Graph Theory

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Contents

Chapter 1

Introduction

1.1 Definitions

A **graph** $G = (V, E)$ is a set V of *vertices* and a set E of unordered pairs of distinct vertices, called *edges*. Write vw or $\{v, w\}$ for the edge joining v and w, and say that v and w are **neighbours** or that they are *adjacent*.

In these notes, unless otherwise stated, graphs are:

- **finite**: $|V| \in \mathbb{N}$.
- **labelled**: vertices are distinguishable, usually $V = [n] := \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.
- **undirected**: edges are *unordered* pairs of vertices.
- **simple**: no loops $\{v, v\}$ or multiple edges (since E is not a multiset).

A graph G with vertex set $\{v_1, \ldots, v_n\}$ has **adjacency matrix** $A(G) = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E, \\ 0 & \text{otherwise.} \end{cases}$$

$A(G)$ is a **symmetric** $n \times n$ 0-1 matrix with zero diagonal.

The **trivial graph** has at most one vertex. Hence it has no edges.

A **subgraph** of a graph $G = (V, E)$ is a graph $H = (W, F)$ such that $W \subseteq V$ and $F \subseteq E$.

We say that H is an **induced subgraph** if for all $v, w \in W$ if $vw \in E(G)$ then $vw \in E(H)$. Write $H = G[W]$, and say that H is the subgraph of G *induced by* the vertex set W.
The number of **vertices** of G, written $|G| = |V(G)|$, is called the *order* of G. The number of **edges** of G, sometimes written $||G|| = |E(G)|$, is called the *size* of G.

Two graphs $G = (V, E)$ and $H = (W, F)$ are **isomorphic** if there exists a *bijection* $\phi : V \to W$ such that $\phi(v)\phi(w) \in F$ if and only if $vw \in E$. The map ϕ is called a *graph isomorphism* or *isomorphism*.

1.2 The Degree of a Vertex

If $v \in e$ where v is a vertex and e is an edge, then we say that e is *incident with* v. The **degree** $d_G(v)$ of vertex v in a graph G is the number of *edges* of G which are *incident with* v. A vertex of degree 0 is an *isolated vertex.*

Let $N_G(v)$ be the set of all **neighbours** of v in G, then $d(v) = |N(v)|$.

Lemma 1.2.1 (The Handshaking Lemma)**.** In any graph, $G = (V, E)$,

$$\sum_{v \in V} d(v) = 2|E|.$$

Let $\delta(G) = \min_{v \in V} d(v)$ be the minimum degree in G, and $\Delta(G) = \max_{v \in V} d(v)$ be the maximum degree in G.

1.2.1 Some Special Graphs

A graph is k**-partite** if there exists a partition of its vertex set

$$V = V_1 \cup V_2 \cup \cdots V_k$$

into k nonempty disjoint subsets (parts) such that there are no edges between vertices in the same part.

The **complete graph** on r vertices, denoted K_r, has all $\binom{r}{2}$ edges present. The **complete bipartite graph** K_r, s has r vertices in one part of the vertex bipartition, s vertices in the other, and all rs present.

A graph is **regular** if every vertex has the same degree. If every vertex of a graph has degree d then we say that the graph is d-regular.

The **complement** of a graph G is the graph $\bar{G} = (V, \bar{E})$ where $vw \in \bar{E}$ if and only if $vw \notin E$. Note that $\bar{K_n}$ is the graph with n vertices and no edges.

If $G = (V, E)$ and $X \subset V$ then $G - X$ denotes the graph obtained from G by deleting all vertices in X and all edges which are incident with vertices in X. If $F \subseteq E$ then $G - F$ denotes the graph $(V, E - F)$ obtained from G by deleting the edges in F.

1.3 Paths and Cycles

A **walk** in the graph G is a sequence of vertices $v_0 v_1 v_2 \cdots v_k$ such that $v_i v_{i+1} \in E$ for $i = 0, 1, \ldots, k-1$. The **length** of this walk is k. The walk is **closed** if $v_0 = v_k$.

An **Euler tour** is a *closed walk* in a graph which uses every edge precisely once. A graph is Eulerian if it has an Euler tour.

Theorem 1.3.1 (Euler, 1736)**.** A connected graph is Eulerian if and only if every vertex has even degree.

A walk is a **path** if it does not visit any vertex more than once. A path is a sequence of *disinct* vertices, with subsequence vertices joined by an edge. A path $v_0 v_1 \ldots v_k$ with k edges is called a k-path and has length k.

If $k \geq 3$ and $P = v_0 v_1 \cdots v_{k-1}$ is a path of length $k-1$ then $C = P + v_0 v_{k-1}$ is a **cycle** of length k, also called a $k-$*cycle*. It is a closed walk which visits no internal vertex more than once.

An edge which joins two vertices of a cycle C, but which is not an edge of C, is called a **chord**. An **induced cycle** is a cycle which has no chords.

Proposition 1.3.2. Every graph G contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G)+1$, if $\delta(G) \geq 2$.

Proof. Let $P = x_0 x_1 \ldots x_k$ be the longest path in G. By maximality of P, all neighbours of x_k lie on P. Hence $\delta(G) \leq d(x_k) \leq k = |\{x_0, x_1, \ldots, x_{k-1}\}|$, which proves the first statement. Let x_i be the smallest-indexed neighbour of x_k in P. Then $C = x_k x_i x_{i+1} \ldots x_{k-1} x_k$ is a cycle of length $\geq \delta(G)+1$ because C contains $d(x_k) \geq \delta(G)$ neighbours of x_k as well as x_k. $\square$

The *minimum length* of a cycle in G is the **girth** of G, denoted by $g(G)$.

Given $x, y \in V$, let $d_G(x,y)$ be the length of a shortest path from x to y in G, called the **distance** from x to y in G. Set $d_G(x,y) = \infty$ if no such path exists.

We say that G is **connected** if $d_G(x,y)$ is finite for all $x, y \in V$.

Let the **diameter** of G be $\operatorname{diam}(G) = \max_{x,y \in V} d_G(x,y)$.

Proposition 1.3.3. Every graph G which contains a cycle satisfies $g(G) \leq 2\operatorname{diam}(G)+1$.

Proof. Let C be a shortst cycle in G, so $|C| = g(G)$. For a contradiction, assume $g(G) \geq 2\operatorname{diam}(G)+2$.

Choose vertices x, y on C with $d_C(x,y) \geq \operatorname{diam}(G)+1$. In G the distance $d_G(x,y)$ is strictly smaller, so any shortest path P from x to y in G is not a subgraph of C. But using P together with the shorter arc of C from x to y gives a closed walk of length $< |C|$. This closed walk contains a shorter cycle than C which is a contradiction. $\square$

1.4 Connectivity

A maximal connected subgraph of G is called a **component** (or **connected component**) of G.

Proposition 1.4.1. The vertices of a connected graph can be labelled $v_1, v_2, \ldots, v_n$ such that $G_n = G$ and $G_i = G[v_1, \ldots, v_i]$ is connected for all i.

Proof. Choose v_1 arbitrarily. Now suppose that we have labelled $v_1, \ldots, v_i$ such that $G_j = G[v_1, \ldots, v_j]$ is connected for all $j = 1, \ldots, i$.

If $i < n$ then $G_i \neq G$, so there exists some $v_j \in \{v_1, \ldots, v_i\}$ with a $w \notin \{v_1, \ldots, v_i\}$ in G. (Otherwise $G_i \neq G$ is a component of G, impossible as G is connected.) Let $v_{i+1} = w$, then $G_{i+1} = G[v_1, \ldots, v_{i+1}]$ is connected. This completes the proof, by induction. $\square$

Let $A, B \subseteq V$ be sets of vertices. An (A, B)**-path** in G is a path $P = x_0x_1 \cdots x_k$ such that

$$P \cap A = \{x_0\}, \quad P \cap B = \{x_k\}.$$

Let $A, B \subseteq V$ and let $X \subseteq V \cup E$ be a set of vertices and edges. We say that X **separates** A and B in G if every $(A, B)-$path in G contains a vertex or edge from X.
Note that we do not assume that A and B are disjoint and if X separates A and B then $A \cap B \subseteq X$.
We say that X **separates** two vertices a, b if $a, b \notin X$ and X separates the sets $\{a\}, \{b\}$.

More generally, we say that X *separates* G, and call X a **separating set** for G, if X separates two vertices of G. That is, X separates G if there exist distinct vertices $a, b \notin X$ such that X separates a and b.

If $X = \{x\}$ is a separating set for G, where $x \in V$, then we say that x is a **cut vertex**.

If $e \in E$ and $G - e$ has more components than G then e is a **bridge**.

The unordered pair (A, B) is a **separation** of G if $A \cup B = V$ and G has no edge between $A - B$ and $B - A$. The second conditions says that $A \cap B$ separates A from B in G. If both $A - B$ and $B - A$ are nonempty then the separation is **proper**. The order of the separation is $|A \cap B|$.

Definition. Let $k \in \mathbb{N}$. The graph G is **k-connected** if $|G| > k$ and $G - X$ is connected for all subsets $X \subseteq V$ with $|X| < k$.

The **connectivity** $\kappa(G)$ of G is defined by

$$\kappa(G) = \max\{k : G \text{ is } k\text{-connected}\}.$$

So, $\kappa(G) = 0$ iff G is trivial or G is disconnected. Also, $\kappa(K_n) = n - 1$ for all positive integers n.

Definition. Let $\ell \in \mathbb{N}$ and let G be a graph with $|G| \geq 2$. If $G - F$ is connected for all $F \subseteq E$ with $|F| < \ell$ then G is ℓ**-edge-connected**.

The **edge connectivity** $\lambda(G)$ is defined by

$$\lambda(G) = \max\{\ell : G \text{ is } \ell\text{-edge-connected}\}.$$

Proposition 1.4.2. If $|G| \geq 2$ then $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Theorem 1.4.3 (Mader, 1973)**.** Let k be a positive integer. Every graph G with average degree at least $4k$ has a $(k + 1)$-connected subgraph H with

$$\frac{|E(H)|}{|V(H)|} > \frac{|E(G)|}{|V(G)|} - k.$$

Proof. We write $|G|$ instead of $|V(G)|$. Let $\gamma = \frac{|E(G)|}{|G|} \geq 2k$. Consider subgraphs G' of G which satisfy:

$$|G'| \geq 2k \quad \text{and} \quad |E(G')| > \gamma(|G'| - k). \tag{1.1}$$

such graphs G' exists as G satisfies 1.1. (Average degree of G is $\frac{2|E(G)|}{|G|} \geq 4k$, so

$|G| \geq 4k$ and $\gamma(|G|-k) = |E(G)|\frac{(|G|-k)}{|G|} < |E(G)|$.)

Now let H be a subgraph of G of smallest order which satisfies 1.1. We continue the proof by proving three claims.

Claim 1. If G' satisfies 1.1 then $|G'| > 2k$.
Proof. If G' satisfies 1.1 and $|G'| = 2k$ then $|E(G')| > \gamma(|G'|-k) \geq 2k^2 > \binom{|G'|}{2}$, contradiction.

Claim 2. $S(H) > \gamma$.
Proof. For a contradiction, suppose that $S(H) \leq \gamma$. Let G' be obtained from H by deleting a vertex of degree $\leq \gamma$. Then $|G'| < |H|$ and G' satisfies 1.1, which is a contradiction. To see this, check:

$$\begin{aligned} |G'| &= |H| - 1 \geq 2k, \quad \text{by Claim 1, and} \\ |E(G')| &\geq |E(H)| - \gamma > \gamma(|H|-k-1), \quad \text{as } H \text{ satisfies 1.1} \\ &= \gamma(|G'|-k). \end{aligned}$$

Hence $S(H) > \gamma$. It follows that $|H| \geq \gamma$. Thus,

$$\frac{|E(H)|}{|H|} > \frac{\gamma(|H|-k)}{|H|}. \qquad \text{(as } H \text{ satisfies 1.1)}$$

Claim 3. H is $(k+1)$-connected.
Proof. By Claim 1, $|H| \geq 2k+1 \geq k+2$ as $k \geq 1$. So H is large enough. For a contradiction, suppose that H is not $(k+1)$-connected. Then H has a proper separation $\{U_1, U_2\}$ of order at most k.

