Higher Theory and Applications of Differential Equations MATH2221 UNSW

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Chapter 1

Linear ODEs

1.1 Introduction

Recall that a first-order ordinary differential equation (ODE) has, in its most general realisation, the form

$$y'(t) = f(t, y(t)).$$

A special case is the equation

$$a(t)y'(t) + b(t)y(t) = f(t),$$

with $a(t) \neq 0$ on some interval $I \in \mathbb{R}$. This special first-order ODE is called a **linear first-order ODE**. Another special case is

$$y'(t) = f(t)g(y),$$

which is known as a **separable first-order ODE**.

For a separable equation the solution is found (at least, implicitly by) writing:

$$\int \frac{1}{g(y)}\, dy = \int f(t)\, dt.$$

Solving Seperable ODEs Consider $y' = t^2 y, y(0) = 3$. This is seperable with $f(t) = t^2$ and $g(y) = y$. Then

$$\int \frac{1}{y}\, dy = \int t^2\, dt$$

so that

$$\ln|y(t)| = \frac{1}{3}t^3 + C.$$

Now apply e^t to both sides to obtain

$$|y(t)| = e^{\frac{1}{3}t^3 + C} = e^C e^{\frac{1}{3}t^3}.$$

Thus, a general solution of the equation is

$$y(t) = Ae^{\frac{1}{3}t^3}.$$

Since $y(0) = 3$, we see that the unique solution is $y(t) = 3e^{\frac{1}{3}t^3}$.

In the case of a linear first-order equation, i.e. $y' + a(t)y = f(t)$, a useful solution method is the integrating factor technique. The idea is to find a function μ so that when we multiply both sides of the equation with μ we find that

$$[\mu y](t)' = \mu(t)(y' + a(t)y) = \mu(t)f(t),$$

for if this happens, then the general solution of the ODE should be

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)f(t)\,dt + \frac{C}{\mu(t)}.$$

Solving Linear First-Order ODE Solve $y' - 2ty = 3t$. We pick

$$\mu(t) = e^{\int -2t\,dt} = e^{-t^2}.$$

Then

$$\begin{aligned}(e^{-t^2}y)' &= 3te^{-t^2} \\ e^{-t^2}y &= \int 3te^{-t^2}\,dt = -\frac{3}{2}e^{-t^2} + C \\ y(t) &= -\frac{3}{2} + Ce^{t^2}.\end{aligned}$$

1.2 Linear Differential Operators

In linear algebra, you have seen the compact notation $A\mathbf{x} = \mathbf{b}$ for system of linear equations. A similar notation when dealing with a linear ordinary differential equations is

$$Lu = f.$$

Here, L is an operator (or transformation) that acts on a function u to create a new function Lu. Given coefficients $a_0(x), a_1(x), \ldots, a_m(x)$ we define the **linear differential operator** L of **order** m,

$$\begin{aligned}Lu(x) &= \sum_{j=0}^{m} a_j(x)D^j u(x) \\ &= a_m D^m u + a_{m-1}D^{m-1}u + \cdots + a_o u,\end{aligned}$$

where $D^j u = d^j u/dx^j$ (with $D^0 u = u$).

We refer to a_m as the **leading coefficient** of L and assume that each $a_j(x)$ is a smooth function of x.

The ODE $Lu = f$ is said to be **singular** with respect to an interval $[a, b]$ if the leading coefficient $a_m(x)$ vanishes for any $x \in [a, b]$.

Example $Lu = (x - 3)u''' - (1 + \cos x)u' + 6u$ is a linear differential of order 3, with leading coefficient $x - 3$. Thus, L is singular on $[1, 4]$, but not singular on $[0, 2]$.

Example $N(u) = u'' + u^2u' - u$ is a nonlinear differential operator of order 2.

Linearity For any constants c_1 and c_2 and any m-times differentiable functions u_1 and u_2,

$$L(c_1 u_1 + c_2 u_2) = c_1 L u_1 + c_2 L u_2.$$

Ordinary differential equations of the form $Lu = 0$ are known as **homogenous**. Those of the form $Lu = f$ are known as **inhomogeneous**.

