Higher Theory and Applications of Differential Equations
MATH2221 UNSW

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Contents

Chapter 1

Linear ODEs

1.1 Introduction

Recall that a first-order ordinary differential equation (ODE) has, in its most general realisation, the form

$$y'(t) = f(t, y(t)).$$

A special case is the equation

$$a(t)y'(t) + b(t)y(t) = f(t),$$

with $a(t) \neq 0$ on some interval $I \in \mathbb{R}$. This special first-order ODE is called a **linear first-order ODE**. Another special case is

$$y'(t) = f(t)g(y),$$

which is known as a **separable first-order ODE**.

For a separable equation the solution is found (at least, implicitly by) writing:

$$\int \frac{1}{g(y)}\, dy = \int f(t)\, dt.$$

Solving Seperable ODEs Consider $y' = t^2 y, y(0) = 3$. This is seperable with $f(t) = t^2$ and $g(y) = y$. Then

$$\int \frac{1}{y}\, dy = \int t^2\, dt$$

so that

$$\ln |y(t)| = \frac{1}{3}t^3 + C.$$

Now apply e^t to both sides to obtain

$$|y(t)| = e^{\frac{1}{3}t^3 + C} = e^C e^{\frac{1}{3}t^3}.$$

Thus, a general solution of the equation is

$$y(t) = Ae^{\frac{1}{3}t^3}.$$

Since $y(0) = 3$, we see that the unique solution is $y(t) = 3e^{\frac{1}{3}t^3}$.

In the case of a linear first-order equation, i.e. $y' + a(t)y = f(t)$, a useful solution method is the integrating factor technique. The idea is to find a function μ so that when we multiply both sides of the equation with μ we find that

$$[\mu y](t)' = \mu(t)(y' + a(t)y) = \mu(t)f(t),$$

for if this happens, then the general solution of the ODE should be

$$y(t) = \frac{1}{\mu(t)}\int \mu(t)f(t)\,dt + \frac{C}{\mu(t)}.$$

Solving Linear First-Order ODE Solve $y' - 2ty = 3t$. We pick

$$\mu(t) = e^{\int -2t\,dt} = e^{-t^2}.$$

Then

$$\begin{aligned}
(e^{-t^2}y)' &= 3te^{-t^2} \\
e^{-t^2}y &= \int 3te^{-t^2}\,dt = -\frac{3}{2}e^{-t^2} + C \\
y(t) &= -\frac{3}{2} + Ce^{t^2}.
\end{aligned}$$

1.2 Linear Differential Operators

In linear algebra, you have seen the compact notation $A\mathbf{x} = \mathbf{b}$ for system of linear equations. A similar notation when dealing with a linear ordinary differential equations is

$$Lu = f.$$

Here, L is an operator (or transformation) that acts on a function u to create a new function Lu. Given coefficients $a_0(x), a_1(x), \ldots, a_m(x)$ we define the **linear differential operator** L of **order** m,

$$\begin{aligned}
Lu(x) &= \sum_{j=0}^{m} a_j(x)D^j u(x) \\
&= a_m D^m u + a_{m-1}D^{m-1}u + \cdots + a_o u,
\end{aligned}$$

where $D^j u = d^j u/dx^j$ (with $D^0 u = u$).

We refer to a_m as the **leading coefficient** of L and assume that each $a_j(x)$ is a smooth function of x.

The ODE $Lu = f$ is said to be **singular** with respect to an interval $[a, b]$ if the leading coefficient $a_m(x)$ vanishes for any $x \in [a, b]$.

Example $Lu = (x-3)u''' - (1 + \cos x)u' + 6u$ is a linear differential of order 3, with leading coefficient $x - 3$. Thus, L is singular on $[1, 4]$, but not singular on $[0, 2]$.

Example $N(u) = u'' + u^2u' - u$ is a nonlinear differential operator of order 2.

Linearity For any constants c_1 and c_2 and any m-times differentiable functions u_1 and u_2,

$$L(c_1u_1 + c_2u_2) = c_1Lu_1 + c_2Lu_2.$$

Ordinary differential equations of the form $Lu = 0$ are known as **homogenous**. Those of the form $Lu = f$ are known as **inhomogeneous**.

When the solution to a differential equation is prescribed at a particular point $x = x_0$, that is

$$u(x_0) = v_0, \quad u'(x_0) = v_1, \quad \dots, \quad u^{(m-1)}(x_0) = v_{m-1},$$

we call it an **initial value problem**. Where a differential equation is order 2 or greater, solutions at 2 or more locations can be prescribed. Such problems are called **boundary value problems**.

