Higher Theory of Statistics
MATH2901 UNSW

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Contents

*With some inspiration from Hussain Nawaz's Notes

1 Probability

1.1 Experiment, Sample Space, Event

Experiment An experiment is any process leading to recorded observations.

Outcome An outcome is a possible result of an experiment.

Sample Space The set Ω of all possible outcomes is the sample space of an experiment. Ω is discrete if it contains a countable (finite or countably infinite) number of outcomes.

Events An event is a set of outcomes, i.e. a subset of Ω. An event occurs if the result of the experiment is one of the outcomes in that event.

Mutual Exclusion Events are mutually exclusive (disjoint) if they have no outcomes in common.

Set Operations If you have trouble remembering the above rules, then one can essentially replace $\cup$ by multiplication and $\cap$ by addition.
(The associative law) If A, B, C are sets then

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

(Distributive Law) If A, B, C are sets then

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

1.2 Sigma-algebra

The $\sigma-$algebra must be defined for rigorously working with probability. The $\sigma-$algebra can be thought of as the family of all possible events in a sample space. Analogously, this may be conceptualised as the power set of the sample space.

Probability A probability is a set function, which is usually denoted by $\mathbb{P}$, that maps events from the $\sigma-$algebra to $[0, 1]$ and satisfies certain properties.

Probability Space The triplet $(\Omega, \mathcal{A}, \mathbb{P})$ is the probability/sample space where

- Ω is the sample space,
- $\mathcal{A}$ is the σ-algebra,
- $\mathbb{P}$ is the probability function.

Properties of Probability Given the probability/sample space $(\Omega, \mathcal{A}, \mathbb{P})$, the probability function $\mathbb{P}$ must satisfy

- For every set $A \in \mathcal{A}$, $\mathbb{P}(A) \geq 0$
- $\mathbb{P}(\Omega) = 1$
- (Countably additive) Suppose the family of sets $(A_i)_{i \in \mathbb{N}}$ are mutually exclusive, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

Probability Lemmas

- Given a family of disjoint sets $(A_i)_{i=1,\ldots,k}$

$$\mathbb{P}\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} \mathbb{P}(A_i)$$

- $\mathbb{P}(\phi) = 0$
- For any $A \in \mathcal{A}, \mathbb{P}(A) \leq 1$ and $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
- Suppose $B, A \in \mathcal{A}$ and $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.

Continuity from Below Given an increasing sequence of events $A_1 \subset A_2 \subset \ldots$ then,

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mathbb{P}(A_n)$$

Continuity from Above Given a decreasing sequence of events $A_1 \supset A_2 \supset \ldots$ then,

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mathbb{P}(A_n)$$

1.3 Conditional Probability and Independence

Conditional Probability The conditional probability that an event A occurs given that an event B has occurred is

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}, \quad \mathbb{P}(B) > 0$$

Independence Events A and B are independent if $\mathbb{P}(A \cup B) = \mathbb{P}(A)\mathbb{P}(B)$.

Lemma - Given two events A and B then $\mathbb{P}(A|B) = \mathbb{P}(A)$ if and only if $\mathbb{P}(B|A) = \mathbb{P}(B)$.

Independence of Sequences

- A countable sequence of event $(A_i)_{i=\mathbb{N}}$ is pairwise independent if $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j)$ for all $i \neq j$.
- A countable sequence of events $(A_i)_{i=\mathbb{N}}$ are independent if for any sub-collection $A_{i_1}, \ldots, A_{i_n}$ we have

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cdots \cap A_{i_n}) = \prod_{j=1}^{n} \mathbb{P}(A_{i_j})$$

Independence implies pairwise independence, but pairwise independence does not imply independence.

Multiplicative Law Given events A and B then

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B),$$

and similarly, if you have events A, B, C then

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_3|A_2 \cap A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_1)$$

Additive Law Let A and B be events then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

Law of Total Probability Suppose $(A_i)_{i=1,\ldots,k}$ are mutually exclusive and exhaustive of Ω, that is $\bigcup_{i=1}^{k} A_i = \Omega$, then for any event B, we have

$$\mathbb{P}(B) = \sum_{i=1}^{k} \mathbb{P}(B|A_i)\mathbb{P}(A_i)$$

Bayes Formula Given sets B, A and a family of disjoint and exhaustive sets $(A_i)_{i=1,\ldots,k}$ then

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\sum_{i=1}^{k} \mathbb{P}(B|A_i)\mathbb{P}(A_i)}$$

1.4 Descriptive Statistics

Categorical Data can be sorted into a finite set of (unordered) categories. e.g. Gender

Quantitative Responses are measured on some sort of scale. e.g. Weight.

