Fourier Transform Notes

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1   Introduction

Similar to the concept of the Taylor Series, all functions can be written as sum of cosine and sine functions with different weightings and frequencies.
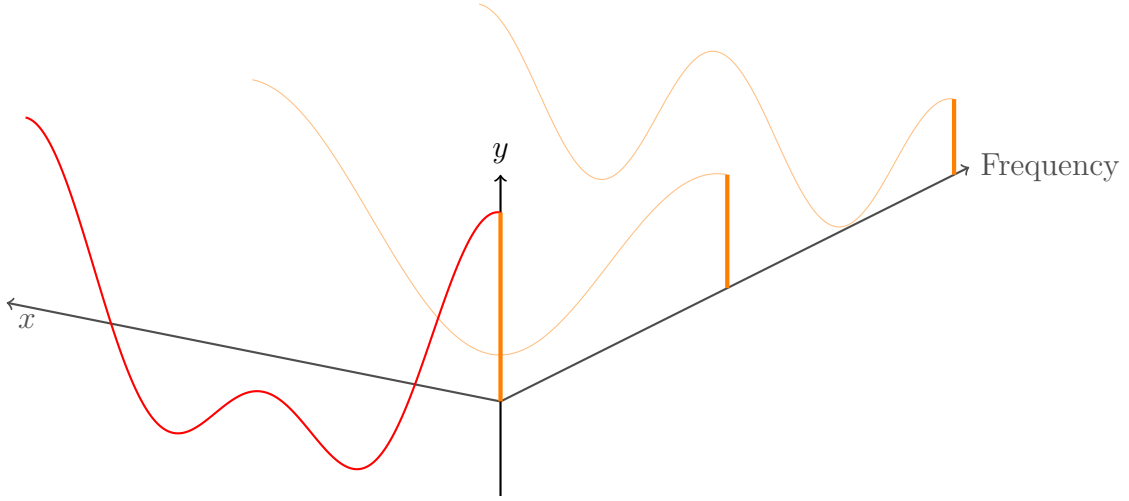

Figure 1: an example of periodic function summation

Figure 1 is a periodic function, and it can be broken down into the sum of two cosine functions $f(x) = 3\cos(x) + 2\cos(2x)$.

This concept of transformation is the fundamental of Fourier Transform as the name suggests, and has great application in many field of study, for example, signal processing. The function of time would be converted into the functions of frequency. It used to be difficult to isolate or manipulate some certain frequencies with the function of time, but we can convert/transform it into the functions of frequency, then do whatever needs to be done. It can be again transformed back into the function of time when needed.

2 Fourier Series

The Fourier Series expresses periodic functions as sums of weighted cosine and sine functions of different frequencies, where the frequencies are integer multiples of the base frequency (frequency of the periodic function). Every non constant term in the Fourier Series are defined as a coefficient multiplied by a cosine or sine function of a frequency.

2.1 Evaluation of Constants and Coefficients

However, most of the time, we only have some information about the function, and we are required to construct the Fourier Series for the function. The major challenge is to find out the coefficients for every frequencies in order to do so.

The general formula of the Trigonometric Fourier Series can be written as

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{2n\pi t}{T}) + \sum_{n=1}^{\infty} b_n \sin(\frac{2n\pi t}{T}) \tag{1}$$

, where T is the period of $f(t)$, and $n \in \mathbb{Z}$.

For a periodic function $f(t)$ with a period of T, we noticed that taking the definite integral of $f(t)$ from the start of a cycle to the end of the cycle then divided by the period (length of a cycle) gives the average value of $f(t)$.

$$\frac{1}{T} \int_T f(t) \mathrm{d}t = \frac{1}{T} \int_T a_0 \mathrm{d}t + \frac{1}{T} \sum_{n=1}^{\infty} a_n \int_T \cos(\frac{2n\pi t}{T}) \mathrm{d}t + \frac{1}{T} \sum_{n=1}^{\infty} b_n \int_T \sin(\frac{2n\pi t}{T}) \mathrm{d}t$$

Evaluate the definite integrals on the right side of the equation:

$$\int_T a_0 \mathrm{d}t = a_0 T$$

$$\int_T \cos(\frac{2n\pi t}{T}) \mathrm{d}t = 0$$

$$\int_T \sin(\frac{2n\pi t}{T}) \mathrm{d}t = 0$$

We sub in the results into the equation and simplify it to get:

$$a_0 = \frac{1}{T} \int_T f(t) \mathrm{d}t \tag{2}$$

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Using this method, we get the average of the periodic function, or in other words the term we evaluated the term a_0. However, we still have to find a way to calculate the rest of the coefficients. Before doing so, we have to know some knowledge about definite integrals of cosine and sine functions. For any constant a_0 and integers m and n where $m \neq n$:

$$\int_{2\pi} a_0 \cos(mx) \mathrm{d}x = 0$$

$$\int_{2\pi} a_0 \sin(mx) \mathrm{d}x = 0$$

$$\int_{2\pi} \cos(mx) \sin(nx) \mathrm{d}x = 0$$

$$\int_{2\pi} \cos(mx) \cos(nx) \mathrm{d}x = 0$$

$$\int_{2\pi} \sin(mx) \sin(nx) \mathrm{d}x = 0$$

$$\int_{2\pi} \cos(mx) \sin(mx) \mathrm{d}x = 0$$

$$\int_{2\pi} \cos^2(mx) \mathrm{d}x = \pi$$

$$\int_{2\pi} \sin^2(mx) \mathrm{d}x = \pi$$

From the above equations, we know that multiplying a cosine or sine function by another cosine or sine function of a different frequency before taking the definite integral does not have a influence on the result. The exception is that when it is multiplied by itself, the result changes from 0 to π. This is useful, we can multiply $f(t)$ by the desired cosine or sine function that contains the desired frequency to get the corresponding coefficient of the term.

