

Higher Theory and Applications of Differential Equations  
MATH2221 UNSW

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# Chapter 1

## Linear ODEs

### 1.1 Introduction

Recall that a first-order ordinary differential equation (ODE) has, in its most general realisation, the form

$$y'(t) = f(t, y(t)).$$

A special case is the equation

$$a(t)y'(t) + b(t)y(t) = f(t),$$

with  $a(t) \neq 0$  on some interval  $I \in \mathbb{R}$ . This special first-order ODE is called a **linear first-order ODE**. Another special case is

$$y'(t) = f(t)g(y),$$

which is known as a **separable first-order ODE**.

For a separable equation the solution is found (at least, implicitly by) writing:

$$\int \frac{1}{g(y)} dy = \int f(t) dt.$$

**Solving Seperable ODEs** Consider  $y' = t^2y, y(0) = 3$ . This is seperable with  $f(t) = t^2$  and  $g(y) = y$ . Then

$$\int \frac{1}{y} dy = \int t^2 dt$$

so that

$$\ln |y(t)| = \frac{1}{3}t^3 + C.$$

Now apply  $e^t$  to both sides to obtain

$$|y(t)| = e^{\frac{1}{3}t^3+C} = e^C e^{\frac{1}{3}t^3}.$$

Thus, a general solution of the equation is

$$y(t) = Ae^{\frac{1}{3}t^3}.$$

Since  $y(0) = 3$ , we see that the unique solution is  $y(t) = 3e^{\frac{1}{3}t^3}$ .

In the case of a linear first-order equation, i.e.  $y' + a(t)y = f(t)$ , a useful solution method is the integrating factor technique. The idea is to find a function  $\mu$  so that when we multiply both sides of the equation with  $\mu$  we find that

$$[\mu y](t)' = \mu(t)(y' + a(t)y) = \mu(t)f(t),$$

for if this happens, then the general solution of the ODE should be

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)f(t) dt + \frac{C}{\mu(t)}.$$

**Solving Linear First-Order ODE** Solve  $y' - 2ty = 3t$ . We pick

$$\mu(t) = e^{\int -2t dt} = e^{-t^2}.$$

Then

$$\begin{aligned} (e^{-t^2}y)' &= 3te^{-t^2} \\ e^{-t^2}y &= \int 3te^{-t^2} dt = -\frac{3}{2}e^{-t^2} + C \\ y(t) &= -\frac{3}{2} + Ce^{t^2}. \end{aligned}$$

## 1.2 Linear Differential Operators

In linear algebra, you have seen the compact notation  $A\mathbf{x} = \mathbf{b}$  for system of linear equations. A similar notation when dealing with a linear ordinary differential equations is

$$Lu = f.$$

Here,  $L$  is an operator (or transformation) that acts on a function  $u$  to create a new function  $Lu$ .

Given coefficients  $a_0(x), a_1(x), \dots, a_m(x)$  we define the **linear differential operator**  $L$  of **order**  $m$ ,

$$\begin{aligned} Lu(x) &= \sum_{j=0}^m a_j(x) D^j u(x) \\ &= a_m D^m u + a_{m-1} D^{m-1} u + \dots + a_0 u, \end{aligned}$$

where  $D^j u = d^j u / dx^j$  (with  $D^0 u = u$ ).

We refer to  $a_m$  as the **leading coefficient** of  $L$  and assume that each  $a_j(x)$  is a smooth function of  $x$ .

The ODE  $Lu = f$  is said to be **singular** with respect to an interval  $[a, b]$  if the leading coefficient  $a_m(x)$  vanishes for any  $x \in [a, b]$ .

**Example**  $Lu = (x - 3)u''' - (1 + \cos x)u' + 6u$  is a linear differential of order 3, with leading coefficient  $x - 3$ . Thus,  $L$  is singular on  $[1, 4]$ , but not singular on  $[0, 2]$ .

**Example**  $N(u) = u'' + u^2 u' - u$  is a nonlinear differential operator of order 2.

**Linearity** For any constants  $c_1$  and  $c_2$  and any  $m$ -times differentiable functions  $u_1$  and  $u_2$ ,

$$L(c_1u_1 + c_2u_2) = c_1Lu_1 + c_2Lu_2.$$

Ordinary differential equations of the form  $Lu = 0$  are known as **homogenous**. Those of the form  $Lu = f$  are known as **inhomogeneous**.

When the solution to a differential equation is prescribed at a particular point  $x = x_0$ , that is

$$u(x_0) = v_0, \quad u'(x_0) = v_1, \quad \dots, \quad u^{(m-1)}(x_0) = v_{m-1},$$

we call it an **initial value problem**. Where a differential equation is order 2 or greater, solutions at 2 or more locations can be prescribed. Such problems are called **boundary value problems**.

**Unique Solution to Linear Initial Problem** For an ODE  $Lu = f$  which is not singular with respect to  $a, b$ , with  $f$  continuous on  $[a, b]$ , the IVP for an  $m$ th-order linear differential operator with  $m$  initial values has a unique solution.

**Solution to  $m$ th Order Problem has Dimension  $m$**  Assume that the linear,  $m$ th-order differential operator  $L$  is not singular on  $[a, b]$ . Then the set of all solutions to the homogenous equation  $Lu = 0$  on  $[a, b]$  is a vector space of dimension  $m$ .

If  $\{u_1, u_2, \dots, u_m\}$  is **any** basis for the solution space of  $Lu = 0$ , then every solution can be written in a unique way as

$$u(x) = c_1u_1(x) + c_2u_2(x) + \dots + c_mu_m(x) \quad \text{for } a \leq x \leq b.$$

We refer to this as the **general solution** of the homogenous equation  $Lu = 0$  on  $[a, b]$ .

**Linear superposition** refers to this technique of constructing a new solution out of a linear combination of old ones.

**Example** The general solution to  $u'' - u' - 2u = 0$  is  $u(x) = c_1e^{-x} + c_2e^{2x}$ .

Consider the inhomogeneous equation  $Lu = f$  on  $[a, b]$ , and fix a particular solution  $u_P$ . For *any* solution  $u$ , the difference  $u - u_P$  is a solution of the homogenous equation because

$$L(u - u_P) = Lu - Lu_P = f - f = 0 \text{ on } [a, b].$$

Hence,  $u(x) - u_P(x) = c_1u_1(x) + \dots + c_mu_m(x)$  for some constants  $c_1, \dots, c_m$  and so

$$u(x) = u_P(x) + \underbrace{c_1u_1(x) + \dots + c_mu_m(x)}_{u_H(x)}, \quad a \leq x \leq b,$$

is the **general solution** of the inhomogeneous equation  $Lu = f$ .

**Example** The inhomogeneous ODE  $u'' - u' - 2u = -2e^x$  has a particular solution  $u_P(x) = e^x$ . The general solution for its homogenous counterpart is  $u_H(x) = c_1e^{-x} + c_2e^{2x}$ . So the general solution of the inhomogeneous ODE is

$$u(x) = u_P(x) + u_H(x) = e^x + c_1e^{-x} + c_2e^{2x}.$$

**Reduction of Order** For  $u = u_1(x) \neq 0$ , a solution to the ODE

$$u'' + p(x)u' + q(x)u = 0,$$

on some interval  $I$ , then a second solution is

$$u = u_1(x) \int \frac{1}{u_1^2 \exp(\int p dx)} dx.$$

**Example** For the ODE  $u'' - 6u' + 9u = 0$ , take  $u_1 = e^{3x}$  and find  $v$ . **Answer**  $xe^{3x}$ .

## 1.3 Differential Operators with Constant Coefficients

If  $L$  has constant coefficients, then the problem of solving  $Lu = 0$  reduces to that of factorising the polynomial having the same coefficients.

Suppose that  $a_j$  is constant for  $0 \leq j \leq m$ , with  $a_m \neq 0$ . We define the associated polynomial of degree  $m$ ,

$$p(z) = \sum_{j=0}^m a_j z^j = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0,$$

so that if

$$Lu = a_m u^{(m)} + a_{m-1} u^{(m-1)} + \cdots + a_1 u' + a_0 u,$$

then formally,  $L = p(D)$ .

By the fundamental theorem of algebra,

$$p(z) = a_m (z - \lambda_1)^{k_1} (z - \lambda_2)^{k_2} \cdots (z - \lambda_r)^{k_r}$$

where  $\lambda_1, \lambda_2, \dots, \lambda_r$  satisfying

$$k_1 + k_2 + \cdots + k_r = m.$$

**Lemma**  $(D - \lambda)x^j e^{\lambda x} = jx^{j-1} e^{\lambda x}$  for  $j \geq 0$ .

**Lemma**  $(D - \lambda)^k x^j e^{\lambda x} = 0$  for  $j = 0, 1, \dots, k-1$ .

**Basic Solutions** If  $(z - \lambda)^k$  is a factor of  $p(z)$  then the function  $u(x) = x^j e^{\lambda x}$  is a solution of  $Lu = 0$  for  $0 \leq j \leq k-1$ .

**General Solution** For the constant-coefficient case, the general solution of the homogenous equation  $Lu = 0$  is

$$u(x) = \sum_{q=1}^r \sum_{l=0}^{k_q-1} c_{ql} x^l e^{\lambda_q x},$$

where the  $c_{ql}$  are arbitrary constants.

**Repeated Real Root** From the factorisation

$$D^4 + 6D^3 + 9D^2 - 4D - 12 = (D - 1)(D + 2)^2(D + 3)$$

we see that the general solution of

$$u'''' + 6u''' + 9u'' - 4u' - 12u = 0$$

is

$$u = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x} + c_4 e^{-3x}.$$

**Complex Root** From the factorisation

$$\begin{aligned} D^3 - 7D^2 + 17D - 15 &= (D^2 - 4D + 5)(D - 3) \\ &= (D - 2 - i)(D - 2 + i)(D - 3) \end{aligned}$$

we see that the general solution of

$$u''' - 7u'' + 17u' - 15u = 0$$

is

$$\begin{aligned} u(x) &= c_1 e^{(2+i)x} + c_2 e^{(2-i)x} + c_3 e^{3x} \\ &= c_4 e^{2x} \cos x + c_5 e^{2x} \sin x + c_3 e^{3x}. \end{aligned}$$

Second-order ODEs arise naturally in classical mechanics for example a harmonic simple oscillator.

## 1.4 Wronskians and Linear Independence

We introduce a function, called the Wronskain that provides us with a way of testing whether a family of solutions to  $Lu = 0$  is linearly independent.

Let  $u_1(x), u_2(x), \dots, u_m(x)$  be functions defined on an interval  $I \in \mathbb{R}$ . The functions  $u_1, \dots, u_m$  are called **linearly dependent** if there exist constant  $a_1, a_2, \dots, a_m$  **not all zero** such that

$$a_1 u_1(x) + a_2 u_2(x) + \dots + a_m u_m(x) = 0 \quad \forall x \in I.$$

If the above equation only holds for

$$a_i = 0, \quad i = 1, 2, \dots, m$$

then the functions are **linearly independent**.

