Higher Theory and Applications of Differential Equations
MATH2221 UNSW

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2024T2

Contents

Chapter 1

Linear ODEs

1.1 Introduction

Recall that a first-order ordinary differential equation (ODE) has, in its most general realisation, the form

$$y'(t) = f(t, y(t)).$$

A special case is the equation

$$a(t)y'(t) + b(t)y(t) = f(t),$$

with $a(t) \neq 0$ on some interval $I \in \mathbb{R}$. This special first-order ODE is called a **linear first-order ODE**. Another special case is

$$y'(t) = f(t)g(y),$$

which is known as a **separable first-order ODE**.

For a separable equation the solution is found (at least, implicitly by) writing:

$$\int \frac{1}{g(y)}\, dy = \int f(t)\, dt.$$

Solving Seperable ODEs Consider $y' = t^2 y, y(0) = 3$. This is seperable with $f(t) = t^2$ and $g(y) = y$. Then

$$\int \frac{1}{y}\, dy = \int t^2\, dt$$

so that

$$\ln|y(t)| = \frac{1}{3}t^3 + C.$$

Now apply e^t to both sides to obtain

$$|y(t)| = e^{\frac{1}{3}t^3 + C} = e^C e^{\frac{1}{3}t^3}.$$

Thus, a general solution of the equation is

$$y(t) = Ae^{\frac{1}{3}t^3}.$$

Since $y(0) = 3$, we see that the unique solution is $y(t) = 3e^{\frac{1}{3}t^3}$.

In the case of a linear first-order equation, i.e. $y' + a(t)y = f(t)$, a useful solution method is the integrating factor technique. The idea is to find a function μ so that when we multiply both sides of the equation with μ we find that

$$[\mu y](t)' = \mu(t)(y' + a(t)y) = \mu(t)f(t),$$

for if this happens, then the general solution of the ODE should be

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)f(t)\,dt + \frac{C}{\mu(t)}.$$

Solving Linear First-Order ODE Solve $y' - 2ty = 3t$. We pick

$$\mu(t) = e^{\int -2t\,dt} = e^{-t^2}.$$

Then

$$\begin{aligned}(e^{-t^2}y)' &= 3te^{-t^2}\\ e^{-t^2}y &= \int 3te^{-t^2}\,dt = -\frac{3}{2}e^{-t^2} + C\\ y(t) &= -\frac{3}{2} + Ce^{t^2}.\end{aligned}$$

1.2 Linear Differential Operators

In linear algebra, you have seen the compact notation $A\mathbf{x} = \mathbf{b}$ for system of linear equations. A similar notation when dealing with a linear ordinary differential equations is

$$Lu = f.$$

Here, L is an operator (or transformation) that acts on a function u to create a new function Lu. Given coefficients $a_0(x), a_1(x), \dots, a_m(x)$ we define the **linear differential operator** L of **order** m,

$$\begin{aligned}Lu(x) &= \sum_{j=0}^{m} a_j(x)D^j u(x)\\ &= a_m D^m u + a_{m-1}D^{m-1}u + \cdots + a_o u,\end{aligned}$$

where $D^j u = d^j u/dx^j$ (with $D^0 u = u$).

We refer to a_m as the **leading coefficient** of L and assume that each $a_j(x)$ is a smooth function of x.

The ODE $Lu = f$ is said to be **singular** with respect to an interval $[a, b]$ if the leading coefficient $a_m(x)$ vanishes for any $x \in [a, b]$.

Example $Lu = (x - 3)u''' - (1 + \cos x)u' + 6u$ is a linear differential of order 3, with leading coefficient $x - 3$. Thus, L is singular on $[1, 4]$, but not singular on $[0, 2]$.

Example $N(u) = u'' + u^2u' - u$ is a nonlinear differential operator of order 2.

Linearity For any constants c_1 and c_2 and any m-times differentiable functions u_1 and u_2,

$$L(c_1 u_1 + c_2 u_2) = c_1 L u_1 + c_2 L u_2.$$

Ordinary differential equations of the form $Lu = 0$ are known as **homogenous**. Those of the form $Lu = f$ are known as **inhomogeneous**.

When the solution to a differential equation is prescribed at a particular point $x = x_0$, that is

$$u(x_0) = v_0, \quad u'(x_0) = v_1, \quad \dots, \quad u^{(m-1)}(x_0) = v_{m-1},$$

we call it an **initial value problem**. Where a differential equation is order 2 or greater, solutions at 2 or more locations can be prescribed. Such problems are called **boundary value problems**.

Unique Solution to Linear Initial Problem For an ODE $Lu = f$ which is not singular with repsect to a, b, with f continuous on $[a, b]$, the IVP for an mth-order linear differential operator with m inital values has a unique solution.

Solution to mth Order Problem has Dimension m Assume that the linear, mth-order differential operator L is not singular on $[a, b]$. Then the set of all solutions to the homogenous equation $Lu = 0$ on $[a, b]$ is a vector space of dimension m.

If $\{u_1, u_2, \dots, u_m\}$ is **any** basis for the solution space of $Lu = 0$, then every solution can be written in a unique way as

$$u(x) = c_1 u_1(x) + c_2 u_2(x) + \cdots + c_m u_m(x) \quad \text{for } a \leq x \leq 4.$$

We refer to this as the **general solution** of the homogenous equation $Lu = 0$ on $[a, b]$.

Linear superposition refers to this technique of constructing a new solution out of a linear combination of old ones.

Example The general solution to $u'' - u' - 2u = 0$ is $u(x) = c_1 e^{-x} + c_2 e^{2x}$.

Consider the inhomogeneous equation $Lu = f$ on $[a, b]$, and fix a particular solution u_P.
For *any* solution u, the difference $u - u_P$ is a solution of the homogenous equation because

$$L(u - u_P) = Lu - Lu_P = f - f = 0 \text{ on } [a, b].$$

Hence, $u(x) - u_P(x) = c_1 u_1(x) + \dots_c m u_m(x)$ for some constants $c_1, \dots, c_m$ and so

$$u(x) = u_P(x) + \underbrace{c_1 u_1(x) + \cdots + c_m u_m(x)}_{u_H(x)}, \quad a \leq x \leq b,$$

is the **general solution** of the inhomogeneous equation $Lu = f$.

Example The inhomogenous ODE $u'' - u' - 2u = -2e^x$ has a particular solution $u_P(x) = e^x$.
The general solution for its homogenous counterpart is $u_H(x) = c_1 e^{-x} + c_2 e^{2x}$.
So the general solution of the inhomogeneous ODE is

$$u(x) = u_P(x) + u_H(x) = e^x + c_1 e^{-x} + c_2 e^{2x}.$$

Reduction of Order For $u = u_1(x) \neq 0$, a solution to the ODE

$$u'' + p(x)u' + q(x)u = 0,$$

on some interval I, then a second solution is

$$u = u_1(x) \int \frac{1}{u_1^2 \exp\left(\int p\, dx\right)}\, dx.$$

Example For the ODE $u'' - 6u' + 9u = 0$, take $u_1 = e^{3x}$ and find v. **Answer** xe^{3x}.

