Structure of CKMV

Colombi, Kumbhakar, Martini, Vittadini (2014)

$$y_{it} = \beta_0 + x'_{it}\beta + \underbrace{b_i}_{effect} - \underbrace{u_{it}}_{short-run} - \underbrace{u_{i0}}_{long-run} + \underbrace{e_{it}}_{noise} \tag{1}$$

CKMV

$$y_{it} = \beta_0 + x_{it}'\beta + \eta_i + \epsilon_{it}$$

Filippini & Greene (2014)

$$y_{it} = \alpha + \beta' x_{it} + \delta_i + \epsilon_{it}$$

Compromise

$$y_{it} = \alpha + \beta' x_{it} + \underbrace{\eta_i}_{w_i - ?} + \underbrace{\epsilon_{it}}_{v_{it} - u_{it}}$$

Three letter identifiers, (T = TRUE), (F = FALSE)

T/F   random effect
T/F   short-run inefficiency (time-varying)
T/F   long-run inefficiency (persistent)

(Table 2)

TTT   4-components (CKMV)
TFT   TRE (Greene 2005)
FTT   Kumbhakar and Heshmati (1995)
TTF   4-comp. without u_{it} $\rightarrow$ 3 comp.
FFT   Pitt&Lee1981, S&S1984, K1987, BC1988
FTF   pooled SF

Assumptions (A1-A4)

Individuals i=1,2,...,n   Periods t=1,2,...,T

(A1a)

"the two random variables"; label: $b_i - u_{i0} = \eta_i$ and $e_{it} - u_{it} = \epsilon_{it}$ are independent in probability.

Dimensions(!): T+1 (per individual), because $\boldsymbol{\epsilon}$ has dimension T, and η has dimension 1. See this from the fact that "the full unconditional log-likelihood function for this model [is] based on the joint distribution of" the vector $(\epsilon_{i1}, ... \epsilon_{iT}, \eta_i)'$ (Filippini & Greene 2014: 10)

(A1b)

these random vectors (here: over n; (!) label as N) are independent in probability:

$\boldsymbol{b}_i$ $N \times 1$

$\boldsymbol{u}_{i0}$ $N \times 1$

$(\boldsymbol{u}_{i1}, ... \boldsymbol{u}_{iT})$ $N \times T$

$(\boldsymbol{e}_{i1}, ... \boldsymbol{e}_{iT})$ $N \times T$

$$\eta_i = b_i - u_{i0}$$

$$\boldsymbol{\epsilon}_i = \boldsymbol{e}_i - \boldsymbol{u}_i$$

(A2)
$u_{i0} \sim N^{+}(0, \sigma^2_{1u})$
$b_i \sim N(0, \sigma^2_b)$

(A3)
$u_{it} \sim N^{+}(0, \sigma^2_{2u,t})$
$e_{it} \sim N(0, \sigma^2_e)$

(A4)
The $\boldsymbol{x}'_{it}$ are vectors of exogenous variables.

Notation

$\mathbf{1}_T$ $T \times 1$ vector of ones
$\mathbf{0}_T$ $T \times 1$ vector of zeros
$\boldsymbol{I}_T$ $T \times T$ identity matrix

$\boldsymbol{y}_i$ $T \times 1$ vector of output
$\boldsymbol{X}_i$ $T \times p$ matrix of inputs

$\boldsymbol{u}_i$ $(T+1) \times 1$ inefficiency vector, i.e. $(\boldsymbol{u}_{i0}, \boldsymbol{u}_{i1}, ...\boldsymbol{u}_{iT})$
$\boldsymbol{e}_i$ $T \times 1$ vector of noise

Model

$$\boldsymbol{y}_i = \mathbf{1}_T(\beta_0 + b_i) + \boldsymbol{X}_i\boldsymbol{\beta} + \boldsymbol{A}\boldsymbol{u}_i + \boldsymbol{e}_i$$

with $A = -[\mathbf{1}_T \quad \boldsymbol{I}_T]$

Matrix $\mathbf{A}$ is $T \times (T+1)$; e.g. T=3

$$\boldsymbol{A} = -\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}_{3\times 4} = \begin{bmatrix} -1 & -1 & -0 & -0 \\ -1 & -0 & -1 & -0 \\ -1 & -0 & -0 & -1 \end{bmatrix}$$

Section 3: joint density function of the random components

$$\mathbf{1}_T b_i + \boldsymbol{A}\boldsymbol{u}_i + \boldsymbol{e}_i$$

Statistical properties

$\phi_q(\boldsymbol{x}, \boldsymbol{\mu}, \boldsymbol{\Omega})$
pdf of q-dimensional normal RV

$\bar{\Phi}_q(\boldsymbol{\mu}, \boldsymbol{\Omega})$
is the probability that a q-variate normal RV belongs to the positive orthant(?)

