

Abstract Algebra and Fundamental Analysis

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1 Transformations and Groups

Definition 1.1. A *transformation* on \mathbb{R}^n is a **bijection** from \mathbb{R}^n to \mathbb{R}^n . We will denote $\mathcal{B}(\mathbb{R}^n)$ the set of all transformations on \mathbb{R}^n .

In particular, a transformation on the Euclidean plane \mathbb{R}^2 is called a **plane transformation**.

Definition 1.2 (Group). A group is a set G equipped with a map

$$* : G \times G \rightarrow G, (g, h) \mapsto g * h = gh,$$

that satisfies the following axioms:

- (G1) **Associativity**, i.e. $g, h, k \in G$, then $(gh)k = g(hk)$.
- (G2) **Existence of identity**, i.e. there is an element denoted by e in G called the *identity* of G such that $eg = g = ge$ for any $g \in G$. (Such e is unique; notation: 1_G .)
- (G3) **Existence of inverse**, i.e. for any $g \in G$, there is an element denoted by $h \in G$ called the inverse of g such that $gh = hg = e$. (h is also unique; notation: g^{-1} .)

A group G is called commutative or abelian if $gh = hg$ for all $g, h \in G$.

Proposition 1.3. *Examples of Transformation Groups*

- (1) *The set $\mathcal{B}(\mathbb{R}^n)$ of all transformations on \mathbb{R}^n together with the operation of composition forms a group.*
- (2) *The set $\mathcal{T}(\mathbb{R}^n)$ of all translations on \mathbb{R}^n together with the operation of composition forms a group.*
- (3) *The set $\mathcal{C}(\mathbb{R}^n)$ of collineations of \mathbb{R}^n together with the operation of composition forms a group.*

Definition 1.4 (Subgroup). Let $(G, *)$ be a group. A nonempty subset $H \subseteq G$ is said to be a subgroup of G , denoted by $H \leq G$, if $(H, *)$ is a group.

Lemma 1.5 (Subgroup Lemma). *A nonempty subset H of a group G is a subgroup if and only if the following two closure conditions are satisfied:*

- (SG1) *Closure under multiplication, i.e. if $h, k \in H$, then $hk \in H$;*
- (SG2) *Closure under inverse, i.e. if $h \in H$, then $h^{-1} \in H$.*

In particular, $1_H = 1_G \in H$.

Definition 1.6 (Group Isomorphisms). For groups G, H , a map $f : G \rightarrow H$ is called a group homomorphism if $f(xy) = f(x)f(y)$ for all $x, y \in G$. A bijective group homomorphism is called an isomorphism. In this case, we say that G is isomorphic to H . Notation $G \cong H$.

2 Subgroups and the Group of Isometries

Lemma 2.1. *If S is a subset of a group $(G, *)$, then $\langle S \rangle = \bigcap_{S \subseteq H \leq G} H$. In other words, $\langle S \rangle$ is the **smallest** subgroup of G that contains all the elements of S .*

Definition 2.2. We call $\langle S \rangle$ the **subgroup of G generated by S** . A group generated by one element is called a **cyclic group**.

Notation:

- space: \mathbb{R}^n ;
- points: A, B, C, P, Q, R, \dots with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{p}, \mathbf{q}, \mathbf{r}, \dots$;
- transformations: $\tau, \pi, \sigma, \delta, \dots$;
- lines: l, m, n, \dots ; line equations in \mathbb{R}^n : $\mathbf{x} = \mathbf{a} + \lambda \mathbf{v}$ for all $\lambda \in \mathbb{R}$;
- planes in \mathbb{R}^n : $\mathbf{x} = \mathbf{a} + \lambda \mathbf{u} + \mu \mathbf{v}$ for all $\lambda, \mu \in \mathbb{R}$;
- **Hyperplanes** through $\mathbf{a} \in \mathbb{R}^n$ with normal $\mathbf{n} \in \mathbb{R}^n = \mathbf{0}$:

$$\mathbb{H}_{\mathbf{n}, \mathbf{a}} = \{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{x} - \mathbf{a}) \cdot \mathbf{n} = 0\} = \langle \mathbf{n} \rangle^\perp + \mathbf{a}.$$