**Example**  $u_1 = \sin 2x$  and  $u_2 = \sin x \cos x$  are linearly dependent.  
 $u_1 = \sin x$  and  $u_2 = \cos x$  are linearly independent.

The **Wronskian** of the functions  $u_1, u_2, \dots, u_m$  is the  $m \times m$  determinant

$$W(x) = W(x; u_1, u_2, \dots, u_m) = \det[D^{i-1}u_j].$$

**Example** The Wronskian of the functions  $u_1 = e^{2x}$ ,  $u_2 = xe^{2x}$  and  $u_3 = e^{-x}$  is

$$W = \begin{vmatrix} e^{2x} & xe^{2x} & e^{-x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} & -e^{-x} \\ 4e^{2x} & 4e^{2x} + 4xe^{2x} & e^{-x} \end{vmatrix} = 9e^{3x}.$$

**Lemma** If  $u_1, \dots, u_m$  are linearly dependent over an interval  $[a, b]$  then  $W(x; u_1, \dots, u_m) = 0$  for  $a \leq x \leq b$ .

**Lemma** If  $u_1, u_2, \dots, u_m$  are solutions of  $Lu = 0$  on the interval  $[a, b]$  then their Wronskian satisfies

$$a_m(x)W'(x) + a_{m-1}(x)W(x) = 0, \quad a \leq x \leq b.$$

**Linear Independence of Solutions** Let  $u_1, u_2, \dots, u_m$  be solutions of a non-singular, linear, homogenous,  $m$ -th order ODE  $Lu = 0$  on the interval  $[a, b]$ .

Either

$W(x) = 0$  for  $a \leq x \leq b$  and the  $m$  solutions are linearly **dependent**,

or else

$W(x) \neq 0$  for  $a \leq x \leq b$  and the  $m$  solutions are linearly **independent**.

## 1.5 Methods for Inhomogeneous Equations

### 1.5.1 Judicious Guessing Method

You would have learned the method of undetermined coefficients for constructing a particular solution  $u_p$  to an inhomogeneous second-order linear ODE  $Lu = f$  in some simple cases. We will study this method systematically for higher-order linear ODEs with constant coefficients.

**Superposition of Solutions** Suppose that  $u_1$  solves  $Lu = e^{3x}$ , and  $u_2$  solves  $Lu = \sin x$ , where  $L$  is a linear differential operator. Then the solution of

$$Lu = e^{3x} + \sin x$$

is

$$u(x) = u_1(x) + u_2(x).$$

And a solution of

$$Lu = \frac{1}{2}e^{3x} - 5\sin x$$

is

$$u(x) = \frac{1}{2}u_1(x) - 5u_2(x).$$

Now we want to investigate some methods for finding particular solutions - i.e., finding a solution of  $Lu = f$ . One such method is the method of judicious guessing. For example:

1. If  $f$  is a polynomial, then guess that  $u_p$  is a polynomial.
2. If  $f$  is a exponential, then guess that  $u_p$  is exponential.



3. If  $f$  is a sine or cosine, then guess that  $u_p$  is a combination of such functions.

One problem with this method: it will only work for the types of functions identified above.

**Example** Suppose that  $u'' - u' = t^2 + 2t$ . Note as before that,

$$u_h(t) = c_1 + c_2 e^t.$$

So guess,

$$u_p(t) = At^3 + Bt^2 + Ct + D.$$

Then

$$t^2 + 2t = u_p'' - u_p' = -3At^2 + (6A - 2B)t + (2B - C).$$

So, equating coefficients of like power terms, we see that

$$A = -\frac{1}{3}, B = -2, C = -4, \text{ and } D \text{ is unrestricted.}$$

Therefore, reabsorbing  $D$  into  $c_1$ , we see that

$$u(t) = u_h(t) + u_p(t) = c_1 + c_2 e^t - \frac{1}{3}t^3 - 2t^2 - 4t.$$

Now we look at this idea of judicious guessing in a more systematic way. Let  $L = p(D)$  be a linear differential operator of order  $m$  with constant coefficients.

**Polynomial Solutions** Assume that  $a_0 = p(0) \neq 0$ . For any integer  $r \geq 0$ , there exists a unique polynomial  $u_P$  of degree  $r$  such that  $Lu_P = x^r$ .

**Exponential Solutions** Let  $L = p(D)$ ,  $M \in \mathbb{R}$  and  $\mu \in \mathbb{C}$ . If  $p(\mu) \neq 0$ , then the function

$$u_P(x) = \frac{Me^{\mu x}}{p(\mu)}$$

satisfies  $Lu_P = Me^{\mu x}$ .

**Example** A particular solution of  $u'' + 4u' - 3i = 3e^{2x}$  is  $u_P = e^{2x}/3$ .

**Product of Polynomial and Exponential** Let  $L = p(D)$  and assume that  $p(\mu) \neq 0$ . For any integer  $r \geq 0$ , there exists a unique polynomial  $v$  of degree  $r$  such that  $u_P = v(x)e^{\mu x}$  satisfies  $Lu_P = x^r e^{\mu x}$ .

## 1.5.2 Annihilator Method

In the previous cases we proposed a solution  $u = u_P$  and showed that it satisfied  $Lu = f$ . The following is a method to derive a particular solution given  $Lu = f$ . If  $f(x)$  is differentiable at least  $n$  times and

$$[a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0]f(x) = 0$$

then  $[a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0]$  **annihilates**  $f$ .

**Example**  $D^n$  annihilates  $x^{m-1}$  for  $m \leq n$ .  
 $(D - \alpha)^n$  annihilates  $x^{m-1}e^{\alpha x}$  for  $m \leq n$ .

**Annihilator Method: Simple Example** Given  $Lu = f$  we can apply the appropriate annihilator to both sides and solving the resulting homogeneous DE.

Let  $Lu = u'' - u'$  and suppose we want a solution such that  $Lu = x^2$ . Annihilating both sides we have

$$D^3(u'' - u') = u^{(5)} - u^{(4)} = 0.$$

Setting  $w = u^{(4)}$ , clearly  $w = Ce^x$  is the general solution. Integrating four times yields

$$u = Ce^x + Ex^3 + Fx^2 + Gx + H.$$

Clearly  $u_h = Ae^x + H$  and the form of the particular solution is  $u_p = x(Ex^2 + Fx + G)$ . Substituting we find  $E = -1/3$ ,  $F = -1$  and  $G = -2$ .

### 1.5.3 Judicious Guessing Method Continued

**Polynomial Solutions: The Remaining Case** Let  $L = p(D)$  and assume  $p(0) = p'(0) = \dots = p^{(k-1)}(0) = 0$  but  $p^{(k)}(0) \neq 0$  where  $1 \leq k \leq m - 1$ . For any integer  $r \geq 0$ , there exists a unique polynomial  $v$  of degree  $r$  such that  $u_p(x) = x^k v(x)$  satisfies  $Lu_p = x^r$ .

**Exponential Times Polynomial: Remaining Case** Let  $L = p(D)$  and assume  $p(\mu) = p'(\mu) = \dots = p^{(k-1)}(\mu) = 0$ . But  $p^{(k)}(\mu) \neq 0$ , where  $1 \leq k \leq m - 1$ . For any integer  $r \geq 0$ , there exists a unique polynomial  $v$  of degree  $r$  such that  $u_p(x) = x^k v(x)e^{\mu x}$  satisfies  $Lu_p = x^r e^{\mu x}$ .

### 1.5.4 Variation of Parameters

**Example** Find the general solution to  $u'' - 4u' + 4u = (x + 1)\exp 2x$ .

Note first that the general solution,  $u_h$ , to  $u'' - 4u' + 4u = 0$  is

$$u(x) = c_1 e^{2x} + c_2 x e^{2x}$$

since the characteristic equation is  $0 = r^2 - 4r + 4 = (r - 2)^2$ . Then

$$W(x) = \begin{vmatrix} e^{2x} & x e^{2x} \\ 2e^{2x} & e^{2x} + 2x e^{2x} \end{vmatrix} = e^{4x} + 2x e^{4x} - 2x e^{4x} = e^{4x}.$$

So by the method of variation of parameters:

$$v_1'(x) = e^{-4x} \cdot -x e^{2x} (x + 1) e^{2x} \text{ and } v_2'(x) = e^{-4x} \cdot e^{2x} (x + 1) e^{2x}.$$

In other words,

$$v_1'(x) = -x^2 - x \text{ and } v_2'(x) = x + 1.$$

Therefore  $u(x) = c_1 e^{2x} + c_2 x e^{2x} - (\frac{1}{3}x^3 + \frac{1}{2}x^2)e^{2x} + (\frac{1}{2}x^2 + x)e^{2x}$ .

## 1.6 Solution via Power Series

**General Case** Consider a general second-order, linear, homogenous ODE

$$Lu = a_2(x)u'' + a_1(x)u' + a_0(x)u = 0.$$

Equivalently,

$$u'' + p(x)u' + q(x)u = 0,$$

where

$$p(x) = \frac{a_1(x)}{a_2(x)} \text{ and } q(x) = \frac{a_0(x)}{a_2(x)}.$$

Assume that  $a_j$  is **analytic** at 0 for  $0 \leq j \leq 2$ . Then  $p$  and  $q$  are analytic at 0, that is, they admit power series expansions

$$p(z) = \sum_{k=0}^{\infty} p_k z^k \text{ and } q(z) = \sum_{k=0}^{\infty} q_k z^k \text{ for } |z| < \rho,$$

for some  $\rho > 0$ .

**Convergence Theorem** If the coefficients  $p(z)$  and  $q(z)$  are analytic for  $|z| < \rho$ , then the formal power series for the solution  $u(z)$ , constructed above, is also analytic for  $|z| < \rho$ .

**Power Series at Zero** Consider

$$Lu = (1 - x^2)u'' - 5xu' - 4u = 0, \quad u(0) = 1, \quad u'(0) = 2.$$

In this case,

$$p(z) = \frac{-5z}{1 - z^2} = -5 \sum_{k=0}^{\infty} z^{2k+1} \text{ and } q(z) = \frac{-4}{1 - z^2} = -4 \sum_{k=0}^{\infty} z^{2k}$$

are analytic for  $|z| < 1$ , so the theorem guarantees that  $u(z)$ , given by the formal power series, is also analytic for  $|z| < 1$ .

**Expansion about a Point other than Zero** Suppose we want a power series expansion about a point  $c \neq 0$ , for instance because the initial conditions are given at  $x = c$ .