Unique Solution to Linear Initial Problem For an ODE $Lu = f$ which is not singular with repsect to a, b, with f continuous on $[a, b]$, the IVP for an mth-order linear differential operator with m inital values has a unique solution.

Solution to mth Order Problem has Dimension m Assume that the linear, mth-order differential operator L is not singular on $[a, b]$. Then the set of all solutions to the homogenous equation $Lu = 0$ on $[a, b]$ is a vector space of dimension m.

If $\{u_1, u_2, \dots, u_m\}$ is **any** basis for the solution space of $Lu = 0$, then every solution can be written in a unique way as

$$u(x) = c_1u_1(x) + c_2u_2(x) + \cdots + c_mu_m(x) \quad \text{for } a \le x \le 4.$$

We refer to this as the **general solution** of the homogenous equation $Lu = 0$ on $[a, b]$.

Linear superposition refers to this technique of constructing a new solution out of a linear combination of old ones.

Example The general solution to $u'' - u' - 2u = 0$ is $u(x) = c_1e^{-x} + c_2e^{2x}$.

Consider the inhomogeneous equation $Lu = f$ on $[a, b]$, and fix a particular solution u_P.
For *any* solution u, the difference $u - u_P$ is a solution of the homogenous equation because

$$L(u - u_P) = Lu - Lu_P = f - f = 0 \text{ on } [a, b].$$

Hence, $u(x) - u_P(x) = c_1u_1(x) + \dots_c mu_m(x)$ for some constants $c_1, \dots, c_m$ and so

$$u(x) = u_P(x) + \underbrace{c_1u_1(x) + \cdots + c_mu_m(x)}_{u_H(x)}, \quad a \le x \le b,$$

is the **general solution** of the inhomogeneous equation $Lu = f$.

Example The inhomogenous ODE $u'' - u' - 2u = -2e^x$ has a particular solution $u_P(x) = e^x$.
The general solution for its homogenous counterpart is $u_H(x) = c_1e^{-x} + c_2e^{2x}$.
So the general solution of the inhomogeneous ODE is

$$u(x) = u_P(x) + u_H(x) = e^x + c_1e^{-x} + c_2e^{2x}.$$

Reduction of Order For $u = u_1(x) \neq 0$, a solution to the ODE

$$u'' + p(x)u' + q(x)u = 0,$$

on some interval I, then a second solution is

$$u = u_1(x) \int \frac{1}{u_1^2 \exp\left(\int p\, dx\right)}\, dx.$$

Example For the ODE $u'' - 6u' + 9u = 0$, take $u_1 = e^{3x}$ and find v. **Answer** xe^{3x}.

1.3 Differential Operators with Constant Coefficients

If L has constant coefficients, then the problem of solving $Lu = 0$ reduces to that of factorising the polynomial having the same coefficients.

Suppose that a_j is constant for $0 \leq j \leq m$, with $a_m \neq 0$. We define the associated polynomial of degree m,

$$p(z) = \sum_{j=0}^{m} a_j z^j = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0,$$

so that if

$$Lu = a_m u^{(m)} + a_{m-1} u^{(m-1)+\cdots+a_1 u + a_0},$$

then formally, $L = p(D)$.

By the fundamental theorem of algebra,

$$p(z) = a_m (z - \lambda_1)^{k_1} (z - \lambda_2)^{k_2} \cdots (z - \lambda)^{k_r}$$

where $\lambda_1, \lambda_@, \ldots, \lambda_r$ satisfying

$$k_1 + k_2 + \cdots + k_r = m.$$

Lemma $(D - \lambda) x^j e^{\lambda x} = j x^{j-1} e^{\lambda x}$ for $j \geq 0$.

Lemma $(D - \lambda)^k x^j e^{\lambda x} = 0$ for $j = 0, 1, \ldots, k - 1$.

Basic Solutions If $(z - \lambda)^k$ is a factor of $p(z)$ then the function $u(x) = x^j e^{\lambda x}$ is a solution of $Lu = 0$ for $0 \leq j \leq k - 1$.

General Solution For the constant-coefficient case, the general solution of the homogenous equation $Lu = 0$ is

$$u(x) = \sum_{q=1}^{r} \sum_{l=0}^{k_q - 1} c_{ql} x^l e^{\lambda_q x},$$

where the c_{ql} are arbitrary constants.

