = \{\mathbf{0}\}$ then we call the sum a direct sum and denote it as $S \oplus T$.

Direct Sum The sum of subspaces S and T is direct if and only if any vector $\mathbf{x} \in S + T$ can be written in a unique way as $\mathbf{x} = \mathbf{a} + \mathbf{b}, \mathbf{a} \in S, \mathbf{b} \in T$.

Dimensions of Sum of Subspaces Suppose S and T are finite dimensional subspaces of vector spaces V. Then

$$\dim(S) + \dim(T) = \dim(S + T) + \dim(S \cap T).$$

For a direct sum of finite dimensional spaces

$$\dim(S) + \dim(T) = \dim(S \oplus T)$$

Complementary Subspace Let V be a finite dimensional vector space and $X \leq V$. Then there is a subspace Y for which $V = X \oplus Y$.

External Direct Sum Let X and Y be two vector spaces over the same field $\mathbb{F}$. The Cartesian product $X \times Y$ can be made into a vector space over $\mathbb{F}$ with the obvious definitions

$$(\mathbf{x}_1, \mathbf{y}_1) + (\mathbf{x}_2, \mathbf{y}_2) = (\mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}_1 + \mathbf{y}_2) \quad \text{and} \quad \lambda(\mathbf{x}_1, \mathbf{y}_1)v = (\lambda \mathbf{x}_1, \lambda \mathbf{y}_1)$$

With this structure we call the Cartesian product the (external) direct sum of X and Y, $X \oplus Y$.

3 Linear Transformations

3.1 Linear Transformations

Linear Transformation Suppose V and W are vector spaces over the field $\mathbb{F}$. A function $T : V \to W$ is a linear transformation or a linear map (or simply linear) if

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$, and
- $T(\lambda \mathbf{v}) = \lambda T(\mathbf{v})$,

for all $\mathbf{u}, \mathbf{v} \in V$ and for all $\lambda \in \mathbb{F}$.

Identity Map, Zero Vector and Negatives Let V and W be vector spaces over the field $\mathbb{F}$.

- The identity map, $\mathrm{id} : V \to V$ defined by $\mathrm{id}(\mathbf{v}) = \mathbf{v}$ is linear.
- If $T : V \to W$ is linear then $T(\mathbf{0}) = \mathbf{0}$ and $T(-\mathbf{v}) = -T(\mathbf{v})$.

Linearity Test Lemma A function $T : V \to W$ between vector spaces over the same field $\mathbb{F}$ is linear if and only if
$$T(\lambda\mathbf{u} + \mathbf{v}) = \lambda T(\mathbf{u}) + T(\mathbf{v})$$
for all $\lambda \in \mathbb{F}$, and $\mathbf{u}, \mathbf{v} \in V$.

Linear Transformations are Vector Spaces Let V and W be two vector spaces over field $\mathbb{F}$. The set $L(V, W)$ of all linear transformations from V to W is a vector space under the operations
$$(S + T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v}), \qquad (\lambda S)(\mathbf{v}) = \lambda S(\mathbf{v}).$$

Composition of Linear Maps Let $T : V \to W$ and $S : W \to X$ be linear maps between vector spaces. Then $S \circ T : V \to X$ is also linear.

Linearity of Inverse Let $T : V \to W$ be an invertible linear map between two vector spaces over field $\mathbb{F}$. Then $T^{-1} : W \to V$ is linear.

Invertible Linear Maps are Groups The invertible linear maps in $L(V, V)$ form a group under compositions. Note that composition of maps is always associative so and the inverse exists by definition of $L(V, V)$, only closure and the identity need to be proved.

Closure exists since composition of linear transformations are vector spaces. The identity map is linear and clearly invertible and so, also exists in the group.

Taking Coordinates is Linear Let V be a (finite-dimensional) vector space over $\mathbb{F}$ with a basis $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Then the function $S : V \to \mathbb{F}^p$ defined by $S(\mathbf{x}) = [\mathbf{x}]_{\mathcal{B}}$ is linear.