A simple change of the independent variable allows us to write

$$u = \sum_{k=0}^{\infty} A_k (z - c)^k = \sum_{k=0}^{\infty} A_k Z^k \text{ where } Z = z - c.$$

Since  $du/dx = du/dZ$  and  $d^2u/dz^2 = d^2u/dZ^2$ , we obtain the translated equation

$$\frac{d^2u}{dZ^2} + p(Z + c) \frac{du}{dZ} + q(Z + c)u = 0.$$

Now compute that  $A_k$  using the series expansions of  $p(Z + c)$  and  $q(Z + c)$  in powers of  $Z$ .

## 1.7 Singular ODEs

In general, we do not want  $L$  to be singular on an interval for which we wish to solve  $Lu = f$ . However, some important applications lead to singular ODEs so we now address this case.

A second-order **Euler-Cauchy ODE** has the form

$$Lu = ax^2u'' + bxu' + cu = f(x),$$

where  $a, b$  and  $c$  are constants with  $a \neq 0$ . This ODE is singular at  $x = 0$ .  
Noticing that

$$Lx^r = [ar(r-1) + br + c]x^r,$$

we see that  $u = x^r$  is a solution of the homogenous equation ( $f = 0$ ) iff

$$ar(r-1) + br + c = 0.$$

**Factorisation** Suppose  $ar(r-1) + br + c = a(r-r_1)(r-r_2)$ . If  $r_1 \neq r_2$  then the general solution of the homogenous equation  $Lu = 0$  is

$$u(x) = C_1x^{r_1} + C_2x^{r_2}, \quad x > 0.$$

**Lemma** If  $r_1 = r_2$  then the general solution of the homogenous Euler-Cauchy equation  $Lu = 0$  is

$$u(x) = C_1x^{r_1} + C_2x^{r_1} \ln x, \quad x > 0.$$

**Euler-Cauchy Equations with Nonreal Indicial Roots** Suppose that  $r_{1,2} = \alpha \pm \beta i$  are the roots of the indicial equation

$$ar(r-1) + br + c = 0$$

associated to the Euler-Cauchy equation

$$at^2u'' + btu' + cu = 0.$$

Then the real-valued solutions can be derived as follows. First note that

$$t^{\alpha+\beta i} = t^\alpha t^{\beta i}$$

is a solution. Then notice that

$$t^{\beta i} = e^{\ln t^{\beta i}} = e^{i \ln t^\beta} = \cos(\ln(t^\beta)) + i \sin(\ln(t^\beta)).$$

So,

$$t^\alpha t^{\beta i} = t^\alpha e^{\ln t^{\beta i}} = t^\alpha e^{i \ln t^\beta} = t^\alpha (\cos(\ln(t^\beta)) + i \sin(\ln(t^\beta)))$$

is a solution. Finally, since each of the real part and the imaginary part is separately a (linear independent) solution, we see that the general solution in this case is (for  $t > 0$ )

$$u(t) = t^\alpha (c_1 \cos(\ln(t^\beta)) + i \sin(\ln(t^\beta))).$$

**Example** Consider  $t^2u'' - tu' + 5u = 0$ . Then the indicial equation is

$$r(r-1) - r + 5 = 0 \implies r = 1 \pm 2i.$$

So the general solution is,

$$u(t) = t(c_1 \cos \ln t^2 + c_2 \sin \ln t^2).$$

A number of important applications lead to ODEs that can be written in the **Frobenius normal form**

$$z^2u'' + zP(z)u' + Q(z)u = 0,$$

where  $P(z)$  and  $Q(z)$  are analytic at  $z = 0$ :

$$P(z) = \sum_{k=0}^{\infty} P_k z^k \text{ and } Q(z) = \sum_{k=0}^{\infty} Q_k z^k, \quad |z| < \rho.$$

Now consider  $z^2 u'' + zP(z)u' + Q(z)u = 0$ . Formal manipulations show that  $Lu(z)$  equals

$$I(r)A_0 z^r + \sum_{k=1}^{\infty} \left( I(k+r)A_k + \sum_{j=0}^{k-1} [(j+r)P_{k-j} + Q_{k-j}] A_j \right) z^{k+r},$$

where  $I(r)$  is the indicial polynomial  $I(r) := r(r-1)P_0r + Q_0$ , so we define  $A_0(r) = 1$  and

$$A_k(r) = \frac{-1}{I(k+r)} \sum_{j=0}^{k-1} [(j+r)P_{k-j} + Q_{k-j}] A_j(r), \quad k \geq 1,$$

provided  $I(k+r) \neq 0$  for all  $k \geq 1$ .

## 1.8 Bessel and Legendre Equations

### 1.8.1 Bessel Equations and Functions

The **Bessel equation with parameter  $\nu$**  is

$$z^2 u'' + zu' + (z^2 - \nu^2)u = 0.$$

This ODE is in Frobenius normal form, with indicial polynomial  $I(r) = (r+\nu)(r-\nu)$ , and we seek a series solution

$$u(z) = \sum_{k=0}^{\infty} A_k z^{k+r}.$$

We assume  $\operatorname{Re} \nu \geq 0$ , so  $r_1 = \nu$  and  $r_2 = -\nu$ .

With the normalisation

$$A_0 = \frac{1}{2^\nu \Gamma(1+\nu)}$$

the series solution is called the **Bessel function of order  $\nu$**  and is denoted

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1+\nu)} \left[ 1 - \frac{(z/2)^2}{1+\nu} + \frac{(z/2)^4}{2!(1+\nu)(2+\nu)} - \cdots \right].$$

From the functional equation  $\Gamma(1+z) = z\Gamma(z)$  we see that

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1+\nu)} - \frac{(z/2)^{\nu+2}}{\Gamma(2+\nu)} + \frac{(z/2)^{\nu+4}}{2!\Gamma(3+\nu)} - \frac{(z/2)^{\nu+6}}{3!\Gamma(4+\nu)} + \cdots$$

and so

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\nu}}{k! \Gamma(k+1+\nu)}.$$

## 1.8.2 Legendre Equation

The **Legendre equation** with parameter  $\nu$  is

$$(1 - z^2)u'' - 2zu' + \nu(\nu + 1)u = 0.$$

This ODE is not singular at  $z = 0$  so the solution has an ordinary Taylor series expansion

$$u = \sum_{k=0}^{\infty} A_k z^k.$$

The  $A_k$  must satisfy

$$(k + 1)(k + 2)A_{k+2} - [k(k + 1) - \nu(\nu + 1)A_k] = 0$$

for  $k \geq 0$ , and since

$$k(k + 1) - \nu(\nu + 1) = (k - \nu)(k + \nu + 1),$$

the recurrence relation is

$$A_{k+1} = \frac{(k - \nu)(k + \nu + 1)}{(k + 1)(k + 2)} A_k \text{ for } k \geq 0.$$

We have

$$u(z) = A_0 u_0(z) + A_1 u_1(z)$$

where

$$u_0(z) = 1 - \frac{\nu(\nu + 1)}{2!} z^2 + \frac{(\nu - 2)\nu(\nu + 1)(\nu + 3)}{4!} z^4 - \dots$$

and

$$u_1(z) = z - \frac{(\nu - 1)(\nu + 2)}{3!} z^3 + \frac{(\nu - 3)(\nu - 1)(\nu + 2)(\nu + 4)}{5!} z^5 - \dots$$

Suppose now that  $\nu = n$  is a non-negative integer. If  $n$  is even the series for  $u_0(z)$  terminates, whereas if  $n$  is odd then the series for  $u_1(z)$  terminates.

The terminating solution is called the **Legendre polynomial** of degree  $n$  and is denoted by  $P_n(z)$  with the normalization

$$P_n(1) = 1.$$

**Legendre Polynomials** The first few Legendre polynomials are

$$\begin{aligned} P_0(z) &= 1, & P_3(z) &= \frac{1}{2}(5z^3 - 3z), \\ P_1(z) &= z, & P_4(z) &= \frac{1}{8}(35z^4 - 30z^2 + 3), \\ P_2(z) &= \frac{1}{2}(3z^2 - 1), & P_5(z) &= \frac{1}{8}(63z^5 - 70z^3 + 15z). \end{aligned}$$

Notice that  $P_n$  is an even or odd function according to whether  $n$  is even or odd.

# Chapter 2

## Dynamical Systems

### 2.1 Terminology

We begin with some examples of how systems of differential equations arise in applications, and see how all such problems can be formulated as a **first-order** system

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}).$$

Such a formulation leads to a natural geometric interpretation of a solution.

**Lotka-Volterra Equations** Simplified ecology with two species:

$$\begin{aligned} F(t) &= \text{number of foxes at time } t, \\ R(t) &= \text{number of rabbits at time } t. \end{aligned}$$

Assume populations large enough that  $F$  and  $R$  can be treated as smoothly varying in time. In the 1920s, Alfred Lotka and Vito Volterra independently proposed the predator-prey model

$$\begin{aligned} \frac{dF}{dt} &= -aF + \alpha FR, & F(0) &= F_0, \\ \frac{dR}{dt} &= bR - \beta FR, & R(0) &= R_0. \end{aligned}$$

Here  $a, \alpha, b$  and  $\beta$  are non-negative constants.

Any first-order system for  $N$  ODEs in the form

$$\begin{aligned} \frac{dx}{dt} &= F_1(x, y, \dots, x_N), & x(0) &= x_{10}, \\ \frac{dy}{dt} &= F_2(x, y, \dots, x_N), & y(0) &= x_{20}, \\ &\vdots & &\vdots \\ \frac{dx_N}{dt} &= F_N(x, y, \dots, x_N), & x_N(0) &= x_{N0}, \end{aligned}$$

can be written in vector notation as

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) \quad \mathbf{x}(0) = \mathbf{x}_0.$$

The system of ODEs is determined by the **vector field**  $\mathbf{F} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ .

A system of ODEs of the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x})$$

is said to be **autonomous**.

In a **non-autonomous** system,  $\mathbf{F}$  will depend explicitly on  $t$ :

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t).$$

It can be shown that it is sufficient (in principle) to develop theory for the autonomous case as a non-autonomous system can be converted into an autonomous system.

**Second-order ODE** Consider an initial-value problem for a general (possibly non-autonomous) second-order ODE

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}, t\right), \text{ with } x = x_0 \text{ and } \frac{dx}{dt} = y_0 \text{ at } t = 0.$$

If  $x = x(t)$  is a solution, and if we let  $y = dx/dt$ , then

$$\frac{dy}{dt} = \frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}, t\right) = f(x, y, t),$$

that is,  $(x, y)$  is a solution of the first-order system

$$\begin{aligned} \frac{dx}{dt} &= y, & x(0) &= x_0, \\ \frac{dy}{dt} &= f(x, y, t) & y(0) &= y_0. \end{aligned}$$

## 2.2 Existence and Uniqueness

The most fundamental question about a dynamical system

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t)$$

is

For a given initial value  $\mathbf{x}_0$ , does a solution  $\mathbf{x}(t)$  satisfying  $\mathbf{x}(0) = \mathbf{x}_0$  exist, and if so is this solution unique?

