Graph Theory

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Chapter 1

Introduction

1.1 Definitions

A **graph** $G = (V, E)$ is a set V of *vertices* and a set E of unordered pairs of distinct vertices, called *edges*. Write vw or $\{v, w\}$ for the edge joining v and w, and say that v and w are **neighbours** or that they are *adjacent*.

In these notes, unless otherwise stated, graphs are:

- **finite**: $|V| \in \mathbb{N}$.
- **labelled**: vertices are distinguishable, usually $V = [n] := \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$.
- **undirected**: edges are *unordered* pairs of vertices.
- **simple**: no loops $\{v, v\}$ or multiple edges (since E is not a multiset).

A graph G with vertex set $\{v_1, \dots, v_n\}$ has **adjacency matrix** $A(G) = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E, \\ 0 & \text{otherwise.} \end{cases}$$

$A(G)$ is a **symmetric** $n \times n$ 0-1 matrix with zero diagonal.

The **trivial graph** has at most one vertex. Hence it has no edges.

A **subgraph** of a graph $G = (V, E)$ is a graph $H = (W, F)$ such that $W \subseteq V$ and $F \subseteq E$.

We say that H is an **induced subgraph** if for all $v, w \in W$ if $vw \in E(G)$ then $vw \in E(H)$. Write $H = G[W]$, and say that H is the subgraph of G *induced by* the vertex set W.
The number of **vertices** of G, written $|G| = |V(G)|$, is called the *order* of G. The number of **edges** of G, sometimes written $||G|| = |E(G)|$, is called the *size* of G.

Two graphs $G = (V, E)$ and $H = (W, F)$ are **isomorphic** if there exists a *bijection* $\phi : V \to W$ such that $\phi(v)\phi(w) \in F$ if and only if $vw \in E$. The map ϕ is called a *graph isomorphism* or *isomorphism*.

1.2 The Degree of a Vertex

If $v \in e$ where v is a vertex and e is an edge, then we say that e is *incident with* v. The **degree** $d_G(v)$ of vertex v in a graph G is the number of *edges* of G which are *incident with* v. A vertex of degree 0 is an *isolated vertex.*

Let $N_G(v)$ be the set of all **neighbours** of v in G, then $d(v) = |N(v)|$.

Lemma 1.2.1 (The Handshaking Lemma)**.** In any graph, $G = (V, E)$,

$$\sum_{v \in V} d(v) = 2|E|.$$

Let $\delta(G) = \min_{v \in V} d(v)$ be the minimum degree in G, and $\Delta(G) = \max_{v \in V} d(v)$ be the maximum degree in G.

1.2.1 Some Special Graphs

A graph is k**-partite** if there exists a partition of its vertex set

$$V = V_1 \cup V_2 \cup \cdots V_k$$

into k nonempty disjoint subsets (parts) such that there are no edges between vertices in the same part.

The **complete graph** on r vertices, denoted K_r, has all $\binom{r}{2}$ edges present. The **complete bipartite graph** K_r, s has r vertices in one part of the vertex bipartition, s vertices in the other, and all rs present.

A graph is **regular** if every vertex has the same degree. If every vertex of a graph has degree d then we say that the graph is d-regular.

The **complement** of a graph G is the graph $\bar{G} = (V, \bar{E})$ where $vw \in \bar{E}$ if and only if $vw \notin E$. Note that $\bar{K_n}$ is the graph with n vertices and no edges.

If $G = (V, E)$ and $X \subset V$ then $G - X$ denotes the graph obtained from G by deleting all vertices in X and all edges which are incident with vertices in X. If $F \subseteq E$ then $G - F$ denotes the graph $(V, E - F)$ obtained from G by deleting the edges in F.

1.3 Paths and Cycles

A **walk** in the graph G is a sequence of vertices $v_0 v_1 v_2 \cdots v_k$ such that $v_i v_{i+1} \in E$ for $i = 0, 1, \ldots, k-1$. The **length** of this walk is k. The walk is **closed** if $v_0 = v_k$.

An **Euler tour** is a *closed walk* in a graph which uses every edge precisely once. A graph is Eulerian if it has an Euler tour.

Theorem 1.3.1 (Euler, 1736)**.** A connected graph is Eulerian if and only if every vertex has even degree.

A walk is a **path** if it does not visit any vertex more than once. A path is a sequence of *disinct* vertices, with subsequence vertices joined by an edge. A path $v_0 v_1 \ldots v_k$ with k edges is called a k-path and has length k.

If $k \geq 3$ and $P = v_0 v_1 \cdots v_{k-1}$ is a path of length $k-1$ then $C = P + v_0 v_{k-1}$ is a **cycle** of length k, also called a $k - cycle$. It is a closed walk which visits no internal vertex more than once.