Let $H_i = H[U_i]$ for $i = 1, 2$. Since any vertex $v \in U_1 - U_2$ has $d_H(v) \geq S(H) > \gamma$ (by Claim 2), and all neighbours of v in H belong to H_1, we have $|H_1| \geq \gamma \geq 2k$. Similarly, $|H_2| \geq 2k$. By minimality of H, neither H_1 nor H_2 satisfies 1.1. Hence $|E(H_i)| \leq \gamma(|H_i| - k)$ for $i = 1, 2$. But then

$$\begin{aligned} |E(H)| &\leq |E(H_1)| + |E(H_2)| \\ &\leq \gamma(|H_1| + |H_2| - 2k) \\ &\leq \gamma(|H| - k), \qquad \text{(by inclusion-exclusion)} \end{aligned}$$

since $|U_1 \cup U_2| \leq k$. This contradicts 1.1 for H. So H is $(k+1)$-connected, completing the proof of Claim 3 and of the theorem. □

1.5 Trees and Forests

A graph with no cycles is a **forest** (also called an acyclic graph). A connected graph with no cycles is a **tree**.

Theorem 1.5.1. The following are equivalent for a graph T:

(i) T is a tree;

(ii) Any two vertices of T are linked by a *unique* path in T;

(iii) T is *minimally connected*: that is, T is connected but $T - e$ is disconnected for every $e \in E(T)$;

(iv) T is *maximally acyclic*: that is, T is acyclic but $T + xy$ has a cycle for any two nonadjacent vertices x, y in T.

Corollary 1.5.2. If G is connected then G has a spanning tree.

Proof. Let be a connected graph and let H be a minimal connected spanning subgraph of G. (Note H exists as G is a connected spanning subgraph of itself.) By theorem 1.5.1, H is a tree. □

Corollary 1.5.3. The vertices of a tree can be labelled as $v_1, \ldots, v_n$ so that for $i \geq 2$, vertex v_i has a unique neighbour in $\{v_1, \ldots, v_{i-1}\}$.

Proof. We use the labelling from Proposition 1.4.1. This labels the vertices of a given tree G as $v_1, \ldots, v_n$ such that $G[v_1, \ldots, v_n]$ is connected. Let $i \geq 1$ then $G[v_1, \ldots, v_i]$ is a tree. Note $G[v_1, \ldots, v_{i+1}]$ is connected by Proposition 1.4.1, so v_{i+1} has at least one neighbour in $G[v_1, \ldots, v_i]$.

For a contradiction, suppose that v_{i+1} has two neighbours z and w in $G[v_1, \ldots, v_i]$. There is a (unique) path P in $G[v_1, \ldots, v_i]$ between z and w, and this path does not visit v_{i+1}. Hence $P \cup \{zv_{i+1}, wv_{i+1}\}$ is a cycle in G, contradiction. □

Corollary 1.5.4. A connected graph with n vertices is a tree if and only if it has $n - 1$ edges.

Proof. Suppose that G is a tree on n vertices. The result is true when $n = 1$. Now suppose the result is true when $n = k$. Let G be a tree on $k + 1$ vertices. Let v be a leaf in G (e.g .take an end vertex of a longest path in G.) Then $G - v$ is a tree on k vertices, so $G - v$ has $k - 1$ edges (inductive hypothesis). Therefore G has k edges as v has degree 1. This concluses the proof, by induction.

Conversely, suppose that $G$is connected with n vertices and $n-1$ edges. Then G contains a spanning tree H, by an earlier corollary. Then H has exactly $n - 1$ edges, since it is a tree on n vertices. Hence $H = G$, so G is a tree. □

Corollary 1.5.5. If T is a tree and G is any graph with $\delta(G) \geq |T| - 1$ then G has a subgraph isomorphic to T.

Chapter 2

Matchings and Hamilton Cycles

Two edges in a graph are called **independent** if they have no vertices in common.
A set M of pairwise independent edges in a graph is called a **matching**.

Given $G = (V, E)$ and $U \subseteq V$, say that $M \subseteq E$ is a **matching of U** if M is matching and every vertex in U is incident with an edge of M. We say that the vertices in U are matched by M, and t hat the vertices not incident with any edge of M are **unmatched**.

A matching M is a **maximal matching** of G if $M \cup \{e\}$ is not a matching for any $e \in E - M$.
A **maximum matching** of G is a matching of G such that no set of edges with size greater than $|M|$ is a matching.

A **perfect matching** of G is a matching of G which matches every vertex of G. Note: a perfect matching is a 1-regular spanning subgraph of G also called a **1-factor** of G.

A k**-factor** is a k-regular spanning subgraph. A **2-factor** in a graph is the union of disjoint cycles which covers all the vertices.

2.1 Matchings in Bipartite Graphs

Let $G = (V, E)$ be a bipartite graph with vertex bipartition $V = A \cup B$. Here A, B are nonempty disjoint sets. We use the convention that all vertices called $a, a', a'', \ldots$ belong to A and similarly for B.

Let M be matching in G. A path in G which starts at an *unmatched* vertex of A and contains, alternately, edges from $E - M$ and from M, is called an **alternating path** with respect to M.
If an alternating path P ends in an unmatched vertex of B then it is called an **augmenting path**.

Definition 2.1.1. A set $U \subseteq V$ is a **cover** (or **vertex cover**) of G if every edge of G is incident with a vertex in U.

Theorem 2.1.2 (König, 1931)**.** Let G be a bipartite graph. The size of a maximum matching in G is equal to the size of the minimum vertex cover of G.

Proof. Let $\hat{U}$ be a cover in G and let Mbe a maximum matching. Then $|\hat{U}| \geq |M|$ as we must cover every edge of M. Hence it suffices to construct a cover U of G with $|U| = |M|$.

We build U be choosing one vertex from each edge of M to place into U, as follows:

- If $ab \in M$ and some alternating path in G with respect to M ends in b. Then put b into U otherwise put a into U.

Let $ab \in E$. If $ab \in M$ then $a \in U$ or $b \in U$ by definition of U. Now assume $abb \notin M$. Since M is maximum, there exists $a'b' \in M$ with $a = a'$ or $b = b'$. If a is unmatched in M then $b = b'$ for some $a'b' \in M$. Hence ab is an alternating path ending in $b = b'$, so we chose b' to go into U from the edge $a'b' \in M$. So the edge ab is covered by U in this case.

Hence we assume that $a = a'$ for some $a'b' \in M$. If $a = a' \in U$ then we are done. Otherwise $b' \in U$, so there is an alternating path P ending in b'. Then $P = a_1b_1a_2b_2 \dots b'$, and we have three cases:

(i) P does not include a or b. Then $Pab = a_1a_2 \dots b'ab$ is an alternating path in G with respect to M. By maximality of M, b is matched or else we have an augmenting path. Hence $b \in U$ as b is the chosen vertex from its matching edge.

(ii) If b is on P before a, or $b \in P$ and $a \notin P$, then $P = a_1b_1a_2 \dots b \dots b'$. Then we let $P' = a_1b_1 \dots b$. This is an alternating path ending in b, so finish proof as case above.

(iii) If a is on P before b, or $a \in P$ and $b \notin P$. Then $P = a_1b_1 \dots a_rb_r \dots b'$ and we take $P' = a_1b_1 \dots ab$. This is an alternating path ending in b, so finish proof as case above.

This proves U is a cover of G and since $|U| = |M|$, this completes the proof. □

For a subset $S \subseteq A$, let $N(S) = \bigcup_{v \in S} N(v)$ be the set of vertices in B which are neighbours of some vertex in S.

Theorem 2.1.3 (Hall, 1935)**.** Let G be a bipartite graph. Then G contains a matching of A if and only if

$$|N(S)| \geq |S| \quad \text{for all } S \subseteq A. \tag{2.1}$$

Proof. We have that this condition is necessary. Now suppose that (2.1) holds. For a contradiction, suppose that G has no matching of A. Then König's Theorem (Theorem 2.1.2) says that G has a cover U with $|U| < |A|$. Suppose that $U = A' \cup B'$ with $A' \subseteq A$ and $B' \subseteq B$. Then $|A'| + |B'| = |U| < |A|$, so $|B'| < |A| - |A'| = |A - A'|$. Since U is a cover, G has no edges from $A - A'$ to $B - B'$. Hence $N(A - A') \subseteq B'$, and so $|N(A - A')| \leq |B'| < |A - A'|$. This contradicts Hall's condition 2.1 for $S = A - A'$. Hence G contains a matching of A. □

Corollary 2.1.4. Let G be a bipartite graph and $d \in \mathbb{N}$. If $|N(S)| \geq |S| - d$ for all $S \subseteq A$ then G has a matching of size $|A| - d$.

Proof. Add d new vertices to B and join each of them by an edge to each vertex of A. Then for all $S \subseteq A$, in the new graph G', $|N_{G'}(S) \geq |S| - d + d = |S|$. Hall's condition is satisfied in G'. Therefore there is a matching M in G' which matches all of A. At least $|A| - d$ edges in M are edges of G. □

Corollary 2.1.5. If G is a k-regular bipartite graph then G has a perfect matching.

Proof. Assume $k \geq 1$. Since G is k-regular, $|E(G) = k|A| = k|B|$, so $|A| = |B|$. Hence it suffices to prove that G contains a matching of A. Every set $S \subseteq A$ is joined to $N(S)$ by a total of $k|S|$ edges. These edges are a subset of the $k|N(S)|$ edges incident with $|N(S)$. Hence $k|S| \leq k|N(S)|$

and diving by k shows that Hall's condition holds. Thus, G has a matching of A. □

Corollary 2.1.6. Every regular graph of positive even degree has a 2-factor.

Proof. Let G be any $2k$-regular graph, $k \geq 1$. Without loss of generality, suppose that G is connected (or apply this argument to each component). By Theorem 1.3.1, G has an Euler tour $v_0 v_1 \dots v_{l-1} v_l$ where $v_l = v_0, e_i = v_i v_{i+1} \in E(G)$ using each edge exactly once.

Replace each vertex $v \in V$ with a pair of vertices v^-, v^+, and replace every edge $e_i = v_i v_{i+1}$ by the edge $v_i^+ v_{i+1}^-$. The resulting graph G' is a k-regular bipartite graph. Hence by Corollary 2.1.5, G' has a perfect matching (1-factor). Collapse every vertex pair (v^-, v^+) back into a single vertex v, for all $v \in V$. The 1-factor of G' becomes a 2-factor of G. □

2.2 Hamilton Cycles

A **Hamilton cycle** is a connected 2-factor. That is, it is a cycle which includes every vertex.

Say G is **Hamiltonian** if it contains a Hamilton cycle. A Hamiltonian graph G must be connected with minimum degree $\delta(G) \geq 2$.

Theorem 2.2.1 (Dirac, 1952)**.** Every graph with $n \geq 3$ vertices and with minimum degree at least $n/2$ has a Hamilton cycle.

Proof. Let G be a graph with minimum degree $\geq n/2$ and $n \geq 3$ vertices. Then G is connected, as otherwise the degree of any vertex in the smaller component must be $< n/2$. Let $P = x_0 \dots x_k$ be a longest path in G. by maximality, all neighbours of x_0 and x_k lie on P. So at least $n/2$ of the vertices $x_0, \dots, x_{k-1}$ are adjacent to x_k and at least $n/2$ of these same vertices satisfy $x_0 x_{i+1} \in E(G)$. By the pigeonhole principle, as $k < n$, there exists $i \in \{0, \dots, k-1\}$ with $x_0 x_{i+1}, x_i x_k \in E(G)$. This gives a cycle $x_0 x_1 \dots x_i x_k \dots x_{i+1} x_0$. We claim this is a Hamilton cycle. If not then, as G is connected, there is some $u \notin C$ with a neighbour $v \in C$. Then we can start at u, go to$v$ then go around C (in some direction) and stop just before we reach v again (i.e. stop at $x \in N_C(v)$). This gives a path which is longer than P, contradiction. □

2.3 Matchings in General Graphs

Given a graph G, let C_G be the set of its components and let $q(G)$ denote the number of odd components (connected components having an odd number of vertices).