$$a_n = \frac{2}{T} \int_T f(t) \cos(\frac{2n\pi t}{T}) \mathrm{d}t \qquad (3)$$

$$b_n = \frac{2}{T} \int_T f(t) \sin(\frac{2n\pi t}{T}) \mathrm{d}t \qquad (4)$$

2.2 Fourier Spectrum

Since a periodic function can be expressed as a Fourier Series, we can use a spectrum to defined a Fourier Series where the x-value would be the frequency of the term instead of time, and the y-value could be the corresponding amplitude/coefficient of the term or the corresponding phase shift of the term.

2.2.1 Single-Sided Spectra

In equation (1), there are four parameters we have to figure out, which is a lot of calculation. We can reduce them, since the sum of cosine and sine function of the same frequency can be expressed in one single term of cosine or sine function using the Sum-to-Product Identities. This would add an extra parameter, phase shift into the equation, however, it removes the amplitude of cosine or sine function, and the corresponding frequency. After combining the cosine terms and the sine terms into only cosine or sine terms, every term in the trigonometric form of a periodic function can be described using three parameters: amplitude, frequency and phase shift. Let ω_0 be the base frequency, and $\omega_0 = \frac{2\pi}{T}$. This can be proved. From (1), we can combine the cosine and sine functions that has the same frequency together into one single term, because a cosine function added to a sine function is a shifted, compressed or stretched cosine or sine function. Let $C_0 = D_0 = a_0$ we obtain the compacted form of the Fourier Series:

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cdot \cos(n\omega_0 t + \phi_n) \tag{5}$$

$$f(t) = D_0 + \sum_{n=1}^{\infty} D_n \cdot \sin(n\omega_0 t + \theta_n) \tag{6}$$

$n\omega_0$ in this equation is the frequency of each term, C_n and D_n are the amplitudes of the corresponding frequency, and ϕ_n is the corresponding phase shift. Both (5) and (6) would work, but (5) is preferred most of the times. We would use (5) in this subsection.

This concept can be proved algebraically.

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cdot \cos(n\omega_0 t + \phi_n)$$

$$= C_0 + \sum_{n=1}^{\infty} C_n \left[\cos(n\omega_0 t)\cos(\phi_n) - \sin(n\omega_0 t)\sin(\phi_n)\right]$$

$$= C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t)\cos(\phi_n) - \sum_{n=1}^{\infty} C_n \sin(n\omega_0 t)\sin(\phi_n)$$

Compare the coefficients of the above equation with (1), we get:

$$\begin{cases} a_0 = C_0 \\ a_n = C_n \cos(\phi_n) \\ b_n = -C_n \sin(\phi_n) \end{cases} \tag{7}$$

Compute $a_n^2 + b_n^2$ using (7) would gives the first equation below, and divide b_n by a_n gives the second equation below, we obtain:

$$\begin{cases} C_0 = a_0 \\ C_n^2 = a_n^2 + b_n^2 \\ \tan(\phi_n) = -\frac{b_n}{a_n} \end{cases} \tag{8}$$

Since we can convert any $a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$ into $C_n \cos(n\omega_0 t + \phi_n)$ using (8), therefore, this concept holds for cosine function. Similar conclusions can be made using the same process with sine function:

$$\begin{cases} D_0 = a_0 \\ D_n^2 = a_n^2 + b_n^2 \\ \tan(\theta_n) = \frac{a_n}{b_n} \end{cases} \tag{9}$$

Extending this concept, for real numbers of k_n, θ and ϕ:

$$k_1 \cos(x) + k_2 \sin(x) + k_3 \cos(x + \phi) + k_4 \sin(x + \theta)$$
$$= k_1 \cos(x) + k_2 \sin(x) + k_3 \cos(\phi)\cos(x) - k_3 \sin(\phi)\sin(x)$$
$$+ k_4 \cos(\phi)\sin(x) + k_4 \sin(\phi)\cos(x)$$
$$= [k_1 + k_3 \cos(\phi) + k_4 \sin(\phi)]\cos(x) + [k_2 - k_3 \sin(\phi) + k_4 \cos(\phi)]\sin(x)$$

Then sub values into (8) and (9) gives the amplitude and phase shift.

After we figure out all of the frequency and their corresponding amplitudes and phase shift, we can create two graphs that describes the Fourier Series: the amplitude spectrum and the phase spectrum. Nevertheless, the spectrum is not continuous due to the frequencies of the terms in the Fourier Series not being continuous. The graphs are only defined at inputs of all whole numbers (zero and all positive integers). Because the Fourier Series only includes positive frequencies.