An edge which joins two vertices of a cycle C, but which is not an edge of C, is called a **chord**. An **induced cycle** is a cycle which has no chords.

Proposition 1.3.2. Every graph G contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G)+1$, if $\delta(G) \geq 2$.

Proof. Let $P = x_0 x_1 \ldots x_k$ be the longest path in G. By maximality of P, all neighbours of x_k lie on P. Hence $\delta(G) \leq d(x_k) \leq k = |\{x_0, x_1, \ldots, x_{k-1}\}|$, which proves the first statement. Let x_i be the smallest-indexed neighbour of x_k in P. Then $C = x_k x_i x_{i+1} \ldots x_{k-1} x_k$ is a cycle of length $\geq \delta(G)+1$ because C contains $d(x_k) \geq \delta(G)$ neighbours of x_k as well as x_k. $\square$

The *minimum length* of a cycle in G is the **girth** of G, denoted by $g(G)$.

Given $x, y \in V$, let $d_G(x,y)$ be the length of a shortest path from x to y in G, called the **distance** from x to y in G. Set $d_G(x,y) = \infty$ if no such path exists.

We say that G is **connected** if $d_G(x,y)$ is finite for all $x, y \in V$.

Let the **diameter** of G be $\operatorname{diam}(G) = \max_{x,y \in V} d_G(x,y)$.

Proposition 1.3.3. Every graph G which contains a cycle satisfies $g(G) \leq 2\operatorname{diam}(G) + 1$.

Proof. Let C be a shortst cycle in G, so $|C| = g(G)$. For a contradiction, assume $g(G) \geq 2\operatorname{diam}(G) + 2$.

Choose vertices x, y on C with $d_C(x,y) \geq \operatorname{diam}(G)+1$. In G the distance $d_G(x,y)$ is strictly smaller, so any shortest path P from x to y in G is not a subgraph of C. But using P together with the shorter arc of C from x to y gives a closed walk of length $< |C|$. This closed walk contains a shorter cycle than C which is a contradiction. $\square$

1.4 Connectivity

A maximal connected subgraph of G is called a **component** (or **connected component**) of G.

Proposition 1.4.1. The vertices of a connected graph can be labelled $v_1, v_2, \ldots, v_n$ such that $G_n = G$ and $G_i = G[v_1, \ldots, v_i]$ is connected for all i.

Proof. Choose v_1 arbitrarily. Now suppose that we have labelled $v_1, \ldots, v_i$ such that $G_j = G[v_1, \ldots, v_j]$ is connected for all $j = 1, \ldots, i$.

If $i < n$ then $G_i \neq G$, so there exists some $v_j \in \{v_1, \ldots, v_i\}$ with a $w \notin \{v_1, \ldots, v_i\}$ in G. (Otherwise $G_i \neq G$ is a component of G, impossible as G is connected.) Let $v_{i+1} = w$, then $G_{i+1} = G[v_1, \ldots, v_i]$ is connected. This completes the proof, by induction. $\square$

Let $A, B \subseteq V$ be sets of vertices. An (A,B)**-path** in G is a path $P = x_0x_1 \cdots x_k$ such that

$$P \cap A = \{x_0\}, \quad P \cap B = \{x_k\}.$$

Let $A, B \subseteq V$ and let $X \subseteq V \cup E$ be a set of vertices and edges. We say that X **separates** A and B in G if every $(A,B)-$path in G contains a vertex or edge from X.
Note that we do not assume that A and B are disjoint and if X separates A and B then $A \cap B \subseteq X$.
We say that X **separates** two vertices a, b if $a, b \notin X$ and X separates the sets $\{a\}, \{b\}$.

More generally, we say that X *separates* G, and call X a **separating set** for G, if X separates two vertices of G. That is, X separates G if there exist distinct vertices $a, b \notin X$ such that X separates a and b.

If $X = \{x\}$ is a separating set for G, where $x \in V$, then we say that x is a **cut vertex**.

If $e \in E$ and $G - e$ has more components than G then e is a **bridge**.

The unordered pair (A, B) is a **separation** of G if $A \cup B = V$ and G has no edge between $A - B$ and $B - A$. The second conditions says that $A \cap B$ separates A from B in G. If both $A - B$ and $B - A$ are nonempty then the separation is **proper**. The order of the separation is $|A \cap B|$.

Definition. Let $k \in \mathbb{N}$. The graph G is **k-connected** if $|G| > k$ and $G - X$ is connected for all subsets $X \subseteq V$ with $|X| < k$.

The **connectivity** $\kappa(G)$ of G is defined by

$$\kappa(G) = \max\{k : G \text{ is } k\text{-connected}\}.$$

So, $\kappa(G) = 0$ iff G is trivial or G is disconnected. Also, $\kappa(K_n) = n - 1$ for all positive integers n.