Repeated Real Root From the factorisation

$$D^4 + 6D^3 + 9D^2 - 4D - 12 = (D-1)(D+2)^2(D+3)$$

we see that the general solution of

$$u'''' + 6u''' + 9'' - 4u' - 12u = 0$$

is

$$u = c_1e^x + c_2e^{-2x} + c_3xe^{-2x} + c_4e^{-3x}.$$

Complex Root From the factorisation

$$\begin{aligned} D^3 - 7D^2 + 17D - 15 &= (D^2 - 4D + 5)(D-3) \\ &= (D-2-i)(D-2+i)(D-3) \end{aligned}$$

we see that the general solution of

$$u''' - 7u'' + 17u' - 15u = 0$$

is

$$\begin{aligned} u(x) &= c_1e^{(2+i)x} + c_2e^{(2-i)x} + c_3e^{3x} \\ &= c_4e^{2x}\cos x + c_5e^{2x}\sin x + c_3e^{3x}. \end{aligned}$$

Second-order ODEs arise naturally in classical mechanics for example a harmonic simple oscillator.

1.4 Wronskians and Linear Independence

We introduce a function, called the Wronskain that provides us with a way of testing whether a family of solutions to $Lu = 0$ is linearly independent.

Let $u_1(x), u_2(x), \dots, u_m(x)$ be functions defined on an interval $I \in \mathbb{R}$. The functions $u_1, \dots, u_m$ are called **linearly dependent** if there exist constant $a_1, a_2, \dots, a_m$ **not all zero** such that

$$a_1u_1(x) + a_2u_2(x) + \cdots + amu_m(x) = 0 \quad \forall x \in I.$$

If the above equation only holds for

$$a_i = 0, \quad i = 1, 2, \dots, m$$

then the functions are **linearly independent**.

Example $u_1 = \sin 2x$ and $u_2 = \sin x \cos x$ are linearly dependent.
$u_1 = \sin x$ and $u_2 = \cos x$ are linearly indepdent.

The **Wronskian** of the functions $u_1, u_2, \dots, u_m$ is the $m \times m$ determinant

$$W(x) = W(x; u_1, u_2, \dots, u_m) = \det\big[D^{i-1}u_j\big].$$

Example The Wronskian of the functions $u_1 = e^{2x}, u_2 = xe^{2x}$ and $u_3 = e^{-x}$ is

$$W = \begin{vmatrix} e^{2x} & xe^{2x} & e^{-x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} & -e^{-x} \\ 4e^{2x} & 4e^{2x} + 4xe^{2x} & e^{-x} \end{vmatrix} = 9e^{3x}.$$

Lemma If $u_1, \ldots, u_m$ are linearly dependent over an interval $[a, b]$ then $W(x; u_1, \ldots, u_m) = 0$ for $a \leq x \leq b$.

Lemma If $u_1, u_2, \ldots, u_m$ are solutions of $Lu = 0$ on the interval $[a, b]$ then their Wronskain satisfies

$$a_m(x)W'(x) + a_{m-1}(x)W(x) = 0, \quad a \leq x \leq b.$$

Linear Independence of Solutions Let $u_1, u_2, \ldots, u_m$ be solutions of a non-singular, linear, homogenous, m-th order ODE $Lu = 0$ on the interval $[a, b]$.
Either
$W(x) = 0$ for $a \leq x \leq b$ and the m solutions are linearly **dependent**,
or else
$W(x) \neq 0$ for $a \leq x \leq b$ and the m solutions are linearly **independent**.

1.5 Methods for Inhomogeneous Equations

1.5.1 Judicious Guessing Method

You would have learned the mthod of undetermined coefficients for constructing a particular solution u_P to an inhomogeneous second-order linear ODE $Lu = f$ in some simple cases. We will study this method systematically for higher-order linear ODEs with constant coefficients.

Superposition of Solutions Suppose that u_1 solves $Lu = e^{3x}$, and u_2 solves $Lu = \sin x$, where L is a linear differential operator. Then the solution of

$$Lu = e^{3x} + \sin x$$

is

$$u(x) = u_1(x) + u_2(x).$$

And a solution of

$$Lu = \frac{1}{2}e^{3x} - 5\sin x$$

is

$$u(x) = \frac{1}{2}u_1(x) - 5u_2(x).$$

Now we want to investigate some methods for finding particular solutions - i.e., finding a solution of $Lu = f$. One such method is the method of judicious guessing. For example:

1. If f is a polynomial, then guess that u_p is a polynomial.

2. If f is a exponential, then guess that u_p is exponential.

3. If f is a sine or cosine, then guess that u_p is a combination of such functions.

One problem with this method: it will only work for the types of functions identified above.

