Graph Theory

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Introduction

1.1 Definitions

A graph G = (V, E) is a set V of vertices and a set E of unordered pairs of distinct vertices, called edges. Write vw or $\{v, w\}$ for the edge joining v and w, and say that v and w are **neighbours** or that they are adjacent.

In these notes, unless otherwise stated, graphs are:

- finite: $|V| \in \mathbb{N}$.
- labelled: vertices are distinguishable, usually $V = [n] := \{1, 2, ..., n\}$ for some $n \in \mathbb{N}$.
- undirected: edges are unordered pairs of vertices.
- simple: no loops $\{v, v\}$ or multiple edges (since E is not a multiset).

A graph G with vertex set $\{v_1, \ldots, v_n\}$ has adjacency matrix $A(G) = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E, \\ 0 & \text{otherwise.} \end{cases}$$

A(G) is a **symmetric** $n \times n$ 0-1 matrix with zero diagonal.

The **trivial graph** has at most one vertex. Hence it has no edges.

A **subgraph** of a graph G = (V, E) is a graph H = (W, F) such that $W \subseteq V$ and $F \subseteq E$.

We say that H is an **induced subgraph** if for all $v, w \in W$ if $vw \in E(G)$ then $vw \in E(H)$. Write H = G[W], and say that H is the subgraph of G induced by the vertex set W.

The number of **vertices** of G, written |G| = |V(G)|, is called the *order* of G. The number of **edges** of G, sometimes written |G| = |E(G)|, is called the *size* of G.

Two graphs G = (V, E) and H = (W, F) are **isomorphic** if there exists a bijection $\phi : V \to W$ such that $\phi(v)\phi(w) \in F$ if and only if $vw \in E$. The map ϕ is called a graph isomorphism or isomorphism.

1.2 The Degree of a Vertex

If $v \in e$ where v is a vertex and e is an edge, then we say that e is incident with v. The **degree** $d_G(v)$ of vertex v in a graph G is the number of edges of G which are incident with v. A vertex of degree 0 is an isolated vertex.

Let $N_G(v)$ be the set of all **neighbours** of v in G, then d(v) = |N(v)|.

Lemma 1.2.1 (The Handshaking Lemma). In any graph, G = (V, E),

$$\sum_{v \in V} d(v) = 2|E|.$$

Let $\delta(G) = \min_{v \in V} d(v)$ be the minimum degree in G, and $\Delta(G) = \max_{v \in V} d(v)$ be the maximum degree in G.

1.2.1 Some Special Graphs

A graph is k-partite if there exists a partition of its vertex set

$$V = V_1 \cup V_2 \cup \cdots V_k$$

into k nonempty disjoint subsets (parts) such that there are no edges between vertices in the same part.

The **complete graph** on r vertices, denoted K_r , has all $\binom{r}{2}$ edges present. The **complete bipartite** graph K_r , s has r vertices in one part of the vertex bipartition, s vertices in the other, and all rs present.

A graph is **regular** if every vertex has the same degree. If every vertex of a graph has degree d then we say that the graph is d-regular.

The **complement** of a graph G is the graph $\bar{G} = (V, \bar{E})$ where $vw \in \bar{E}$ if and only if $vw \notin E$. Note that \bar{K}_n is the graph with n vertices and no edges.

If G = (V, E) and $X \subset V$ then G - X denotes the graph obtained from G by deleting all vertices in X and all edges which are incident with vertices in X. If $F \subseteq E$ then G - F denotes the graph (V, E - F) obtained from G by deleting the edges in F.

1.3 Paths and Cycles

A walk in the graph G is a sequence of vertices $v_0v_1v_2\cdots v_k$ such that $v_iv_{i+1}\in E$ for $i=0,1,\ldots,k-1$. The length of this walk is k. The walk is closed if $v_0=v_k$.

An **Euler tour** is a *closed walk* in a graph which uses every edge precisely once. A graph is Eulerian if it has an Euler tour.

Theorem 1.3.1 (Euler, 1736). A connected graph is Eulerian if and only if every vertex has even degree.

A walk is a **path** if it does not visit any vertex more than once. A path is a sequence of *disinct* vertices, with subsequence vertices joined by an edge. A path $v_0v_1...v_k$ with k edges is called a k-path and has length k.

If $k \geq 3$ and $P = v_0 v_1 \cdots v_{k-1}$ is a path of length k-1 then $C = P + v_0 v_{k-1}$ is a **cycle** of length k, also called a k-cycle. It is a closed walk which visits no internal vertex more than once.

An edge which joins two vertices of a cycle C, but which is not an edge of C, is called a **chord**. An **induced cycle** is a cycle which has no chords.

Proposition 1.3.2. Every graph G contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G) + 1$, if $\delta(G) \geq 2$.

Proof. Let $P = x_0 x_1 \dots x_k$ be the longest path in G. By maximality of P, all neighbours of x_k lie on P. Hence $\delta(G) \leq d(x_k) \leq k = |\{x_0, x_1, \dots, x_{k-1}\}|$, which proves the first statement. Let x_i be the smallest-indexed neighbour of x_k in P. Then $C = x_k x_i x_{i+1} \dots x_{k-1} x_k$ is a cycle of length $\geq \delta(G) + 1$ because C contains $d(x_k) \geq \delta(G)$ neighbours of x_k as well as x_k .

The minimum length of a cycle in G is the girth of G, denoted by q(G).

Given $x, y \in V$, let $d_G(x, y)$ be the length of a shortest path from x to y in G, called the **distance** from x to y in G. Set $d_G(x, y) = \infty$ if no such path exists.

We say that G is **connected** if $d_G(x, y)$ is finite for all $x, y \in V$.

Let the **diameter** of G be $diam(G) = \max_{x,y \in V} d_G(x,y)$.

Proposition 1.3.3. Every graph G which contains a cycle satisfies $g(G) \leq 2 \operatorname{diam}(G) + 1$.

Proof. Let C be a shortst cycle in G, so |C| = g(G). For a contradiction, assume $g(G) \ge 2 \operatorname{diam}(G) + 2$.

Choose vertices x, y on C with $d_C(x, y) \ge \operatorname{diam}(G) + 1$. In G the distance $d_G(x, y)$ is strictly smaller, so any shortest path P from x to y in G is not a subgraph of C. But using P together with the shorter arc of C from x to y gives a closed walk of length < |C|. This closed walk contains a shorter cycle than C which is a contradiction.

1.4 Connectivity

A maximal connected subgraph of G is called a **component** (or **connected component**) of G.

Proposition 1.4.1. The vertices of a connected graph can be labelled v_1, v_2, \ldots, v_n such that $G_n = G$ and $G_i = G[v_1, \ldots, v_i]$ is connected for all i.

Proof. Choose v_1 arbitrarily. Now suppose that we have labelled v_1, \ldots, v_i such that $G_j = G[v_1, \ldots, v_j]$ is connected for all $j = 1, \ldots, i$.

If i < n then $G_i \neq G$, so there exists some $v_j \in \{v_1, \ldots, v_i\}$ with a $w \notin \{v_1, \ldots, v_i\}$ in G. (Otherwise $G_i \neq G$ is a component of G, impossible as G is connected.) Let $v_{i+1} = w$, then $G_{i+1} = G[v_1, \ldots, v_{i+1}]$ is connected. This completes the proof, by induction.