Numerical Summaries of the Quantitative Data Given observations $x = (x_1, \ldots, x_n)$. The sample mean (estimated mean) or average is given by

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Sample variance (estimated variance)

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

2 Random Variables

2.1 Random Variables

Random Variables A random variable (r.v) X is a function from Ω to $\mathbb{R}$ such that $\forall \mathbf{x} \in \mathbb{R}$, the set $A_{\mathbf{x}} = \{\omega \in \Omega, X(\omega) \leq \mathbf{x}\}$ belongs to the σ-algebra $\mathcal{A}$.

Cumulative Distribution Function The cumulative distribution function of a r.v X is defined by

$$F_X(\mathbf{x}) := \mathbb{P}(\{\omega : X(\omega) \leq \mathbf{x}\}) = \mathbb{P}(X \leq \mathbf{x})$$

Cumulative Distribution Theorems Suppose F_X is a cumulative distribution function of X, then

- it is bounded between zero and one, and

$$\lim_{x \downarrow -\infty} F_X(x) = 0 \quad \text{and} \quad \lim_{x \uparrow \infty} F_X(x) = 1$$

- it is non-decreasing, that is if $x \leq y$ then $F_X(x) \leq F_X(y)$
- for any $x < y$,

$$\mathbb{P}(x < X \leq y) = \mathbb{P}(X \leq y) - \mathbb{P}(X \leq x) = F_X(y) - F_X(x)$$

- it is right continuous, that is

$$\lim_{n \uparrow \infty} F_X(x + \frac{1}{n}) = F_X(x)$$

- it has finite left limit and

$$\mathbb{P}(X < x) = \lim_{n \to \infty} F_X(x - \frac{1}{n})$$

  which we denote by $F_X(x-)$.

Discrete Random Variables A r.v X is said to be discrete if the image of X consists of countable many values x, for which $\mathbb{P}(X = x) > 0$.

Discrete Probability Function The probability function of a discrete r.v X is the function $\boldsymbol{\nabla} F_X(x) = \mathbb{P}(X = x)$ and satisfies

$$\sum_{\text{all possible } x} \mathbb{P}(X = x) = 1$$

Continuous Random Variables A r.v X is said to be continuous if the image of X takes a continuum of values.

Continuous Probability Density Function The probability density function of a continuous r.v is a real-valued function f_X on $\mathbb{R}$ with the property that

$$\mathbb{P}(X \in A) = \int_A f_X(y)\, dy$$

for any 'Borel' subset of $\mathbb{R}$.

For a function $f : \mathbb{R} \to \mathbb{R}$ to be a valid density function, the function f must satisfty the following properties.

1. for all $x \in \mathbb{R}, f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Useful Properties (for continuous random variable) For any continuous random variable X with the density f_X,

1. by taking $A = (-\infty, x], \mathbb{P}(X \in (-\infty, x]) = \mathbb{P}(X \leq x)$ and
$$F_X(x) = \int_{-\infty}^{x} f_X(y)\, dy$$
2. For any $a < b \in \mathbb{R}$, one can compute $\mathbb{P}(a < X \leq b)$ by
$$F_X(b) - F_X(a) = \int_a^b f_X(x)\, dx$$
3. From the fundamental theorem of calculus and 1, we have
$$F_X'(x) = \frac{d}{dx} \int_{-\infty}^{x} f_X(y)\, dy = f_X(x).$$

2.2 Expectation and Variance

Expectation The expectation of a r.v X is denoted by $\mathbb{E}(X)$ and it is computed by

1. Let X be a discrete r.v. then

$$\mathbb{E}(X) := \sum_{\text{all possible } x} x\mathbb{P}(X = x) = \sum_{\text{all possible} x} x\boldsymbol{\nabla} F_X(x)$$

2. Let X be a continuous r.v. with density function $f_X(x)$ then

$$\mathbb{E}(x) := \int_{-\infty}^{\infty} x f_X(x)\, dx$$

Expectation of Transformed Random Variables Suppose $g : \mathbb{R} \to \mathbb{R}$, then the expectation of the transformed r.v $g(X)$ is

$$\mathbb{E}(g(X)) = \begin{cases} \int_{\mathbb{R}} g(x) f_X(x)\, dx & \text{continuous} \\ \sum_x g(x)\mathbb{P}(X = x) & \text{discrete} \end{cases}$$

usually one is interested in computing $\mathbb{E}(X^r)$ for $r \in \mathbb{N}$, which is called the r-th moment of X.