Example:

The following is an example of the graphs of $f(t) = 4\cos(10\pi t + \frac{\pi}{4}) - 6\cos(20\pi t + \frac{\pi}{6}) + 2\sin(40\pi t + \frac{\pi}{3})$.

$$f(t) = 4\cos(10\pi t + \frac{\pi}{4}) + 6\cos(20\pi t + \frac{7}{6}\pi) + 2\cos(40\pi t - \frac{\pi}{6})$$

In this example, the base frequency, ω_0 is 10π. From the Fourier Series expansion above, we can get the two spectrum:
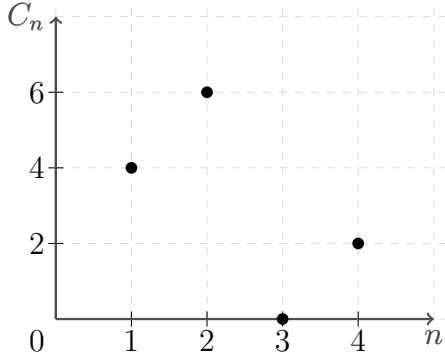

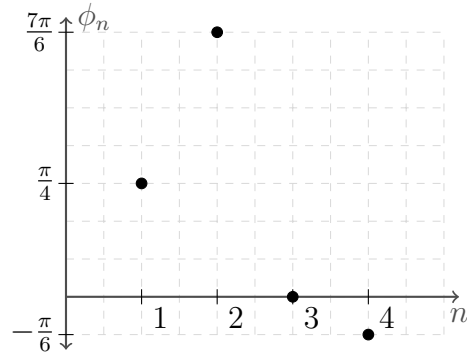Figure 1: the amplitude spectrum

Figure 2: the phase spectrum

2.2.2   Two-Sided Spectra

From Euler's formula $e^{ix} = \cos x + i\sin x$, we can deduce to get:

$$\begin{cases} \cos(n\omega_0 t) = \frac{1}{2}\left[e^{in\omega_0 t} + e^{-in\omega_0 t}\right] \\ \sin(n\omega_0 t) = -\frac{i}{2}\left[e^{in\omega_0 t} - e^{-in\omega_0 t}\right] \end{cases} \tag{10}$$

Substitute (10) into (1), we obtain:

$$
\begin{aligned}
f(t) &= a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2} \left[e^{in\omega_0 t} + e^{-in\omega_0 t} \right] + \sum_{n=1}^{\infty} \frac{b_n}{2i} \left[e^{in\omega_0 t} - e^{-in\omega_0 t} \right] \\
&= a_0 + \sum_{n=1}^{\infty} \left[\frac{a_n - ib_n}{2} e^{in\omega_0 t} + \frac{a_n + ib_n}{2} e^{-in\omega_0 t} \right] \\
&= a_0 + \sum_{n=1}^{\infty} \left[\frac{a_n - ib_n}{2} e^{in\omega_0 t} \right] + \sum_{n=-\infty}^{-1} \left[\frac{a_{-n} + ib_{-n}}{2} e^{in\omega_0 t} \right]
\end{aligned}
$$

By substituting $-n$ into (3) and (4), we obtain:

$$
\begin{cases}
a_n = a_{-n} \\
b_n = -b_{-n}
\end{cases}
\tag{11}
$$

Let $F_n = \frac{a_n - ib_n}{2}$, and $F_0 = a_0$:

$$
\begin{aligned}
f(t) &= a_0 + \sum_{n=1}^{\infty} \left[\frac{a_n - ib_n}{2} e^{in\omega_0 t} \right] + \sum_{n=-\infty}^{-1} \left[\frac{a_n - ib_n}{2} e^{in\omega_0 t} \right] \\
&= F_0 e^{0 in\omega_0 t} + \sum_{n=1}^{\infty} F_n e^{in\omega_0 t} + \sum_{n=-\infty}^{-1} F_n e^{in\omega_0 t}
\end{aligned}
$$

Continue simplifying this equation, we deduced the exponential Fourier Series:

$$
f(t) = \sum_{n=-\infty}^{\infty} F_n e^{in\omega_0 t}
\tag{12}
$$

Expand F_n with (3) and (4):

$$
\begin{aligned}
F_n &= \frac{a_n - ib_n}{2} \\
&= \frac{1}{2} \cdot \frac{2}{T} \int_T f(t) \cos(n\omega_0 t) dt - \frac{i}{2} \cdot \frac{2}{T} \int_T f(t) \sin(n\omega_0 t) dt
\end{aligned}
$$

Apply Euler's formula:

$$
F_n = \frac{1}{T} \int_T f(t) e^{-in\omega_0 t} dt
\tag{13}
$$

We had removed another parameter from the equation, combining the amplitudes and phase shifts information into a single complex number, F_n. The magnitude of F_n describes amplitude, and the argument of it describes phase shifts. Since $F_n = |F_n| e^{i\phi_n}$ is a complex number, we can calculate the magnitude and argument:

$$\begin{cases} |F_0| = C_0 = D_0 \\ |F_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2} = \frac{1}{2}C_n = \frac{1}{2}D_n, n > 0 \\ \tan(\phi_n) = -\frac{b_n}{a_n} = \tan(\phi_n) = -\cot(\theta_n), n > 0 \end{cases}$$

Since our Fourier Series included negative values, we can sub in (11) into the above equation, this suggests that the amplitude, $|F_n|$ is an even function of n, while ϕ_n is an odd function of n.

$$\begin{cases} |F_n| = |F_{-n}| \\ \phi_n = -\phi_{-n} \end{cases}$$

Example:

$$f(t) = 4\cos(10\pi t + \frac{\pi}{4}) - 6\cos(20\pi t + \frac{\pi}{6}) + 2\sin(40\pi t + \frac{\pi}{3})$$