Let $A, B \subseteq V$ be sets of vertices. An (A, B)-path in G is a path $P = x_0 x_1 \cdots x_k$ such that

$$P \cap A = \{x_0\}, \quad P \cap B = \{x_k\}.$$

Let $A, B \subseteq V$ and let $X \subseteq V \cup E$ be a set of vertices and edges. We say that X separates A and B in G if every (A, B)-path in G contains a vertex or edge from X.

Note that we do not assume that A and B are disjoint and if X separates A and B then $A \cap B \subseteq X$. We say that X separates two vertices a, b if $a, b \notin X$ and X separates the sets $\{a\}, \{b\}$.

More generally, we say that X separates G, and call X a **separating set** for G, if X separates two vertices of G. That is, X separates G if there exist distinct vertices $a, b \notin X$ such that X separates a and b.

If $X = \{x\}$ is a separating set for G, where $x \in V$, then we say that x is a **cut vertex**.

If $e \in E$ and G - e has more components than G then e is a **bridge**.

The unordered pair (A, B) is a **separation** of G if $A \cup B = V$ and G has no edge between A - B and B - A. The second conditions says that $A \cap B$ separates A from B in G. If both A - B and B - A are nonempty then the separation is **proper**. The order of the separation is $|A \cap B|$.

Definition. Let $k \in \mathbb{N}$. The graph G is **k-connected** if |G| > k and G - X is connected for all subsets $X \subseteq V$ with |X| < k.

The **connectivity** $\kappa(G)$ of G is defined by

$$\kappa(G) = \max\{k : G \text{ is } k\text{-connected}\}.$$

So, $\kappa(G) = 0$ iff G is trivial or G is disconnected. Also, $\kappa(K_n) = n - 1$ for all positive integers n.

Definition. Let $\ell \in \mathbb{N}$ and let G be a graph with $|G| \geq 2$. If G - F is connected for all $F \subseteq E$ with $|F| < \ell$ then G is ℓ -edge-connected.

The **edge connectivity** $\lambda(G)$ is defined by

$$\lambda(G) = \max\{\ell : G \text{ is } \ell\text{-edge-connected}\}.$$

Proposition 1.4.2. If $|G| \ge 2$ then $\kappa(G) \le \lambda(G) \le \delta(G)$.

Theorem 1.4.3 (Mader, 1973). Let k be a positive integer. Every graph G with average degree at least 4k has a (k+1)-connected subgraph H with

$$\frac{|E(H)|}{|V(H)|} > \frac{|E(G)|}{|V(G)|} - k.$$

Proof. We write |G| instead of |V(G)|. Let $\gamma = \frac{|E(G)|}{|G|} \ge 2k$. Consider subgraphs G' of G which satisfy:

$$|G'| \ge 2k$$
 and $|E(G')| > \gamma(|G'| - k)$. (1.1)

such graphs G' exists as G satisfies 1.1. (Average degree of G is $\frac{2|E(G)|}{|G|} \geq 4k$, so

$$|G| \ge 4k$$
 and $\gamma(|G| - k) = |E(G)| \frac{(|G| - k)}{|G|} < |E(G)|$.)

Now let H be a subgraph of G of smallest order which satisfies 1.1. We continue the proof by proving three claims.

Claim 1. If G' satisfies 1.1 then |G'| > 2k.

Proof. If G' satisfies 1.1 and |G'| = 2k then $|E(G')| > \gamma(|G'| - k) \ge 2k^2 > {|G'| \choose 2}$, contradiction.

Claim 2. $S(H) > \gamma$.

Proof. For a contradiction, suppose that $S(H) \leq \gamma$. Let G' be obtained from H by deleting a vertex of degree $\leq \gamma$. Then |G'| < |H| and G' satisfies 1.1, which is a contradiction. To see this, check:

$$|G'| = |H| - 1 \ge 2k$$
, by Claim 1, and $|E(G')| \ge |E(H)| - \gamma > \gamma(|H| - k - 1)$, as H satisfies 1.1 $= \gamma(|G'| - k)$.

Hence $S(H) > \gamma$. It follows that $|H| \ge \gamma$. Thus,

$$\frac{|E(H)|}{|H|} > \frac{\gamma(|H| - k)}{|H|}.$$
 (as H satisfies 1.1)

Claim 3. H is (k+1)-connected.

Proof. By Claim 1, $|H| \ge 2k + 1 \ge k + 2$ as $k \ge 1$. So H is large enough. For a contradiction, suppose that H is not (k+1)-connected. Then H has a proper separation $\{U_1, U_2\}$ of order at most k.

Let $H_i = H[U_i]$ for i = 1, 2. Since any vertex $v \in U_1 - U_2$ has $d_H(v) \ge S(H) > \gamma$ (by Claim 2), and all neighbours of v in H belong to H_1 , we have $|H_1| \ge \gamma \ge 2k$. Similarly, $|H_2| \ge 2k$. By minimality of H, neither H_1 nor H_2 satisfies 1.1. Hence $|E(H_i)| \le \gamma(|H_i| - k)$ for i = 1, 2. But then

$$|E(H)| \le |E(H_1)| + |E(H_2)|$$

$$\le \gamma(|H_1| + |H_2| - 2k)$$

$$\le \gamma(|H| - k),$$
 (by inclusion-exclusion)

since $|U_1 \cup U_2| \le k$. This contradicts 1.1 for H. So H is (k+1)-connected, completing the proof of Claim 3 and of the theorem.

1.5 Trees and Forests

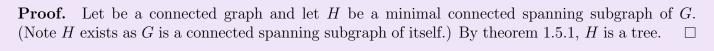
A graph with no cycles is a **forest** (also called an acyclic graph). A connected graph with no cycles is a **tree**.

Theorem 1.5.1. The following are equivalent for a graph T:

- (i) T is a tree;
- (ii) Any two vertices of T are linked by a unique path in T;
- (iii) T is minimally connected: that is, T is connected but T-e is disconnected for every $e \in E(T)$;

(iv) T is maximally acyclic: that is, T is acyclic but T + xy has a cycle for any two nonadjacent vertices x, y in T.

Corollary 1.5.2. If G is connected then G has a spanning tree.



Corollary 1.5.3. The vertices of a tree can be labelled as v_1, \ldots, v_n so that for $i \geq 2$, vertex v_i has a unique neighbour in $\{v_1, \ldots, v_{i-1}\}$.

Proof. We use the labelling from Proposition 1.4.1. This labels the vertices of a given tree G as v_1, \ldots, v_n such that $G[v_1, \ldots, v_n]$ is connected. Let $i \geq 1$ then $G[v_1, \ldots, v_i]$ is a tree. Note $G[v_1, \ldots, v_{i+1}]$ is connected by Proposition 1.4.1, so v_{i+1} has at least one neighbour in $G[v_1, \ldots, v_i]$. For a contradiction, suppose that v_{i+1} has two neighbours z and w in $G[v_1, \ldots, v_i]$. There is a (unique)

For a contradiction, suppose that v_{i+1} has two neighbours z and w in $G[v_1, \ldots, v_i]$. There is a (unique) path P in $G[v_1, \ldots, v_i]$ between z and w, and this path does not visit v_{i+1} . Hence $P \cup \{zv_{i+1}, wv_{i+1}\}$ is a cycle in G, contradiction.

