Shortest Path with negative edge weights

<u>Inputs</u>: Directed graph $G=(V,E)$

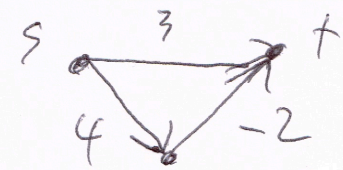
$\forall e \in E$, $c_e \in \mathbb{R}$, c_e is the weight of e
can be < 0

G has no negative cycles

$t \in V$

<u>Output</u>: Shortest path from every $s \in V$ to t

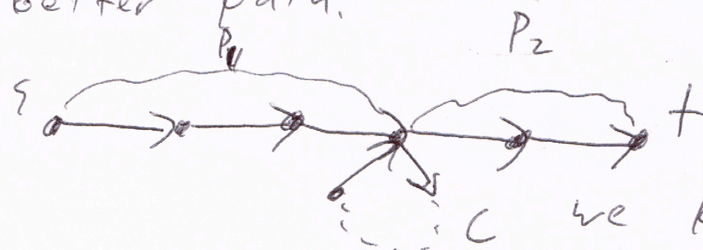
- Dijkstra's won't work



Bellman-Ford

<u>proposition</u>: If G has no negative cycles the $\exists$ a shortest path from s to t that is simple. (no cycles)

<u>proof</u>: assume path p from s-to-t has a cycle... take the cycle out to get a better path.



$P_1 \, C \, P_2 = P_{s\text{-}t}$

$wt(P_{s\text{-}t}) = wt(P_1) + wt(C) + wt(P_2)$

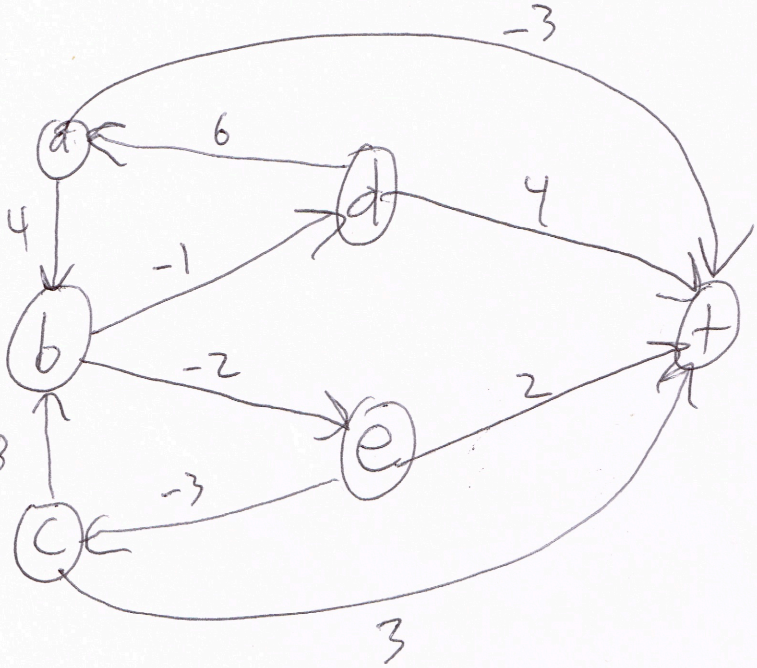
we know $wt(C) \geq 0$

remove C... $P_{st}' = P_1 P_2$ $\quad wt(P_{st}') = wt(P_1) + \ldots$

$wt(P_{st}') \leq wt(P_{st})$ ■

- only need paths of length $\leq n-1$ since there are no cycles.

ex:



$OPT(v,i) =$

shortest path from v to t using at most i edges.

$OPT(d,0) = \infty$ $\quad OPT(d,1) = 4$ $\quad OPT(d,2) = 3$

$OPT(d,3) = 3$ $\quad OPT(d,4) = 2$ $\quad OPT(d,5) = 0$

Claim $OPT(v,i) \geq OPT(v, i+1)$ can't get worse with more choices

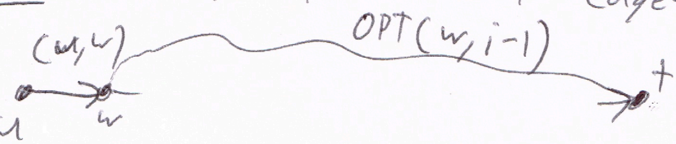
$d(s,t) = OPT(s, n-1)$

$$\boxed{OPT(u,i) = \min\left(OPT(u,i-1), \min_{(u,w)\in E} c_{u,w} + OPT(w,i-1)\right) \quad u \in V, 1 \leq i \leq n-1}$$