Example Suppose that $u'' - u' = t^2 + 2t$. Note as before that,

$$u_h(t) = c_1 + c_2 e^t.$$

So guess,

$$u_p(t) = At^3 + Bt^2 + Ct + D.$$

Then

$$t^2 + 2t = u_p'' - u' = -3At^2 + (6A - 2B)t + (2B - C).$$

So, equating coefficients of like power terms, we see that

$$A = -\frac{1}{3}, B = -2, C = -4, \text{ and } D \text{ is unrestricted.}$$

Therefore, reabsorbing D into c_1, we see that

$$u(t) = u_h(t) + u_p(t) = c_1 + c_2 e^t - \frac{1}{3}t^3 - 2t^2 - 4t.$$

Now we look at this idea of judicious guessing in a more systematic way. Let $L = p(D)$ be a linear differential operator of order m with constant coefficients.

Polynomial Solutions Assume that $a_0 = p(0) \neq 0$. For any integer $r \geq 0$, there exists a unique polynomial u_P of degree r such that $Lu_P = x^r$.

Exponential Solutions Let $L = p(D), M \in \mathbb{R}$ and $\mu \in \mathbb{C}$. If $p(\mu) \neq 0$, then the function

$$u_P(x) = \frac{Me^{\mu x}}{p(\mu)}$$

satisfies $Lu_P = Me^{\mu x}$.

Example A particular solution of $u'' + 4u' - 3i = 3e^{2x}$ is $u_P = e^{2x}/3$.

Product of Polynomial and Exponential Let $L = p(D)$ and assume that $p(\mu) \neq 0$. For any integer $r \geq 0$, there exists a unique polynomial v of degree r such that $u_P = v(x)e^{\mu x}$ satisfies $Lu_P = x^r e^{\mu x}$.

1.5.2 Annihilator Method

In the previous cases we proposed a solution $u = u_P$ and showed that it satisfied $Lu = f$. The following is a method to derive a particular solution given $Lu = f$. If $f(x)$ is differentiable at least n times and

$$[a_n D^n + a_{n-1} D^{n-1} + \cdots a_1 D^1 + a_0] f(x) = 0$$

then $[a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D^1 + a_0]$ **annihilates** f.

Example D^n annhihilates x^{m-1} for $m \leq n$.
$(D-\alpha)^n$ annihilates $x^{m-1}e^{\alpha x}$ for $m \leq n$.

Annhilator Method: Simple Example Given $Lu = f$ we can apply the appropriate annhiliator to both sides and solving the resutling homogenous DE.
Let $Lu = u'' - u'$ and suppose we want a solution such that $Lu = x^2$. Annihilating both sides we have

$$D^3(u'' - u') = u^{(5)} - u^{(4)} = 0.$$

Setting $w = u^{(4)}$, clearly $w = Ce^x$ is the general solution. Integrating four times yields

$$u = Ce^x + Ex^3 + Fx^2 + Gx + H.$$

Clearly $u_h = Ae^x + H$ and the form of the particular solution is $u_P = x(Ex^2 + Fx + G$. Substituting find $E = -1/3, F = -1$ and $G = -2$.

1.5.3 Judicious Guessing Method Continued

Polynomial Solutions: The Reamining Case Let $L = p(D)$ and assume $p(0) = p'(0) = \cdots = p^{(k-1)}(0) = 0$ but $p^{(k)}(0) \neq 0$ where $1 \leq k \leq m-1$. For any integer $r \geq 0$, there exists a unique polynomial v of degree $r$such that $u_P(x) = x^K v(x)$ satisfies $Lu_P = x^r$.

Exponential Times Polynomial: Remaining Case Let $L = p(D)$ and assume $p(\mu) = p'(\mu) = \cdots = p^{(k-1)}(\mu) = 0$. But $p^{(k)}(\mu) \neq 0$, where $1 \leq k \leq m-1$. For any integer $r \geq 0$, there exists a unique polynomial v of degree r such that $u_P(x) = x^k v(x) e^{\mu x}$ satisfies $Lu_P = x^r e^{\mu x}$.