- For points P, Q in \mathbb{R}^n , we may also define the **perpendicular bisector** of the line segment \overline{PQ} to be the hyperplane \mathbb{H} that passes through the midpoint of \overline{PQ} and perpendicular to \overline{PQ} . So \mathbb{H} has the equation $(\mathbf{x} - \mathbf{m}) \cdot (\mathbf{p} - \mathbf{q}) = 0$ where $\mathbf{m} = \frac{1}{2}(\mathbf{p} + \mathbf{q})$.
- It is clear that, for all $X \in \mathbb{H}$,

$$d(X, P) = \sqrt{\|\mathbf{x} - \mathbf{m}\|^2 + \|\mathbf{p} - \mathbf{m}\|^2} = \sqrt{\|\mathbf{x} - \mathbf{m}\|^2 + \|\mathbf{q} - \mathbf{m}\|^2} = d(X, Q).$$

The Euclidean space \mathbb{R}^n

- Length of a vector: $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$;
- Distance between two points P, Q : $d(P, Q) := \|\mathbf{p} - \mathbf{q}\|$;
- Projection of \mathbf{a} on \mathbf{b} : $\text{proj}_{\mathbf{b}}(\mathbf{a}) = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b}$;
- Angle between \mathbf{a} and \mathbf{b} : $\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|}$;
- Orthogonality: $\mathbf{a} \perp \mathbf{b} \iff \mathbf{a} \cdot \mathbf{b} = 0$;

Definition 2.3. An *isometry* on \mathbb{R}^n is a map $\tau : \mathbb{R}^n \rightarrow \mathbb{R}^n$ which preserves distance between points: $d(P, Q) = d(\tau(P), \tau(Q))$, $\forall P, Q \in \mathbb{R}^n$.

Lemma 2.4. *The set of isometries which fix the zero vector is equal to the set of (linear) maps that represent multiplication by an orthogonal matrix.*

Theorem 2.5. *An isometry can be decomposed into a translation multiplied by a linear transformation, which can be represented by an orthogonal matrix. In other words, for every $\tau \in \mathcal{I}(\mathbb{R}^n)$, there exist an orthogonal $n \times n$ matrix Q and a vector $\mathbf{b} \in \mathbb{R}^n$ such that $\tau = T_{Q, \mathbf{b}} = T_{I, \mathbf{b}} \circ T_{Q, \mathbf{0}}$. In particular, an isometry is a **transformation**.*

Theorem 2.6. *The group of Isometries*

- (1) *The set $\mathcal{I}(\mathbb{R}^n)$ of all isometries forms a subgroup of the group $\mathcal{B}(\mathbb{R}^n)$ of all transformations.*
- (2) *The group $\mathcal{I} = \mathcal{I}(\mathbb{R}^n)$ contains two subgroups: the group \mathcal{T} of translations and the group \mathcal{O} of all orthogonal linear transformations. Moreover, we have $\mathcal{I} = \mathcal{T}\mathcal{O} := \{\tau\sigma \mid \tau \in \mathcal{T}, \sigma \in \mathcal{O}\}$.*

3 Reflections and Isometries

Definition 3.1. Let \mathbb{H} be a hyperplane. The reflection $\sigma_{\mathbb{H}}$ in \mathbb{H} is the mapping defined by:

$$\sigma_{\mathbb{H}}(P) = \begin{cases} P & \text{if } P \in \mathbb{H}; \\ P' & \text{if } P \text{ is off } \mathbb{H} \text{ and } \mathbb{H} \text{ is the perpendicular bisector of } P\bar{P}'. \end{cases}$$

(in the sense that $d(P, X) = d(P', X)$ for all $X \in \mathbb{H}$.)

Proposition 3.2. Let \mathbb{H} be a hyperplane.

- (1) A reflection $\sigma_{\mathbb{H}}$ is an isometry satisfying $\sigma_{\mathbb{H}}^2 = 1$.
- (2) $\sigma_{\mathbb{H}}$ fixes a line $m \not\subseteq \mathbb{H}$ if and only if $m \perp \mathbb{H}$.
- (3) $\sigma_{\mathbb{H}}$ fixes a line **pointwise** if and only if $m \subseteq \mathbb{H}$.