Convert the example from the preceding sub-subsection using (10):

$$\begin{aligned} f(t) =& 4\cos(10\pi t + \frac{\pi}{4}) + 6\cos(20\pi t + \frac{7}{6}\pi) + 2\cos(40\pi t - \frac{\pi}{6}) \\ =& 2e^{i\pi(10t+\frac{1}{4})} + 3e^{i\pi(20t+\frac{7}{6})} + e^{i\pi(40t-\frac{1}{6})} \\ &+ 2e^{-i\pi(10t+\frac{1}{4})} + 3e^{-i\pi(20t+\frac{7}{6})} + e^{-i\pi(40t-\frac{1}{6})} \end{aligned}$$

This also works if $f(t)$ is the sum of sine functions. With the same example, since $\cos(x) = \sin(x - \frac{\pi}{2})$, we can easily convert all of the cosine terms into sine to use the same example. Simply by replacing $-i$ with $e^{\frac{i\pi}{2}}$ would give the exact same result.

$$f(t) = 4\sin(10\pi t + \frac{3\pi}{4}) + 6\sin(20\pi t + \frac{5}{3}\pi) + 2\sin(40\pi t + \frac{\pi}{3})$$
$$= \text{i}2\left(e^{-\text{i}\pi(10t+\frac{3}{4})} - e^{\text{i}\pi(10t+\frac{3}{4})}\right) + \text{i}3\left(e^{-\text{i}\pi(10t+\frac{5}{3})} - e^{\text{i}\pi(20t+\frac{5}{3})}\right) +$$
$$\text{i}\left(e^{-\text{i}\pi(40t+\frac{1}{3})} - e^{\text{i}\pi(40t+\frac{1}{3})}\right)$$
$$= 2\left(e^{\text{i}\pi(10t+\frac{1}{4})} + e^{-\text{i}\pi(10t+\frac{1}{4})}\right) + 3\left(e^{\text{i}\pi(20t+\frac{7}{6})} + e^{-\text{i}\pi(20t+\frac{7}{6})}\right) +$$
$$e^{\text{i}\pi(40t-\frac{1}{6})} + e^{-\text{i}\pi(40t-\frac{1}{6})}$$
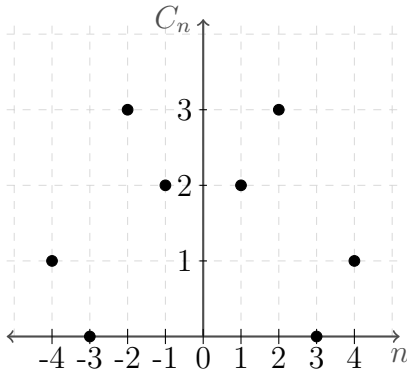

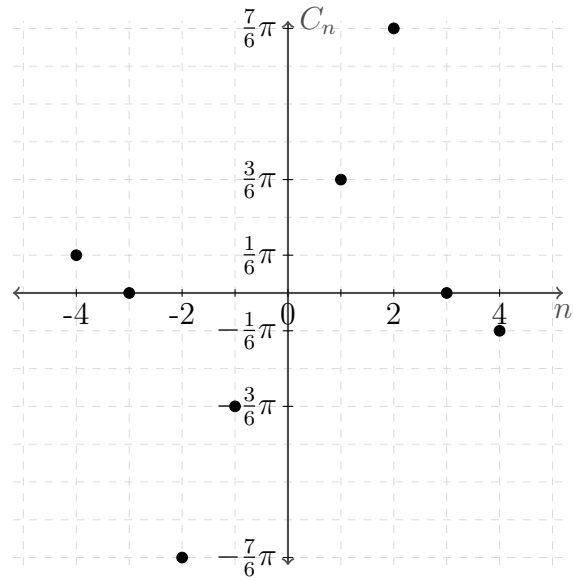Figure 3: the amplitude spectrum

Figure 4: the phase spectrum

The amplitude spectrum is symmetrical over the C_n-axis, while the phase spectrum is a 180° rotation.

Comparing these two graphs with the single-sided spectra, it is obvious that they look nearly identical for $n \geq 0$.

If the phase shift is factored out, the function can be expressed with only

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the amplitudes; however, the amplitude becomes a complex number:

$$f(t) = 2(e^{i\pi(10t+\frac{1}{4})} + e^{-i\pi(10t+\frac{1}{4})}) + 3(e^{i\pi(20t+\frac{7}{6})} + e^{-i\pi(20t+\frac{7}{6})}) +$$
$$e^{i\pi(40t-\frac{1}{6})} + e^{-i\pi(40t-\frac{1}{6})}$$
$$= 2e^{i\frac{\pi}{4}}e^{i10\pi t} + 2e^{-i\frac{\pi}{4}}e^{-i10\pi t} + 3e^{i\frac{7\pi}{6}}e^{i20\pi t} + 3e^{-i\frac{7\pi}{6}}e^{-i20\pi t} +$$
$$e^{-i\frac{\pi}{6}}e^{i40\pi t} + e^{i\frac{\pi}{6}}e^{-i40\pi t}$$

n	-4	-3	-2	-1	0	1	2	3	4
C_n	$e^{\frac{\pi}{6}i}$	0	$3e^{-\frac{7\pi}{6}i}$	$2e^{-\frac{\pi}{4}i}$	0	$2e^{i\frac{\pi}{4}}$	$3e^{\frac{7\pi}{6}i}$	0	$e^{-\frac{\pi}{6}i}$

3 Fourier Transform (Fourier Integral)

For continuous periodic functions, we had showed the transform by using the Fourier series, we are able to describe the continuous periodic function using amplitude and phase shift instead of time. Not only that, the process of conversion was also showed.