3.2 Kernel and Image

Kernel Let $T : V \to W$ be a linear transformation. The kernel (or nullspace) of T is the set
$$\ker T = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}.$$

Image If $U \leq V$ then the image of U is the set
$$T(U) = \{T(\mathbf{u}) : \mathbf{u} \in U\}.$$
We also define the image of T (or range of T), $\mathrm{im}(T)$ as the image of all of V : $\mathrm{im}(T) = T(V)$.

Kernal and Image of Linear Transformations Let $T : V \to W$ be a linear transformation between vector spaces over $\mathbb{F}$ and $U \leq V$. Then

1. $\ker T$ is a subspace of V.
2. $T(U)$ is a subspace of W, and so $\mathrm{im}(T) \leq W$.
3. If U is finite-dimensional, so is $T(U)$, so if V is finite dimensional, so is $\mathrm{im}(T)$.

Rank and Nullity If T is a linear transformation, then the dimension of the kernel of T is called the nullity of T, and the dimension of its image is called the rank of T.

Nullity - One to One A linear map $T : V \to W$ is one-to-one if and only if $\mathrm{nullity}(T) = 0$.

Rank-Nullity Theorem If V is a finite dimensional vector space over $\mathbb{F}$ and $T : V \to W$ is linear then

$$\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim(V).$$

Bijective, Injective, Surjective Let V, W be vector spaces over $\mathbb{F}$ with $\dim(V) = \dim(W)$ finite and $T : V \to W$ be linear. The following are equivalent:

- T is invertible (bijective).
- T is one-ton-one (injective) i.e. $\mathrm{nullity}(T) = 0$.
- T is onto (subjective) i.e. $\mathrm{rank}(T) = \dim(V)$.

Isomorphism An invertible linear map $T : V \to W$ is called an isomorphism of the vector spaces V and W.

Isomorphism + Dimensions Finite dimension vector spaces V and W over $\mathbb{F}$ are isomorphic if and only if they have the same dimension.

3.3 Spaces Associated with Matrices

Kernel, Image, Nullity and Rank Let A be a $p \times q$ matrix over field $\mathbb{F}$, and define a map $T : \mathbb{F}^q \to \mathbb{F}^p$ by $T(\mathbf{x}) = A\mathbf{x}$. The kernel, image, nullity and rank of A are by definition the same as those of this map T.

Column Space Suppose A has columns $\mathbf{c}_1, \ldots, \mathbf{c}_q$ (all in $\mathbb{F}^p$). Then

$$\begin{aligned} \mathrm{im}(A) &= \{A\mathbf{x} : \mathbf{x} \in \mathbb{F}^q\} \\ &= \{x_1\mathbf{c}_1 + \cdots x_q\mathbf{c}_q : x_i \in \mathbb{F}\} \\ &= \qquad\qquad (\{\mathbf{c}_1, \ldots, \mathbf{c}_q\}) \end{aligned}$$

That is, $\mathrm{im}(A)$ is the space spanned by the columns of A: the column space of $A, \mathrm{col}(A)$, a subspace of $\mathbb{F}^p$. The rank of A is thus the dimension of the column space of A.

Rank-Nullity Theorem for Matrices For $A \in M_{p,q}(\mathbb{F}), \operatorname{rank}(A) + \operatorname{nullity}(A) = q$, the number of columns of A.

Row Space The row space of $A, \operatorname{row}(A)$, is defined similarly as the space spanned by the rows: it is a subspace of $\mathbb{F}^q$. Note that $\operatorname{row}(A) = \operatorname{col}(A^T) = \operatorname{im}(A^T)$.

Row and Col Spaces Let $A \in M_{p,q}(\mathbb{F})$. The spaces $\operatorname{row}(A)$ and $\operatorname{col}(A)$ have the same dimension.

