

# Higher Theory of Statistics

## Math2901 UNSW

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2023T2

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\*With some inspiration from Hussain Nawaz's Notes

# 1 Probability

## 1.1 Experiment, Sample Space, Event

**Experiment** An experiment is any process leading to recorded observations.

**Outcome** An outcome is a possible result of an experiment.

**Sample Space** The set  $\Omega$  of all possible outcomes is the sample space of an experiment.  $\Omega$  is discrete if it contains a countable (finite or countably infinite) number of outcomes.

**Events** An event is a set of outcomes, i.e. a subset of  $\Omega$ . An event occurs if the result of the experiment is one of the outcomes in that event.

**Mutual Exclusion** Events are mutually exclusive (disjoint) if they have no outcomes in common.

**Set Operations** If you have trouble remembering the above rules, then one can essentially replace  $\cup$  by multiplication and  $\cap$  by addition.  
(The associative law) If  $A, B, C$  are sets then

$$\begin{aligned}(A \cup B) \cup C &= A \cup (B \cup C) \\ (A \cap B) \cap C &= A \cap (B \cap C)\end{aligned}$$

(Distributive Law) If  $A, B, C$  are sets then

$$\begin{aligned}A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C)\end{aligned}$$

## 1.2 Sigma-algebra

The  $\sigma$ -algebra must be defined for rigorously working with probability. The  $\sigma$ -algebra can be thought of as the family of all possible events in a sample space. Analogously, this may be conceptualised as the power set of the sample space.

**Probability** A probability is a set function, which is usually denoted by  $\mathbb{P}$ , that maps events from the  $\sigma$ -algebra to  $[0, 1]$  and satisfies certain properties.

**Probability Space** The triplet  $(\Omega, \mathcal{A}, \mathbb{P})$  is the probability/sample space where

- $\Omega$  is the sample space,
- $\mathcal{A}$  is the  $\sigma$ -algebra,
- $\mathbb{P}$  is the probability function.

**Properties of Probability** Given the probability/sample space  $(\Omega, \mathcal{A}, \mathbb{P})$ , the probability function  $\mathbb{P}$  must satisfy

- For every set  $A \in \mathcal{A}$ ,  $\mathbb{P}(A) \geq 0$
- $\mathbb{P}(\Omega) = 1$
- (Countably additive) Suppose the family of sets  $(A_i)_{i \in \mathbb{N}}$  are mutually exclusive, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

### Probability Lemmas

- Given a family of disjoint sets  $(A_i)_{i=1, \dots, k}$

$$\mathbb{P}\left(\bigcup_{i=1}^k A_i\right) = \sum_{i=1}^k \mathbb{P}(A_i)$$

- $\mathbb{P}(\emptyset) = 0$
- For any  $A \in \mathcal{A}$ ,  $\mathbb{P}(A) \leq 1$  and  $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
- Suppose  $B, A \in \mathcal{A}$  and  $A \subseteq B$ , then  $\mathbb{P}(A) \leq \mathbb{P}(B)$ .

**Continuity from Below** Given an increasing sequence of events  $A_1 \subset A_2 \subset \dots$  then,

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} \mathbb{P}(A_n)$$

**Continuity from Above** Given a decreasing sequence of events  $A_1 \supset A_2 \supset \dots$  then,

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} \mathbb{P}(A_n)$$

## 1.3 Conditional Probability and Independence

**Conditional Probability** The conditional probability that an event  $A$  occurs given that an event  $B$  has occurred is

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}, \quad \mathbb{P}(B) > 0$$

**Independence** Events  $A$  and  $B$  are independent if  $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ .

Lemma - Given two events  $A$  and  $B$  then  $\mathbb{P}(A|B) = \mathbb{P}(A)$  if and only if  $\mathbb{P}(B|A) = \mathbb{P}(B)$ .

## Independence of Sequences

- A countable sequence of event  $(A_i)_{i \in \mathbb{N}}$  is pairwise independent if  $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j)$  for all  $i \neq j$ .
- A countable sequence of events  $(A_i)_{i \in \mathbb{N}}$  are independent if for any sub-collection  $A_{i_1}, \dots, A_{i_n}$  we have

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cdots \cap A_{i_n}) = \prod_{j=1}^n \mathbb{P}(A_{i_j})$$

Independence implies pairwise independence, but pairwise independence does not imply independence.