Theorem 3.3. If $\mathbb{H} = \mathbb{H}_{\mathbf{n}, \mathbf{a}}$, then there exist $Q = I - \frac{2}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} \mathbf{n}^T \in O_n(\mathbb{R})$ and $\mathbf{b} = 2 \frac{\mathbf{a} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n}$ such that

$$\sigma_{\mathbb{H}}(\mathbf{x}) = Q\mathbf{x} + \mathbf{b}.$$

Corollary 3.4. In \mathbb{R}^2 , if line ℓ has equation $aX + bY + c = 0$, then the reflection σ_{ℓ} in ℓ has equation:

$$\begin{aligned} \sigma_{\ell}(\mathbf{x}) &= \frac{1}{a^2 + b^2} \begin{bmatrix} b^2 - a^2 & -2ab \\ -2ab & a^2 - b^2 \end{bmatrix} \mathbf{x} + \frac{1}{a^2 + b^2} \begin{bmatrix} -2ac \\ -2bc \end{bmatrix} \\ &= \begin{pmatrix} x \\ y \end{pmatrix} - 2 \frac{(ax + by + c)}{a^2 + b^2} \begin{pmatrix} a \\ b \end{pmatrix}. \end{aligned}$$

Definition 3.5 (Points in Generic Position). We say that m points $P_1(\mathbf{p}_1), P_2(\mathbf{p}_2), \dots, P_m(\mathbf{p}_m)$ in \mathbb{R}^n are in **generic position** if the vectors $\mathbf{p}_i - \mathbf{p}_1$, for $i = 2, 3, \dots, m$, are linearly independent. In particular, $n + 1$ points in \mathbb{R}^n are in generic position if every hyperplane contains at most n of the $n + 1$ points.

Theorem 3.6. (1) An isometry on \mathbb{R}^n that fixes $n + 1$ points in generic position is the identity map.

(2) An isometry on \mathbb{R}^n that fixes n points in generic position is a reflection **or** the identity.

(3) An isometry that fixes $n - 1$ but not n points in generic position is a product of two **reflections**.

(4) Every isometry (in \mathbb{R}^n) is a product of **at most** $n + 1$ reflections.

Corollary 3.7. The group $\mathcal{I}(\mathbb{R}^n)$ is generated by reflections $\mathbb{H}_{\mathbf{n}, \mathbf{a}}$ for all $\mathbf{0} \neq \mathbf{n}, \mathbf{a} \in \mathbb{R}^n$.

Corollary 3.8. (1) A plane isometry that fixes three vertices of a triangle is the identity map.

(2) Every plane isometry $\tau \in \mathcal{I}(\mathbb{R}^2)$ is a product of at most three reflections in three lines.

4 Translations and Rotations on \mathbb{R}^2

Theorem 4.1. *An isometry τ in \mathbb{R}^n is a **translation** if and only if τ is the product of two reflections in parallel hyperplanes.*

Corollary 4.2. *A plane isometry is a translation if and only if it is a product of two reflections in parallel lines.*

Definition 4.3. A **rotation** on \mathbb{R}^2 about a point C , through angle θ , is the transformation that fixes C and otherwise sends a point P to a point P' , where $d(C, P) = d(C, P')$, and the angle from \vec{CP} to $\vec{CP'}$ is θ (in anti-clockwise direction) if $\theta > 0$, and clockwise if $\theta < 0$). We denote this transformation by $\rho_{C,\theta}$.

Theorem 4.4. *A plane isometry is a **rotation** if and only if it is the product of two reflections in intersecting lines. Further we have*

- (1) *if lines l, m intersect at C , and the directed angle from l to m is $\frac{\theta}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, then $\sigma_m \sigma_l = \rho_{C,\theta}$;*
- (2) *if lines p, q, r are concurrent, then there exists a line l such that $\sigma_r \sigma_q \sigma_p = \sigma_l$.*

Corollary 4.5. (1) *A non-identity rotation (on \mathbb{R}^2) fixes exactly one point.*

- (2) *A rotation with centre C fixes every circle with centre C .*
- (3) *The set of all rotations about a particular point (i.e., with centre at a particular point) is a subgroup of the group $\mathcal{I}(\mathbb{R}^2)$ of isometries; further still, it is a **commutative** subgroup. In other words,*

$$\mathcal{R}_C := \{\rho_{C,\theta} : \theta \in \mathbb{R}\} \leq \mathcal{I}(\mathbb{R}^2) \text{ and } \rho\rho' = \rho'\rho, \forall \rho, \rho' \in \mathcal{R}_C.$$