Answer is **yes**, whenever the vector field  $\mathbf{F}$  is **Lipschitz**.

The number  $L$  is a **Lipschitz constant** for a function  $f : [a, b] \rightarrow \mathbb{R}$  if

$$|f(x) - f(y)| \leq L|x - y| \quad \text{for all } x, y \in [a, b].$$

**Example** Consider  $f(x) = 2x^2 - x + 1$  for  $0 \leq x \leq 1$ . Since

$$\begin{aligned} f(x) - f(y) &= 2(x^2 - y^2) - (x - y) = 2(x + y)(x - y) - (x - y) \\ &= (2x + 2y - 1)(x - y) \end{aligned}$$



we have  $|f(x) - f(y)| = |2x + 2y - 1||x - y|$  so a Lipschitz constant is

$$L = \max_{x,y \in [0,1]} |2x + 2y - 1| = 3.$$

We say that the function  $f : [a, b] \rightarrow \mathbb{R}$  is Lipschitz if a Lipschitz constant for  $f$  exists.

**Lipschitz Continuity** If  $f$  is Lipschitz then  $f$  is (uniformly) continuous.

**Continuous does not imply Lipschitz** Consider the (uniformly) continuous function

$$f(x) = 3 + \sqrt{x} \text{ for } 0 \leq x \leq 4.$$

In this case, if  $x, y \in (0, 4]$  then

$$f(x) - f(y) = \sqrt{x} - \sqrt{y} = \left( \sqrt{x} - \sqrt{y} \times \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} \right) = \frac{x - y}{\sqrt{x} + \sqrt{y}}$$

so if a Lipschitz constant  $L$  exists then

$$L \geq \frac{|f(x) - f(y)|}{|x - y|} = \frac{1}{\sqrt{x} + \sqrt{y}}$$

for arbitrarily small  $x$  and  $y$ , a contradiction.

A function  $f : I \rightarrow \mathbb{R}$  is  $C^k$  if  $f, f', f'', \dots, f^{(k)}$  all exist and are continuous on the interval  $I$ .

**Theorem** For any closed and bounded interval  $I = [a, b]$ , if  $f$  is  $C^1$  on  $I$  then  $L = \max_{x \in I} |f'(x)|$  is a Lipschitz constant for  $f$  on  $I$ .

A vector field  $\mathbf{F} : S \subseteq \mathbb{R}^N$  is Lipschitz on  $S \subseteq \mathbb{R}^N$  if

$$\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\| \quad \text{for all } \mathbf{x}, \mathbf{y} \in S$$

Here,

$$\|\mathbf{x}\| = \left( \sum_{j=1}^N x_j^2 \right)^{\frac{1}{2}}$$

denotes the **Euclidean norm** of the vector  $\mathbf{x} \in \mathbb{R}^N$ .

We say that  $\mathbf{F}(\mathbf{x}, t)$  is **Lipschitz in  $\mathbf{x}$**  if

$$\|\mathbf{F}(\mathbf{x}, t) - \mathbf{F}(\mathbf{y}, t)\| \leq L\|\mathbf{x} - \mathbf{y}\|.$$

**Local Existence and Uniqueness** Let  $\mathbf{x}_0 \in \mathbb{R}^N$ , fix  $r > 0$  and  $\tau > 0$ , and put

$$S = \{(\mathbf{x}, t) \in \mathbb{R}^N \times \mathbb{R} : \|\mathbf{x} - \mathbf{x}_0\| \leq r \text{ and } |t| \leq \tau\}.$$

If  $\mathbf{F}(\mathbf{x}, t)$  is Lipschitz in  $\mathbf{x}$  for  $\mathbf{x}, t \in S$ , and if

$$\|\mathbf{F}(\mathbf{x}, t)\| \leq M \quad \text{for } (\mathbf{x}, t) \in S,$$

then there exists a unique  $C^1$  function  $bf x(t)$  satisfying

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t) \quad \text{for } |t| \leq \min\{r/M, \tau\}, \quad \text{with } \mathbf{x}(0) = \mathbf{x}_0.$$

## 2.3 Linear Dynamical Systems

Linear differential equations are generally much easier to solve than nonlinear ones. Fortunately, linear DEs suffice for describing many important applications.

We say that the  $N \times N$ , first order system of ODEs

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t)$$

is **linear** if the RHS has the form

$$\mathbf{F}(\mathbf{x}, t) = A(t)\mathbf{x} + \mathbf{b}(t)$$

for some  $N \times N$  matrix-valued function  $A(t) = [a_{ij}(t)]$  and a vector-valued function  $\mathbf{b} = [b_i(t)]$ .

The system is autonomous precisely when  $A$  and  $\mathbf{b}$  are constant.

**Global Existence and Uniqueness** If the elements of  $A(t)$  and components of  $\mathbf{b}$  are continuous for  $0 \leq t \leq T$ , then the linear initial-value problem

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x} + \mathbf{b}(t) \quad \text{for } 0 \leq t \leq T, \quad \text{with } \mathbf{x}(0) = \mathbf{x}_0,$$

has a unique solution  $\mathbf{x}(t)$  for  $0 \leq t \leq T$ .

We now investigate the special case when  $A$  is constant and  $\mathbf{b}(t) = \mathbf{0}$ :

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

**General Solution via Eigensystem** If  $\mathbf{v}$  is a constant vector and  $A\mathbf{v} = \lambda\mathbf{v}$ , we define  $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ . Then

$$\frac{d\mathbf{x}}{dt} = \lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}(\lambda\mathbf{v}) = e^{\lambda t}(A\mathbf{v}) = A(e^{\lambda t}\mathbf{v}) = A\mathbf{x}$$

that is,  $\mathbf{x}$  is a solution of  $d\mathbf{x}/dt = A\mathbf{x}$ . If  $A\mathbf{v}_j = \lambda_j\mathbf{v}_j$  for  $1 \leq j \leq N$ , then the linear combination

$$\mathbf{x}(t) = \sum_{j=1}^N c_j e^{\lambda_j t} \mathbf{v}_j$$

is also a solution because the ODE is linear and homogenous. Provided the  $\mathbf{v}_j$  are linearly independent, then the above equation is a **general solution** because given any  $\mathbf{x}_0 \in \mathbb{R}^N$  there exist unique  $c_j$  such that

$$\mathbf{x}(0) = \sum_{j=1}^N c_j \mathbf{v}_j = \mathbf{x}_0.$$

**Example** Consider

$$\begin{aligned}\frac{dx}{dt} &= -5x + 2y, & x(0) &= 5, \\ \frac{dy}{dt} &= -6x + 3y & y(0) &= 7.\end{aligned}$$

Note that the initial value problem can be written in the vector form

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), \mathbf{x}(0) = \mathbf{x}_0,$$

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ -6 & 3 \end{bmatrix} \quad \text{and } \mathbf{x}_0 := \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Solving the system, using the eigenpair approach, we would need to find the eigenvectors and eigenvalues.

Characteristic equation is

$$0 = |\mathbf{A} - \lambda \mathbf{I}| = (-5 - \lambda)(3 - \lambda) + 12 \implies \lambda_1 := -3 \text{ and } \lambda_2 = 1.$$

Next we find the associated eigenvectors.

$$\lambda_1 = -3 : (\mathbf{A} + 3\mathbf{I})\mathbf{v} = \mathbf{0} \implies \mathbf{v}_1 := \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \lambda_2 = 1 : (\mathbf{A} - \mathbf{I})\mathbf{v} = \mathbf{0} \implies \mathbf{v}_2 := \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

This means that a general solution of the system  $\mathbf{x}' = \mathbf{A}\mathbf{x}$  is

$$\mathbf{x}(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Applying the initial value we can see that the unique solution is  $x(t) = 4e^{-3t} + e^t$  and  $y(t) = 4e^{-3t} + 3e^t$ .

A square matrix  $A \in \mathbb{C}^{N \times N}$  is **diagonalisable** if there exists a non-singular matrix  $Q \in \mathbb{C}^{N \times N}$  such that  $Q^{-1}AQ$  is diagonal.

**Theorem** A square matrix  $A \in \mathbb{C}^{N \times N}$  is diagonalisable if and only if there exists a basis  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N\}$  for  $\mathbb{C}^N$  consisting of eigenvectors of  $A$ . Indeed if,

$$A\mathbf{v}_j = \lambda_j \mathbf{v}_j \text{ for } j = 1, 2, \dots, N,$$

and we put  $Q = [\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_N]$  then  $Q^{-1}AQ = \Lambda$  where

$$A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix}$$

Consider a diagonalisable matrix  $A$ . Since  $Q^{-1}AQ = \Lambda$ , it follows that  $A$  has an eigenvalue decomposition

$$A = Q\Lambda Q^{-1}.$$

In general, we see by induction on  $k$  that

$$A^k = Q\Lambda^k Q^{-1} \text{ for } k = 0, 1, 2, \dots$$

**Example**

$$A = \begin{bmatrix} -5 & 2 \\ -6 & 3 \end{bmatrix}$$

then

$$\Lambda = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}, \quad Q^{-1} = \frac{1}{2} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

so

$$A^k = Q\Lambda^k Q^{-1} = \frac{1}{2} \begin{bmatrix} (-1)^k \times 3^{k+1} - 1 & (-1)^{k+1} \times 3^k + 1 \\ (-1)^k \times 3^{k+1} - 3 & (-1)^{k+1} \times 3^k + 3 \end{bmatrix}.$$

For any polynomial

$$p(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_m z^m$$

and any square matrix  $A$ , we define

$$p(A) = c_0 I + c_1 A + c_2 A^2 + \cdots + c_m A^m.$$

When  $A$  is diagonalisable,  $A^k = Q\Lambda^k Q^{-1}$  so

$$\begin{aligned} p(A) &= c_0 Q I Q^{-1} + c_1 Q \Lambda Q^{-1} + \cdots + c_m Q \Lambda^m Q^{-1} \\ &\vdots \\ &= Q p(\Lambda) Q^{-1} \end{aligned}$$

**Lemma** For any polynomial  $p$  and any diagonal matrix  $\Lambda$ ,

$$p(\Lambda) = \begin{bmatrix} p(\lambda_1) & & \\ & \ddots & \\ & & p(\lambda_N) \end{bmatrix}$$

**Theorem** If two polynomials  $p$  and  $q$  are equal on the spectrum of a diagonalisable matrix  $A$ , that is, if

$$p(\lambda_j) = q(\lambda_j) \text{ for } j = 1, 2, \dots, N,$$

then  $p(A) = q(A)$ .