<u>Case 1</u>: Shortest path $\leq i$ edges uses $\leq i-1$ edges

$OPT(u,i) = OPT(u,i-1)$

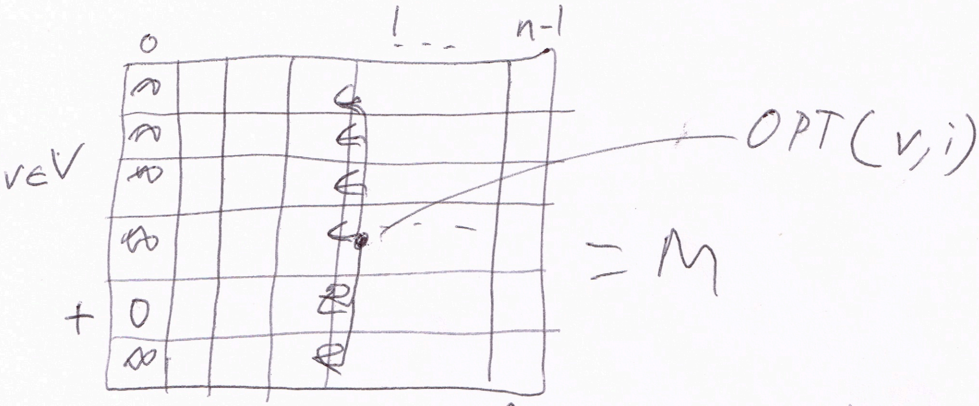
<u>Case 2</u>: Shortest path $\leq i$ edges uses exactly i edges



$c_{(u,w)} + OPT(w,i-1) = OPT(u,i)$

To compute $OPT(v,i)$ we need to know $OPT(*, i-1)$ for all $* \in V$

Build a matrix for $OPT(v,i)$ for all $v \in V$, $0 \le i \le n-1$



shortest paths to be outputed

Shortest-Path(s,t)

$M : V \times \{0, \dots, n-1\}$

$M \equiv OPT$

initialize/Base case

1.) $M[t,0] = 0$, $M[u,0] = \infty \;\; \forall u \neq t$

2.) For $i = 0, \dots, n-1$

2a.) For $\forall v \in V$

2b.) $M[v,i] = \min\left(M[v,i-1], \min\limits_{(v,w) \in E} c_{v,w} + M[w,i-1]\right)$

3.) Output $M[n-1,s] \;\; \forall s \in V$

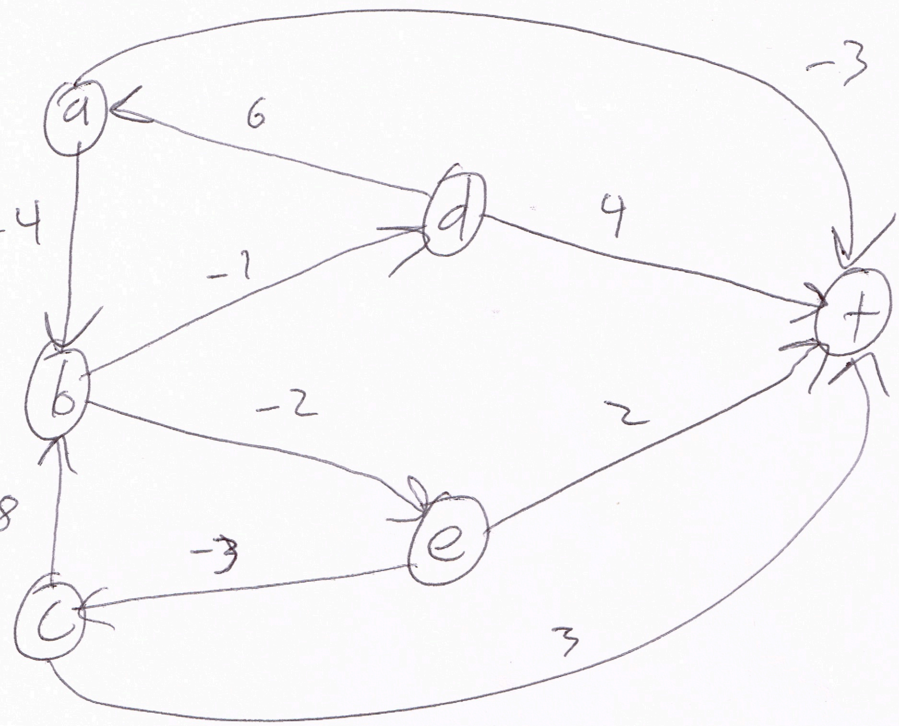
runtime

1	2	2a	2b	3
$O(n)$	$O(n)$	$\cancel{O(n)}$ $O(m)$		$O(n)$

look at all edges

Correctness: by induction

runtime $= O(n) + O(n) \cdot O(m) + O(n) = O(n \cdot m)$



Outgoing edges

a: t, b

b: d, e

c: t, b

d: t, a

e: t, c

t: {}

	0	1	2	3	4	5
a	∞	-3	-3	-4	-6	-6
b	∞	∞	0	-2	-2	-2
c	∞	3	3	3	3	3
d	∞	4	3	3	2	0
e	∞	2	0	0	0	0
t	0	0	0	0	0	0

↑

Values of shortest paths to t.