CSN definitions
Random vector $\boldsymbol{z}$ has a (o, q) CSN distribution with parameters $\boldsymbol{\mu}, \boldsymbol{\Gamma}, \boldsymbol{D}, \boldsymbol{\nu}, \boldsymbol{\Delta}$ and pdf:

$$f(\boldsymbol{z}, \boldsymbol{\mu}, \boldsymbol{\Gamma}, \boldsymbol{D}, \boldsymbol{\nu}, \boldsymbol{\Delta}, o, q) = \frac{\phi_o(\boldsymbol{z}, \boldsymbol{\mu}, \boldsymbol{\Gamma})\,\bar{\Phi}_q(\boldsymbol{D}(\boldsymbol{z}-\boldsymbol{\mu})-\boldsymbol{\nu}, \boldsymbol{\Delta})}{\bar{\Phi}_q(-\boldsymbol{\nu}, \boldsymbol{D}+\boldsymbol{D}\boldsymbol{\Gamma}\boldsymbol{D}')} \tag{2}$$

Moment-generating function of $\boldsymbol{z}$

$$E\left(exp\{\boldsymbol{t}'\boldsymbol{z}\}\right) = \frac{\bar{\Phi}_q\left(\boldsymbol{D}\Gamma\boldsymbol{t} - \nu, \boldsymbol{\Delta} + \boldsymbol{D}\Gamma\boldsymbol{D}'\right)}{\bar{\Phi}_q\left(-\nu, \boldsymbol{\Delta} + \boldsymbol{D}\Gamma\boldsymbol{D}'\right)} exp\{\boldsymbol{t}'\mu + \frac{1}{2}\boldsymbol{t}'\Gamma\boldsymbol{t}\} \tag{3}$$

CSN in the TTT model

Relevant matrices

$$\boldsymbol{V} = \begin{bmatrix} \sigma^2_{1u} & \mathbf{0}'_T \\ \mathbf{0}_T & \boldsymbol{\Psi} \end{bmatrix}_{(T+1)\times(T+1)}$$

"...where $\boldsymbol{\Psi}$ is the diagonal matrix with the variances $\sigma^2_{2u,t}$ $(t = 1, 2, ..., T)$ on the main diagonal."

$$\boldsymbol{\Psi} = \begin{bmatrix} \sigma^2_{2u,1} & 0 & 0 \\ 0 & \sigma^2_{2u,2} & 0 \\ 0 & 0 & \sigma^2_{2u,3} \end{bmatrix}_{3\times 3}$$

$$\boldsymbol{\Sigma} = \sigma_e^2 \boldsymbol{I}_T + \sigma_b^2 \mathbf{1}_T \mathbf{1}_T'$$

dim $T \times T$

$$\begin{aligned}\boldsymbol{\Lambda} &= \boldsymbol{V} - \boldsymbol{VA}' \left(\boldsymbol{\Sigma} + \boldsymbol{AVA}'\right)^{-1} \boldsymbol{AV} \\ &= \left(\boldsymbol{V}^{-1} + \boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A}\right)^{-1}\end{aligned}$$

dim $(T+1) \times (T+1)$

$$\begin{aligned}\boldsymbol{R} &= \boldsymbol{VA}' \left(\boldsymbol{\Sigma} + \boldsymbol{AVA}'\right)^{-1} \\ &= \boldsymbol{\Lambda A}'\boldsymbol{\Sigma}^{-1}\end{aligned}$$

dim $(T+1) \times (T)$

Examples, T=3

$$\boldsymbol{V} = \begin{bmatrix} \sigma^2_{1u} & 0 & 0 & 0 \\ 0 & \sigma^2_{2u,1} & 0 & 0 \\ 0 & 0 & \sigma^2_{2u,2} & 0 \\ 0 & 0 & 0 & \sigma^2_{2u,3} \end{bmatrix}_{4\times 4}$$

$$\begin{aligned} \boldsymbol{\Sigma} &= \begin{bmatrix} \sigma^2_e & 0 & 0 \\ 0 & \sigma^2_e & 0 \\ 0 & 0 & \sigma^2_e \end{bmatrix} + \begin{bmatrix} \sigma^2_b & \sigma^2_b & \sigma^2_b \\ \sigma^2_b & \sigma^2_b & \sigma^2_b \\ \sigma^2_b & \sigma^2_b & \sigma^2_b \end{bmatrix} \\ &= \begin{bmatrix} \sigma^2_e + \sigma^2_b & \sigma^2_b & \sigma^2_b \\ \sigma^2_b & \sigma^2_e + \sigma^2_b & \sigma^2_b \\ \sigma^2_b & \sigma^2_b & \sigma^2_e + \sigma^2_b \end{bmatrix}_{3\times 3} \end{aligned}$$