Nevertheless, the Fourier series can only deal with periodic functions and it cannot pick out amplitude and phase shift information of continuous frequencies. This is the problem that Fourier transform solves. The Fourier integral solves both problems by setting the cycle to $-\infty$ to ∞, and expand the definition of n to the real number set. The entire function becomes one cycle, resulting in the base frequency, ω_0 to approach 0. Due to the fact that $\frac{1}{T} = \frac{\omega_0}{2\pi}$, and ω_0 equals to the difference between the two nearby frequency($n\omega_0$ and $(n+1)\omega_0$) in the Fourier series.

The equation for the Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} \left[e^{in\omega_0 t} \cdot \frac{\omega_0}{2\pi} \int^{T} f(t)e^{-in\omega_0 t}\mathrm{d}t \right]$$

becomes:

$$f(t) = \lim_{\omega_0 \to 0} \sum_{n=-\infty}^{\infty} \left[e^{in\omega_0 t} \cdot \frac{\omega_0}{2\pi} \int_{-\infty}^{\infty} f(t)e^{-in\omega_0 t}\mathrm{d}t \right]$$

$$= \int_{-\infty}^{\infty} \left[e^{i\omega t} \cdot \frac{\mathrm{d}\omega}{2\pi} \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\mathrm{d}t \right]$$

Re-write the above equation properly:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\mathrm{d}t \right] e^{i\omega t}\mathrm{d}\omega \tag{1}$$

Since the frequency became continuous, so the coefficients (what used to be F_n) became a function of frequency, or:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\mathrm{d}t \tag{2}$$

We get the equation for the inverse Fourier transform by subbing (2) into (1):

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\mathrm{d}\omega \tag{3}$$

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The above equations transform between function of time and function of frequency. Equation (2) is called the Forward Fourier Transform. It transforms the function about time into the function about frequency. On the other hand, equation (3) does the opposite. It is called the Reverse Fourier Transform, and it transforms the function about frequency into the function about time.

Unlike the Fourier Seires, the coefficient $\frac{1}{2\pi}$ which corresponds to $\frac{1}{T}$ in the Fourier Series had become a part of the (3) instead of (2). This difference does not really matter other than scaling the Fourier coefficients by a constant, since we don't really care about the unit of the Fourier coefficients. But it is still important to specify the definition of the equations to avoid any mistakes. We would stick with the ones that are more commonly used.

The process of transformation can be written down as $f(t) \xleftrightarrow{\mathcal{F}} F(\omega)$.

The Continuous Time Fourier Transform allows the transformation of continuous non-periodic functions to be done. With the help of the Fourier Series, picking out frequency information of any continuous functions becomes possible.

4 Dirac Delta Function

Since the function constructed only with the analytical discrete data only has definitions on some points, we can use the Dirac Delta Function:

$$\delta(t) = \begin{cases} +\infty & \text{if } t = 0, \\ 0 & \text{if } t \neq 0. \end{cases}$$

In addition,

$$\int_{-\infty}^{+\infty} \delta(t)\mathrm{d}t = 1$$

Sampling of a signal occurs every time interval, while at times between two sampling occurs, the sampling function of continuous time that we get is 0. This means that the constructed sampling function has a very similar property with impulse response functions. If we have to sample the continuous time function, $f(t)$, we can get a sampling function of continuous time using the Dirac Delta Function (impulse response function), $x[n]$ where n is the time of the sampling occurs:

$$x[n] = f(t)\delta(t - n)$$

Take samples repeatedly with a sampling time gap of T_s, we would get:

$$x_p(t) = \sum_{n=-\infty}^{+\infty} f(t)\delta(t - nT_s)$$

For the function in frequency form, we have to derive a impulse frequency function. We cannot simply use $F(\omega)\delta(\omega - \theta)$, because the frequency is symmetrical. Consider the function

$$p(t) = \cos(\omega_0 t)$$

Then after Fourier transform

$$\mathcal{F}\{p(t)\} = P(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

We now have the impulse frequency function, $P(t)$

$$X_p(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega)P(\omega - \theta)\mathrm{d}\theta$$

For any functions with discrete frequencies, they must be periodic. Because the frequency approaches zero for non-periodic functions, and we could not approach zero with discrete numbers.

5 Discrete Time Fourier Transform

There are times when we have finite amount of points on a function, and we wanted to curve fit the data points with cosine and sine functions. This is the problem that the Discrete Time Fourier Transform (DTFT) and the Discrete Fourier Transform (DFT) solved. The difference between them is that DTFT outputs a continuous function of frequency while DFT outputs a discrete one.

Due to that time is discrete in DTFT, we have to replace the integral with summation:

$$F(\omega) = \sum_{t=-\infty}^{+\infty} f[t]e^{-i\omega t} \tag{1}$$

From (1), we can tell that $F(\omega)$ is a function of sum of cosine and sine functions. Which means that $F(\omega)$ is a periodic function. Compare the fourier series with the DTFT, it again reveals the symmetrical property of the Fourier transform.

