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github.com/jhh37/varpro

Revisiting the Variable Projection Method for Separable Nonlinear Least Squares Problems

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1. Separable Nonlinear Least Squares (SNLS) problems

A general SNLS problem has the form

$$\min_{\mathbf{u},\mathbf{v}} \|\boldsymbol{\varepsilon}(\mathbf{u},\mathbf{v})\|_2^2 = \min_{\mathbf{u},\mathbf{v}} \|\mathbf{G}(\mathbf{u})\mathbf{v} - \mathbf{z}(\mathbf{u})\|_2^2 \quad (1)$$

where $\mathbf{u} \in \mathbb{R}^p$ and $\mathbf{v} \in \mathbb{R}^q$ are the model parameters
$\mathbf{z}: \mathbb{R}^q \to \mathbb{R}^s$ and $\mathbf{G}: \mathbb{R}^p \to \mathbb{R}^{s\times q}$ are functions of $\mathbf{u}$

Examples: Bundle adjustment (BA) with weak-perspective cameras [1]

$$\min_{\{\mathbf{q}_i\},\{\mathbf{x}_j\}} \sum_{(i,j)\in\Omega} \|\mathbf{KP}(\mathbf{q}_i)\tilde{\mathbf{x}}_j - \mathbf{m}_{ij}\|_2^2$$

BA with affine cameras [2]:

$$\min_{\{\mathbf{P}_i\},\{\mathbf{x}_j\}} \textstyle\sum_{(i,j)\in\Omega} \|\mathbf{P}_i\tilde{\mathbf{x}}_j - \mathbf{m}_{ij}\|_2^2$$

Other problems which can be cast as matrix factorization [3]
e.g. non-rigid SfM, recommender systems: $\min_{\mathbf{U},\mathbf{V}} \|\mathbf{W} \odot (\mathbf{UV} - \mathbf{M})\|_F^2$

2. Known approaches

A. Joint optimization [4]

- Minimize (1) over $\mathbf{u}$ and $\mathbf{v}$ simultaneously using a 2nd order optimizer, i.e. form a stacked vector $\mathbf{x} = [\mathbf{u};\mathbf{v}]$ and solve $\min_{\mathbf{x}=[\mathbf{u};\mathbf{v}]} \|\boldsymbol{\varepsilon}(\mathbf{u},\mathbf{v})\|_2^2$
- Embedded Point Iterations (**EPI**) [5,6]: after each joint update, one may perform further update on $\mathbf{v}$, i.e. $\mathbf{v}^*(\mathbf{u}+\Delta\mathbf{u}) = \arg\min_{\mathbf{v}} \|\mathbf{G}(\mathbf{u}+\Delta\mathbf{u})\mathbf{v} - \mathbf{z}(\mathbf{u}+\Delta\mathbf{u})\|_2^2$

B. Variable Projection (VarPro) [7]

- Eliminate $\mathbf{v}$ optimally from (1) and solve the reduced problem using a 2nd order optimizer, i.e. $\mathbf{v}^*(\mathbf{u}) \coloneqq \arg\min_{\mathbf{v}} \|\mathbf{G}(\mathbf{u})\mathbf{v} - \mathbf{z}(\mathbf{u})\|_2^2 = \mathbf{G}(\mathbf{u})^\dagger \mathbf{z}(\mathbf{u})$. Now solve

$$\min_{\mathbf{u}} \|\boldsymbol{\varepsilon}^*(\mathbf{u})\|_2^2 \coloneqq \min_{\mathbf{u}} \|\boldsymbol{\varepsilon}(\mathbf{u},\mathbf{v}^*(\mathbf{u}))\|_2^2 = \min_{\mathbf{u}} \left\|\mathbf{G}(\mathbf{u})\,\mathbf{G}(\mathbf{u})^\dagger \mathbf{z}(\mathbf{u}) - \mathbf{z}(\mathbf{u})\right\|_2^2$$

