# Higher Several Variable Calculus
# Math2111 UNSW

Jeremy Le
(Based of Hussain Nawaz's Notes)

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## Contents

# 1 Introduction

**Real one-variable calculus** $f : \mathbb{R} \to \mathbb{R}$

- limits
- continuity
- differentiability
- integrability

**Important Theorems**

- Min-max theorem
  A continuous function on a closed interval attains a max and min value.
- Intermediate Value Theorem
  A continuous function on $[a, b]$ attains all values in $[f(a), f(b)]$.
- Mean Value Theorem
  Connects the instantaneous rate of change of differentiable function to its change over a finite closed interval.

**Mutivariable Calclus Applications** $f : \mathbb{R}^n \to \mathbb{R}^m$

- Fluid dynamics
- Black Scholes Options Pricing Model

# 2 Curves and Surfaces

## 2.1 Curves

The parameterisation of a curve in $\mathbb{R}^n$ is a vector-valued function

$$\boldsymbol{c} : \boldsymbol{I} \to \mathbb{R}^n$$

where $I$ is an interval on $\mathbb{R}$.

- A multiple point is a point through which the curve passes more than once.
- If $\boldsymbol{I} = [a, b]$ then $\boldsymbol{c}(a)$ and $\boldsymbol{c}(b)$ are called end points.
- A curve is closed if its end points are the same point, $\boldsymbol{c}(a) = \boldsymbol{c}(b)$.

## 2.2 Limits and Calculus for Curves

For an interval $\boldsymbol{I} \subset \mathbb{R}$ and curve $\boldsymbol{c} : \boldsymbol{I} \to \mathbb{R}^n$ with

$$\boldsymbol{c}(t) = (c_1(t), c_2(t), \ldots c_n(t)),$$

the functions $c_i : \boldsymbol{I} \to \mathbb{R}, i = 1, 2, \ldots, n$ are called the components of $\boldsymbol{c}$.

- If $\lim_{t\to a} c_i(t)$ exists for all $i$, then $\lim_{t\to a} \boldsymbol{c}(t)$ and

$$\lim_{t\to a} \boldsymbol{c}(t) = \left(\lim_{t\to a} c_1(t), \lim_{t\to a} c_2(t), \ldots \lim_{t\to a} c_n(t)\right)$$

- If $c_i'(t)$ exists for all $i$, then

$$\boldsymbol{c}'(t) = (c_1'(t), c_2'(t), \ldots, c_n'(t))$$

## 2.3 Surfaces

You have seen surfaces in $\mathbb{R}^3$ described in 3 ways.

- Graph: $z = f(x, y)$
- Implicitly: $x^2 + y^2 + z^2 = 1$
- Parametrically: $\mathbf{x} = \mathbf{x}_0 + \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2$

# 3 Analysis

## 3.1 Formal Definition of a Limit

**1-variable Calculus** Recall that $\lim_{x\to a} f(x) = L$ requires that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $|x - a| < \delta$ then

$$|f(x) - L| < \epsilon.$$

## 3.2 Distance Functions (metrics)

A function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ which satisfies the following three properties is called a metric.

- **Positive Definite**: for all $x, y \in \mathbb{R}^n$, $d(x, y) > 0$ and $d(x, y) = 0$ iff $x = y$.
- **Symmetric**: for all $x, y \in \mathbb{R}^n$, $d(x, y) = d(y, x)$.
- **Triangle Inequality** for all $x, y, z \in \mathbb{R}^n$, $d(x, y) + d(y, z) \geq d(x, z)$.

**Euclidean Distance** The Euclidean distance between $x$ and $y$ defined by

$$d(x, y) = ||x - y|| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

is a metric.

**Equivalent Metrics** Two metrics $d$ and $\delta$ are considered equal if there exists constants $0 < c < C < \infty$ such that

$$c\delta(x, y) \leq d(x, y) \leq C\delta(x, y).$$

## 3.3 Limits of Sequences

**Ball** A ball around $\mathbf{a} \in \mathbb{R}^n$ of radius $\epsilon > 0$ is the set

$$B(\mathbf{a}, \epsilon) = \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{a}, \mathbf{x}) < \epsilon\}.$$

**Limit of Sequences** For a sequence $\{\mathbf{x}_i\}$ of points in $\mathbb{R}^n$ we say that $\mathbf{x}$ is the limit of the sequence if and only if

$$\forall \epsilon > 0 \exists N \text{ such that } n \geq N \implies d(\mathbf{x}, \mathbf{x}_n) < \epsilon$$

or equivalently

$$\forall \epsilon > 0 \exists N \text{ such that } n \geq N \implies \mathbf{x}_n \in B(\mathbf{x}, \epsilon).$$

If $\mathbf{x}$ is the limit of the sequence $\{\mathbf{x}_i\}$ then for each positive $\epsilon$ there is a point in the sequence beyond which all points of the sequence are inside $B(\mathbf{x}, \epsilon)$.

**Convergence**

$$\begin{aligned}&\text{A sequence } \mathbf{x}_k \text{ converges to a limit } \mathbf{x}\\ &\quad\Leftrightarrow \text{the components of } \mathbf{x}_k \text{ converge to the components of } \mathbf{x}\\ &\quad\Leftrightarrow d(\mathbf{x}_k, \mathbf{x}) \to 0.\end{aligned}$$

**Cauchy Sequences** A sequence $\{\mathbf{x}_k\}$ in $\mathbb{R}^n$ is a Cauchy sequence if

$$\forall \epsilon > 0 \exists K \text{ such that } k, l > K \implies d(\mathbf{x}_k, \mathbf{x}_l) < \epsilon.$$

A sequence $\{\mathbf{x}_k\}$ converges in $\mathbb{R}^n$ to a limit if and only if $\{\mathbf{x}_k\}$ is a Cauchy sequence.

## 3.4 Open and Closed Sets

**Definitions** Consider $x_k$

- $x_0 \in \Omega$ is an <u>interior point</u> of $\Omega$ if there is a ball around $x_0$ completely contained in $\Omega$. That is, there exists a $\epsilon > 0$ such that $B(x_0, \epsilon) \subseteq \Omega$.
- $\Omega$ is <u>open</u> if every point of $\Omega$ is an interior point.
- $\Omega$ is <u>closed</u> if its complement is open.
- $x_0 \in \Omega$ is a <u>boundary point</u> of $\Omega$ if every ball around $x_0$ contains points in $\Omega$ and points not in $\Omega$.

**Closed Sets** $\Omega \subset \mathbb{R}^n$ is closed if and only if it contains all of its boundary points.

**Union and Intersection**

- A finite union/intersection of open sets is open.
- A finite union/intersection of closed sets is closed.

**Limit Points and Sets** $\mathbf{x}_0$ is a limit point (or accumulation point) of $\Omega$ if there is a sequence $\{\mathbf{x}_i\}$ in $\Omega$ with limit $\mathbf{x}_0$ and $\mathbf{x}_i \neq \mathbf{x}$.

- Every interior points of $\Omega$ is a limit point of $\Omega$.
- $\mathbf{x}_0$ is not necessarily in $\Omega$.
- A set is closed $\Leftrightarrow$ it contains all of its limit points.

**Variations of a Set** Consider the set $\Omega \in \mathbb{R}^n$.

- The <u>interior</u> of $\Omega$ is the set of all its interior points (denoted $\text{Int}(\Omega)$).
- The <u>boundary</u> of $\Omega$ is the set of all its boundary points (denoted $\partial\Omega$).
- The <u>closure</u> of $\Omega$ is $\Omega \cup \partial\Omega$ (denoted by $\bar{\Omega}$).

The interior is the largest open subset and the closure is the smallest closed set containing $\Omega$.