**Example** Recall that

$$A = \begin{bmatrix} -5 & 2 \\ -6 & 3 \end{bmatrix}$$

has eigenvalues  $\lambda_1 = -3$  and  $\lambda_2 = 1$ . Let

$$p(z) = z^2 - 4 \quad \text{and} \quad q(z) = -2z - 1,$$

and observe

$$p(-3) = 5 = q(-3) \text{ and } p(1) = -3 = q(1).$$

We find

$$p(A) = A^2 - 4I = \begin{bmatrix} 9 & -4 \\ 12 & -7 \end{bmatrix} = -2A - I = q(A).$$

**Exponential of a Diagonalisable Matrix** If  $A = Q\Lambda Q^{-1}$  is diagonalisable, then

$$e^A = Qe^\Lambda Q^{-1} \text{ and } e^\Lambda = \begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_N} \end{bmatrix}$$

**Fundamental Matrix** A fundamental matrix  $\Phi$  for the linear homogenous vector equation

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$$

satisfies the following two properties.

1. The columns of  $\mathbf{X}$  are linearly independent vector functions so that, in particular,  $|\mathbf{X}(t)| \neq 0$ ; and
2.  $\Phi$  solves the matrix equation  $\mathbf{X}'(t) = \mathbf{A}\mathbf{X}(t)$ .

**Theorem** Suppose that  $\Phi$  is a fundamental matrix for the vector equations

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

Then every solution of this equation has the form

$$\Phi \mathbf{c}$$

for some constant vector  $\mathbf{c}$ .

**Nilpotent Matrix** A matrix is nilpotent if there exists a positive integer  $k$  such that  $\mathbf{A}^k = \mathbf{O}$ , where  $\mathbf{O}$  denotes the zero matrix.

**Example**

$$\mathbf{A}^2 = \mathbf{A}\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O}.$$

Therefore  $\mathbf{A}$  is nilpotent and, in particular,

$$e^t \mathbf{A} = \mathbf{I} + t\mathbf{A} = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$$

## 2.4 Stability

In many applications we are interested to know how the solution  $\mathbf{x}(t)$  behaves as  $t \rightarrow \infty$ , and might not care much about the precise details of the transient behaviour for finite  $t$ . We say that  $\mathbf{a} \in \mathbb{R}^N$  is an equilibrium point for the dynamical system  $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x})$  if

$$\mathbf{F}(\mathbf{a}) = \mathbf{0}.$$

Thus the solution of

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) \quad \text{for all } t, \text{ with } \mathbf{x}(0) = \mathbf{a}$$

is just the constant function  $\mathbf{x}(t) = \mathbf{a}$ .

An equilibrium point  $\mathbf{a}$  is **stable** if for every  $\epsilon > 0$  there exists  $\delta > 0$  such that whenever  $\|\mathbf{a}_0 - \mathbf{a}\| < \delta$  the solution of

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) \quad \text{for } t > 0, \text{ with } \mathbf{x}(0) = \mathbf{x}_0$$

satisfies

$$\|\mathbf{x}(t) - \mathbf{a}\| < \epsilon \text{ for all } t > 0.$$

Let  $D$  be an open subset of  $\mathbb{R}^N$  that contains an equilibrium point  $\mathbf{a}$ . We say that  $\mathbf{a}$  is **asymptotically stable** in  $D$  if  $\mathbf{a}$  is stable and, whenever  $\mathbf{a}_0 \in D$ , the solution of

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) \quad \text{for } t > 0, \text{ with } \mathbf{x}(0) = \mathbf{x}_0$$

satisfies

$$\mathbf{a}(t) \rightarrow \mathbf{a} \text{ as } t \rightarrow \infty.$$

In this case  $D$  is called a **domain of attraction** for  $\mathbf{a}$ .

**Criteria for Stability** Let  $A$  be a diagonalisable matrix with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_N$ . The equilibrium point  $\mathbf{a} = -A^{-1}\mathbf{b}$  is of

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{b} \text{ with } \mathbf{x}(0) = \mathbf{x}_0 \text{ and } \det(A) \neq 0.$$

1. **stable** if and only if  $\operatorname{Re} \lambda_j \leq 0$  for all  $j$
2. **asymptotically stable** if and only if  $\operatorname{Re} \lambda_j < 0$  for all  $j$ .

In the second case, the domain of attraction is the whole of  $\mathbb{R}^N$ .

## 2.5 Classification of 2D Linear Systems with $\det A \neq 0$

The equilibrium point  $\mathbf{a} = 0$  may be asymptotically stable, stable or unstable but may also have various other properties.

### 2.5.1 Case 1: Real Eigenvalues and Linearly Independent Eigenvectors

Suppose you have real eigenvalues  $\lambda_1$  and  $\lambda_2$  and two linearly independent eigenvectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . General solution:

$$\mathbf{x} = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2.$$

Canonical form:

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

**Stable Node Example** ( $\lambda_2 < \lambda_1 < 0$ )

$$\frac{dx}{dt} = -x, \quad \frac{dy}{dt} = -2y, \quad A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

Eigenvalues and eigenvectors:

$$\lambda_1 = -1, \lambda_2 = -2, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The general solution is

$$\mathbf{x} = c_1 e^{-t} \mathbf{v}_1 + c_2 e^{-2t} \mathbf{v}_2 = \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{-2t} \end{pmatrix}.$$

Solution of the initial value problem:

$$\begin{pmatrix} x(y) \\ y(t) \end{pmatrix} = \begin{pmatrix} x(0)e^{-t} \\ y(0)e^{-2t} \end{pmatrix}.$$

**Unstable Node Example** ( $0 < \lambda_1 < \lambda_2$ )

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = 2y, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Eigenvalues and eigenvectors:

$$\lambda_1 = 1, \lambda_2 = 2, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Solution of the initial value problem:

$$\begin{pmatrix} x(y) \\ y(t) \end{pmatrix} = \begin{pmatrix} x(0)e^t \\ y(0)e^{2t} \end{pmatrix}.$$

All trajectories (except  $\mathbf{x}(t) = \mathbf{0}$ ) are repelled from equilibrium point which is unstable.

**(Un)stable stars:**  $\lambda_1 = \lambda_2 \neq 0$

$$\frac{dx}{dt} = \lambda_1 x, \quad \frac{dy}{dt} = \lambda_1 y, \quad A = \lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

All vectors are eigenvectors.

The general solution is

$$\mathbf{x} = e^{\lambda_1 t} \mathbf{v}.$$

Solution of the initial value problem:

$$\mathbf{x}(t) = e^{\lambda_1 t} \mathbf{x}(0).$$

All orbits (except  $\mathbf{x}(t) = \mathbf{0}$ ) are oriented half-lines which are either attracted ( $\lambda_1 < 0$ ) or repelled ( $\lambda_1 > 0$ ) by the equilibrium point.

**Saddle Node Example (unstable:**  $\lambda_2 < 0 < \lambda_1$ )

$$\frac{dx}{dt} = x + 2y, \quad \frac{dy}{dt} = 3x + 2y, \quad A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

Eigenvalues and eigenvectors:

$$\lambda_1 = -1, \lambda_2 = 4, \mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Solution of the initial value problem:

$$\begin{pmatrix} x(y) \\ y(t) \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Here the first solution is repelling and the section is attracting, so the solution is unstable.

### Nonreal eigenstuff for $A$

$$\begin{aligned} e^{\lambda_1 t} \mathbf{v}_1 &= e^{(\alpha + \beta i)t} (\mathbf{p} + i\mathbf{q}) \\ &= e^{\alpha t} (\cos(\beta t) + i \sin(\beta t)) (\mathbf{p} + i\mathbf{q}) \\ &= \underbrace{e^{\alpha t} (\cos(\beta t) \mathbf{p} - \sin(\beta t) \mathbf{q})}_{:= \mathbf{x}_{\text{Re}}(t)} + i \underbrace{e^{\alpha t} (\sin(\beta t) \mathbf{p} + \cos(\beta t) \mathbf{q})}_{:= \mathbf{x}_{\text{Im}}(t)} \end{aligned}$$

So, a basis for the solution space is then

$$\mathcal{B} := \{\mathbf{x}_{\text{Re}}, \mathbf{x}_{\text{Im}}\}.$$

The general solution is,

$$\mathbf{x}(t) := c_1 \mathbf{x}_{\text{Re}}(t) + c_2 \mathbf{x}_{\text{Im}}(t)$$

for arbitrary constants  $c_1, c_2 \in \mathbb{R}$ .

### 2.5.2 Case 2: Complex Conjugate Eigenvalues

Suppose you have complex conjugate eigenvalues  $\lambda_1 = \bar{\lambda}_2 \notin \mathbb{R}$ .

General solution:

$$\mathbf{x} = c_1 \text{Re}(e^{\lambda_1 t} \mathbf{v}_1) + c_2 \text{Im}(e^{\lambda_1 t} \mathbf{v}_1).$$

Canonical form:

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad \lambda_1 = \alpha + i\beta.$$

### Interpretation

$$\mathbf{x}(t) = e^{\alpha t} R(t) \mathbf{x}(0), \quad R(t) = \begin{pmatrix} \cos \beta t & \sin \beta t \\ -\sin \beta t & \cos \beta t \end{pmatrix}$$

Thus, the initial vector  $\mathbf{x}(0)$  is rotated by the rotation matrix  $R(t)$  and scaled by the factor  $e^{\alpha t}$ .

**Centre Example (stable:  $\text{Re}(\lambda_1) = 0$ )**

$$\frac{dx}{dt} = -2y, \quad \frac{dy}{dt} = 2x, \quad A = \lambda_1 \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

Eigenvalues:

$$\lambda_1 = \bar{\lambda}_2 = -2i$$

Solution of the initial value problem:

$$\begin{pmatrix} x(y) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} y(0) \\ y(0) \end{pmatrix}.$$

The solution constitutes orbits which are oriented circles. These are stable (but not asymptotically stable).



**Stable Foci Example** ( $\operatorname{Re}(\lambda_1) < 0$ )

$$\frac{dx}{dt} = -x - 2y, \quad \frac{dy}{dt} = 2x - y, \quad A = \lambda_1 \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix}$$

Eigenvalues:

$$\lambda_1 = \bar{\lambda}_2 = -1 - 2i$$

Solution of the initial value problem:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t} \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} \rightarrow \mathbf{0} \text{ as } t \rightarrow \infty.$$

Orbits are oriented spirals which are attracted to the asymptotically stable equilibrium point.

## 2.6 Final Remarks on Nonlinear DEs

A function  $G : \mathbb{R}^N \rightarrow \mathbb{R}$  is a **first integral** (or constant of the motion) for the system of ODEs

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x})$$

if  $G(\mathbf{x}(t))$  is constant for every solution  $\mathbf{x}(t)$ .