Linearity of Expectation The expectation $\mathbb{E}$ is linear, i.e., for any constants $a, b \in \mathbb{R}$,

$$\mathbb{E}(aX + b) = a\mathbb{E}(X) + b.$$

Variance Let X be a r.v and we set $\mu = \mathbb{E}(X)$. The variance is X is denoted by $\mathrm{Var}(X)$ and

$$\mathrm{Var}(X) := \mathbb{E}((X - \mu)^2)$$

and the standard deviation of X is the square root of the variance.

Properties of Variance Given a random variable X then for any constant $a, b \in \mathbb{R}$,

1. $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$
2. $\mathrm{Var}(ax) = a^2 \mathrm{Var}(X)$
3. $\mathrm{Var}(X + b) = \mathrm{Var}(X)$
4. $\mathrm{Var}(b) = 0$

2.3 Moment Generating Functions

Moments A moment of the random variable is denoted by

$$\mathbb{E}[X^r], \quad r = 1, 2, \ldots$$

Moments measure mean, variance, skewness, and kurtosis, all ways of looking at the shape of the distribution.

Moment Generating Function The moment generating function (MGF) of a r.v X is denoted by

$$M_X(u) := \mathbb{E}(e^{uX})$$

and we say that the MGF of X exists if $M_X(u)$ is finite in some interval containing zero.

The moment generating function of X exists if there exists $h > 0$ such that the $M_X(x)$ is finite for $x \in [-h, h]$.

Calculating Raw Moments Suppose the moment generating function of a r.v X exists then

$$\mathbb{E}(X^r) = \lim_{u \to 0} M_X^{(r)}(u) = \lim_{u \to 0} \frac{d^r}{du} M_X(u)$$

Equivalence of Moment Generating Functions Let X and Y be two r.vs such that the moment generating function of X and Y exists and $M_Y(u) = M_X(u)$ for all u in some interval containing zero then $F_X(x) = F_Y(x)$ for all $x \in \mathbb{R}$.

This theorem tells you that if the moment generating function exists then it uniquely characterises the cumulative distribution function of the random variable.

2.3.1 Useful Inequalities

The Markov Inequality (Chebychev's First Inequality) For any non-negative r.v X and $a > 0$,

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}$$

Chebychev's Second Inequality Suppose X is any r.v with $\mathbb{E}(X) = \mu, \text{Var}(X) = \sigma^2$ and $k > 0$ then

$$\mathbb{P}(|X - \mu| > k\sigma) \leq \frac{1}{k^2}$$

Convex (Concave) Functions A function h is convex (concave) if for any $\lambda \in [0, 1]$ and x_1 and x_2 in the domain of h, we have

$$h(\lambda x_1 + (1 - \lambda)x_2) \leq (\geq) \lambda h(x_1) + (1 - \lambda)h(x_2)$$

Jensen's Inequality Suppose h is a convex (concave) function and X is a r.v then

$$h(\mathbb{E}(X)) \leq (\geq) \mathbb{E}(h(X))$$

By using Jensen's inequality, one can show

Arithmetic Mean $\geq$ Geometric Mean $\geq$ Harmonic Mean.

That is given a sequence of number $(a_i)_{i=1,\ldots,n}$, we have

$$\frac{1}{n} \sum_{i=1}^{n} a_i \geq \left(\prod_{i=1}^{n} a_i \right)^{\frac{1}{n}} \geq n \left(\sum_{i=1}^{n} a_i^{-1} \right)^{-1}$$

3 Common Distributions

3.1 Common Discrete Distributions

Bernoulli Trail A Bernoulli trial is an experiment with two possible outcomes. The outcomes are often labelled 'success' and 'failure'. A Bernoulli trial defines a random variable X, given by

$$X = \begin{cases} 1 & \text{if the trail is a success} \\ 0 & \text{if the trail is a failure} \end{cases}$$

- Let $p \in [0, 1]$ be the probability of success
- We write $X \sim \text{Bernoulli}(p)$
- The probability function is given by $\mathbb{P}(X = 1) = p$ and $\mathbb{P}(X = 0) = 1 - p$
- $\mathbb{E}(X) = p$
- $\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = p(1 - p)$

Binomial Distribution Consider a sequence of n independent Bernoulli trials each with probability of success p. Let

$$X := \text{ total number of successes}$$

then X is a Binomial r.v with parameter n and p, and we write $X \sim \text{Bin}(n, p)$.