Theorem 4.6 (Equation of a rotation). (1) *The rotation $\rho_{\mathbf{0},\theta} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ about the origin $\mathbf{0}$ and through angle θ is the linear isomorphism $T_{Q,\mathbf{0}}(\mathbf{x}) = Q\mathbf{x}$, where Q is the following matrix:*

$$Q = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

- (2) *If \mathbf{c} is the position vector of C , then $\rho_{C,\theta} = T_{\mathbf{c}}(\rho_{\mathbf{0},\theta})T_{-\mathbf{c}}$. Hence, $\rho_{C,\theta}$ has the equation $\rho_{C,\theta}(\mathbf{x}) = Q\mathbf{x} + \mathbf{b}$, where Q defines $\rho_{\mathbf{0},\theta}$ as in (1) and $\mathbf{b} = (I - Q)\mathbf{c}$. At the group level, we have $\mathcal{R}_C = T_{\mathbf{c}}\mathcal{R}_{\mathbf{0}}T_{-\mathbf{c}}$. Call the group \mathcal{R}_C is **conjugate** to the group $\mathcal{R}_{\mathbf{0}}$.*

Half-turn A rotation of the form $\rho_C := \rho_{C,\pi}$ is called a half-turn. A half-turn has the equation

$$\mathbf{x}' = -\mathbf{x} + 2\mathbf{c},$$

where \mathbf{c} is the position vector of C .

Definition 4.7. A figure $F_1 \subseteq \mathbb{R}^n$ is **congruent** to a figure $F_2 \subseteq \mathbb{R}^n$ if one can be mapped onto the other by an isometry; i.e. if there exists an isometry τ such that $\tau(F_1) = F_2$. **Notation:** $F_1 \cong F_2$ means F_1 is congruent to F_2 .

Theorem 4.8. *If $\triangle ABC \cong \triangle A'B'C'$ in \mathbb{R}^2 (same side lengths), then there exists a **unique** plane isometry τ such that*

$$\tau(A) = A', \tau(B) = B', \tau(C) = C'.$$

5 Classification of Plane Isometries

Definition 5.1. A plane isometry τ is called a **glide reflection** with axis c (a line) if there exist distinct lines a, b which are perpendicular to c such that $\tau = \sigma_c \sigma_b \sigma_a (= \sigma_b \sigma_a \sigma_c)$.

Proposition 5.2. (1) *A glide reflection is a composition of a reflection in line a and a halfturn centred at a point off a .*

(2) *A glide reflection is a translation followed by a reflection.*

(3) *A glide reflection fixes no points.*

(4) *A glide reflection fixes exactly one line, the axis, c .*

(5) *The midpoint of any point and its image under a glide reflection lies on its axis (c).*

Theorem 5.3. *Distinct lines p, q, r are neither concurrent, nor parallel, if and only if $\sigma_r \sigma_q \sigma_p$ is a glide reflection.*

Definition 5.4. An isometry that is a product of an even (resp., odd) number of reflections is said to be even (resp., odd) isometry.

Theorem 5.5. 1. *The set \mathcal{E} of even isometries in \mathbb{R}^n forms a subgroup of \mathcal{I} .*

2. *If \mathcal{E}' denotes the set of odd isometries, then $\mathcal{E} \cap \mathcal{E}' = \emptyset$.*

3. *If $\sigma = \sigma_{\mathbb{H}}$ is a reflection, then $\mathcal{E}' = \sigma \mathcal{E} := \{\sigma \pi \mid \pi \in \mathcal{E}\}$.*

4. *We also have $\sigma \mathcal{E} = \mathcal{E} \sigma$ and $\mathcal{I} = \mathcal{E} \sqcup \sigma \mathcal{E}$.*

Corollary 5.6. *For any non-identity plane isometries, it is either even or odd. All even isometries are either translations or rotations. All odd isometries are reflections or glide reflections.*

Theorem 5.7. *A product of 4 reflections in \mathbb{R}^2 is a product of 2 reflections.*

Definition 5.8. Let $\Omega \subseteq \mathbb{R}^n$ be a geometric figure (or a subset). A **symmetry** of Ω is an isometry τ such that $\tau(\Omega) = \Omega$.

All the symmetries of Ω form a group $\text{sym}(\Omega)$, the **symmetry group** of Ω .

