

# Higher Several Variable Calculus

## Math2111 UNSW

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# 1 Introduction

**Real one-variable calculus**  $f : \mathbb{R} \rightarrow \mathbb{R}$

- limits
- continuity
- differentiability
- integrability

## Important Theorems

- Min-max theorem  
A continuous function on a closed interval attains a max and min value.
- Intermediate Value Theorem  
A continuous function on  $[a, b]$  attains all values in  $[f(a), f(b)]$ .
- Mean Value Theorem  
Connects the instantaneous rate of change of differentiable function to its change over a finite closed interval.

**Multivariable Calculus Applications**  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

- Fluid dynamics
- Black Scholes Options Pricing Model

# 2 Curves and Surfaces

## 2.1 Curves

The parameterisation of a curve in  $\mathbb{R}^n$  is a vector-valued function

$$\mathbf{c} : I \rightarrow \mathbb{R}^n$$

where  $I$  is an interval on  $\mathbb{R}$ .

- A multiple point is a point through which the curve passes more than once.
- If  $I = [a, b]$  then  $\mathbf{c}(a)$  and  $\mathbf{c}(b)$  are called end points.
- A curve is closed if its end points are the same point,  $\mathbf{c}(a) = \mathbf{c}(b)$ .

## 2.2 Limits and Calculus for Curves

For an interval  $I \subset \mathbb{R}$  and curve  $\mathbf{c} : I \rightarrow \mathbb{R}^n$  with

$$\mathbf{c}(t) = (c_1(t), c_2(t), \dots, c_n(t)),$$

the functions  $c_i : I \rightarrow \mathbb{R}, i = 1, 2, \dots, n$  are called the components of  $\mathbf{c}$ .

- If  $\lim_{t \rightarrow a} c_i(t)$  exists for all  $i$ , then  $\lim_{t \rightarrow a} \mathbf{c}(t)$  and

$$\lim_{t \rightarrow a} \mathbf{c}(t) = \left( \lim_{t \rightarrow a} c_1(t), \lim_{t \rightarrow a} c_2(t), \dots, \lim_{t \rightarrow a} c_n(t) \right)$$

- If  $c'_i(t)$  exists for all  $i$ , then

$$\mathbf{c}'(t) = (c'_1(t), c'_2(t), \dots, c'_n(t))$$

## 2.3 Surfaces

You have seen surfaces in  $\mathbb{R}^3$  described in 3 ways.

- Graph:  $z = f(x, y)$
- Implicitly:  $x^2 + y^2 + z^2 = 1$
- Parametrically:  $\mathbf{x} = \mathbf{x}_0 + \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2$

# 3 Analysis

## 3.1 Formal Definition of a Limit

**1-variable Calculus** Recall that  $\lim_{x \rightarrow a} f(x) = L$  requires that for all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that if  $|x - a| < \delta$  then

$$|f(x) - L| < \epsilon.$$

## 3.2 Distance Functions (metrics)

A function  $d : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  which satisfies the following three properties is called a metric.

- **Positive Definite:** for all  $x, y \in \mathbb{R}^n$ ,  $d(x, y) > 0$  and  $d(x, y) = 0$  iff  $x = y$ .
- **Symmetric:** for all  $x, y \in \mathbb{R}^n$ ,  $d(x, y) = d(y, x)$ .
- **Triangle Inequality** for all  $x, y, z \in \mathbb{R}^n$ ,  $d(x, y) + d(y, z) \geq d(x, z)$ .

**Euclidean Distance** The Euclidean distance between  $x$  and  $y$  defined by

$$d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

is a metric.

**Equivalent Metrics** Two metrics  $d$  and  $\delta$  are considered equal if there exists constants  $0 < c < C < \infty$  such that

$$c\delta(x, y) \leq d(x, y) \leq C\delta(x, y).$$

### 3.3 Limits of Sequences

**Ball** A ball around  $\mathbf{a} \in \mathbb{R}^n$  of radius  $\epsilon > 0$  is the set

$$B(\mathbf{a}, \epsilon) = \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{a}, \mathbf{x}) < \epsilon\}.$$

**Limit of Sequences** For a sequence  $\{\mathbf{x}_i\}$  of points in  $\mathbb{R}^n$  we say that  $\mathbf{x}$  is the limit of the sequence if and only if

$$\forall \epsilon > 0 \exists N \text{ such that } n \geq N \implies d(\mathbf{x}, \mathbf{x}_n) < \epsilon$$

or equivalently

$$\forall \epsilon > 0 \exists N \text{ such that } n \geq N \implies \mathbf{x}_n \in B(\mathbf{x}, \epsilon).$$

If  $\mathbf{x}$  is the limit of the sequence  $\{\mathbf{x}_i\}$  then for each positive  $\epsilon$  there is a point in the sequence beyond which all points of the sequence are inside  $B(\mathbf{x}, \epsilon)$ .

#### Convergence

A sequence  $\mathbf{x}_k$  converges to a limit  $\mathbf{x}$

$\Leftrightarrow$  the components of  $\mathbf{x}_k$  converge to the components of  $\mathbf{x}$

$\Leftrightarrow d(\mathbf{x}_k, \mathbf{x}) \rightarrow 0$ .

**Cauchy Sequences** A sequence  $\{\mathbf{x}_k\}$  in  $\mathbb{R}^n$  is a Cauchy sequence if

$$\forall \epsilon > 0 \exists K \text{ such that } k, l > K \implies d(\mathbf{x}_k, \mathbf{x}_l) < \epsilon.$$

A sequence  $\{\mathbf{x}_k\}$  converges in  $\mathbb{R}^n$  to a limit if and only if  $\{\mathbf{x}_k\}$  is a Cauchy sequence.

### 3.4 Open and Closed Sets

**Definitions** Consider  $x_k$

- $x_0 \in \Omega$  is an interior point of  $\Omega$  if there is a ball around  $x_0$  completely contained in  $\Omega$ . That is, there exists a  $\epsilon > 0$  such that  $B(x_0, \epsilon) \subseteq \Omega$ .
- $\Omega$  is open if every point of  $\Omega$  is an interior point.
- $\Omega$  is closed if its complement is open.
- $x_0 \in \Omega$  is a boundary point of  $\Omega$  if every ball around  $x_0$  contains points in  $\Omega$  and points not in  $\Omega$ .

**Closed Sets**  $\Omega \subset \mathbb{R}^n$  is closed if and only if it contains all of its boundary points.

## Union and Intersection

- A finite union/intersection of open sets is open.
- A finite union/intersection of closed sets is closed.

**Limit Points and Sets**  $\mathbf{x}_0$  is a limit point (or accumulation point) of  $\Omega$  if there is a sequence  $\{\mathbf{x}_i\}$  in  $\Omega$  with limit  $\mathbf{x}_0$  and  $\mathbf{x}_i \neq \mathbf{x}_0$ .

- Every interior points of  $\Omega$  is a limit point of  $\Omega$ .
- $\mathbf{x}_0$  is not necessarily in  $\Omega$ .
- A set is closed  $\Leftrightarrow$  it contains all of its limit points.

**Variations of a Set** Consider the set  $\Omega \in \mathbb{R}^n$ .

- The interior of  $\Omega$  is the set of all its interior points (denoted  $\text{Int}(\Omega)$ ).
- The boundary of  $\Omega$  is the set of all its boundary points (denoted  $\partial\Omega$ ).
- The closure of  $\Omega$  is  $\Omega \cup \partial\Omega$  (denoted by  $\bar{\Omega}$ ).

The interior is the largest open subset and the closure is the smallest closed set containing  $\Omega$ .