On the other hand, frequency is continuous, which we could use the same equation for the inverse transformation as the Fourier transform derived for the continuous Fourier transform, only with a limited amount of changes. Time is discrete, which means that we have to limit/round the input of the function $f(t)$ to integers. To do so, simply replace $f(t)$ with $f[t]$ would be fine. On top of that, due to that $F(\omega)$ is a periodic function, setting any interval of 2π as the limits of the integral does exactly what we wanted it to do (it diverges if we take the integral between the interval $(-\infty, \infty)$). We can get the equation for the inverse discrete time Fourier transform (IDTFT):

$$f[t] = \frac{1}{2\pi} \int_{2\pi} F(\omega)e^{i\omega t}\mathrm{d}\omega \tag{2}$$

6   The Discrete Fourier Transform (DFT)

Similar to the Fourier Series, the frequency is discrete. However, the time is also discrete for DFT. These two properties give an opportunity for the application of fourier transform on computers. Computers can only record state at any moment of time discretely and repeatedly within a time period, this process is called sampling. In addition, it cannot record frequency as a continuous function. Instead, they treat the functions as finite amount of discrete points which can be easily stored and processed.

Since the frequency for the Fourier series is also discrete, we can use the same equation of accumulating all Fourier terms for DFT, which is also the inverse discrete Fourier transform (IDFT):

$$f[j] = \frac{1}{n} \sum_{k=0}^{n-1} F[k] e^{\frac{\mathrm{i}2\pi jk}{n}} \tag{1}$$

where n is the number of data points, and ω becomes $\frac{2k\pi}{n}$.

However, we have to get a equation for the forward transformation. Since time is discrete for DFT, we can simply replace integral in the continuous Fourier transform into a summation:

$$F[k] = \sum_{j=0}^{n-1} f[j] e^{-\frac{\mathrm{i}2\pi jk}{n}} \tag{2}$$

It maybe obvious that the negative frequencies do not appear in (2), which was a result of the introduction of imaginary numbers in the Fourier series. The negative frequencies were used to cancel out the imaginary parts of the results. Keep in mind that $e^{\mathrm{i}(2\pi+\theta)} = e^{\mathrm{i}\theta}$ because cosine and sine functions have a period of 2π, we get:

$$F[n-k] = \sum_{j=0}^{n-1} f[j] e^{-\frac{2j(n-k)\pi i}{n}}$$

$$= \sum_{j=0}^{n-1} f[j] e^{-ij(2\pi - \frac{2k\pi}{n})}$$

$$= \sum_{j=0}^{n-1} f[j] e^{\frac{i2\pi jk}{n}}$$

$$= F[-k]$$

The above steps shows that frequencies from $\frac{n}{2}$ to n takes the job of negative frequencies. All the imaginary numbers cancel out at the end if the inputs were all real numbers.

We have a vector of collected data, f_n, and a vector of Fourier coefficients F_n. The collected multiplied by the Discrete Fourier Transformation matrix would give the Fourier coefficients. Rewrite the above equation for the forward transform gives:

$$\begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & W_n & W_n^2 & \dots & W_n^n \\ 1 & W_n^2 & W_n^4 & \dots & W_n^{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_n^{n-1} & W_n^{2(n-1)} & \dots & W_n^{(n-1)^2} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{bmatrix} \tag{3}$$

where $W_n = e^{-\frac{2\pi i}{n}}$.

6.1   The Fast Fourier Transform (FFT)

To compute all of the Fourier coefficients is quite expensive. The time complexity for computing such matrix multiplication is $\mathcal{O}(n^2)$. The algorithm FFT reduces the cost significantly, improves the time complexity to $\mathcal{O}(n \log n)$.

The derivation of FFT starts off at finding any n points on a polynomial with a degree of $n-1$, because given any n points, we can find the coefficients of a polynomial with a degree of $n-1$.

$$P(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n$$

Separate the polynomial into odd and even parts:

$$P(x) = P_e(x^2) + x P_o(x^2)$$

where $P_e(x) = p_0 + p_2 x + p_4 x^2 + \ldots$, and $P_o(x) = p_1 + p_3 x + p_5 x^2 + \ldots$.

We can utilize the property that odd functions satisfies $f_o(-x) = -f_o(x)$, and even functions satisfies $f_e(-x) = f_e(x)$ to reduce the amount of calculation needed by a half. $P_e(x^2)$ and $P_o(x^2)$ have a line of symmetrical at $x = 0$, which means that if we know $P_e(x_i^2)$ and $P_o(x_i^2)$, we also know the value of $P_o((-x_i)^2)$ and $P_e((-x_i)^2)$ without doing any calculations. By carefully selecting the points to be positive-negative pairs, we successfully reduced the amount of calculation by a half. In comparison, values of $P(x)$ have to be evaluated at $x = \pm x_1, \pm x_2, \pm x_3, \ldots, \pm x_n$ without applying this method. But now, we only have to evaluate values of $P_o(x)$ and $P_e(x)$ at $x = x_1^2, x_2^2, x_3^2, \ldots, x_{\frac{n}{2}}^2$.

$$\begin{cases} P(x_i) = P_e(x_i^2) + x_i P_o(x_i^2) \\ P(-x_i) = P_e(x_i^2) - x_i P_o(x_i^2) \end{cases}$$

We can continue to apply this method to continue reduce the amount of calculations required. Noticed that $P_o(x)$ and $P_e(x)$ are also a polynomial function of their own, which means that we can recursively apply this method to further reduce the work.