1.5.4 Variation of Parameters

Example Find the general solution to $u'' - 4u' + 4u = (x+1)\exp 2x$.
Note first that the general solution, u_h, to $u'' - 4u' + 4u = 0$ is

$$u(x) = c_1 e^{2x} + c_2 x e^{2x}$$

since the characteristic equation is $0 = r^2 - 4r + 4 = (r-2)^2$. Then

$$W(x) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x} + 2xe^{4x} - 2xe^{4x} = e^{4x}.$$

So by the method of variation of parameters:

$$v_1'(x) = e^{-4x} \cdot -xe^{2x}(x+1)e^{2x} \text{ and } v_2'(x) = e^{-4x} \cdot e^{2x}(x+1)e^{2x}.$$

In other words,

$$v_1'(x) = -x^2 - x \text{ and } v_2'(x) = x + 1.$$

Therefore $u(x) = c_1 e^{2x} + c_2 x e^{2x} - (\frac{1}{3}x^3 + \frac{1}{2}x^2)e^{2x} + (\frac{1}{2}x^2 + x)xe^{2x}$.

1.6 Solution via Power Series

General Case Consider a general second-order, linear, homogenous ODE

$$Lu = a_2(x)u'' + a_1(x)u' + a_0(x)u = 0.$$

Equivalently,

$$u'' + p(x)u' + q(x)u = 0,$$

where

$$p(x) = \frac{a_1(x)}{a_2(x)} \text{ and } q(x) = \frac{a_0(x)}{a_2(x)}.$$

Assume that a_j is **analytic** at 0 for $0 \leq j \leq 2$. Then p and q are analytic at 0, that is, they admit power series expansions

$$p(z) = \sum_{k=0}^{\infty} p_k z^k \text{ and } q(z) = \sum_{k=0}^{\infty} q_k z^k \text{ for } |z| < \rho,$$

for some $\rho > 0$.

Convergence Theorem If the coefficients $p(z)$ and $q(z)$ are analytic for $|z| < \rho$, then the formal power series for the solution $u(z)$, constructed above, is also analytic for $|z| < \rho$.

Power Series at Zero Consider

$$Lu = (1 - x^2)u'' - 5xu' - 4u = 0, \quad u(0) = 1, \quad u'(0) = 2.$$

In this case,

$$p(z) = \frac{-5z}{1 - z^2} = -5\sum_{k=0}^{\infty} z^{2k+1} \text{ and } q(z) = \frac{-4}{1 - z^2} = -4\sum_{k=0}^{\infty} z^{2k}$$

are analytic for $|z| < 1$, so the theorem guarantees that $u(z)$, given by the formal power series, is also analytic for $|z| < 1$.

Expansion about a Point other than Zero Suppose we want a power series expansion about a point $c \neq 0$, for instance because the initial conditions are given at $x = c$.
A simple change of the independent variable allows us to write

$$u = \sum_{k=0}^{\infty} A_k(z - c)^k = \sum_{k=0}^{\infty} A_k Z^k \text{ where } Z = z - c.$$

Since $du/dx = du/dZ$ and $d^2u/dz^2 = d^2u/dZ^2$, we obtain the translated equation

$$\frac{d^2u}{dZ^2} + p(Z + c)\frac{du}{dZ} + q(Z + c)u = 0.$$

Now compute that A_k using the series expansions of $p(Z + c)$ and $q(Z + c)$ in powers of Z.

1.7 Singular ODEs

In general, we do not want L to be singular on an interval for which we wish to solve $Lu = f$. However, some important applications lead to singular ODEs so we now address this case.

A second-order **Euler-Cauchy ODE** has the form

$$Lu = ax^2u'' + bxu' + cu = f(x),$$

where a, b and c are constants with $a \neq 0$. This ODE is singular at $x = 0$.
Noticing that

$$Lx^r = [ar(r-1) + br + c]x^r,$$

we see that $u = x^r$ is a solution of the homogenous equation ($f = 0$) iff

$$ar(r-1) + br + c = 0.$$

Factorisation Suppose $ar(r-1) + br + c = a(r - r_1)(r - r_2)$. If $r_1 \neq r_2$ then the general solution of the homogenous equation $Lu = 0$ is

$$u(x) = C_1x^{r_1} + C_2x^{r_2}, \quad x > 0.$$

Lemma If $r_1 = r_2$ then the general solution of the homogenous Euler-Cauchy equation $Lu = 0$ is