## 3.5 Limits

**Limit of a Function at a Point** Let  $\mathbf{b} \in \mathbb{R}^m, \Omega \subseteq \mathbb{R}^n, \mathbf{a} \in \bar{\Omega}$  and let  $\mathbf{f} : \Omega \rightarrow \mathbb{R}^m$  be a function. We say that  $\mathbf{f}(\mathbf{x})$  converges to  $\mathbf{b}$  as  $\mathbf{x} \rightarrow \mathbf{a}$  if

$$\forall \epsilon > 0 \exists \delta > 0 \text{ such that for } \mathbf{x} \in \Omega :$$

$$0 < d(\mathbf{x}, \mathbf{a}) < \delta \implies d(\mathbf{f}(\mathbf{x}), \mathbf{b}) < \epsilon.$$

or alternatively

$$\mathbf{x} \in B(\mathbf{a}, \delta) \cap \Omega \implies \mathbf{f}(\mathbf{x}) \in B(\mathbf{b}, \epsilon).$$

If such  $\mathbf{b}$  exists, then it is unique and we write

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}.$$

**Useful Limit Theorems** Let  $\mathbf{b} \in \mathbb{R}^m, \Omega \subseteq \mathbb{R}^n, \mathbf{a} \in \bar{\Omega}$  and let  $\mathbf{f} : \Omega \rightarrow \mathbb{R}^m$  be a function. Then

$$\begin{aligned} \lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b} &\iff \lim_{\mathbf{x} \rightarrow \mathbf{a}} f_i(\mathbf{x}) = b_i \text{ for all } i = 1, \dots, m \\ \lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b} &\iff \lim_{k \rightarrow \infty} \mathbf{f}(\mathbf{x}_k) = \mathbf{b} \end{aligned}$$

for every sequence  $\{\mathbf{x}_k\}_{k=1}^{\infty} \subseteq \Omega$  with  $\lim_{k \rightarrow \infty} \mathbf{x}_k = \mathbf{a}$ .

The first theorem is useful to show that a limit exists whilst the second is useful to show the limit does not exist.

**Algebra of limits** Given that,  $\lim_{x \rightarrow x_0} f(x) = a$  and  $\lim_{x \rightarrow x_0} g(x) = b$ , then,

$$\begin{aligned}\lim_{x \rightarrow x_0} (f + g)(x) &= a + b \\ \lim_{x \rightarrow x_0} (fg)(x) &= ab \\ \lim_{x \rightarrow x_0} \left(\frac{f}{g}\right)(x) &= \frac{a}{b}, \text{ given } b \neq 0.\end{aligned}$$

**Pinching Principle** Let  $\Omega \subset \mathbb{R}^n$ , let  $\mathbf{a}$  be a limit point of  $\Omega$  and let  $f, g, h : \Omega \rightarrow \mathbb{R}$  be functions such that there exists  $\epsilon > 0$  such that

$$g(\mathbf{x}) \leq f(\mathbf{x}) \leq h(\mathbf{x}) \quad \forall \mathbf{x} \in B(\mathbf{a}, \epsilon) \cap \Omega.$$

Then

$$\lim_{x \rightarrow \mathbf{a}} g(\mathbf{x}) = \mathbf{b} = \lim_{x \rightarrow \mathbf{a}} h(\mathbf{x}) \implies \lim_{x \rightarrow \mathbf{a}} f(\mathbf{x}) = \mathbf{b}.$$

### 3.6 Continuity

Continuity is like an extension to limits. It first requires that the limit exists and that the limit equals the actual value at that point.

**Definition** Let  $\mathbf{a} \in \Omega \subseteq \mathbb{R}^n$  and let  $f : \Omega \rightarrow \mathbb{R}^m$  be a function. Then  $f$  is continuous at  $\mathbf{a}$  if and only if

$$\lim_{x \rightarrow \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$$

$f$  is said to be continuous on  $\Omega$  if it is continuous at  $\mathbf{a}$  for every  $\mathbf{a} \in \Omega$ .

#### Epsilon-Delta Interpretation

For all  $\epsilon > 0$  there exists  $\delta > 0$  such that if  $x \in B(\mathbf{a}, \delta) \cap \Omega \implies f(x) \in B(f(\mathbf{a}), \epsilon)$ .

**Continuity by Components** All component functions  $f_i : \Omega \rightarrow \mathbb{R}$  are continuous at  $\mathbf{a}$ .

**Continuity through Sequences** For every sequence  $\{\mathbf{x}_k\}_{k=1}^{\infty}$  with  $\mathbf{x}_k \in \Omega$  for all  $k$ , if  $\{\mathbf{x}_k\}_{k=1}^{\infty}$  has limit  $\mathbf{a}$  then  $\{f(\mathbf{x}_k)\}_{k=1}^{\infty}$  converges to  $f(\mathbf{a})$ .

**Elementary Functions** If  $f : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  is an elementary function, then  $f$  is continuous on  $\Omega$ .

**Preimage** Suppose that  $\Omega \subseteq \mathbb{R}^n$  and  $f : \Omega \rightarrow \mathbb{R}^m$  is a function. The preimage of a set  $U \subseteq \mathbb{R}^m$  is defined by

$$f^{-1}(U) = \{x \in \mathbb{R}^n : f(x) \in U\}.$$

**Continuity - Using Preimage** Suppose that  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ . The following two statements are equivalent.

- $f$  is continuous on  $\Omega$ .
- $f^{-1}(U)$  is open in  $\mathbb{R}^n$  for every open subset  $U$  of  $\mathbb{R}^m$ .

### 3.7 Path Connected Sets

**Definition** A set  $\Omega \subseteq \mathbb{R}^n$  is said to be path connected if for any  $\mathbf{x}, \mathbf{y} \in \Omega$ , there is a continuous function  $\varphi$  such that  $\varphi(t) \in \Omega$  for all  $t \in [0, 1]$  and  $\varphi(0) = \mathbf{x}$  and  $\varphi(1) = \mathbf{y}$ .

**Theorem** Let  $\Omega \subseteq \mathbb{R}^n$  and  $\mathbf{f} : \Omega \rightarrow \mathbb{R}^m$  be continuous. Then

$$B \subseteq \Omega \text{ and } B \text{ path connected} \implies \mathbf{f}(B) \text{ path connected.}$$

### 3.8 Compact Sets

**Bounded** A set  $\Omega \subseteq \mathbb{R}^n$  is bounded if there is an  $M \in \mathbb{R}$  such that  $d(\mathbf{x}, \mathbf{0}) \leq M$  for all  $\mathbf{x} \in \Omega \iff \Omega \subseteq B(\mathbf{0}, M)$ .

**Compact** A set  $\Omega \subseteq \mathbb{R}^n$  is compact if it is closed and bounded.

**Theorem** Let  $\Omega \subseteq \mathbb{R}^n$  and  $f : \Omega \rightarrow \mathbb{R}^m$  be continuous. Then

$$K \subseteq \Omega \text{ and } K \text{ compact} \implies f(K) \text{ compact.}$$

### 3.9 Bolzano-Weierstrass Theorem

For  $\Omega \subseteq \mathbb{R}^n$ , the following are equivalent.

1.  $\Omega$  is compact.
2. Every sequence in  $\Omega$  has a subsequence that converges to an element of  $\Omega$ .

## 4 Differentiation

### 4.1 Differentiability, Derivatives and Affine Approximations

**Differentiability in  $\mathbb{R}$**   $f : \mathbb{R} \rightarrow \mathbb{R}$  is differentiable at some  $a \in \mathbb{R}$  means there is a *good* straight-line approximation to  $f$  near  $a$  called a tangent line. This approximating function is given by

$$T(x) = f(a) + f'(a)(x - a) = f(a) - f'(a)a + f'(a)x = y_0 + L(x)$$

where for all  $a$ ,  $y_0 = f(a) - f'(a)a$  is a fixed number and  $L : \mathbb{R} \rightarrow \mathbb{R} = f'(a)x$  is the linear map.

Recall that

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$



**Linear Maps** A function  $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is called linear iff for all  $x, y \in \mathbb{R}^n$  for all  $\lambda \in \mathbb{R}$  :

$$L(x + y) = L(x) + L(y) \text{ and } L(\lambda x) = \lambda L(x).$$

**Affine Maps** A function  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is affine means there is  $y_0 \in \mathbb{R}^m$  and a linear map (ie matrix)  $\mathbf{L} : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that

$$T(\mathbf{x}) = \mathbf{y}_0 + \mathbf{L}(\mathbf{x}).$$

A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is affine iff  $f(x) = ax + b$ , for some  $a, b \in \mathbb{R}$ .