**Multiplicative Law** Given events  $A$  and  $B$  then

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B),$$

and similarly, if you have events  $A, B, C$  then

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_3|A_2 \cap A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_1)$$

**Additive Law** Let  $A$  and  $B$  be events then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

**Law of Total Probability** Suppose  $(A_i)_{i=1, \dots, k}$  are mutually exclusive and exhaustive of  $\Omega$ , that is  $\bigcup_{i=1}^k A_i = \Omega$ , then for any event  $B$ , we have

$$\mathbb{P}(B) = \sum_{i=1}^k \mathbb{P}(B|A_i)\mathbb{P}(A_i)$$

**Bayes Formula** Given sets  $B, A$  and a family of disjoint and exhaustive sets  $(A_i)_{i=1, \dots, k}$  then

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\sum_{i=1}^k \mathbb{P}(B|A_i)\mathbb{P}(A_i)}$$

## 1.4 Descriptive Statistics

**Categorical** Data can be sorted into a finite set of (unordered) categories. e.g. Gender

**Quantitative** Responses are measured on some sort of scale. e.g. Weight.

**Numerical Summaries of the Quantitative Data** Given observations  $x = (x_1, \dots, x_n)$ . The sample mean (estimated mean) or average is given by

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

Sample variance (estimated variance)

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$$

## 2 Random Variables

### 2.1 Random Variables

**Random Variables** A random variable (r.v)  $X$  is a function from  $\Omega$  to  $\mathbb{R}$  such that  $\forall \mathbf{x} \in \mathbb{R}$ , the set  $A_{\mathbf{x}} = \{\omega \in \Omega, X(\omega) \leq \mathbf{x}\}$  belongs to the  $\sigma$ -algebra  $\mathcal{A}$ .

**Cumulative Distribution Function** The cumulative distribution function of a r.v  $X$  is defined by

$$F_X(\mathbf{x}) := \mathbb{P}(\{\omega : X(\omega) \leq \mathbf{x}\}) = \mathbb{P}(X \leq \mathbf{x})$$

**Cumulative Distribution Theorems** Suppose  $F_X$  is a cumulative distribution function of  $X$ , then

- it is bounded between zero and one, and

$$\lim_{x \downarrow -\infty} F_X(x) = 0 \quad \text{and} \quad \lim_{x \uparrow \infty} F_X(x) = 1$$

- it is non-decreasing, that is if  $x \leq y$  then  $F_X(x) \leq F_X(y)$
- for any  $x < y$ ,

$$\mathbb{P}(x < X \leq y) = \mathbb{P}(X \leq y) - \mathbb{P}(X \leq x) = F_X(y) - F_X(x)$$

- it is right continuous, that is

$$\lim_{n \uparrow \infty} F_X(x + \frac{1}{n}) = F_X(x)$$

- it has finite left limit and

$$\mathbb{P}(X < x) = \lim_{n \rightarrow \infty} F_X(x - \frac{1}{n})$$

which we denote by  $F_X(x-)$ .

**Discrete Random Variables** A r.v  $X$  is said to be discrete if the image of  $X$  consists of countable many values  $x$ , for which  $\mathbb{P}(X = x) > 0$ .

**Discrete Probability Function** The probability function of a discrete r.v  $X$  is the function  $\nabla F_X(x) = \mathbb{P}(X = x)$  and satisfies

$$\sum_{\text{all possible } x} \mathbb{P}(X = x) = 1$$

**Continuous Random Variables** A r.v  $X$  is said to be continuous if the image of  $X$  takes a continuum of values.

**Continuous Probability Density Function** The probability density function of a continuous r.v is a real-valued function  $f_X$  on  $\mathbb{R}$  with the property that

$$\mathbb{P}(X \in A) = \int_A f_X(y) dy$$

for any 'Borel' subset of  $\mathbb{R}$ .

For a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  to be a valid density function, the function  $f$  must satisfy the following properties.