**Simple Example** The function  $G(x, y) = x^2 + y^2$  is a first integral of the linear system of ODEs

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x.$$

In fact, putting

$$\mathbf{F}(x, y) = \begin{bmatrix} -y \\ x \end{bmatrix}$$

we have

$$\nabla \cdot \mathbf{F} = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \cdot \begin{bmatrix} -y \\ x \end{bmatrix} = (2x)(-y) + (2y)(x) = 0,$$

or equivalently,

$$\frac{dG}{dt} = \frac{\partial G}{\partial x} \frac{dx}{dt} + \frac{\partial G}{\partial y} \frac{dy}{dt} = (2x)(-y) + (2y)(x) = 0.$$

**Cayley-Hamilton** Let  $\mathbf{A} \in \mathbb{R}^{n \times n}$ . Then  $\mathbf{A}$  satisfies its characteristic equation.

**Putzer's Algorithm** Let  $\{\lambda_j\}_{j=1}^n$  be the collection of  $n$  not necessarily distinct eigenvalues of a given matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$ . Then

$$e^{\mathbf{A}t} = \sum_{k=0}^{n-1} p_{k+1}(t) \mathbf{M}_k,$$

where

$$\mathbf{M}_0 := \mathbf{I} \text{ and } \mathbf{M}_k := \prod_{j=1}^k (\mathbf{A} - \lambda_j \mathbf{I}), 1 \leq k \leq n,$$

and the vector-valued function  $\mathbf{p}(t) := (p_1(t), \dots, p_n(t))$  satisfies the vectorial equation

$$\mathbf{p}'(t) = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 1 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \lambda_n \end{bmatrix} \mathbf{p}(t), \quad \mathbf{p}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

So in the case in which  $n = 2$ , i.e., a two-dimensional vector equation, Putzer's algorithm reduces to

$$e^{\mathbf{A}t} = p_1(t)\mathbf{I} + p_2(t)(\mathbf{A} - \lambda_1\mathbf{I}),$$

where

$$\begin{bmatrix} p_1'(t) \\ p_2'(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}, \quad \mathbf{p}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Similarly, in the case in which  $n = 3$ , i.e., a three-dimensional vector equation, Putzer's algorithm reduces to

$$e^{\mathbf{A}t} = p_1(t)\mathbf{I} + p_2(t)(\mathbf{A} - \lambda_1\mathbf{I}) + p_3(t)(\mathbf{A} - \lambda_1\mathbf{I})(\mathbf{A} - \lambda_2\mathbf{I}),$$

where

$$\begin{bmatrix} p_1'(t) \\ p_2'(t) \\ p_3'(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 1 & \lambda_2 & 0 \\ 0 & 1 & \lambda_3 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix}, \quad \mathbf{p}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

# Chapter 3

## Initial-Boundary Value Problems in 1D

We have seen that an initial-value problem for a (nonsingular) linear ODE  $Lu = f$  always has a unique solution. However, matters are not so simple for a **boundary-value problem**: a solution might not exist, or if one exists it might not be unique.

### 3.1 Two-Point Boundary Value Problems

In an  $m$ th order **initial-value problem** we specify  $m$  initial conditions at the left end of the interval. In an  $m$ th order **boundary-value problem**, we again specify  $m$  conditions involving the solution and its derivatives, but some apply at the left end and some at the right end.

**Boundary Conditions** Consider the second-order ODE

$$u'' + u' = 0 \quad \text{for } 0 < x < \pi$$

whose general solution is

$$u(x) = A \cos x + B \sin x.$$

A **unique solution**  $u(x) = \sin x$  exists satisfying

$$u'(0) = 1 \text{ and } u(\pi) = 0.$$

**No solution** exists satisfying

$$u'(0) = 0 \text{ and } u(\pi) = 1.$$

**Infinitely many solutions**  $u(x) = C \sin x$  exists satisfying

$$u'(0) = 0 \text{ and } u(\pi) = 0.$$

We want to solve (Inhomogeneous BVP):

$$Lu = f \quad \text{for } a < x < b, \quad \text{with } B_1 u = \alpha_1 \text{ and } B_2 u = \alpha_2,$$

where

$$Lu = a_2 u'' + a_1 u' + a_0 u$$

is a 2nd-order linear differential operator, and the **boundary operators** have the form

$$\begin{aligned} B_1 u &= b_{11} u'(a) + b_{10} u(a), \\ B_2 u &= b_{21} u'(b) + b_{20} u(b). \end{aligned}$$

### Linear Two-Point Boundary Value

$$\begin{aligned}
u'' - u &= x - 1 && \text{for } 0 < x < \log 2, \\
u &= 2 && \text{at } x = 0, \\
u' - 2u &= 2 \log 2 - 4 && \text{at } x = \log 2.
\end{aligned}$$

## 3.2 Existence and Uniqueness

Since  $L$ ,  $B_1$  and  $B_2$  are all linear, the solutions of the **homogenous BVP**

$$Lu = 0 \quad \text{for } a < x < b, \quad \text{with } B_1 u = 0 \text{ and } B_2 u = 0,$$

form a vector space: if  $u_1$  and  $u_2$  are solutions of the inhomogeneous BVP then so is  $u = c_1 u_1 + c_2 u_2$  for any constants  $c_1$  and  $c_2$ .

**Uniqueness** The inhomogenous BVP has **at most** one solution iff the homogenous BVP has only the **trivial solution**  $u \equiv 0$ .

**Exactly One Solution** If the homogenous problem has only the trivial solution, then for every choice of  $f, \alpha_1$  and  $\alpha_2$  the inhomogenous problem

$$Lu = f \quad \text{for } a < x < b, \quad \text{with } B_1 u = \alpha_1 \text{ and } B_2 u = \alpha_2,$$

has a unique solution.

## 3.3 Inner Products and Norms of Functions

If a homogenous initial boundary value problem admits non-trivial solutions, then the inhomogeneous problem might or might not have any solutions, depending on the forcing term and boundary values.

To formulate a condition that guarantees existence we require a short digression that introduces some ideas from functional analysis.

The **inner product**  $\langle f, g \rangle$  of a pair of continuous functions  $f, g : [a, b] \rightarrow \mathbb{R}$  is defined by

$$\langle f, g \rangle = \int_a^b f(x)g(x) dx.$$

The corresponding **norm** of  $f$  is defined by

$$\|f\| = \sqrt{\langle f, f \rangle} = \left( \int_a^b [f(x)]^2 \right)^{1/2}.$$

We say that  $f$  and  $g$  are **orthogonal** if  $\langle f, g \rangle = 0$ .

**Inner Product and Norms** If

$$[a, b] = [-1, 1], \quad f(x) = x, \quad g(x) = \cos \pi x,$$

then

$$\langle f, g \rangle = \int_{-1}^1 x \cos \pi x dx = 0, \quad \|f\| = \sqrt{\frac{2}{3}}, \quad \|g\| = 1.$$

Thus,  $f$  and  $g$  are orthogonal over the interval  $[-1, 1]$ .

**Cauchy-Schwarz Inequality**  $|\langle f, g \rangle| \leq \|f\| \|g\|$ .

**Triangle Inequality**  $\|f + g\| \leq \|f\| + \|g\|$ .

### 3.4 Self-Adjoint Differential Operators

Define the **formal adjoint** as

$$\begin{aligned} L^*v &= (a_2v)'' - (a_1v)' + a_0v \\ &= a_2v'' + (2a_2' - a_1)v' + (a_2'' - a_1' + a_0)v \end{aligned}$$

and the **bilinear concomitant**

$$P(u, v) = u'(a_2v) - u(a_2v)' + u(a_1v),$$

we have the **Lagrange identity**

$$\langle Lu, v \rangle = \langle u, L^*v \rangle +$$

**Adjoint Operators and Lagrange identity** If

$$Lu = 3xu'' - (\cos x)u' + e^xu$$

then

$$\begin{aligned} L^*v &= (3xv)'' + [(\cos x)v]' + e^xv \\ &= 3xv'' + (6 + \cos x)v' + (e^x - \sin x)v \end{aligned}$$

and

$$\begin{aligned} P(u, v) &= u'(3xv) - u(3xv)' - uv \cos x \\ &= 3x(u'v - uv') - (3 + \cos x)uv. \end{aligned}$$

Then  $(Lu)v = uL^*v + \frac{d}{dx}P(u, v)$ .

The operator  $L$  is **formally self-adjoint** if  $L^* = L$ .

**Formally Self-Adjoint Condition** A second-order, linear differential operator  $L$  is formally self-adjoint iff it can be written in the form

$$Lu = -(pu')' + qu = -pu'' - p'u' + qu,$$

in which case the Lagrange identity takes the form

$$(Lu)v - u(Lv) = -(p(x)(u'v - uv'))',$$

or in other words, the bilinear concomitant is

$$P(u, v) = -p(x)(u'v - uv').$$

**Bessel and Legendre are Self-Adjoint** Consider the Bessel equation

$$x^2 u'' + xu' + (x^2 - \nu^2)u = f(x).$$

Dividing both sides by  $x$  gives  $Lu = -x^{-1}f(x)$  where

$$Lu = -(xu')' + (\nu^2 x^{-1} - x)u.$$

The Legendre equation

$$(1 - x^2)u'' - 2xu' + \nu(\nu + 1)u = f(x)$$

has the form  $Lu = -f(x)$  with  $Lu = -[(1 - x^2)u']' - \nu(\nu + 1)u$ .

**Transforming to Formally Self-Adjoint Form** If we can evaluate the integrating factor

$$p(x) = \exp \left( \int \frac{a_1(x)}{a_2(x)} dx \right),$$

then we can transform an ODE of the form  $a_2 u'' + a_1 u' + a_0 u = f(x)$  to formally self-adjoint form:

$$\begin{aligned} -pu'' - \frac{pa_1}{a_2}u' - \frac{pa_0}{a_2}u &= \frac{-pf(x)}{a_2}, \\ -(pu'' + p'u') - \frac{pa_0}{a_2}u &= \frac{-pf(x)}{a_2}, \\ -(pu')' + qu &= f(x) \end{aligned}$$

where  $q = -pa_0/a_2$  and  $f = -pf/a_2$ .

**Euler-Cauchy ODE** Write the Euler-Cauchy ODE  $ax^2 u'' + bxu' + cu = f(x)$  in formally self-adjoint form. Note that here  $a_2(x) = ax^2$ ,  $a_1(x) = bx$  and  $a_0(x) \equiv c$ .