**Affine approximation** The function  $f : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$  has an affine approximation at a point  $a \in \Omega$  if and only if there exists a matrix  $A \in M_{m \times n}(\mathbb{R})$  such that

$$\lim_{x \rightarrow a} \frac{d(f(x) - f(a), A(x - a))}{d(x, a)} = 0$$

If  $f$  has an affine approximation at a point  $a \in \Omega$ , then the matrix  $A$  in the definition is called the derivative of  $f$  at  $a$  and is denoted by  $Df(a)$  (or  $Daf$ ).

The function  $T_a f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  defined by

$$T_a f(x) = Df(a)(x - a) + f(a)$$

is called the best affine approximation of  $f$  at  $a$ .

**Differentiability in  $\mathbb{R}^n \rightarrow \mathbb{R}^m$**  A function  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  is differentiable for some  $a \in \Omega$  if there exists a linear map  $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that

$$\lim_{x \rightarrow a} \frac{\|f(x) - f(a) - L(x - a)\|}{\|L(x - a)\|} = 0.$$

Notation: the matrix of the linear map  $L$ , the derivative of  $f$  at  $a$  is denoted by  $D_a f$ .

**Delta Epsilon Definition of Differentiability** A function  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  is differentiable on  $a \in \Omega$  if there is a linear map  $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that  $\forall \epsilon > 0 \exists \delta > 0$  such that for all  $x \in \Omega$

$$\|x - a\| < \delta \rightarrow \|f(x) - f(a) - L(x - a)\| < \epsilon \|x - a\|.$$

## 4.2 Partial Derivatives

Let  $\mathbf{a} \in \mathbb{R}^n$  and  $f : \Omega \rightarrow \mathbb{R}$  be a function with coordinates  $x_i$  and standard basis vectors  $\mathbf{e}_i, i \in \{1, \dots, n\}$ . The partial derivative of  $f$  in direction  $i$  is defined as

$$\frac{\partial f}{\partial x_i} = \lim_{h \rightarrow 0} \frac{f(\mathbf{a} + h\mathbf{e}_i) - f(\mathbf{a})}{h}$$

assuming the limit exists.

**Claiaut's Theorem** If  $f, \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j}, \frac{\partial^2 f}{\partial x_i x_j}, \frac{\partial^2 f}{\partial x_j x_i}$  all exist and are continuous on an open set around  $\mathbf{a}$  then

$$\frac{\partial^2 f}{\partial x_i x_j}(\mathbf{a}) = \frac{\partial^2 f}{\partial x_j x_i}(\mathbf{a}).$$

That is the partial derivatives commute.

### 4.3 Jacobian Matrix

**Definition** If all partial derivatives of  $\mathbf{f} : \Omega \rightarrow \mathbb{R}^m$  exists at  $\mathbf{a} \in \omega \subseteq \mathbb{R}^n$ , then the Jacobian matrix of  $\mathbf{f}$  at  $\mathbf{a}$  is

$$J_{\mathbf{a}} \mathbf{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \frac{\partial f_1}{\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{a}) & \frac{\partial f_2}{\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \frac{\partial f_m}{\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{pmatrix}.$$

**Theorem** Let  $\Omega \subseteq \mathbb{R}^n, \mathbf{a} \in \Omega$  be an interior point and  $\mathbf{f} : \Omega \rightarrow \mathbb{R}^m$  be a function. If  $\mathbf{f}$  is differentiable at  $\mathbf{a}$  then all partial derivatives  $\frac{\partial f_j}{\partial x_i}$  exist at  $\mathbf{a}$  and

$$D\mathbf{f}(\mathbf{a}) = J\mathbf{f}(\mathbf{a}).$$

Best affine approximation:  $T_{\mathbf{a}} f(x) = Jf(\mathbf{a})(x - \mathbf{a}) + f(\mathbf{a})$ .

### 4.4 Differentiable and Continuous

**Limit at 0** For  $\mathbf{x} \in \mathbb{R}^n$  and  $L$  an  $m \times n$  matrix,

$$\lim_{\mathbf{x} \rightarrow \mathbf{0}} \|L\mathbf{x}\| = 0.$$

**Open Sets** Let  $\Omega \subseteq \mathbb{R}^n$  be open and let  $f : \Omega \rightarrow \mathbb{R}^m$  be a function that is differentiable on  $\Omega$ . Then  $f$  is continuous on  $\Omega$ .

**Partial Derivatives + Continuity** Let  $\Omega \subseteq \mathbb{R}^n$  be open and let  $f : \Omega \rightarrow \mathbb{R}^m$  be a function. If for all  $i = 1, \dots, n$  and all  $j = 1, \dots, m$  the partial derivative  $\frac{\partial f_j}{\partial x_i}$  exists and is continuous on  $\Omega$  then  $f$  is differentiable on  $\Omega$ .

### 4.5 Chain Rule, Gradient, Directional Derivatives, Tangent Planes

**Chain Rule** Let  $\Omega \subseteq \mathbb{R}^n, \Omega' \subseteq \mathbb{R}^m$  and let  $\mathbf{a} \in \Omega$ . Suppose  $\mathbf{f} : \Omega \rightarrow \mathbb{R}^m$  and  $\mathbf{g} : \Omega' \rightarrow \mathbb{R}^k$  are functions such that  $\mathbf{f}(\Omega) \subseteq \Omega'$ . If  $\mathbf{f}$  is differentiable at  $\mathbf{a}$  and  $\mathbf{g}$  is differentiable at  $\mathbf{f}(\mathbf{a})$ , then  $\mathbf{g} \circ \mathbf{f}$  is differentiable at  $\mathbf{a}$  and

$$D(\mathbf{g} \circ \mathbf{f})(\mathbf{a}) = D\mathbf{g}(\mathbf{f}(\mathbf{a}))D\mathbf{f}(\mathbf{a}).$$

**Gradient** For  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ , if the Jacobian exists, then it is given by the  $1 \times n$  matrix

$$Jf = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}.$$

This is equivalent to the gradient of  $f$ . That is,

$$\text{grad}(f) = \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}.$$

**Directional Derivative** The directional derivative of  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$  in the direction of the unit vector  $\hat{\mathbf{u}}$  at  $\mathbf{a} \in \Omega$  is

$$D_{\hat{\mathbf{u}}}f(\mathbf{a}) = f'_{\hat{\mathbf{u}}}(\mathbf{a}) = \lim_{h \rightarrow 0} \frac{f(\mathbf{a} + h\hat{\mathbf{u}}) - f(\mathbf{a})}{h}.$$

if the limit exists.

Equivalently, if  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable at  $a$  then for a unit vector  $u$

$$D_u f(a) = f'_u(a) = \nabla f(a) \cdot u.$$

Alternatively, allowing  $\theta$  to be the angle between  $\nabla f(a)$  and  $u$ ,

$$D_u f(a) = |\nabla f(a)| \cdot |u| \cdot \cos \theta.$$

**Affine Approximation** Allow  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$  to be a differentiable function at  $a \in \Omega$ . The best affine approximation to  $f$  at  $a$  may be written in terms of the gradient vector as

$$T(x) = f(a) + \nabla f(a) \cdot (x - a).$$

**Tangent Planes** The tangent plane to a function  $z = f(x, y)$  is given by

$$z = T(x, y).$$

## 4.6 Taylor Series and Theorem

**Taylor's Theorem** For all continuous and differentiable functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$f(x) \approx P_{k,a}(x) = \sum_{n=0}^k \frac{f^{(n)}(a)}{n!} (x-a)^n + R_{k,a}(x)$$

where the remainder  $R$  is

$$R_{k,a}(x) = \frac{f^{(k+1)}(z)}{(k+1)!} (x-a)^{k+1}$$

for some  $z$  between  $x$  and  $a$ .

$P_{0,a}, P_{1,a}, P_{2,a}, P_{3,a}$  are the best constant, affine, quadratic, cubic approximations.