1. for all  $x \in \mathbb{R}$ ,  $f(x) \geq 0$
2.  $\int_{-\infty}^{\infty} f(x) dx = 1$

**Useful Properties (for continuous random variable)** For any continuous random variable  $X$  with the density  $f_X$ ,

1. by taking  $A = (-\infty, x]$ ,  $\mathbb{P}(X \in (-\infty, x]) = \mathbb{P}(X \leq x)$  and

$$F_X(x) = \int_{-\infty}^x f_X(y) dy$$

2. For any  $a < b \in \mathbb{R}$ , one can compute  $\mathbb{P}(a < X \leq b)$  by

$$F_X(b) - F_X(a) = \int_a^b f_X(x) dx$$

3. From the fundamental theorem of calculus and 1, we have

$$F'_X(x) = \frac{d}{dx} \int_{-\infty}^x f_X(y) dy = f_X(x).$$

## 2.2 Expectation and Variance

**Expectation** The expectation of a r.v  $X$  is denoted by  $\mathbb{E}(X)$  and it is computed by

1. Let  $X$  be a discrete r.v. then

$$\mathbb{E}(X) := \sum_{\text{all possible } x} x \mathbb{P}(X = x) = \sum_{\text{all possible } x} x \nabla F_X(x)$$

2. Let  $X$  be a continuous r.v. with density function  $f_X(x)$  then

$$\mathbb{E}(x) := \int_{-\infty}^{\infty} x f_X(x) dx$$

**Expectation of Transformed Random Variables** Suppose  $g : \mathbb{R} \rightarrow \mathbb{R}$ , then the expectation of the transformed r.v  $g(X)$  is

$$\mathbb{E}(g(X)) = \begin{cases} \int_{\mathbb{R}} g(x) f_X(x) dx & \text{continuous} \\ \sum_x g(x) \mathbb{P}(X = x) & \text{discrete} \end{cases}$$

usually one is interested in computing  $\mathbb{E}(X^r)$  for  $r \in \mathbb{N}$ , which is called the  $r$ -th moment of  $X$ .

**Linearity of Expectation** The expectation  $\mathbb{E}$  is linear, i.e., for any constants  $a, b \in \mathbb{R}$ ,

$$\mathbb{E}(aX + b) = a\mathbb{E}(X) + b.$$

**Variance** Let  $X$  be a r.v and we set  $\mu = \mathbb{E}(X)$ . The variance of  $X$  is denoted by  $\text{Var}(X)$  and

$$\text{Var}(X) := \mathbb{E}((X - \mu)^2)$$

and the standard deviation of  $X$  is the square root of the variance.

**Properties of Variance** Given a random variable  $X$  then for any constant  $a, b \in \mathbb{R}$ ,

1.  $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$
2.  $\text{Var}(ax) = a^2 \text{Var}(X)$
3.  $\text{Var}(X + b) = \text{Var}(X)$
4.  $\text{Var}(b) = 0$

## 2.3 Moment Generating Functions

**Moments** A moment of the random variable is denoted by

$$\mathbb{E}[X^r], \quad r = 1, 2, \dots$$

Moments measure mean, variance, skewness, and kurtosis, all ways of looking at the shape of the distribution.

**Moment Generating Function** The moment generating function (MGF) of a r.v  $X$  is denoted by

$$M_X(u) := \mathbb{E}(e^{uX})$$

and we say that the MGF of  $X$  exists if  $M_X(u)$  is finite in some interval containing zero.

The moment generating function of  $X$  exists if there exists  $h > 0$  such that the  $M_X(x)$  is finite for  $x \in [-h, h]$ .

**Calculating Raw Moments** Suppose the moment generating function of a r.v  $X$  exists then

$$\mathbb{E}(X^r) = \lim_{u \rightarrow 0} M_X^{(r)}(u) = \lim_{u \rightarrow 0} \frac{d^r}{du} M_X(u)$$

**Equivalence of Moment Generating Functions** Let  $X$  and  $Y$  be two r.vs such that the moment generating function of  $X$  and  $Y$  exists and  $M_Y(u) = M_X(u)$  for all  $u$  in some interval containing zero then  $F_X(x) = F_Y(x)$  for all  $x \in \mathbb{R}$ .

This theorem tells you that if the moment generating function exists then it uniquely characterises the cumulative distribution function of the random variable.