If $(Y_i)_{i=1,\ldots,n}$ is a sequence of independent Bernoulli(p) random variable then $X := \sum_{i=1}^{n} Y_i$ is $\text{Bin}(n, p)$. The expectation of a Binomial random variable.

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} \mathbb{E}(Y_i) = np$$

Poisson Distribution A r.v X is said to follow the Poisson distribution with parameter λ, if it's probability function is given

$$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \qquad k = 0, 1, \ldots$$

where $\lambda = \mathbb{E}(X) = \text{Var}(X)$.

Hypergeometric Distribution A random variable has hypergeometric distribution with parameter N, m, n and we write $X \sim \text{Hyp}(n, m, N)$ if

$$\mathbb{P}(X = x) = \frac{C_x^m C_{n-x}^{n-m}}{C_n^N} \qquad x = 1, \ldots, n$$

3.2 Continuous Distribution

Normal Random Variable A random variable X is said to be a normal random variable with parameters μ and σ^2 if its probability density function is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}$$

and we write $X \sim \mathcal{N}(\mu, \sigma^2)$.

Linear Transform Let X be a r.v with probability density function f_X, let $Y := a + bX$ then for $b > 0$ and $a \in \mathbb{R}$,

$$f_Y(x) = \frac{1}{b} f_X\left(\frac{x-a}{b}\right)$$

Linear Transform of Normally Distributed Random Variable Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$ and $a \in \mathbb{R}$ and $b > 0$. The random variable $Y := a + bX$ is also normally distributed with parameter $(a + b\mu, b^2\sigma^2)$, i.e. $Y \sim \mathcal{N}(a + b\mu, b^2\sigma^2)$.

Standardisation Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$ then

$$Z := \frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$$

Exponential Distribution A random variable X is said to be exponentially distributed with parameter $\lambda > 0$ if its probability density function is given by

$$f_X(x) = \frac{1}{\lambda} e^{-\frac{1}{\lambda}x}, \qquad x > 0$$

and we write $X \sim \exp(\lambda)$. Then $\mathbb{E}(x) = \lambda$ and $\mathrm{Var}(X) = \lambda^2$.

Gamma Distribution A random variable X is said to be Gamma distributed with parameter $\alpha, \beta > 0$ if its probability density function is given by

$$f_X(x; \alpha, \beta) = \frac{e^{\frac{-x}{\beta}} x^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha}, \qquad x > 0$$

and we write $X \sim \mathrm{Gamma}(\alpha, \beta)$ where $\mathbb{E}(X) = \alpha\beta$ and $\mathrm{Var}(X) = \alpha\beta^2$.

Beta Distribution The Beta function is given by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \qquad x, y > 0$$

and the Beta and Gamma functions satisfies the following relationship

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad x, y > 0$$

A random variable is said to follow a Beta distribution with parameters $\alpha, \beta > 0$ if its density function is given by

$$f_X(x; \alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}, \qquad x \in (0,1)$$

and we write $X \sim \text{Beta}(\alpha, \beta)$.

3.2.1 QQ-plot

Quantile Suppose X is a continuous random variable with CDF given by F_X. The $k\%$-th quantile of X is given by

$$Q_X(k) := F_X^{-1}(k), \qquad k \in [0,1]$$

where F_X^{-1} is the inverse of the CDF F_X.

Quantile Plot Given continuous r.vs X and Y, the theoretical qunatile plot of X against Y is the graph

$$(Q_X(k), Q_Y(k)), \qquad k \in [0,1]$$

Suppose we are given X and $Y = aX + b$ for some $a > 0, b \in \mathbb{R}$ then the quantile plot of X against Y is a straight line.

Given r.v.s X and Y and suppose that the quantile plot of X against Y is a straight line. Then the distribution of X is equal to the distribution of a linear transform of Y.