When the solution to a differential equation is prescribed at a particular point $x = x_0$, that is

$$u(x_0) = v_0, \quad u'(x_0) = v_1, \quad \dots, \quad u^{(m-1)}(x_0) = v_{m-1},$$

we call it an **initial value problem**. Where a differential equation is order 2 or greater, solutions at 2 or more locations can be prescribed. Such problems are called **boundary value problems**.

Unique Solution to Linear Initial Problem For an ODE $Lu = f$ which is not singular with repsect to a, b, with f continuous on $[a, b]$, the IVP for an mth-order linear differential operator with m inital values has a unique solution.

Solution to mth Order Problem has Dimension m Assume that the linear, mth-order differential operator L is not singular on $[a, b]$. Then the set of all solutions to the homogenous equation $Lu = 0$ on $[a, b]$ is a vector space of dimension m.

If $\{u_1, u_2, \dots, u_m\}$ is **any** basis for the solution space of $Lu = 0$, then every solution can be written in a unique way as

$$u(x) = c_1 u_1(x) + c_2 u_2(x) + \cdots + c_m u_m(x) \quad \text{for } a \leq x \leq 4.$$

We refer to this as the **general solution** of the homogenous equation $Lu = 0$ on $[a, b]$.

Linear superposition refers to this technique of constructing a new solution out of a linear combination of old ones.

Example The general solution to $u'' - u' - 2u = 0$ is $u(x) = c_1 e^{-x} + c_2 e^{2x}$.

Consider the inhomogeneous equation $Lu = f$ on $[a, b]$, and fix a particular solution u_P.
For *any* solution u, the difference $u - u_P$ is a solution of the homogenous equation because

$$L(u - u_P) = Lu - Lu_P = f - f = 0 \text{ on } [a, b].$$

Hence, $u(x) - u_P(x) = c_1 u_1(x) + \dots_c m u_m(x)$ for some constants $c_1, \dots, c_m$ and so

$$u(x) = u_P(x) + \underbrace{c_1 u_1(x) + \cdots + c_m u_m(x)}_{u_H(x)}, \quad a \leq x \leq b,$$

is the **general solution** of the inhomogeneous equation $Lu = f$.

Example The inhomogenous ODE $u'' - u' - 2u = -2e^x$ has a particular solution $u_P(x) = e^x$.
The general solution for its homogenous counterpart is $u_H(x) = c_1 e^{-x} + c_2 e^{2x}$.
So the general solution of the inhomogeneous ODE is

$$u(x) = u_P(x) + u_H(x) = e^x + c_1 e^{-x} + c_2 e^{2x}.$$

Reduction of Order For $u = u_1(x) \neq 0$, a solution to the ODE

$$u'' + p(x)u' + q(x)u = 0,$$

on some interval I, then a second solution is

$$u = u_1(x) \int \frac{1}{u_1^2 \exp\left(\int p\, dx\right)}\, dx.$$

Example For the ODE $u'' - 6u' + 9u = 0$, take $u_1 = e^{3x}$ and find v. **Answer** xe^{3x}.

1.3 Differential Operators with Constant Coefficients

If L has constant coefficients, then the problem of solving $Lu = 0$ reduces to that of factorising the polynomial having the same coefficients.

Suppose that a_j is constant for $0 \leq j \leq m$, with $a_m \neq 0$. We define the associated polynomial of degree m,

$$p(z) = \sum_{j=0}^{m} a_j z^j = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0,$$

so that if

$$Lu = a_m u^{(m)} + a_{m-1} u^{(m-1) + \cdots + a_1 u + a_0},$$

then formally, $L = p(D)$.

By the fundamental theorem of algebra,

$$p(z) = a_m (z - \lambda_1)^{k_1} (z - \lambda_2)^{k_2} \cdots (z - \lambda)^{k_r}$$

where $\lambda_1, \lambda_@, \ldots, \lambda_r$ satisfying

$$k_1 + k_2 + \cdots + k_r = m.$$

Lemma $(D - \lambda) x^j e^{\lambda x} = j x^{j-1} e^{\lambda x}$ for $j \geq 0$.