### 2.3.1 Useful Inequalities

**The Markov Inequality (Chebychev's First Inequality)** For any non-negative r.v  $X$  and  $a > 0$ ,

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}$$

**Chebychev's Second Inequality** Suppose  $X$  is any r.v with  $\mathbb{E}(X) = \mu$ ,  $\text{Var}(X) = \sigma^2$  and  $k > 0$  then

$$\mathbb{P}(|X - \mu| > k\sigma) \leq \frac{1}{k^2}$$

**Convex (Concave) Functions** A function  $h$  is convex (concave) if for any  $\lambda \in [0, 1]$  and  $x_1$  and  $x_2$  in the domain of  $h$ , we have

$$h(\lambda x_1 + (1 - \lambda)x_2) \leq (\geq) \lambda h(x_1) + (1 - \lambda)h(x_2)$$

**Jensen's Inequality** Suppose  $h$  is a convex (concave) function and  $X$  is a r.v then

$$h(\mathbb{E}(X)) \leq (\geq) \mathbb{E}(h(X))$$

By using Jensen's inequality, one can show

$$\text{Arithmetic Mean} \geq \text{Geometric Mean} \geq \text{Harmonic Mean}.$$

That is given a sequence of number  $(a_i)_{i=1, \dots, n}$ , we have

$$\frac{1}{n} \sum_{i=1}^n a_i \geq \left( \prod_{i=1}^n a_i \right)^{\frac{1}{n}} \geq n \left( \sum_{i=1}^n a_i^{-1} \right)^{-1}$$



## 3 Common Distributions

### 3.1 Common Discrete Distributions

**Bernoulli Trail** A Bernoulli trial is an experiment with two possible outcomes. The outcomes are often labelled 'success' and 'failure'. A Bernoulli trial defines a random variable  $X$ , given by

$$X = \begin{cases} 1 & \text{if the trial is a success} \\ 0 & \text{if the trial is a failure} \end{cases}$$

- Let  $p \in [0, 1]$  be the probability of success
- We write  $X \sim \text{Bernoulli}(p)$
- The probability function is given by  $\mathbb{P}(X = 1) = p$  and  $\mathbb{P}(X = 0) = 1 - p$
- $\mathbb{E}(X) = p$
- $\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = p(1 - p)$

**Binomial Distribution** Consider a sequence of  $n$  independent Bernoulli trials each with probability of success  $p$ . Let

$$X := \text{total number of successes}$$

then  $X$  is a Binomial r.v with parameter  $n$  and  $p$ , and we write  $X \sim \text{Bin}(n, p)$ .

If  $(Y_i)_{i=1, \dots, n}$  is a sequence of independent  $\text{Bernoulli}(p)$  random variable then  $X := \sum_{i=1}^n Y_i$  is  $\text{Bin}(n, p)$ . The expectation of a Binomial random variable.

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{i=1}^n Y_i\right) = \sum_{i=1}^n \mathbb{E}(Y_i) = np$$

**Poisson Distribution** A r.v  $X$  is said to follow the Poisson distribution with parameter  $\lambda$ , if it's probability function is given

$$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad k = 0, 1, \dots$$

where  $\lambda = \mathbb{E}(X) = \text{Var}(X)$ .

**Hypergeometric Distribution** A random variable has hypergeometric distribution with parameter  $N, m, n$  and we write  $X \sim \text{Hyp}(n, m, N)$  if

$$\mathbb{P}(X = x) = \frac{C_x^m C_{n-x}^{n-m}}{C_n^N} \quad x = 1, \dots, n$$

## 3.2 Continuous Distribution

**Normal Random Variable** A random variable  $X$  is said to be a normal random variable with parameters  $\mu$  and  $\sigma^2$  if its probability density function is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}$$

and we write  $X \sim \mathcal{N}(\mu, \sigma^2)$ .

**Linear Transform** Let  $X$  be a r.v with probability density function  $f_X$ , let  $Y := a + bX$  then for  $b > 0$  and  $a \in \mathbb{R}$ ,

$$f_Y(x) = \frac{1}{b} f_X\left(\frac{x-a}{b}\right)$$

**Linear Transform of Normally Distributed Random Variable** Suppose  $X \sim \mathcal{N}(\mu, \sigma^2)$  and  $a \in \mathbb{R}$  and  $b > 0$ . The random variable  $Y := a + bX$  is also normally distributed with parameter  $(a + b\mu, b^2\sigma^2)$ , i.e.  $Y \sim \mathcal{N}(a + b\mu, b^2\sigma^2)$ .