$$u(x) = C_1x^{r_1} + C_2x^{r_1} \ln x, \quad x > 0.$$

Euler-Cauchy Equations with Nonreal Indicial Roots Suppose that $r_{1,2} = \alpha \pm \beta i$ are the roots of the indicial equation

$$ar(r-1) + br + c = 0$$

associated to the Euler-Cauchy equation

$$at^2u'' + btu' + cu = 0.$$

Then the real-valued solutions can be derived as follows. First note that

$$t^{\alpha+\beta i} = t^\alpha t^{\beta i}$$

is a solution. Then notice that

$$t^{\beta i} = e^{\ln t^{\beta i}} = e^{i \ln t^\beta} = \cos\big(\ln\big(t^\beta\big)\big) + i \sin\big(\ln\big(t^\beta\big)\big).$$

So,

$$t^\alpha t^{\beta i} = t^\alpha e^{\ln t^{\beta i}} = t^\alpha e^{i \ln t^\beta} = t^\alpha \left(\cos\big(\ln\big(t^\beta\big)\big) + i \sin\big(\ln\big(t^\beta\big)\big)\right)$$

is a solution. Finally, since each of the real part an the imaginary part is .separately a (linear independent) solution, we see that the general solution in this case is (for $t > 0$)

$$u(t) = t^\alpha \left(c_1 \cos\big(\ln\big(t^\beta\big)\big) + i \sin\big(\ln\big(t^\beta\big)\big)\right).$$

Example Consider $t^2u'' - tu' + 5u = 0$. Then the indicial equation is

$$r(r-1) - r + 5 = 0 \implies r = 1 \pm 2i.$$

So the general solution is,

$$u(t) = t(c_1 \cos \ln t^2 + c_2 \sin \ln t^2).$$

A number of important applications lead to ODEs that can be written in the **Frobenious normal form**

$$z^2u'' + zP(z)u' + Q(z)u = 0,$$

where $P(z)$ and $Q(z)$ are analytic at $z=0$:

$$P(z)=\sum_{k=0}^{\infty} P_k z^k \text{ and } Q(z)=\sum_{k=0}^{\infty} Q_k z^k, \quad |z|<\rho.$$

Now consider $z^2u''+zP(z)u'+Q(z)u=0$. FOrmal manipulations show that $Lu(z)$ equals

$$I(r)A_0z^r+\sum_{k=1}^{\infty}\left(I(k+r)A_k+\sum_{j=0}^{k-1}\left[(j+r)P_{k-j}+Q_{k-j}\right]A_j\right)z^{k+r},$$

where $I(r)$ is the indicial polynomial $I(r):=r(r-1)P_0r+Q_0$, so we define $A_0(r)=1$ and

$$A_k(r)=\frac{-1}{I(k+r)}\sum_{j=0}^{k-1}[(j+r)P_{k-j}+Q_{k-j}]A_j(r), \quad k\geq 1,$$

provided $I(k+r)\neq 0$ for all $k\geq 1$.

1.8 Bessel and Legendre Equations

1.8.1 Bessel Equations and Functions

The **Bessel equation with parameter** ν is

$$z^2u''+zu'+(z^2-\nu^2)u=0.$$

This ODE is in Frobenius normal form, with indicial polynomial $I(r)=(r+\nu)(r-\nu)$, and we seek a series solution

$$u(z)=\sum_{k=0}^{\infty}A_kz^{k+r}.$$

We assume $\operatorname{Re}\nu\geq 0$, so $r_1=\nu$ and $r_2=-\nu$.
With the normalisation

$$A_0=\frac{1}{2^{\nu}\Gamma(1+\nu)}$$

the series solution is called the **Bessel function of order** ν and is denoted

$$J_\nu(z)=\frac{(z/2)^\nu}{\Gamma(1+\nu)}\left[1-\frac{(z/2)^2}{1+\nu}+\frac{(z/2)^4}{2!(1+\nu)(2+v)}-\cdots\right].$$

From the functional equation $\Gamma(1+z)=z\Gamma(z)$ we see that

$$J_\nu(z)=\frac{(z/2)^\nu}{\Gamma(1+v)}-\frac{(z/2)^{\nu+2}}{\Gamma(2+\nu)}+\frac{(z/2)^{\nu+4}}{2!\Gamma(3+v)}-\frac{(z/2)^{\nu+6}}{3!\Gamma(4+\nu)}+\cdots$$

and so

$$J_\nu(z)=\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{2k+\nu}}{k!\Gamma(k+1+\nu)}.$$