## 3.5 Limits

**Limit of a Function at a Point** Let $\mathbf{b} \in \mathbb{R}^m, \Omega \subseteq \mathbb{R}^n, \mathbf{a} \in \bar{\Omega}$ and let $\mathbf{f} : \Omega \to \mathbb{R}^m$ be a function. We say that $\mathbf{f}(\mathbf{x})$ converges to $\mathbf{b}$ as $\mathbf{x} \to \mathbf{a}$ if

$$\forall \epsilon > 0 \; \exists \delta > 0 \text{ such that for } \mathbf{x} \in \Omega :$$

$$0 < d(\mathbf{x}, \mathbf{x}_0) < \delta \implies d(\mathbf{f}(\mathbf{x}), \mathbf{b}) < \epsilon.$$

or alternatively

$$\mathbf{x} \in B(\mathbf{a}, \delta) \cap \Omega \implies \mathbf{f}(\mathbf{x}) \in B(\mathbf{b}, \epsilon).$$

If such $\mathbf{b}$ exists, then it is unique and we write

$$\lim_{x \to a} \mathbf{f}(\mathbf{x}) = \mathbf{b}.$$

**Useful Limit Theorems** Let $\mathbf{b} \in \mathbb{R}^m, \Omega \subseteq \mathbb{R}^n, \mathbf{a} \in \bar{\Omega}$ and let $\mathbf{f} : \Omega \to \mathbb{R}^m$ be a function. Then

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b} \iff \lim_{\mathbf{x} \to \mathbf{a}} f_i(\mathbf{x}) = b_i \text{ for all } i = 1, \ldots, m$$
$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b} \iff \lim_{k \to \infty} \mathbf{f}(\mathbf{x}_k) = \mathbf{b}$$

for every sequence $\{\mathbf{x}_k\}_{k=1}^{\infty} \subseteq \Omega$ with $\lim_{k \to \infty} \mathbf{x}_k = \mathbf{a}$.

The first theorem is useful to show that a limit exists whilst the second is useful to show the limit does not exist.

**Algebra of limits** Given that, $\lim_{x\to x_0} f(x) = a$ and $\lim_{x\to x_0} g(x) = b$, then,

$$\lim_{x\to x_0} (f+g)(x) = a+b$$
$$\lim_{x\to x_0} (fg)(x) = ab$$
$$\lim_{x\to x_0} (\frac{f}{g})(x) = \frac{a}{b}, \text{ given } b \neq 0.$$

**Pinching Principle** Let $\Omega \subset \mathbb{R}^n$, let $\mathbf{a}$ be a limit point of $\Omega$ and let $f, g, h : \Omega \to \mathbb{R}$ be functions such that there exists $\epsilon > 0$ such that

$$g(\mathbf{x}) \leq f(\mathbf{x}) \leq h(\mathbf{x}) \qquad \forall \mathbf{x} \in B(\mathbf{a}, \epsilon) \cap \Omega.$$

Then

$$\lim_{x\to\mathbf{a}} g(\mathbf{x}) = \mathbf{b} = \lim_{x\to\mathbf{a}} h(\mathbf{x}) \implies \lim_{x\to\mathbf{a}} f(\mathbf{x}) = \mathbf{b}.$$

## 3.6 Continuity

Continuity is like an extension to limits. It first requires that the limit exists and that the limit equals the actual value at that point.

**Definition** Let $\mathbf{a} \in \Omega \subseteq \mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}^m$ be a function. Then $f$ is continuous at $\mathbf{a}$ if and only if

$$\lim_{x\to a} f(\mathbf{x}) = f(\mathbf{a})$$

$f$ is said to be continuous on $\Omega$ if it is continuous at $\mathbf{a}$ for every $\mathbf{a} \in \Omega$.

**Epsilon-Delta Interpretation**

For all $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in B(\mathbf{a}, \delta) \cap \Omega \implies f(x) \in B(f(\mathbf{a}), \epsilon)$.

**Continuity by Components** All component functions $f_i : \Omega \to \mathbb{R}$ are continuous at $\mathbf{a}$.

**Continuity through Sequences** For every sequence $\{\mathbf{x}_k\}_{k=1}^{\infty}$ with $\mathbf{x}_k \in \Omega$ for all $k$, if $\{\mathbf{x}_k\}_{k=1}^{\infty}$ has limit $\mathbf{a}$ then $\{f(\mathbf{x}_k)\}_{k=1}^{\infty}$ converges to $f(a)$.

**Elementary Functions** If $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is an elementary function, then $f$ is continuous on $\Omega$.

**Preimage** Suppose that $\Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^m$ is a function. The preimage of a set $U \subseteq \mathbb{R}^m$ is defined by

$$f^{-1}(U) = \{x \in \mathbb{R}^n : f(x) \in U\}.$$

**Continuity - Using Preimage** Suppose that $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$. The following two statements are equivalent.

- $f$ is continuous on $\Omega$.
- $f^{-1}(U)$ is open in $\mathbb{R}^n$ for every open subset $U$ of $\mathbb{R}^m$.

## 3.7 Path Connected Sets

**Definition** A set $\Omega \subseteq \mathbb{R}^n$ is said to be path connected if for any $\mathbf{x}, \mathbf{y} \in \Omega$, there is a continuous function $\varphi$ such that $\varphi(t) \in \Omega$ for all $t \in [0,1]$ and $\varphi(0) = \mathbf{x}$ and $\varphi(1) = \mathbf{y}$.

**Theorem** Let $\Omega \subseteq \mathbb{R}^n$ and $\mathbf{f} : \Omega \to \mathbb{R}^m$ be continuous. Then

$$B \subseteq \Omega \text{ and } B \text{ path connected} \implies \mathbf{f}(B) \text{ path connected.}$$

## 3.8 Compact Sets

**Bounded** A set $\Omega \subseteq \mathbb{R}^n$ is bounded if there is an $M \in \mathbb{R}$ such that $d(\mathbf{x}, \mathbf{0}) \leq M$ for all $\mathbf{x} \in \Omega \iff \Omega \subseteq B(\mathbf{0}, M)$ .

**Compact** A set $\Omega \subseteq \mathbb{R}^n$ is compact if it is closed and bounded.

**Theorem** Let $\Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^m$ be continuous. Then

$$K \subseteq \Omega \text{ and } K \text{ compact} \implies f(K) \text{ compact.}$$

## 3.9 Bolzano-Weierstrass Theorem

For $\Omega \subseteq \mathbb{R}^n$, the following are equivalent.

1. $\Omega$ is compact.
2. Every sequence in $\Omega$ has a subsequence that converges to an element of $\Omega$.

# 4 Differentiation

## 4.1 Differentiability, Derivatives and Affine Approximations

**Differentiability in $\mathbb{R}$** $f : \mathbb{R} \to \mathbb{R}$ is differentiable at some $a \in \mathbb{R}$ means there is a *good* straight-line approximation to $f$ near $a$ called a tangent line. This approximating function is given by

$$T(x) = f(a) + f'(a)(x - a) = f(a) - f'(a)a + f'(a)x = y_0 + L(x)$$

where for all $a$, $y_0 = f(a) - f'(a)a$ is a fixed number and $L : \mathbb{R} \to \mathbb{R} = f'(a)x$ is the linear map.

Recall that

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

**Linear Maps** A function $L : \mathbb{R}^n \to \mathbb{R}^m$ is called linear iff for all $x, y \in \mathbb{R}^n$ for all $\lambda \in \mathbb{R}$ :

$$L(x + y) = L(x) + L(y) \text{ and } L(\lambda x) = \lambda L(x).$$

**Affine Maps** A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is affine means there is $y_0 \in \mathbb{R}^m$ and a linear map (ie matrix) $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$T(\mathbf{x}) = \mathbf{y}_0 + \mathbf{L}(\mathbf{x}).$$

A function $f : \mathbb{R} \to \mathbb{R}$ is affine iff $f(x) = ax + b$, for some $a, b \in \mathbb{R}$.

**Affine approximation** The function $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ has an affine approximation at a point $a \in \Omega$ if and only if there exists a matrix $A \in M_{m\times n}(\mathbb{R})$ such that

$$\lim_{x\to a} \frac{d(f(x) - f(a), A(x - a))}{d(x, a)} = 0$$

If $f$ has an affine approximation at a point $a \in \Omega$, then the matrix $A$ in the definition is called the derivative of $f$ at $a$ and is denoted by $Df(a)$ (or $Daf$).

The function $T_a f : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T_a f(x) = Df(a)(x - a) + f(a)$$

is called the best affine approximation of $f$ at $a$.

