

Higher Theory and Applications of Differential Equations
MATH2221 UNSW

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Chapter 1

Linear ODEs

1.1 Introduction

Recall that a first-order ordinary differential equation (ODE) has, in its most general realisation, the form

$$y'(t) = f(t, y(t)).$$

A special case is the equation

$$a(t)y'(t) + b(t)y(t) = f(t),$$

with $a(t) \neq 0$ on some interval $I \in \mathbb{R}$. This special first-order ODE is called a **linear first-order ODE**. Another special case is

$$y'(t) = f(t)g(y),$$

which is known as a **separable first-order ODE**.

For a separable equation the solution is found (at least, implicitly by) writing:

$$\int \frac{1}{g(y)} dy = \int f(t) dt.$$

Solving Seperable ODEs Consider $y' = t^2y, y(0) = 3$. This is seperable with $f(t) = t^2$ and $g(y) = y$. Then

$$\int \frac{1}{y} dy = \int t^2 dt$$

so that

$$\ln |y(t)| = \frac{1}{3}t^3 + C.$$

Now apply e^t to both sides to obtain

$$|y(t)| = e^{\frac{1}{3}t^3+C} = e^C e^{\frac{1}{3}t^3}.$$

Thus, a general solution of the equation is

$$y(t) = Ae^{\frac{1}{3}t^3}.$$

Since $y(0) = 3$, we see that the unique solution is $y(t) = 3e^{\frac{1}{3}t^3}$.

In the case of a linear first-order equation, i.e. $y' + a(t)y = f(t)$, a useful solution method is the integrating factor technique. The idea is to find a function μ so that when we multiply both sides of the equation with μ we find that

$$[\mu y](t)' = \mu(t)(y' + a(t)y) = \mu(t)f(t),$$

for if this happens, then the general solution of the ODE should be

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)f(t) dt + \frac{C}{\mu(t)}.$$

Solving Linear First-Order ODE Solve $y' - 2ty = 3t$. We pick

$$\mu(t) = e^{\int -2t dt} = e^{-t^2}.$$

Then

$$\begin{aligned} (e^{-t^2}y)' &= 3te^{-t^2} \\ e^{-t^2}y &= \int 3te^{-t^2} dt = -\frac{3}{2}e^{-t^2} + C \\ y(t) &= -\frac{3}{2} + Ce^{t^2}. \end{aligned}$$

1.2 Linear Differential Operators

In linear algebra, you have seen the compact notation $A\mathbf{x} = \mathbf{b}$ for system of linear equations. A similar notation when dealing with a linear ordinary differential equations is

$$Lu = f.$$

Here, L is an operator (or transformation) that acts on a function u to create a new function Lu .

Given coefficients $a_0(x), a_1(x), \dots, a_m(x)$ we define the **linear differential operator** L of **order** m ,

$$\begin{aligned} Lu(x) &= \sum_{j=0}^m a_j(x) D^j u(x) \\ &= a_m D^m u + a_{m-1} D^{m-1} u + \dots + a_0 u, \end{aligned}$$

where $D^j u = d^j u / dx^j$ (with $D^0 u = u$).

We refer to a_m as the **leading coefficient** of L and assume that each $a_j(x)$ is a smooth function of x .

The ODE $Lu = f$ is said to be **singular** with respect to an interval $[a, b]$ if the leading coefficient $a_m(x)$ vanishes for any $x \in [a, b]$.

Example $Lu = (x - 3)u''' - (1 + \cos x)u' + 6u$ is a linear differential of order 3, with leading coefficient $x - 3$. Thus, L is singular on $[1, 4]$, but not singular on $[0, 2]$.

Example $N(u) = u'' + u^2 u' - u$ is a nonlinear differential operator of order 2.

Linearity For any constants c_1 and c_2 and any m -times differentiable functions u_1 and u_2 ,

$$L(c_1u_1 + c_2u_2) = c_1Lu_1 + c_2Lu_2.$$

Ordinary differential equations of the form $Lu = 0$ are known as **homogenous**. Those of the form $Lu = f$ are known as **inhomogeneous**.

When the solution to a differential equation is prescribed at a particular point $x = x_0$, that is

$$u(x_0) = v_0, \quad u'(x_0) = v_1, \quad \dots, \quad u^{(m-1)}(x_0) = v_{m-1},$$

we call it an **initial value problem**. Where a differential equation is order 2 or greater, solutions at 2 or more locations can be prescribed. Such problems are called **boundary value problems**.

Unique Solution to Linear Initial Problem For an ODE $Lu = f$ which is not singular with respect to a, b , with f continuous on $[a, b]$, the IVP for an m th-order linear differential operator with m initial values has a unique solution.

Solution to m th Order Problem has Dimension m Assume that the linear, m th-order differential operator L is not singular on $[a, b]$. Then the set of all solutions to the homogenous equation $Lu = 0$ on $[a, b]$ is a vector space of dimension m .

If $\{u_1, u_2, \dots, u_m\}$ is **any** basis for the solution space of $Lu = 0$, then every solution can be written in a unique way as

$$u(x) = c_1u_1(x) + c_2u_2(x) + \dots + c_mu_m(x) \quad \text{for } a \leq x \leq b.$$

We refer to this as the **general solution** of the homogenous equation $Lu = 0$ on $[a, b]$.

Linear superposition refers to this technique of constructing a new solution out of a linear combination of old ones.

Example The general solution to $u'' - u' - 2u = 0$ is $u(x) = c_1e^{-x} + c_2e^{2x}$.

Consider the inhomogeneous equation $Lu = f$ on $[a, b]$, and fix a particular solution u_P . For *any* solution u , the difference $u - u_P$ is a solution of the homogenous equation because

$$L(u - u_P) = Lu - Lu_P = f - f = 0 \text{ on } [a, b].$$

Hence, $u(x) - u_P(x) = c_1u_1(x) + \dots + c_mu_m(x)$ for some constants c_1, \dots, c_m and so

$$u(x) = u_P(x) + \underbrace{c_1u_1(x) + \dots + c_mu_m(x)}_{u_H(x)}, \quad a \leq x \leq b,$$

is the **general solution** of the inhomogeneous equation $Lu = f$.

Example The inhomogeneous ODE $u'' - u' - 2u = -2e^x$ has a particular solution $u_P(x) = e^x$. The general solution for its homogenous counterpart is $u_H(x) = c_1e^{-x} + c_2e^{2x}$. So the general solution of the inhomogeneous ODE is

$$u(x) = u_P(x) + u_H(x) = e^x + c_1e^{-x} + c_2e^{2x}.$$