Higher Linear Algebra
MATH2621 UNSW

Jeremy Le *

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Contents

*With some inspiration from Hussain Nawaz's Notes

1 Assumed Knowledge

- the definition of complex numbers,
- their arithmetic,
- Cartesian and polar representations,
- the Argand diagram,
- de Moivre's theorem, and
- extracting nth roots of complex numbers.

2 Inequalities and Sets of Complex Numbers

2.1 Equalities and Inequalities

Modulus Squared of a Sum For all complex numbers w and z,

$$|w+z|^2 = |w|^2 + 2\,\mathrm{Re}(w\bar{z}) + |z|^2.$$

Triangle Inequality For all complex numbers w and z,

$$|w+z| \leq |w| + |z| \qquad \forall w, z, \in \mathbb{C}.$$

Circle Inequality For all complex numbers w and z,

$$||w| - |z|| \leq |w - z|.$$

Modulus of e^z If $z \in \mathbb{C}$, then

$$|e^z| = e^{\mathrm{Re}(z)}.$$

Modolus of $e^z - 1$ inequality For all real numbers θ,

$$|e^{i\theta} - 1| \leq |\theta|.$$

2.2 Properties of Sets

Open Ball The open ball with centre z_0 and radius ϵ, written $\mathrm{B}(z_0, \epsilon)$, is the set $\{z \in \mathbb{C} : |z - z_0| < \epsilon\}$.

Punctured Open Ball The punctured open ball with centre z_0 and radius ϵ, written $B^{\circ}(z_0, \epsilon)$, is the set $\{z \in \mathbb{C} : 0 < |z - z_0| < \epsilon\}$.

Interior, Exterior and Boundary Points Suppose that $S \subseteq \mathbb{C}$. For any point z_0 in $\mathbb{C}$, there are three mutually exclusive and exhaustive possibilities:

(1) When the positive real number ϵ is sufficiently small, $\mathrm{B}(z_0, \epsilon)$ is a subset of S, that is, $\mathrm{B}(z_0, \epsilon) \cap S = \mathrm{B}(z_0, \epsilon)$. In this case, z_0 is an interior point of S.

(2) When the positive real number *epsilon* is sufficiently small, $\mathrm{B}(z_0, \epsilon)$ does not meet S, that is, $\mathrm{B}(z_0, \epsilon)$ $capS = \emptyset$. In this case, z_0 is an exterior point of S.

(3) No matter how small the positive real number ϵ is, neither of the above holds, that is, $\emptyset \subset \mathrm{B}(z_0, \epsilon) \cap S \subset \mathrm{B}(z_0, \epsilon)$. In this case, z_0 is a boundary point of S.

Open, Closed, Closure, Bounded, Compact, Region Sets Suppose that $S \subseteq \mathbb{C}$.

(1) The set S is open if all its points are interior points.

(2) The set S is closed if it contains all of its boundary points, or equivalently, if its complement $\mathbb{C} \setminus S$ is open.

(3) The closure of the set S is the set consisting of the points of S together with the boundary points of S.

(4) The set is bounded if $S \subseteq \mathrm{B}(0, R)$ for some $R \in \mathbb{R}^+$

(5) The set S is compact if it is both closed and bounded.

(6) The set S is a region if it is an open set together with none, some, or all of its boundary points.

2.3 Arcs

Polygonal Arc A polygonal arc is a finite sequence of finite directed line segments, where the end point of one line segment is the initial point of the next one.

Simple Closed Polygonal Arc A simple closed polygonal arc is a polygonal arc that does not cross itself, but the final point of the last segment is the initial point of the first segment.

Interior and Exterior Arc The complement of a simple closed polygonal arc is made up of two pieces: one, the interior of the arc, is bounded, and the other, exterior is not.

Polygonally Path-connectedness Let $X \subseteq \mathbb{C}$ be a subset of the complex plane.

(1) The set X is polygonally path-connected if any two points of X can be joined by a polygonal arc lying inside X.

(2) The set X is simply polygonally connected if it is polygonally path-connected and if the interior of every simple closed polygonal arc in X lies in X, that is, if "X has no holes".

(3) The set X is a domain if it is open and polygonally path-connected.

3 Functions of a Complex Variable

Complex Function A complex function is one whose domain, or whose range, or both, is a subset of the complex plane $\mathbb{C}$ that is not a subset of the real line $\mathbb{R}$.

Complex Polynomial A complex polynomial is a function $p : \mathbb{C} \to \mathbb{C}$ of the form

$$p(z) = a_d z^d + \cdots + a_1 z + a_0,$$

where $a_d, \ldots, a_1, a_0 \in \mathbb{C}$. If $a_d \neq 0$, we say that p is of degree d. A rational function is a quotient of polynomials.