**Hessian Matrix** For  $\Omega \subseteq \mathbb{R}^n$  and  $f : \Omega \rightarrow \mathbb{R}$ , the *Hessian matrix* of  $f$  at a point  $a \in \Omega$  is the  $n \times n$  matrix

$$Hf(a) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(a) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(a) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(a) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(a) & \frac{\partial^2 f}{\partial x_2^2}(a) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(a) & \frac{\partial^2 f}{\partial x_2 \partial x_n}(a) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(a) \end{pmatrix}.$$

assuming the 2<sup>nd</sup> order partial derivatives exist.

**Class** A function  $f : \Omega \rightarrow \mathbb{R}, \Omega \subseteq \mathbb{R}^n$  open, is called (of class)  $C^r$  if all partial derivatives of  $f$  of order  $\leq r$  exist and are continuous.

**Taylor Polynomials** Let  $\Omega \subseteq \mathbb{R}^n$  be open, let  $a \in \Omega$ , and let  $f : \Omega \rightarrow \mathbb{R}$  be a function of class  $C^2$ . The polynomial

$$P_{1,a}(x) = f(a) + \nabla f(a) \cdot (x-a)$$

is called the Taylor polynomial of order 1 about  $a$  and the polynomial

$$P_{2,a}(x) = f(a) + \nabla f(a) \cdot (x-a) + \frac{1}{2}(x-a) \cdot Hf(a)(x-a)$$

is called the Taylor Polynomial of order 2 about  $a$ .

In general, if  $f : \Omega \rightarrow \mathbb{R}$  is  $C^r, \Omega$  open,  $a \in \Omega$ :

$$\begin{aligned} P_{r,a}(x) &= f(a) + \nabla f(a) \cdot (x-a) + \frac{1}{2}(x-a) \cdot Hf(a)(x-a) \\ &+ \cdots + \frac{1}{r!} \sum_{i_1, \dots, i_r=1}^n \frac{\partial^r f}{\partial x_{i_1} \dots \partial x_{i_r}}(a) (x_{i_1} - a_{i_1}) \cdots (x_{i_r} - a_{i_r}). \end{aligned}$$

**Taylor's Theorem (1<sup>st</sup> order)** Let  $\Omega \subseteq \mathbb{R}^n$  be open, let  $f : \Omega \rightarrow \mathbb{R}$  be a function of class  $C^2$ . Let  $x, a \in \Omega$  s.t. the line segment between  $x$  and  $a$  is contained in  $\Omega$ . Then there exist  $z$  on this line segment such that

$$f(x) = f(a) + \nabla f(a) \cdot (x - a) + R_{1,a}(x)$$

where  $R_{1,a}(x) = \frac{1}{2}(x - a) \cdot (Hf(z)(x - a))$ .

**Taylor's Theorem (2<sup>nd</sup> order)** Let  $\Omega \subseteq \mathbb{R}^n$  be open, let  $f : \Omega \rightarrow \mathbb{R}$  be a function of class  $C^3$ . Let  $x, a \in \Omega$  s.t. the line segment between  $x$  and  $a$  is contained in  $\Omega$ . Then there exist  $z$  on this line segment such that

$$f(x) = f(a) + \nabla f(a) \cdot (x - a) + \frac{1}{2}(x - a)Hf(a)(x - a) + R_{2,a}(x)$$

where  $R_{2,a}(x) : \Omega \rightarrow \mathbb{R}$  is a function such that  $\frac{|R_{2,a}(x)|}{|x - a|^2} \rightarrow 0$  as  $x \rightarrow a$ .

## 4.7 Maxima, Minima and Saddle Points

**Definitions** Let  $a \in \Omega \subseteq \mathbb{R}^n$  and  $f : \Omega \rightarrow \mathbb{R}$  be a function. Then

- $a$  is an *absolute or global maximum* of  $f$  if  $f(a) \geq f(x)$  for all  $x \in \Omega$ .
- $a$  is an *absolute or global minimum* of  $f$  if  $f(a) \leq f(x)$  for all  $x \in \Omega$ .
- $a$  is a *local maximum* of  $f$  if there is an open  $A \subseteq \Omega$  containing  $a$  such that  $f(a) \geq f(x)$  for all  $x \in A$ .
- $a$  is a *local minimum* of  $f$  if there is an open  $A \subseteq \Omega$  containing  $a$  such that  $f(a) \leq f(x)$  for all  $x \in A$ .
- $a$  is a *stationary point* of  $f$  if  $f$  is differentiable at  $a$  and  $\nabla f(a) = 0$ .
- $a$  is a *saddle point* of  $f$  if  $a$  is a stationary point of  $f$  but it's neither a local max nor a local minimum of  $f$ .

**Critical Points** Let  $a \in \Omega \subseteq \mathbb{R}^n$  and  $f : \Omega \rightarrow \mathbb{R}$  be a function. If  $a$  is a local maximum or a local minimum then

1.  $a$  is a stationary, or
2.  $a \in \partial\Omega \iff a$  is a boundary pt, or
3.  $f$  is not differentiable at  $a$ .

Points satisfying 1, 2 or 3 are called critical points.

## 4.8 Classification of Stationary Points

**Definition:** An  $n \times n$  matrix  $H$  is

- positive definite  $\iff$  all eigenvalues are  $> 0$
- positive semi-definite  $\iff$  all eigenvalues are  $\geq 0$
- negative definite  $\iff$  all eigenvalues are  $< 0$
- negative semi-definite  $\iff$  all eigenvalues are  $\leq 0$

**Criterion for Local Extrema** Let  $\Omega \subseteq \mathbb{R}^n$  be open,  $a \in \Omega$  and let  $f : \Omega \rightarrow \mathbb{R}$  be a function such that all partial derivatives of  $f$  of order at most 2 exists on  $\Omega$  and  $\nabla f(a) = 0$ . Then

- $Hf(a)$  is positive definite  $\implies f$  has a local minimum at  $a$ ;
- $Hf(a)$  is negative definite  $\implies f$  has a local maximum at  $a$ ;
- $f$  has a local minimum at  $a \implies Hf(a)$  is positive semi-definite;
- $f$  has a local maximum at  $a \implies Hf(a)$  is negative semi-definite;

**Sylvester's Criterion** If  $H_k$  is the upper  $k \times k$  matrix of  $H$  and  $\Delta_k = \det(H_k)$ , then

- $H$  is positive definite  $\iff \Delta_k > 0$  for all  $k$
- $H$  is positive semi-definite  $\implies \Delta_k \geq 0$  for all  $k$
- $H$  is negative definite  $\iff \Delta_k < 0$  for all odd  $k$  and  $\Delta_k > 0$  for all even  $k$
- $H$  is negative semi-definite  $\implies \Delta_k \leq 0$  for all odd  $k$  and  $\Delta_k \geq 0$  for all even  $k$

## 4.9 Lagrange Multipliers, Implicit and Inverse Function Theorems

**Lagrange Multipliers** Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  are differentiable and  $S = \{x \in \mathbb{R}^n : \varphi(x) = c\}$  defines a smooth surface on  $\mathbb{R}^n$ . If  $f$  attains a local maximum or minimum at a point  $a \in S$  then  $\nabla f(a)$  and  $\nabla \varphi(a)$  are parallel. If  $\nabla \varphi(a) \neq 0$ , there exist a Lagrange multiplier  $\lambda \in \mathbb{R}$  such that

$$\nabla f(a) = \lambda \nabla \varphi(a).$$

**Inverse Function Theorem for  $f : \mathbb{R} \rightarrow \mathbb{R}$**  If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is differentiable on an open interval  $I \subseteq \mathbb{R}$  and  $f'(x) \neq 0$  for all  $x \in I$ , then  $f$  is invertible on  $I$  and the inverse  $f^{-1} : f(I) \rightarrow \mathbb{R}$  is differentiable with

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

**Generalising the Inverse Function Theorem** Let  $\Omega \subseteq \mathbb{R}^n$  be open,  $f : \Omega \rightarrow \mathbb{R}^n$  be  $C^1$  and suppose  $a \in \Omega$ . If  $Df(a)$  is invertible (as a matrix) then  $f$  is invertible on an open set  $U$  containing  $a$ . That is,

$$f^{-1} : f(U) \rightarrow U$$

exists. Furthermore,  $f^{-1}$  is  $C^1$  and for  $x \in U$ ,

$$D_{f(x)}f^{-1} = (D_x f)^{-1}.$$

## 5 Integration

### 5.1 Riemann Integral

**Riemann Integral** For a bounded function  $f : R \rightarrow \mathbb{R}$ , if there exists a unique number  $I$  such that

$$\underline{\mathcal{S}}_{\mathcal{P}_1, \mathcal{P}_2}(f) \leq I \leq \overline{\mathcal{S}}_{\mathcal{P}_1, \mathcal{P}_2}(f)$$

for every pair of partitions  $\mathcal{P}_1, \mathcal{P}_2$  of  $R$ , then  $f$  is Riemann integrable on  $R$  and

$$I = \int \int_R f = \int \int_R f(x, y) dA.$$

$I$  is called the Riemann integral of  $f$  over  $R$ .