1.3 Differential Operators with Constant Coefficients

If L has constant coefficients, then the problem of solving $Lu = 0$ reduces to that of factorising the polynomial having the same coefficients.

Suppose that a_j is constant for $0 \leq j \leq m$, with $a_m \neq 0$. We define the associated polynomial of degree m,

$$p(z) = \sum_{j=0}^{m} a_j z^j = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0,$$

so that if

$$Lu = a_m u^{(m)} + a_{m-1} u^{(m-1) + \cdots + a_1 u + a_0},$$

then formally, $L = p(D)$.

By the fundamental theorem of algebra,

$$p(z) = a_m (z - \lambda_1)^{k_1} (z - \lambda_2)^{k_2} \cdots (z - \lambda)^{k_r}$$

where $\lambda_1, \lambda_@, \ldots, \lambda_r$ satisfying

$$k_1 + k_2 + \cdots + k_r = m.$$

Lemma $(D - \lambda) x^j e^{\lambda x} = j x^{j-1} e^{\lambda x}$ for $j \geq 0$.

Lemma $(D - \lambda)^k x^j e^{\lambda x} = 0$ for $j = 0, 1, \ldots, k - 1$.

Basic Solutions If $(z - \lambda)^k$ is a factor of $p(z)$ then the function $u(x) = x^j e^{\lambda x}$ is a solution of $Lu = 0$ for $0 \leq j \leq k - 1$.

General Solution For the constant-coefficient case, the general solution of the homogenous equation $Lu = 0$ is

$$u(x) = \sum_{q=1}^{r} \sum_{l=0}^{k_q - 1} c_{ql} x^l e^{\lambda_q x},$$

where the c_{ql} are arbitrary constants.

Repeated Real Root From the factorisation

$$D^4 + 6D^3 + 9D^2 - 4D - 12 = (D-1)(D+2)^2(D+3)$$

we see that the general solution of

$$u'''' + 6u''' + 9'' - 4u' - 12u = 0$$

is

$$u = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x} + c_4 e^{-3x}.$$

Complex Root From the factorisation

$$\begin{aligned} D^3 - 7D^2 + 17D - 15 &= (D^2 - 4D + 5)(D-3) \\ &= (D-2-i)(D-2+i)(D-3) \end{aligned}$$

we see that the general solution of

$$u''' - 7u'' + 17u' - 15u = 0$$

is

$$\begin{aligned} u(x) &= c_1 e^{(2+i)x} + c_2 e^{(2-i)x} + c_3 e^{3x} \\ &= c_4 e^{2x} \cos x + c_5 e^{2x} \sin x + c_3 e^{3x}. \end{aligned}$$

Second-order ODEs arise naturally in classical mechanics for example a harmonic simple oscillator.

1.4 Wronskians and Linear Independence

We introduce a function, called the Wronskain that provides us with a way of testing whether a family of solutions to $Lu = 0$ is linearly independent.

Let $u_1(x), u_2(x), \dots, u_m(x)$ be functions defined on an interval $I \in \mathbb{R}$. The functions $u_1, \dots, u_m$ are called **linearly dependent** if there exist constant $a_1, a_2, \dots, a_m$ **not all zero** such that

$$a_1 u_1(x) + a_2 u_2(x) + \cdots + am u_m(x) = 0 \quad \forall x \in I.$$

If the above equation only holds for

$$a_i = 0, \quad i = 1, 2, \dots, m$$

then the functions are **linearly independent**.

Example $u_1 = \sin 2x$ and $u_2 = \sin x \cos x$ are linearly dependent.
$u_1 = \sin x$ and $u_2 = \cos x$ are linearly indepdent.

The **Wronskian** of the functions $u_1, u_2, \dots, u_m$ is the $m \times m$ determinant

$$W(x) = W(x; u_1, u_2, \dots, u_m) = \det\big[D^{i-1} u_j\big].$$

Example The Wronskian of the functions $u_1 = e^{2x}, u_2 = xe^{2x}$ and $u_3 = e^{-x}$ is

$$W = \begin{vmatrix} e^{2x} & xe^{2x} & e^{-x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} & -e^{-x} \\ 4e^{2x} & 4e^{2x} + 4xe^{2x} & e^{-x} \end{vmatrix} = 9e^{3x}.$$

Lemma If $u_1, \dots, u_m$ are linearly dependent over an interval $[a, b]$ then $W(x; u_1, \dots, u_m) = 0$ for $a \leq x \leq b$.

Lemma If $u_1, u_2, \dots, u_m$ are solutions of $Lu = 0$ on the interval $[a, b]$ then their Wronskain satisfies

$$a_m(x)W'(x) + a_{m-1}(x)W(x) = 0, \quad a \leq x \leq b.$$

Linear Independence of Solutions Let $u_1, u_2, \dots, u_m$ be solutions of a non-singular, linear, homogenous, m-th order ODE $Lu = 0$ on the interval $[a, b]$.
Either
  $W(x) = 0$ for $a \leq x \leq b$ and the m solutions are linearly **dependent**,
or else
  $W(x) \neq 0$ for $a \leq x \leq b$ and the m solutions are linearly **independent**.

1.5 Methods for Inhomogeneous Equations

1.5.1 Judicious Guessing Method

You would have learned the mthod of undetermined coefficients for constructing a particular solution u_P to an inhomogeneous second-order linear ODE $Lu = f$ in some simple cases. We will study this method systematically for higher-order linear ODEs with constant coefficients.

Superposition of Solutions Suppose that u_1 solves $Lu = e^{3x}$, and u_2 solves $Lu = \sin x$, where L is a linear differential operator. Then the solution of

$$Lu = e^{3x} + \sin x$$

is

$$u(x) = u_1(x) + u_2(x).$$

And a solution of

$$Lu = \frac{1}{2}e^{3x} - 5\sin x$$

is

$$u(x) = \frac{1}{2}u_1(x) - 5u_2(x).$$

Now we want to investigate some methods for finding particular solutions - i.e., finding a solution of $Lu = f$. One such method is the method of judicious guessing. For example:

1. If f is a polynomial, then guess that u_p is a polynomial.

2. If f is a exponential, then guess that u_p is exponential.

3. If f is a sine or cosine, then guess that u_p is a combination of such functions.

One problem with this method: it will only work for the types of functions identified above.