6 Similarities

Definition 6.1. A transformation $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called a **similarity of ratio** $r > 0$ if

$$d(\alpha(P), \alpha(Q)) = rd(P, Q), \text{ for all } P, Q \in \mathbb{R}^n.$$

Proposition 6.2. (1) *An isometry is a similarity of ratio 1.*

(2) *A similarity fixing two points is an isometry.*

(3) *A similarity fixing $n + 1$ points in generic position is the identity.*

(4) *The set of all similarities in \mathbb{R}^n forms a group, denote this set by \mathcal{S} or $\mathcal{S}(\mathbb{R}^n)$.*

Definition 6.3. A **stretch of ratio** $r > 0$ about point C is a transformation $\delta_{C,r}$ that fixes C and otherwise sends a point P to a point P' , where P' is the unique point on the **ray** from C through P such that $d(C, P') = r \cdot d(C, P)$.

Theorem 6.4. *Decomposition of a similarity* If α is a similarity of ratio $r > 0$, and P is any **fixed** point, then $\alpha = \tau \delta_{P,r} = \delta_{P,r} \tau'$, for some isometries τ, τ' . Moreover, we have

$$\mathcal{S} = \bigsqcup_{r>0} \mathcal{S} S_{P,r} = \bigsqcup_{r>0} S_{P,r} \mathcal{S} \text{ (disjoint unions),}$$

where $\mathcal{S} S_{P,r} = \{\tau S_{P,r} \mid \tau \in \mathcal{S}\}$ and $S_{P,r} \mathcal{S} = \{S_{P,r} \tau \mid \tau \in \mathcal{S}\}$.

Corollary 6.5. *A similarity is a **collineation** that preserves betweenness, midpoints, angles, perpendicularity, etc.*

Definition 6.6. (1) A **point reflection** about $C(\mathbf{c})$ is the isometry $\rho_C : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by

$$\rho_C(\mathbf{x}) = -(\mathbf{x} - \mathbf{c}) + \mathbf{c} = -\mathbf{x} + 2\mathbf{c}.$$

(2) A **dilation** about the point C is a stretch transformation $\delta_{C,r}$ ($r > 0$) about C , or it is a stretch transformation followed by a point reflection both about C (i.e., $\rho_C \delta_{C,r}$).

Lemma 6.7. (1) *A point reflection is an isometry.*

(2) *The product of two point reflections is a translation.*

(3) *The product of a translation and a point reflection is a point reflection.*

Proposition 6.8. *All point reflections generate a subgroup \mathcal{H} (of \mathcal{S}). Moreover, \mathcal{H} is a (disjoint) union of the set \mathcal{T} of all translations and the set of all point reflections: for a fixed C ,*

$$\mathcal{H} = \mathcal{T} \sqcup \rho_C \mathcal{T} = \mathcal{T} \sqcup \mathcal{T} \rho_C = \mathcal{T} \sqcup \{\rho_P \mid P \in \mathbb{R}^n\}.$$

Proposition 6.9. *The dilation $\tau = \rho_C \delta_{C,r}$ ($r > 0$) has the following equation:*

$$\tau(\mathbf{x}) = (-r)\mathbf{x} + (1+r)\mathbf{c}.$$

Lemma 6.10. *Let $\mathbb{R}^\times = \{r \in \mathbb{R} \mid r \neq 0\}$. For any $r, s \in \mathbb{R}^\times$, and any point $P(\mathbf{p})$, we have*

$$(1) \delta_{P,-r} = \rho_O \delta_{P,r};$$

$$(2) \delta_{P,1} = 1, \delta_{P,-1} = \rho_P;$$

$$(3) \delta_{P,r} \delta_{P,s} = \delta_{P,rs};$$

$$(4) \delta_{P,r}^{-1} = \delta_{P,r^{-1}}.$$

Proposition 6.11. *The set $\{\delta_{C,r} \mid r \in \mathbb{R}^\times (:= \mathbb{R} - 0)\}$ forms a group that is isomorphic to the group $(\mathbb{R}^\times, \cdot)$.*

7 Dilatations

Definition 7.1. A collineation δ on \mathbb{R}^n is called a **dilatation** if, for every line ℓ in \mathbb{R}^n , $\ell \parallel \delta(\ell)$.

Proposition 7.2. The set $\mathcal{D}(\mathbb{R}^n)$ of all dilatations in \mathbb{R}^n forms a subgroup of $\mathcal{C}(\mathbb{R}^n)$.