Theorem 2.3.1 (Tutte, 1947)**.** A graph G has a perfect matching if and only if

$$q(G - S) \leq |S| \quad \text{for all } S \subseteq V(G). \tag{2.2}$$

Proof. We have seen that the condition (2.2) is necessary: if G has a perfect matching then (2.2) holds. Now suppose that G has no perfect matching. We want to find a "bad" set S_0 which fails condition (2.2). If $|G|$ is odd then, $S_0 = \emptyset$ is bad. So assume $|G|$ is even.

Claim 1. If G' is obtained from G by adding edges and $S_0 \subseteq V$ is bad for G' then S_0 is bad for G.

Proof. If S_0 bad for G' then $q(G - S_0) > |S_0|$. But each odd component of $G' - S$ is a disjoint union of components of $G - S$, at least one of which must be odd. So $q(G - S) \geq q(G' - S)$.

Hence by Claim 1, we can assume that G has no perfect matching but adding any edge to G gives a graph G' which has a perfect matching.

Claim 2. S is a bad set for G if and only if all components of $G - S$ are complete and every vertex in S is adjacent to all other vertices in G.
Proof. For proof, call the second half of the claim (*). If S is bad for G but does satisfy (*) then we can add an edge to G to get a graph G' with S still bad for G'. This contradicts our assumption on the maximality of G. Conversely suppose S satisfies (*) but S is not bad. Then we can form a perfect matching since $|G|$ is even. This is a contradiction as G has no perfect matching. Hence S is bad.

Define $S_0 = \{v \in V : d_G(v) = n - 1\}$ to be the set of all vertices v in G which are adjacent to every other vertex $w \neq v$.

Claim 3. S_0 is bad.
Proof. We need to show that S_0 satisfies (*). For a contradiction, suppose that S_0 does not satisfy (*). Then $G - S_0$ has a component K which is not complete. Let $a, a' \in V(K)$ with $aa' \notin E(G)$. Fix a shortest path from a to a' in K which starts $abc \ldots a'$. Such a path has length ≥ 2 and $ac \notin E(G)$. Note $b \in K$, so $b \in S_0$, so there is some $d \in V$ with $bd \notin E$. By maximality of G, there is a perfect matching M_1 in $G + ac$ and a perfect matching M_2 in $G + bd$. Take a maximal path P in G, starting at d with an edge from M_1, and taking alternately edges from M_1 and M_2. Say $P = d \ldots v$.

- If the last edge of P is in M_1 then $v = b$ or we could extend P. Let $C = P + bd$ (cycle in $G + bd$).
- If the last edge of P is in M_2 then $v \in \{a, c\}$ as the M_1 edge incident with v must be ac. Let C be the cycle $d \ldots vbd$.

In each case, C is an alternating (even length) cycle in $G + bd$ which contains bd. Form M_2' from M_2 by replacing $M_2 \cap C$ by $C - M_2$. This gives a perfect matching of G, contradiction. Hence S_0 satisfies (*), so Claim 3 holds and the proof is complete. $\square$

Corollary 2.3.2 (Petersen, 1891)**.** Every bridge cubic (3-regular) graph has a perfect matching.

Proof. Let G be a bridgeless cubic graph. We prove that G satisfies Tutte's condition. Let $S \subseteq V(G)$ be given and consider an odd component C of $G - S$. The sum of the degrees of vertices in C is $3|C|$, which is an odd number. Every edge with both end vertices in C contributes an even number to this sum. Hence the number of edges from C to S is odd.

As G has no bridge, there must be at least 3 edges from S to C. Therefore the number of edges from S to $G - S$ is at least $3q(G - S)$. But the number of edges from S to $G - S$ is bounded above by the sum of the degrees of vertices in S, which is $3|S|$ as G is cubic. Hence $3q(G - S) \leq$ # edges from S to G - S $\leq 3|S|$ and thus $q(G - S) \leq |S|$. Therefore by Tutte's Theorem, G has a perfect matching. $\square$

Chapter 3

The Probabilistic Method

This chapter assumes knowledge of elementary probability knowledge. Content from first year is sufficient.

Example 3.0.1. Let Ω be the set of all graphs on the vertex set $\{1, 2, \ldots, n\}$. Then $|\Omega| = 2^{\binom{n}{2}}$. Define $\pi(G) = 2^{\binom{n}{2}}$ for all $G \in \Omega$. This is the *uniform model of random graphs*.

Lemma 3.0.2. The expected number of edges in a uniformly chosen graph on the vertex set $\{1, 2, \ldots n\}$ is $\frac{1}{2}\binom{n}{2}$.

Proof. (From Definition) For $0 \leq m \leq \binom{n}{2} = N$, there $\binom{N}{m}$ are exactly of graphs on vertex set $\{1, \ldots, n\}$ with m edges. Let X be the number of edges in the random graph. Then

$$\begin{aligned}
EX &= \sum_{m=0}^{N} \Pr(X = m) \cdot m \\
&= \sum_{m=0}^{N} \frac{\binom{N}{m}}{2^N} \cdot m \\
&= \frac{N}{2^N} \sum_{m=1}^{N} \frac{(N-1)!}{(m-1)!(N-m)!} \\
&= \frac{N}{2^N} \sum_{j=0}^{N-1} \binom{N-1}{j} && \text{(j = m -1)} \\
&= \frac{N}{2^N} 2^{N-1} && \text{(by the binomal theorem)} \\
&= \frac{N}{2} = \frac{1}{2}\binom{n}{2}.
\end{aligned}$$

□

Let $A \subseteq \Omega$ be an event. The indicator variable I_A for $A \subseteq \Omega$ is

$$I_A(z) = \begin{cases} 1 & \text{if } z \in A \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.0.3 (Linearity of Expectation)**.** Let $X_1, \ldots, X_k$ be random variables on Ω and let $c_1, \ldots, c_k \in \mathbb{R}$. Define the random variable $X = c_1 X_1 + \cdots + c_k X_k$. Then

$$\mathbb{E}[X] = c_1\mathbb{E}[X_1] + c_2\mathbb{E}[X_2] + \cdot + c_k\mathbb{E}[X_k].$$

Definition 3.0.4 (Markov's Inequality)**.** SUppose that $X : \Omega \to [0, \infty)$ is a nonnegative random variable on Ω and let $k > 0$. Then

$$\Pr(X \geq k) \leq \frac{\mathbb{E}[X]}{k}.$$

In particular, if X is a nonnegative integer-valued random variable then

$$\Pr(X \neq 0) \leq \mathbb{E}[X].$$

Let $k \geq 2$ be an integer. Events $A_1, \dots, A_k$ in Ω are **mutually independent** if for all $j, \ell_1, \dots, \ell_j$ with $2 \leq j \leq k$ and $1 \leq \ell_1 < \ell_2 < \cdots < \ell_j \leq k$,

$$\Pr\left(\bigcap_{i=1}^{j} A_{\ell_i}\right) = \prod_{i=1}^{j} \Pr(A_{\ell_i}).$$

Lemma 3.0.5. Let Ω be the set of all subsets of some given set S, where $|S| = n$. Define a random set $X \subseteq S$ by setting $\Pr(x \in X) = \frac{1}{2}$, independently for each $x \in S$. Then $\Pr(X = A) = 2^{-n}$ for all $A \subseteq S$, so this gives the uniform probability space on Ω.

Proof. Fix $A \subseteq \Omega$. Then

$$\begin{aligned}
\Pr(X = A) &= \prod_{x \in A} \Pr(\text{heads}) \cdot \prod_{x \notin A} \Pr(\text{tails}) && \text{(using independence)} \\
&= \left(\frac{1}{2}\right)^{|A|} \cdot \left(\frac{1}{2}\right)^{n-|A|} \\
&= \left(\frac{1}{2}\right)^{n} = 2^{-n}
\end{aligned}$$

as claimed. □

Theorem 3.0.6 (Alon & Spencer, Theorem 2.2.1)**.** Let G be a graph with n vertices and m edges. Then G contains a bipartite subgraph with at least $m/2$ edges.

Proof. Let Ω be the set of all subsets of $V(G)$. Then $|\Omega| = 2^n$. Consider the uniform probability space on Ω. Let $A \subseteq V$ be a randomly chosen element of Ω and define $B = V - A$. Call $xy \in E(G)$ a crossing edge if exactly one of x, y belongs to A. Let X be the number of crossing edges. Finally, for each edge $e \in E(G)$ define the indicator variable

$$X_e = \begin{cases} 1 & \text{if } e \text{ is a crossing edge,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $X = \sum_{e \in E(G)} X_e$. For any $e = xy \in E(G)$, we have,

$$\begin{aligned}
\Pr(x \in A \text{ and } y \notin A) &= \Pr(x \in A) \Pr(y \in A) && \text{(using independence)} \\
&= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbb{E}X_e &= \Pr((x \in A \text{ and } y \notin A) \text{ or } (x \notin A \text{ and } y \in A)) \\
&= \Pr(x \in A \text{ and } y \notin A) + \Pr(x \notin A \text{ and } y \in A) && \text{(events are disjoint)} \\
&= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.
\end{aligned}$$

Hence, by linearity of expectation,

$$\mathbb{E}X = \sum_{e \in E(G)} \mathbb{E}X_e = \frac{m}{2}.$$

Thus there exists a fixed set $A_0 \subseteq V(G)$ which has at least $\frac{m}{2}$ crossing edges. The corresponding bipartition $(A_0, V(G) - A_0)$ defines a bipartite subgraph consisting of the $\geq \frac{m}{2}$ crossing edges. $\square$

An **independent set** in a graph G is a subset $U \subseteq V$ such that if $v, w \in U$ then $vw \in E(G)$. Let $\alpha(G)$ be the size of a maximum independent set in G, called the **independence number**.

Theorem 3.0.7. Let G have n vertices and $nd/2$ edges, where $d \geq 1$. Then $\alpha(G) \geq \frac{n}{25T1d}$. Note d, is the average degree of G.

Proof. Define the random subset $S \subseteq V(G)$ by $\Pr(v \in S) = p$, independently for all $v \in V$. Here $p \in [0, 1]$ which we will fix later.

Let $X = |S|$ and let Y be the number of edges of G with both endvertices in S. Then $\mathbb{E}X = pn$. For $e \in E(G)$ let Y_e be the indicator variable for the event $e \subseteq S$. Then for every $e = xy \in E(G)$,

$$\begin{aligned} \mathbb{E}Y_e &= \Pr(x \in S \text{ and } y \in S) \\ &= \Pr(x \in S) \cdot \Pr(y \in S) && \text{(by independence)} \\ &= p^2. \end{aligned}$$

Therefore, by linearity of expectation and the fact that $Y = \sum_{e \in E(G)}$ we have

$$\mathbb{E}Y = \sum_{e \in E(G)} \mathbb{E}Y_e = \frac{nd}{2}p^2.$$

By linearity of expectation,

$$\mathbb{E}(X - Y) = \mathbb{E}X - \mathbb{E}Y = pn - p^2\frac{nd}{2}.$$

Want to choose p to maximise this, so $p = \frac{1}{d}$ and $p \in [0, 1]$. Substituting gives $\mathbb{E}(X - Y) = \frac{n}{2d}$. Hence there exists a fixed set $S_0 \subseteq V(G)$ with $|S_0| - (\#\text{ edges in} S_0) \geq \frac{n}{2d}$. Delete one vertex from each edge within S_0 to give a set S^* of at least $\frac{n}{2d}$ vertices which is an independent set. $\square$

Chapter 4

Graph Colourings

A **vertex colouring** of a graph $G = (V, E)$ is a function $c : V \to S$ such that $c(u) \neq c(v)$ whenever $uv \in E$. Here S is the set of available colours, usually $S = \{1, 2, \ldots, k\}$ for some positive integer k.

A k**-colouring** of G is a colouring $c : V \to \{1, 2, \ldots, k\}$. Often we want the smallest value of k for which a k-colouring of G exists. This smallest value of k is called the **chromatic number** of G, denoted $\chi(G)$.