Example Suppose that $u'' - u' = t^2 + 2t$. Note as before that,

$$u_h(t) = c_1 + c_2 e^t.$$

So guess,

$$u_p(t) = At^3 + Bt^2 + Ct + D.$$

Then

$$t^2 + 2t = u_p'' - u' = -3At^2 + (6A - 2B)t + (2B - C).$$

So, equating coefficients of like power terms, we see that

$$A = -\frac{1}{3}, B = -2, C = -4, \text{ and } D \text{ is unrestricted.}$$

Therefore, reabsorbing D into c_1, we see that

$$u(t) = u_h(t) + u_p(t) = c_1 + c_2 e^t - \frac{1}{3}t^3 - 2t^2 - 4t.$$

Now we look at this idea of judicious guessing in a more systematic way. Let $L = p(D)$ be a linear differential operator of order m with constant coefficients.

Polynomial Solutions Assume that $a_0 = p(0) \neq 0$. For any integer $r \geq 0$, there exists a unique polynomial u_P of degree r such that $Lu_P = x^r$.

Exponential Solutions Let $L = p(D), M \in \mathbb{R}$ and $\mu \in \mathbb{C}$. If $p(\mu) \neq 0$, then the function

$$u_P(x) = \frac{Me^{\mu x}}{p(\mu)}$$

satisfies $Lu_P = Me^{\mu x}$.

Example A particular solution of $u'' + 4u' - 3i = 3e^{2x}$ is $u_P = e^{2x}/3$.

Product of Polynomial and Exponential Let $L = p(D)$ and assume that $p(\mu) \neq 0$. For any integer $r \geq 0$, there exists a unique polynomial v of degree r such that $u_P = v(x)e^{\mu x}$ satisfies $Lu_P = x^r e^{\mu x}$.

1.5.2 Annihilator Method

In the previous cases we proposed a solution $u = u_P$ and showed that it satisfied $Lu = f$. The following is a method to derive a particular solution given $Lu = f$. If $f(x)$ is differentiable at least n times and

$$[a_n D^n + a_{n-1} D^{n-1} + \cdots a_1 D^1 + a_0] f(x) = 0$$

then $[a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D^1 + a_0]$ **annihilates** f.

Example D^n annhihilates x^{m-1} for $m \leq n$.
$(D-\alpha)^n$ annihilates $x^{m-1}e^{\alpha x}$ for $m \leq n$.

Annhilator Method: Simple Example Given $Lu = f$ we can apply the appropriate annhiliator to both sides and solving the resutling homogenous DE.
Let $Lu = u'' - u'$ and suppose we want a solution such that $Lu = x^2$. Annihilating both sides we have

$$D^3(u'' - u') = u^{(5)} - u^{(4)} = 0.$$

Setting $w = u^{(4)}$, clearly $w = Ce^x$ is the general solution. Integrating four times yields

$$u = Ce^x + Ex^3 + Fx^2 + Gx + H.$$

Clearly $u_h = Ae^x + H$ and the form of the particular solution is $u_P = x(Ex^2 + Fx + G$. Substituting find $E = -1/3, F = -1$ and $G = -2$.

1.5.3 Judicious Guessing Method Continued

Polynomial Solutions: The Reamining Case Let $L = p(D)$ and assume $p(0) = p'(0) = \cdots = p^{(k-1)}(0) = 0$ but $p^{(k)}(0) \neq 0$ where $1 \leq k \leq m-1$. For any integer $r \geq 0$, there exists a unique polynomial v of degree $r$such that $u_P(x) = x^K v(x)$ satisfies $Lu_P = x^r$.

Exponential Times Polynomial: Remaining Case Let $L = p(D)$ and assume $p(\mu) = p'(\mu) = \cdots = p^{(k-1)}(\mu) = 0$. But $p^{(k)}(\mu) \neq 0$, where $1 \leq k \leq m-1$. For any integer $r \geq 0$, there exists a unique polynomial v of degree r such that $u_P(x) = x^k v(x)e^{\mu x}$ satisfies $Lu_P = x^r e^{\mu x}$.

1.5.4 Variation of Parameters

Example Find the general solution to $u'' - 4u' + 4u = (x+1)\exp 2x$.
Note first that the general solution, u_h, to $u'' - 4u' + 4u = 0$ is

$$u(x) = c_1 e^{2x} + c_2 x e^{2x}$$

since the characteristic equation is $0 = r^2 - 4r + 4 = (r-2)^2$. Then

$$W(x) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x} + 2xe^{4x} - 2xe^{4x} = e^{4x}.$$

So by the method of variation of parameters:

$$v_1'(x) = e^{-4x} \cdot -xe^{2x}(x+1)e^{2x} \text{ and } v_2'(x) = e^{-4x} \cdot e^{2x}(x+1)e^{2x}.$$

In other words,

$$v_1'(x) = -x^2 - x \text{ and } v_2'(x) = x + 1.$$

Therefore $u(x) = c_1 e^{2x} + c_2 x e^{2x} - (\frac{1}{3}x^3 + \frac{1}{2}x^2)e^{2x} + (\frac{1}{2}x^2 + x)xe^{2x}$.

1.6 Solution via Power Series

General Case Consider a general second-order, linear, homogenous ODE

$$Lu = a_2(x)u'' + a_1(x)u' + a_0(x)u = 0.$$

Equivalently,

$$u'' + p(x)u' + q(x)u = 0,$$

where

$$p(x) = \frac{a_1(x)}{a_2(x)} \text{ and } q(x) = \frac{a_0(x)}{a_2(x)}.$$

Assume that a_j is **analytic** at 0 for $0 \leq j \leq 2$. Then p and q are analytic at 0, that is, they admit power series expansions

$$p(z) = \sum_{k=0}^{\infty} p_k z^k \text{ and } q(z) = \sum_{k=0}^{\infty} q_k z^k \text{ for } |z| < \rho,$$

for some $\rho > 0$.

Convergence Theorem If the coefficients $p(z)$ and $q(z)$ are analytic for $|z| < \rho$, then the formal power series for the solution $u(z)$, constructed above, is also analytic for $|z| < \rho$.

Power Series at Zero Consider

$$Lu = (1 - x^2)u'' - 5xu' - 4u = 0, \quad u(0) = 1, \quad u'(0) = 2.$$

In this case,

$$p(z) = \frac{-5z}{1 - z^2} = -5\sum_{k=0}^{\infty} z^{2k+1} \text{ and } q(z) = \frac{-4}{1 - z^2} = -4\sum_{k=0}^{\infty} z^{2k}$$

are analytic for $|z| < 1$, so the theorem guarantees that $u(z)$, given by the formal power series, is also analytic for $|z| < 1$.