Define  $p$  by

$$p(x) = \exp \left( \int \frac{bx}{ax^2} dx \right) = \exp \left( \frac{b}{a} \int \frac{1}{x} dx \right) = e^{\frac{b}{a} \ln x} = x^{\frac{b}{a}}.$$

Then recalling that

$$q = -\frac{pa_0}{a_2} \text{ and } \tilde{f} = \frac{pf}{a_2},$$

the formally self-adjoint form is

$$-(x^{\frac{b}{a}} u')' - \frac{c}{ax^2} x^{\frac{b}{a}} u = -\frac{1}{ax^2} x^{\frac{b}{a}} f(x).$$

Any formally self-adjoint operator  $L = -(pu')' + qu$  satisfies the identity

$$\langle Lu, v \rangle - \langle u, Lv \rangle = \sum_{i=1}^2 (B_i u R_i v - R_i u B_i v),$$

for all  $v$  and  $v$  where

$$R_1 u = \frac{p(a)u(a)}{b_{11}} \text{ or } R_1 u = -\frac{p(a)u'(a)}{b_{10}}$$

and

$$R_2 u = -\frac{p(b)u(b)}{b_{21}} \text{ or } R_2 u = \frac{p(N)u'(b)}{b_{20}}.$$

**Necessary Condition for Existence** If  $u$  is a solution of the inhomogeneous BVP, and if  $v$  is a solution of the homogenous problem

$$\begin{aligned} Lv &= 0 & \text{for } a < x < b, \\ B_1 v &= 0 & \text{at } x = a, \\ B_2 v &= 0 & \text{at } x = b. \end{aligned}$$

then on the one hand

$$\langle Lv, v \rangle - \langle u, Lv \rangle = \langle f, v \rangle - \langle u, 0 \rangle = \langle f, v \rangle$$

and on the other hand,

$$\langle Lv, v \rangle - \langle u, Lv \rangle = \underbrace{\alpha_1}_{=B_1 u} R_1 v - R_1 u \times \underbrace{0}_{=B_1 v} + \underbrace{\alpha_2}_{=B_2 u} R_2 v - R_2 u \times \underbrace{0}_{=B_2 v}.$$

then the data  $f, \alpha_1$  and  $\alpha_2$  must satisfy

$$\langle f, v \rangle = \alpha_1 R_1 v + \alpha_2 R_2 v.$$

**Fredholm Alternative** Either the homogenous BVP has only the trivial solution  $v \equiv 0$ , in which case

the inhomogeneous BVP has a unique solution  $u$  for every choice of  $f, \alpha_1$  and  $\alpha_2$ ,

OR else the homogenous BVP admits non-trivial solutions, in which case

the inhomogeneous BVP has a solution  $u$  iff  $f, \alpha_1$  and  $\alpha_2$  satisfy  $\langle f, v \rangle = \alpha_1 R_1 v + \alpha_2 R_2 v$  for every solution  $v$  of the homogenous BVP.

In the latter case,  $u + Cv$  is also a solution of the inhomogeneous BVP for any constant  $C$ .

# Chapter 4

## Generalised Fourier Series

This chapter will cover how if we generalise the concept of Fourier expansions that include the familiar trigonometric Fourier series allows us to solve a range of partial differential equations by separating variables in curvilinear coordinates.

### 4.1 Separation of Variables for Linear PDEs

As an example of the **separation of variables technique for linear PDEs** consider the one-dimensional heat PDE, which is

$$u_t = c^2 u_{xx},$$

where  $c$  is the thermal diffusivity of the material. We specifically consider as an example the following problem.

#### 4.1.1 The Diffusion PDE

$$\begin{aligned} u_t &= u_{xx}, & 0 \leq x \leq 1, t \geq 0 \\ u(0, t) &= 0 = u(1, t), & t > 0 \\ u(x, 0) &= f(x), & 0 < x < 1 \end{aligned}$$

Let  $u(x, t) = X(x)T(t)$ , so that

$$\begin{aligned} XT' &= X''T & \text{for } 0 \leq x \leq 1, t \geq 0, \\ X(0) &= X(1) = 0. \end{aligned}$$

Now we obtain:

$$\frac{X''}{X} = \frac{T'}{T}$$

and we set this equal to a separation constant  $-\lambda$  that will help us find a basis for the solutions.

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda \implies X'' = -\lambda X, \quad T' = -\lambda T.$$

Rearranging, we have:

$$X'' + \lambda X = 0 \quad T' + \lambda T = 0.$$

There are three cases for  $\lambda$ : zero, positive, negative.

**Case 1:**  $\lambda = 0$ . Then  $X'' = 0$ , which gives us  $X = Ax + B$ . The boundary conditions  $X(0) = X(1) = 0$  imply that  $B = 0$  and  $A = 0$ , which gives us  $X = 0$ .



**Case 2:**  $\lambda < 0$ . So  $\lambda = -k^2$  for some  $k > 0$ . Then  $X'' - k^2X = 0$ , which results in  $X(x) = Ae^{kx} + Be^{-kx}$ . However, the boundary conditions  $X(0) = X(1) = 0$  imply that  $A + B = 0$  and  $Ae^k + Be^{-k} = 0$ . This results in  $A = -B$  so  $A(e^k - e^{-k}) = 0$ , and so  $A = B = 0$ . Thus again,  $X \equiv 0$ .

**Case 3:**  $\lambda > 0$ . So  $\lambda = k^2$  for some  $k > 0$ . Then  $X'' + k^2X = 0$ , which means  $X(x) = A \cos(kx) + B \sin(kx)$ . The boundary conditions  $X(0) = X(1) = 0$  imply that  $A = 0$  and  $B \sin(k) = 0$ . This time, we can get non-trivial solutions, when  $k$  is a multiple of  $\pi$  i.e.  $k = n\pi$  for some  $n \in \mathbb{Z}^+$ .

Thus we are interested in  $\lambda = n^2\pi^2$ ,  $X(x) = B \sin(n\pi x)$  for  $n \in \mathbb{Z}^+$ .

Now we deal with  $T$ . Since we know  $\lambda$  now, we have

$$T' + n^2\pi^2T = 0$$

for some  $n \in \mathbb{Z}^+$ . We can solve this 1st order ODE:

$$T(t) = Ce^{-n^2\pi^2t}.$$

So for each  $n$ , we combine  $T(t)$  with  $X(x)$  to get

$$u_n(x, t) = A_n e^{-n^2\pi^2t} \sin n\pi x,$$

for some constant  $A_n$ . We then superimpose these solutions so,

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2t} \sin n\pi x.$$

Finally, we can use the initial conditions, yielding

$$u(x, 0) = \sum_{n=1}^{\infty} A_n \sin n\pi x = f(x).$$

This is the half-range Fourier sine series of  $f$ , so

$$A_n = 2 \int_0^1 f(x) \sin n\pi x \, dx.$$

If we were given an explicit  $f$ , we could evaluate this to get the final solution for  $u$ .

### 4.1.2 Wave Equation

Our second example of the application of Fourier series methods is to the partial differential equation describing a vibrating string, such as in a musical instrument like a piano.

Put  $c = \sqrt{T_0/\rho}$  (which has the dimensions of length / time and is called the wave speed). Now suppose that the string is initially at rest with a known deflection  $u_0(x)$ , then

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} &= 0, & 0 < x < \ell, t > 0, \\ u(0, t) &= 0, & t > 0, \\ u(\ell, t) &= 0, & t > 0, \\ u(x, 0) &= u_0(x), & 0 < x < \ell, \\ \frac{\partial u}{\partial t}(x, 0) &= 0, & 0 < x < \ell. \end{aligned}$$

Then the separation of variables technique used in the diffusion example follows almost exactly to solve for  $u$  in the wave equation.

## 4.2 Complete Orthogonal Systems

Expanding a function as a linear combination of orthogonal function leads naturally to the notion of a generalised Fourier series.

If  $w : (a, b) \rightarrow \mathbb{R}$  satisfies

$$w(x) > 0 \quad \text{for } a < x < b,$$

then we define the inner product with **weight function**  $w$  by

$$\langle f, g \rangle_w = \langle f, gw \rangle = \int_a^b f(x)g(x)w(x) dx,$$

and the corresponding norm by

$$\|f\|_w = \sqrt{\langle f, f \rangle_w} = \sqrt{\int_a^b [f(x)]^2 w(x) dx}.$$

Two functions  $f$  and  $g$  are **orthogonal with respect to  $w$  over the interval  $(a, b)$**  if  $\langle f, g \rangle_w = 0$ .

A set of functions  $S \subseteq L_2(a, b, w)$  is said to be **orthogonal** if every pair of functions in  $S$  is orthogonal and if no function is identically zero on  $(a, b)$ .

We say that  $S$  is **orthonormal** if, in addition, each function has norm 1.

**Orthogonal implies Independent** If  $S$  is orthogonal then  $S$  is linearly independent.

**Generalised Pythagorus Theorem** If  $\{\phi_1, \dots, \phi_N\}$  is orthogonal then, for any  $C_1, \dots, C_N \in \mathbb{R}$ ,

$$\left\| \sum_{j=1}^N C_j \phi_j \right\|_w^2 = \sum_{j=1}^N C_j^2 \|\phi_j\|_w^2.$$

**Generalised Fourier Coefficients** If  $f$  is in the span of an orthogonal set of functions  $\{\phi_1, \phi_2, \dots, \phi_N\}$  in  $L_2(a, b, w)$ , then the coefficients in the representation

$$f(x) = \sum_{j=1}^N A_j \phi_j(x)$$

are given by

$$A_j = \frac{\langle f, \phi_j \rangle_w}{\|\phi_j\|_w^2} \quad \text{for } 1 \leq j \leq N.$$

We call  $A_j$  the  $j$ th **Fourier coefficient** of  $f$  with respect to the given orthogonal set of functions.

Consider **approximating** a function  $f \in L_2(a, b, w)$  by a function in the span of an orthogonal set  $\{\phi_1, \phi_2, \dots, \phi_N\}$ , that is finding coefficients  $C_j$  such that

$$f(x) \approx \sum_{j=1}^N C_j \phi_j(x) \quad \text{for } a < x < b.$$

We seek to choose the  $C_j$  so that the **weighted mean-square error**

$$\left\| f - \sum_{j=1}^N C_j \phi_j \right\|_w^2 = \int_a^b \left( f(x) - \sum_{j=1}^N C_j \phi_j(x) \right)^2 w(x) dx$$

is as small as possible.

**Least-Squares Approximation** For all  $C_1, C_2, \dots, C_N$ , the weighted mean-square error satisfies

$$\left\| f - \sum_{j=1}^N C_j \phi_j \right\|_w^2 = \|f\|_w^2 - \sum_{j=1}^N A_j^2 \|\phi_j\|_w^2 + \sum_{j=1}^N (C_j - A_j)^2 \|\phi_j\|_w^2.$$

The Fourier coefficients satisfy Bessel's inequality, which is

$$\sum_{j=1}^{\infty} A_j^2 \|\phi_j\|_w^2 \leq \|f\|_w^2.$$

An orthogonal set  $S$  is **complete** if there is no non-trivial function in  $L_2(a, b, w)$  orthogonal to every function in  $S$ , i.e. if the condition

$$\langle f, \phi \rangle_w = 0 \quad \text{for every } \phi \in S$$

implies that

$$\|f\|_w = 0.$$

In particular, if  $S$  is a complete orthogonal set, then every proper subset of  $S$  **fails** to be complete.

