Higher Linear Algebra
MATH2621 UNSW

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Contents

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[*]With some inspiration from Hussain Nawaz's Notes

1 Assumed Knowledge

- the definition of complex numbers,

- their arithmetic,

- Cartesian and polar representations,

- the Argand diagram,

- de Moivre's theorem, and

- extracting nth roots of complex numbers.

2 Inequalities and Sets of Complex Numbers

2.1 Equalities and Inequalities

Modulus Squared of a Sum   For all complex numbers w and z,

$$|w + z|^2 = |w|^2 + 2\operatorname{Re}(w\bar{z}) + |z|^2.$$

Triangle Inequality   For all complex numbers w and z,

$$|w + z| \le |w| + |z| \qquad \forall w, z, \in \mathbb{C}.$$

Circle Inequality   For all complex numbers w and z,

$$||w| - |z|| \le |w - z|.$$

Modulus of e^z   If $z \in \mathbb{C}$, then
$$|e^z| = e^{\operatorname{Re}(z)}.$$

Modolus of $e^z - 1$ inequality   For all real numbers θ,

$$|e^{i\theta} - 1| \le |\theta|.$$

2.2 Properties of Sets

Open Ball   The open ball with centre z_0 and radius ϵ, written $\mathrm{B}(z_0, \epsilon)$, is the set $\{z \in \mathbb{C} : |z - z_0| < \epsilon\}$.

Punctured Open Ball   The punctured open ball with centre z_0 and radius ϵ, written $B^\circ(z_0, \epsilon)$, is the set $\{z \in \mathbb{C} : 0 < |z - z_0| < \epsilon\}$.

Interior, Exterior and Boundary Points Suppose that $S \subseteq \mathbb{C}$. For any point z_0 in $\mathbb{C}$, there are three mutually exclusive and exhaustive possibilities:

(1) When the positive real number ϵ is sufficiently small, $\mathrm{B}(z_0, \epsilon)$ is a subset of S, that is, $\mathrm{B}(z_0, \epsilon) \cap S = \mathrm{B}(z_0, \epsilon)$. In this case, z_0 is an interior point of S.

(2) When the positive real number *epsilon* is sufficiently small, $\mathrm{B}(z_0, \epsilon)$ does not meet S, that is, $\mathrm{B}(z_0, \epsilon) \, capS = \emptyset$. In this case, z_0 is an exterior point of S.

(3) No matter how small the positive real number ϵ is, neither of the above holds, that is, $\emptyset \subset \mathrm{B}(z_0, \epsilon) \cap S \subset \mathrm{B}(z_0, \epsilon)$. In this case, z_0 is a boundary point of S.

Open, Closed, Closure, Bounded, Compact, Region Sets Suppose that $S \subseteq \mathbb{C}$.

(1) The set S is open if all its points are interior points.

(2) The set S is closed if it contains all of its boundary points, or equivalently, if its complement $\mathbb{C} \setminus S$ is open.

(3) The closure of the set S is the set consisting of the points of S together with the boundary points of S.

(4) The set is bounded if $S \subseteq \mathrm{B}(0, R)$ for some $R \in \mathbb{R}^+$

(5) The set S is compact if it is both closed and bounded.

(6) The set S is a region if it is an open set together with none, some, or all of its boundary points.

2.3   Arcs

Polygonal Arc   A polygonal arc is a finite sequence of finite directed line segments, where the end point of one line segment is the initial point of the next one.

Simple Closed Polygonal Arc   A simple closed polygonal arc is a polygonal arc that does not cross itself, but the final point of the last segment is the initial point of the first segment.

Interior and Exterior Arc   The complement of a simple closed polygonal arc is made up of two pieces: one, the interior of the arc, is bounded, and the other, exterior is not.

Polygonally Path-connectedness   Let $X \subseteq \mathbb{C}$ be a subset of the complex plane.

(1) The set X is polygonally path-connected if any two points of X can be joined by a polygonal arc lying inside X.

(2) The set X is simply polygonally connected if it is polygonally path-connected and if the interior of every simple closed polygonal arc in X lies in X, that is, if "X has no holes".

(3) The set X is a domain if it is open and polygonally path-connected.

3   Functions of a Complex Variable

Complex Function   A complex function is one whose domain, or whose range, or both, is a subset of the complex plane $\mathbb{C}$ that is not a subset of the real line $\mathbb{R}$.

Complex Polynomial   A complex polynomial is a function $p : \mathbb{C} \to \mathbb{C}$ of the form

$$p(z) = a_d z^d + \cdots + a_1 z + a_0,$$

where $a_d, \ldots, a_1, a_0 \in \mathbb{C}$. If $a_d \neq 0$, we say that p is of degree d. A rational function is a quotient of polynomials.

The Fundamental Theorem of Algebra   Every nonconstant complex polynomial p of degree d factorizes: there exists $\alpha_1, \alpha_2, \ldots, \alpha_d$ and c in $\mathbb{C}$ such that

$$p(z) = c \prod_{j=1}^{d} (z - a_j).$$

Polynomial Division and Partial Fractions   Suppose that p and q are polynomials. Then

$$\frac{p(z)}{q(z)} = s(z) + \frac{r(z)}{q(z)},$$

where r and s are polynomials, and the degree of r is strictly less than the degree of q. Further, if

$$q(z) = c \prod_{j=1}^{e} (z - \beta_j)^{m_j},$$

then we may decompose the term r/q into partial fractions:

$$\frac{r(z)}{q(z)} = \sum_{j=1}^{e} \sum_{k=1}^{m_j} \frac{a_{jk}}{(z - \beta_j)^k}.$$

Real and Imaginary Parts   To a function $f : S \to \mathbb{C}$, where $S \subseteq \mathbb{C}$, we associate two real-valued functions u and v of two real variables:

$$f(x + iy) = u(x, y) + iv(x, y).$$

Then $u(x, y) = \operatorname{Re} f(x + iy)$ and $v(x, y) = \operatorname{Im} f(x + iy)$.

3.1   The function $w = 1/z$

Consider the mapping $w = 1/z$.

(1) The image of a line through 0 (with the origin removed) is a line through 0 (with the origin removed).

(2) The image of a line that does not pass through 0 is a circle (with the origin removed). If p is the closest point on the line to 0, then the line segment between 0 and 1/p is a diameter of the circle.

(3) The image of a circle that passes through 0 is a line. If q is the furthest point on the circle from 0, then the closest point on the line to 0 is $1/q$.

(4) The image of a circle that does not pass through 0 is a circle. If p and q are the closest and furthest point on the circle from 0, then the closest and furthest point on the image circle to 0 are $1/q$ and $1/p$.

3.2 Fractional Linear Transformations

Factorising Matrices Every 2×2 complex matrix with determinant 1 may be written as a product f at most three matrices of the following special types:

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$