Expansion about a Point other than Zero Suppose we want a power series expansion about a point $c \neq 0$, for instance because the initial conditions are given at $x = c$.
A simple change of the independent variable allows us to write

$$u = \sum_{k=0}^{\infty} A_k (z - c)^k = \sum_{k=0}^{\infty} A_k Z^k \text{ where } Z = z - c.$$

Since $du/dx = du/dZ$ and $d^2u/dz^2 = d^2u/dZ^2$, we obtain the translated equation

$$\frac{d^2u}{dZ^2} + p(Z + c)\frac{du}{dZ} + q(Z + c)u = 0.$$

Now compute that A_k using the series expansions of $p(Z + c)$ and $q(Z + c)$ in powers of Z.

1.7 Singular ODEs

In general, we do not want L to be singular on an interval for which we wish to solve $Lu = f$. However, some important applications lead to singular ODEs so we now address this case.

A second-order **Euler-Cauchy ODE** has the form

$$Lu = ax^2u'' + bxu' + cu = f(x),$$

where a, b and c are constants with $a \neq 0$. This ODE is singular at $x = 0$.
Noticing that

$$Lx^r = [ar(r-1) + br + c]x^r,$$

we see that $u = x^r$ is a solution of the homogenous equation ($f = 0$) iff

$$ar(r-1) + br + c = 0.$$

Factorisation Suppose $ar(r-1) + br + c = a(r - r_1)(r - r_2)$. If $r_1 \neq r_2$ then the general solution of the homogenous equation $Lu = 0$ is

$$u(x) = C_1x^{r_1} + C_2x^{r_2}, \quad x > 0.$$

Lemma If $r_1 = r_2$ then the general solution of the homogenous Euler-Cauchy equation $Lu = 0$ is

$$u(x) = C_1x^{r_1} + C_2x^{r_1} \ln x, \quad x > 0.$$

Euler-Cauchy Equations with Nonreal Indicial Roots Suppose that $r_{1,2} = \alpha \pm \beta i$ are the roots of the indicial equation

$$ar(r-1) + br + c = 0$$

associated to the Euler-Cauchy equation

$$at^2u'' + btu' + cu = 0.$$

Then the real-valued solutions can be derived as follows. First note that

$$t^{\alpha+\beta i} = t^\alpha t^{\beta i}$$

is a solution. Then notice that

$$t^{\beta i} = e^{\ln t^{\beta i}} = e^{i \ln t^\beta} = \cos\big(\ln\big(t^\beta\big)\big) + i \sin\big(\ln\big(t^\beta\big)\big).$$

So,

$$t^\alpha t^{\beta i} = t^\alpha e^{\ln t^{\beta i}} = t^\alpha e^{i \ln t^\beta} = t^\alpha \left(\cos\big(\ln\big(t^\beta\big)\big) + i \sin\big(\ln\big(t^\beta\big)\big)\right)$$

is a solution. Finally, since each of the real part an the imaginary part is .separately a (linear independent) solution, we see that the general solution in this case is (for $t > 0$)

$$u(t) = t^\alpha \left(c_1 \cos\big(\ln\big(t^\beta\big)\big) + i \sin\big(\ln\big(t^\beta\big)\big)\right).$$

Example Consider $t^2u'' - tu' + 5u = 0$. Then the indicial equation is

$$r(r-1) - r + 5 = 0 \implies r = 1 \pm 2i.$$

So the general solution is,

$$u(t) = t(c_1 \cos \ln t^2 + c_2 \sin \ln t^2).$$

A number of important applications lead to ODEs that can be written in the **Frobenious normal form**

$$z^2u'' + zP(z)u' + Q(z)u = 0,$$

where $P(z)$ and $Q(z)$ are analytic at $z = 0$:

$$P(z) = \sum_{k=0}^{\infty} P_k z^k \text{ and } Q(z) = \sum_{k=0}^{\infty} Q_k z^k, \quad |z| < \rho.$$

Now consider $z^2 u'' + zP(z)u' + Q(z)u = 0$. FOrmal manipulations show that $Lu(z)$ equals

$$I(r)A_0 z^r + \sum_{k=1}^{\infty} \left(I(k+r)A_k + \sum_{j=0}^{k-1} \left[(j+r)P_{k-j} + Q_{k-j} \right] A_j \right) z^{k+r},$$

where $I(r)$ is the indicial polynomial $I(r) := r(r-1)P_0 r + Q_0$, so we define $A_0(r) = 1$ and

$$A_k(r) = \frac{-1}{I(k+r)} \sum_{j=0}^{k-1} [(j+r)P_{k-j} + Q_{k-j}]A_j(r), \quad k \geq 1,$$

provided $I(k+r) \neq 0$ for all $k \geq 1$.

1.8 Bessel and Legendre Equations

1.8.1 Bessel Equations and Functions

The **Bessel equation with parameter** ν is

$$z^2 u'' + zu' + (z^2 - \nu^2)u = 0.$$

This ODE is in Frobenius normal form, with indicial polynomial $I(r) = (r+\nu)(r-\nu)$, and we seek a series solution

$$u(z) = \sum_{k=0}^{\infty} A_k z^{k+r}.$$

We assume $\operatorname{Re} \nu \geq 0$, so $r_1 = \nu$ and $r_2 = -\nu$.
With the normalisation

$$A_0 = \frac{1}{2^\nu \Gamma(1+\nu)}$$

the series solution is called the **Bessel function of order** ν and is denoted

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1+\nu)} \left[1 - \frac{(z/2)^2}{1+\nu} + \frac{(z/2)^4}{2!(1+\nu)(2+v)} - \cdots \right].$$

From the functional equation $\Gamma(1+z) = z\Gamma(z)$ we see that

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1+v)} - \frac{(z/2)^{\nu+2}}{\Gamma(2+\nu)} + \frac{(z/2)^{\nu+4}}{2!\Gamma(3+v)} - \frac{(z/2)^{\nu+6}}{3!\Gamma(4+\nu)} + \cdots$$

and so

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\nu}}{k!\Gamma(k+1+\nu)}.$$

1.8.2 Legendre Equation

The **Legendre equation** with parameter nu is

$$(1 - z^2)u'' - 2zu' + \nu(\nu + 1)u = 0.$$

This ODE is not singular at $z = 0$ so the solution has an ordinary Taylor series expansion

$$u = \sum_{k=0}^{\infty} A_k z^k.$$

The A_k must satisfy

$$(k + 1)(k + 2)A_{k+2} - [k(k + 1) - \nu(\nu + 1)A_k] = 0$$

for $k \geq 0$, and since

$$k(k + 1) - \nu(\nu + 1) = (k - \nu)(k + \nu + 1),$$

the recurrence relation is

$$A_{k+1} = \frac{(k - \nu)(k + \nu + 1)}{(k + 1)(k + 2)} A_k \text{ for } k \geq 0.$$

We have

$$u(z) = A_0 u_0(z) + A_1 u_1(z)$$

where

$$u_0(z) = 1 - \frac{\nu(\nu + 1)}{2!} z^2 + \frac{(\nu - 2)\nu(\nu + 1)(\nu + 3)}{4!} z^4 - \cdots$$

and

$$u_1(z) = z - \frac{(\nu - 1)(\nu + 2)}{3!} z^3 + \frac{(\nu - 3)(\nu - 1)(\nu + 2)(\nu + 4)}{5!} z^5 - \cdots .$$

Suppose now that $\nu = n$ is a non-negative integer. If n is even the series for $u_0(z)$ terminates, whereas if n is odd then the series for $u_1(z)$ terminates.