**Example** The set  $S = \{\sin jx : j \geq 1 \text{ and } j \neq 7\}$  is **not** complete in  $L_2(0, \pi)$  because  $\sin 7x$  is orthogonal to every function in  $S$ .

**Equivalent Definitions of Completeness** If  $S = \{\phi_1, \phi_2, \dots\}$  is orthogonal in  $L_2(a, b, w)$ , then the following properties are equivalent:

1.  $S$  is complete;
2. for each  $f \in L_2(a, b, w)$  if  $A_j$  denotes the  $j$ th Fourier coefficient of  $f$  then

$$\left\| f - \sum_{j=1}^N A_j \phi_j \right\|_w \rightarrow 0 \text{ as } N \rightarrow \infty;$$

3. each function  $f \in L_2(a, b, w)$  satisfies **Parseval's identity**:

$$\|f\|_w^2 = \sum_{j=1}^{\infty} A_j^2 \|\phi_j\|_w^2.$$

**Least-squares Error** If  $S = \{\phi_1, \phi_2, \phi_3, \dots\}$  is a complete orthogonal sequence in  $L_2(a, b, w)$ ,

then for any  $f \in L_2(a, b, w)$ ,

$$\|e_N\|^2 = \sum_{j=N+1}^{\infty} A_j \|\phi_j\|_w^2.$$

### 4.3 Sturm-Liouville Problems

An ODE of the form

$$[p(x)u']' + [\lambda r(x) - q(x)]u = 0, \quad a < x < b,$$

is called a **Sturm-Liouville** equation. The coefficients  $p, q, r$  must all be real-valued with

$$p(x) > 0 \text{ and } r > 0 \text{ for } a < x < b.$$

Defining the formally self-adjoint differential operator

$$Lu = -[p(x)u']' + q(x)u,$$

we can write the ODE as

$$Lu = \lambda ru \quad \text{on } (a, b).$$

Any non-trivial (possibly complex-valued) solution  $u$  satisfying  $Lu = \lambda ru$  on  $(a, b)$  (plus appropriate boundary conditions) is said to be an **eigenfunction** of  $L$  with **eigenvalue**  $\lambda$ . In this case, we refer to  $(\phi, \lambda)$  as an **eigenpair**.

**Legendre's Equation** Legendre's equation

$$(1 - x^2)u'' - 2xu' + \nu(\nu + 1)u = 0$$

is equivalent to

$$[(1 - x^2)u']' + \nu(\nu + 1)u = 0$$

which is of the Sturm-Liouville form

$$p(x) = 1 - x^2, \quad q(x) = 0, \quad r(x) = 1, \quad \lambda = \nu(\nu + 1).$$

Assume as before that  $p, q, r$  are real-valued with  $p(x) > 0$  and  $r(x) > 0$  for  $a < x < b$ . A **regular Sturm-Liouville eigenproblem** is of the form

$$\begin{aligned} Lu &= \lambda ru & \text{for } a < x < b, \\ B_1 u &= b_{11}u' + b_{10}u = 0 & \text{at } x = a, \\ B_2 u &= b_{21}u' + b_{20}u = 0 & \text{at } x = b. \end{aligned}$$

where  $a$  and  $b$  are finite with

$$p(a) \neq 0 \text{ and } p(b) \neq 0,$$

and where  $b_{10}, b_{11}, b_{20}, b_{21}$  are real with

$$|b_{10}| + |b_{11}| \neq 0 \text{ and } |b_{20}| + |b_{21}| \neq 0.$$

**Eigenfunctions are Orthogonal** Let  $L$  be a Sturm-Liouville differential operator. If  $u, v : [a, b] \rightarrow \mathbb{C}$  satisfy

$$Lu = \lambda ru \text{ on } (a, b), \quad \text{with } B_1 u = 0 = B_2 u,$$

and

$$Lv = \mu rv \text{ on } (a, b), \quad \text{with } B_1 v = 0 = B_2 v,$$

and if  $\lambda \neq \mu$ , then  $u$  and  $v$  are orthogonal on the interval  $(a, b)$  with respect to the weight function  $r(x)$ , i.e.,

$$\langle u, v \rangle_r = \int_a^b u(x)v(x)r(x) dx = 0.$$

**Eigenvalues are Real** Let  $L$  be a Sturm-Liouville differential operation. If  $u : [a, b] \rightarrow \mathbb{C}$  is not identically zero and satisfies

$$Lu = \lambda ru \text{ on } (a, b), \quad \text{with } B_1 u = 0 = B_2 u,$$

then  $\lambda$  is real.

**Completeness of the Eigenfunctions** The regular Sturm-Liouville problem has a infinite sequence of eigenfunctions  $\phi_1, \phi_2, \phi_3, \dots$  with corresponding eigenvalues  $\lambda_1, \lambda_2, \lambda_3, \dots$  and moreover:

1. the eigenfunctions  $\phi_1, \phi_2, \phi_3, \dots$  form a complete orthogonal system on the interval  $(a, b)$  with respect to the weight function  $r(x)$ ;
2. the eigenvalues satisfy  $\lambda_1 < \lambda_2 < \lambda_3 < \dots$  with  $\lambda_j \rightarrow \infty$  as  $j \rightarrow \infty$ .

## 4.4 Elliptic Differential Operators

We now return to the study of PDEs by briefly introducing some concepts related to a more general study of second-order PDEs - namely, ellipticity and divergence form PDEs.

**Vector Calculus Notation** Partial derivative operator  $\partial_j = \partial/\partial x_j$ .

For a scalar field  $u : \mathbb{R}^d \rightarrow \mathbb{R}$ , the **gradient** is the vector field  $\text{grad } u : \mathbb{R}^d \rightarrow \mathbb{R}^d$  defined by

$$\text{grad } u = \nabla u = \sum_{j=1}^d \partial_j u \mathbf{e}_j = \begin{bmatrix} \partial_1 u \\ \partial_2 u \\ \dots \\ \partial_d u \end{bmatrix}$$

For a vector field  $\mathbf{F} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , the **divergence** is the scalar field  $\text{div } \mathbf{F} : \mathbb{R}^d \rightarrow \mathbb{R}$  defined by

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \sum_{j=1}^d \partial_j F_j = \partial_1 F_1 + \partial_2 F_2 + \dots + \partial_d F_d.$$

**Second-order Linear PDEs in  $\mathbb{R}^d$**  The most general second-order linear partial differential operator in  $\mathbb{R}^d$  has the form

$$Lu = - \sum_{j=1}^d \sum_{k=1}^d a_{jk}(\mathbf{x}) \partial_j \partial_k u + \sum_{k=1}^d b_k(\mathbf{x}) \partial_k u + c(\mathbf{x})u.$$

**Laplacian** The **Laplacian** is defined by  $\nabla^2 u = \nabla \cdot (\nabla u) = \text{div}(\text{grad } u)$ , that is,

$$\nabla^2 u = \sum_{j=1}^d \partial_j^2 u = \partial_1^2 u + \partial_2^2 u + \cdots + \partial_d^2 u.$$

Thus,  $-\nabla^2 u$  has the form of the second-order linear PDE with

$$a_{jk}(\mathbf{x}) = \delta_{jk}, \quad b_k(\mathbf{x}) = 0, \quad c(\mathbf{x}) = 0.$$

We call

$$L_0 u = \sum_{j=1}^d \sum_{k=1}^d a_{jk}(\mathbf{x}) \partial_j \partial_k u$$

the **principal part** of the partial differential operator.

A second-order linear partial differential operator is uniformly **elliptic** in a subset  $\omega \subseteq \mathbb{R}^d$  if there exists a positive constant  $c$  such that

$$\xi^T A(\mathbf{a}) \xi \geq c \|\xi\|^2 \quad \text{for all } \mathbf{a} \in \Omega \text{ and } \xi \in \mathbb{R}^d.$$

**Elliptic** The operator  $L = -\nabla^2$  is elliptic (with  $c = 1$ ) on any  $\Omega \subseteq \mathbb{R}^d$ , since

$$\sum_{j=1}^d \sum_{k=1}^d \delta_{jk} \xi_j \xi_k = \sum_{k=1}^d \xi_k^2 = \|\xi\|^2.$$

**Not Elliptic** The operator  $L = -(\partial_1^2 + 2\partial_2^2 - \partial_3^2)$  is **not** elliptic in  $\mathbb{R}^3$  since in this case the quadratic form

$$\xi^T A \xi = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \xi_1^2 + 2\xi_2^2 - \xi_3^2$$

is negative if  $\xi_1 = \xi_2 = 0$  and  $\xi_3 \neq 0$ .

**Symmetry and Skew-Symmetry** Put

$$a_{jk}^{sy} = \frac{1}{2}(a_{jk} + a_{kj}) = \text{symmetric part of } a_{jk}$$

$$a_{jk}^{sk} = \frac{1}{2}(a_{jk} - a_{kj}) = \text{skew-symmetric part of } a_{jk},$$

so that

$$a_{jk} = a_{jk}^{sy} + a_{jk}^{sk}, \quad a_{kj}^{sy} = a_{jk}^{sy}, \quad a_{kj}^{sk} = -a_{jk}^{sk}$$

When investigating if  $L$  is elliptic, it suffices to look at  $a_{jk}^{sy}$ .

**Lemma**

$$\sum_{j=1}^d \sum_{k=1}^d a_{jk}(\mathbf{x}) \xi_j \xi_k = \sum_{j=1}^d a_{jj}^{sy}(\mathbf{x}) \xi_j^2$$

**Theorem** Denote the eigenvalues of the real symmetric matrix  $[a_{jk}^s]$  by  $\lambda_j(x)$  for  $1 \leq j \leq d$ . The operator of Second-order Linear PDEs is elliptic on  $\Omega$  if and only if there exists a positive constant  $c$  such that

$$\lambda_j(\mathbf{x}) \geq c \quad \text{for } 1 \leq j \leq d \text{ and all } \mathbf{x} \in \Omega.$$

**Elliptic**

- $L = -(3\partial_1^2 + 2\partial_1\partial_2 + 2\partial_2^2)$  is elliptic.
- $L = -(\partial_1^2 - 4\partial_1\partial_2 + \partial_2^2)$  is not elliptic.

The Laplacian occurs in three of the most well studied PDEs:

1. **Poisson equation (Laplace's equation if  $f \equiv 0$ )** (elliptic):

$$-\nabla^2 u = f.$$

2. **Diffusion equation or heat equation** (parabolic):

$$\frac{\partial u}{\partial t} - \nabla^2 u = f.$$

3. **Wave equation** (hyperbolic):

$$\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = f.$$