Here comes the problem, numbers $[x_1^2, x_2^2, x_3^2, \ldots, x_{\frac{n}{2}}^2]$ have to be positive-negative pairs. This could not be done without the use of imaginary numbers. If we let $\omega_n = e^{\frac{2\pi i}{n}}$, where n is powers of 2 for easier calculations. and choose the points as $x = \omega_n^0, \omega_n^1, \omega_n^2, \ldots, \omega_n^{n-1}$, the problem would be solved. The reason is that $\omega_n^{j+\frac{n}{2}} = -\omega_n^j$, which means that $(\omega_n^j, \omega_n^{j+\frac{n}{2}})$ are positive-negative pairs.

The problem is now simplified. The essential part of FFT is to choose the points to be positive-negative pairs to take advantage of the symmetrical

properties. The problem that we solved now is to quickly calculate the values of $P(\omega_n^i)$ in the matrix below.

$$
\begin{bmatrix}
P(\omega_n^0) \\
P(\omega_n^1) \\
P(\omega_n^2) \\
\vdots \\
P(\omega_n^{n-1})
\end{bmatrix}
=
\begin{bmatrix}
1 & 1 & 1 & \dots & 1 \\
1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\
1 & \omega_n^2 & \omega_n^4 & \dots & \omega_n^{2(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)^2}
\end{bmatrix}
\begin{bmatrix}
p_0 \\
p_1 \\
p_2 \\
\vdots \\
p_{n-1}
\end{bmatrix}
$$

where $\omega_n = e^{\frac{2\pi i}{n}}$.

There is a lot of similarities compared to the DFT, the only difference is that the exponent of the values in the transformation matrix is positive in the above equation, while it is negative in the forward DFT. The concept still applies.

6.1.1 Choice of n

Every time the FFT is recursively called, the inputs are squared. $P(x)$ becomes $P_e(x^2) + x_o^P(x^2)$. Continue recusing, $P_e(x^2) \to P_{ee}(x^4) + x^2 P_{eo}(x^4)$, $P_{ee}(x^4) \to P_{eee}(x^8) + x^4 P_{eeo}(x^8)$, so on and so forth. The exponents of the inputs are increasing in powers of 2. For a polynomial of degree n, then $n+1$ points needed to be calculated, and $\lceil \log_2(n) \rceil$ times of recursion has to take place. This means that the input x would be eventually raised to $x^{(2^{\lceil \log_2(n) \rceil})}$. Sub in $x = e^{\frac{2\pi i}{n}}$, the exponential part cancels out very nicely if $n = 2^{\lceil \log_2(n) \rceil}$, or by other words, it would be much easier to deal with if n is powers of 2. At the end of the recursion, all we have to calculate is $P_{...}(1)$ instead of a complex number as the input.

If $P(x)$ is spitted into 3 terms instead of 2, then with a similar process, we can derive that it would work best if $n = 3^{\lceil \log_3(n) \rceil}$.

7 Laplace Transform

The Fourier transform only works with functions that do not diverge to infinity as the inputs approach infinity. Which means that it does not work with functions such as exponential or polynomials. The Laplace Transform is a generalized transformation of the Fourier transform. It can transform functions that increase as rapidly as exponential functions, and functions that grow slower. It expanded the set of functions that the transformation can be applied. However, it does not included functions that increase faster than exponential functions, such as the Gamma function, or double exponential functions ($f(x) = a^{(b^x)}$), etc... Another limitation to the use of the Laplace transform is that the function is defined for only positive values of input ($t \geq 0$).

Let's say the function, $f(t)$, that we wanted to transform is bounded by e^{bt}, where a is a positive constant. We can multiply this function by a term which decays faster than e^{bt}. That term can be e^{-at}, where $b \geq a$.

$$g(t) = f(t)e^{-at}$$

We now have a function that converges as time approaches infinity. If time is negative, it does not converges, but we do not need to consider negative values of time in most cases. We can apply Fourier transform to this new function. Let $s = a + i\omega$:

$$G(s) = \int_0^\infty f(t)e^{-at}e^{-i\omega t}\mathrm{d}t$$
$$= \int_0^\infty f(t)e^{-st}\mathrm{d}t$$

Even though time has to be greater than 0, the frequency can still be negative. By substituting $s = a + \omega i$, the inverse transform is:

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^\infty G(s)e^{i\omega t}\mathrm{d}\omega$$
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^\infty G(s)e^{t(a+i\omega)}\mathrm{d}\omega$$
$$f(t) = \frac{1}{2\pi i}\int_{a-\infty}^{a+\infty} G(s)e^{st}\mathrm{d}s$$

Which can be written as:

$$\begin{cases} \mathcal{L}\{f(t)\} = F(s) = \displaystyle\int_0^\infty f(t)e^{-st}\mathrm{d}t \\ \mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi\mathrm{i}} \displaystyle\int_{a-\infty}^{a+\infty} F(s)e^{st}\mathrm{d}s \end{cases} \tag{1}$$

8 Properties of Laplace Transform

Since the Fourier transform is a special case of the Laplace transform where $a = 0$, the properties of the Laplace transform can be found on the Fourier transform.

8.1 Linearity

The first property is that the Laplace Transform is linear, which means that if a and b are constants:

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} \tag{1}$$

The property of linearity also holds for the inverse Laplace transform.

$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\} \tag{2}$$

This holds because integration is a linear process.