The terminating solution is called the **Legendre polynomial** of degree n and is denoted by $P_n(z)$ with the normalization

$$P_n(1) = 1.$$

Lengdre Polynomials The first few Legendre polynomials are

$$\begin{aligned} P_0(z) &= 1, & P_3(z) &= \frac{1}{2}(5z^3 - 3z), \\ P_1(z) &= z, & P_4(z) &= \frac{1}{8}(35z^4 - 30z^2 + 3), \\ P_2(z) &= \frac{1}{2}(3z^2 - 1), & P_5(z) &= \frac{1}{8}(63z^5 - 70z^3 + 15z). \end{aligned}$$

Notice that P_n is an even or odd function according to whether n is even or odd.

Chapter 2

Dynamical Systems

2.1 Terminology

We begin with some examples of how systems of differential equations arise in applications, and see how all such problems can be formulas as a **first-order** system

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}).$$

Such a formulation leads to a natural geometric interpretation of a solution.

Lotka-Volterra Equations Simplified ecology with two species:

$$\begin{aligned} F(t) &= \text{ number of foxes at time t}, \\ R(t) &= \text{ number of rabbits at time t}. \end{aligned}$$

Assume populations large enough at F and R can be treated as smoothly varying in time.
In the 1920s, Alfred Lotka and Vito Volterra independently proposed the predator-prey model

$$\begin{aligned} \frac{dF}{dt} &= -aF + \alpha FR, \quad F(0) = F_0, \\ \frac{dR}{dt} &= bR - \beta FR, \qquad R(0) = R_0. \end{aligned}$$

Here a, α, b and β are non-negative constants.

Any first-order system for N ODEs in the form

$$\begin{aligned} \frac{dx}{dy} &= F_1(x, y, \ldots, x_N), \qquad x(0) = x_{10}, \\ \frac{dy}{dt} &= F_2(x, y, \ldots, x_N), \qquad y(0) = x_{20}, \\ &\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ \frac{dx_N}{dt} &= F_N(x, y, dots, x_N), \quad x_N(0) = x_{N0}, \end{aligned}$$

can be written in vector notation as

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) \quad \mathbf{x}(0) = \mathbf{x}_0.$$

The system of ODEs is determined by the **vector field** $\mathbf{F} : \mathbb{R}^N \to \mathbb{R}^N$.
A system of ODEs of the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x})$$

is said to be **autonomous**.
In a **non-autonomous** system, $\mathbf{F}$ will depend explicitly on t:

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t).$$

It can be shown that it is sufficient (in principle) to develop theory for the autonomous case as a non-autonomous system can be converted into an autonomous system.

Second-order ODE Consider an initial-value problem for a general (possibly non-autonomous) second-order ODE

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}, t\right), \text{ with } x = x_0 \text{ and } \frac{dx}{dt} = y_0 \text{ at } t = 0.$$

If $x = x(t)$ is a solution, and if we let $y = dx/dt$, then

$$\frac{dy}{dt} = \frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}, t\right) = f(x, y, t),$$

that is, (x, y) is a solution of the first-order system

$$\begin{aligned} \frac{dx}{dt} &= y, & x(0) &= x_0, \\ \frac{dy}{dt} &= f(x, y, t) & y(0) &= y_0. \end{aligned}$$

2.2 Existence and Uniqueness

The most fundamental question about a dynamical system

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t)$$

is

> For a given initial value $\mathbf{x}_0$, does a solution $\mathbf{x}(t)$ satisfying $\mathbf{x}(0) = \mathbf{x}_0$ exist, and if so is this solution unique?

Answer is **yes**, whenever the vector field $\mathbf{F}$ is **Lipschitz**.

The number L is a **Lipschitz constant** for a function $f : [a, b] \to \mathbb{R}$ if

$$|f(x) - f(y)| \leq L|x - y| \quad \text{for all } x, y \in [a, b].$$

Example Consider $f(x) = 2x^2 - x + 1$ for $0 \leq x \leq 1$. Since

$$\begin{aligned} f(x) - f(y) = 2(x^2 - y^2) - (x - y) &= 2(x + y)(x - y) - (x - y) \\ &(2x + 2y - 1)(x - y) \end{aligned}$$

we have $|f(x)-f(y)| = |2x+2y-1||x-y|$ so a Lipschitz constant is

$$L = \max_{x,y\in[0,1]} |2x+2y-1| = 3.$$

We say that the function $f : [a,b] \to \mathbb{R}$ is Lipschitz if a Lipschitz constant for f exists.

Lipschitz Continuity If f is Lipshictz then f is (uniformly) continuous.

Continous does not imply Lipschitz Consider the (uniformly) continous function

$$f(x) = 3 + \sqrt{x} \text{ for } 0 \leq x \leq 4.$$

In this case, if $x, y \in (0,4]$ then

$$f(x) - f(y) = \sqrt{x} - \sqrt{y} = \left(\sqrt{x} - \sqrt{y} \times \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right) \qquad = \frac{x-y}{\sqrt{x}+\sqrt{y}}$$

so if a Lipschitz constant L exists then

$$L \geq \frac{|f(x)-f(y)|}{|x-y|} = \frac{1}{\sqrt{x}+\sqrt{y}}$$

for arbitrarily small x and y, a contradiction.

A function $f : I \to \mathbb{R}$ is C^k if $f, f', f'', \ldots, f^{(k)}$ all exist and are continuos on the interval I.

Theorem For any closed and bounded interval $I = [a,b]$, if f is C^1 on I then $L = \max_{x\in I} |f'(X)|$ is a Lipschitz constant for f on I.