If $\chi(G) = k$ then G is said to be k-chromatic.
If $\chi(G) \leq k$ then G is said to be k-colourable.

The set of all vertices in G with a given colour under c is called a **colour class**. Each colour class is an independent set. k-colouring is a partition of $V(G)$ into k independent sets.

A **clique** in a graph G is a complete subgraph of G. The order of the largest clique in G is called the **clique number** of G, denoted $\omega(G)$.

Fact: $\chi(G) \geq \omega(G)$ and $\chi(G) \geq n/\alpha(G)$.

An **edge colouring** of G is a map $c : E \to S$ such that $c(e) \neq c(f)$ whenever e and f share an endvertex. If $S = \{1, 2, \ldots, k\}$ then c is a k**-edge-colouring** and G is k-edge-colourable.

Let $\chi'(G)$ be the smallest positive integer k for which G is k-edge-colourable. We call $\chi'(G)$ the **chromatic index** of G.

A **colour class** in an edge colouring is a matching of G. Hence an edge colouring displays $E(G)$ as a union of disjoint matchings.

The **line graph**, denoted $L(G)$, has vertex set $E(G)$ and $e, f \in E(G)$ form an edge of $L(G)$ if and only if e, f share an endvertex in G. Every edge-colouring of G is a vertex colour of $L(G)$ and vice-versa. So $\chi'(G) = \chi(L(G))$.

4.1 Vertex Colourings

Proposition 4.1.1. If graph G has m edges then $\chi(G) \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}$.

Proof. Fix a k-colouring of G with $k = \chi(G)$ colours. Then Ghas at least one edge between any two distinct colour classes, or we could merge them to give a colouring of G with $\leq k-1$ colours. Hence $m \geq \binom{k}{2} = \frac{1}{2}(k)(k-1)$ then solve for k to complete the proof. $\square$

Greedy Algorithm Given a graph G, fix an ordering $v_1, v_2, \ldots, v_n$ on the vertices of G and colour them one by one in this order using the first available colour (least positive integer) as you go along. Since v_i has at most $\Delta(G)$ neighbours, this produces a k-colouring of G with $k \leq \Delta(G)+1 \implies \chi(G) \leq \Delta(G)+1$.

Fact: $\chi(G) = \Delta(G) + 1$ if G is a complete graph or an odd cycle.

Theorem 4.1.2 (Brooks, 1941)**.** Let G be a connected graph. If G is neither complete nor a n odd cycle then $\chi(G) \leq \Delta(G)$. In fact we will prove the following restatement of Brooks Theorem, due to Zajac (2018):

Let $k \geq 3$ be an integer and let G be a graph with $\Delta(G) \leq k$. If G does not contain K_{k+1} as a subgraph then G is k-colourable.

We call this the "new" version of Brooks Theorem and prove that this implies Brooks Theorem.

Proof. Suppose that G is a graph which satisfies the assumptions of Brooks Theorem. That is, G be a connected graph which is not an odd cycle and which is not complete. Let $\Delta = \Delta(G)$ be the maximum degree of G. We want to show that $\chi(G) \leq \Delta$, as this is the conclusion required for Brooks Theorem.

First suppose that $\Delta \leq 2$. Then G is either a path or an even cycle, as G is connected. Hence G is bipartite and so $\chi(G) \leq 2 = \Delta$, as required.

Now suppose that $\Delta \geq 3$. We wish to apply the new version of Brooks Theorem with $k = \Delta$, so we must check that G does not contain $K_{\Delta+1}$ as a subgraph. For a contradiction, suppose that G does have a subgraph H which is isomorphic to $K_{\Delta+1}$. Then H is Δ-regular, and G has maximum degree Δ, so there is no edge from a vertex of H to a vertex of $G - V(H)$. It follows that H is a component of G. But G is connected, so the only possibility is that $G = H$. BUt this contradicts our assumption that G is not complete.

Therefore, G satisfies the assumptions of the new version of Brooks Theorem, and by applying this result we find that G is Δ-colourable. From this we conclude that $\chi(G) \leq \Delta$, as required.

In both cases, the conclusion of Brooks Theorem holds, completing the proof. $\square$

We now prove that this "new" version is true.

Proof. First an obversation, let G be a graph with maximum degree $\Delta(G) \leq k$, where $\{1, \ldots, k\}$ will be our set of colours. Suppose that G is partially coloured. Let $P = v_1 v_2 \ldots v_j$ be a path in G such that all vertices of P are uncoloured. Then we can colour vertices $v_1, v_2, \ldots, v_{j-1}$ in this order, since at the moment that we colour $v_i (1 \leq i \leq j-1)$, we know that $v_i$has an uncoloured neighbour v_{i+1} and hence aat most $\Delta - 1$ neighbours. Call this procedure `PATHCOLOUR`$(v_1, \ldots, v_{j-1}; v_j)$. Note that this procedure colours $v_1, \ldots, v_{j-1}$ but it leaves v_j uncoloured. In particular if $j = 1$ then `PATHCOLOUR`(v_1) leaves the graph unchanged.

Proof by induction on $n = |G|$, where G is a graph with $\Delta(G) \leq k$ and $k \geq 3$. If $n \leq k$ then we can k-colour G by giving each vertex a distinct colour.

> **Claim.** If G has a vertex of degree $< k$ then G is k-colourable.
>
> **Proof.** Let v be a vertex of degree $< k$ and let $G' = G - v$. By the inductive hypothesis we can k-colour G'. Fix one such colouring C. Then at most $k-1$ colours are used by C on neighbours of v, so we have an available colour which we can use to colour v.

Now we assume that G is k-regular. Let v be a vertex of G and consider $G[\{v\} \cup N(v)]$. Since G has no subgraph isomorphic to K_{k+1}, we know that v has two neighbours x, y which are not adjacent. Let $v_1 = x, v_2 = v, v_3 = y$, and extend the path $v_1v_2v_3$ to a maximal length path in $G, P = v_1v_2v_3 \ldots v_r$ which starts with $v_1v_2v_3$.

Case 1. Suppose that $r = n$. This means that all vertices of G lie on P (Hamilton Path). Let v_j be any neighbour of v_2 other than v_2 and v_3. Since G is k-regular and $k \geq 3$ we can choose such a vertex v_j. We now describe how to k-colour G.

- First colour v_1 and v_3 the same colour.
- Next apply $\texttt{PATHCOLOUR}(v_4, v_5, \ldots, v_{j-1}; v_j)$ which colours $v_4, \ldots, v_{j-1}$ and leaves v_j uncoloured.
- Next apply $\texttt{PATHCOLOUR}(v_n, v_{n-1}, \ldots, v_j; v_2)$ which will colour all remaining vertices of G except v_2.
- Finally we have an available colour for v_2 since two of its neighbour (v_1 and v_3) have the same colour. Colour v_2 with an available colour.

Case 2. Now suppose that $r < n$. Recall that all neighbours of v_r lie on P. Let v_j be the neighbour of v_r with the smallest index. Then $C = v_jv_{j+1} \ldots v_rv_j$ is a cycle in G. Let $G' = G - V(C)$. We can k-colour G' by induction. If there is no edge between G' and C then we can also k-colour $G[V(C)]$, by induction and we are done. Otherwise ($G[V(C)]$ is not a component of G): let v_ℓ be the vertex on C with largest index which has a neighbour in G', and let u be a neighbour of v_ℓ in G'. Note, v_ℓ is well defined as v_j has a neighbour in G' if $j \geq 2$. Note $\ell \leq r-1$ since all neighbours of $v_r$belong to $V(C)$. Also $v_{\ell+1}$ has no neighbours outside C, by choice of v_ℓ. We now describe how to k-colour vertices of C, giving a k-colouring of G.

- First, colour $v_{\ell+1}$ with the colour assigned to u.
- Next, apply $\texttt{PATHCOLOUR}(v_{\ell+2}, \ldots, v_r, v_j, v_{j+1}, \ldots, v_{\ell-1}; v_\ell)$ which colours all remaining vertices of G except v_ℓ.
- Finally, colour v_ℓ with an available colour which exists because v_ℓ has two neighbours with the same colour.

This completes thr proof in Case 2, by mathematical induction. □

4.2 Edge Colourings

By considering a vertex of maximum degree, we see that the chromatic index $\chi'(G)$ satisfies $\chi'(G) \geq \Delta(G)$for all graphs G.

Proposition 4.2.1 (Köning, 1916)**.** If G is bipartite then $\chi'(G) = \Delta(G)$.

Proof. We prove t his by induction on $m = |E(G)|$. If $m = 0$ then the result is trivially true. So, assume that $m \geq 1$ and that the result holds for all bipartite graphs with at most $m - 1$ edges.

Let $\Delta = \Delta(G)$, choose $xy \in E$ and let $G' = G - xy$. By induction, we can fix a Δ-edge-colouring of G'. We call edges coloured α, "α-edges", etc. In G', vertices x, y both have degree $\Delta - 1$. So there are colours $\alpha, \beta \in \{1, 2, \ldots, \Delta\}$ such that x is not incident with an α-edge, and y is not incident with a β-edge.

If $\alpha = \beta$ then we can colour the edge xy with colour α to give a Δ-edge-colouring of G, and we are done. Now assume that $\alpha \neq \beta$. Without loss of generality, we can assume that x is incident with a β-edge xu. Extend the β-edge xu to a maximal walk W whose edges are coloured α, β alternately. Since no such walk can contain a vertex colour twice, W is a path.

> **Claim.** W does not contain y.
>
> **Proof.** For a contradiction, suppose that y lies on W. Then y must be an endvertex of W, and the edge of W incident with y must be an α-edge. Hence W has even length, and so $W + xy$ is an odd cycle in the bipartite graph G. This is a contradiction.

By maximality of W, we can swap the colours α and β on all edges of W. This gives a new Δ-edge-colouring of G' such that β does not appear on any edge incident with x. Since y does not lie on W, there is still no β-edge incident with y. FInally we can colour edge xy with colour β in G, giving a Δ-edge-colouring of G. This completes the proof, by induction. □

Theorem 4.2.2 (Vizing, 1964)**.** Every graph G satisfies

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

Chapter 5

Connectivity

5.1 2-Connected Graphs

Let G be a graph. A maximal connected subgraph of G with no cut vertex is called a **block**. Every block of G is either a maximal 2-connected subgraph of G or a bridge or an isolated vertex.

By maximality, different blocks of G overlap in at most one vertex, which must be a **cut vertex** in G. Hence every edge of G lies in a unique block, and G is the union of its blocks.

Let A be the set of cut vertices in G and let $\mathcal{B}$ be the set of blocks in G. Form the bipartite graph on $A \cup \mathcal{B}$ with edge set

$$\{aB : a \in A, B \in \mathcal{B} \text{ and } a \in B\}.$$

Lemma 5.1.1. The block graph of a connected graph is a tree.

Let H be a subgraph of a graph G. An H**-path** is a path in G which intersects H only in its endvertices.

Proposition 5.1.2. A graph is 2-connected if and only if it can be constructed from a cycle by successively adding H-paths to graphs H already constructed.

Proof. Every graph constructs in this way is 2-connected. Conversely, let G be 2-connected. Then $|G| \geq 3$ and G contains a cycle. Hence G has a maximal subgraph H which is constructible using the method described in the proposition stated.

If $H = G$, then we are done. For a contradiction, suppose that $H \neq G$. Since any edge $xy \in E(G) - E(H)$ with $x, y \in H$ is an H-path, by maximality we see that every $xy \in E(G)$ with $x, y \in H$ must belong to $E(H)$. Hence, H is an induced subgraph of G.

By the fact that G is connected, there is an edge vw with $v \in G - H, w \in H$. Since G is 2-connected we know that $G - W$ is connected. Let P be the shortest path from v to H in $G - w$. Then wvP is a H-path in G, and $H \cup wvP$ is a larger constructible subgraph than H, contradicting the maximality of H. $\square$

5.2 3-Connected Graphs

Let $e = xy \in E(G)$. Define the graph $G/e = (V', E')$ where $V' = (V - \{x, y\}) \cup \{v_e\}$,

$$E' = \{uw \in E(G) : \{u, w\} \cup \{x, y\} = \emptyset\} \cup \{v_e w : xw \in E(G) \text{ or } yw \in E(G)\}.$$

We say that G/e is formed by **contradicting** the edge e in G. This creates a new vertex v_e which replaces the endvertices of e.