3.4 The Matrix of a Linear Map

Matrices of Linear Maps Let V, W be two finite dimensional vector spaces over $\mathbb{F}$. Suppose $\dim(V) = q$ and V has basis $\mathcal{B}$ and also $\dim(W) = p$ and W has basis $\mathcal{C}$. If $T : V \to W$ is linear then there is a unique $A \in M_{p,q}(\mathbb{F})$ with

$$[T(\mathbf{v})]_{\mathcal{C}} = A[\mathbf{v}]_{\mathcal{B}}.$$

Conversely, for any $A \in M_{p,q}(\mathbb{F})$, the equation defines a unique linear map from V to W.

Notation We call A in the above theorem the matrix of T with respect to $\mathcal{B}$ and $\mathcal{C}$. A useful notation is to denote this matrix by $[T]_{\mathcal{C}}^{\mathcal{B}}$ and then the equation takes the form

$$[T(\mathbf{v})]_{\mathcal{C}} = [T]_{\mathcal{C}}^{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}.$$

Dimension of Linear Map If $\dim(V) = q$ and $\dim(W) = p$ then $\dim(L(V, W)) = pq$.

Composition of Linear Maps as Matrices Let $T : V \to W$ and $S : W \to X$ be linear maps between vector spaces and suppose V, W and X have bases $\mathcal{A}, \mathcal{B}, \mathcal{C}$ respectively. Then, the matrix $S \circ T : V \to X$ is the product of the matrices of T and S, all taken with respect to the appropriate bases:

$$[S \circ T]_{\mathcal{C}}^{\mathcal{A}} = [S]_{\mathcal{C}}^{\mathcal{B}} \cdot [T]_{\mathcal{B}}^{\mathcal{A}}.$$

Inverting Matricies as Transformations If $T : V \to W$ is linear and invertible, the matrix of T^{-1} is the inverse of the matrix of T. Thus the group of invertible linear maps on an n-dimensional vector space over $\mathbb{F}$ is isomorphic to $\mathrm{GL}(n, \mathbb{F})$. Formally,

$$[T^{-1}]_{\mathcal{B}}^{\mathcal{C}} = ([T]_{\mathcal{C}}^{\mathcal{B}})^{-1}.$$

Change of Basis Matrix If vector space V has two bases $\mathcal{B}$ and $\mathcal{C}$, the matrix $[\mathrm{id}]_{\mathcal{C}}^{\mathcal{B}}$ of the identity map is called the change of basis matrix (from $\mathcal{B}$ to $\mathcal{C}$). This can be used to change coordinates:

$$[\mathbf{v}]_{\mathcal{C}} = [\mathrm{id}]_{\mathcal{C}}^{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}.$$

Rank and Nullity of Matrices Let $T : V \to W$ be a linear map between finite dimensional vector spaces over $\mathbb{F}$ and A its matrix with respect to any two bases in V and W. Then

$$\text{nullity}(A) = \text{nullity}(T) \quad \text{and} \quad \text{rank}(A) = \text{rank}(T).$$

Invariant Subspace Let V be a vector space over $\mathbb{F}$ and $T : V \to V$ a linear map. If $X \leq V$ is such that $T(X) \leq X$, we call X and invariant subspace of T.

Linear Maps of Invariant Subspaces Let $T : V \to V$ be a linear map on a finite dimensional vector space. Suppose $V = X \oplus Y$ with both X and Y invariant subspaces of T with dimensions p and q respectively. Then there is a basis $\mathcal{B}$ for V in which the matrix $[T]^{\mathcal{B}}_{\mathcal{B}}$ of T is of the form

$$[T]^{\mathcal{B}}_{\mathcal{B}} = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix}$$

with A a $p \times p$ and B a $q \times q$ matrix.

3.5 Similarity

Definition Matrices A and B in $M_{p,p}(\mathbb{F})$ are similar if there exists a matrix $P \in \text{GL}(p, \mathbb{F})$ such that $B = P^{-1}AP$.