**Differentiability in** $\mathbb{R}^n \to \mathbb{R}^n$ A function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable for some $a \in \Omega$ if there exists a linear map $L : \mathbb{R}n \to \mathbb{R}^m$ such that

$$\lim_{x\to a} \frac{||f(x) - f(a) - L(x - a)||}{||L(x - a)||} = 0.$$

Notation: the matrix of the linear map $L$, the derivative of $f$ at $a$ is denoted by $D_a f$.

**Delta Epsilon Definition of Differentiability** A function $f : \Omega \subset \mathbb{R} \to \mathbb{R}^m$ is differentiable on $a \in \Omega$ if there is a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that $\forall \epsilon > 0 \exists \delta > 0$ such that for all $x \in \Omega$

$$||x - a|| < \delta \to ||f(x) - f(a) - L(x - a)|| < \epsilon \, ||x - a|| \, .$$

## 4.2 Partial Derivatives

Let $\mathbf{a} \in \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ be a function with coordinates $x_i$ and standard basis vectors $\mathbf{e}_i, i \in \{1, \dots, n\}$. The partial derivative of $f$ in direction $i$ is defined as

$$\frac{\partial f}{\partial x_i} = \lim_{h\to 0} \frac{f(\mathbf{a} + h\mathbf{e}_i - f(\mathbf{a})}{h}$$

assuming the limit exists.

**Claiaut's Theorem** If $f, \dfrac{\partial f}{\partial x_i}, \dfrac{\partial f}{\partial x_j}, \dfrac{\partial^2 f}{\partial x_i x_j}, \dfrac{\partial^2 f}{\partial x_j x_i}$ all exist and are continuous on an open set around $\mathbf{a}$ then

$$\frac{\partial^2 f}{\partial x_i x_j}(\mathbf{a}) = \frac{\partial^2 f}{\partial x_j x_i}(\mathbf{a}).$$

That is the partial derivatives commute.

## 4.3 Jacobian Matrix

**Definition** If all partial derivatives of $\mathbf{f} : \Omega \to \mathbb{R}^m$ exists at $\mathbf{a} \in \omega \subseteq \mathbb{R}^n$, then the Jacobian matrix of $\mathbf{f}$ at $\mathbf{a}$ is

$$J_a f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \frac{\partial f_1}{\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{a}) & \frac{\partial f_2}{\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\mathbf{a}) & \frac{\partial f_n}{\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{pmatrix}.$$

**Theorem** Let $\Omega \subseteq \mathbb{R}^n, \mathbf{a} \in \Omega$ be an interior point and $\mathbf{f} : \Omega \to \mathbb{R}^m$ be a function. If $\mathbf{f}$ is differentiable at $\mathbf{a}$ then all partial derivatives $\dfrac{\partial f_j}{\partial x_i}$ exist at $\mathbf{a}$ and

$$D\mathbf{f}(\mathbf{a}) = J\mathbf{f}(\mathbf{a}).$$

Best affine approximation: $T_a f(x) = Jf(a)(x - a) + f(a)$.

## 4.4 Differentiable and Continuous

**Limit at 0** For $\mathbf{x} \in \mathbb{R}^n$ and $L$ an $m \times n$ matrix,

$$\lim_{x \to \mathbf{0}} ||L\mathbf{x}|| = 0.$$

**Open Sets** Let $\Omega \in \mathbb{R}^n$ be open and let $f : \Omega \to \mathbb{R}^m$ be a function that is differentiable on $\Omega$. Then $f$ is continuous on $\Omega$.

**Partial Derivatives + Continuity** Let $\Omega \subseteq \mathbb{R}^n$ be open and let $f : \Omega \to \mathbb{R}^m$ be a function. If for all $i = 1, \ldots, n$ and all $j = 1, \ldots, m$ the partial derivative $\dfrac{\partial f_j}{\partial x_i}$ exists and is continuous on $\Omega$ then $f$ is differentiable on $\Omega$.

## 4.5 Chain Rule, Gradient, Directional Derivatives, Tangent Planes

**Chain Rule** Let $\Omega \subseteq \mathbb{R}^n, \Omega' \subseteq \mathbb{R}^m$ and let $\mathbf{a} \in \Omega$. Suppose $\mathbf{f} : \Omega \to \mathbb{R}^m$ and $\mathbf{g} : \Omega' \to \mathbb{R}^k$ are functions such that $\mathbf{f}(\Omega) \subseteq \Omega'$. If $\mathbf{f}$ is differentiable at $\mathbf{a}$ and $\mathbf{g}$ is differentiable at $\mathbf{f}(\mathbf{a})$, then $\mathbf{g} \circ \mathbf{f}$ is differentiable at $\mathbf{a}$ and

$$D(\mathbf{g} \circ \mathbf{f})(\mathbf{a}) = D\mathbf{g}(\mathbf{f}(\mathbf{a}))D\mathbf{f}(\mathbf{a}).$$

**Gradient** For $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, if the Jacobian exists, then it is given by the $1 \times n$ matrix

$$Jf = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \cdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}.$$

This is equivalent to the gradient of $f$. That is,

$$\text{grad}(f) = \boldsymbol{\nabla} f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \cdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}.$$

**Directional Derivative** The directional derivative of $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ in the direction of the unit vector $\hat{\mathbf{u}}$ at $\mathbf{a} \in \Omega$ is

$$D_{\hat{\mathbf{u}}} f(\mathbf{a}) = f'_{\hat{\mathbf{u}}}(\mathbf{a}) = \lim_{h \to 0} \frac{f(\mathbf{a} + h\hat{\mathbf{u}}) - f(\mathbf{a})}{h}.$$

if the limit exists.

Equivalently, if $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at $a$ then for a unit vector $u$

$$D_u f(a) = f'_u(a) = \boldsymbol{\nabla} f(a) \cdot u.$$

Alternatively, allowing $\theta$ to be the angle between $\boldsymbol{\nabla} f(a)$ and $u$,

$$D_u f(a) = |\boldsymbol{\nabla} f(a)| \cdot |u| \cdot \cos\theta.$$

**Affine Approximation** Allow $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ to be a differentiable function at $a \in \Omega$. The best affine approximation to $f$ at $a$ may be written in terms of the gradient vector as

$$T(x) = f(a) + \boldsymbol{\nabla} f(a) \cdot (x - a).$$

**Tangent Planes** The tangent plane to a function $z = f(x, y)$ is given by

$$z = T(x, y).$$

## 4.6 Taylor Series and Theorem

**Taylor's Theorem** For all continuous and differentiable functions $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) \approx P_{k,a}(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{k,a}(x)$$

where the remainder $R$ is

$$R_{k,a}(x) = \frac{f^{(k+1)}(z)}{(k+1)!}(x-a)^{k+1}$$

for some $z$ between $x$ and $a$.

$P_{0,a}, P_{1,a}, P_{2,a}, P_{3,a}$ are the best constant, affine, quadratic, cubic approximations.

**Hessian Matrix** For $\Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$, the *Hessain matrix of $f$ at a point $a \in \Omega$* is the $n \times n$ matrix

$$Hf(a) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(a) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(a) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(a) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(a) & \frac{\partial^2 f}{\partial x_2^2}(a) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(a) & \frac{\partial^2 f}{\partial x_2 \partial x_n}(a) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(a) \end{pmatrix}.$$

assuming the $2^{\text{nd}}$ order partial derivatives exist.

**Class** A function $f : \Omega \to \mathbb{R}, \Omega \subseteq \mathbb{R}^n$ open, is called (of class) $C^r$ if all partial derivatives of $f$ of order $\leq r$ exist and are continuous.

**Taylor Polynomials** Let $\Omega \subseteq \mathbb{R}^n$ be open, let $a \in \Omega$, and let $f : \Omega \to \mathbb{R}$ be a function of class $C^2$. The polynomial

$$P_{1,a}(x) = f(a) + \boldsymbol{\nabla} f(a) \cdot (x-a)$$

is called the Taylor polynomial of order 1 about $a$ and the polynomial

$$P_{2,a}(x) = f(a) + \boldsymbol{\nabla} f(a) \cdot (x-a) + \frac{1}{2}(x-a) \cdot Hf(a)(x-a)$$

is called the Taylor Polynomial of order 2 about $a$.