Lemma 5.2.1. Let G be a 3-connected graph with $|G| \geq 5$. Then G has an edge e such that G/e is 3-connected.

Proof. For a contradiction, suppose that no such edge exists. For any edge $xy \in E(G)$, the graph G/xy is not 3-connected, but $|G/xy| = |G| - 4 \geq 4$ by assumption that $|G| \geq 5$. Hence G/xy has a separating set S with $|S| \leq 2$. Since G is 3-connected, the contracted vertex v_{xy} must belong to S, and $|S| = 2$, or we would have a separating set in G with ≤ 2 vertices. So there is some $z \in V(G), z \notin \{x, y\}$ such that $S = \{v_{xy}, z\}$. Any two vertices separated in G/xy by S are also separated in G by the set $T = \{x, y, z\}$.

`FACT`: Since no proper subset of T separates G, by the 3-connectivity of G, every vertex in T has a neighbour in every component C of $G - T$.

Choose the edge xy, vertex z, and component C of $G - \{x, y, z\}$ such that $|C|$ is as small as possible. Let v be a neighbour of z in C, which we know must exists by our `FACT`. By assumption, G/zv is not 3-connected, and $|G/zv| = |G| - 1 \geq 4$. Hence (by our earlier argument) there is a vertex $w \notin \{v, z\}$ such that $\{v, w, z\}$ separates G. Also by our `FACT`, every vertex in $\{v, w, z\}$ has a neighbour in every component of $G - \{v, w, z\}$.

Since x and y are adjacent, $G - \{z, v, w\}$ has a component D such that $D \cap \{x, y\} = \emptyset$. By our `FACT` we know that v has a neighbour in D. Recall that $v \in C$ in $G - \{x, y, z\}$. Since D is connected and $(\{v\} \cup V(D)) \cap \{x, y, z\}$, it follows that $\{v\} \cup V(D) \subseteq V(C)$. Hence D is a proper subgraph of C, as $v \notin V(D)$. Therefore $|D| < |C|$, contradicting the minimality of C.

Hence G/e is 3-connected for some $e \in E(G)$. $\square$

Reversing this, we can construct all 3-connected graphs starting with K_4 and "uncontracting" edges.

Theorem 5.2.2. A graph G is 3-connected if and only if there exists a sequence $G_0, G_1, \ldots, G_r$ of graphs such that

(i) $G_0 = K_4$ and $G_r = G$,

(ii) G_{i+1} has an edge xy with degrees $d(x), d(y) \geq 3$ such that $G_i = G_{i+1}/xy$, for $i = 0, \ldots, r-1$.

5.3 Menger's Theorem

A set $S \subset V$ separating A from B in G is called an (A, B)**-separator**. This means that every (A, B)-path intersects S, and in particular $A \cap B \subseteq S$.

Let $\mathcal{P}, \mathcal{Q}$ be sets of **disjoint** (A, B)**-paths** in G. Say that $\mathcal{Q}$ **exceeds** $\mathcal{P}$ if the set of vertices in A which belong to paths in $\mathcal{P}$ is a *proper subset* of the set of vertices in A which belong to paths in $\mathcal{Q}$ and similarly for B.

If $P = x_0 x_1 \cdots x_k$ then we write P_{x_i} for the subpath $x_0 \cdots x_i$ and we write $x_i P$ for the subpath $x_i x_{i+1} \cdots x_k$.

Theorem 5.3.1 (Menger's Theorem, 1927)**.** Let $G = (V, E)$ be a graph and $A, B \subseteq V$. Then the minimum number of vertices separating A from B in G equals the maximum number of disjoint (A, B)-paths in G.

Proof. Let $k = k(G, A, B)$ be the minimum number of vertices separating A and B in G. (That is, $k = |S|$ where $S \subseteq V$ is a smallest (A, B)-separating set.) Then k is an upper bound on the maximum number of disjoint (A, B)-paths or else we could not separate A and B by deleting any set of k vertices. So it suffices to prove that a set of k disjoint (A, B)-paths exists. In fact, we will prove a stronger statement:

> If $\mathcal{P}$ is any set of $< k$ disjoint (A, B)-paths, then there is a set $\mathcal{Q}$ of $|\mathcal{P}| + 1$ disjoint (A, B)-paths in G which exceeds $\mathcal{P}$.

We will keep G and A fixed and let B vary, applying induction on the number of vertices in $\bigcup_{P \in \mathcal{P}} P$.

Base Case: If $\mathcal{P} = \emptyset$ then $|\bigcup_{P \in \mathcal{P}} P| = 0$. We can let $\mathcal{Q} = \{\mathcal{P}\}$ for any (A, B)-path P. Then $\mathcal{Q}$ exceeds $\mathcal{P}$.

Inductive Step: Let $\mathcal{P}$ be a set of $< k$ disjoint (A, B)-paths, and $B_0 \subseteq B$ be the set of end vertices of paths in $\mathcal{P}$ ("start vertices" are in A, "endvertices" are in B). Since $|B_0| \leq k - 1, B_0$ is not an (A, B)-separating set and hence there is an (A, B)-path in $G - B_0$. Call this (A, B)-path R. So R is disjoint from B_0. If R avoids all vertices in $\bigcup_{P \in \mathcal{P}}$ then $\mathcal{Q} = \mathcal{P} \cup \{R\}$ exceeds $\mathcal{P}$, as required. Otherwise, let x be the last vertex of R (traversing R from A to B) that lies on some path $P \in \mathcal{P}$. Note that $x \notin B$, by choice of R, so Px is shorter than P.

Let $B' = B \cup V(xP \cup xR)$ and let $\mathcal{P}' = (\mathcal{P} - \{P\}) \cup \{Px\}$. Then $\mathcal{P}'$ is a set of disjoint (A, B')-paths. Also $|\mathcal{P}'| = |\mathcal{P}|$, but the union of paths in $\mathcal{P}'$ is strictly smaller than $|\bigcup_{\hat{P} \in \mathcal{P}} \hat{P}|$. Also, $B \subseteq B'$, so an (A, B')-separating set is also an (A, B)-separating set. Hence $k(G, A, B') \geq k(G, A, B)$. So $|\mathcal{P}' < k(G, A, B) \leq k(G, A, B')$. Applying the inductive hypothesis to $G, A, B', \mathcal{P}'$, we conclude that there is a set $\mathcal{Q}'$ of $|\mathcal{P}| + 1$ disjoint (A, B')-paths in G which exceeds $\mathcal{P}$! Now $\mathcal{Q}'$ contains a path Q which ends in x, and a unique path Q' whose last vertex y is not among the last vertices of the paths in $\mathcal{P}$! In particular, $y \neq x$.

> *Case 1:* $y \in B$. If $y \in B$, then define $\mathcal{Q} = (\mathcal{Q}' - \{Q\}) \cup \{QxP\}$
>
> *Case 2:* $y \notin B$ and $y \in xR$. If $y \in xR$ and $y \notin B$, then $y \notin xP$, and we define $\mathcal{Q} = (\mathcal{Q}' - \{Q, Q'\}) \cup \{QxP, Q'yR\}$.
>
> *Case 3:* $y \notin B$ and $y \in xP$. If $y \in xP$ and $y \notin$ then $y \notin xR$, and we define $\mathcal{Q} = (\mathcal{Q}' - \{Q, Q'\} \cup \{QxR, Q'yP\})$.

In all cases, Q is a set of $|\mathcal{P}| + 1$ disjoint (A, B)-paths which exceeds $\mathcal{P}$, proving the inductive step. Hence there is a set of k disjoint (A, B)-paths in G, as required. □

Corollary 5.3.2. Let a, b be distinct vertices of G.

(i) If $ab \notin E$ then the minimum number of vertices (distinct from a and b) separating a from b is equal to the maximum number independent (a, b)-paths in G.

(ii) The minimum number of edges separating a from b in G equals the maximum number of edge-disjoint (a, b)-paths in G.

Proof.

(i) Apply Menger's Theorem with $A = N(A), B = N(b)$. Note that a set of k disjoint (A, B)-path corresponds to a set of independent (a, b)-paths by adding vertex a at the start and vertex b to the end.

(ii) Apply Menger's Theorem to the line graph $L(G)$ of G with $A = E(a)$, the set of edges of G incident with a, $B = E(b)$, the set of edges of G incident with b.

$\square$

Theorem 5.3.3 (Global version of Menger's Theorem)**.**

(i) A graph is k-connected if and only if it has order at least 2 and there are k independent paths between any two distinct vertices.

(ii) A graph is k-edge-connected if and only if it has at least two vertices and k edge-disjoint paths between any two distinct vertices.

Proof.

(i) Suppose that G is a graph and $|G| \geq 2$. Now suppose that G has k independent paths between any two distinct vertices $a, b \in V$. Then $|G| \geq k$, as there are at least $k - 1$ paths of length at least two between a and b. Also, G cannot be disconnected by deleting a set of $\leq k - 1$ vertices. Hence G is k-connected.

For the converse, suppose that G is k-connected and assume for a contradiction that there are distinct vertices a, b with at most $k - 1$ independent (a, b)-paths. Since G is k-connected we have $|G| \geq k + 1$. By Corollary 5.3.2, we must have $ab \in E$. Let $G' = G - ab$. Then G' has at most $k - 2$ independent (a, b)-paths. Hence by Corollary 5.3.2, there is an (a, b)-separating set $X \subseteq V$ with $|X| \leq k - 2$. Since $|G| \geq k + 1$, there is at least one more vertex $v \notin X \cup \{a, b\}$ in G. Now X separates v from at least one of a or b, say from a (since a, b lie in distinct components of $G' - X$). But then $X \cup \{b\}$ is a set of at most $k - 1$ vertices which separates v from a in G. This contradicts the fact that G is k-connected.

Hence G has at least k independent (a, b)-paths in G, completing the proof.

(ii) Follows immediately from Corollary 5.3.2.

$\square$

Chapter 6

Planar Graphs

A graph which is drawn in the plane so that no edges meet except at common endvertices is called a **plane graph**. An abstract graph which can be drawn in this way is called **planar**.

A graph is drawn in the Euclidean plane $\mathbb{R}^2$ by representing each vertex by a point and each edge by a curve between two distinct points.

6.1 Plane Graphs

An **arc** (or **polygonal arc**) is a subset of $\mathbb{R}^2$ composed of the union of finitely many straight line segments, which is homeomorphic to $[0, 1]$.

A **plane graph** is a pair (V, E) of finite sets (with elements of V called vertices and elements of E called edges) such that

(i) $V \subseteq \mathbb{R}^2$;

(ii) Every edge is an arc between two distinct vertices (no loops);

(iii) Different edges have different sets of endvertices (no repeated edges);

(iv) The interior of an edge contains no vertex and no point of any other edge.

Here the **interior** of an edge/arc e, denoted $\mathring{e}$, is the arc minus its endpoints: if e is the arc from x to y then $\mathring{e} = e - \{x, y\}$.

A **plane graph** defines a graph G in a natural way. We use the name G for abstract graph, the plane graph and the **point set**

$$V \cup \left(\bigcup_{e \in E} e \right) \subseteq \mathbb{R}^2.$$

The point set of a plane graph G is a closed set in $\mathbb{R}^2$, and $\mathbb{R}^2 - G$ is open. Two points in an open set O are equivalent if they are equal or they can be linked by an arc in O. This is an equivalence relation.

The equivalence classes of $\mathbb{R}^2 - G$ are open connected regions, call the **faces** of G. Since G is bounded (that is, it lies within some sufficiently large disc $D \subseteq R^2$), exactly **one** face of G is unbounded: it is the face that contains $\mathbb{R}^2 - D$. We call the unbounded faces the **outer face** of G. All other faces of G are called inner faces.

Let $F(G)$ be the set of faces of G. The **boundary** of a face f is called the **frontier** of f. It is the set of all points $y \in \mathbb{R}^2$ such that every neighbourhood of y meets both f and $\mathbb{R}^2 - f$.