**Properties of the Riemann Integral** For a function of one variable, the Riemann integral is interpreted as the (signed) area bounded by the graph  $y = f(x)$  and the  $x$ -axis over the interval  $[a, b]$ . For a function of two variables  $\int \int_R f$  is the (signed) volume bounded by the graph  $z = f(x, y)$  and the  $xy$ -plane over the rectangle  $R$ . If  $f$  and  $g$  are integrable on  $R$ ,

- Linearity:  $\int \int_R \alpha f + \beta g = \alpha \int \int_R f + \beta \int \int_R g$ ,  $\alpha, \beta \in \mathbb{R}$ .
- Positivity (monotonicity): If  $f(x) \leq g(x), \forall x \in R$  then  $\int \int_R f \leq \int \int_R g$
- $|\int \int_R f| \leq \int \int_R |f|$
- If  $R = R_1 \cup R_2$  and  $(\text{interior } R_1) \cap (\text{interior } R_2) = \emptyset$  then

$$\int \int_R f = \int \int_{R_1} f + \int \int_{R_2} f.$$

### 5.2 Fubini's Theorem

**Fubini's Theorem - Rectangles** Let  $f : R \rightarrow \mathbb{R}$  be continuous on a rectangular domain  $R = [a, b] \times [c, d]$ . Then  $f$  is a bounded function and is integrable over  $R$ . Moreover,

$$\int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy = \int \int_R f.$$

**Fubini's Theorem - Discontinuous** Let  $f : R \rightarrow \mathbb{R}$  be bounded on a rectangular domain  $R = [a, b] \times [c, d]$  with the discontinuities of  $f$  confined to a finite union of graphs of continuous functions. If the integral  $\int_c^d f(x, y) dy$  exists for each  $x \in [a, b]$  then

$$\int \int_R f = \int_a^b \left( \int_c^d f(x, y) dy \right) dx.$$

Similarly, if the integral  $\int_a^b f(x, y) dx$  exists for each  $y \in [c, d]$ , then

$$\int \int_R f = \int_c^d \left( \int_a^b f(x, y) dx \right) dy.$$

**Iterated Integrals for Elementary Regions** Suppose  $D$  is a  $y$ -simple region bounded by  $x = a, x = b, y = \varphi_1(x)$  and  $y = \varphi_2(x)$  and  $f : D \rightarrow \mathbb{R}$  is continuous. Then

$$\int \int_D f = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dx dy.$$

A similar result holds for integrals over  $x$ -simple regions.

### 5.3 Leibniz' Rule

**Basic Version** Let  $a, b, c, d \in \mathbb{R}$ . If  $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$  and  $\frac{\partial f}{\partial x}$  are continuous on the rectangle  $[a, b] \times [c, d]$ . Then

$$g(x) = \int_c^d f(x, y) dy.$$

is differentiable and has derivative

$$g'(x) = \frac{d}{dx} \left[ \int_c^d f(x, y) dy \right] = \int_c^d \frac{\partial f}{\partial x}(x, y) dy \quad \text{for } a \leq x \leq b.$$

**With variable limits** Let  $a, b \in \mathbb{R}$  with  $a \leq b$ , let  $\varphi_1, \varphi_2 : [a, b] \rightarrow \mathbb{R}$  be continuously differentiable functions such that  $\varphi_1(x) \leq \varphi_2(x)$  for all  $x \in [a, b]$ . If  $f : D_1 \rightarrow \mathbb{R}$  and  $\frac{\partial f}{\partial x}$  are continuous on the region  $D_1$  with

$$D_1 = \{(x, y) : x \in [a, b] \text{ and } \varphi_1(x) \leq y \leq \varphi_2(x)\}$$

then the function  $g(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy$  is differentiable and

$$g'(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial f}{\partial x}(x, y) dy + f(x, \varphi_2(x))\varphi_2'(x) - f(x, \varphi_1(x))\varphi_1'(x).$$

Note: If  $\varphi_1(x) \equiv c, \varphi_2(x) \equiv d$  where  $c, d$  are constants. Then  $g'(x) = \int_c^d \frac{\partial f}{\partial x} dy$  (reduced to the previous version).



## 5.4 Change of Variable

Let  $\Omega \subseteq \mathbb{R}^n$  and  $F : \Omega \rightarrow \mathbb{R}^n$  be an injective and continuously differentiable function such that  $\det JF(x) \neq 0$  for all  $x \in \Omega$ . If  $f$  is any function that is integrable on  $\Omega' = F(\Omega)$  then

$$\int \int_{\Omega'} (f \circ F) |\det JF|.$$

## 6 Fourier Series

**Fourier Series** A Fourier series is the approximation of simple periodic functions by the sum of period functions of the form  $\sin(x), \cos(x)$ . Note that unlike Taylor series, a function  $f$  may be discontinuous. However, any lack of continuity leads to an infinite sum in the Fourier series.

### 6.1 Inner Products and Norms

**Inner Products** Let  $V$  be a (real) vector space. An inner product on  $V$  is a map that assigns each  $f, g \in V$  a real number  $\langle f, g \rangle$  in such a way that

- $\langle f, f \rangle \geq 0$ ,
- $\langle f, f \rangle = 0$  if and only if  $f$  is zero,
- $\langle \lambda f + \mu g, h \rangle = \lambda \langle f, h \rangle + \mu \langle g, h \rangle$ ,
- $\langle g, f \rangle = \langle f, g \rangle$ .

for all functions  $f, g, h \in V$  and all real constants  $\lambda, \mu$ .

#### Usual Inner Products

- The vector space  $\mathbb{R}^n$  consisting of all  $n$ -dimensional vector admits the following inner product

$$\langle v, w \rangle = v \cdot w = \sum_{i=1}^n v_i w_i.$$

- The vector space  $C[a, b]$  consisting of all continuous function defined on the interval  $[a, b]$  admits the following inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x) dx.$$

**Norms** A norm on  $V$  is a map that assigns each  $f \in V$  a real number  $\|f\|$  in such a way that

- $\|f\| > 0$ ,
- $\|f\| = 0$  if and only if  $f = 0$ ,
- $\|\lambda f\| = |\lambda| \|f\|$ ,
- $\|f + g\| \leq \|f\| + \|g\|$  (triangle inequality)

for all functions  $f, g \in V$  and all real constant  $\lambda$ .