**Standardisation** Suppose  $X \sim \mathcal{N}(\mu, \sigma^2)$  then

$$Z := \frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$$

**Exponential Distribution** A random variable  $X$  is said to be exponentially distributed with parameter  $\lambda > 0$  if its probability density function is given by

$$f_X(x) = \frac{1}{\lambda} e^{-\frac{1}{\lambda}x}, \quad x > 0$$

and we write  $X \sim \exp(\lambda)$ . Then  $\mathbb{E}(x) = \lambda$  and  $\text{Var}(X) = \lambda^2$ .

**Gamma Distribution** A random variable  $X$  is said to be Gamma distributed with parameter  $\alpha, \beta > 0$  if its probability density function is given by

$$f_X(x; \alpha, \beta) = \frac{e^{-\frac{x}{\beta}} x^{\alpha-1}}{\Gamma(\alpha) \beta^\alpha}, \quad x > 0$$

and we write  $X \sim \text{Gamma}(\alpha, \beta)$  where  $\mathbb{E}(X) = \alpha\beta$  and  $\text{Var}(X) = \alpha\beta^2$ .

**Beta Distribution** The Beta function is given by

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x, y > 0$$

and the Beta and Gamma functions satisfies the following relationship

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y > 0$$

A random variable is said to follow a Beta distribution with parameters  $\alpha, \beta > 0$  if its density function is given by

$$f_X(x; \alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}, \quad x \in (0, 1)$$

and we write  $X \sim \text{Beta}(\alpha, \beta)$ .

### 3.2.1 QQ-plot

**Quantile** Suppose  $X$  is a continuous random variable with CDF given by  $F_X$ . The  $k\%$ -th quantile of  $X$  is given by

$$Q_X(k) := F_X^{-1}(k), \quad k \in [0, 1]$$

where  $F_X^{-1}$  is the inverse of the CDF  $F_X$ .

**Quantile Plot** Given continuous r.v.s  $X$  and  $Y$ , the theoretical quantile plot of  $X$  against  $Y$  is the graph

$$(Q_X(k), Q_Y(k)), \quad k \in [0, 1]$$

Suppose we are given  $X$  and  $Y = aX + b$  for some  $a > 0, b \in \mathbb{R}$  then the quantile plot of  $X$  against  $Y$  is a straight line.

Given r.v.s  $X$  and  $Y$  and suppose that the quantile plot of  $X$  against  $Y$  is a straight line. Then the distribution of  $X$  is equal to the distribution of a linear transform of  $Y$ .

### 3.2.2 Indicator Functions

- A indicator function of a set  $A$  is defined by

$$I_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \in A^c \end{cases}$$

- Indicator function of an interval is given as

$$I_{[a, b]}(x) = I_{\{a \leq x \leq b\}} \quad \text{or} \quad I_{(a, b]}(x) = I_{\{a < x \leq b\}}$$

- The indicator unifies expectation  $\mathbb{E}$  and probability  $\mathbb{P}$  notation since, the probability is the expectation of the indicator function. Therefore, it may be written that

$$\mathbb{P}(X \in A) = \int_A f_X(x) dx = \int_{-\infty}^{\infty} I_A(x) f_X(x) dx = \mathbb{E}(I_A(X)).$$

## 4 Bivariate Distribution

The joint density function of two continuous random variables  $X$  and  $Y$  is given by a bivariate function  $f_{X,Y}$  with the properties

1. For all  $x, y \in \mathbb{R}^2$ ,  $f_{X,Y}(x, y) \geq 0$ .

2. The double integral over  $\mathbb{R}^2$  is equal to one, that is

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1.$$

3. For any (measurable) set  $A, B \in \mathbb{R}$

$$\int_B \int_A f_{X,Y}(x, y) dx dy = \mathbb{P}(X \in A, Y \in B).$$

**Min and Max** We write  $a \vee b = \max(a, b)$  and  $a \wedge b = \min(a, b)$ .

**Tonelli's Theorem** Suppose  $f : \mathbb{R}^2 \rightarrow \mathbb{R}_+$  then

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y) dx dy = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y) dy dx$$

**Fubini - Tonelli's Theorem** Suppose  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ , if either

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x, y)| dx dy < \infty \text{ or } \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x, y)| dy dx < \infty$$

then

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y) dx dy = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y) dy dx$$