Corollary 1.5.4. A connected graph with n vertices is a tree if and only if it has n-1 edges.

Proof. Suppose that G is a tree on n vertices. The result is true when n = 1. Now suppose the result is true when n = k. Let G be a tree on k + 1 vertices. Let G be a leaf in G (e.g. take an end vertex of a longest path in G.) Then G - v is a tree on K vertices, so G - v has K - 1 edges (inductive hypothesis). Therefore G has K edges as K has degree 1. This concluses the proof, by induction.

Conversely, suppose that G is connected with n vertices and n-1 edges. Then G contains a spanning tree H, by an earlier corollary. Then H has exactly n-1 edges, since it is a tree on n vertices. Hence H=G, so G is a tree.

Corollary 1.5.5. If T is a tree and G is any graph with $\delta(G) \geq |T| - 1$ then G has a subgraph isomorphic to T.

Matchings and Hamilton Cycles

Two edges in a graph are called **independent** if they have no vertices in common. A set M of pairwise independent edges in a graph is called a **matching**.

Given G = (V, E) and $U \subseteq V$, say that $M \subseteq E$ is a **matching of U** if M is matching and every vertex in U is incident with an edge of M. We say that the vertices in U are matched by M, and t hat the vertices not incident with any edge of M are **unmatched**.

A matching M is a **maximal matching** of G if $M \cup \{e\}$ is not a matching for any $e \in E - M$. A **maximum matching** of G is a matching of G such that no set of edges with size greater than |M| is

A maximum matching of G is a matching of G such that no set of edges with size greater than |M| is a matching.

A **perfect matching** of G is a matching of G which matches every vertex of G. Note: a perfect matching is a 1-regular spanning subgraph of G also called a **1-factor** of G.

A k-factor is a k-regular spanning subgraph. A **2-factor** in a graph is the union of disjoint cycles which covers all the vertices.

2.1 Matchings in Bipartite Graphs

Let G = (V, E) be a bipartite graph with vertex bipartition $V = A \cup B$. Here A, B are nonempty disjoint sets. We use the convention that all vertices called a, a', a'', \ldots belong to A and similarly for B.

Let M be matching in G. A path in G which starts at an *unmatched* vertex of A and contains, alternately, edges from E-M and from M, is called an **alternating path** with respect to M.

If an alternating path P ends in an unmatched vertex of B then it is called an **augmenting path**.

Definition 2.1.1. A set $U \subseteq V$ is a **cover** (or **vertex cover**) of G if every edge of G is incident with a vertex in U.

Theorem 2.1.2 (König, 1931). Let G be a bipartite graph. The size of a maximum matching in G is equal to the size of the minimum vertex cover of G.

Proof. Let \hat{U} be a cover in G and let M be a maximum matching. Then $|\hat{U}| \geq |M|$ as we must cover every edge of M. Hence it suffices to construct a cover U of G with |U| = |M|.

We build U be choosing one vertex from each edge of M to place into U, as follows:

• If $ab \in M$ and some alternating path in G with respect to M ends in b. Then put b into U otherwise put a into U.

Let $ab \in E$. If $ab \in M$ then $a \in U$ or $b \in U$ by definition of U. Now assume $abb \notin M$. Since M is maximum, there exists $a'b' \in M$ with a = a' or b = b'. If a is unmatched in M then b = b' for some $a'b' \in M$. Hence ab is an alternating path ending in b = b', so we chose b' to go into U from the edge $a'b' \in M$. So the edge ab is covered by U in this case.

Hence we assume that a = a' for some $a'b' \in M$. If $a = a' \in U$ then we are done. Otherwise $b' \in U$, so there is an alternating path P ending in b'. Then $P = a_1b_1a_2b_2...b'$, and we have three cases:

- (i) P does not include a or b. Then $Pab = a_1 a_2 \dots b'ab$ is an alternating path in G with respect to M. By maximality of M, b is matched or else we have an augmenting path. Hence $b \in U$ as b is the chosen vertex from its matching edge.
- (ii) If b is on P before a, or $b \in P$ and $a \notin P$, then $P = a_1b_1a_2...b...b'$. Then we let $P' = a_1b_1...b$. This is an alternating path ending in b, so finish proof as case above.
- (iii) If a is on P before b, or $a \in P$ and $b \notin P$. Then $P = a_1b_1 \dots a_rb_r \dots b'$ and we take $P' = a_1b_1 \dots ab$. This is an alternating path ending in b, so finish proof as case above.

This proves U is a cover of G and since |U| = |M|, this completes the proof.

For a subset $S \subseteq A$, let $N(S) = \bigcup_{v \in S} N(v)$ be the set of vertices in B which are neighbours of some vertex in S.

Theorem 2.1.3 (Hall, 1935). Let G be a bipartite graph. Then G contains a matching of A if and only if

$$|N(S)| \ge |S|$$
 for all $S \subseteq A$. (2.1)

Proof. We have that this condition is necessary. Now suppose that (2.1) holds. For a contradiction, suppose that G has no matching of A. Then König's Theorem (Theorem 2.1.2) says that G has a cover U with |U| < |A|. Suppose that $U = A' \cup B'$ with $A' \subseteq A$ and $B' \subseteq B$. Then |A'| + |B'| = |U| < |A|, so |B'| < |A| - |A'| = |A - A'|. Since U is a cover, G has no edges from A - A' to B - B'. Hence $N(A - A') \subseteq B'$, and so $|N(A - A')| \le |B'| < |A - A'|$. This contradicts Hall's condition 2.1 for S = A - A'. Hence G contains a matching of A.

Corollary 2.1.4. Let G be a bipartite graph and $d \in \mathbb{N}$. If $|N(S)| \ge |S| - d$ for all $S \subseteq A$ then G has a matching of size |A| - d.

Proof. Add d new vertices to B and join each of them by an edge to each vertex of A. Then for all $S \subseteq A$, in the new graph G', $|N_{G'}(S)| \ge |S| - d + d = |S|$. Hall's condition is satisfied in G'. Therefore there is a matching M in G' which matches all of A. At least |A| - d edges in M are edges of G.

Corollary 2.1.5. If G is a k-regular bipartite graph then G has a perfect matching.

Proof. Assume $k \ge 1$. Since G is k-regular, |E(G) = k|A| = k|B|, so |A| = |B|. Hence it suffices to prove that G contains a matching of A. Every set $S \subseteq A$ is joined to N(S) by a total of k|S| edges. These edges are a subset of the k|N(S)| edges incident with |N(S)|. Hence $k|S| \le k|N(S)|$

and diving by k shows that Hall's condition holds. Thus, G has a matching of A.

Corollary 2.1.6. Every regular graph of positive even degree has a 2-factor.