3. Levenberg-Marquardt (LM) algorithm [8]

LM: 2nd-order optimization algorithm for solving general nonlinear least squares problem

$$\min_{\mathbf{x}} \|\boldsymbol{\varepsilon}(\mathbf{x})\|_2^2$$

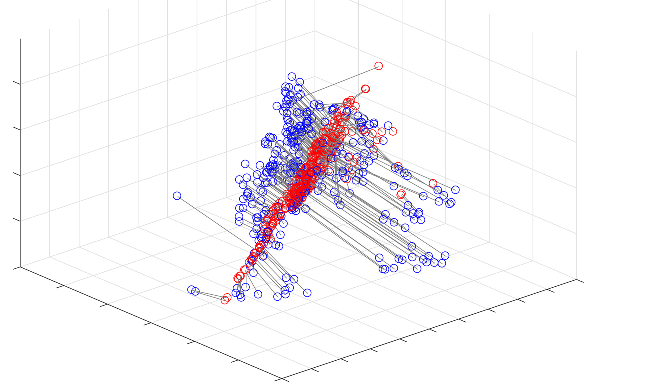
- It assumes the residual vector $\boldsymbol{\varepsilon}(\mathbf{x})$ to be locally linear. i.e. $\boldsymbol{\varepsilon}(\mathbf{x}+\Delta\mathbf{x}) \approx \boldsymbol{\varepsilon}(\mathbf{x}) + \frac{\partial\boldsymbol{\varepsilon}(\mathbf{x})}{\partial\mathbf{x}}$
- Damping (λ) penalizes large updates
  It implicitly decides the trust region radius

inputs: $\mathbf{x} \in \mathbb{R}^n$
$\lambda \leftarrow 10^{-4}$
repeat
  repeat
    $\Delta\mathbf{x} \leftarrow \arg\min_{\Delta\mathbf{x}} \left\|\boldsymbol{\varepsilon}(\mathbf{x}) + \frac{\partial\boldsymbol{\varepsilon}(\mathbf{x})}{\partial\mathbf{x}}\Delta\mathbf{x}\right\|_2^2 + \lambda\|\Delta\mathbf{x}\|_2^2$
    $\lambda \leftarrow 10\lambda$
  until $\|\boldsymbol{\varepsilon}(\mathbf{x}+\Delta\mathbf{x})\|_2^2 < \|\boldsymbol{\varepsilon}(\mathbf{x})\|_2^2$
  $\mathbf{x} \leftarrow \mathbf{x} + \Delta\mathbf{x}$
  $\lambda \leftarrow 0.01\lambda$
until convergence
output: $\mathbf{x} \in \mathbb{R}^n$

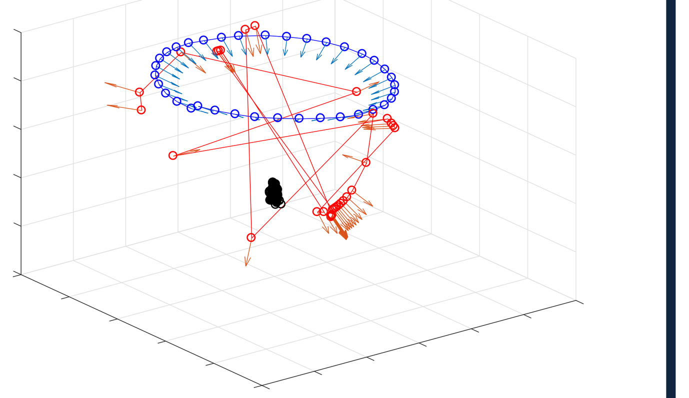
4. VarPro vs Joint optimization on SNLS problems

It is widely known that VarPro (blue) outperforms Joint optimization (red) [3,9,10]

(a) *Trimmed dinosaur* [3,4]



(b) 3D structure



(c) Affine cameras

A. Common misconceptions

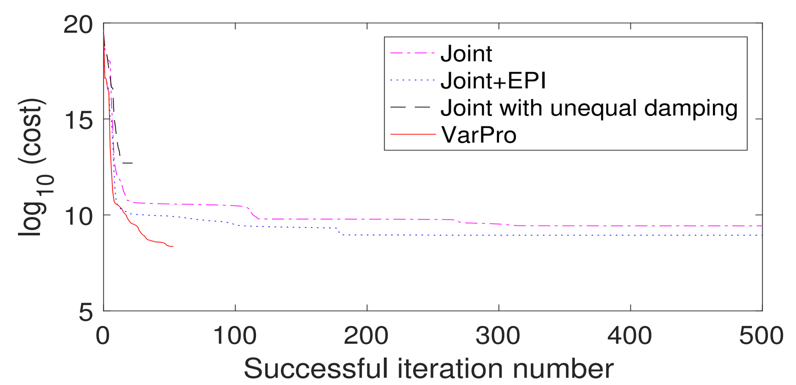
- VarPro has higher iteration complexity than Joint optimization (Joint) and is not scalable
- VarPro = Joint Optimization plus EPI (Joint+EPI) [6]

B. Our comparison strategy and results

- Analysis of the update equations of $\mathbf{u}$ and $\mathbf{v}$ (for the LM subproblem) for each method
- More details in the paper