A vector field $\mathbf{F} : S \in \mathbb{R}^N$ is Lipschitz on $S \subseteq \mathbb{R}^N$ if

$$\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})\| \leq L\|\mathbf{x}-\mathbf{y}\| \qquad \text{for all} \mathbf{x}, \mathbf{y} \in S$$

Here,

$$\|\mathbf{x}\| = \left(\sum_{j=1}^{N} x_j^2\right)^{\frac{1}{2}}$$

denotes the **Euclidean norm** of the vector $\mathbf{x} \in \mathbb{R}^N$.
We say that $\mathbf{F}(\mathbf{x}, t)$ is **Lipschitz in x** if

$$\|\mathbf{F}(\mathbf{x},t) - \mathbf{F}(\mathbf{y},t)\| \leq L\|\mathbf{x}-\mathbf{y}\|.$$

Local Existence and Uniquness Let $\mathbf{x}_0 \in \mathbb{R}^N$, fix $r > 0$ and $\tau > 0$, and put

$$S = \{(\mathbf{x}, t) \in \mathbb{R}^N \times \mathbb{R} : \|\mathbf{x} - \mathbf{x}_0\| \leq r \text{ and } |t| \leq \tau\}.$$

If $\mathbf{F}(\mathbf{x}, t)$ is Lipschitz in $\mathbf{x}$ for $\mathbf{x}, t \in S$, and if

$$\|\mathbf{F}(\mathbf{x}, t)\| \leq M \qquad \text{for } (\mathbf{x}, t) \in S,$$

then there exists a unique C^1 function $bfx(t)$ satisfying

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t) \quad \text{for } |t| \leq \min\{r/M, \tau\}, \text{ with } \mathbf{x}(0) = \mathbf{x}_0.$$

2.3 Linear Dynamical Systems

Linear differential equations are generally much easier to solve than nonlinear ones. Fortunately, linear DEs suffice for describing many important applications.
We say that the $N \times N$, first order system of ODEs

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t)$$

is **linear** if the RHS has the form

$$\mathbf{F}(\mathbf{x}, t) = A(t)\mathbf{x} + \mathbf{b}(t)$$

for some $N \times N$ matrix-valued function $A(t) = [a_{ij}(t)]$ and a vector-valued function $\mathbf{b} = [b_i(t)]$.

The system is autonomous precisely when A and $\mathbf{b}$ are constant.

Global Existence and Uniqueness If the elements of $A(t)$ and components of $\mathbf{b}$ are continuous for $0 \leq t \leq T$, then the linear initial-value problem

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x} + \mathbf{b}(t) \quad \text{for } 0 \leq t \leq T, \quad \text{with } \mathbf{x}(0) = \mathbf{x}_0,$$

has a unique solution $\mathbf{x}(t)$ for $0 \leq t \leq T$.

We now investigate the special case when A is constant and $\mathbf{b}(t) = \mathbf{0}$:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

General Solution via Eigensystem If $\mathbf{v}$ is a constant vector and $A\mathbf{v} = \lambda\mathbf{v}$, we define $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$. Then

$$\frac{d\mathbf{x}}{dt} = \lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}(\lambda\mathbf{v}) = e^{\lambda t}(A\mathbf{v}) = A(e^{\lambda t}\mathbf{v} = A\mathbf{x})$$

that is, $\mathbf{x}$ is a solution of $dvx/dt = A\mathbf{x}$. If $A\mathbf{v}_j = \lambda_j\mathbf{v}_j$ for $1 \leq j \leq N$, then the linear combination

$$\mathbf{x}(t) = \sum_{j=1}^{N} c_j e^{\lambda_j} t\mathbf{v}_j$$

is also a solution because the ODE is linear and homogenous. Provided the $\mathbf{v}_j$ are linearly independent, then the above equation is a **general solution** because given any $x_0 \in \mathbb{R}^N$ there exist unique c_j such that

$$\mathbf{x}(0) = \sum_{j=1}^{N} c_j\mathbf{v}_j = \mathbf{x}_0.$$

Example Consider

$$\frac{dx}{dt} = -5x + 2y, \quad x(0) = 5,$$
$$\frac{dy}{dt} = -6x + 3y \quad\ \ y(0) = 7.$$

Note that the initial value problem can be written in the vector form

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), \mathbf{x}(0) = \mathbf{x}_0,$$

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ -6 & 3 \end{bmatrix} \quad \text{and } \mathbf{x}_0 := \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Solving the system, using the eigenpair approach, we would need to find the eigenvectors and eigenvalues.
Characteristic equation is

$$0 = |\mathbf{A} - \lambda \mathbf{I}| = (-5-\lambda)(3-\lambda) + 12 \implies \lambda_1 := -3 \text{ and } \lambda_2 = 1.$$

Next we find the associated eigenvectors.

$$\lambda_1 = -3 : (\mathbf{A} + 3\mathbf{I})\mathbf{v} = \mathbf{0} \implies \mathbf{v}_1 := \begin{bmatrix} 1 \\ 1 \end{bmatrix} \lambda_2 \qquad = 1 : (\mathbf{A} + \mathbf{I})\mathbf{v} = \mathbf{0} \implies \mathbf{v}_1 := \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

This means that a general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Applying the initial value we can see that the unique solution is $x(t) = 4e^{-3t} + e^t$ and $y(t) = 4e^{-3t} + 3e^t$.

A square matrix $A \in \mathbb{C}^{N\times N}$ is **diagonalisable** if there exists a non-singular matrix $Q \in \mathbb{C}^{N\times N}$ such that $Q^{-1}AQ$ is diagonal.

Theorem A square matrix $A \in \mathbb{C}^{N\times N}$ is diagonalisable if and only if there exists a basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N\}$ for $\mathbb{C}^N$ consisting of eigenvectors of A. Indeed if,

$$A\mathbf{v}_j = \lambda_j \mathbf{v}_j \text{ for } j = 1, 2, \ldots, N,$$

and we put $Q = \begin{bmatrix} \mathbf{v}_1 & vv_2 & \cdots \mathbf{v}_N \end{bmatrix}$ then $Q^{-1}AQ = A$ where

$$A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix}$$

Consider a diagonalisable matrix A. Since $Q^{-1}AQ = \Lambda$, it follows that A has an eigenvalue decomposition

$$A = Q\Lambda Q^{-1}.$$

In general, we see by induction on k that

$$A^k = Q\Lambda^k Q^{-1} \text{ for } k = 0, 1, 2, \ldots$$

Example

$$A = \begin{bmatrix} -5 & 2 \\ -6 & 3 \end{bmatrix}$$

then

$$\Lambda = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}, \quad Q^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

so

$$A^k = Q\Lambda^k Q^{-1} = \frac{1}{2}\begin{bmatrix} (-1)^k \times 3^{k+1} - 1 & (-1)^{k+1} \times 3^k + 1 \\ (-1)^k \times 3^{k+1} - 3 & (-1)^{k+1} \times 3^k + 3 \end{bmatrix}.$$

For any polynomial

$$p(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_m z^m$$

and any square matrix A, we define

$$p(A) = c_0 I + c_1 A + c_2 A^2 + \cdots + c_m A^m.$$

When A is diagonalisable, $A^k = Q\Lambda^k Q^{-1}$ so

$$\begin{aligned} p(A) &= c_0 Q I Q^{-1} + c_1 Q \Lambda Q^{-1} + \cdots + c_m Q \Lambda^m Q^{-1} \\ &\vdots \\ &= Q p(\Lambda) Q^{-1} \end{aligned}$$

Lemma For any polynomial p and any diagonal matrix Λ,

$$p(A) = \begin{bmatrix} p(\lambda_1) & & \\ & \ddots & \\ & & p(\lambda_N) \end{bmatrix}$$

Theorem If two polynomials p and q are equal on the specturm of a diagonlisable matrix A, that is, if

$$p(\lambda_j) = q(\lambda_j) \text{ for } j = 1, 2, \ldots, N,$$

then $p(A) = q(A)$.