Lemma 6.1.1. Let G be a plane graph with subgraph $H \subseteq G$ and face $f \in F(G)$.

(i) There is a face $f' \in F(H)$ which contains f (that is, $f \subseteq f'$).

(ii) If the frontier of f lies in H then $f' = f$.

Proof.

(i) Points in f are also equivalent in $\mathbb{R}^2 - H$, so they belong to an equivalence class f' of $\mathbb{R}^2 - H$. That is, $f \subseteq f'$ and $f' \in F(H)$.

(ii) We prove the contrapositive. Suppose that f is a proper subset of $f'(f \subsetneq f')$. Choose points $a \in f$ and $b \in f' - f$. Both a and b belong to f in $\mathbb{R}^2 - H$, so there is an arc between them in $\mathbb{R}^2 - H$.

But a and b are not equivalent in $\mathbb{R}^2 - G$ as $a \in f$ and $b \notin f$. So the arc must meet a point x on the frontier X of f, and $x \notin H$ as $x \in f' \subseteq \mathbb{R}^2 - H$. Therefore $X \notin H$.

□

Lemma 6.1.2. Let G be a plane graph and let e be an edge of G.

(i) If X is the frontier of a face of G then either $e \subseteq X$ or $X \cap \mathring{e} = \emptyset$.

(ii) If e lies on a cycle $C \subseteq G$ then e lies on the frontier of exactly two faces of G, and these are contained in the distinct faces of C.

(iii) If e does not lie on a cycle then e lies on the frontier of exactly one face of G.

Corollary 6.1.3. The frontier of a face of a plane graph G is always the point set of a subgraph of G.

The subgraph of G whose point set is the frontier of a face f is said to bound f and is called the **boundary** of f. Denote this subgraph by $G[f]$. A face is said to be **incident** with the vertices and edges of its boundary. By Lemma 6.1.1 (ii), every face of G is also a face of it's boundary.

Proposition 6.1.4. A plane forest has exactly one face.

Lemma 6.1.5. If a plane graph has two distinct faces with the same boundary then the graph is a cycle.

Proof. Let G be a plane graph and let f_1, f_2 be distinct fac es of G with the same boundary $H = G[f_1] = G[f_2]$. Since f_1, f_2 are also faces of H, the above proposition imples that H contains a cycle C.

We claim $H = C$. For a contradiction, suppose that H has a vertex or edge which is not in C. This additional vertex or edge of H lies in one of the faces of C and hence cannot lie on the boundary of whichever f_i is contained in the other face of C.

Thus f_1 and f_2 are exactly the two distinct faces of C. Hence $f_1 \cup C \cup f_2 = \mathbb{R}^2$. But $f_1 \cup C \cup f_2 \subseteq$

$f_1 \cup G \cup f_2 \cup \{\text{other faces of } G\} = \mathbb{R}^2$ and therefore $G = C$. □

Proposition 6.1.6. In a 2-connected plane graph, every face is bounded by a cycle.

Proof. Let f be a face in a 2-connected plane graph G. We proceed by induction using Proposition 5.1.2. If G is a cycle then the result is true. Now assume that G is not a cycle. Then by Proposition 5.1.2, there is a 2-connected plane subgraph H of G and a plane H-path P such that $G = H \cup P$. By the inductive hypothesis, every face of H is bounded by a cycle.

The interior of P lies in the face f' of H, and f' is bounded by a cycle C. If f is a face of H then we are done. Otherwise, the frontier of f intersects $P - H$, so $f \subseteq f$! Therefore f is a face of $C \cup P$ and hence f is bounded by a cycle, by observation. □

Theorem 6.1.7 (Euler's Formula, 1752)**.** Let G be a connected plane graph with n vertices, m edges and ℓ faces. Then

$$n - m + \ell = 2.$$

Proof. Fix n and apply induction on m. For $m \leq n - 1$ then, as G is connected we must have $m = n - 1$ and G is a tree. Then the result follows Proposition 6.1.4.

Now suppose that $m \geq n$. Then G has an edge e which belongs to a cycle. Let $G' = G - e$ which is a connected plane graph. By Lemma 6.1.2 (ii), e lies on the boundary of exactly two distinct faces f_1 and f_2 of G. There is a face f_e of G' which contains $\mathring{e}$, since all points of $\mathring{e}$ are equivalent in $\mathbb{R}^2 - G'$.

> **Claim.** We claim the following result, $F(G) - \{f_1, f_2\} = F(G') - \{f_e\}$.
>
> **Proof.** First let $f \in F(G) - \{f_1, f_2\}$. By Lemma 6.1.2 $G[f] \subseteq G - \mathring{e} = G'$ and hence $f \in F(G')$ by Lemma 6.1.2 (ii). Also $f \neq f_e$ as $\mathring{e} \subseteq f_e$ but $\mathring{e} \cap f = \emptyset$. So $f \in F(G') - \{f_e\}$ proving "$\subseteq''$ part of the claim.
>
> Next let $f' \in F(G') - \{f_e\}$. Then $f' \neq f_1, f_2$ (as open sets): for any $x \in \mathring{e}$, any open set around x intersects both f_1 and f_2. But there are open sets containing x which are disjoint from f', as $\mathring{e} \in f_e, f_e$ open, $f'$and f_e are disjoint.
>
> Also $f' \cap \mathring{e} = \emptyset$ as $\mathring{e} \subseteq f_e$ and f_e is disjoint from f'. Hence every pair of points in f' belong to $\mathbb{R}^2 - G$, and they are equivalent in $\mathbb{R}^2 - G$. Thus, G has a face f which contains f'. By Lemma 6.1.2 (i), f is contained in a face f'' of G'. Hence $f' \subseteq f \subseteq f''$. Therefore $f' = f''$ (faces of G' which overlap must be equal) and $f' = f \in F(G)$. So $f' \in F(G) - \{f_1, f_2\}$, completing the proof of the claim.

Then G' has exactly one less face and exactly one less edge than G. So the result for G follows by the formula for G', which holds by induction: $n - (m - 1) + (\ell - 1) = 2$. □

Corollary 6.1.8. The graphs $K_5, K_{3,3}$ are not planar.

Proof. For a contradiction, suppose that K_5 is planar. Any planar embedding of K_5 must have ℓ faces where $5 - 10 + \ell = 2$ by Euler's Formula (note that K_5 is connected). Rearranging gives $\ell = 7$. But K_5 is 2-connected and hence every face is bounded by a cycle (of length at least 3), by Proposition 6.1.6. Also, every edge of G lies on the boundary of exactly two faces, as K_5 has no bridges and using Lemma 6.1.2 (ii). We will double count elements of the set

$S = \{(e, f) : e \in E(K_5), f \in F(K_5), e \subseteq G[f]\}$ (incident edge-face pairs). We get $3\ell \leq |S| = 2 \times 10$. Hence $\ell \leq 20/3 < 7$, contradiction. So K_5 is not planar.

Similarly, as $K_{3,3}$ is connected, any planar embedding of $K_{3,3}$ would have ℓ faces, where $6 - 9 + \ell = 2$ by Eulers formula. So $\ell = 5$. Also, every face is bounded by a cycle of length at least 4, as $K_{3,3}$ is 2-connected and bipartite (using Proposition 6.1.6) and every edge is incident with exactly 2 faces, as above.

Double counting incident (edge, face) pairs gives $4\ell \leq 2 \times 9$, so $\ell \leq \frac{9}{2} < 5$. This contradiction shows that $K_{3,3}$ is not planar. $\square$

A **subdivision** of a graph G is obtained by replacing each edge of G by an independent path between its endvertices.

Kuratowski's Theorem (1930) says that a graph G is planar if and only if no subgraph of G is a subdivision of K_5 or $K_{3,3}$.

A plane graph G is **maximally plane** (or just **maximal**) if we cannot add a new edge to form a new plane graph G' with $V(G') = V(G)$ such that $E(G')$ strictly contains $E(G)$.

Call G a **plane triangulation** if every face of G (including the outer face) is bounded by a triangle.

Proposition 6.1.9. A plane graph of order at least 3 is maximally plane if and only if it is a plane triangulation.

Proof. Let G be a plane graph with $|G| \geq 3$. First suppose that G is a plane triangulation. Then G is maximally plane, any additional edge e would have its interior completely within a face f of G, and the endvertices of e would lie on the boundary of f. But all these edges are already present as $G[f] \cong K_3$ which is complete, and repeated edges are not allowed.

For the converse, suppose that G is maximally plane. Let $f \in F(G)$ be a face and let $H = G[f]$.

> **Claim 1.** The induced subgraph $G[H]$ is complete. If not, say vertices x, y of $G[H]$ are not adjacent in G. But we can add an edge through the face f between x and y, giving a plane graph with more edges than G. This contradicts maximality of G.

Hence $G[H] = K_r$ for some r. Then $r \leq 4$ as K_5 is not planar. Note: H might not be complete (that is, it might not be a induced subgraph of G).

> **Claim 2.** H contains a cycle. If not, then H is a forest. Either $r \geq 3$, and $H \subsetneq K_r = G[H] \subseteq G$ or $r = 2$ and $|G| \geq 3$ while $|H| = r = 2$. In either case, $H \neq G$. But by Proposition 6.1.4, H has exactly one face f and hence $f \cup H = \mathbb{R}^2$. Therefore $G = H$, contradiction.
>
> **Claim 3.** $r = 3$, and hence $H = K_3$. We know that $r \leq 4$ and by Claim 2 we have $r \geq 3$. So it is enough to rule out $r = 4$. For a contradiction, suppose that $r = 4$ and let $V(H) = \{v_1, v_2, v_3, v_4\}$. Without loss of generality let $C = v_1v_2v_3v_4v_1$ be a cycle in H (note, H contains a cycle by Claim 2: how do we know it is a 4-cycle?).
>
> Since $C \subseteq G$, by Lemma 6.1.1 (i), the face f is contained within a face f_c of C. let f_c' be the other face of C.

FACT. Edges v_1v_3 and v_2v_4 lie in different faces of C. If not, we can add a new vertex u in the face of C which does not contain these edges, and add edges uv_1, uv_2, uv_3, uv_4 giving a plane embedding of K_5, contradiction.

But, since v_1 and v_3 lie on $G[f]$, they can be linked by an arc whose interior lies in f_c which avoids G. Hence the plane edge v_2v_4 of $G[H]$ goes through f_c', not f_c.

Similarly, since v_2 and v_4 lie on $G[f]$, they can be linked by an arc whose interior lies in f_C and which avoids G. Hence the plane edge v_1v_3 of $G[H]$ runs through f_c', not f_c. This contradicts our *FACT*. Hence $r \neq 4$ so $r = 3$ and Claim 3 holds.

So every face of G is bounded by a 3-cycle. □

Corollary 6.1.10. A plane graph with $n \geq 3$ vertices has at most $3n-6$ edges. Every plane triangulation has $3n-6$ edges.

Proof. By Proposition 6.1.9 it suffices to prove the second statement. Let G be a plane triangulation. If G was disconnected then at least one face of G must have a disconnected boundary. But all faces of G are bounded by 3-cycles, so G is connected.

Next, every edge lies on the boundary of some face, which is a 3-cycle. So every edge of G belongs to a cycle and hence lies on the boundary of exactly two faces. Furthermore, every face boundary has exactly 3 edges. Let $n = |G|, m = |E(G)|$ and $\ell = |F(G)|$. Double-counting incident (edge - face) pairs gives $3\ell = 2m$. Thus $\ell = \frac{2m}{3}$. Substituting this into Euler's formula, as G is connected gives $n - m + \frac{2m}{3} = 2$. Hence $m = 3(n-2) = 3n-6$ as required. □

6.2 Colouring Maps

Theorem 6.2.1 (Four Colour Theorem)**.** Every planar graph is 4-colourable. (That is, there exists a proper 4-colouring of the vertices of any planar graph.)

Proposition 6.2.2. Every planar graph is 5-colourable.