Similar Matrices over Different Bases Matrices A_1 and A_2 are similar if and only if they are the matrices of the same linear transformation with respect to two choices of bases.

Similarity Invariant A property of matrices is called a similarity invariant if it is the same for all similar matrices.

The determinant, rank, nullity and trace of matrices are all similarity invariants.

3.6 Multilinear Maps

Bilinear Let V_1, V_2 and W be vector spaces over field $\mathbb{F}$. A map $T : V_1 \times V_2 \to W$ is bilinear if it is linear in each argument, that is

$$\begin{aligned} T(\lambda \mathbf{v}_1 + \mathbf{v}_1', \mathbf{v}_2) &= \lambda T(\mathbf{v}_1, \mathbf{v}_2) + T(\mathbf{v}_1', \mathbf{v}_2) \\ T(\mathbf{v}_1, \lambda \mathbf{v}_2 + \mathbf{v}_2') &= \lambda T(\mathbf{v}_1, \mathbf{v}_2) + T(\mathbf{v}_1, \mathbf{v}_2') \end{aligned}$$

for all suitable vectors and scalars. If $V_2 = V_1$ we call T bilinear on V_1.

Symmertric and Alternating Multilinear Maps A multilinear map T on V is said to be symmetric if its value on any ordered set of vectors is unchanged when any two of the vectors are swapped. If such a swap always simply changes the sign of the value, T is called alternating.

4 Inner Product Spaces

4.1 The Dot product in $\mathbb{R}^p$

Positive Definite A bilinear $\mathbb{F}$-valued map, T, on $\mathbb{F}^p$ is positive definite if for all $\mathbf{a} \in \mathbb{F}^p, T(\mathbf{a}, \mathbf{a}) \geq 0$ and $T(\mathbf{a}, \mathbf{a}) = 0$ if and only if $\mathbf{a} = 0$.

Cauchy-Schawrz Inequality in $\mathbb{R}^p$ For any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^p$ we have

$$-||\mathbf{a}||||\mathbf{b}|| \leq \mathbf{a} \cdot \mathbf{b} \leq ||\mathbf{a}||||\mathbf{b}||.$$

If $\mathbf{a} \neq 0, \mathbf{b} \neq 0$ then

$$-1 \leq \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} \leq 1.$$

Angle between Two Vectors If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ are non-zero then the angle θ between $\mathbf{a}$ and $\mathbf{b}$ is defined by

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||||\mathbf{b}||}, \quad \theta \in [0, \pi]$$

We call non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ orthogonal if $\mathbf{a} \cdot \mathbf{b} = 0$.

Orthogonal Complement Let $X \leq \mathbb{R}^p$ for some p. The space

$$Y = \{\mathbf{y} \in \mathbb{R}^p : \mathbf{y} \cdot \mathbf{x} = 0 \text{ for all } \mathbf{x} \in X\}$$

is called the orthogonal complement of $X, X^{\perp}$

Orthogonal Sets A set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^p$ of non-zero vectors is orthogonal if $\mathbf{v}_i \cdot \mathbf{j} = 0, i \neq j$. We say S is orthonomral if $\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & \text{else} \end{cases}$.

An orthogonal set S in $\mathbb{R}^p$ is linearly independent.

The Triangle Inequality For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^p$ - $||\mathbf{a} + \mathbf{b}|| \leq ||\mathbf{a}|| + ||\mathbf{b}||$.

4.2 Dot product in $\mathbb{C}^p$

Dot Product The standard dot product on $\mathbb{C}^p$ is defined by

$$\mathbf{a} \cdot \mathbf{b} \sum_{i=1}^{p} \overline{a_i} b_i = \overline{\mathbf{a}}^T \mathbf{b}.$$

Notation We will use $\mathbf{a}^*$ as a useful shorthand for $\overline{\mathbf{a}}^T$ from now on, so that $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^* \mathbf{b}$.