The Fundamental Theorem of Algebra Every nonconstant complex polynomial p of degree d factorizes: there exists $\alpha_1, \alpha_2, \ldots, \alpha_d$ and c in $\mathbb{C}$ such that

$$p(z) = c \prod_{j=1}^{d} (z - a_j).$$

Polynomial Division and Partial Fractions Suppose that p and q are polynomials. Then

$$\frac{p(z)}{q(z)} = s(z) + \frac{r(z)}{q(z)},$$

where r and s are polynomials, and the degree of r is strictly less than the degree of q. Further, if

$$q(z) = c \prod_{j=1}^{e} (z - \beta_j)^{m_j},$$

then we may decompose the term r/q into partial fractions:

$$\frac{r(z)}{q(z)} = \sum_{j=1}^{e} \sum_{k=1}^{m_j} \frac{a_{jk}}{(z - \beta_j)^k}.$$

Real and Imaginary Parts To a function $f : S \to \mathbb{C}$, where $S \subseteq \mathbb{C}$, we associate two real-valued functions u and v of two real variables:

$$f(x + iy) = u(x, y) + iv(x, y).$$

Then $u(x, y) = \operatorname{Re} f(x + iy)$ and $v(x, y) = \operatorname{Im} f(x + iy)$.

3.1 The function $w = 1/z$

Consider the mapping $w = 1/z$.

(1) The image of a line through 0 (with the origin removed) is a line through 0 (with the origin removed).

(2) The image of a line that does not pass through 0 is a circle (with the origin removed). If p is the closest point on the line to 0, then the line segment between 0 and 1/p is a diameter of the circle.

(3) The image of a circle that passes through 0 is a line. If q is the furthest point on the circle from 0, then the closest point on the line to 0 is $1/q$.

(4) The image of a circle that does not pass through 0 is a circle. If p and q are the closest and furthest point on the circle from 0, then the closest and furthest point on the image circle to 0 are $1/q$ and $1/p$.

3.2 Fractional Linear Transformations

Factorising Matrices Every 2×2 complex matrix with determinant 1 may be written as a product f at most three matrices of the following special types:

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Image of Lines and Circles Let T_M be a fractional linear transformation. Then the image of a line under T_M is a line or a circle, and the image of a circle under T_M is also a line or a circle.

4 Limits and Continuity

4.1 Limits

Definition of a Limit Suppose that f is a complex function and that z_0 is in Domain$(f)^-$. We say that $f(z)$ tends to ℓ as z tends to z_0, or that ℓ is the limit of $f(z)$ as z tends to z_0, and we write $f(z) \to \ell$ as $z \to z_0$, or

$$\lim_{z \to z_0} f(z) = \ell,$$

if, for every $\epsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that $|f(z) - \ell| < \epsilon$ provided that z is in Domain(f) and $0 < |z - z_0| < \delta$.

Limit within a Subset Suppose also S is a subset of Domain(f) and that $z_0 \in \bar{S}$. We say that $f(z)$ tends to ℓ as z tends to z_0 in S, or that ℓ is the limit of $f(z)$ as z tends to z_0 in S, and write $f(z) \to \ell$ as $z \to z_0$ in S, or

$$\lim_{\substack{z \to z_0 \\ z \in S}} f(z) = \ell,$$

if, for every $\epsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that $|f(z) - \ell| < \epsilon$ provided that $z \in S$ and $0 < |z - z_0| < \delta$.

Limits at Infinity Suppose that f is a complex function, that $\ell \in \mathbb{C} \subset \{\infty\}$, and that either $z_0 \in \mathrm{Domain}(f)^-$ or $Domain(f)$ is not bounded and $z_0 = \infty$. We say that $f(z)$ tends to ℓ as z tends to z_0, or that ℓ is the limit of $f(z)$ as z tends to z_0, and we write $f(z) \to \ell$ as $z \to z_0$, or

$$\lim_{z \to z_0} f(z) = \ell,$$

if for all $\epsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that $f(z) \in \mathrm{B}(\ell, \epsilon)$ provided that $z \in \mathrm{B}^\circ(z_0, \delta)$.