Example Recall that

$$A = \begin{bmatrix} -5 & 2 \\ -6 & 3 \end{bmatrix}$$

has eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 1$. Let

$$p(z) = z^2 - 4 \quad \text{and} \quad q(z) = -2z - 1,$$

and observe

$$p(-3) = 5 = q(-3) \text{ and } p(1) = -3 = q(1).$$

We find

$$p(A) = A^2 - 4I = \begin{bmatrix} 9 & -4 \\ 12 & -7 \end{bmatrix} = -2A - I = q(A).$$

Exponential of a Diagonalisable Matrix If $A = Q\Lambda Q^{-1}$ is diagonalisable, then

$$e^A = Qe^\Lambda Q^{-1} \text{ and } e^\Lambda = \begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_N} \end{bmatrix}$$

Fundamental Matrix A fundamental matrix Φ for the linear homogenous vector equation

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$$

satisfies the following two properties.

1. The columns of $\mathbf{X}$ are linearly independent vector functions so that, in particular, $|\mathbf{X}(t) \neq 0$; and
2. Φ solves the matrix equation $\mathbf{X}'(t) = \mathbf{A}\mathbf{X}(t)$.

Theorem SUppose that $\boldsymbol{\Phi}$ is a fundamental matrix for the vector equations

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

Then every solution of this equation has the form

$$\boldsymbol{\Phi}\mathbf{c}$$

for some constant vector $\mathbf{c}$.

Nilpotent Matrix A matrix is nilpotent if there exists a positive integer k such that $\mathbf{A}^k = \mathbf{O}$, where $\mathbf{O}$ denotes the zero matrix.

Example

$$\mathbf{A}^2 = \mathbf{A}\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O}.$$

Therefore $\mathbf{A}$ is nilpotent and, in particular,

$$e^t\mathbf{A} = \mathbf{I} + t\mathbf{A} = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$$

2.4 Stability

In many applications we are interested to know how the solution $\mathbf{x}(t)$ behaves as $t \to \infty$, and might not care much about the precise details of the transient behaviour for finite t. We say that $\mathbf{a} \in \mathbb{R}^N$ is an equilibrium point for the dynamical system $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x})$ if

$$\mathbf{F}(\mathbf{a}) = \mathbf{0}.$$

Thus the solution of

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) \quad \text{for all } t, \text{ with } \mathbf{x}(0) = \mathbf{a}$$

is just the constant function $\mathbf{x}(t) = \mathbf{a}$.

An equilibrium point $\mathbf{a}$ is **stable** if for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever $\|\mathbf{a}_0 - \mathbf{a}\| < \delta$ the solution of

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) \quad \text{for } t > 0, \text{ with } \mathbf{x}(0) = \mathbf{x}_0$$

satisfies

$$\|\mathbf{x}(t) - \mathbf{a}\| < \epsilon \text{ for all } t > 0.$$

Let D be an open subset of $\mathbb{R}^N$ that contains an equilibrium point $\mathbf{a}$. We say that $\mathbf{a}$ is **asymptotically stable** in D if $\mathbf{a}$ is stable and, whenever $\mathbf{a}_0 \in D$, the solution of

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) \quad \text{for } t > 0, \text{ with } \mathbf{x}(0) = \mathbf{x}_0$$

satisfies

$$\mathbf{a}(t) \to \mathbf{a} \text{ as } t \to \infty.$$

In this case D is called a **domain of attraction** for $\mathbf{a}$.

Criteria for Stability Let A be a diagonlisable matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$. The equilibrium point $\mathbf{a} = -A^{-1}\mathbf{b}$ is of

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{b} \text{ with } \mathbf{x}(0) = \mathbf{x}_0 \text{ and } \det(A) \neq 0.$$

1. **stable** if and only if $\operatorname{Re}\lambda_j \leq 0$ for all j
2. **asymptotically stable** if and only if $\operatorname{Re}\lambda_j < 0$ for all j.

In the second case, the domain of attraction is the whole of $\mathbb{R}^N$.

2.5 Classification of 2D Linear Systems with $\det A \neq 0$

The equilibrium point $\mathbf{a} = 0$ may be asymptotically stable, stable or unstable but may also have various other properties.

2.5.1 Case 1: Real Eigenvalues and Linearly Independent Eigenvectors

Suppose you have real eigenvalues λ_1 and λ_2 and two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. General solution:

$$\mathbf{x} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2.$$

Canonical form:

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Stable Node Example $(\lambda_2 < \lambda_1 < 0)$

$$\frac{dx}{dt} = -x, \quad \frac{dy}{dt} = -2y, A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

Eigenvalues and eigenvectors:

$$\lambda_1 = -1, \lambda_2 = -2, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The general solution is

$$\mathbf{x} = c_1 e^{-t}\mathbf{v}_1 + c_2 e^{-2t}\mathbf{v}_2 = \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{-2t} \end{pmatrix}.$$

Solution of the initial value problem:

$$\begin{pmatrix} x(y) \\ y(t) \end{pmatrix} = \begin{pmatrix} x(0)e^{-t} \\ y(0)e^{-2t} \end{pmatrix}.$$

Unstable Node Example $(0 < \lambda_1 < \lambda_2)$

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = 2y, A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Eigenvalues and eigenvectors:

$$\lambda_1 = 1, \lambda_2 = 2, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Solution of the initial value problem:

$$\begin{pmatrix} x(y) \\ y(t) \end{pmatrix} = \begin{pmatrix} x(0)e^{t} \\ y(0)e^{2t} \end{pmatrix}.$$

All trajectories (except $\mathbf{x}(t) = \mathbf{0}$) are repelled from equilibrium point which is unstable.

(Un)stable stars: $\lambda_1 = \lambda_2 \neq 0$

$$\frac{dx}{dt} = \lambda_1 x, \quad \frac{dy}{dt} = \lambda_1 y, A = \lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

All vectors are eigenvectors.
The general solution is

$$\mathbf{x} = e^{\lambda_1 t}\mathbf{v}.$$

Solution of the initial value problem:

$$\mathbf{x}(t) = e^{\lambda_1 t}\mathbf{x}(0).$$

All orbits (except $vx(t) = \mathbf{0}$) are oriented half-lines which are either attracted $\lambda_1 < 0$ or repelled $(\lambda_1 > 0)$ by the equilibrium point.