**Usual Norms** Consider a vector space  $C[a, b]$  consisting of all continuous functions on  $[a, b]$ .

- The 2-norm ( $L^2$ -norm) is a norm on  $C[a, b]$ :

$$\|f\|_2 = \sqrt{\int_a^b f(x)^2 dx}$$

- The max norm is a norm on  $C[a, b]$ :

$$\|f\|_\infty = \max_{a \leq x \leq b} \{|f(x)|\}$$

**Theorem** Every inner product on a vector space  $V$  induces a norm given by

$$\|f\| = \sqrt{\langle f, f \rangle},$$

and the Cauchy-Schwartz inequality holds:

$$|\langle f, g \rangle| \leq \|f\| \|g\| \text{ for all } f, g \in V.$$

## 6.2 Fourier Coefficients and Fourier Series

**Fourier Series** Suppose that a given function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a  $2\pi$ -periodic and is square integrable (i.e.,  $\int_{-\pi}^{\pi} f(x)^2 dx < \infty$ ). Its Fourier series is given by

$$S_f(x) = \frac{a_0}{2} + \sum_{k=1}^n [a_k \cos(kx) + b_k \sin(kx)]$$

where

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx, \quad k = 0, 1, 2, \dots$$

and

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx, \quad k = 1, 2, \dots$$

## 6.3 Pointwise Convergence of Fourier Series

**Piecewise Continuous Functions** Consider a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  and a point  $c \in \mathbb{R}$ . Suppose that the one-sided limits  $f(c^+) = \lim_{x \rightarrow c^+} f(x)$  and  $f(c^-) = \lim_{x \rightarrow c^-} f(x)$  exists.

- If  $f(c^+) = f(c^-) = f(c)$ , then  $f$  is continuous at  $c$ .
- If  $f(c^+) = f(c^-) \neq f(c)$  or if  $f(c^+) = f(c^-)$  but  $f(c)$  is undefined, then  $f$  has a removable discontinuity at  $c$ .
- If  $f(c^+) \neq f(c^-)$ , then  $f$  has a jump discontinuity at  $c$ .

A function  $f : [a, b] \rightarrow \mathbb{R}$  is piecewise continuous on  $[a, b]$  if and only if

- (1) For each  $x \in [a, b]$ ,  $f(x^+)$  exists;
- (2) For each  $x \in (a, b]$ ,  $f(x^-)$  exists;
- (3)  $f$  is continuous on  $(a, b)$  except at (most) a finite number of points.

Note that if  $f$  is only piecewise continuous then the partial sum of the Fourier series does not necessarily converge to  $f$  for all  $x$ .

**Piecewise Differentiable Functions** Consider a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  and a point  $c \in \mathbb{R}$ . We write

$$D^+ f(c) = \lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c^+)}{h}$$

if this one-sided limit exists. Likewise,

$$D^- f(c) = \lim_{h \rightarrow 0^-} \frac{f(c+h) - f(c^-)}{h}.$$

A function  $f$  is differentiable at  $c$  if and only if  $f(c^+) = f(c) = f(c^-)$  and  $D^+ f(c) = D^- f(c)$ . A function  $f$  is piecewise differentiable on  $[a, b]$  if and only if

- (1) For each  $x \in [a, b]$ ,  $D^+ f(x)$  exists;
- (2) For each  $x \in (a, b]$ ,  $D^- f(x)$  exists;
- (3)  $f$  is differentiable on  $(a, b)$  except at (most) a finite number of points.

**Pointwise Convergence** Let  $c \in \mathbb{R}$  and suppose that a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  has the following properties:

1.  $f$  is  $2\pi$ -periodic;
2.  $f$  is piecewise continuous on  $[-\pi, \pi]$ ;
3.  $D^+ f(c)$  and  $D^- f(c)$  exists.

If  $f$  is continuous at  $c$  then,

$$S_f(c) = f(c).$$

If  $f$  has a jump/removable discontinuity at  $c$ , then

$$S_f(c) = \frac{1}{2}[f(c^+) + f(c^-)].$$

## 6.4 General Periodic, Half Range + Odd and Even Functions

**General Periodic Functions** Suppose that  $f$  has period  $2L$ , instead of  $2\pi$ :

$$f(x + 2L) = f(x) \text{ for } x \in \mathbb{R}.$$

Note that  $\cos\left(\frac{\pi}{L}x\right)$  and  $\sin\left(\frac{\pi}{L}x\right)$  are periodic functions with period  $2L$ . So, the decomposition becomes

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left( a_k \cos\left(\frac{k\pi}{L}x\right) + b_k \sin\left(\frac{k\pi}{L}x\right) \right)$$

where

$$a_k = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{k\pi}{L}x\right) dx, \quad k = 0, 1, 2, \dots$$

and

$$b_k = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{k\pi}{L}x\right) dx, \quad k = 1, 2, \dots$$

**Half Range Expansion** Let  $f$  be defined on  $[0, L]$ . We can extend  $f$  to an even function (or odd function) on  $[-L, L]$  and calculate its Fourier Series.

**Odd and Even Functions** We define an odd and even functions by the conditions  $f(-x) = -f(x)$  and  $f(-x) = f(x)$  respectively for a function  $f$ . The following elementary properties hold:

- Odd  $\times$  Even = Odd
- Odd  $\times$  Odd = Even
- Even  $\times$  Even = Even
- $\int_{-L}^L$  Odd = 0

**Odd and Even Functions for Fourier Series** If  $f$  is odd, then

$$a_k = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{k\pi}{L}x\right) dx = 0$$

and

$$b_k = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{k\pi}{L}x\right) dx = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{k\pi}{L}x\right) dx.$$

So the Fourier series becomes

$$S_f(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{L}x\right). \quad (\text{Fourier Sine Series})$$

If  $f$  is even, then

$$a_k = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{k\pi x}{L}\right) dx = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{k\pi x}{L}\right) dx.$$

and

$$b_k = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{k\pi x}{L}\right) dx = 0$$

So the Fourier series becomes

$$S_f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{L}\right). \quad (\text{Fourier Cosine Series})$$

## 6.5 Convergence of Sequences

**Pointwise Convergence** Let  $f_k : \mathbb{R} \rightarrow \mathbb{R}$ . We say  $f_k$  converges to  $f$  on  $[a, b]$  pointwisely iff, for every  $x \in [a, b]$ ,  $f_k(x) \rightarrow f(x)$  as  $k \rightarrow \infty$ . In this case,  $f$  is called the pointwise limit. In terms of  $\epsilon - \delta$  language:

For every  $x \in [a, b]$ ,  $\epsilon > 0$ , there exists an  $K$  (depends on  $\epsilon$  and  $x$ ), such that

$$|f_k(x) - f(x)| \leq \epsilon \text{ for all } k \geq K.$$

**Uniform Convergence** Let  $f_k : \mathbb{R} \rightarrow \mathbb{R}$ . We say  $f_k$  converges to  $f$  on  $[a, b]$  uniformly iff for every  $\epsilon > 0$ , there exists an  $K$  (depends on  $\epsilon$  only), such that

$$\sup_{x \in [a, b]} |f_k(x) - f(x)| \leq \epsilon \text{ for all } k \geq K.$$

**Uniform Convergence Theorem** If  $f_k : \mathbb{R} \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  for all  $k$  if:

- $f_k \rightarrow f$  uniformly on  $[a, b]$  then  $f$  is continuous on  $[a, b]$ .
- $f$  has at least one discontinuity on  $[a, b]$ ,  $f_k$  cannot converge uniformly to  $f$  on  $[a, b]$ .

**Weierstrass Test** Let  $f_k : \mathbb{R} \rightarrow \mathbb{R}$  be a sequence of function defined on  $[a, b]$ . Suppose that there exists a sequence of numbers  $c_k$  such that

$$|f_k(x)| \leq c_k \text{ for all } x \in [a, b]$$

and  $\sum_{k=1}^{\infty} c_k$  converges (or exists as a real number). Then  $\sum_{k=1}^{\infty} f_k$  converges uniformly to a function  $f$  on  $[a, b]$ .

Note that this test also holds for functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  for  $x \in \Omega$  where  $\Omega$  is a closed bounded set in  $\mathbb{R}^n$ .

**Norm Convergence** Consider the supremum norm  $\|f\| = \sup_{x \in [a,b]} |f(x)|$ . The definition of uniform convergence can be equivalently written as: for every  $\epsilon > 0$ , there exists an  $K$  such that

$$\|f_k - f\| \leq \epsilon \text{ for all } k \geq K.$$

Equivalently,

$$\lim_{k \rightarrow \infty} \|f_k - f\| = 0.$$

Here, the norm is defined as the supremum norm. Extending this idea, we can define norm convergence for any arbitrary norm.