Standard Limits Suppose that $\alpha, c \in \mathbb{C}$. Then

$$\begin{aligned} \lim_{z \to \alpha} c &= c & \lim_{z \to \infty} c &= c \\ \lim_{z \to \alpha} z - c &= \alpha - c & \lim_{z \to \infty} z - \alpha &= \infty \\ \lim_{z \to \alpha} \frac{1}{z - \alpha} &= \infty & \lim_{z \to \alpha} \frac{1}{z - \alpha} &= 0 \end{aligned}$$

Lemmas on Limits

1. Suppose that f is a complex function, that $T \subseteq S \subseteq \mathrm{Domain}(f)$, and that $z_0 \in \bar{T}$. If $\lim_{\substack{z \to z_0 \\ z \in S}} f(z)$ exists, then so does $\lim_{\substack{z \to z_0 \\ z \in T}} f(z)$, and they are equal.
2. Suppose that f is a complex function, and that $z_0 \in \mathrm{Domain}(f)^-$. If $\lim_{z \to z_0} f(z)$ exists, then it is unique.

Algebra of Limits Suppose that f and g are complex functions and that $c \in \mathbb{C}$. Then

$$\begin{aligned} \lim_{z \to z_0} cf(z) &= c \lim_{z \to z_0} f(z) \\ \lim_{z \to z_0} f(z) + g(z) &= \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z) \\ \lim_{z \to z_0} f(z)g(z) &= \lim_{z \to z_0} f(z) \lim_{z \to z_0} g(z) \\ \lim_{z \to z_0} \frac{f(z)}{g(z)} &= \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)}, \end{aligned}$$

in the sense that if the right hand side exists, then so does the left hand size and they are equal. In particular, for the quotient, we require that $\lim_{z \to z_0} g(z) \neq 0$.

Limits and Complex Conjugation Suppose that f is a complex function and that either $\mathrm{Domain}(f)$ is unbounded and $z_0 = \infty$ or $z_0 \in \mathrm{Domain}(f)^-$. Then

$$\begin{aligned} \lim_{z \to z_0} \overline{f(z)} &= \overline{\lim_{z \to z_0} f(z)} \\ \lim_{z \to z_0} \mathrm{Re}(f(z)) &= \mathrm{Re} \lim_{z \to z_0} f(z) \\ \lim_{z \to z_0} \mathrm{Im}(f(z)) &= \mathrm{Im} \lim_{z \to z_0} f(z) \\ \lim_{z \to z_0} f(z) &= \lim_{z \to z_0} \mathrm{Re}(f(z)) + i \lim_{z \to z_0} \mathrm{Im}(f(z)), \end{aligned}$$

in the sense that if the right hand side exists, then so does the left hand size, and they are equal. In particular, $f(z)$ tends to ℓ as z tends to z_0 if and only if $\mathrm{Re}(f(z))$ tends to $\mathrm{Re}(\ell)$ and $\mathrm{Im}(f(z))$ tends to $\mathrm{Im}(\ell)$ as z tends to z_0.

4.2 Continuity

Definition Suppose that the complex function f is defined in a set $S \subseteq \mathbb{C}$, and that $z_0 \in S$. We say that f is continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0);$$

that is, the limit exists, $f(z_0)$ exists, and they are equal.

We say that f is continuous in S if it is continuous at all points of S, and continuous if it is continuous in its domain.

Properties of Continuous Functions

- Suppose that $c \in \mathbb{C}$, and that $f : S \to \mathbb{C}$ and $g : S \to \mathbb{C}$ are continuous complex functions in $S \subseteq \mathbb{C}$. Then $cf, f+g, |f|, \bar{f}, \mathrm{Re}\, f, \mathrm{Im}\, f$ and fg are continuous in S, as is f/g provided that $g(z) \neq 0$ for any z in S.

- Suppose that $f : S \to \mathbb{C}$ and $g : T \to \mathbb{C}$ are continuous complex functions in $S \subseteq \mathbb{C}$ and $T \subseteq \mathbb{C}$. Then $f \circ g$ is continuous where it is defined, that is, in $\{z \in T, g(z) \in S\}$.

Continuity and Boundedness Suppose that the set $S \subseteq \mathbb{C}$ is compact (i.e., closed and bounded) and that f is a continuous complex function defined on S. Then there exists a point z_0 in S such that

$$|f(z_0)| = \max\{|f(z)| : z \in S\}.$$

The Log Function The function $\mathrm{Log} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is defined by

$$\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z).$$

5 Differentiability

Definition Suppose that $S \subseteq \mathbb{C}$ and that $f : S \to \mathbb{C}$ is a complex function. Then we say that f is differentiable at the point z_0 in S if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \qquad \text{or equivalently} \qquad \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h},$$

exists. If it exists, it is called the derivative of f at z_0, and written $f'(z_0)$ or $\dfrac{df(z_0)}{dz}$.