Lemma $(D - \lambda)^k x^j e^{\lambda x} = 0$ for $j = 0, 1, \ldots, k - 1$.

Basic Solutions If $(z - \lambda)^k$ is a factor of $p(z)$ then the function $u(x) = x^j e^{\lambda x}$ is a solution of $Lu = 0$ for $0 \leq j \leq k - 1$.

General Solution For the constant-coefficient case, the general solution of the homogenous equation $Lu = 0$ is

$$u(x) = \sum_{q=1}^{r} \sum_{l=0}^{k_q - 1} c_{ql} x^l e^{\lambda_q x},$$

where the c_{ql} are arbitrary constants.

Repeated Real Root From the factorisation

$$D^4 + 6D^3 + 9D^2 - 4D - 12 = (D-1)(D+2)^2(D+3)$$

we see that the general solution of

$$u'''' + 6u''' + 9'' - 4u' - 12u = 0$$

is

$$u = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x} + c_4 e^{-3x}.$$

Complex Root From the factorisation

$$\begin{aligned} D^3 - 7D^2 + 17D - 15 &= (D^2 - 4D + 5)(D-3) \\ &= (D-2-i)(D-2+i)(D-3) \end{aligned}$$

we see that the general solution of

$$u''' - 7u'' + 17u' - 15u = 0$$

is

$$\begin{aligned} u(x) &= c_1 e^{(2+i)x} + c_2 e^{(2-i)x} + c_3 e^{3x} \\ &= c_4 e^{2x}\cos x + c_5 e^{2x}\sin x + c_3 e^{3x}. \end{aligned}$$

Second-order ODEs arise naturally in classical mechanics for example a harmonic simple oscillator.

1.4 Wronskians and Linear Independence

We introduce a function, called the Wronskain that provides us with a way of testing whether a family of solutions to $Lu = 0$ is linearly independent.

Let $u_1(x), u_2(x), \ldots, u_m(x)$ be functions defined on an interval $I \in \mathbb{R}$. The functions $u_1, \ldots, u_m$ are called **linearly dependent** if there exist constant $a_1, a_2, \ldots, a_m$ **not all zero** such that

$$a_1 u_1(x) + a_2 u_2(x) + \cdots + a m u_m(x) = 0 \quad \forall x \in I.$$

If the above equation only holds for

$$a_i = 0, \quad i = 1, 2, \ldots, m$$

then the functions are **linearly independent**.

Example $u_1 = \sin 2x$ and $u_2 = \sin x \cos x$ are linearly dependent.
$u_1 = \sin x$ and $u_2 = \cos x$ are linearly indepdent.

The **Wronskian** of the functions $u_1, u_2, \ldots, u_m$ is the $m \times m$ determinant

$$W(x) = W(x; u_1, u_2, \ldots, u_m) = \det\left[D^{i-1} u_j\right].$$

Example The Wronskian of the functions $u_1 = e^{2x}, u_2 = xe^{2x}$ and $u_3 = e^{-x}$ is

$$W = \begin{vmatrix} e^{2x} & xe^{2x} & e^{-x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} & -e^{-x} \\ 4e^{2x} & 4e^{2x} + 4xe^{2x} & e^{-x} \end{vmatrix} = 9e^{3x}.$$

Lemma If $u_1, \ldots, u_m$ are linearly dependent over an interval $[a, b]$ then $W(x; u_1, \ldots, u_m) = 0$ for $a \leq x \leq b$.

Lemma If $u_1, u_2, \ldots, u_m$ are solutions of $Lu = 0$ on the interval $[a, b]$ then their Wronskain satisfies

$$a_m(x)W'(x) + a_{m-1}(x)W(x) = 0, \quad a \leq x \leq b.$$

Linear Independence of Solutions Let $u_1, u_2, \ldots, u_m$ be solutions of a non-singular, linear, homogenous, m-th order ODE $Lu = 0$ on the interval $[a, b]$.
Either

$W(x) = 0$ for $a \leq x \leq b$ and the m solutions are linearly **dependent**,

or else

$W(x) \neq 0$ for $a \leq x \leq b$ and the m solutions are linearly **independent**.