	$\lambda_{\mathbf{v}} \neq 0$	$\lambda_{\mathbf{v}} = 0$
EPI off	Joint (4%)	(Joint+zero $\lambda_{\mathbf{v}}$) (0%)
EPI on	Joint+EPI (24%)	**VarPro** (94 %)

(d) Comparison of methods on (a)



(e) Iteration plots of methods in (d)

D. Unified pseudocode

Joint	Joint+EPI	VarPro	
			inputs: $\mathbf{u} \in \mathbb{R}^p, \mathbf{v} = \arg\min_{\mathbf{v}} \|\boldsymbol{\varepsilon}(\mathbf{u},\mathbf{v})\|_2^2 \in \mathbb{R}^q$
•	•	•	$[\lambda_{\mathbf{u}} ; \lambda_{\mathbf{v}}] \leftarrow [10^{-4} ; 10^{-4}]$
		•	$\lambda_{\mathbf{v}} \leftarrow 0$
•	•	•	**repeat**
•	•	•	**repeat**
•	•	•	$\mathbf{g} \leftarrow \mathtt{J}_{\mathbf{u}}(\mathbf{u},\mathbf{v})^\top(\mathtt{I} - \mathtt{J}_{\mathbf{v}}(\mathbf{u})\mathtt{J}_{\mathbf{v}}(\mathbf{u})^{-\lambda_{\mathbf{v}}})\boldsymbol{\varepsilon}(\mathbf{u},\mathbf{v})$
•	•	•	$\Delta\mathbf{u} \leftarrow -\left(\mathtt{J}_{\mathbf{u}}(\mathbf{u},\mathbf{v})^\top(\mathtt{I} - \mathtt{J}_{\mathbf{v}}(\mathbf{u})\mathtt{J}_{\mathbf{v}}(\mathbf{u})^{-\lambda_{\mathbf{v}}})\mathtt{J}_{\mathbf{u}}(\mathbf{u},\mathbf{v}) + \lambda_{\mathbf{u}}\mathtt{I}\right)^{-1}\mathbf{g}$
•			$\Delta\mathbf{v} \leftarrow \arg\min_{\Delta\mathbf{v}} \|\boldsymbol{\varepsilon}(\mathbf{u},\mathbf{v}) + \mathtt{J}_{\mathbf{u}}(\mathbf{u},\mathbf{v})\Delta\mathbf{u} + \mathtt{J}_{\mathbf{v}}(\mathbf{u})\Delta\mathbf{v}\|_2^2 + \lambda_{\mathbf{v}}\|\Delta\mathbf{v}\|_2^2$
	•	•	$\Delta\mathbf{v} \leftarrow \arg\min_{\Delta\mathbf{v}} \|\boldsymbol{\varepsilon}(\mathbf{u}+\Delta\mathbf{u},\mathbf{v}) + \mathtt{J}_{\mathbf{v}}(\mathbf{u}+\Delta\mathbf{u})\Delta\mathbf{v}\|_2^2$
•	•	•	$[\lambda_{\mathbf{u}} ; \lambda_{\mathbf{v}}] \leftarrow 10\,[\lambda_{\mathbf{u}} ; \lambda_{\mathbf{v}}]$
•	•	•	**until** $\|\boldsymbol{\varepsilon}(\mathbf{u}+\Delta\mathbf{u},\mathbf{v}+\Delta\mathbf{v})\|_2^2 < \|\boldsymbol{\varepsilon}(\mathbf{u},\mathbf{v})\|_2^2$
•	•	•	$[\mathbf{u} ; \mathbf{v}] \leftarrow [\mathbf{u} ; \mathbf{v}] + [\Delta\mathbf{u} ; \Delta\mathbf{v}]$
•	•	•	$[\lambda_{\mathbf{u}} ; \lambda_{\mathbf{v}}] \leftarrow 0.01\,[\lambda_{\mathbf{u}} ; \lambda_{\mathbf{v}}]$
•	•	•	**until** convergence
			output: $\mathbf{u} \in \mathbb{R}^p, \mathbf{v} \in \mathbb{R}^q$

$$\mathbf{J}_{\mathbf{u}}(\mathbf{u},\mathbf{v}) \coloneqq \frac{\partial\boldsymbol{\varepsilon}(\mathbf{u},\mathbf{v})}{\partial\mathbf{u}}$$

$$\mathbf{J}_{\mathbf{v}}(\mathbf{u}) \coloneqq \frac{\partial\boldsymbol{\varepsilon}(\mathbf{u},\mathbf{v})}{\partial\mathbf{v}} = \mathbf{G}(\mathbf{u})$$