Let  $V$  be a vector space of functions  $f$  equipped with a norm  $\|f\|$ . We say a sequence of functions  $f_1, \dots, f_k, \dots$ , (norm) converges to  $f$  in  $V$  if  $f \in V$  and

$$\lim_{k \rightarrow \infty} \|f_k - f\| = 0.$$

As such, the  $L^2$  norm convergence, also known as mean square convergence is equivalent to the following

$$\lim_{k \rightarrow \infty} \int_a^b [f_k(x) - f(x)]^2 dx = 0.$$

**Parseval Theorem** Let  $f$  be  $2\pi$  periodic, bounded and  $\int_{-\pi}^{\pi} f(x)^2 dx < +\infty$ . Then, the Fourier series of  $f$  converges to  $f$  in the mean square sense. Moreover, the following Parseval's identity holds:

$$\int_{-\pi}^{\pi} f^2(x) dx = \|f\|_2^2 = \frac{\pi}{2} a_0^2 + \pi \sum_{k=1}^{\infty} (a_k^2 + b_k^2).$$

This identity continues to hold for  $2L$  periodic functions integrated over  $[-L, L]$ .

## 7 Vector Fields

### 7.1 Vector Fields and Flow

**Vector Fields** A vector field in 3D space has components that are functions and is of the type

$$\begin{aligned} \mathbf{F}(\mathbf{x}) &= \mathbf{F}(x, y, z) \\ &= (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)) \\ &= F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}. \end{aligned}$$

A vector field in 2D has components that are functions and is of the type

$$\begin{aligned} \mathbf{F}(\mathbf{x}) &= \mathbf{F}(x, y) \\ &= (F_1(x, y), F_2(x, y)) \\ &= F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}. \end{aligned}$$

**Flow Lines** If  $\mathbf{F}$  is a vector field, a *flow line* for  $\mathbf{F}$  is a path  $\mathbf{c}(t)$  such that

$$\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t)).$$

That is,  $\mathbf{F}$  yields the velocity field of the path  $\mathbf{c}(t)$ .

**The Del  $\nabla$  operator** The vector differential operator  $\nabla$  is not a vector, but an operator. It may be considered a symbolic vector. The differential operator may be written as

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

**Divergence** If  $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ , the divergence of  $\mathbf{F}$  is the scalar field

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Divergence may be thought as a type of derivative that describes the measure at which a vector field *spreads away* from a certain point. If the divergence is positive, then there is a net outflow while there is net inflow if the divergence is negative.

Observe that the divergence of a vector field will be real-valued.

**Curl** If  $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ , the curl of  $\mathbf{F}$  is the vector field

$$\begin{aligned} \operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \mathbf{i} + \left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \mathbf{j} + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathbf{k}. \end{aligned}$$

Curl is also analogous to a type of derivative for vector fields. The curl may be thought as the measure at which the vector field *swirls* around a point. A positive swirl can be thought of as a counterclockwise rotation.

Observe that the curl of a vector field is also a vector field.

## 7.2 Vector Identities

### Basic Vector Identities

1.  $\nabla(f + g) = \nabla f + \nabla g$
2.  $\nabla(\lambda f) = \lambda \nabla f$  where  $\lambda \in \mathbb{R}$
3.  $\nabla(fg) = f\nabla g + g\nabla f$ . You may draw analogies to the product.
4.  $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$  where  $g \neq 0$ . This is analogous to the quotient rule.

5.  $\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$
6.  $\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$
7.  $\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f$
8.  $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$
9.  $\nabla \cdot (\nabla \times \mathbf{F}) = 0$
10.  $\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} = \nabla f \times \mathbf{F}$
11.  $\nabla \times (\nabla f) = 0$
12.  $\nabla^2(fg) = f\nabla^2g + 2(\nabla f \cdot \nabla g) + g\nabla^2f$
13.  $\nabla \cdot (\nabla f \times \nabla g) = 0$
14.  $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2g - g\nabla^2f$

## 8 Path Integrals

### 8.1 Path Integrals

**Path (Scalar Line) Integrals** We say that a vector-valued function  $\mathbf{c}(t)$  parametrises a curve  $C$  for  $a < t < b$  if the image of  $\mathbf{c}$  traces out the curve  $C$ .

**Computing a Scalar Line Integral** Let  $\mathbf{c}(t)$  be a parametrisation of a curve  $C \in \mathbb{R}^3$  for  $a < t < b$ . Assume that  $f(x, y, z)$  and  $\mathbf{c}'(t)$  are continuous. Then

$$\int_C f(x, y, z) ds = \int_a^b f(\mathbf{c}(t)) \|\mathbf{c}'(t)\| dt$$

The value of the integral on the right does not depend on the choice of parametrisation. For  $f(x, y, z) = 1$ , we obtain the length of  $C$ :

$$\text{Length of } C = \int_C \|\mathbf{c}'(t)\| dt$$

where  $\|\mathbf{c}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$  for  $\mathbf{c}(t) = x(t), y(t), z(t)$ .

### Elementary Properties of Path Integral

- $\int_C f_1 ds + \int_C f_2 ds = \int_C (f_1 + f_2) ds$
- $\int_C \lambda f ds = \lambda \int_C f ds, \quad \lambda \in \mathbb{R}$



## 8.2 Applications of Path Integrals

**Mass** Suppose that  $\delta = \delta(x, y, z)$  which is a density function.

$$M = \int_C \delta(x, y, z) dz$$

**First Moments About the Coordinate Planes**

$$M_{yz} = \int_C x \delta ds, \quad M_{xz} = \int_C y \delta ds, \quad M_{xy} = \int_C z \delta ds$$

**Coordinates of the Center of Mass**

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}$$

**Moments of Inertia about Axes**

$$I_x = \int_C (y^2 + z^2) \delta dx, \quad I_y = \int_C (x^2 + z^2) \delta ds, \quad I_z = \int_C (x^2 + y^2) \delta ds$$

## 9 Vector Line Integrals

### 9.1 Vector Line Integrals

**Vector Line Integrals** There is an important distinction between vector and scalar line integrals. To define a vector line integral we must specify a direction along the path or curve  $C$ .

A curve  $C$  can be traversed in one of two directions. We say that  $C$  is oriented if one of these two directions is specified. We refer to the specified direction as the forward direction along the curve.

**Computing a Line Integral** Let  $\mathbf{c}(t)$  be a parameterisation of an oriented curve  $C$  for  $a \leq t \leq b$ . The line integral of a vector field  $\mathbf{F}$  along  $C$  is defined by

$$\int_C \mathbf{F} \cdot ds = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt.$$

**Link with the path integral** Let  $\mathbf{c}(t)$  be a parametrisation of an oriented smooth curve  $C$  and let  $\hat{\mathbf{T}}$  denotes the unit tangent vector pointing in the forward direction of  $C$ .

$$\hat{\mathbf{T}}(\mathbf{c}(t)) = \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|}$$

Then, the line integral of a vector field  $\mathbf{F}$  over the oriented curve  $C$  is the path integral of the tangential component of  $\mathbf{F}$  along  $C$ , that is

$$\int_C \mathbf{F} \cdot ds = \int_C \mathbf{F} \cdot \hat{\mathbf{T}} ds.$$

**Summing Paths** Let  $C_i, i = 1, \dots, m$  be curves with continuous differentiable parameterisations. Let  $C = C_1 + C_2 + \dots + C_m$ , that is,  $C$  is the union of curves  $C_i$ , which are joined end-to-end. Then, we define

$$\int_C \mathbf{F} \cdot ds = \sum_{i=1}^m \int_{C_i} \mathbf{F} \cdot ds.$$

**Work notation** Denote  $\mathbf{c}(t) = (x(t), y(t), z(t))$  and  $\mathbf{F} = (M, N, P) = M\mathbf{i}, N\mathbf{j}, P\mathbf{k}$ . Then, we can denote work as any of the following notations:

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot ds \\ &= \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt && \text{(Definition)} \\ &= \int_a^b \left( M \frac{dx}{dt} + N \frac{dy}{dt} + P \frac{dz}{dt} \right) dt \\ &= \int_C M dx + N dy + P dz. && \text{(Alternative form)} \end{aligned}$$

**Properties of Line Integrals** Let  $C$  be a smooth oriented curve and let  $\mathbf{F}$  and  $\mathbf{G}$  be vector fields.