In general, if $f : \Omega \to \mathbb{R}$ is $C^r, \Omega$ open , $a \in \Omega$:

$$\begin{aligned} P_{r,a}(x) = f(a) &+ \boldsymbol{\nabla} f(a) \cdot (x-a) + \frac{1}{2}(x-a) \cdot Hf(a)(x-a) \\ &+ \cdots + \frac{1}{r!} \sum_{i_1,\ldots,i_r=1}^{n} \frac{\partial^r f}{\partial x_{i_1} \ldots \partial x_{i_r}}(a)(x_{i_1} - a_{i_1}) \cdot \cdots \cdot (x_{i_r} - a_{i_r}). \end{aligned}$$

**Taylor's Theorem ($1^{\text{st}}$ order)** Let $\Omega \in \mathbb{R}^n$ be open, let $f : \Omega \to \mathbb{R}$ be a function of class $C^2$. Let $x, a \in \Omega$ s.t. the line segment between $x$ and $a$ is contained in $\Omega$. Then there exist $z$ on this line segment such that

$$f(x) = f(a) + \nabla f(a) \cdot (x - a) + R_{1,a}(x)$$

where $R_{1,a}(x) = \frac{1}{2}(x - a) \cdot (Hf(z)(z - a))$.

**Taylor's Theorem (2 $^{\text{nd}}$ order)** Let $\Omega \in \mathbb{R}^n$ be open, let $f : \Omega \to \mathbb{R}$ be a function of class $C^3$. Let $x, a \in \Omega$ s.t. the line segment between $x$ and $a$ is contained in $\Omega$. Then there exist $z$ on this line segment such that

$$f(x) = f(a) + \nabla f(a) \cdot (x - a) + \frac{1}{2}(x - a)Hf(a)(x - a) + R_{2,a}(x)$$

where $R_{2,a}(x) : \Omega \to \mathbb{R}$ is a function such that $\frac{R_{2,a}(x)|}{|x-a|^2} \to 0$ as $x \to a$.

## 4.7 Maxima, Minima and Saddle Points

**Definitions** Let $a \in \Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ be a function. Then

- $a$ is an absolute or *global maximum* of $f$ if $f(a) \geq f(x)$ for all $x \in \Omega$.
- $a$ is an absolute or *global minimum* of $f$ if $f(a) \leq f(x)$ for all $x \in \Omega$.
- $a$ is a *local maximum* of $f$ if there is an open $A \subseteq \Omega$ containing $a$ such that $f(a) \geq f(x)$ for all $x \in A$.
- $a$ is a *local minimum* of $f$ if there is an open $A \subseteq \Omega$ containing $a$ such that $f(a) \leq f(x)$ for all $x \in A$.
- $a$ is a *stationary point* of $f$ if $f$ is differentiable at $a$ and $\nabla f(a) = 0$.
- $a$ is a *saddle point* of $f$ if $a$ is a stationary point of $f$ but it's neither a local max nor a local minimum of $f$.

**Critical Points** Let $a \in \Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ be a function. If $a$ is a local maximum or a local minimum then

1. $a$ is a stationary, or
2. $a \in \partial\Omega \Longleftrightarrow a$ is a boundary pt, or
3. $f$ is not differentiable at $a$.

Points satisfying $1, 2$ or $3$ are called critical points.

## 4.8 Classification of Stationary Points

**Definition:** An $n \times n$ martix $H$ is

- positive definite $\iff$ all eigenvalues are $> 0$
- positive semi-definite $\iff$ all eigenvalues are $\geq 0$
- positive definite $\iff$ all eigenvalues are $< 0$
- positive semi-definite $\iff$ all eigenvalues are $\leq 0$

**Criterion for Local Extrema** Let $\Omega \subseteq \mathbb{R}^n$ be open, $a \in \Omega$ and let $f : \Omega \to \mathbb{R}$ be a function such that all paritial derivaitves of $f$ of order at most 2 exists on $\Omega$ and $\boldsymbol{\nabla} f(a) = 0$. Then

- $Hf(a)$ is positive definite $\implies$ $f$ has a local minimum at $a$;
- $Hf(a)$ is negative definite $\implies$ $f$ has a local maximum at $a$;
- $f$ has a local minimum at $a$ $\implies$ $Hf(a)$ is positive semi-definite;
- $f$ has a local maximum at $a$ $\implies$ $Hf(a)$ is negative semi-definite;

**Sylvesetr's Criterion** If $H_k$ is the upper $k \times k$ matrix of $H$ and $\Delta_k = det(H_k)$, then

- $H$ is positive definite $\iff$ $\Delta_k > 0$ for all $k$
- $H$ is positive semi-definite $\implies$ $\Delta_k \geq 0$ for all $k$
- $H$ is negative definite $\iff$ $\Delta_k < 0$ for all odd k and $\Delta_k > 0$ for all even k
- $H$ is negative semi-definite $\implies$ $\Delta_k \leq 0$ for all odd k and $\Delta_k \geq 0$ for all even k

## 4.9 Lagrange Multipliers, Implicit and Inverse Function Theorems

**Lagrange Multipliers** Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$ are differentiable and $S = \{x \in \mathbb{R}^n : \varphi(x) = c\}$ defines a smooth surface on $\mathbb{R}^n$. If $f$ attains a local maximum or minimum at a point $a \in S$ then $\boldsymbol{\nabla} f(a)$ and $\boldsymbol{\nabla}\varphi(a)$ are parallel. If $\boldsymbol{\nabla}\varphi(a) \neq 0$, there exist a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$\boldsymbol{\nabla} f(a) = \lambda \boldsymbol{\nabla}\varphi(a).$$

**Inverse Function Theorem for $f : \mathbb{R} \to \mathbb{R}$** If $f : \mathbb{R} \to \mathbb{R}$ is differentiable on an open interval $I \in \mathbb{R}$ and $f'(x) \neq 0$ for all $x \in I$, then $f$ is invertible on $I$ and the inverse $f^{-1} : f(I) \to \mathbb{R}$ is differentiable with

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

**Generalising the Inverse Function Theorem** Let $\Omega \subseteq \mathbb{R}^n$ be open, $f : \Omega \to \mathbb{R}^n$ be $C^1$ and suppose $a \in \Omega$. If $Df(a)$ is invertible (as a matrix) then $f$ is invertible on an open set $U$ containing $a$. That is,

$$f^{-1} : f(U) \to U$$

exists. Furthermore, $f^{-1}$ is $C^1$ and for $x \in U$,

$$D_{f(x)} f^{-1} = (D_x f)^{-1}.$$

# 5 Integration

## 5.1 Riemann Integral

**Riemann Integral** For a bounded function $f : R \to \mathbb{R}$, if there exists a unique numder $I$ such that

$$\underline{\mathcal{S}}_{\mathcal{P}_1, \mathcal{P}_2}(f) \leq I \leq \overline{\mathcal{S}}_{\mathcal{P}_1, \mathcal{P}_2}(f)$$

for every pair of partitions $\mathcal{P}_1, \mathcal{P}_2$ of $R$, then $f$ is Riemann integrable on $R$ and

$$I = \iint_R f = \iint_R f(x, y) dA.$$

$I$ is called the Riemann integral of $f$ over $R$.