Saddle Node Example (unstable: $\lambda_2 < 0 < \lambda_1$**)**

$$\frac{dx}{dt} = x + 2y, \quad \frac{dy}{dt} = 3x + 2y, A = \lambda_1 \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

Eigenvalues and eigenvectors:

$$\lambda_1 = -1, \lambda_2 = 4, \mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Solution of the initial value problem:

$$\begin{pmatrix} x(y) \\ y(t) \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Here the first solution is repelling and the section is attracting, so the solution is unstable.

Nonreal eigenstuff for A

$$\begin{aligned} e^{\lambda_1 t}\mathbf{v}_1 &= e^{(\alpha+\beta i)t}(\mathbf{p}+i\mathbf{q}) \\ &= e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))(\mathbf{p}+i\mathbf{q}) \\ &= \underbrace{e^{\alpha t}(\cos(\beta t)\mathbf{p} - \sin(\beta t)\mathbf{q})}_{:=\mathbf{x}_{\mathrm{Re}}(t)} + i\,\underbrace{e^{\alpha t}(\sin(\beta t)\mathbf{p} + \cos(\beta t)\mathbf{q})}_{:=\mathbf{x}_{\mathrm{Im}}(t)} \end{aligned}$$

So, a basis for the solution space is then

$$\mathcal{B} := \{\mathbf{x}_{\mathrm{Re}}, \mathbf{x}_{\mathrm{Im}}\}.$$

The general solution is,

$$\mathbf{x}(t) := c_1\mathbf{x}_{\mathrm{Re}}(t) + c_2\mathbf{x}_{\mathrm{Im}}(t)$$

for arbitrary constants $c_1, c_2 \in \mathbb{R}$.

2.5.2 Case 2: Complex Conjugate Eigenvalues

Suppose you have complex conjugate eigenvalues $\lambda_1 = \bar{\lambda}_2 \notin \mathbb{R}$.
General solution:

$$\mathbf{x} = c_1 \operatorname{Re}(e^{\lambda_1 t}\mathbf{v}_1) + c_2 \operatorname{Im}(e^{\lambda_1 t}\mathbf{v}_1).$$

Canonical form:

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad \lambda_1 = \alpha + i\beta.$$

Interpretation

$$\mathbf{x}(t) = e^{\alpha t}R(t)\mathbf{x}(0), \quad R(t) = \begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}$$

Thus, the initial vector $\mathbf{x}(0)$ is rotated by the rotation matrix $R(t)$ and scaled by the factor $e^{\alpha t}$.

Centre Example (stable: $\operatorname{Re}(\lambda_1) = 0$**)**

$$\frac{dx}{dt} = -2y, \quad \frac{dy}{dt} = 2x, A = \lambda_1 \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

Eigenvalues:

$$\lambda_1 = \bar{\lambda}_2 = -2i$$

Solution of the initial value problem:

$$\begin{pmatrix} x(y) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} y(0) \\ y(0) \end{pmatrix}.$$

The solution constitutes orbits which are oriented circles. These are stable (but not asymptotically stable).

Stable Foci Example ($\mathrm{Re}(\lambda_1) < 0$)

$$\frac{dx}{dt} = -x - 2y, \quad \frac{dy}{dt} = 2x - y, A = \lambda_1 \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix}$$

Eigenvalues:

$$\lambda_1 = \bar{\lambda_2} = -1 - 2i$$

Solution of the initial value problem:

$$\begin{pmatrix} x(y) \\ y(t) \end{pmatrix} = e^{-t} \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} \to \mathbf{0} \text{ as } t \to \infty.$$

Orbits are oriented spirals which are attracted to the asymptotically stable equilibrium point.

2.6 Final Remarks on Nonlinear DEs

A function $G : \mathbb{R}^N \to \mathbb{R}$ is a **first integral** (or constant of the motion) for the system of ODEs

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x})$$

if $G(\mathbf{x}(t))$ is constant for every solution $\mathbf{x}(t)$.

Simple Example The function $G(x, y) = x^2 + y^2$ is a first integral of the linear system of ODEs

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x.$$

In fact, putting

$$\mathbf{F}(x, y) = \begin{bmatrix} -y \\ x \end{bmatrix}$$

we have

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \cdot \begin{bmatrix} -y \\ x \end{bmatrix} = (2x)(-y) + (2y)(x) = 0,$$

or equivalently,

$$\frac{dG}{dt} = \frac{\partial G}{\partial x}\frac{dx}{dt} + \frac{\partial G}{\partial y}\frac{dy}{dt} = (2x)(-y) + (2y)(x) = 0.$$

Cayley-Hamilton Let $\mathbf{A} \in \mathbb{R}^{n \times n}$. Then $\mathbf{A}$ satisfies its characteristic equation.

Putzer's Algorithm Let $\{\lambda_j\}_{j=1}^n$ be the collection of n not necessarily distinct eigenvalues of a given matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$. Then

$$e^{\mathbf{A}t} = \sum_{k=0}^{n-1} p_{k+1}(t)\mathbf{M}_k,$$

where

$$\mathbf{M}_0 := \mathbf{I} \text{ and } \mathbf{M}_k := \Pi_{j=1}^{k}(\mathbf{A} - \lambda_j \mathbf{I}), 1 \leq k \leq n,$$

and the vector-valued function $\mathbf{p}(t) := (p_1(t), \ldots, p_n(t))$ satisfies the vectorial equation

$$\mathbf{p}'(t) = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & 0\cdots & 0 & \\ 0 & 1 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \lambda_n \end{bmatrix} \mathbf{p}(t), \quad \mathbf{p}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

So in the case in which $n = 2$, i.e., a two-dimensional vector equation, Putzer's algorithm reduces to

$$e^{\mathbf{A}t} = p_1(t)\mathbf{I} + p_2(t)(\mathbf{A} - \lambda_1\mathbf{I}),$$

where

$$\begin{bmatrix} p_1'(t) \\ p_2'(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}, \quad \mathbf{p}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Similarly, in the case in which $n = 3$, i.e., a three-dimensional vector equation, Putzer's algorithm reduces to

$$e^{\mathbf{A}t} = p_1(t)\mathbf{I} + p_2(t)(\mathbf{A} - \lambda_1\mathbf{I}) + p_3(t)(\mathbf{A} - \lambda_1\mathbf{I})(\mathbf{A} - \lambda_2\mathbf{I}),$$

where

$$\begin{bmatrix} p_1'(t) \\ p_2'(t) \\ p_3'(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 1 & \lambda_2 & 0 \\ 0 & 1 & \lambda_3 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix}, \quad \mathbf{p}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$