E. Scalable implementation using MINRES

- Iteration complexity of VarPro is roughly equal to that of Joint+EPI
- Any indirect solvers such as Preconditioned Conjugate Gradient (PCG) may be used to solve the LM subproblem of VarPro
- MINRES (similar to PCG) shows better numerical stability and is thus selected

5. Experimental results

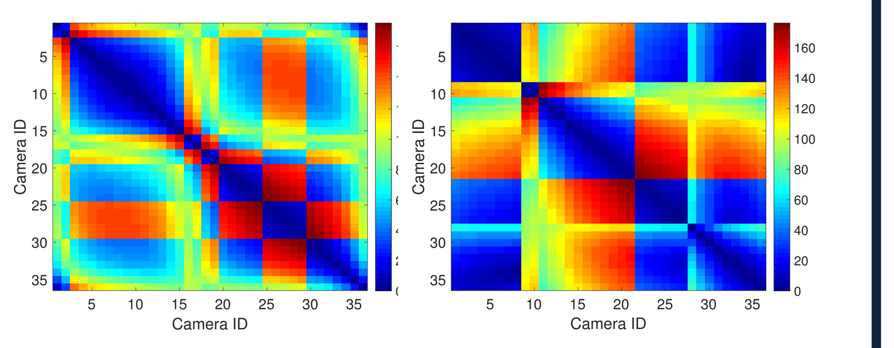
Dataset	f	n	Missing (%)	Joint	Joint+EPI	VarPro	VarPro-MINRES
Blue teddy bear (trimmed)	196	827	80.7	10 (238)	20 (155)	**88 (22.3)**	76 (21.9)
Corridor	11	737	50.2	40 (8.71)	4 (14.8)	100 (1.07)	**100 (0.78)**
Dinosaur (trimmed)	36	319	76.9	4 (5.95)	24 (9.38)	**94 (1.55)**	**99 (3.96)**
Dinosaur including outliers	36	4983	90.8	0 (28.6)	0 (62.1)	**100 (13.9)**	36 (38.9)
House	10	672	57.7	44 (4.90)	8 (9.71)	**100 (0.30)**	**100 (0.41)**
Road scene #47	11	150	47.1	44 (1.88)	32 (3.00)	**100 (0.16)**	**100 (0.17)**
Stockholm Guildhall (trimmed)	43	1000	18.0	92 (45.1)	48 (35.7)	100 (22.8)	**100 (3.12)**
Wilshire	190	411	60.7	38 (409)	94 (9.90)	100 (7.64)	**100 (1.96)**
Ladybug (skeleton)	49	7776	91.6	0 (77.3)	0 (155)	**50 (49.7)**	0 (155)
Trafalgar Square (skeleton)	21	11315	84.7	0 (76.2)	0 (160)	**100 (14.7)**	100 (56.4)
Dubrovnik (skeleton)	16	22106	76.3	38 (159)	0 (346)	**100 (23.6)**	100 (32.9)
Venice (skeleton)	52	64053	89.6	0 (913)	0 (1495)	**80 (123)**	60 (329)

Table 1: Success rates of different methods with median times in parenthesis

- VarPro-MINRES implemented less efficiently in MATLAB
- VarPro implemented using a patched version of Ceres Solver [6]

6. Why does Joint opt. perform poorly on SNLS problems?

- Joint+EPI and VarPro behave similarly unless $\mathbf{v}^*(\mathbf{u})$ (*triangulated 3D points*) is sensitive to changes in $\mathbf{u}$ (*cameras*)
- $\mathbf{G}(\mathbf{u})$ is poorly conditioned in this case
  Near parallel cameras in affine BA
- Damping on the eliminated parameters $\mathbf{v}$ ($\lambda_{\mathbf{v}}$) prevents large updates on $\mathbf{v}$ (*3D points*)
- Conversely, VarPro allows large updates of $\mathbf{v}$, therefore improving the update direction $\Delta\mathbf{u}$



(f) VarPro (94%) (g) Joint+EPI

Angles between pairs of affine camera directions in (c)

7. Conclusion

- Joint optimization and VarPro are algorithmically very similar and can be unified
- The most important difference between Joint optimization and VarPro is the unbalanced trust-region assumption in VarPro
- VarPro can be in principle implemented as efficiently as standard Joint optimization
- VarPro has much higher success rates than Joint optimization in affine BA due to its tolerance to large updates in 3D points when cameras are near parallel

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