3.2.2 Indicator Functions

- A indicator function of a set A is defined by

$$I_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \in A^c \end{cases}$$

- Indicator function of an interval is given as

$$I_[a,b](x) = I_{\{a \leq x \leq b\}} \qquad \text{or} \qquad I_{(a,b]}(x) = I_{\{a < x \leq b\}}$$

- The indicator unifies expectation $\mathbb{E}$ and probability $\mathbb{P}$ notation since, the probability is the expectation of the indicator function. Therefore, it may be written that

$$\mathbb{P}(X \in A) = \int_A f_X(x)dx = \int_{-\infty}^{\infty} I_A(x) f_X(x) dx = \mathbb{E}(I_A(X)).$$

4 Bivariate Distribution

The joint density function of two continuous random variables X and Y is given by a bivariate function $f_{X,Y}$ with the properties

1. For all $x, y \in \mathbb{R}^2$, $f_{X,Y}(x,y) \geq 0$.
2. The double integral over $\mathbb{R}^2$ is equal to one, that is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dxdy = 1.$$

3. For any (measurable) set $A, B \in \mathbb{R}$

$$\int_B \int_A f_{X,Y}(x,y)\,dxdy = \mathbb{P}(X \in A, Y \in B).$$

Min and Max We write $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$.

Tonelli's Theorem Suppose $f : \mathbb{R}^2 \to \mathbb{R}_+$ then

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\,dxdy = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\,dydx$$

Fubini - Tonelli's Theorem Suppose $f : \mathbb{R}^2 \to \mathbb{R}$, if either

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |f(x,y)\,dxdy < \infty \text{ or } \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x,y)\,dydx < \infty$$

then

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\,dxdy = \int_{\mathbb{R}}\int \_\mathbb{R} f(x,y)\,dydx$$

Expected Value of Bounded Borel Functions For any (bounded Borel) function $g : \mathbb{R}^2 \to \mathbb{R}$ and random variables X and Y, then (given these integrals/sum are finite)

$$\mathbb{E}(g(X,Y)) = \begin{cases} \sum_{\forall x}\sum_{\forall x} g(x,y)\mathbb{P}(X = x, Y = y) & \text{discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dxdy, & \text{continuous} \end{cases}$$

Marginal Probability/Density Function The marginal densities are given by

$$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y)\,dy$$
$$f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x,y)\,dx$$

and similarly for discrete random variables X and Y.

$$\mathbb{P}(X = x) = \sum_y \mathbb{P}(X = x, Y = y)$$
$$\mathbb{P}(Y = y) = \sum_x \mathbb{P}(X = x, Y = y)$$

4.1 Independence

Independence Two random discrete variables X and Y are independent if for all outcomes x and y,

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y)$$

or if X are Y are continuous random variable with joint probability density $f_{X,Y}$ then

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

for all (x, y) in the domain of $f_{X,Y}$.

Independence - Generalised X and Y are independent if and only if for all $x, y \in \mathbb{R}^2$,

$$\mathbb{P}(X \leq x, Y \leq y) = F_X(x)F_Y(y)$$

and in general, for any bounded functions $g, f : \mathbb{R} \to \mathbb{R}$

$$\mathbb{E}(g(X)f(Y)) = \mathbb{E}(g(X))\mathbb{E}(f(Y))$$

4.2 Conditional Probability

Conditional Probability Suppose X and Y are

1. discrete random variables, then the conditional probability function of X are given the set $\{Y = y\}$ is given by

$$\mathbb{P}(X = x \mid Y = y) := \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)}$$

2. continuous random variables, then the conditional probability density function of X given the set Y is given by

$$f_{X|Y}(x \mid y) := \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

Multivariate Gaussian A random vector $X = (X_1, X_2)$ is said to be Gaussian with $\mu_X = (\mu_{X_1}, \mu_{X_2})$ and Covariance matrix V if

$$f_X(x) = \frac{1}{\sqrt{(2\pi)^d |V|}} \exp\left(-\frac{1}{2}(X - \mu_X)^T V^{-1}(X - \mu_X) \right).$$

Here $d = 2$ (dimension), V^{-1} is the matrix inverse of V and $|V|$ i the determinant of V.

Variance Matrix The variance matrix is a symmetric matrix with entries

$$V_{ij} = \mathrm{Cov}(X_i, X_j) \qquad \text{where } i = 1, \ldots, d \text{ and } j = 1, \ldots, d.$$

If $X = (X_1, X_2)$ is multivariate Gaussian then X_i for $i = 1, 2$ must be one-dimensional Gaussian but the converse is not true.