Proof. Let G be a plane graph with n vertices and m edges. If $n \leq 5$ then 5-colouring is easy. So we assume that $n \geq 6$. Assume by induction that every plane graph with at most $n-1$ vertices can be 5-coloured. By Corollary 6.1.10, the average degree of G satisfies

$$\bar{d}(G) = \frac{2m}{n} \leq \frac{2(3n-6)}{n} < 6.$$

Hence G has at least one vertex of degree ≤ 5. Let v be a vertex of G with degree ≤ 5. If $d_G(v) \leq 4$ then by induction we can 5-colour $G - v$ and extend this colouring to a 5-colouring of G by choosing a colour for v which does not appear on $N(v)$. So we can assume that $d_G(v) = 5$.

Note, some pair of distinct neighbours $u, w \in N(v)$ must not be adjacent, as K_5 is not planar. Contract the edge uv and then contract the edge vw, preserving planarity. This gives a plane graph $\hat{G}$ with $n-2$ vertices. By induction, $\hat{G}$ is 5-colourable. Let $\hat{c}$ be a 5-colouring of $\hat{G}$. We define a 5-colouring c of $G - v$ by

$$c(x) = \begin{cases} \hat{c}(x) & \text{if } x \notin \{u, w\}, \\ \hat{c}(uwv) & \text{if } x \in \{u, w\}. \end{cases}$$

Now at most 4 colours appear on $N(v)$ under c, so we can colour v with a missing colour to give a 5-colouring of G. This completes the proof, by induction. □

Theorem 6.2.3. Every planar graph which does not contain a triangle is 3-colourable.

Chapter 7

Ramsey Theory

For integers $s, t \geq 2$, let $R(s,t)$ be the least positive integer n such that any red-blue colouring of K_n has either a red copy of K_s or a blue copy of K_t.

The numbers $R(s,t)$ are called **Ramsey numbers**. Write $R(s)$ instead of $R(s,s)$ (this is the *diagonal* case).

7.1 Upper Bounds

Theorem 7.1.1 (Erdős & Szekeres, 1935)**.** For all integers $s, t \geq 2$, the Ramsey number $R(s,t)$ is finite. If $s > 2$ and $t > 2$ then

$$R(s,t) \leq R(s-1,t) + R(s,t-1) \tag{7.1}$$

and hence

$$R(s,t) \leq \binom{s+t-2}{s-1}. \tag{7.2}$$

Proof. We know that $R(s,2) = R(2,s)$ for all $s \geq 2$. Assume by induction that $R(s-1,t)$ and $R(s,t-1)$ are both finite. Let $n = R(s-1,t) + R(s,t-1)$. Consider any red-blue colouring of the edges of K_n. Let x be a vertex of K_n. Then x has degree $n-1 = R(s-1,t)+R(s,t-1)-1$.

By the pingeonhole principle, either

- there are at least $n_1 = R(s-1,t)$ red edges incident with x
- there are at least $n_2 = R(s,t-1)$ blue edges incident with x.

Without loss of generality, assume the former. Consider the subgraph K_{n_1} spanned by a set of n_1 vertices which are joined to x by red edges.

- If K_{n_1} contains a blue copy of K_t then we are done.
- Otherwise, K_{n_1} contains a red copy of K_{s-1}, since $n_1 = R(s-1,t)$.

Together with x this gives a red copy of K_s, completing the proof of (7.1). Then we use induction on $s+t$ to prove (7.2). $\square$

7.2 Lower Bounds

Theorem 7.2.1 (Erdős, 1947)**.** If $\binom{n}{s}2^{1-\binom{s}{2}} < 1$ then $R(s) > n$. Hence $R(s) > \lfloor 2^{s/2} \rfloor$ for $s \geq 3$.

Proof. Take a random red-blue colouring of the edges of K_n, where each edge is coloured independently red or blue, each with proability 1/2. For any fixed set R of s vertices, let A_R b e the event that the induced subgraph $K_n[R]$ is monochromatic. Then, using independence,

$$Pr(A_R) = (\frac{1}{2})^{\binom{s}{2}} + (\frac{1}{2})^{\binom{s}{2}} = \frac{2}{2^{\binom{s}{2}}},$$

since there are $\binom{s}{2}$ edges in $K_n[R]$ and the events "all red" and "all blue" on $K_n[R]$ are disjoint. Let X be the number of monochromatic copies of K_s in the random red-blue colouring. Then $X = \sum_{R \leq [n], |R|=s} A_R$, where $[n] = \{1, 2, \ldots, n\} = V(K_n)$ and $\mathbb{I}(A_R)$ is the indicator variable for the event A_R.

Hence, by linearity of expectation,

$$\mathbb{E}X = \sum_{R \subseteq [n], |R|=s} \mathbb{E}(\mathbb{I}(A_R)) = \sum_{R \subseteq [n], |R|=s} Pr(A_s) = \binom{n}{s} \frac{2}{2^{\binom{s}{2}}}.$$

By the assumption we have $\mathbb{E}X = \binom{n}{s}2^{1-\binom{s}{2}} < 1$. Therefore there is a fixed red-blue colouring of the edges of K_n with no monochromatic copy of K_s. Hence $R(s) > n$. This proves the first statement.

Now suppose that $s \geq 3$ and $n = \lfloor 2^{s/2} \rfloor = \lfloor \sqrt{2}^3 \rfloor$. Then

$$\binom{n}{s} 2^{1-\binom{s}{2}} \leq \frac{2^{1+s/2-s^2/2} n^s}{s!} \leq \frac{2^{1+s/2-s^2/2} 2^{s^2/2}}{s!} \leq \frac{2^{1+s/2}}{s!} < 1$$

(as $n^s \leq 2^{s^2/2}$ by choice of n) and this holds for $s \geq 3$. □

7.3 Graph Ramsey Theory

Let H_1, H_2 be fixed graphs with no isolated vertices, and let $R(H_1, H_2)$ be the least positive integer n such that in every red-blue colouring of the edges of K_n, then there is either a red copy of H_1 or a blue copy of H_2.

Write $R(H) = R(H, H)$ and note that $R(K_s, K_t) = R(s, t)$, the Ramsey numbers.

Theorem 7.3.1. Write ℓK_2 for a set of ℓ independent edges. For $\ell \geq 1$ and $p \geq 2$,

$$R(\ell K_2, K_p) = 2\ell + p - 2.$$

Proof. First consider $K_{2\ell+p-3}$. We colour the edges of $K_{2\ell+p-3}$ so that there is a red $K_{2\ell-1}$ and all other edges are blue. Then we cannot find ℓ independent red edges as this would require 2ℓ vertices which are incident with red edges, but we only have $2\ell - 1$. That is, there is no red copy of ℓK_2.

Next, the largest blue complete subgraph $2\ell + p - 3 - (2\ell - 2) = p - 1$ vertices, noting that we can keep exactly one vertex which is incident with a red edge. Hence there is no blue K_p, so

$R(\ell K_2, K_p) \geq 2\ell + p - 2$.

Next, take any red-blue colouring of the edges of K_n, where $n = 2\ell + p - 2$. If we can find a red ℓK_2 then we are done. So suppose that there are at most s independent red edges, where $s \leq \ell - 1$. Then the set of $n - 2s \geq 2\ell + p - 2 - 2(\ell - 1) = p$ vertices which are not incident with these red edges must span a blue complete subgraph: if not, we can find a larger red matching, contradicting the definition of s.

Hence $R(\ell K_2, K_p) \leq 2\ell + p - 2$, so $R(\ell K_2, K_p) = 2\ell + p - 2$ as claimed. □

For a graph G, let $c(G)$ be the number of vertices in the largest component of G, and let $u(G)$ be the **chromatic surplus** of G, which is the maximum size of the smallest colour class of G, taken over all $\chi(G)$-colourings of G. Note that $u(C_{2k}) = k$ and $u(C_{2k+1}) = 1$.

Theorem 7.3.2. For all graphs H_1, H_2 with no isolated vertices, we have

$$R(H_1, H_2) \geq (\chi(H_1) - 1)(c(H_2) - 1) + u(H_1).$$

In particular, if H_2 is connected then

$$R(H_1, H_2) \geq (\chi(H_1) - 1)(|H_2| - 1) + 1.$$

Proof. Let $k = \chi(H_1), u = u(H_1)$ and $c = c(H_2)$. Then

$$R(H_1, H_2) \geq R(H_1, K_2) = |H_1| \geq \chi(H_1)u(H_1) = ku.$$

Hence if $c \leq u$ then

$$R(H_1, H_2) \geq ku \geq (k - 1)c + u \geq (k - 1)(c - 1) + u,$$

as required. Now suppose that $c > u$ and let $n = (k - 1)(c - 1) + u - 1$. Partition the vertices of K_n into parts $A_1, A_2, \ldots, A_{k-1}, B$ where $|A_j| = c - 1$ for $j = 1, \ldots, k - 1$and $|B| = u - 1$.

Let $K_n[A_i]$ be a blue K_{c-1} for all $i = 1, \ldots, k - 1$ and let $K_n[B]$ be a blue K_{u-1}. Colour all remaining edges red.

The largest component in H_2 has order c, but the largest component of the blue subgraph of K_n has order $c - 1$, since $c > u$. Hence there is no blue copy of H_2.

Next, if there is a red copy of H_1 then the k-partite sets $A_1, \ldots, A_{k-1}, B$ induce a k-colouring (proper vertex colouring) of H_1. Furthermore, $k = \chi(H_1)$ and the smallest colour class in this vertex colouring contains $u - 1$ vertices, as $u < c$. But this contradicts the definition of $u = u(H_1)$. Hence there is no red H_1 either, so $R(H_1, H_2) > n$. So $R(H_1, H_2) \geq n + 1 = (k - 1)(c - 1) + u$.

The second statement follows as $u(H_1) \geq 1$ for all graphs H_1 with no isolated vertices, and $c(H_2) = |H_2|$ if H_2 is connected. □

Chapter 8

Random Graphs

We define the uniform model of random graphs in a similar manner to what was done in the Probabilistic Method chapter.

For some probability $p \in [0,1]$, each pair of distinct vertices $\{i,j\}$ let $\Pr(ij \in E) = p$ independently for each $i \neq j$. This gives a random graph model called the binomial model denoted $G(n,p)$. Note $G(n,\frac{1}{2})$ is the uniform model.

We write $G \in G(n,p)$ to mean that G is a random graph chosen from the binomial model. For a fixed $G_0 \in \Omega_n$, the probability that the random graph G equals G_0 is

$$\Pr(G = G_0) = p^{|E(G_0)|}(1-p)^{\binom{n}{2}-|E(G_0)|}$$

which depends only on $|E(G_0)|$ using independence.

For $G \in G(n,p)$, the expected number of edges of G is $p\binom{n}{2}$.

For fixed $p \in [0,1]$, we have a **sequence** of probability spaces,

$$(G(n,p))_{n\in\mathbb{Z}^+}.$$

We can also let p be a function of n, where $p(n) \in [0,1]$ for all $n \in \mathbb{Z}^+$. This gives the sequence of probability spaces

$$(G(n,p(n)))_{n\in\mathbb{Z}^+}.$$

Recall that $\omega(G)$ is the clique number of G, and $\alpha(G)$ is the independence number of G.

Lemma 8.0.1. Let $G \in G(n,p)$. Then for any integer $k \geq 2$,

$$\Pr(\omega(G) \geq k) \leq \binom{n}{k} p^{\binom{k}{2}},$$
$$\Pr(\alpha(G) \geq k) \leq \binom{n}{k} (1-p)^{\binom{k}{2}}.$$

Proof. Let $G \in G(n,p)$. If G has a clique of order $\geq k$ then G has a clique of order k. For a set S of k vertices, let A_s be the event "$G[S]$ is a clique". Then $\Pr[A_s] = p^{\binom{k}{2}}$, using independence, since

$\binom{k}{2}$ edges within in S must be present. Hence

$$\Pr(\omega(G) \geq k) = \Pr\left(\bigcup_{|S|=k} A_s\right) \leq \sum_{|S|=k} \Pr(A_s) = \binom{n}{k} p^{\binom{k}{2}}$$

(using the union bound), the result as required. □

For $a \in \mathbb{R}$ and $r \in \mathbb{N}$, let

$$(a)_r = a(a-1)\cdots(a-r+1)$$

denote the **falling factorials**.