$$\boldsymbol{\Lambda} = \left(\begin{bmatrix} \sigma^2_{1u} & 0 & 0 & 0 \\ 0 & \sigma^2_{2u,1} & 0 & 0 \\ 0 & 0 & \sigma^2_{2u,2} & 0 \\ 0 & 0 & 0 & \sigma^2_{2u,3} \end{bmatrix}^{-1} \right.$$

$$\left. + \begin{bmatrix} -1 & -1 & -1 \\ -1 & -0 & -0 \\ -0 & -1 & -0 \\ -0 & -0 & -1 \end{bmatrix} \begin{bmatrix} \sigma^2_e + \sigma^2_b & \sigma^2_b & \sigma^2_b \\ \sigma^2_b & \sigma^2_e + \sigma^2_b & \sigma^2_b \\ \sigma^2_b & \sigma^2_b & \sigma^2_e + \sigma^2_b \end{bmatrix}^{-1} \begin{bmatrix} -1 & -1 & -0 & -0 \\ -1 & -0 & -1 & -0 \\ -1 & -0 & -0 & -1 \end{bmatrix} \right)^{-1}$$

$$\boldsymbol{R} = \boldsymbol{\Lambda} \begin{bmatrix} -1 & -1 & -1 \\ -1 & -0 & -0 \\ -0 & -1 & -0 \\ -0 & -0 & -1 \end{bmatrix} \begin{bmatrix} \sigma^2_e + \sigma^2_b & \sigma^2_b & \sigma^2_b \\ \sigma^2_b & \sigma^2_e + \sigma^2_b & \sigma^2_b \\ \sigma^2_b & \sigma^2_b & \sigma^2_e + \sigma^2_b \end{bmatrix}^{-1}$$

Proposition 1

"...$\boldsymbol{y}_i$ has a $(T, T+1)$ CSN distribution with the parameters:

Location $\quad \boldsymbol{\mu} = \mathbf{1}_T\beta_0 + \boldsymbol{X}_i\boldsymbol{\beta}$

Scale $\quad \boldsymbol{\Gamma} = \boldsymbol{\Sigma} + \boldsymbol{AVA}'$

Skewness $\quad \boldsymbol{D} = \boldsymbol{R}$

Mean in $\bar{\Phi}$ $\quad \boldsymbol{\nu} = 0$

Covariance in $\bar{\Phi}$ $\quad \boldsymbol{\Delta} = \boldsymbol{\Lambda}$

Examples?

Conditional density of $\boldsymbol{y}_i$

$$f(\boldsymbol{y}_i) = \phi_T\left(\boldsymbol{y}_i;\, \mathbf{1}_T\beta_0 + \boldsymbol{X}_i\boldsymbol{\beta},\, \Sigma + \boldsymbol{AVA}'\right) \frac{\bar{\Phi}_{T+1}\left(\boldsymbol{R}\left(\boldsymbol{y}_i - \boldsymbol{X}_i\boldsymbol{\beta} - \mathbf{1}_T\beta_0\right),\, \Lambda\right)}{2^{-(T+1)}}$$

Note that

$$\begin{aligned} \boldsymbol{y}_i - \boldsymbol{X}_i\boldsymbol{\beta} - \mathbf{1}_T\beta_0 &= \mathbf{1}_T b_i + \boldsymbol{Au}_i + \boldsymbol{e}_i \\ &= \eta_i + \boldsymbol{\epsilon}_i \end{aligned}$$

Moment-generating function (dimensions t and k ?)

$$E\left(exp\{\boldsymbol{t}'\boldsymbol{y}_i\}\right) = \frac{\bar{\Phi}_{(T+1)}\left(\boldsymbol{V}^{\frac{1}{2}}\boldsymbol{A}'\boldsymbol{t}, \boldsymbol{I}\right)}{2^{-(T+1)}} exp\{\boldsymbol{t}'(\mathbf{1}_T\beta_0 + \boldsymbol{X}_i\boldsymbol{\beta}) + \frac{1}{2}\boldsymbol{t}'(\boldsymbol{\Sigma} + \boldsymbol{AVA}')\boldsymbol{t}\} \tag{4}$$

shows that these parameters can be identified:

$$-\boldsymbol{AV}^{\frac{1}{2}} = \left[\sigma_{1u}\mathbf{1}_T \quad \boldsymbol{\Psi}^{\frac{1}{2}}\right]$$

$$\boldsymbol{\Sigma} + \boldsymbol{AVA}' = \boldsymbol{\Gamma} \qquad (scale\ pdf)_{TxT}$$

$$\hat{\boldsymbol{y}}_i = \mathbf{1}_T\beta_0 + \boldsymbol{X}_i\boldsymbol{\beta}$$

