Stable Matching Problem

Input: n men $M = \{m_1, m_2, \ldots, m_n\}$
n women $W = \{w_1, w_2, \ldots, w_n\}$

-and-

Preference lists for each man and woman

ex. $P_{m_1} = (w_n, w_{n-1}, \ldots, w_2, w_1)$

Output: A stable matching of men and women.
call it S

- $\forall_{m \in M} ! \exists_{w \in W} \; (m, w) \in S$
- $\forall_{w \in W} ! \exists_{m \in M} \; (m, w) \in S$
- no instabilities

$$\neg \exists_{(m,w) \notin S} \left(\begin{array}{l} (m, w') \in S \wedge (m', w) \in S \wedge \\ m \text{ prefers } w \text{ over } w' \\ \wedge\ w \text{ prefers } m \text{ over } m' \end{array} \right)$$

Gale-Shapley Algorithm

Outline

- To start: Everyone is free
- women make proposals to men
  - men say yes or no
- End with n married couples

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(0) Everyone's free

(1) Pick a free woman $w \in W$ to propose
- she proposes to her most prefered man $m \in$
- (m, w) get engaged since m is free
  - If a more preferable woman w' proposes to m later, ~~he~~ he will take the upgrade.

loop:

(2) General Case
- Some enganged, some free
- Pick a free woman w
  - w proposes to her most prefered man m whom she has not yet proposed to.

Case 1: m is free (m, w) get enganged.

Case 2: m is enganged to w' ie. (m, w') are togeth

- If ~~he~~ m prefers w' over w, he says no, w remains free, (m, w') remain engage
- If m prefers w over w', he says yes, (m, w) get enganged, w' becomes free

(3) If all men and women are engaged, they get married.

Observation 1: Once a man gets engaged, he becomes engaged to better women and never becomes free again.

Observation 2: A woman keeps getting engaged to worse men.

Q1) Does the algorithm terminate?

Q2) Does the algorithm always output a stable matching?

Theorem 1: There are $\leq n^2$ proposals during a run of the GS algorithm

Proof: $P(t)$ = # of proposals made after t iterations

- Each iteration of the algo has 1 proposal
  $\therefore P(t+1) = P(t) + 1$
- A woman never proposes to the same man more than once.
- if all n woman propose to all n men, then $P(t) = n^2$ so $\forall_t\ P(t) \leq n^2$ ■

Theorem 2: The GS algorithm always outputs a stable matching.

Call the matching S.

Lemma 1: S is a perfect matching

Lemma 2: S contains no instabilities

Proof of Lemma 1: Proof by Contradiction using Pigeon-Hole Principle (PHP)

- assume S is not a perfect matching
- Show that this causes a contradiction.

p = S is a perfect matching

$p \vee \neg p$ True

$p \vee \neg p$

if $\neg p$ is False T F

- If S is not a perfect matching, ~~there exists an instability.~~ there is a w that is free at the end of the algo.
- This implies w proposed to all n men and was either rejected or dumped by all of them.
- That means all n men are engaged (since men remain enganged)
- There are only ~~se~~ $n-1$ other women

n engaged men

$\leq n-1$ engaged women

$\therefore$ S is a perfect matching

Proof of Lemma 2 ; S contains no instabilities.

Proof by contradiction (again)

assume ∃ an instability and show a contracictio

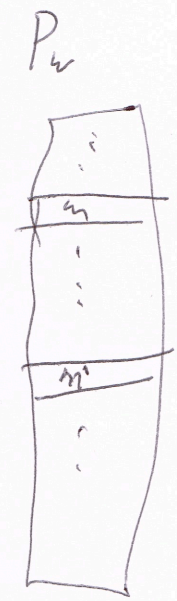
Define the instability with two pairs in S

$(m, w) \in S$

$(m', w') \in S$

s.t. ~~m~~ m prefers w' over w

and w' prefers m over m'



~~(m, w')~~ (m, w') would get engaged.

- Did w' propose to m?

- what did m say?

Case 1 - yes? (m, w') were engaged and m dumped her for someone better.

~~no? then~~ men keep getting engaged to better women. m prefers w' over w whom he ended up with. ⊥

Case 2 : no? He's already with someone better.

Ended up with w whom he prefers less than w' and therefor prefers ~~w~~ w less that his partner when w' proposed.

men Can't get worse engagement. ⊥

∴ the matching is stable. ∎