Proof. Let G be any 2k-regular graph, $k \geq 1$. Without loss of generality, suppose that G is connected (or apply this argument to each component). By Theorem 1.3.1, G has an Euler tour $v_0v_1 \ldots v_{l-1}v_l$ where $v_l = v_0, e_i = v_iv_{i+1} \in E(G)$ using each edge exactly once.

Replace each vertex $v \in V$ with a pair of vertices v^-, v^+ , and replace every edge $e_i = v_i v_{i+1}$ by the edge $v_i^+ v_{i+1}^-$. The resulting graph G' is a k-regular bipartite graph. Hence by Corollary 2.1.5, G' has a perfect matching (1-factor). Collapse every vertex pair (v^-, v^+) back into a single vertex v, for all $v \in V$. The 1-factor of G' becomes a 2-factor of G.

2.2 Hamilton Cycles

A **Hamilton cycle** is a connected 2-factor. That is, it is a cycle which includes every vertex.

Say G is **Hamiltonian** if it contains a Hamilton cycle. A Hamiltonian graph G must be connected with minimum degree $\delta(G) \geq 2$.

Theorem 2.2.1 (Dirac, 1952). Every graph with $n \ge 3$ vertices and with minimum degree at least n/2 has a Hamilton cycle.

Proof. Let G be a graph with minimum degree $\geq n/2$ and $n \geq 3$ vertices. Then G is connected, as otherwise the degree of any vertex in the smaller component must be < n/2. Let $P = x_0 \dots x_k$ be a longest path in G. by maximality, all neighbours of x_0 and x_k lie on P. So at least n/2 of the vertices x_0, \dots, x_{k-1} are adjacent to x_k and at least n/2 of these same vertices satisfy $x_0x_{i+1} \in E(G)$. By the pigeonhole principle, as k < n, there exists $i \in \{0, \dots, k-1\}$ with $x_0x_{i+1}, x_ix_k \in E(G)$. This gives a cycle $x_0x_1 \dots x_ix_k \dots x_{i+1}x_0$. We claim this is a Hamilton cycle. If not then, as G is connected, there is some $u \notin C$ with a neighbour $v \in C$. Then we can start at u, go to v then go around v (in some direction) and stop just before we reach v again (i.e. stop at v and v and v aparts which is longer than v contradiction.

2.3 Matchings in General Graphs

Given a graph G, let C_G be the set of its components and let q(G) denote the number of odd components (connected components having an odd number of vertices).

Theorem 2.3.1 (Tutte, 1947). A graph G has a perfect matching if and only if

$$q(G-S) \le |S|$$
 for all $S \subseteq V(G)$. (2.2)

Proof. We have seen that the condition (2.2) is necessary: if G has a perfect matching then (2.2) holds. Now suppose that G has no perfect matching. We want to find a "bad" set S_0 which fails condition (2.2). If |G| is odd then, $S_0 = \emptyset$ is bad. So assume |G| is even.

Claim 1. If G' is obtained from G by adding edges and $S_0 \subseteq V$ is bad for G' then S_0 is bad for G.

Proof. If S_0 bad for G' then $q(G - S_0) > |S_0|$. But each odd component of G' - S is a disjoint union of components of G - S, at least one of which must be odd. So $q(G - S) \ge q(G' - S)$.

Hence by Claim 1, we can assume that G has no perfect matching but adding any edge to G gives a graph G' which has a perfect matching.

Claim 2. S is a bad set for G if and only if all components of G - S are complete and every vertex in S is adjacent to all other vertices in G.

Proof. For proof, call the second half of the claim (*). If S is bad for G but does satisfy (*) then we can add an edge to G to get a graph G' with S still bad for G'. This contradicts our assumption on the maximality of G. Conversely suppose S satisfies (*) but S is not bad. Then we can form a perfect matching since |G| is even. This is a contradiction as G has no perfect matching. Hence S is bad.

Define $S_0 = \{v \in V : d_G(v) = n - 1\}$ to be the set of all vertices v in G which are adjacent to every other vertex $w \neq v$.

Claim 3. S_0 is bad.

Proof. We need to show that S_0 satisfies (*). For a contradiction, suppose that S_0 does not satisfy (*). Then $G - S_0$ has a component K which is not complete. Let $a, a' \in V(K)$ with $aa' \notin E(G)$. Fix a shortest path from a to a' in K which starts $abc \dots a'$. Such a path has length ≥ 2 and $ac \notin E(G)$. Note $b \in K$, so $b \in S_0$, so there is some $d \in V$ with $bd \notin E$. By maximality of G, there is a perfect matching M_1 in G + ac and a perfect matching M_2 in G + bd. Take a maximal path P in G, starting at d with an edge from M_1 , and taking alternately edges from M_1 and M_2 . Say $P = d \dots v$.

- If the last edge of P is in M_1 then v = b or we could extend P. Let C = P + bd (cycle in G + bd).
- If the last edge of P is in M_2 then $v \in \{a, c\}$ as the M_1 edge incident with v must be ac. Let C be the cycle $d \dots vbd$.

In each case, C is an alternating (even length) cycle in G + bd which contains bd. Form M'_2 from M_2 by replacing $M_2 \cap C$ by $C - M_2$. This gives a perfect matching of G, contradiction. Hence S_0 satisfies (*), so Claim 3 holds and the proof is complete.

Corollary 2.3.2 (Petersen, 1891). Every bridge cubic (3-regular) graph has a perfect matching.

Proof. Let G be a bridgeless cubic graph. We prove that G satisfies Tutte's condition. Let $S \subseteq V(G)$ be given and consider an odd component C of G - S. The sum of the degrees of vertices in C is 3|C|, which is an odd number. Every edge with both end vertices in C contributes an even number to this sum. Hence the number of edges from C to S is odd.

As G has no bridge, there must be at least 3 edges from S to G. Therefore the number of edges from S to G-S is at least 3q(G-S). But the number of edges from S to G-S is bounded above by the sum of the degrees of vertices in S, which is 3|S| as G is cubic. Hence $3q(G-S) \le \#$ edges from S to $G-S \le 3|S|$ and thus $q(G-S) \le |S|$. Therefore by Tutte's Theorem, G has a perfect matching.

The Probabilistic Method

This chapter assumes knowledge of elementary probability knowledge. Content from first year is sufficient.

Example 3.0.1. Let Ω be the set of all graphs on the vertex set $\{1, 2, ..., n\}$. Then $|\Omega| = 2^{\binom{n}{2}}$. Define $\pi(G) = 2^{\binom{n}{2}}$ for all $G \in \Omega$. This is the *uniform model of random graphs*.

Lemma 3.0.2. The expected number of edges in a uniformly chosen graph on the vertex set $\{1, 2, \dots n\}$ is $\frac{1}{2} \binom{n}{2}$.