Proposition 2

Log-likelihood function of NT observations (following from assumption A1b and Proposition 1) of the N independent CSN RVs $(\boldsymbol{y}_i - \boldsymbol{X}_i\beta)$.

$$
\begin{aligned}
ln\, L = {} & N(T+1)\ \ ln\,(2) \\
& + \sum_i \left[ln\, \phi_T \left(\boldsymbol{y}_i - \boldsymbol{X}_i\beta ; \mathbf{1}_T\beta_0, \boldsymbol{\Sigma} + \boldsymbol{AVA}' \right) \right] \\
& + \sum_i \left[ln\, \bar{\Phi}_{T+1} \left(\boldsymbol{R} \left(\boldsymbol{y}_i - \boldsymbol{X}_i\boldsymbol{\beta} - \mathbf{1}_T\beta_0 \right) ; \boldsymbol{\Lambda} \right) \right]
\end{aligned}
\tag{5}
$$

$$
ln\, L = N(T+1)\ \ ln\,(2) + \sum_i \left[ln\, \phi_T \left(\eta_i + \boldsymbol{\epsilon}_i + \mathbf{1}_T\beta_0 ;\ \ \mathbf{1}_T\beta_0, \boldsymbol{\Sigma} + \boldsymbol{AVA}' \right) \right] + \sum_i \left[ln\, \bar{\Phi}_{T+1} \left(\boldsymbol{R} \left(\eta_i + \boldsymbol{\epsilon}_i \right) ;\ \boldsymbol{\Lambda} \right) \right]
$$

PDF

$\phi_q\left(\boldsymbol{x}, \boldsymbol{\mu}, \boldsymbol{\Omega}\right)$

$$ln\,\phi_T(\underbrace{\boldsymbol{y}_i - \boldsymbol{X}_i\boldsymbol{\beta}}_{\boldsymbol{w}_i};\ \ \mathbf{1}_T\beta_0,\ \ \underbrace{\boldsymbol{\Sigma} + \boldsymbol{AVA}'}_{\boldsymbol{\Gamma}})$$

$$\boldsymbol{y}_i - \boldsymbol{X}_i\boldsymbol{\beta} - \mathbf{1}_T\beta_0 = \eta_i + \boldsymbol{\epsilon}_i$$

$$\underbrace{\boldsymbol{y}_i - \boldsymbol{X}_i\boldsymbol{\beta}}_{label\ as\ \boldsymbol{w}_i} = \eta_i + \boldsymbol{\epsilon}_i + \mathbf{1}_T\beta_0$$

Multivariate normal pdf, $p > 1$

$$\phi = f(v) = (2\pi)^{-\frac{p}{2}} \; |\Sigma|^{-\frac{1}{2}} \, exp \left\{ - \left(\frac{1}{2} \right) \; (v - \mu)' \; \Sigma^{-1} \; (v - \mu) \right\}$$

$$ln \, \phi = ln \, f(v) = p \, \left(-\frac{1}{2} \right) \, ln \, (2\pi) - \frac{1}{2} \, ln \, |\Sigma| \, \left\{ - \left(\frac{1}{2} \right) \; (v - \mu)' \; \Sigma^{-1} \; (v - \mu) \right\}$$

log pdf of Proposition 2

$$ln \, \phi_T(\underbrace{\eta_i + \boldsymbol{\epsilon}_i + \mathbf{1}_T \beta_0}_{\textit{label as } \boldsymbol{w}_i}; \;\; \mathbf{1}_T \beta_0, \;\; \mathbf{\Sigma} + \boldsymbol{AVA}')$$

$$ln \, \phi_T(w_i) = \; T \, \left(-\frac{1}{2} \right) \, ln \, (2\pi) - \frac{1}{2} \, ln \, |\mathbf{\Gamma}| \, \left\{ - \left(\frac{1}{2} \right) \; (w_i - \mu)' \; \mathbf{\Gamma}^{-1} \; (w_i - \mu) \right\}$$

Note: $w_i - \mu = (\eta_i + \boldsymbol{\epsilon}_i + \mathbf{1}_T \beta_0) - \mathbf{1}_T \beta_0$

CDF

$$\bar{\Phi}_{T+1}\left(\boldsymbol{R}\left(\boldsymbol{y}_i - \boldsymbol{X}_i\boldsymbol{\beta} - \mathbf{1}_T\beta_0\right); \boldsymbol{\Lambda}\right)$$

$\boldsymbol{R}(\boldsymbol{y}_i - \boldsymbol{X}_i\boldsymbol{\beta} - \mathbf{1}_T\beta_0)$ $\quad ((T+1)\times T)(T\times 1) \rightarrow \quad (T+1)\times 1$

$\boldsymbol{\Lambda}$ $\quad (T+1)\times(T+1)$

$\bar{\Phi}_{T+1}(.)$ $\quad 1\times 1$