Lemma 8.0.2. Let $k \geq 3$ be an integer. The expected number of k-cycle in $G \in G(n,p)$ is

$$\frac{(n)_k}{2k} p^k.$$

Proof. Let X be the number of k-cycles in $G \in G(n,p)$. Given a sequence $(v_1, v_2, \ldots, v_k)$ of k distinct vertices, the probability that this sequence describes a walk around a k-cycle is

$$\Pr(v_1 v_2, v_2, v_3, \ldots, v_{k-1} v_k, v_k v_1 \in E(G)) = p^k, \text{ using independence}$$

There are $(n)_k$ ways to choose this sequence of k distinct vertices. Each cycle in G corresponds to exactly $2k$ such sequences corresponding to the choice of start vertex and direction.

Hence, by linearity of expectation, $\mathbb{E}X = \frac{(n)_k}{2k} p^k$, as claimed. □

If $\Pr(G \in \mathcal{P}) \to 1$ as $n \to \infty$, for some graph property $\mathcal{P}$, we say that $G \in P$ holds **asymptotically almost surely**, abbreviated to "a.a.s.".

Proposition 8.0.3. For fixed $p \in (0,1)$ and every graph H, a.a.s. $G \in G(n,p)$ has an induced subgraph which is isomorphic to H.

Proof. Let $k = |V(H)|$. Suppose that $n \geq k$ and let $\mathcal{U} \subseteq \{1, 2, \ldots, n\}$ be a fixed set of k vertices. The probability that $G[\mathcal{U}] \cong H$ is some fixed constant $r \in (0,1)$ which depends only on H and P but not on n.

Now we can find $\lfloor \frac{n}{k} \rfloor$ disjoint sets of k vertices, $\mathcal{U}_1, \ldots, \mathcal{U}_{\lfloor \frac{n}{k} \rfloor}$, within $V(G) = [n]$. The probability that none of $\mathcal{U}_1, \ldots, \mathcal{U}_{\lfloor \frac{n}{k} \rfloor}$ induces a copy of H is $(1-r)^{\lfloor \frac{n}{k} \rfloor}$, since the $\mathcal{U}_j$ are disjoint and hence the events $G[\mathcal{U}_j] \ncong H$ are independent of each other (for $j = 1, \ldots, \lfloor \frac{n}{k} \rfloor$).

But $(1-r)^{\lfloor \frac{n}{k} \rfloor} \to 0$ as $n \to \infty$, since $1 - r \in (0,1)$ and $\lfloor \frac{n}{k} \rfloor \to \infty$ as $n \to \infty$. Hence a.a.s., one of $\mathcal{U}_1, \ldots, \mathcal{U}_{\lfloor \frac{n}{k} \rfloor}$ induces a copy of H. □

Given $i, j \in \mathbb{N}$, let $\mathcal{P}_{ij}$ be the property that given any disjoint vertex set U, W with $|U| \leq i$ and $|W| \leq j$, the graph contains a vertex $v \in U \cup W$ that is adjacent to all vertices in U but none in W.

Lemma 8.0.4. For every constant $p \in (0,1)$ and all $i, j \in \mathbb{N}$, let $G \in G(n,p)$. Then a.a.s. $G \in \mathcal{P}_{ij}$.

Proof. Assume that $n \geq i + j + 1$. For fixed disjoint set $U, W \subseteq [n]$ and $v \in [n] - (U \cup W)$, the probability that v is adjacent to all vertices of U and to no vertices of W is $p^{|U|}(1-p)^{|W|} \geq p^i(1-p)^j$ using independence. To simplify notation we write $q = 1 - p$. Hence the probability that no such v exists for the given sets U and W is

$$(1 - p^{|U|}q^{|W|})^{n-|U|-|W|}$$

since these events are independent for distinct $v \in U \cup W$ (no edge/non-edge choices are considered in more than one of these events). Now

$$\begin{aligned}(1 - p^{|U|}q^{|W|})^{n-|U|-|W|} &\leq (1 - p^j q^j)^{n-|U|-|W|} \\ &\leq (1 - p^i q^j)^{n-i-j}.\end{aligned}$$

There are at most n^{i+j+2} pairs of disjoint sets U, W with $|U| \leq i$ and $|W| \leq j$, as

$$\sum_{s=0}^{i} \binom{n}{s} \leq \sum_{s=0}^{i} n^s = \frac{n^{i+1} - 1}{n - 1} \leq n^{i+1},$$

and similarly for W. Hence the probability that some U, W has no suitable v is at most

$$n^{i+j+2}(1 - p^i q^j)^{n-i-j} \to 0 \text{ as } n \to \infty$$

since $1 - p^i q^j \in (0, 1)$. Hence a.a.s. $\mathcal{P}_{ij}$ holds, as required. □

Corollary 8.0.5. For every constant $p \in (0, 1)$ and all $k \in \mathbb{N}$, a.a.s. $G \in G(n, p)$ is k-connected.

Proof. By Lemma 8.0.4, it is enough to show that every graph in $\mathcal{P}_{2,k-1}$ is k-connected when n is sufficiently large. Assume that $n \geq k + 2$ (one more than is needed for k-connectivity). Let W be any set of at most $k - 1$ vertices. We want to prove that $G - W$ is still connected. So let x, y be distinct vertices in $[n] - W$ and define $U = \{x, y\}$. By definition of $\mathcal{P}_{2,k-1}$ there is a vertex v in $[n] - (U \cup W)$ such that v is adjacent to both x and y. Hence xvy is a path between x and y in $G - W$, proving that $G - W$ is connected. □

Proposition 8.0.6. For every constant $p \in (0, 1)$ and all $\epsilon > 0$, a.a.s. $G \in G(n, p)$ satisfies

$$\chi(G) \geq \frac{\ln(1/q)n}{(2 + \epsilon)\ln n}$$

where $q = 1 - p$.

Proof. Let a be any fixed integer, $2 \leq a \leq n$. Then by Lemma 8.0.1

$$\begin{aligned}\Pr(\alpha(G) \geq a) &\leq \binom{n}{a}(1-p)^{\binom{a}{2}} \\ &\leq n^a(1-p)^{\binom{a}{2}} \\ &= q^{a\frac{\ln n}{\ln q} + \frac{a(a-1)}{2}} \\ &= q^{\frac{a}{2}(\frac{2\ln n}{\ln q} + a - 1)} \\ &= q^{\frac{a}{2}(a - 1 - \frac{2\ln n}{\ln(1/q)})}\end{aligned}$$

Set $a = \lceil \frac{(2+\epsilon)\ln n}{\ln(1/q)} \rceil$. Then

$$\lim_{n\to\infty} \frac{a}{2}\left(a - 1 - \frac{2\ln n}{\ln(1/q)}\right) \geq \lim_{n\to\infty} \frac{(2+\epsilon)\ln n}{2\ln(1/q)}\left(\frac{\epsilon \ln n}{\ln(1/q)} - 1\right) = \infty$$

Hence $\Pr(\alpha(G) \geq a) \to \infty$ as $n \to \infty$, since $q \in (0,1)$.

This shows that a.a.s. $G \in G(n,p)$ has no independent set of order $\lceil \frac{(2+\epsilon)\ln n}{\ln(1/q)} \rceil$, and hence a.a.s. $\alpha(G) < \frac{(2+\epsilon)\ln n}{\ln(1/q)}$. Therefore a.a.s. for $G \in G(n,p)$,

$$\chi(G) \geq \frac{n}{\alpha(G)} > \frac{\ln(1/q)n}{(2+\epsilon)\ln n}.$$

□

Lemma 8.0.7. Let k be a positive integer and let $p = p(n)$ be a function of n such that $p(n) \in (0,1)$ and

$$p(n) \geq \frac{6k\ln n}{n}$$

for sufficiently large n. Then for $G \in G(n,p)$, a.a.s.

$$\alpha(G) < \frac{n}{2k}.$$

Proof. Let $n, r \in \mathbb{Z}, n \geq r \geq 2$. By Lemma 8.0.1 for $G \in G(n,p)$ we have

$$\begin{aligned}\Pr(\alpha(G) \geq r) &\leq \binom{n}{r}(1-p)^{\binom{r}{2}} \\ &\leq n^r(1-p)^{\binom{r}{2}} \\ &= (n(1-p)^{\frac{r-1}{2}})^r \\ &\leq (ne^{\frac{-p(r-1)}{2}})^r\end{aligned}$$

since $1 - p \leq e^{-p}$. If $p \geq \frac{6k\ln n}{n}$ and $r \geq \frac{n}{2k}$ then

$$\begin{aligned}ne^{-\frac{p(r-1)}{2}} &= ne^{-\frac{pr}{2}+\frac{p}{2}} \\ &\leq ne^{-\frac{3}{2}\ln n+\frac{p}{2}} \\ &\leq ne^{-\frac{3}{2}\ln n+\frac{1}{2}} \\ &= n \cdot n^{-\frac{3}{2}} && \text{(since } p \leq 1) \\ &= \sqrt{\frac{e}{n}} \to 0 \text{ as } n \to \infty.\end{aligned}$$

Since $p = p(n) \geq \frac{6k\ln n}{n}$ for sufficiently large n, we take $r = \lceil \frac{n}{2k} \rceil$ to conclude that

$$\lim_{n\to\infty} \Pr\Big(\alpha(G) \geq \frac{n}{2k}\Big) = \lim_{n\to\infty} \Pr(\alpha(G) \geq r) = 0.$$

□

Recall that the **girth** of a graph is the length of its smallest cycle and Markov's inequality: if $X : \Omega \to \mathbb{N}$ is a nonnegative integer-valued random variable on a set Ω, and $k > 0$, then

$$\Pr(X \geq k) \leq \frac{\mathbb{E}X}{k}.$$

Theorem 8.0.8 (Erdős, 1959)**.** For every integer $k \geq 3$ there exists a graph H with girth $g(H) > k$ and chromatic number $\chi(H) > k$.

Proof. Fix ϵ with $0 < \epsilon < \frac{1}{k}$ and let $p = p(n) = n^{\epsilon - 1}$. Let $X(G)$ be the number of cycles in $G \in G(n,p)$ with length $\leq k$. By Lemma 8.0.2 and linearity of expectation,

$$\mathbb{E}X = \sum_{i=3}^{k} \frac{(n)_i}{2i} p^i \leq \frac{1}{2} \sum_{i=3}^{k} (np)^i \leq \frac{k-2}{2}(np)^k,$$

as $np = n^\epsilon > 1$. Using Markov's inequality

$$\Pr\Big(X \geq \frac{n}{2}\Big) \leq \frac{\mathbb{E}X}{\frac{n}{2}} \leq (k-2)n^{k-1}p^k = (k-2)n^{k-1}n^{(\epsilon-1)k} = (k-2)n^{k\epsilon - 1}.$$

Note $k\epsilon < 1$ by choice of ϵ. Hence

$$\lim_{n\to\infty} \Pr\Big(X \geq \frac{n}{2}\Big) = 0.$$

That is, a.a.s. $X(G) < \frac{n}{2}$. Note also that, $p = n^\epsilon - 1 \geq \frac{6k \ln n}{n}$ for large enough n, as k is constant. Hence by Lemma 8.0.7, we can choose n large enough so that

$$\Pr\Big(X \geq \frac{n}{2}\Big) < \frac{1}{2} \text{ and } \Pr\Big(\alpha(G) \geq \frac{n}{2k}\Big) < \frac{1}{2}.$$

This shows that for some fixed graph G_0 on n vertices we have $\alpha(G) < \frac{n}{2k}$ and G_0 has fewer than $\frac{n}{2}$ cycles of length $\leq k$. Construct H from G_0 by deleting one vertex from every cycle in G_0 of length $\leq k$. Then $|H| \geq \frac{n}{2}$ and by construction, $g(H) > k$.

Also $\alpha(H) \leq \alpha(G_0) < \frac{n}{2k}$ since every independent set in H is also an independent set in G_0. Therefore

$$\chi(H) \geq \frac{|H|}{\alpha(H)} \geq \frac{\frac{n}{2}}{\alpha(H)} > \frac{\frac{n}{2}}{\frac{n}{(2k)}} = k,$$

completing the proof. $\square$