Conditional Expectations and Variance Given any bound (Borel) function g, the conditional expectation of $g(X)$ given the set $\{Y = y\}$ is

$$\mathbb{E}(g(X) \mid Y = y) = \begin{cases} \sum_x g(x)\mathbb{P}(X = x \mid Y = y) & \text{discrete} \\ \int_{-\infty}^{\infty} g(x) f_{X|Y}(x \mid y)\, dx & \text{continuous} \end{cases}$$

The conditional variance of X given the set $\{Y = y\}$ is

$$\operatorname{Var}(X \mid Y = y) = \mathbb{E}(X^2 \mid Y = y) - (\mathbb{E}(X \mid Y = y))^2$$

Independent Conditional Expectation and Variance Suppose the random variables X and Y are independent then

1. The conditional expectation of X given Y is simply the expectation of X,

$$\mathbb{E}(X \mid Y = y) = \mathbb{E}(X)$$

2. The conditional variance of X is simply the variance of X.

$$\operatorname{Var}(X \mid Y = y) = \operatorname{Var}(X)$$

Bounded Borel Conditional Expectation Given random variables X and Y a (bounded Borel) function $g : \mathbb{R}^2 \to \mathbb{R}$

$$\mathbb{E}(g(X, Y)) = \int_{\mathbb{R}} \mathbb{E}(g(X, y) \mid Y = y) f_Y(y)\, dy$$

where we define

$$\mathbb{E}(g(X, y) \mid Y = y) := \int_{\mathbb{R}} g(x, y) f_{X|Y}(x \mid y)\, dx$$

4.3 Covariance and Correlation

Covariance Given two random variables X and Y, the covariance of X and Y is given by

$$\operatorname{Cov}(X, Y) = \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y)))$$

Properties of Covariance The covaiance satisfies the following properties. For random variables X and Y

1. $\operatorname{Cov}(X, X) = \operatorname{Var}(X)$,
2. $\operatorname{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$,
3. if X and Y are independent then $\operatorname{Cov}(X, Y) = 0$
4. The covariance is symmetric, i.e. $\operatorname{Cov}(X, Y) = \operatorname{Cov}(Y, X)$.
5. The covariance is a bilinear function, i.e. for all $a, b \in \mathbb{R}$ and random variables X, Y and Z

$$\operatorname{Cov}(aX + bY, Z) = a\operatorname{Cov}(X, Z) + b\operatorname{Cov}(Y, Z)$$
$$\operatorname{Cov}(X, aY + bZ) = a\operatorname{Cov}(X, Y) + b\operatorname{Cov}(X, Z)$$

Correlation The correlation between two random variable X and Y is defined to be

$$\operatorname{Corr}(X,Y) := \frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}}$$

Properties of Correlation Given two random variable X and Y, the following property holds for the correlation function

1. $|\operatorname{Corr}(X,Y)| \leq 1$
2. $\operatorname{Corr}(X,Y) = -1$ iff there exists $a \in \mathbb{R}$ and $b < 0$ such that $\mathbb{P}(Y = a + bX) = 1$
3. $\operatorname{Corr}(X,Y) = 1$ iff there exists $a \in \mathbb{R}$ and $b < 0$ such that $\mathbb{P}(Y = a + bX) = 1$

4.4 Bivariate Transforms

Montone Probability Density Let X be a random variable with density f_X, if h is monotone over the set $\{x : f_X(x) > 0\}$ then the probability density of $Y := h(X)$ is given by

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = f_X \circ h^{-1}(y)\left|\frac{dh^{-1}(y)}{dy}\right|$$

CDF Strictly Increasing Suppose X has density f_X and its CDF F_X strictly increasing (once it is greater than zero) then $Y := F_X(X) \sim \text{Uniform}[0,1]$.

Bivariate Transforms Given random variable X and Y, suppose U and V are transforms of X and Y taking value in $\mathbb{R}$, then

$$f_{U,V}(u,v) = f_{X,Y}(x,y)|\det(J)|$$

where $\det(J)$ is the determintant of the Jacobian (of the inverse)

$$J = \begin{pmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{pmatrix}$$

5 Sum of Variables

Sum of Independent Random Variable If X and Y are independent discrete random variabels then

$$\mathbb{P}(X+Y=z) = \sum_y \mathbb{P}(X = z-y)\mathbb{P}(Y=y)$$

where the sum is taken over all possible outcomes of Y.