1.8.2 Legendre Equation

The **Legendre equation** with parameter nu is

$$(1-z^2)u'' - 2zu' + \nu(\nu+1)u = 0.$$

This ODE is not singular at $z=0$ so the solution has an ordinary Taylor series expansion

$$u = \sum_{k=0}^{\infty} A_k z^k.$$

The A_k must satisfy

$$(k+1)(k+2)A_{k+2} - [k(k+1) - \nu(\nu+1)A_k] = 0$$

for $k \geq 0$, and since

$$k(k+1) - \nu(\nu+1) = (k-\nu)(k+\nu+1),$$

the recurrence relation is

$$A_{k+1} = \frac{(k-\nu)(k+\nu+1)}{(k+1)(k+2)} A_k \text{ for } k \geq 0.$$

We have

$$u(z) = A_0 u_0(z) + A_1 u_1(z)$$

where

$$u_0(z) = 1 - \frac{\nu(\nu+1)}{2!} z^2 + \frac{(\nu-2)\nu(\nu+1)(\nu+3)}{4!} z^4 - \cdots$$

and

$$u_1(z) = z - \frac{(\nu-1)(\nu+2)}{3!} z^3 + \frac{(\nu-3)(\nu-1)(\nu+2)(\nu+4)}{5!} z^5 - \cdots.$$

Suppose now that $\nu = n$ is a non-negative integer. If n is even the series for $u_0(z)$ terminates, whereas if n is odd then the series for $u_1(z)$ terminates.

The terminating solution is called the **Legendre polynomial** of degree n and is denoted by $P_n(z)$ with the normalization

$$P_n(1) = 1.$$

Lengdre Polynomials The first few Legendre polynomials are

$$\begin{aligned} P_0(z) &= 1, & P_3(z) &= \frac{1}{2}(5z^3 - 3z), \\ P_1(z) &= z, & P_4(z) &= \frac{1}{8}(35z^4 - 30z^2 + 3), \\ P_2(z) &= \frac{1}{2}(3z^2 - 1), & P_5(z) &= \frac{1}{8}(63z^5 - 70z^3 + 15z). \end{aligned}$$

Notice that P_n is an even or odd function according to whether n is even or odd.

Chapter 2

Dynamical Systems

2.1 Terminology

We begin with some examples of how systems of differential equations arise in applications, and see how all such problems can be formulas as a **first-order** system

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}).$$

Such a formulation leads to a natural geometric interpretation of a solution.

Lotka-Volterra Equations Simplified ecology with two species:

$$\begin{aligned} F(t) &= \text{ number of foxes at time t}, \\ R(t) &= \text{ number of rabbits at time t}. \end{aligned}$$

Assume populations large enough at F and R can be treated as smoothly varying in time.
In the 1920s, Alfred Lotka and Vito Volterra independently proposed the predator-prey model

$$\begin{aligned} \frac{dF}{dt} &= -aF + \alpha FR, \quad F(0) = F_0, \\ \frac{dR}{dt} &= bR - \beta FR, \qquad R(0) = R_0. \end{aligned}$$

Here a, α, b and β are non-negative constants.

Any first-order system for N ODEs in the form

$$\begin{aligned} \frac{dx}{dy} &= F_1(x, y, \ldots, x_N), \qquad x(0) = x_{10}, \\ \frac{dy}{dt} &= F_2(x, y, \ldots, x_N), \qquad y(0) = x_{20}, \\ &\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ \frac{dx_N}{dt} &= F_N(x, y, dots, x_N), \quad x_N(0) = x_{N0}, \end{aligned}$$

can be written in vector notation as

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) \quad \mathbf{x}(0) = \mathbf{x}_0.$$

The system of ODEs is determined by the **vector field** $\mathbf{F} : \mathbb{R}^N \to \mathbb{R}^N$.
A system of ODEs of the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x})$$

is said to be **autonomous**.
In a **non-autonomous** system, $\mathbf{F}$ will depend explicitly on t:

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t).$$

It can be shown that it is sufficient (in principle) to develop theory for the autonomous case as a non-autonomous system can be converted into an autonomous system.