Properties of the Dot Product The standard dot product on $\mathbb{C}^p$ has the following properties:

1. $\mathbf{a} \cdot (\lambda \mathbf{b} + \mathbf{c}) = \lambda \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ for $\lambda \in \mathbb{C}$.
2. $\mathbf{b} \cdot \mathbf{a} = \overline{\mathbf{a} \cdot \mathbf{b}}$.
3. $(\lambda \mathbf{b} + \mathbf{c}) \cdot \mathbf{a} = \overline{\lambda} \mathbf{b} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{a}$ for $\lambda \in \mathbb{C}$.
4. $||\mathbf{a}|| \geq 0$ and $||\mathbf{a}|| = 0 \iff \mathbf{a} = 0$.

4.3 Inner Product Spaces

Inner Product If V is a vector space over $\mathbb{F}$ then an inner product on V is a function $\langle , \rangle : V \times V \to \mathbb{F}$, that is, for all $\mathbf{u}, \mathbf{v} \in V \langle \mathbf{u}, \mathbf{v} \rangle \in \mathbb{F}$, such that

IP1 $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$.

IP2 $\langle \mathbf{u}, \alpha \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$.

IP3 $\langle \mathbf{v}, \mathbf{u} \rangle = \overline{\langle \mathbf{u}, \mathbf{v} \rangle}$.

IP4 $\langle \mathbf{v}, \mathbf{v} \rangle$ is real and > 0 if $\mathbf{v} \neq 0$ and $= 0$ if $\mathbf{v} = 0$.

We call V with $\langle , \rangle$ an inner product space.

The norm of the vector is then $\|\mathbf{v}\| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2}$.

Properties of the Inner Product Let V be an inner product space. Then for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\alpha \in \mathbb{C}$:

1. $\|\mathbf{u}\| > 0$ if and only if $\mathbf{u} \neq 0$.
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$.
3. $\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \overline{\alpha} \langle \mathbf{u}, \mathbf{v} \rangle$ and $\|\alpha \mathbf{u}\| = |\alpha| \|\mathbf{u}\|$.
4. $\langle \mathbf{x}, \mathbf{u} \rangle = 0$ for all $\mathbf{u}$ if and only if $\mathbf{x} = 0$.
5. $|\langle \mathbf{u}, \mathbf{v} \rangle \leq \|\mathbf{u}\| \|\mathbf{v}\|$ (Cauchy - Schwarz inequality).
6. $\|\mathbf{u}\| \mathbf{v} \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ (The Triangle Inequality).

4.4 Orthogonality and Orthonormality

Orthogonal Let V be an inner product space. Non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\langle u, v \rangle = 0$; we will use the notation $\mathbf{u} \perp \mathbf{v}$ for this.

A set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq V$ of non-zero vectors is orthogonal if $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0, i \neq j$.

Orthonormal We say S is orthonormal if $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

Projection In inner product space V let $\mathbf{v} \neq \mathbf{0}$. The projection of $\mathbf{u} \in V$ onto $\mathbf{v}$ is defined as

$$\operatorname{proj}_{\mathbf{v}}(\mathbf{u}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}.$$

Note: $\mathbf{u} - \alpha\mathbf{v} \perp \mathbf{v} \iff \mathbf{u} - \alpha\mathbf{v} = \mathbf{u} - \operatorname{proj}_{\mathbf{v}}\mathbf{u}$.

Orthogonal and Orthonomral Sets If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal set of non-zero vectors in inner product space V and $\mathbf{v} \in \operatorname{span}(S)$ then $\mathbf{v} = \sum_{i=1}^{k} \operatorname{proj}_{\mathbf{v}_i} \mathbf{v}$.

If S is an orthonormal set $\mathbf{v} = \sum_{i=1}^{k} \langle \mathbf{v}_i, \mathbf{v} \rangle \mathbf{v}_i$.

Orthonormal Basis If $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is an orthonormal basis for V then $\mathbf{v} = \sum_{i=1}^{n} \langle \mathbf{e}_i, \mathbf{v} \rangle \mathbf{e}_i$.