Lemma 7.3. A dilatation that fixes two points is the identity map. Hence, for dilatations δ_1, δ_2 and distinct point A, B , if $\delta_1(A) = \delta_2(A)$ and $\delta_1(B) = \delta_2(B)$, then $\delta_1 = \delta_2$.

Lemma 7.4. (1) If A, B, C are collinear, distinct, with $\frac{CB}{CA} = r \neq 0$, then $\delta_{C,r}(A) = B$.

(2) For collinear points A, B, P, P' , if $\frac{AP}{PB} = \frac{AP'}{P'B}$, then $P = P'$.

(3) Let τ be a dilatation and let $\tau(P) = P'$ for every point P . If there exist points A, B such that \overrightarrow{AB} and $\overrightarrow{A'B'}$ have the same (resp., opposite) direction, then, for any points C, D , \overrightarrow{CD} and $\overrightarrow{C'D'}$ have the same (resp., opposite) direction.

Corollary 7.5. If points A, B, C are sent to A', B', C' under a dilatation, then

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}.$$

Theorem 7.6. A dilatation is either a translation or a dilation. Hence, every dilatation is a similarity.

8 Classification of Plane Similarities

Definition 8.1. We say that figure $f_1 \subseteq \mathbb{R}^n$ and figure $f_2 \subseteq \mathbb{R}^n$ are **similar** if there is a similarity α such that $\alpha(f_1) = f_2$.

Theorem 8.2. If $\triangle ABC \sim \triangle A'B'C'$ in \mathbb{R}^2 , then there exists a **unique** plane similarity α such that

$$\alpha(A) = A', \alpha(B) = B', \alpha(C) = C'.$$

Theorem 8.3 (Equations of Similarities). If α is a similarity in \mathbb{R}^n , then there exist $Q \in O_n(\mathbb{R})$, $\mathbf{b} \in \mathbb{R}^n$ and $r \in \mathbb{R}_{>0}$ such that

$$\alpha(\mathbf{x}) = rQ\mathbf{x} + \mathbf{b}, \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Lemma 8.4. A similarity without a fixed point is an isometry.

Definition 8.5. (1) A **stretch reflection** in \mathbb{R}^2 is a non-identity stretch about some point C followed by a reflection about a line through C .

(2) A **stretch rotation** in \mathbb{R}^2 is a non-identity stretch about some point C followed by a non-identity rotation about C .

Theorem 8.6. A non-identity plane similarity is exactly one of the following:

Isometry, Stretch of ratio $r \neq 1$, Stretch reflections, Stretch rotation.

Theorem 8.7. In the equation of similarities, the algebraic classification is as follows:

1. α is an isometry if $r = 1$;
2. α is a stretch (of ratio $r \neq 1$) if $r \neq 1$ and $Q = I$;
3. α is a stretch reflection if $r \neq 1, Q \neq I$ and $\det(Q) = -1$;
4. α is a stretch rotation if $r \neq 1, Q \neq I$ and $\det(Q) = 1$;

9 Normal Subgroups

Definition 9.1. A subgroup K of a group G is called a **normal subgroup** if $g^{-1}Kg \leq K$ (equivalently, $g^{-1}Kg = K$, or $gK = Kg$) for all $g \in G$. **Notation:** $K \trianglelefteq G$.

Theorem 9.2. Suppose $\alpha \in \mathcal{S}$ is a similarity, and $G \in \{\mathcal{I}, \mathcal{E}, \mathcal{D}, \mathcal{H}, \mathcal{T}\}$. Then $\alpha\tau\alpha^{-1} \in G$, for all $\tau \in G$. In other words, each of the groups $\mathcal{I}, \mathcal{E}, \mathcal{D}, \mathcal{H}, \mathcal{T}$ is a normal subgroup of \mathcal{S} .

Corollary 9.3. For $\alpha \in \mathcal{S}$, a point C and a hyperplane \mathbb{H} in \mathbb{R}^n , we have

$$\alpha\sigma_{\mathbb{H}}\alpha^{-1} = \sigma_{\alpha(\mathbb{H})}, \quad \alpha\rho_C\alpha^{-1} = \rho_{\alpha(C)}, \quad \alpha\delta_{C,r}\alpha^{-1} = \delta_{\alpha(C),r}.$$

In particular, in \mathbb{R}^2 , $\alpha\rho_{C,\theta}\alpha^{-1} = \rho_{\alpha(C),\pm\theta}$.