**Properties of the Riemann Integral** For a function of one variable, the Riemann integral is interpreted as the (signed) area bounded by the graph $y = f(x)$ and the $x$-axis over the interval $[a, b]$. For a function of two variables $\iint_R f$ is the (signed) volume bounded by the graph $z = f(x, y)$ and the $xy$-plane over the rectangle $R$. If $f$ and $g$ are integrable on $R$,

- Linearity: $\iint_R \alpha f + \beta g = \alpha \iint_R f + \beta \int \int_R g, \qquad \alpha, \beta \in \mathbb{R}$.
- Positivity (monotonicity): If $f(x) \leq g(x), \forall x \in R$ then $\iint_R f \leq \iint_R g$
- $\left| \iint_R f \right| \leq \iint_R |f|$
- If $R = R_1 \cup R_2$ and (interior $R_1$) $\cap$ (interior $R_2$) $= \emptyset$ then

$$\iint_R f = \iint_{R_1} f + \iint_{R_2} f.$$

## 5.2 Fubini's Theorem

**Fubini's Theorem - Rectangles** Let $f : R \to \mathbb{R}$ be continuous on a rectangular domain $R = [a, b] \times [c, d]$. Then $f$ is a bounded function and is integerable over $R$. Moreover,

$$\int_a^b \int_c^d f(x, y)\, dy dx = \int_c^d \int_a^b f(x, y)\, dx dy = \iint_R f.$$

**Fubini's Theorem - Discontinuous** Let $f : R \to \mathbb{R}$ be bounded on a rectangular domain $R = [a, b] \times [c, d]$ with the discontinuities of $f$ confined to a finite union of graphs of continuous functions. If the integral $\int_c^d f(x, y)\, dy$ exists for each $x \in [a, b]$ then

$$\iint_R f = \int_a^b \left( \int_c^d f(x, y)\, dy \right) dx.$$

Similarly, if the integral $\int_a^b f(x, y)\, dx$ exists for each $y \in [c, d]$, then

$$\iint_R f = \int_c^d \left( \int_a^b f(x, y)\, dx \right) dy.$$

**Iterated Integrals for Elementary Regions** Suppose $D$ is a $y$-simple region bounded by $x = a, x = b, y = \varphi_1(x)$ and $y = \varphi_2(x)$ and $f : D \to \mathbb{R}$ is continuous. Then

$$\iint_D f = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dxdy.$$

A simile result hols for integrals over $x$-simple regions.

## 5.3 Leibniz' Rule

**Basic Version** Let $a, b, c, d \in \mathbb{R}$ If $f : [a, b] \times [c, d] \to \mathbb{R}$ and $\frac{\partial f}{\partial x}$ are continuous on the rectangle $[a, b] \times [c, d]$. Then

$$g(x) = \int_c^d f(x, y)\, dy.$$

is differentiable and has derivative

$$g'(x) = \frac{d}{dx} \left[ \int_c^d f(x, y)\, dy \right] = \int_c^d \frac{\partial f}{\partial x}(x, y) dy \quad \text{for } a \le x \le b.$$

**With variable limits** Let $a, b \in \mathbb{R}$ with $a \le b$, let $\varphi_1, \varphi_2 : [a, b] \to \mathbb{R}$ be continuously differientiable functions such that $\varphi_1(x) \le \varphi_2(x)$ for all $x \in [a, b]$. If $f : D_1 \to \mathbb{R}$ and $\frac{\partial f}{\partial x}$ are continuous on the region $D_1$ with

$$D_1 = \{(x, y) : x \in [a, b] \text{ and } \varphi_1(x) \le y \le \varphi_2(x)\}$$

then the function $g(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy$ is differntiable and

$$g'(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial f}{\partial x}(x, y)\, dy + f(x, \varphi_2(x))\varphi_2'(x) - f(x, \varphi_1(x))\varphi_1'(x).$$

Note: If $\varphi_1(x) \equiv c, \varphi_2(x) \equiv d$ where $c, d$ are constants. Then $g'(x) = \int_c^d \frac{\partial f}{\partial x}\, dy$ (reduced to the previous version).

## 5.4 Change of Variable

Let $\Omega \subseteq \mathbb{R}^n$ and $F : \Omega \to \mathbb{R}^n$ be an injective and continuously differentiable function such that $\det JF(x)) \neq 0$ for all $x \in \Omega$. If $f$ is any function that is integrable on $\Omega' = F(\Omega)$ then

$$\iint_{\Omega'} (f \circ F)|\det JF|.$$

# 6 Fourier Series

**Fourier Series** A Fourier series is the approximation of simple periodic functions by the sum of period functions of the form $\sin(x), \cos(x)$. Note that unlike Taylor series, a function $f$ may be discontinuous. However, any lack of continuity leads to an infinite sum in the Fourier series.

## 6.1 Inner Products and Norms

**Inner Products** Let $V$ be a (real) vector space. An inner product on $V$ is a map that assigns each $f, g \in V$ a real number $\langle f, g\rangle$ in such a way that

- $\langle f, f\rangle \geq 0$,
- $\langle f, f\rangle = 0$ if and only if $f$ is zero,
- $\langle \lambda f + \mu g, h\rangle, = \lambda\langle f, h\rangle + \mu\langle g, h\rangle$,
- $\langle g, f\rangle = \langle f, g\rangle$.

for all functions $f, g, h \in V$ and all real constants $\lambda, \mu$.

### Usual Inner Products

- The vector space $\mathbb{R}^n$ consisting of all $n$-dimensional vector admits the following inner product
$$\langle v, w\rangle = v \cdot w = \sum_{i=1}^{n} v_i w_i.$$
- The vector space $C[a, b]$ consisting of all continuous function defined on the interval $[a, b]$ admits the following inner product
$$\langle f, g\rangle = \int_a^b f(x)g(x)\, dx.$$

**Norms** A norm on $V$ is a map that assigns each $f \in V$ a real number $||f||$ in such a way that

- $||f|| > 0$,
- $||f|| = 0$ if and only if $f = 0$,
- $||\lambda f|| = |\lambda| \, ||f||$,
- $||f + g|| \leq ||f|| + ||g||$ (triangle inequality)

for all functions $f, g \in V$ and all real constant $\lambda$.

**Usual Norms** Consider a vector space $C[a, b]$ consisting of all continuous functions on $[a, b]$.

- The 2-norm ($L^2$-norm) is a norm on $C[a, b]$:

$$||f||_2 = \sqrt{\int_a^b f(x)^2 \, dx}$$

- The max norm is a norm on $C[a, b]$:

$$||f||_\infty = \max_{a \leq x \leq b} \{|f(x)|\}$$

**Theorem** Every inner product on a vector space $V$ induces a norm given by

$$||f|| = \sqrt{\langle f, f \rangle},$$

and the Cauchy-Schwartz inequality holds:

$$|\langle f, g \rangle| \leq ||f|| \, ||g|| \text{ for all } f, g \in V.$$

## 6.2 Fourier Coefficients and Fourier Series

**Fourier Series** Suppose that a given function $f : \mathbb{R} \to \mathbb{R}$ is a $2\pi$-periodic and is square integrable (i.e., $\int_{-\pi}^{\pi} f(x)^2 \, dx < \infty$). Its Fourier series is given by

$$S_f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} [a_k \cos(kx) + b_k \sin(kx)]$$

where

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) \, dx, \quad k = 0, 1, 2, \ldots$$

and

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) \, dx, \quad k = 1, 2, \ldots$$

## 6.3 Pointwise Convergence of Fourier Series

**Piecewise Continuous Functions** Consider a function $f : \mathbb{R} \to \mathbb{R}$ and a point $c \in \mathbb{R}$. Suppose that the one-sided limits $f(c^+) = \lim_{x\to c^+} f(x)$ and $f(c^-) = \lim_{x\to c^-} f(x)$ exists.

- If $f(c^+) = f(c^-) = f(c)$, then $f$ is continuous at $c$.
- If $f(c^+) = f(c^-) \neq f(c)$ or if $f(c^+) = f(c^-)$ but $f(c)$ is undefined, then $f$ has a removable discontinuity at $c$.
- If $f(c^+) \neq f(c^-)$, then $f$ has a jump discontinuity at $c$.

A function $f : [a, b] \to \mathbb{R}$ is piecewise continuous on $[a, b]$ if and only if

(1) For each $x \in [a, b), f(x^+)$ exists;

(2) For each $x \in (a, b], f(x^-)$ exists;

(3) $f$ is continuous on $(a, b)$ except at (most) a finite number of points.

Note that if $f$ is only piecewise continuous then the partial sum of the Fourier series does not necessarily converge to $f$ for all $x$.

**Piecewise Differentiable Functions** Consider a function $f : \mathbb{R} \to \mathbb{R}$ and a point $c \in \mathbb{R}$. We write

$$D^+f(c) = \lim_{h\to 0^+} \frac{f(c+h) - f(c^+)}{h}$$

if this one-sided limit exists. Likewise,

$$D^-f(c) = \lim_{h\to 0^-} \frac{f(c+h) - f(c^-)}{h}.$$

A function $f$ is differentiable at $c$ if and only if $f(c^+) = f(c) = f(c^-)$ and $D^+f(c) = D^-f(c)$. A function $f$ is piecewise differentiable on $[a, b]$ if and only if

(1) For each $x \in [a, b), D^+f(x)$ exists;

(2) For each $x \in (a, b], D^-f(x)$ exists;

(3) $f$ is differentiable on $(a, b)$ except at (most) a finite number of points.