Definition. Let $\ell \in \mathbb{N}$ and let G be a graph with $|G| \geq 2$. If $G - F$ is connected for all $F \subseteq E$ with $|F| < \ell$ then G is ℓ**-edge-connected**.

The **edge connectivity** $\lambda(G)$ is defined by

$$\lambda(G) = \max\{\ell : G \text{ is } \ell\text{-edge-connected}\}.$$

Proposition 1.4.2. If $|G| \geq 2$ then $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Theorem 1.4.3 (Mader, 1973)**.** Let k be a positive integer. Every graph G with average degree at least $4k$ has a $(k+1)$-connected subgraph H with

$$\frac{|E(H)|}{|V(H)|} > \frac{|E(G)|}{|V(G)|} - k.$$

Proof. We write $|G|$ instead of $|V(G)|$. Let $\gamma = \frac{|E(G)|}{|G|} \geq 2k$. Consider subgraphs G' of G which satisfy:

$$|G'| \geq 2k \quad \text{and} \quad |E(G')| > \gamma(|G'| - k). \tag{1.1}$$

such graphs G' exists as G satisfies 1.1. (Average degree of G is $\frac{2|E(G)|}{|G|} \geq 4k$, so

$|G| \geq 4k$ and $\gamma(|G|-k) = |E(G)|\frac{(|G|-k)}{|G|} < |E(G)|$.)

Now let H be a subgraph of G of smallest order which satisfies 1.1. We continue the proof by proving three claims.

Claim 1. If G' satisfies 1.1 then $|G'| > 2k$.
Proof. If G' satisfies 1.1 and $|G'| = 2k$ then $|E(G')| > \gamma(|G'|-k) \geq 2k^2 > \binom{|G'|}{2}$, contradiction.

Claim 2. $S(H) > \gamma$.
Proof. For a contradiction, suppose that $S(H) \leq \gamma$. Let G' be obtained from H by deleting a vertex of degree $\leq \gamma$. Then $|G'| < |H|$ and G' satisfies 1.1, which is a contradiction. To see this, check:

$$\begin{aligned} |G'| &= |H| - 1 \geq 2k, \quad \text{by Claim 1, and} \\ |E(G')| &\geq |E(H)| - \gamma > \gamma(|H| - k - 1), \quad \text{as } H \text{ satisfies 1.1} \\ &= \gamma(|G'| - k). \end{aligned}$$

Hence $S(H) > \gamma$. It follows that $|H| \geq \gamma$. Thus,

$$\frac{|E(H)|}{|H|} > \frac{\gamma(|H|-k)}{|H|}. \qquad \text{(as } H \text{ satisfies 1.1)}$$

Claim 3. H is $(k+1)$-connected.
Proof. By Claim 1, $|H| \geq 2k+1 \geq k+2$ as $k \geq 1$. So H is large enough. For a contradiction, suppose that H is not $(k+1)$-connected. Then H has a proper separation $\{U_1, U_2\}$ of order at most k.

Let $H_i = H[U_i]$ for $i = 1, 2$. Since any vertex $v \in U_1 - U_2$ has $d_H(v) \geq S(H) > \gamma$ (by Claim 2), and all neighbours of v in H belong to H_1, we have $|H_1| \geq \gamma \geq 2k$. Similarly, $|H_2| \geq 2k$. By minimality of H, neither H_1 nor H_2 satisfies 1.1. Hence $|E(H_i)| \leq \gamma(|H_i| - k)$ for $i = 1, 2$. But then

$$\begin{aligned} |E(H)| &\leq |E(H_1)| + |E(H_2)| \\ &\leq \gamma(|H_1| + |H_2| - 2k) \\ &\leq \gamma(|H| - k), \qquad \text{(by inclusion-exclusion)} \end{aligned}$$

since $|U_1 \cup U_2| \leq k$. This contradicts 1.1 for H. So H is $(k+1)$-connected, completing the proof of Claim 3 and of the theorem. $\square$

1.5 Trees and Forests

A graph with no cycles is a **forest** (also called an acyclic graph). A connected graph with no cycles is a **tree**.

Theorem 1.5.1. The following are equivalent for a graph T:

(i) T is a tree;

(ii) Any two vertices of T are linked by a *unique* path in T;

(iii) T is *minimally connected*: that is, T is connected but $T - e$ is disconnected for every $e \in E(T)$;

(iv) T is *maximally acyclic*: that is, T is acyclic but $T + xy$ has a cycle for any two nonadjacent vertices x, y in T.

Corollary 1.5.2. If G is connected then G has a spanning tree.

Proof. Let be a connected graph and let H be a minimal connected spanning subgraph of G. (Note H exists as G is a connected spanning subgraph of itself.) By theorem 1.5.1, H is a tree. $\square$