(i) Linearity:

$$\begin{aligned} \int_C (\mathbf{F} + \mathbf{G}) \cdot ds &= \int_C \mathbf{F} \cdot ds + \int_C \mathbf{G} \cdot ds \\ \int_C k\mathbf{F} \cdot ds &= k \int_C \mathbf{F} \cdot ds \quad (k \text{ a constant}) \end{aligned}$$

(ii) Reversing orientation:

$$\int_{-C} \mathbf{F} \cdot ds = - \int_C \mathbf{F} \cdot ds$$

(iii) Additivity: If  $C$  is a union of  $n$  smooth curves  $C_1 + \dots + C_n$ , then

$$\int_C \mathbf{F} \cdot ds = \int_{C_1} + \dots + \int_{C_n} \mathbf{F} \cdot ds$$

## 9.2 Other Applications

**Flow Integral, Circulation** If  $\mathbf{r}(t)$  is a smooth curve in the domain of a continuous velocity field  $\mathbf{F}$ , the flow along the curve from  $t = a$  to  $t = b$  is

$$\text{Flow} = \int_a^b \mathbf{F} \cdot \hat{\mathbf{T}} ds$$

The integral in this case is called a flow integral. If the curve is a closed loop, the flow is called the circulation around the curve.

**Flux Across a Closed Curve in the Plane** If  $C$  is a smooth closed curve in the domain of a continuous vector field  $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$  in the plane and if  $\hat{\mathbf{n}}$  is the outward-pointing unit normal vector on  $C$ , the flux of  $\mathbf{F}$  across  $C$  is

$$\text{Flux of } \mathbf{F} \text{ across } C = \int_C \mathbf{F} \cdot \hat{\mathbf{n}} \, ds.$$

**Calculating Flux Across a Smooth Closed Plane Curve**

$$(\text{Flux of } \mathbf{F} = M\mathbf{i} + N\mathbf{j} \text{ across } C) = \oint_C M \, dy - N \, dx$$

The integral can be evaluated from any smooth parametrisation  $x = g(t), y = h(t), a \leq t \leq b$ , that traces  $C$  counterclockwise exactly once.

### 9.3 Fundamental Theorem of Line Integrals

**(Second) Fundamental Theorem of Calculus in One Variable** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ . If  $f(x) = \varphi'(x)$ , then

$$\int_a^b \varphi'(x) \, dx = \int_a^b f(x) \, dx = \varphi(b) - \varphi(a).$$

**Gradient Fields** A vector field  $\mathbf{F}$  is called a gradient vector field if there exists a real-valued function  $\varphi$  such that  $\mathbf{F} = \nabla\varphi$ . That is,  $(M, N, P) = (\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z})$ . A vector field  $\mathbf{F}$  with this property is called conservative and  $\varphi$  is called the potential function of  $\mathbf{F}$ .

**Fundamental Theorem for Gradient Vector Fields** If  $\mathbf{F} = \nabla\varphi$  on a domain  $\mathcal{D}$ , then for every oriented smooth curve  $C$  in  $\mathcal{D}$  with initial point  $P$  and terminal point  $Q$ .

$$\int_C \mathbf{F} \cdot ds = \varphi(Q) - \varphi(P)$$

If  $C$  is closed (i.e., if  $P = Q$ ), then  $\oint_C \mathbf{F} \cdot ds = 0$ .

**Cross Partial of a Gradient Vector Field are Equal** Let  $\mathbf{F} = (F_1, F_2, F_3)$  be a gradient vector field whose components have continuous partial derivatives. Then the cross partials are equal:

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$$

Similarly, if the vector field in the plane  $\mathbf{F} = (F_1, F_2)$  is the gradient vector field, then  $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ . Equivalently,  $\nabla \times \mathbf{F} = \mathbf{0}$ .

### 9.4 Green's Theorem

Green's Theorem connects double integrals with line integrals and is very useful for line integrals over complicated vector fields with simpler partial derivatives.

**Green's Theorem (Flux-divergence or Normal Form)** Let  $D$  be a bounded simple region in  $\mathbb{R}^2$  with nonempty interior, whose boundary consists of a finite number of smooth curves. Let  $C$  be the boundary of  $D$  with a positive (counter-clockwise) direction. Let  $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$  be a vector field which is continuously differentiable on  $D$ . Then, the outward flux of  $\mathbf{F}$  across the curve  $C$  equals the double integral of divergence  $\nabla \cdot \mathbf{F}$  over  $D$ , that is

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{n}} \, ds = \oint_C -N \, dx + M \, dy = \int \int_D \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \, dx \, dy$$

Three key assumptions:

- The region  $D$  is bounded and simple region with nonempty interior.
- The boundary  $C$  is oriented in the positive (counter-clockwise) direction, and is a finite union of smooth curves.
- The vector field  $\mathbf{F}$  is continuously differentiable on  $D$ .

**Green's Theorem (Circulation-curl or Tangential Form)** Let  $D$  be a bounded simple region in  $\mathbb{R}^2$  with nonempty interior, whose boundary consists of a finite number of smooth curves. Let  $C$  be the boundary of  $D$  with a positive (counter-clockwise) direction. Let  $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$  be a vector field which is continuously differentiable on  $D$ . Then, the counter-clockwise circulation of  $\mathbf{F}$  around  $C$  equals the double integral  $\nabla \times \mathbf{F} \cdot \mathbf{k}$  over  $D$ , that is

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{T}} \, ds = \oint_C M \, dx + N \, dy = \int \int_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy$$

**Area of a Region** Let  $D$  be a simple and bounded region with non-empty interior and let  $C$  be its boundary with positive (counter-clockwise) direction which is a finite union of smooth curves. Then, the area of  $D$  can be calculated by

$$\text{Area}(D) = \frac{1}{2} \oint_C (-y \, dx + x \, dy).$$

## 10 Surface Integrals

### 10.1 Parametrised Surfaces

**Parametrised Surface** A parametrised surface is a function  $\phi : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3$ , where  $D$  is some domain in  $\mathbb{R}^2$ , that is,

$$\phi(u, v) = (x(u, v), y(u, v), z(u, v)).$$

The surface  $S$  corresponding to the function  $\phi$  is its image:  $S = \phi(D)$ . If  $\phi$  is differentiable (resp. continuously differentiable), then we call  $S$  a differentiable (resp. continuously differentiable) surface.

**Cone** The cone  $z^2 = x^2 + y^2$  has the parametrisation

$$\phi(u, v) = (u \cos v, u \sin v, u), \quad 0 \leq v \leq 2\pi, u \in \mathbb{R}.$$

**Cylinder** The cylinder of radius  $R$ ,  $x^2 + y^2 = R^2$  has the parametrisation

$$\phi(\theta, z) = (R \cos \theta, R \sin \theta, z), \quad 0 \leq \theta \leq 2\pi, z \in \mathbb{R}.$$

**Sphere** The sphere of radius  $R$ ,  $x^2 + y^2 + z^2 = R^2$  has the parametrisation

$$\Phi(\theta, \phi) = (R \cos \theta \sin \phi, R \sin \theta \sin \phi, R \cos \phi), \quad 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi.$$

## 10.2 Surface Area

In the rest of this section we consider smooth parametrised surfaces and also piecewise smooth parametrised surfaces.

**Area of a Surface** Let  $\Phi(u, v)$  be parametrisation of a smooth surface  $S$  with parameter domain  $D$ . The area of the surface  $S$  is

$$\text{Area}(S) = \int \int_D \|\mathbf{T}_u \times \mathbf{T}_v\| \, du dv.$$

Sometimes we write

$$\|\mathbf{n}(u, v)\| = \|\mathbf{T}_u \times \mathbf{T}_v\|.$$

Note that this  $\mathbf{n}(u, v)$  is not necessarily a unit vector and neither are the tangent vectors.