Proof. (From Definition) For $0 \le m \le \binom{n}{2} = N$, there $\binom{N}{m}$ are exactly of graphs on vertex set $\{1, \ldots, n\}$ with m edges. Let X be the number of edges in the random graph. Then

$$EX = \sum_{m=0}^{N} \Pr(X = m) \cdot m$$

$$= \sum_{m=0}^{N} \frac{\binom{N}{m}}{2^{N}} \cdot m$$

$$= \frac{N}{2^{N}} \sum_{m=1}^{N} \frac{(N-1)!}{(m-1)!(N-m)!}$$

$$= \frac{N}{2^{N}} \sum_{j=0}^{N-1} \binom{N-1}{j}$$

$$= \frac{N}{2^{N}} 2^{N-1}$$

$$= \frac{N}{2} = \frac{1}{2} \binom{n}{2}.$$
(by the binomal theorem)
$$= \frac{N}{2} = \frac{1}{2} \binom{n}{2}.$$

Let $A \subseteq \Omega$ be an event. The indicator variable I_A for $A \subseteq \Omega$ is

$$I_A(z) = \begin{cases} 1 & \text{if } z \in A \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.0.3 (Linearity of Expectation). Let X_1, \ldots, X_k be random variables on Ω and let $c_1, \ldots, c_k \in \mathbb{R}$. Define the random variable $X = c_1 X_1 + \cdots + c_k X_k$. Then

$$\mathbb{E}[X] = c_1 \mathbb{E}[X_1] + c_2 \mathbb{E}[X_2] + \dots + c_k \mathbb{E}[X_k].$$

Definition 3.0.4 (Markov's Inequality). SUppose that $X : \Omega \to [0, \infty)$ is a nonnegative random variable on Ω and let k > 0. Then

$$\Pr(X \ge k) \le \frac{\mathbb{E}[X]}{k}.$$

In particular, if X is a nonnegative integer-valued random variable then

$$\Pr(X \neq 0) \leq \mathbb{E}[X].$$

Let $k \geq 2$ be an integer. Events A_1, \ldots, A_k in Ω are **mutually independent** if for all $j, \ell_1, \ldots, \ell_j$ with $2 \leq j \leq k$ and $1 \leq \ell_1 < \ell_2 < \cdots < \ell_j \leq k$,

$$\Pr\left(\bigcap_{i=1}^{j} A_{\ell_i}\right) = \prod_{i=1}^{j} \Pr(A_{\ell_i}).$$

Lemma 3.0.5. Let Ω be the set of all subsets of some given set S, where |S| = n. Define a random set $X \subseteq S$ by setting $\Pr(x \in X) = \frac{1}{2}$, independently for each $x \in S$. Then $\Pr(X = A) = 2^{-n}$ for all $A \subseteq S$, so this gives the uniform probability space on Ω .

Proof. Fix $A \subseteq \Omega$. Then

$$\Pr(X = A) = \prod_{x \in A} \Pr(\text{heads}) \cdot \prod_{x \notin A} \Pr(\text{tails})$$
 (using independence)
$$= \left(\frac{1}{2}\right)^{|A|} \cdot \left(\frac{1}{2}\right)^{n-|A|}$$

$$= \left(\frac{1}{2}\right)^{n} = 2^{-n}$$

as claimed.

Theorem 3.0.6 (Alon & Spencer, Theorem 2.2.1). Let G be a graph with n vertices and m edges. Then G contains a bipartite subgraph with at least m/2 edges.

Proof. Let Ω be the set of all subsets of V(G). Then $|\Omega| = 2^n$. Consider the uniform probability space on Ω . Let $A \subseteq V$ be a randomly chosen element of Ω and define B = V - A. Call $xy \in E(G)$ a crossing edge if exactly one of x, y belongs to A. Let X be the number of crossing edges. Finally, for each edge $e \in E(G)$ define the indicator variable

$$X_e = \begin{cases} 1 & \text{if } e \text{ is a crossing edge,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $X = \sum_{e \in E(G)} X_e$. For any $e = xy \in E(G)$, we have,

$$\Pr(x \in A \text{ and } y \notin A) = \Pr(x \in A) \Pr(y \in A)$$
 (using independence)
= $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Therefore

$$\mathbb{E}X_e = \Pr((x \in A \text{ and } y \notin A) \text{ or } (x \notin A \text{ and } y \in A))$$

$$= \Pr(x \in A \text{ and } y \notin A) + \Pr(x \notin A \text{ and } y \in A)$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$
(events are disjoint)

Hence, by linearity of expectation,

$$\mathbb{E}X = \sum_{e \in E(G)} \mathbb{E}X_e = \frac{m}{2}.$$

Thus there exists a fixed set $A_0 \subseteq V(G)$ which has at least $\frac{m}{2}$ crossing edges. The corresponding bipartition $(A_0, V(G) - A_0)$ defines a bipartite subgraph consisting of the $\geq \frac{m}{2}$ crossing edges. \square

An **independent set** in a graph G is a subset $U \subseteq V$ such that if $v, w \in U$ then $vw \in E(G)$. Let $\alpha(G)$ be the size of a maximum independent set in G, called the **independence number**.

Theorem 3.0.7. Let G have n vertices and nd/2 edges, where $d \ge 1$. Then $\alpha(G) \ge \frac{n}{25T1d}$. Note d, is the average degree of G.

Proof. Define the random subset $S \subseteq V(G)$ by $\Pr(v \in S) = p$, independently for all $v \in V$. Here $p \in [0,1]$ which we will fix later.

Let X = |S| and let Y be the number of edges of G with both endvertices in S. Then $\mathbb{E}X = pn$. For $e \in E(G)$ let Y_e be the indicator variable for the event $e \subseteq S$. Then for every $e = xy \in E(G)$,

$$\mathbb{E}Y_e = \Pr(x \in S \text{ and } y \in S)$$

$$= \Pr(x \in S) \cdot \Pr(y \in S)$$

$$= p^2.$$
 (by independence)

Therefore, by linearity of expectation and the fact that $Y = \sum_{e \in E(G)}$ we have

$$\mathbb{E}Y = \sum_{e \in E(G)} \mathbb{E}Y_e = \frac{nd}{2}p^2.$$

By linearity of expectation,

$$\mathbb{E}(X - Y) = \mathbb{E}X - \mathbb{E}Y = pn - p^2 \frac{nd}{2}.$$

Want to choose p to maximise this, so $p = \frac{1}{d}$ and $p \in [0,1]$. Substituting gives $\mathbb{E}(X - Y) = \frac{n}{2d}$. Hence there exists a fixed set $S_0 \subseteq V(G)$ with $|S_0| - (\# \text{ edges in } S_0) \ge \frac{n}{2d}$. Delete one vertex from each edge within S_0 to give a set S^* of at least $\frac{n}{2d}$ vertices which is an independent set. \square

Graph Colourings

A vertex colouring of a graph G = (V, E) is a function $c : V \to S$ such that $c(u) \neq c(v)$ whenever $uv \in E$. Here S is the set of available colours, usually $S = \{1, 2, ..., k\}$ for some positive integer k.

A k-colouring of G is a colouring $c: V \to \{1, 2, ..., k\}$. Often we want the smallest value of k for which a k-colouring of G exists. This smallest value of k is called the **chromatic number** of G, denoted $\chi(G)$.

If $\chi(G) = k$ then G is said to be k-chromatic.

If $\chi(G) \leq k$ then G is said to be k-colourable.

The set of all vertices in G with a given colour under c is called a **colour class**. Each colour class is an independent set. k-colouring is a partition of V(G) into k independent sets.