8.2 Laplace Transform of Derivatives

The property about differentiation is that

$$\mathcal{L}\{f^{(n)}(t)\} = \int_0^\infty f^{(n)}(t)e^{-st}\mathrm{d}t$$

$$= \left[f^{(n-1)}(t)e^{-st}\right]_{t=0}^{t=\infty} + s\int_0^\infty f^{(n-1)}(t)e^{-st}\mathrm{d}t$$

Since e^{st} grows faster than any orders of derivative of $f(t)$, then:

$$\lim_{t\to\infty} f^{(n-1)}(t)e^{-st} = 0$$

hence:

$$\mathcal{L}\{f^{(n)}(t)\} = f^{(n-1)}(0) + s\mathcal{L}\{f^{(n-1)}(t)\}$$
$$= f^{(n-1)}(0) + sf^{(n-2)}(0) + s^2\mathcal{L}\{f^{(n-2)}(t)\}$$
$$= f^{(n-1)}(0) + sf^{(n-2)}(0) + s^2 f^{(n-3)}(0)$$
$$+ \cdots + s^{n-1}f(0) + s^n\mathcal{L}\{f(t)\}$$

Hence, we can generalized the Laplace transform of derivatives:

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} + \sum_{m=1}^{n} s^{m-1} f^{(n-m)}(0) \tag{3}$$

All terms other than $s^n \mathcal{L}\{f(t)\}$ does not contain the variable t, we have successfully removed all derivatives in respect of t from the expression. This is useful in the solutions of differential equations.

With the same process, we can obtain the inverse Laplace transform of derivatives:

$$
\begin{aligned}
\mathcal{L}^{-1}\{F^{(n)}(s)\} =& \frac{1}{2\pi\mathrm{i}} \int_{a-\infty}^{a+\infty} F^{(n)}(s) e^{st} \mathrm{d}s \\
=& \frac{1}{2\pi\mathrm{i}} \left[F^{(n-1)}(s) e^{st} \right]_{a-\infty}^{s=a+\infty} - \frac{t}{2\pi\mathrm{i}} \int_{a-\infty}^{a+\infty} F^{(n-1)}(s) e^{st} \mathrm{d}s \\
=& \frac{1}{2\pi\mathrm{i}} \left[F^{(n-1)}(s) e^{st} \right]_{a-\infty}^{s=a+\infty} - t\mathcal{L}^{-1}\{F^{(n-1)}(s)\} \\
=& \frac{1}{2\pi\mathrm{i}} \left[F^{(n-1)}(s) e^{st} \right]_{a-\infty}^{s=a+\infty} - \frac{t}{2\pi\mathrm{i}} \left[F^{(n-2)}(s) e^{st} \right]_{a-\infty}^{s=a+\infty} \\
& + \frac{t^2}{2\pi\mathrm{i}} \left[F^{(n-3)}(s) e^{st} \right]_{a-\infty}^{s=a+\infty} + \cdots + \frac{(-t)^{n-1}}{2\pi\mathrm{i}} \left[F(s) e^{st} \right]_{a-\infty}^{s=a+\infty} \\
& + (-t)^n \mathcal{L}^{-1}\{F(s)\}
\end{aligned}
$$

The inverse Laplace transform of derivatives can be generalized as the following:

$$\mathcal{L}^{-1}\{F^{(n)}(s)\} = (-t)^n \mathcal{L}^{-1}\{F(s)\} + \sum_{m=1}^{n} \frac{(-t)^{m-1}}{2\pi\mathrm{i}} \left[F^{(n-m)}(s) e^{st} \right]_{a-\infty}^{s=a+\infty} \tag{4}$$

8.3   The Convolution Theorem

The convolution of two function $f(t)$ and $g(t)$ is written as $(f * g)(t)$, and convolution is defined as:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) \mathrm{d}\tau \tag{5}$$

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This is also called the convolution integral. By substituting $u = t - \tau$, the commutativity property of convolution.

$$f * g = g * f \tag{6}$$

If the function that $f(t)$ is convolution with is the Delta Distribution $\delta(t)$, then:

$$f * \delta = f \tag{7}$$

The convolution are very connected with the Laplace transform. Let $t = x + v$:

$$F(s)G(s) = \left[\int_0^\infty f(x)e^{-sx}\mathrm{d}x\right]\left[\int_0^\infty g(v)e^{-sv}\mathrm{d}v\right]$$

$$= \int_0^\infty \int_v^\infty e^{-st}f(t-v)g(v)\mathrm{d}t\mathrm{d}v$$

$$= \int_0^\infty e^{-st} \int_0^t f(t-v)g(v)\mathrm{d}v\mathrm{d}t$$

$$F(s)G(s) = \mathcal{L}\{(f * g)(t)\}$$

Therefore,

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$$

A similar conclusion can be drawn with the same process:

$$F(s) * G(s) = \mathcal{L}\{f(t) \cdot g(t)\}$$

$$\mathcal{L}^{-1}\{F(s) * G(s)\} = f(t) \cdot g(t)$$

9 Gabor Transform

The Laplace transform does not give any information about every frequencies at any given specific moment of time. In order to do so, we can multiply the cosine and sine functions by a Gaussian function $(e^{-\pi x^2})$ to get the Gabor transform.

With that being said, the forward Gabor transform can be defined as:

$$F(s, \tau) = \int_0^\infty f(t)e^{-st}e^{-\pi(t-\tau)^2}\mathrm{d}t \tag{1}$$

and the inverse Gabor transform can be defined as:

$$f(t) = \frac{1}{2\pi\mathrm{i}} \int_0^\infty e^{\pi(t-\tau)^2} \left[\int_0^\infty F(t)e^{st}\mathrm{d}t\right] \mathrm{d}\tau \tag{2}$$