Sum of Independent Continuous Random Variables (Convolution formula) Suppose X and Y are independent continuous r.vs with density f_X and f_Y. Let $Z = X + Y$ then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy$$

Moment Generating Function Approach If X and Y are independent random variables for which the moment generating function exists then

$$M_{X+Y}(t) = M_X(t) M_Y(t)$$

In general if $(X_i)_i$ is an independent sequence of random variables, then

$$M_{\sum_{i=1}^n X_i}(t) = \prod_{i=1}^n M_{X_i}(t)$$

.

Useful Results Using the method of moment generating function method one can show the following. Suppose $(X_i)_{i=1,\ldots,n}$ be a =n independent identically distributed (iid) sequence of random variables and we set $Y := \sum_{i=1}^n X_i$ then if

- $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ then $Y \sim \mathcal{N}(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2$
- $X_i \sim \exp(\lambda)$ or $\text{Gamma}(1, \lambda)$ then $Y \sim \text{Gamma}(n, \lambda)$
- $X_i \sim \text{Gamma}(\alpha_i, \beta)$ then $Y \sim \text{Gamma}(\sum_{i=1}^n \alpha_i, \beta)$
- $X_i \sim \text{Poisson}(\lambda_i)$ then $Y \sim \text{Poisson}(\sum_{i=1}^n \lambda_i)$.
- $X_i \sim \text{Bernoulli}(p_i)$ then $Y \sim \text{Binomial}(n, p)$.
- $X_i \sim \text{Binomial}(n_i, p)$ then $Y \sim \text{Binomial}(\sum_{i=1}^n n_i, p)$

6 Central Limit Theorem

6.1 Central Limit Theorem

Central Limit Theorem Let $(X_n)_{n\in\mathbb{N}_+}$ be an independent identically distributed sequence of random variables with common mean $\mu = \mathbb{E}(X_1)$ and variance $\sigma^2 = \text{Var}(X_1) < \infty$. Let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ then

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} Z \sim \mathcal{N}(0,1)$$

6.2 Convergences

Convergence in Distribution Let $(X_i)_{i\in\mathbb{N}_+}$ be a sequence of random variables, we say that X_n converges to X in distribution if for all x, for which $F_X(x)$ is continuous

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x).$$

In this case, we write $X_n \xrightarrow{d} X$.

Convergence of Moment Generating Functions and Existence of CDF Let $(X_n)_{n\in\mathbb{N}_+}$ be a sequence of r.v each with moment generating function $M_{X_n}(t)$. Suppose that

$$M(t) = \lim_{n\to\infty} M_{X_n}(t)$$

exists then there exists an unique valid cumulative distribution function F and r.v X such that $F_X = F$.

Convergence of Random Variables A sequence of random variables $(X_n)_{n=1,\ldots,}$ converges in probability to a r.v X if for all $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \epsilon) = 0$$

and we write $X_n \xrightarrow{\mathbb{P}} X$.

Law of Large Numbers Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent r.vs with mean μ and finite variance σ^2, we set $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, then

$$\overline{X}_n \xrightarrow{\mathbb{P}} \mu$$

(Strong Version): Same Thing but using *almost surely* probability for convergence.

Equal Almost Surely Two random variables X and Y are said to be equal almost surely if $\mathbb{P}(Y = X) = 1$ and we write $X = Y$ a.s.

Almost Surely Convergence Given a random variable X, a sequence $(X_n)_{n\in\mathbb{N}}$ converges to almost surely to X, if

$$\mathbb{P}(\lim_{n\to 0} X_n = X) = 1$$

and we write $X \xrightarrow{a.s} X$.

Convergence in L^p A sequence of random variables $(X_i)_{i\in\mathbb{N}_+}$ is said to converge in L^p to another random variable X if for $p \geq 1$,

$$\lim_{n\to\infty} \mathbb{E}(|X_n - X|^p) = 0$$

in particular, if $p = 2$, we say that X_n converges to X in the mean square sense.