A clique in a graph G is a complete subgraph of G. The order of the largest clique in G is called the clique number of G, denoted $\omega(G)$.

Fact: $\chi(G) \ge \omega(G)$ and $\chi(G) \ge n/\alpha(G)$.

An **edge colouring** of G is a map $c: E \to S$ such that $c(e) \neq c(f)$ whenever e and f share an endvertex. If $S = \{1, 2, ..., k\}$ then c is a k-edge-colouring and G is k-edge-colourable.

Let $\chi'(G)$ be the smallest positive integer k for which G is k-edge-colourable. We call $\chi'(G)$ the **chromatic** index of G.

A colour class in an edge colouring is a matching of G. Hence an edge colouring displays E(G) as a union of disjoint matchings.

The **line graph**, denoted L(G), has vertex set E(G) and $e, f \in E(G)$ form an edge of L(G) if and only if e, f share an endvertex in G. Every edge-colouring of G is a vertex colour of L(G) and vice-versa. So $\chi'(G) = \chi(L(G))$.

4.1 Vertex Colourings

Proposition 4.1.1. If graph G has m edges then $\chi(G) \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}$.

Proof. Fix a k-colouring of G with $k = \chi(G)$ colours. Then G has at least one edge between any two distinct colour classes, or we could merge them to give a colouring of G with $\leq k-1$ colours. Hence $m \geq {k \choose 2} = \frac{1}{2}(k)(k-1)$ then solve for k to complete the proof.

Greedy Algorithm Given a graph G, fix an ordering v_1, v_2, \ldots, v_n on the vertices of G and colour them one by one in this order using the first available colour (least positive integer) as you go along. Since v_i has at most $\Delta(G)$ neighbours, this produces a k-colouring of G with $k \leq \Delta(G)+1 \implies \chi(G) \leq \Delta(G)+1$.

Fact: $\chi(G) = \Delta(G) + 1$ if G is a complete graph or an odd cycle.

Theorem 4.1.2 (Brooks, 1941). Let G be a connected graph. If G is neither complete nor a n odd cycle then $\chi(G) \leq \Delta(G)$. In fact we will prove the following restatement of Brooks Theorem, due to Zajac (2018):

Let $k \geq 3$ be an integer and let G be a graph with $\Delta(G) \leq k$. If G does not contain K_{k+1} as a subgraph then G is k-colourable.

We call this the "new" version of Brooks Theorem and prove that this implies Brooks Theorem.

Proof. Suppose that G is a graph which satisfies the assumptions of Brooks Theorem. That is, G be a connected graph which is not an odd cycle and which is not complete. Let $\Delta = \Delta(G)$ be the maximum degree of G. We want to show that $\chi(G) \leq \Delta$, as this is the conclusion required for Brooks Theorem.

First suppose that $\Delta \leq 2$. Then G is either a path or an even cycle, as G is connected. Hence G is bipartite and so $\chi(G) \leq 2 = \Delta$, as required.

Now suppose that $\Delta \geq 3$. We wish to apply the new version of Brooks Theorem with $k = \Delta$, so we must check that G does not contain $K_{\Delta+1}$ as a subgraph. For a contradiction, suppose that G does have a subgraph H which is isomorphic to $K_{\Delta+1}$. Then H is Δ -regular, and G has maximum degree Δ , so there is no edge from a vertex of H to a vertex of G - V(H). It follows that H is a component of G. But G is connected, so the only possibility is that G = H. But this contradicts our assumption that G is not complete.

Therefore, G satisfies the assumptions of the new version of Brooks Theorem, and by applying this result we find that G is Δ -colourable. From this we conclude that $\chi(G) \leq \Delta$, as required.

In both cases, the conclusion of Brooks Theorem holds, completing the proof.

We now prove that this "new" version is true.

Proof. First an obversation, let G be a graph with maximum degree $\Delta(G) \leq k$, where $\{1, \ldots, k\}$ will be our set of colours. Suppose that G is partially coloured. Let $P = v_1 v_2 \ldots v_j$ be a path in G such that all vertices of P are uncoloured. Then we can colour vertices $v_1, v_2, \ldots, v_{j-1}$ in this order, since at the moment that we colour $v_i (1 \leq i \leq j-1)$, we know that v_i has an uncoloured neighbour v_{i+1} and hence aat most $\Delta - 1$ neighbours. Call this procedure PATHCOLOUR $(v_1, \ldots, v_{j-1}; v_j)$. Note that this procedure colours v_1, \ldots, v_{j-1} but it leaves v_j uncoloured. In particular if j = 1 then PATHCOLOUR (v_1) leaves the graph unchanged.

Proof by induction on n = |G|, where G is a graph with $\Delta(G) \leq k$ and $k \geq 3$. If $n \leq k$ then we can k-colour G by giving each vertex a distinct colour.

Claim. If G has a vertex of degree < k then G is k-colourable.

Proof. Let v be a vertex of degree < k and let G' = G - v. By the inductive hypothesis we can k-colour G'. Fix one such colouring C. Then at most k - 1 colours are used by C on neighbours of v, so we have an available colour which we can use to colour v.

Now we assume that G is k-regular. Let v be a vertex of G and consider $G[\{v\} \cup N(v)]$. Since G has no subgraph isomorphic to K_{k+1} , we know that v has two neighbours x, y which are not adjacent. Let $v_1 = x, v_2 = v, v_3 = y$, and extend the path $v_1v_2v_3$ to a maximal length path in $G, P = v_1v_2v_3 \dots v_r$ which starts with $v_1v_2v_3$.

Case 1. Suppose that r = n. This means that all vertices of G lie on P (Hamilton Path). Let v_j be any neighbour of v_2 other than v_2 and v_3 . Since G is k-regular and $k \geq 3$ we can choose such a vertex v_j . We now describe how to k-colour G.

- First colour v_1 and v_3 the same colour.
- Next apply PATHCOLOUR $(v_4, v_5, \ldots, v_{j-1}; v_j)$ which colours v_4, \ldots, v_{j-1} and leaves v_j uncoloured.
- Next apply PATHCOLOUR $(v_n, v_{n-1}, \dots, v_j; v_2)$ which will colour all remaining vertices of G except v_2 .
- Finally we have an available colour for v_2 since two of its neighbour $(v_1 \text{ and } v_3)$ have the same colour. Colour v_2 with an available colour.

Case 2. Now suppose that r < n. Recall that all neighbours of v_r lie on P. Let v_j be the neighbour of v_r with the smallest index. Then $C = v_j v_{j+1} \dots v_r v_j$ is a cycle in G. Let G' = G - V(C). We can k-colour G' by induction. If there is no edge between G' and C then we can also k-colour G[V(C)], by induction and we are done. Otherwise (G[V(C)]) is not a component of G: let v_ℓ be the vertex on C with largest index which has a neighbour in G', and let u be a neighbour of v_ℓ in G'. Note, v_ℓ is well defined as v_j has a neighbour in G' if $j \geq 2$. Note $\ell \leq r - 1$ since all neighbours of v_r belong to V(C). Also $v_{\ell+1}$ has no neighbours outside C, by choice of v_ℓ . We now describe how to k-colour vertices of C, giving a k-colouring of G.