**Expected Value of Bounded Borel Functions** For any (bounded Borel) function  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  and random variables  $X$  and  $Y$ , then (given these integrals/sum are finite)

$$\mathbb{E}(g(X, Y)) = \begin{cases} \sum_{\forall x} \sum_{\forall y} g(x, y) \mathbb{P}(X = x, Y = y) & \text{discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dx dy, & \text{continuous} \end{cases}$$

**Marginal Probability/Density Function** The marginal densities are given by

$$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x, y) dy$$
$$f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x, y) dx$$

and similarly for discrete random variables  $X$  and  $Y$ .

$$\mathbb{P}(X = x) = \sum_y \mathbb{P}(X = x, Y = y)$$
$$\mathbb{P}(Y = y) = \sum_x \mathbb{P}(X = x, Y = y)$$

## 4.1 Independence

**Independence** Two random discrete variables  $X$  and  $Y$  are independent if for all outcomes  $x$  and  $y$ ,

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y)$$

or if  $X$  and  $Y$  are continuous random variable with joint probability density  $f_{X,Y}$  then

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

for all  $(x, y)$  in the domain of  $f_{X,Y}$ .

**Independence - Generalised**  $X$  and  $Y$  are independent if and only if for all  $x, y \in \mathbb{R}^2$ ,

$$\mathbb{P}(X \leq x, Y \leq y) = F_X(x)F_Y(y)$$

and in general, for any bounded functions  $g, f : \mathbb{R} \rightarrow \mathbb{R}$

$$\mathbb{E}(g(X)f(Y)) = \mathbb{E}(g(X))\mathbb{E}(f(Y))$$

## 4.2 Conditional Probability

**Conditional Probability** Suppose  $X$  and  $Y$  are

1. discrete random variables, then the conditional probability function of  $X$  are given the set  $\{Y = y\}$  is given by

$$\mathbb{P}(X = x \mid Y = y) := \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)}$$

2. continuous random variables, then the conditional probability density function of  $X$  given the set  $Y$  is given by

$$f_{X|Y}(x \mid y) := \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

**Multivariate Gaussian** A random vector  $X = (X_1, X_2)$  is said to be Gaussian with  $\mu_X = (\mu_{X_1}, \mu_{X_2})$  and Covariance matrix  $V$  if

$$f_X(x) = \frac{1}{\sqrt{(2\pi)^d |V|}} \exp\left(-\frac{1}{2}(X - \mu_X)^T V^{-1}(X - \mu_X)\right).$$

Here  $d = 2$  (dimension),  $V^{-1}$  is the matrix inverse of  $V$  and  $|V|$  is the determinant of  $V$ .

**Variance Matrix** The variance matrix is a symmetric matrix with entries

$$V_{ij} = \text{Cov}(X_i, X_j) \quad \text{where } i = 1, \dots, d \text{ and } j = 1, \dots, d.$$

If  $X = (X_1, X_2)$  is multivariate Gaussian then  $X_i$  for  $i = 1, 2$  must be one-dimensional Gaussian but the converse is not true.

**Conditional Expectations and Variance** Given any bound (Borel) function  $g$ , the conditional expectation of  $g(X)$  given the set  $\{Y = y\}$  is

$$\mathbb{E}(g(X) | Y = y) = \begin{cases} \sum_x g(x) \mathbb{P}(X = x | Y = y) & \text{discrete} \\ \int_{-\infty}^{\infty} g(x) f_{X|Y}(x | y) dx & \text{continuous} \end{cases}$$

The conditional variance of  $X$  given the set  $\{Y = y\}$  is

$$\text{Var}(X | Y = y) = \mathbb{E}(X^2 | Y = y) - (\mathbb{E}(X | Y = y))^2$$

**Independent Conditional Expectation and Variance** Suppose the random variables  $X$  and  $Y$  are independent then

1. The conditional expectation of  $X$  given  $Y$  is simply the expectation of  $X$ ,

$$\mathbb{E}(X | Y = y) = \mathbb{E}(X)$$

2. The conditional variance of  $X$  is simply the variance of  $X$ .

$$\text{Var}(X | Y = y) = \text{Var}(X)$$

**Bounded Borel Conditional Expectation** Given random variables  $X$  and  $Y$  a (bounded Borel) function  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\mathbb{E}(g(X, Y)) = \int_{\mathbb{R}} \mathbb{E}(g(X, y) | Y = y) f_Y(y) dy$$

where we define

$$\mathbb{E}(g(X, y) | Y = y) := \int_{\mathbb{R}} g(x, y) f_{X|Y}(x | y) dx$$

### 4.3 Covariance and Correlation

**Covariance** Given two random variables  $X$  and  $Y$ , the covariance of  $X$  and  $Y$  is given by

$$\text{Cov}(X, Y) = \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y)))$$