Second-order ODE Consider an initial-value problem for a general (possibly non-autonomous) second-order ODE

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}, t\right), \text{ with } x = x_0 \text{ and } \frac{dx}{dt} = y_0 \text{ at } t = 0.$$

If $x = x(t)$ is a solution, and if we let $y = dx/dt$, then

$$\frac{dy}{dt} = \frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}, t\right) = f(x, y, t),$$

that is, (x, y) is a solution of the first-order system

$$\begin{aligned} \frac{dx}{dt} &= y, & x(0) &= x_0, \\ \frac{dy}{dt} &= f(x, y, t) & y(0) &= y_0. \end{aligned}$$

2.2 Existence and Uniqueness

The most fundamental question about a dynamical system

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t)$$

is

> For a given initial value $\mathbf{x}_0$, does a solution $\mathbf{x}(t)$ satisfying $\mathbf{x}(0) = \mathbf{x}_0$ exist, and if so is this solution unique?

Answer is **yes**, whenever the vector field $\mathbf{F}$ is **Lipschitz**.

The number L is a **Lipschitz constant** for a function $f : [a, b] \to \mathbb{R}$ if

$$|f(x) - f(y)| \leq L|x - y| \quad \text{for all } x, y \in [a, b].$$

Example Consider $f(x) = 2x^2 - x + 1$ for $0 \leq x \leq 1$. Since

$$\begin{aligned} f(x) - f(y) &= 2(x^2 - y^2) - (x - y) = 2(x + y)(x - y) - (x - y) \\ &(2x + 2y - 1)(x - y) \end{aligned}$$

we have $|f(x) - f(y)| = |2x + 2y - 1||x - y|$ so a Lipschitz constant is

$$L = \max_{x,y \in [0,1]} |2x + 2y - 1| = 3.$$

We say that the function $f : [a, b] \to \mathbb{R}$ is Lipschitz if a Lipschitz constant for f exists.

Lipschitz Continuity If f is Lipshictz then f is (uniformly) continuous.

Continous does not imply Lipschitz Consider the (uniformly) continous function

$$f(x) = 3 + \sqrt{x} \text{ for } 0 \leq x \leq 4.$$

In this case, if $x, y \in (0, 4]$ then

$$f(x) - f(y) = \sqrt{x} - \sqrt{y} = \left(\sqrt{x} - \sqrt{y} \times \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right) \qquad = \frac{x - y}{\sqrt{x} + \sqrt{y}}$$

so if a Lipschitz constant L exists then

$$L \geq \frac{|f(x) - f(y)|}{|x - y|} = \frac{1}{\sqrt{x} + \sqrt{y}}$$

for arbitrarily small x and y, a contradiction.

A function $f : I \to \mathbb{R}$ is C^k if $f, f', f'', \ldots, f^{(k)}$ all exist and are continuos on the interval I.

Theorem For any closed and bounded interval $I = [a, b]$, if f is C^1 on I then $L = \max_{x \in I} |f'(X)|$ is a Lipschitz constant for f on I

A vector field $\mathbf{F} : S \in \mathbb{R}^N$ is Lipschitz on $S \subseteq \mathbb{R}^N$ if

$$\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\| \qquad \text{for all} \mathbf{x}, \mathbf{y} \in S$$

Here,

$$\|\mathbf{x}\| = \left(\sum_{j=1}^{N} x_j^2\right)^{\frac{1}{2}}$$

denotes the **Euclidean norm** of the vector $\mathbf{x} \in \mathbb{R}^N$.
We say that $\mathbf{F}(\mathbf{x}, t)$ is **Lipschitz in $\mathbf{x}$** if

$$\|\mathbf{F}(\mathbf{x}, t) - \mathbf{F}(\mathbf{y}, t)\| \leq L\|\mathbf{x} - \mathbf{y}\|.$$

Local Existence and Uniquness Let $\mathbf{x}_0 \in \mathbb{R}^N$, fix $r > 0$ and $\tau > 0$, and put

$$S = \{(\mathbf{x}, t) \in \mathbb{R}^N \times \mathbb{R} : \|\mathbf{x} - \mathbf{x}_0\| \leq r \text{ and } |t| \leq \tau\}.$$

If $\mathbf{F}(\mathbf{x}, t)$ is Lipschitz in $\mathbf{x}$ for $\mathbf{x}, t \in S$, and if

$$\|\mathbf{F}(\mathbf{x}, t)\| \leq M \qquad \text{for } (\mathbf{x}, t) \in S,$$

then there exists a unique C^1 function $bfx(t)$ satisfying

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t) \quad \text{for } |t| \leq \min\{r/M, \tau\}, \text{ with } \mathbf{x}(0) = \mathbf{x}_0.$$