Convergence in L^p and Probability Suppose $(X_n)_{n\in\mathbb{N}}$ is a sequence of r.vs converging to X in L^p for $p \geq 1$, then X_n converges to X in probability.

$$X_n \xrightarrow{L^p} X \implies X_n \xrightarrow{\mathbb{P}} X$$

Convergence in Probability and Distribution Convergence in probability implies convergence in distribution. That is given X and a sequence $(X_n)_{n\in\mathbb{N}_+}$,

$$X_n \xrightarrow{a.s.} X \implies X_n \xrightarrow{\mathbb{P}} X \implies X_n \xrightarrow{d} X$$

Convergence Remark We have shown the following implications

$$X_n \xrightarrow{L^p} X \implies X_n \xrightarrow{\mathbb{P}} X \implies X_n \xrightarrow{d} X$$

6.3 Applications of the Central Limit Theorem

Normal approximation to Binomial Distribution Suppose $X \sim \text{Binomial}(n, p)$ then

$$\frac{X - np}{\sqrt{np(1-p)}} \xrightarrow{d} \mathcal{N}(0, 1)$$

Convergence to Constant in Distribution and Probability Suppose the sequence of r.vs $(X_n)_{n\in\mathbb{N}}$ converges to a constant c in distribution, then $(X_n)_{n\in\mathbb{N}}$ converges to a constant c in probability. That is

$$X_n \xrightarrow{d} c \implies X_n \xrightarrow{\mathbb{P}} c$$

Continuous Mapping Lemma Suppose $X_n \xrightarrow{\mathbb{P}} X$ in probability then for any continuous function, g, $g(X_n) \xrightarrow{\mathbb{P}} g(X)$.

Slutsky' Theorem Let $(X_n)_{n\in\mathbb{N}_+}$ be a sequence of r.vs converging to X in distribution and $(Y_i)_{i\in\mathbb{N}_+}$ is another sequence of r.vs that converges in probability to a constnat c, then

1. $X_n + Y_n \xrightarrow{d} X + c$
2. $X_n Y_n \xrightarrow{d} Xc$

6.4 Delta Method

Delta Method Let $\frac{(X_n - \theta)}{\sigma - \sqrt{n}} \xrightarrow{d} Z \sim \mathcal{N}(0, 1)$ and g is differentiable in a neighbourhood of θ and $g'(\theta) \neq 0$ then

$$\sqrt{n}(g(X_n) - g(\theta)) \xrightarrow{d} \mathcal{N}(0, \sigma^2[g'(\theta)]^2)$$

Extend Delta Method Let $\frac{(X_n-\theta)}{\sigma-\sqrt{n}} \xrightarrow{d} Z \sim \mathcal{N}(0,1)$ and g is k-times differentiable in a neighbourhood of θ and $g^{(r)}(\theta) = 0$ for all $r < k \in \mathbb{N}$ then

$$n^{\frac{k}{2}}(g(Y_n) - g(\theta)) \xleftarrow{d} \frac{1}{k!} g^{(k)}(\theta) Z^k$$

As a special case, for $k = 2$, we have that the limiting distribution is $\mathcal{X}^2$.

7 Statistical Inference

7.1 Data and Models

Samples and Data We have a sequence of (random) observations $(X_1, \ldots, X_n)$ which is called a set of random samples and $(x_1, \ldots, x_n)$ the sample data. The aim is usually to find appropriate models to describe this sequence of random observations.

Parametric Models and Space A parametric model for a random sample $(X_1, \ldots, X_n)$ is a family of probability/density functions $f(x : \theta)$ where $\theta \in \Theta$, where $\Theta \subset \mathbb{R}^d$ is called the parameter space.

7.2 Estimators

Estimators Suppose $(X_1, \ldots, X_n) \sim \{f_X(x; \theta), \theta \in \Theta\}$. An estimator of θ, denoted by $\hat{\theta}_n$ is any real valued function of $X_1, \ldots, X_n$, that is

$$\hat{\theta}_n = \hat{\theta}_n(X_1, \ldots, X_n) = g(X_1, \ldots, X_n)$$

where $g : \mathbb{R}^n \to \mathbb{R}$.

- An estimator of a parameter is a random variable! It is a function of the random variables $(X_1, \ldots, X_n)$.
- An estimator also has its own probability distribution and can be computed from the distribution of $(X_1, \ldots, X_n)$.

Bias Let $\hat{\theta}$ be an estimator of the parameter θ. The bias of the estimator $\hat{\theta}$s defined to be

$$\text{Bias}(\hat{\theta}) = \mathbb{E}(\hat{\theta}) - \theta.$$

If $\text{Bias}(\hat{\theta}) = 0$, then $\hat{\theta}$ is aid to be an unbiased estimator of θ.

Student t-distribution A random variable T is said to have t-distribution with degree of freedom ν, if its probability density function

$$f_T(x) = \frac{\Gamma(\frac{\nu}{2})}{\Gamma(\nu/2)\Gamma(1/2)} \nu^{-1/2} \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2}, \quad x \in (-\infty, \infty)$$