Compressible Navier-Stokes Equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \tag{1}$$

$$\frac{\partial}{\partial t}(\rho u_j) + \frac{\partial}{\partial x_i}(\rho u_i u_j - P_{ij}) = 0 \tag{2}$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_i}(u_i E - u_j P_{ij} + q_i) = 0 \tag{3}$$

where ρ is the mass density, $\mathbf{u}$ is the velocity, P_{ij} is the stress tensor, E is the energy density, and $\mathbf{q}$ is the heat flux. Indices $i,\ j$ equal 1, 2, 3, and repeated indices are summed over. Eq. (1) expresses conservation of mass, Eq. (2) expresses conservation of momentum, and Eq. (3) expresses conservation of energy.

The stress tensor and energy density are

$$P_{ij} = -nk_B T\delta_{ij} - \frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{4}$$

$$E = \frac{3}{2}nk_B T + \frac{1}{2}\rho u^2 \tag{5}$$

where the density $n = \rho/m$, m is the mass, and T is the temperature. Note that μ has units of $\rho\nu$, where ν has units of L^2/t.

The heat flux is usually approximated by the Fourier law $\mathbf{q} = -\kappa\nabla T$.

Euler Equations of Ideal Gas Dynamics

The **Nonlinear Conservation Laws** of gas dynamics (neglect viscosity and heat conduction) are

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \tag{6}$$

$$\frac{\partial}{\partial t}(\rho u_j) + \frac{\partial}{\partial x_i}(\rho u_i u_j + P\delta_{ij}) = 0 \tag{7}$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_i}(u_i(E + P)) = 0 \tag{8}$$

For a "gamma law gas" the pressure P is given by the equation of state

$$P = (\gamma - 1)\left(E - \frac{1}{2}\rho u^2\right) \tag{9}$$

The Euler equations are hyperbolic.

Incompressible Navier-Stokes Equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{u} \tag{10}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{11}$$

where P is the pressure. The incompressible Navier-Stokes equations are incompletely parabolic (parabolic + elliptic).

Hydrodynamic Model for Semiconductor Devices

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_i}(n u_i) = 0 \tag{12}$$

$$\frac{\partial}{\partial t}(m n u_j) + \frac{\partial}{\partial x_i}(m n u_i u_j) = -\frac{\partial P}{\partial x_j} - n \frac{\partial V}{\partial x_j} - \frac{m n u_j}{\tau_p} \tag{13}$$

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x_i}(u_i(W + P) + q_i) = -n u_i \frac{\partial V}{\partial x_i} - \frac{\left(W - \frac{3}{2} n k_B T_0\right)}{\tau_w} \tag{14}$$

$$\nabla \cdot (\epsilon \nabla V) = e^2 (N - n) \tag{15}$$

where n is the electron density (we could also add a set of equations for holes), $\mathbf{u}$ is the velocity, m is the effective mass, $P = n k_B T$ is the pressure, T is the temperature, $V = -e\phi$ is the electrostatic potential energy, ϕ is the electrostatic potential, $e > 0$ is the electronic charge, $W = 3 n k_B T/2 + m n u^2/2$ is the energy density, $\mathbf{q}$ is the heat flux, and T_0 is the temperature of the semiconductor lattice. Eq. (12) expresses conservation of electron number, Eq. (13) expresses conservation of momentum, and Eq. (14) expresses conservation of energy. The last terms in Eqs. (13) and (14) represent electron scattering, which is modeled by the standard relaxation time approximation, with momentum and energy relaxation times $\tau_p = \tau_p(T)$ and $\tau_w = \tau_w(T)$. In Poisson's equation (15), $N = N(\mathbf{x})$ is the background doping density in the semiconductor device.

The hydrodynamic model PDEs have hyperbolic, parabolic, and elliptic modes.