**Pointwise Convergence** Let $c \in \mathbb{R}$ and suppose that a function $f : \mathbb{R} \to \mathbb{R}$ has the following properties:

1. $f$ is $2\pi$-periodic;
2. $f$ is piecewise continuous on $[-\pi, \pi]$;
3. $D^+f(c)$ and $D^-f(c)$ exists.

If $f$ is continuous at $c$ then,

$$S_f(c) = f(c).$$

If $f$ has a jump/removable discontinuity at $c$, then

$$S_f(c) = \frac{1}{2}[f(c^+) + f(c^-)].$$

## 6.4 General Periodic, Half Range + Odd and Even Functions

**General Periodic Functions** Suppose that $f$ has period $2L$, instead of $2\pi$:

$$f(x+2L) = f(x) \text{ for } x \in \mathbb{R}.$$

Note that $\cos\left(\frac{\pi}{L}x\right)$ and $\sin\left(\frac{\pi}{L}x\right)$ are periodic functions with period $2L$. So, the decomposition becomes

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos\left(\frac{k\pi}{L}x\right) + b_k \sin\left(\frac{k\pi}{L}x\right)\right)$$

where

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{k\pi x}{L}\right)\,dx, \quad k = 0, 1, 2, \ldots$$

and

$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right)\,dx, \quad k = 1, 2, \ldots$$

**Half Range Expansion** Let $f$ be defined on $[0, L]$. We can extend $f$ to an even function (or odd function) on $[-L, L]$ and calculate its Fourier Series.

**Odd and Even Functions** We define an odd and even functions by the conditions $f(-x) = -f(x)$ and $f(-x) = f(x)$ respectively for a function $f$. The following elementary properties hold:

- Odd $\times$ Even = Odd
- Odd $\times$ Odd = Even
- Even $\times$ Even = Even
- $\int_{-L}^{L}$ Odd $= 0$

**Odd and Even Functions for Fourier Series** If $f$ is odd, then

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{k\pi x}{L}\right)\,dx = 0$$

and

$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right)\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right)\,dx.$$

So the Fourier series becomes

$$S_f(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{L}\right). \qquad \text{(Fourier Sine Series)}$$

If $f$ is even, then

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{k\pi x}{L}\right)\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{k\pi x}{L}\right)\,dx.$$

and

$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right)\,dx = 0$$

So the Fourier series becomes

$$S_f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{L}\right). \qquad \text{(Fourier Cosine Series)}$$

## 6.5 Convergence of Sequences

**Pointwise Convergence** Let $f_k : \mathbb{R} \to \mathbb{R}$. We say $f_k$ converges to $f$ on $[a,b]$ pointwisely iff, for every $x \in [a,b]$, $f_k(x) \to f(x)$ as $k \to \infty$. In this case, $f$ is called the pointwise limit. In terms of $\epsilon - \delta$ language:

For every $x \in [a,b], \epsilon > 0$, there exists an $K$ (depends on $\epsilon$ and $x$), such that

$$|f_k(x) - f(x)| \le \epsilon \text{ for all } k \ge K.$$

**Uniform Convergence** Let $f_k : \mathbb{R} \to \mathbb{R}$. We say $f_k$ converges to $f$ on $[a,b]$ uniformly iff for every $\epsilon > 0$, there exists an $K$ (depends on $\epsilon$ only), such that

$$\sup_{x\in[a,b]} |f_k(x) - f(x)| \le \epsilon \text{ for all } k \ge K.$$

**Uniform Convergence Theorem** If $f_k : \mathbb{R} \to \mathbb{R}$ is continuous on $[a,b]$ for all $k$ if:

- $f_k \to f$ uniformly on $[a,b]$ then $f$ is continuous on $[a,b]$.
- $f$ has at least one discontinuity on $[a,b]$, $f_k$ cannot converge uniformly to $f$ on $[a,b]$.

**Weierstrass Test** Let $f_k : \mathbb{R} \to \mathbb{R}$ be a sequence of function defined on $[a,b]$. Suppose that there exists a sequence of numbers $c_k$ such that

$$|f_k(x)| \le c_k \text{ for all } x \in [a,b]$$

and $\sum_{k=1}^{\infty} c_k$ converges (or exists as a real number). Then $\sum_{k=1}^{\infty} f_k$ converges uniformly to a function $f$ on $[a,b]$.

Note that this test also holds for functions $f : \mathbb{R}^n \to \mathbb{R}$ for $x \in \Omega$ where $\Omega$ is a closed bounded set in $\mathbb{R}^n$.

**Norm Convergence** Consider the supremum norm $||f|| = \sup_{x\in[a,b]} |f(x)|$. The definition of uniform convergence can be equivalently written as: for every $\epsilon > 0$, there exists an $K$ such that

$$||f_k - f|| \leq \epsilon \text{ for all } k \geq K.$$

Equivalently,

$$\lim_{k\to\infty} ||f_k - f|| = 0.$$

Here, the norm is defined as the supremum norm. Extending this idea, we can define norm convergence for any arbitrary norm.

Let $V$ be a vector space of functions $f$ equipeed with a norm $||f||$. We say a sequence of functions $f_1, \ldots f_k, \ldots,$ (norm) converges to $f$ in $V$ if $f \in V$ and

$$\lim_{k\to\infty} ||f_k - f|| = 0.$$

As such, the $L^2$ norm convergence, also known as mean square convergence is equivalent to the following

$$\lim_{k\to\infty} \int_a^b [f_k(x) - f(x)]^2 dx = 0.$$

**Parseval Theorem** Let $f$ be $2\pi$ periodic, bounded and $\int_{-\pi}^{\pi} f(x)^2\, dx < +\infty$. Then, the Fourier series of $f$ converges to $f$ in the mean square sense. Moreover, the following Parseval's identity holds:

$$\int_{-\pi}^{\pi} f^2(x)\, dx = ||f||_2^2 = \frac{\pi}{2} a_0^2 + \pi \sum_{k=1}^{\infty} (a_k^2 + b_k^2).$$

This identity continues to hold for $2L$ periodic functions integrated over $[-L, L]$.

# 7 Vector Fields

## 7.1 Vector Fields and Flow

**Vector Fields** A vector field in 3D space has components that are functions and is of the type

$$\begin{aligned}
\mathbf{F}(\mathbf{x}) &= \mathbf{F}(x, y, z) \\
&= (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)) \\
&= F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}.
\end{aligned}$$

A vector field in 2D has components that are functions and is of the type

$$\begin{aligned}
\mathbf{F}(\mathbf{x}) &= \mathbf{F}(x, y) \\
&= (F_1(x, y), F_2(x, y)) \\
&= F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}.
\end{aligned}$$

**Flow Lines** If $\mathbf{F}$ is a vector field, a *flow line* for $\mathbf{F}$ is a path $\mathbf{c}(t)$ such that

$$\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t)).$$

That is, $\mathbf{F}$ yields the velocity field of the path $\mathbf{c}(t)$.

**The Del $\nabla$ operator** The vector differential operator $\nabla$ is not a vector, but an operator. It may be considered a symbolic vector. The differential operator may be written as

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

**Divergence** If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, the divergence of $\mathbf{F}$ is the scalar field

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Divergence may be thought as a type of derivative that describes the measure at which a vector field *spreads away* from a certain point. If the divergence is positive, then there is a net outflow while there is net inflow if the divergence is negative.

Observe that the divergence of a vector field will be real-valued.

**Curl** If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, the curl of $\mathbf{F}$ is the vector field

$$\begin{aligned}\text{curl } \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.\end{aligned}$$

Curl is also analogous to a type of derivative for vector fields. The curl may be thought as the measure at which the vector field *swirls* around a point. A positive swirl can be thought of as a counterclockwise rotation.

Observe that the curl of a vector field is also a vector field.