Proposition 9.4. 1. If $H \leq G$, then G is a disjoint union of **cosets** $gH, g \in G$.

2. If $K \trianglelefteq G$, then $G/K := \{gK \mid g \in G\}$ is a group with the subset multiplication. (G/K is called the **quotient group** of G by K).

10 Collineations

Theorem 10.1. A transformation is a collineation in \mathbb{R}^n if and only if the images of collinear points are themselves collinear.

Lemma 10.2. If α is a collineation in \mathbb{R}^n , and l, m are parallel lines, then $\alpha(l)$ and $\alpha(m)$ are parallel.

Theorem 10.3. A collineation takes the midpoint of points A, B to the midpoints of points $\alpha(A), \alpha(B)$.

Corollary 10.4. For a collineation α , if $n+1$ points P_0, P_1, \dots, P_n divide the segment $\overline{P_0P_n}$ into n congruent segments $\overline{P_{i-1}P_i}$, and $P'_i = \alpha(P_i)$, then the $n+1$ points P'_0, \dots, P'_n divide the segment $\overline{P'_0P'_n}$ into n congruent segments $\overline{P'_{i-1}P'_i}$.

In particular, if a point P is between A and B , and $\frac{AP}{PB} = r$ is **rational**, then $P' = \alpha(P)$ is between $\alpha(A)$ and $\alpha(B)$ and $\frac{A'P'}{P'B'} = r$.

11 Darboux's Theorem

Lemma 11.1. Let $t > 0, t \neq 1$, and P, Q be points on $\ell(A, B)$ such that $\frac{AP}{PB} = t, \frac{AQ}{QB} = -t$. Then C is the midpoint of P, Q if and only if $\frac{AC}{CB} = -t^2$.

Theorem 11.2. If α is a collineation, and point P is between points A, B then $\alpha(P)$ is between $\alpha(A), \alpha(B)$.

Corollary 11.3. A collineation on \mathbb{R}^n fixing two points on a line fixes the line pointwise.

12 Affine Transformations

Theorem 12.1. *A collineation in \mathbb{R}^n fixing $n + 1$ points in generic position is the identity.*

Definition 12.2. An **affine transformation** $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is one that has an equation of the form $\alpha(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for all $\mathbf{x} \in \mathbb{R}^n$, where $A \in GL_n(\mathbb{R})$, $\mathbf{b} \in \mathbb{R}^n$. (In other words, $\alpha = T_{A,\mathbf{b}}$.)

Lemma 12.3. *The set \mathcal{A} of all affine transformations in \mathbb{R}^n forms a group. Moreover, it contains the similarity group \mathcal{S} as a subgroup of \mathcal{A} .*

Theorem 12.4. *Let τ be a transformation. Then the following are equivalent:*

1. τ is an affine transformation;
2. τ is a collineation.

Proposition 12.5. *A (non-degenerate) conic section*

$$aX^2 + bXY + cY^2 + dX + eY + f = 0$$

*is affine equivalent to one of the following **affine standard form**:*

$$Y = X^2, \quad X^2 + Y^2 = 1, \quad XY = 1.$$

Definition 12.6. An affine transformation α with equation $\alpha(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ is called an **equi-affine transformation** if $\det(\alpha) := \det(A) = \pm 1$. An equi-affine transformation in \mathbb{R}^2 is called an **equiareal transformation**.

Proposition 12.7 (The group of equi-affine transformations). *The set \mathcal{Q} of all equi-affine transformations forms a subgroup of \mathcal{A} that has $\mathcal{Q}^+ = \{\alpha \in \mathcal{Q} \mid \det(\alpha) = 1\}$ as a normal subgroup.*

Theorem 12.8. (1) *Let α be an affine transformation in \mathbb{R}^2 with equation $\alpha(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ and let $\alpha(P) = P'$, etc., then $\text{area}(\triangle P'Q'R') = |\det A| \text{area}(\triangle PQR)$.*

(2) *If Ω is the parallelepiped spanned by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in \mathbb{R}^3 and α is an affine transformation in \mathbb{R}^3 , then $\text{vol}(\alpha\Omega) = |\det(A)| \text{vol}(\Omega)$.*