- First, colour $v_{\ell+1}$ with the colour assigned to u.
- Next, apply PATHCOLOUR $(v_{\ell+2}, \ldots, v_r, v_j, v_{j+1}, \ldots, v_{\ell-1}; v_\ell)$ which colours all remaining vertices of G except v_ℓ .
- Finally, colour v_{ℓ} with an available colour which exists because v_{ℓ} has two neighbours with the same colour.

This completes the proof in Case 2, by mathematical induction.

4.2 Edge Colourings

By considering a vertex of maximum degree, we see that the chromatic index $\chi'(G)$ satisfies $\chi'(G) \ge \Delta(G)$ for all graphs G.

Proposition 4.2.1 (Köning, 1916). If G is bipartite then $\chi'(G) = \Delta(G)$.

Proof. We prove t his by induction on m = |E(G)|. If m = 0 then the result is trivially true. So, assume that $m \ge 1$ and that the result holds for all bipartite graphs with at most m - 1 edges.

Let $\Delta = \Delta(G)$, choose $xy \in E$ and let G' = G - xy. By induction, we can fix a Δ -edge-colouring of G'. We call edges coloured α , " α -edges", etc. In G', vertices x, y both have degree $\Delta - 1$. So there are colours $\alpha, \beta \in \{1, 2, ..., \Delta\}$ such that x is not incident with an α -edge, and y is not incident with a β -edge.

If $\alpha = \beta$ then we can colour the edge xy with colour α to give a Δ -edge-colouring of G, and we are done. Now assume that $\alpha \neq \beta$. Without loss of generality, we can assume that x is incident with a β -edge xu. Extend the β -edge xu to a maximal walk W whose edges are coloured α, β alternately. Since no such walk can contain a vertex colour twice, W is a path.

Claim. W does not contain y.

Proof. For a contradiction, suppose that y lies on W. Then y must be an endvertex of W, and the edge of W incident with y must be an α -edge. Hence W has even length, and so W + xy is an odd cycle in the bipartite graph G. This is a contradiction.

By maximality of W, we can swap the colours α and β on all edges of W. This gives a new Δ -edge-colouring of G' such that β does not appear on any edge incident with x. Since y does not lie on W, there is still no β -edge incident with y. Finally we can colour edge xy with colour β in G, giving a Δ -edge-colouring of G. This completes the proof, by induction.

Theorem 4.2.2 (Vizing, 1964). Every graph G satisfies

$$\Delta(G) \le \chi'(G) \le \Delta(G) + 1.$$

Connectivity

5.1 2-Connected Graphs

Let G be a graph. A maximal connected subgraph of G with no cut vertex is called a **block**. Every block of G is either a maximal 2-connected subgraph of G or a bridge or an isolated vertex.

By maximality, different blocks of G overlap in at most one vertex, which must be a **cut vertex** in G. Hence every edge of G lies in a unique block, and G is the union of its blocks.

Let A be the set of cut vertices in G and let \mathcal{B} be the set of blocks in G. Form the bipartite graph on $A \cup \mathcal{B}$ with edge set

$$\{aB : a \in A, B \in \mathcal{B} \text{ and } a \in B\}.$$

Lemma 5.1.1. The block graph of a connected graph is a tree.

Let H be a subgraph of a graph G. An H-path is a path in G which intersects H only in its endvertices.

Proposition 5.1.2. A graph is 2-connected if and only if it can be constructed from a cycle by successively adding H-paths to graphs H already constructed.

Proof. Every graph constructs in this way is 2-connected. Conversely, let G be 2-connected. Then $|G| \geq 3$ and G contains a cycle. Hence G has a maximal subgraph H which is constructible using the method described in the proposition stated.

If H = G, then we are done. For a contradiction, suppose that $H \neq G$. Since any edge $xy \in E(G) - E(H)$ with $x, y \in H$ is an H-path, by maximality we see that every $xy \in E(G)$ with $x, y \in H$ must belong to E(H). Hence, H is an induced subgraph of G.

By the fact that G is connected, there is an edge vw with $v \in G - H$, $w \in H$. Since G is 2-connected we know that G - W is connected. Let P be the shortest path from v to H in G - w. Then wvP is a H-path in G, and $H \cup wvP$ is a larger constructible subgraph than H, contradicting the maximality of H.

5.2 3-Connected Graphs

Let $e = xy \in E(G)$. Define the graph G/e = (V', E') where $V' = (V - \{x, y\}) \cup \{v_e\}$,

$$E' = \{uw \in E(G) : \{u, w\} \cup \{x, y\} = \emptyset\} \cup \{v_e w : xw \in E(G) \text{ or } yw \in E(G)\}.$$

We say that G/e is formed by **contradicting** the edge e in G. This creates a new vertex v_e which replaces the endvertices of e.

Lemma 5.2.1. Let G be a 3-connected graph with $|G| \ge 5$. Then G has an edge e such that G/e is 3-connected.

Proof. For a contradiction, suppose that no such edge exists. For any edge $xy \in E(G)$, the graph G/xy is not 3-connected, but $|G/xy| = |G| - 4 \ge 4$ by assumption that $|G| \ge 5$. Hence G/xy has a separating set S with $|S| \le 2$. Since G is 3-connected, the contracted vertex v_{xy} must belong to S, and |S| = 2, or we would have a separating set in G with ≤ 2 vertices. So there is some $z \in V(G), z \notin \{x,y\}$ such that $S = \{v_{xy},z\}$. Any two vertices separated in G/xy by S are also separated in G by the set $T = \{x,y,z\}$.

FACT: Since no proper subset of T separates G, by the 3-connectivity of G, every vertex in T has a neighbour in every component C of G - T.

Choose the edge xy, vertex z, and component C of $G - \{x, y, z\}$ such that |C| is as small as possible. Let v be a neighbour of z in C, which we know must exists by our FACT. By assumption, G/zv is not 3-connected, and $|G/zv| = |G| - 1 \ge 4$. Hence (by our earlier argument) there is a vertex $w \notin \{v, z\}$ such that $\{v, w, z\}$ separates G. Also by our FACT, every vertex in $\{v, w, z\}$ has a neighbour in every component of $G - \{v, w, z\}$.

Since x and y are adjacent, $G - \{z, v, w\}$ has a component D such that $D \cap \{x, y\} = \emptyset$. By our FACT we know that v has a neighbour in D. Recall that $v \in C$ in $G - \{x, y, z\}$. Since D is connected and $(\{v\} \cup V(D)) \cap \{x, y, z\}$, it follows that $\{v\} \cup V(D) \subseteq V(C)$. Hence D is a proper subgraph of C, as $v \notin V(D)$. Therefore |D| < |C|, contradicting the minimality of C.

Hence G/e is 3-connected for some $e \in E(G)$.

Reversing this, we can construct all 3-connected graphs starting with K_4 and "uncontracting" edges.

Theorem 5.2.2. A graph G is 3-connected if and only if there exists a sequence G_0, G_1, \ldots, G_r of graphs such that

- (i) $G_0 = K_4$ and $G_r = G$,
- (ii) G_{i+1} has an edge xy with degrees $d(x), d(y) \ge 3$ such that $G_i = G_{i+1}/xy$, for $i = 0, \ldots, r-1$.