## 7.2 Vector Identities

### Basic Vector Identities

1. $\nabla(f + g) = \nabla f + \nabla g$
2. $\nabla(\lambda f) = \lambda \nabla f$ where $\lambda \in \mathbb{R}$
3. $\nabla(fg) = f\nabla g + g\nabla f$. You may draw analogies to the product.
4. $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$ where $g \neq 0$. This is analogous to the quotient rule.

5. $\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$

6. $\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$

7. $\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f$

8. $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

9. $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

10. $\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} = \nabla f \times \mathbf{F}$

11. $\nabla \times (\nabla f) = 0$

12. $\nabla^2(fg) = f\nabla^2 g + 2(\nabla f \cdot \nabla g) + g\nabla^2 f$

13. $\nabla \cdot (\nabla f \times \nabla g) = 0$

14. $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla f^2$

# 8 Path Integrals

## 8.1 Path Integrals

**Path (scalar line) Integrals** We say that a vector-valued function $\mathbf{c}(t)$ parametrises a curve $C$ for $a < t < b$ if the image of $\mathbf{c}$ traces out the curve $C$.

**Computing a Scalar Line Integral** Let $\mathbf{c}(t)$ be a parametrisation of a curve $C \in \mathbb{R}^3$ for $a < t < b$. Assume that $f(x, y, z)$ and $\mathbf{c}'(t)$ are continuous. Then

$$\int_C f(x,y,z)\, ds = \int_a^b f(\mathbf{c}(t))||\mathbf{c}'(t)||\, dt$$

The value of the integral on the right does not depend on the choice of parametrisation. For $f(x, y, z) = 1$, we obtain the length of $C$:

$$\text{Length of } C = \int_C ||\mathbf{c}'(t)||\, dt$$

where $||\mathbf{c}'(t)|| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$ for $\mathbf{c}(t) = x(t), y(t), z(t))$.

**Elementary Properties of Path Integral**

- $\int_C f_1\, ds + \int_C f_2\, ds = \int_C (f_1 + f_2)\, ds$
- $\int_C \lambda f\, ds = \lambda \int_C f\, ds, \quad \lambda \in \mathbb{R}$

## 8.2 Applications of Path Integrals

**Mass** Suppose that $\delta = \delta(x, y, z)$ which is a density function.

$$M = \int_C \delta(x, y, z)\, dz$$

**First Moments About the Coordinate Planes**

$$M_{yz} = \int_C x\delta\, ds, \qquad M_{xz} = \int_C y\delta\, ds, \qquad M_{xy} = \int_C z\delta\, ds$$

**Coordinates of the Center of Mass**

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

**Moments of Inertia about Axes**

$$I_x = \int_C (y^2 + z^2)\delta\, dx, \qquad I_y = \int_C (x^2 + z^2)\delta\, ds, \qquad I_z = \int_C (x^2 + y^2)\delta\, ds$$

# 9 Vector Line Integrals

## 9.1 Vector Line Integrals

**Vector Line Integrals** There is an important distinction between vector and scalar line integrals. To define a vector line integral we must specify a direction along the path or curve $C$.

A curve $C$ can be traversed in one of two directions. We say that $C$ is oriented if one of these two directions is specified. We refer to the specified direction as the forward direction along the curve.

**Computing a Line Integral** Let $\mathbf{c}(t)$ be a parameterisation of an oriented curve $C$ for $a \leq t \leq b$. The line integral of a vector field $\mathbf{F}$ along $C$ is the defined by

$$\int_C \mathbf{F} \cdot ds = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt.$$

**Link with the path integral** Let $\mathbf{c}(t)$ be a parametrisation of an oriented smooth curve $C$ and let $\hat{\mathbf{T}}$ denotes the unit tangent vector pointing in the forward direction of $C$.

$$\hat{\mathbf{T}}(\mathbf{c}(t)) = \frac{\mathbf{c}'(t)}{||\mathbf{c}'(t)||}$$

Then, the line integral of a vector field $\mathbf{F}$ over the oriented curve $C$ is the path integral of the tangential component of $\mathbf{F}$ along $C$, that is

$$\int_c \mathbf{F} \cdot ds = \int_C \mathbf{F} \cdot \hat{\mathbf{T}}\, ds.$$

**Summing Paths** Let $C_i, i = 1, \dots, m$ be curves with continuous differentiable parameterisations. Let $C = C_1 + C_2 + \cdots + C_m$, that is, $C$ is the union of curves $C_i$, which are joined end-to-end. Then, we define

$$\int_C \mathbf{F} \cdot ds = \sum_{i=1}^{m} \int_{C_i} \mathbf{F} \cdot ds.$$

**Work notation** Denote $\mathbf{c}(t) = (x(t), y(t), z(t))$ and $\mathbf{F} = (M, N, P) = M\mathbf{i}, N\mathbf{j}, P\mathbf{k}$. Then, we can denote work as any of the following notations:

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot ds \\ &= \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt && \text{(Definition)} \\ &= \int_a^b \left( M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt} \right) dt \\ &= \int_a^b M dx + N dy + P dz. && \text{(Alternative form)} \end{aligned}$$

**Properties of Line Integrals** Let $C$ be a smooth oriented curve and let $\mathbf{F}$ and $\mathbf{G}$ be vector fields.

(i) Linearity:

$$\int_C (\mathbf{F} + \mathbf{G}) \cdot ds = \int_C \mathbf{F} \cdot ds + \int_C \mathbf{G} \cdot ds$$

$$\int_C k\mathbf{F} \cdot ds = k \int_C \mathbf{F} \cdot ds \qquad (k \text{ a constant})$$

(ii) Reversing orientation:

$$\int_{-C} \mathbf{F} \cdot ds = -\int_C \mathbf{F} \cdot ds$$

(iii) Additivity: If $C$ is a union of $n$ smooth curves $C_1 + \cdots + C_n$, then

$$\int_C \mathbf{F} \cdot ds = \int_{C_1} + \cdots + \int_{C_n} \mathbf{F} \cdot ds$$

## 9.2 Other Applications

**Flow Integral, Circulation** If $\mathbf{r}(t)$ is a smooth curve in the domain of a continuous velocity field $\mathbf{F}$, the flow along the curve from $t = a$ to $t = b$ is

$$\text{Flow} = \int_a^b \mathbf{F} \cdot \hat{\mathbf{T}}\, ds$$

The integral in this case is called a flow integral. If the curve is a closed loop, the flow is called the circulation around the curve.

**Flux Across a Closed Curve in the Plane** If $C$ is a smooth closed curve in the domain of a continuous vector field $\mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ in the plane and if $\hat{\mathbf{n}}$ is the outward-pointing unit normal vector on $C$, the flux of $\mathbf{F}$ across $C$ is

$$\text{Flux of } \mathbf{F} \text{ across } C = \int_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds.$$

**Calculating Flux Across a Smooth Closed Plane Curve**

$$(\text{Flux of } \mathbf{F} = M\mathbf{i} + N\mathbf{j} \text{ across } C) = \oint_C M\, dy - N\, dx$$

The integral can be evaluated from any smooth parametrisation $x = g(t), y = h(t), a \leq t \leq b$, that traces $C$ counterclockwise exactly once.

## 9.3 Fundamental Theorem of Line Integrals

**(Second) Fundamental Theorem of Calculus in One Vairable** Let $f : \mathbb{R} \to \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$. If $f(x) = \varphi'(x)$, then

$$\int_a^b \varphi'(x)\, dx = \int_a^b f(x)\, dx = \varphi(b) - \varphi(a).$$

**Gradient Fields** A vector field $\mathbf{F}$ is called a gradient vector field if there exists a real-valued function $\varphi$ such that $\mathbf{F} = \boldsymbol{\nabla}\varphi$. That is, $(M, N, P) = (\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z})$. A vector field $\mathbf{F}$ with this property is called conservative and $\varphi$ is called the potential function of $\mathbf{F}$.

**Fundamental Theorem for Gradient Vector Fields** If $\mathbf{F} = \boldsymbol{\nabla}\varphi$ on a domain $\mathcal{D}$, then for every oriented smooth curve $C$ in $\mathcal{D}$ with initial point $P$ and terminal point $Q$.

$$\int_C \mathbf{F} \cdot ds = \varphi(Q) - \varphi(P)$$

If $C$ is closed (i.e., if $P = Q$), then $\oint_C \mathbf{F} \cdot ds = 0$.