**Properties of Covariance** The covariance satisfies the following properties. For random variables  $X$  and  $Y$

1.  $\text{Cov}(X, X) = \text{Var}(X)$ ,
2.  $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$ ,
3. if  $X$  and  $Y$  are independent then  $\text{Cov}(X, Y) = 0$
4. The covariance is symmetric, i.e.  $\text{Cov}(X, Y) = \text{Cov}(Y, X)$ .
5. The covariance is a bilinear function, i.e. for all  $a, b \in \mathbb{R}$  and random variables  $X, Y$  and  $Z$

$$\text{Cov}(aX + bY, Z) = a\text{Cov}(X, Z) + b\text{Cov}(Y, Z)$$

$$\text{Cov}(X, aY + bZ) = a\text{Cov}(X, Y) + b\text{Cov}(X, Z)$$

**Correlation** The correlation between two random variable  $X$  and  $Y$  is defined to be

$$\text{Corr}(X, Y) := \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

**Properties of Correlation** Given two random variable  $X$  and  $Y$ , the following property holds for the correlation function

1.  $|\text{Corr}(X, Y)| \leq 1$
2.  $\text{Corr}(X, Y) = -1$  iff there exists  $a \in \mathbb{R}$  and  $b < 0$  such that  $\mathbb{P}(Y = a + bX) = 1$
3.  $\text{Corr}(X, Y) = 1$  iff there exists  $a \in \mathbb{R}$  and  $b > 0$  such that  $\mathbb{P}(Y = a + bX) = 1$

## 4.4 Bivariate Transforms

**Montone Probability Density** Let  $X$  be a random variable with density  $f_X$ , if  $h$  is monotone over the set  $\{x : f_X(x) > 0\}$  then the probability density of  $Y := h(X)$  is given by

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right| = f_X \circ h^{-1}(y) \left| \frac{dh^{-1}(y)}{dy} \right|$$

**CDF Strictly Increasing** Suppose  $X$  has density  $f_X$  and its CDF  $F_X$  strictly increasing (once it is greater than zero) then  $Y := F_X(X) \sim \text{Uniform}[0, 1]$ .

**Bivariate Transforms** Given random variable  $X$  and  $Y$ , suppose  $U$  and  $V$  are transforms of  $X$  and  $Y$  taking value in  $\mathbb{R}$ , then

$$f_{U,V}(u, v) = f_{X,Y}(x, y) |\det(J)|$$

where  $\det(J)$  is the determinant of the Jacobian (of the inverse)

$$J = \begin{pmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{pmatrix}$$

## 5 Sum of Variables

**Sum of Independent Random Variable** If  $X$  and  $Y$  are independent discrete random variables then

$$\mathbb{P}(X + Y = z) = \sum_y \mathbb{P}(X = z - y) \mathbb{P}(Y = y)$$

where the sum is taken over all possible outcomes of  $Y$ .

**Sum of Independent Continuous Random Variables** (Convolution formula) Suppose  $X$  and  $Y$  are independent continuous r.v.s with density  $f_X$  and  $f_Y$ . Let  $Z = X + Y$  then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)f_Y(y) dy$$

**Moment Generating Function Approach** If  $X$  and  $Y$  are independent random variables for which the moment generating function exists then

$$M_{X+Y}(t) = M_X(t)M_Y(t)$$

In general if  $(X_i)_i$  is an independent sequence of random variables, then

$$M_{\sum_{i=1}^n X_i}(t) = \prod_{i=1}^n M_{X_i}(t)$$

**Useful Results** Using the method of moment generating function method one can show the following. Suppose  $(X_i)_{i=1,\dots,n}$  be a =n independent identically distributed (iid) sequence of random variables and we set  $Y := \sum_{i=1}^n X_i$  then if