5.3 Menger's Theorem

A set $S \subset V$ separating A from B in G is called an (A, B)-separator. This means that every (A, B)-path intersects S, and in particular $A \cap B \subseteq S$.

Let \mathcal{P} , \mathcal{Q} be sets of **disjoint** (A, B)-paths in G. Say that \mathcal{Q} exceeds \mathcal{P} if the set of vertices in A which belong to paths in \mathcal{P} is a *proper subset* of the set of vertices in A which belong to paths in \mathcal{Q} and similarly for B.

If $P = x_0 x_1 \cdots x_k$ then we write P_{x_i} for the subpath $x_0 \cdots x_i$ and we write $x_i P$ for the subpath $x_i x_{i+1} \cdots x_k$.

Theorem 5.3.1 (Menger's Theorem, 1927). Let G = (V, E) be a graph and $A, B \subseteq V$. Then the minimum number of vertices separating A from B in G equals the maximum number of disjoint (A, B)-paths in G.

Proof. Let k = k(G, A, B) be the minimum number of vertices separating A and B in G. (That is, k = |S| where $S \subseteq V$ is a smallest (A, B)-separating set.) Then k is an upper bound on the maximum number of disjoint (A, B)-paths or else we could not separate A and B by deleting any set of k vertices. So it suffices to prove that a set of k disjoint (A, B)-paths exists. In fact, we will prove a stronger statement:

If \mathcal{P} is any set of $\langle k \text{ disjoint } (A, B)\text{-paths}$, then there is a set \mathcal{Q} of $|\mathcal{P}| + 1$ disjoint (A, B)-paths in G which exceeds \mathcal{P} .

We will keep G and A fixed and let B vary, applying induction on the number of vertices in $\bigcup_{P \in \mathcal{P}} P$.

Base Case: If $\mathcal{P} = \emptyset$ then $|\bigcup_{P \in \mathcal{P}} P| = 0$. We can let $\mathcal{Q} = \{\mathcal{P}\}$ for any (A, B)-path P. Then \mathcal{Q} exceeds \mathcal{P} .

Inductive Step: Let \mathcal{P} be a set of < k disjoint (A, B)-paths, and $B_0 \subseteq B$ be the set of end vertices of paths in \mathcal{P} ("start vertices" are in A, "endvertices" are in B). Since $|B_0| \le k - 1$, B_0 is not an (A, B)-separating set and hence there is an (A, B)-path in $G - B_0$. Call this (A, B)-path R. So R is disjoint from B_0 . If R avoids all vertices in $\bigcup_{P \in \mathcal{P}}$ then $\mathcal{Q} = \mathcal{P} \cup \{R\}$ exceeds \mathcal{P} , as required. Otherwise, let x be the last vertex of R (traversing R from A to B) that lies on some path $P \in \mathcal{P}$. Note that $x \notin B$, by choice of R, so Px is shorter than P.

Let $B' = B \cup V(xP \cup xR)$ and let $\mathcal{P}' = (\mathcal{P} - \{P\}) \cup \{Px\}$. Then \mathcal{P}' is a set of disjoint (A, B')-paths. Also $|\mathcal{P}'| = |\mathcal{P}|$, but the union of paths in \mathcal{P}' is strictly smaller than $|\bigcup_{\hat{P} \in \mathcal{P}} \hat{P}|$. Also, $B \subseteq B'$, so an (A, B')-separating set is also an (A, B)-separating set. Hence $k(G, A, B') \geq k(G, A, B)$. So $|\mathcal{P}'| < k(G, A, B) \leq k(G, A, B')$. Applying the inductive hypothesis to (G, A, B', \mathcal{P}') , we conclude that there is a set \mathcal{Q}' of $|\mathcal{P}| + 1$ disjoint (A, B')-paths in (G, A, B') which exceeds (G, A, B', \mathcal{P}') is not among the last vertices of the paths in (G, A, B'). In particular, (G, A, B') whose last vertex (G, A, B') is not among the last vertices of the paths in (G, A, B').

Case 1: $y \in B$. If $y \in B$, then define $Q = (Q' - \{Q\}) \cup \{QxP\}$

Case 2: $y \notin B$ and $y \in xR$. If $y \in xR$ and $y \notin B$, then $y \notin xP$, and we define $Q = (Q' - \{Q, Q'\}) \cup \{QxP, Q'yR\}$.

Case 3: $y \notin B$ and $y \in xP$. If $y \in xP$ and $y \notin then <math>y \notin xR$, and we define $Q = (Q' - \{Q, Q'\} \cup \{QxR, Q'yP\})$.

In all cases, Q is a set of $|\mathcal{P}| + 1$ disjoint (A, B)-paths which exceeds \mathcal{P} , proving the inductive step. Hence there is a set of k disjoint (A, B)-paths in G, as required.

Corollary 5.3.2. Let a, b be distinct vertices of G.

- (i) If $ab \notin E$ then the minimum number of vertices (distinct from a and b) separating a from b is equal to the maximum number independent (a, b)-paths in G.
- (ii) The minimum number of edges separating a from b in G equals the maximum number of edge-disjoint (a, b)-paths in G.

Proof.

(i) Apply Menger's Theorem with A = N(A), B = N(b). Note that a set of k disjoint (A, B)-path

corresponds to a set of independent (a, b)-paths by adding vertex a at the start and vertex b to the end.

(ii) Apply Menger's Theorem to the line graph L(G) of G with A = E(a), the set of edges of G incident with a, B = E(b), the set of edges of G incident with b.

Theorem 5.3.3 (Global version of Menger's Theorem).

- (i) A graph is k-connected if and only if it has order at least 2 and there are k independent paths between any two distinct vertices.
- (ii) A graph is k-edge-connected if and only if it has at least two vertices and k edge-disjoint paths between any two distinct vertices.

Proof.

(i) Suppose that G is a graph and $|G| \ge 2$. Now suppose that G has k independent paths between any two distinct vertices $a, b \in V$. Then $|G| \ge k$, as there are at least k-1 paths of length at least two between a and b. Also, G cannot be disconnected by deleting a set of $\le k-1$ vertices. Hence G is k-connected.

For the converse, suppose that G is k-connected and assume for a contradiction that there are distinct vertices a, b with at most k-1 independent (a, b)-paths. Since G is k-connected we have $|G| \geq k+1$. By Corollary 5.3.2, we must have $ab \in E$. Let G' = G - ab. Then G' has at most k-2 independent (a, b)-paths. Hence by Corollary 5.3.2, there is an (a, b)-separating set $X \subseteq V$ with $|X| \leq k-2$. Since $|G| \geq k+1$, there is at least one more vertex $v \notin X \cup \{a,b\}$ in G. Now X separates v from at least one of v0, say from v1 vertices which separates v2 from v2. But then v3 is a set of at most v4 vertices which separates v5 from v6. This contradicts the fact that v6 is v5-connected.

Hence G has at least k independent (a, b)-paths in G, completing the proof.

(ii) Follows immediately from Corollary 5.3.2.