**Cross Partials of a Gradient Vector Field are Equal** Let $\mathbf{F} = (F_1, F_2, F_3)$ be a gradient vector field whose components have continuous partial derivatives. Then the cross partials are equal:

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$$

Similarly, if the vector field in the plane $\mathbf{F} = (F_1, F_2)$ is the gradient vector field, then $\dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}$. Equivalently, $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$.

## 9.4 Green's Theorem

Green's Theorem connects double integrals with line integrals and is very useful for line integrals over complicated vector fields with simpler partial derivatives.

**Green's Theorem (Flux-divergence or Normal Form)** Let $D$ be a bounded simple region in $\mathbb{R}^2$ with nonempty interior, whose boundary consists of a finite number of smooth curves. Let $C$ be the boundary of $D$ with a positive (counter-clockwise) direction. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field which is continuously differentiable on $D$. Then, the outward flux of $\mathbf{F}$ across the curve $C$ equals the double integral of divergence $\boldsymbol{\nabla} \cdot F$ over $D$, that is

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \oint_C -N\, dx + M\, dy = \iint_D \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

Three key assumptions:

- The region $D$ is bounded and simple region with nonempty interior.
- The boundary $C$ is oriented in the positive (counter-clockwise) direction, and is a finite union of smooth curves.
- The vector field $\mathbf{F}$ is continuously differentiable on $D$.

**Green's Theorem (Circulation-curl or Tangential Form)** Let $D$ be a bounded simple region in $\mathbb{R}^2$ with nonempty interior, whose boundary consists of a finite number of smooth curves. Let $C$ be the boundary of $D$ with a positive (counter-clockwise) direction. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field which is continuously differentiable on $D$. Then, the counter-clockwise circulation of $\mathbf{F}$ around $C$ equals the double integral $\boldsymbol{\nabla} \times F \cdot k$ over $D$, that is

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{T}}\, ds = \oint_C M\, dx + N\, dy = \iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

**Area of a Region** Let $D$ be a simple and bounded region with non-empty interior and let $C$ be its boundary with positive (counter-clockwise) direction which is a finite union of smooth curves. Then, the area of $D$ can be calculated by

$$\text{Area}(D) = \frac{1}{2} \oint_C (-y\, dx + x\, dy).$$

# 10 Surface Integrals

## 10.1 Parametrised Surfaces

**Parametrised Surface** A parametrised surface is a function $\phi : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, where $D$ is some domain in $\mathbb{R}^2$, that is,

$$\phi(u, v) = (x(u, v), y(u, v), z(u, v)).$$

The surface $S$ corresponding to the function $\phi$ is its image: $S = \phi(D)$. If $\phi$ is differentiable (resp. continuously differentiable), then we call $S$ a differentiable (resp. continuously differentiable) surface.

**Cone** The cone $z^2 = x^2 + y^2$ has the parametrisation

$$\phi(u, v) = (u \cos v, u \sin v, u), \qquad 0 \le v \le 2\pi,\ u \in \mathbb{R}.$$

**Cylinder** The cylinder of radius $R, x^2 + y^2 = R^2$ has the parametrisation

$$\phi(\theta, z) = (R \cos\theta, R \sin\theta, z), \qquad 0 \le \theta \le 2\pi,\ z \in \mathbb{R}.$$

**Sphere** The sphere of radius $R, x^2 + y^2 + z^2 = R^2$ has the parametrisation

$$\Phi(\theta, \phi) = (R \cos\theta \sin\theta, R \sin\theta \sin\theta, R \cos\theta), \qquad 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.$$

## 10.2 Surface Area

In the rest of this section, we consider smooth parametrised surfaces and also piecewise smooth parametrised surfaces.

**Area of a Surface** Let $\Phi(u, v)$ be parametrisation of a smooth surface $S$ with parameter domain $D$. The area of the surface $S$ is

$$\text{Area}(S) = \iint_D ||\mathbf{T}_u \times \mathbf{T}_v||\, dudv.$$

Sometimes we write

$$||\mathbf{n}(u, v)|| = ||\mathbf{T}_u \times \mathbf{T}_v||.$$

Note that this $\mathbf{n}(u, v)$ is not necessarily a unit vector and neither are the tangent vectors.

## 10.3 Surface Integral

Let $\Phi(u, v)$ be a parametrisation of a smooth parametrised surface $S$ with parameter domain $D$. The surface integral of $f$ over $S$ is

$$\begin{aligned} &\iint_S f(x, y, z)\, dS \\ &= \iint_D f(\Phi(u, v))||\mathbf{T_u} \times \mathbf{T_v}||\, dudv \\ &= \iint_D f(\Phi(u, v))||\mathbf{n}(u, v)||\, dudv. \end{aligned}$$

If $S$ is piecewise smooth parameterised surface $S$ which are made up of finitely many smooth surface $S_i, i = 1, \ldots, m$, then, the surface integral of $f$ over $S$ is

$$\iint_S f(x, y, z)\, dS = \sum_{i=1}^{m} \iint_{S_i} f(x, y, z)\, dS.$$

## 10.4 Surface Integrals of Vector-Valued Functions

The surface integral of a vector field $\mathbf{F}$ over an oriented smooth parametrerised surface $S$ is defined as

$$\iint_S \mathbf{F} \cdot dS = \iint_S (\mathbf{F} \cdot \hat{\mathbf{n}})\, dS.$$

More generally, for a piecewise smooth parametrised surface $S$ formed by finite union of oriented smooth surfaces $S_i, i = 1, \ldots, m$, then

$$\iint_S \mathbf{F} \cdot dS = \sum_{i=1}^{m} \iint_{S_i} \mathbf{F} \cdot dS.$$

If $S$ is a smooth parametrised oriented surface and $\Phi$ parameterises the surface $S$ (i.e., $\hat{\mathbf{n}}$ in the normal direction specificed by the orientation of $S$) then,

$$\begin{aligned}
\iint_S \mathbf{F} \cdot dS &= \iint_S (\mathbf{F} \cdot \hat{\mathbf{n}})\, dS \\
&= \iint_D \left( \mathbf{F}(\Phi(u, v)) \cdot \frac{\mathbf{T_u} \times \mathbf{T_v}}{||\mathbf{T_u} \times \mathbf{T_v}||} \right) ||\mathbf{T_u} \times \mathbf{T_v}||\, dudv \\
&= \iint_D \mathbf{F}(\Phi(u, v)) \cdot (\mathbf{T_u} \times \mathbf{T_v})\, dudv
\end{aligned}$$

# 11 Integral Theorems

## 11.1 Stokes Theorem

Stokes theorem gives the relationship between a surface integral over a surface $S$ and a linear integral around the boundary curve of $S$.

Let $S$ be a smooth oriented surface defined by a one-to-one parametrisation $\Phi : D \subset R^2 \to S$, where $D$ is a region to which Green's theorem applies. Let $\partial S$ denote the oriented boundary of $S$ and let $\mathbf{F}$ be a $C^1$ vector field on $S$. Then

$$\iint_S (\boldsymbol{\nabla} \times F) \cdot dS = \int_{\partial S} F \cdot ds.$$

## 11.2 (Gauss) Divergence Theorem

The divergence theorem gives the relationship between a triple integral over a region $W$ and a surface integral over its boundary surface $S$.

Let $W \subseteq \mathbb{R}^3$ be a bounded, solid and simple region, and let $\mathbf{F}$ be a vector field in $\mathbb{R}^3$ which is continuously differentiable on $W$. Let $S$ be the boundary of $W$ which is a piece-wise smooth parameterised surface formed by a finite union of oriented smooth surfaces (say $S_i$). Then, the outward flux of $\mathbf{F}$ across the surface $S$ equals the triple integral of divergence div$\mathbf{F}$ over $W$, that is

$$\iint_S \mathbf{F} \cdot dS = \iiint_W \boldsymbol{\nabla} \cdot \mathbf{F}\, dV$$

where $\iint_S \mathbf{F} \cdot dS = \sum \iint_{S_i} \mathbf{F} \cdot dS$ and the surface are oriented such that the normal vector points outwards.