- $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$  then  $Y \sim \mathcal{N}(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2)$
- $X_i \sim \exp(\lambda)$  or  $\text{Gamma}(1, \lambda)$  then  $Y \sim \text{Gamma}(n, \lambda)$
- $X_i \sim \text{Gamma}(\alpha_i, \beta)$  then  $Y \sim \text{Gamma}(\sum_{i=1}^n \alpha_i, \beta)$
- $X_i \sim \text{Poisson}(\lambda_i)$  then  $Y \sim \text{Poisson}(\sum_{i=1}^n \lambda_i)$ .
- $X_i \sim \text{Bernoulli}(p_i)$  then  $Y \sim \text{Binomial}(n, p)$ .
- $X_i \sim \text{Binomial}(n_i, p)$  then  $Y \sim \text{Binomial}(\sum_{i=1}^n n_i, p)$

## 6 Central Limit Theorem

### 6.1 Central Limit Theorem

**Central Limit Theorem** Let  $(X_n)_{n \in \mathbb{N}_+}$  be an independent identically distributed sequence of random variables with common mean  $\mu = \mathbb{E}(X_1)$  and variance  $\sigma^2 = \text{Var}(X_1) < \infty$ . Let  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$  then

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} Z \sim \mathcal{N}(0, 1)$$



## 6.2 Convergences

**Convergence in Distribution** Let  $(X_i)_{i \in \mathbb{N}_+}$  be a sequence of random variables, we say that  $X_n$  converges to  $X$  in distribution if for all  $x$ , for which  $F_X(x)$  is continuous

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x).$$

In this case, we write  $X_n \xrightarrow{d} X$ .

**Convergence of Moment Generating Functions and Existence of CDF** Let  $(X_n)_{n \in \mathbb{N}_+}$  be a sequence of r.v each with moment generating function  $M_{X_n}(t)$ . Suppose that

$$M(t) = \lim_{n \rightarrow \infty} M_{X_n}(t)$$

exists then there exists an unique valid cumulative distribution function  $F$  and r.v  $X$  such that  $F_X = F$ .

**Convergence of Random Variables** A sequence of random variables  $(X_n)_{n=1, \dots, \infty}$ , converges in probability to a r.v  $X$  if for all  $\epsilon > 0$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P}(|X_n - X| > \epsilon) = 0$$

and we write  $X_n \xrightarrow{\mathbb{P}} X$ .

**Law of Large Numbers** Let  $(X_n)_{n \in \mathbb{N}}$  be a sequence of independent r.vs with mean  $\mu$  and finite variance  $\sigma^2$ , we set  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ , then

$$\bar{X}_n \xrightarrow{\mathbb{P}} \mu$$

(Strong Version): Same Thing but using *almost surely* probability for convergence.

**Equal Almost Surely** Two random variables  $X$  and  $Y$  are said to be equal almost surely if  $\mathbb{P}(Y = X) = 1$  and we write  $X = Y$  a.s.

**Almost Surely Convergence** Given a random variable  $X$ , a sequence  $(X_n)_{n \in \mathbb{N}}$  converges to almost surely to  $X$ , if

$$\mathbb{P}(\lim_{n \rightarrow \infty} X_n = X) = 1$$

and we write  $X \xrightarrow{a.s.} X$ .

**Convergence in  $L^p$**  A sequence of random variables  $(X_i)_{i \in \mathbb{N}_+}$  is said to converge in  $L^p$  to another random variable  $X$  if for  $p \geq 1$ ,

$$\lim_{n \rightarrow \infty} \mathbb{E}(|X_n - X|^p) = 0$$

in particular, if  $p = 2$ , we say that  $X_n$  converges to  $X$  in the mean square sense.

**Convergence in  $L^p$  and Probability** Suppose  $(X_n)_{n \in \mathbb{N}}$  is a sequence of r.v.s converging to  $X$  in  $L^p$  for  $p \geq 1$ , then  $X_n$  converges to  $X$  in probability.

$$X_n \xrightarrow{L^p} X \implies X_n \xrightarrow{\mathbb{P}} X$$

**Convergence in Probability and Distribution** Convergence in probability implies convergence in distribution. That is given  $X$  and a sequence  $(X_n)_{n \in \mathbb{N}_+}$ ,

$$X_n \xrightarrow{a.s.} X \implies X_n \xrightarrow{\mathbb{P}} X \implies X_n \xrightarrow{d} X$$

**Convergence Remark** We have shown the following implications

$$X_n \xrightarrow{L^p} X \implies X_n \xrightarrow{\mathbb{P}} X \implies X_n \xrightarrow{d} X$$