Supplementary Material for *Optimization Landscape of Policy Gradient Methods for Discrete-time Static Output Feedback*

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S.I. Proof of Lemma 6

The performance difference lemma, also known as almost smoothness, is the basis for deriving the gradient domination condition.

Lemma 1 (Performance difference lemma)**.** Suppose K, $K' \in \mathbb{K}$. It holds that:

$$\begin{aligned}J(K') - J(K) = 2\mathrm{Tr}\big(\Sigma_{K'}(K'C-KC)^\top E_K\big)+\\ + \mathrm{Tr}\big(\Sigma_{K'}(K'C-KC)^\top(R+B^\top P_K B)(K'C-KC)\big).\end{aligned}$$

Proof. Let x'_t and u'_t be the state and action sequences generated by K', and $c'_t = x_t'^\top Q x'_t + u_t'^\top R u'_t$. Then, one has

$$\begin{aligned}&J(K')-J(K)\\ &\quad=\mathbb{E}_{x_0\sim\mathcal{D}}\Big[\sum_{t=0}^{\infty} c'_t - V_K(x_0)\Big]\\ &\quad=\mathbb{E}_{x_0\sim\mathcal{D}}\Big[\sum_{t=0}^{\infty} (c'_t + V_K(x'_t) - V_K(x'_t)) - V_K(x_0)\Big]\\ &\quad=\mathbb{E}_{x_0\sim\mathcal{D}}\Big[\sum_{t=0}^{\infty} (c'_t + V_K(x'_{t+1}) - V_K(x'_t))\Big],\end{aligned}$$

where the last step utilizes the fact that $x_0 = x'_0$.

Let $A_K(x_t, K') = c_t + V_K(x_{t+1}) - V_K(x_t)|_{u_t=-K'Cx_t}$, which can be expanded as

$$\begin{aligned}&A_K(x_t,K')\\ &=x_t^\top(Q+C^\top K'^\top R K' C)x_t + x_t^\top \mathcal{A}_{K'}^\top P_K \mathcal{A}_{K'} x_t - V_K(x_t)\\ &=x_t^\top(Q+(K'C-KC+KC)^\top R(K'C-KC+KC))x_t\\ &\quad + x_t^\top(A-B(K'C-KC+KC))^\top P_K(A\\ &\qquad\qquad - B(K'C-KC+KC))x_t - V_K(x_t)\\ &=2x_t^\top(K'C-KC)^\top((R+B^\top P_K B)KC - B^\top P_K A)x_t\\ &\quad + x_t^\top(K'C-KC)^\top(R+B^\top P_K B)(K'C-KC)x_t\\ &=2x_t^\top(K'C-KC)^\top E_K x_t\\ &\quad + x_t^\top(K'C-KC)^\top(R+B^\top P_K B)(K'C-KC)x_t.\end{aligned}$$

Then, we get that

$$\begin{aligned}&J(K')-J(K)\\ &=\mathbb{E}_{x_0\sim\mathcal{D}}\Big[\sum_{t=0}^{\infty} A_K(x'_t,K')\Big]\\ &=\mathbb{E}_{x_0\sim\mathcal{D}}\Big[\sum_{t=0}^{\infty}\Big(2\mathrm{Tr}\big(x'_t x_t'^\top(K'C-KC)^\top E_K\big)+\\ &\quad\mathrm{Tr}\big(x'_t x_t'^\top(K'C-KC)^\top(R+B^\top P_K B)(K'C-KC)\big)\Big)\Big]\\ &=2\mathrm{Tr}\big(\Sigma_{K'}(K'C-KC)^\top E_K\big)+\\ &\quad\mathrm{Tr}\big(\Sigma_{K'}(K'C-KC)^\top(R+B^\top P_K B)(K'C-KC)\big).\end{aligned}$$

□

Next, we show the main proof of Lemma 6.

Proof. Let $X = (R+B^\top P_K B)^{-1}E_K\Sigma_{K'}C^\top\mathcal{L}_{K'}^{-1}$. From Lemma 1, we find that

$$\begin{aligned}&J(K')-J(K)\\ &=2\mathrm{Tr}\big(\Sigma_{K'}(K'C-KC)^\top E_K\big)\\ &\quad+\mathrm{Tr}\big(\Sigma_{K'}(K'C-KC)^\top(R+B^\top P_K B)(K'C-KC)\big)\\ &=\mathrm{Tr}\big(\Sigma_{K'}C^\top(\Delta K+X)^\top(R+B^\top P_K B)(\Delta K+X)C\big)\\ &\quad-\mathrm{Tr}\big(\Sigma_{K'}C^\top\mathcal{L}_{K'}^{-1}C\Sigma_{K'}E_K^\top(R+\\ &\qquad\qquad B^\top P_K B)^{-1}E_K\Sigma_{K'}C^\top\mathcal{L}_{K'}^{-1}C\big)\\ &\geq-\mathrm{Tr}\big(\mathcal{L}_{K'}^{-1}C\Sigma_{K'}E_K^\top(R+B^\top P_K B)^{-1}E_K\Sigma_{K'}C^\top\big),\end{aligned}\tag{S1}$$

where $\Delta K = K' - K$ and the equality holds when $K' = K - X$.

Then, one has

$$\begin{aligned}&J(K)-J(K^*)\\ &\leq\mathrm{Tr}\big(\mathcal{L}_{K^*}^{-1}C\Sigma_{K^*}E_K^\top(R+B^\top P_K B)^{-1}E_K\Sigma_{K^*}C^\top\big)\\ &\leq\|\Sigma_{K^*}C^\top\mathcal{L}_{K^*}^{-1}C\Sigma_{K^*}\|\mathrm{Tr}\big(E_K^\top(R+B^\top P_K B)^{-1}E_K\big)\\ &\leq\|\Sigma_{K^*}C^\top\mathcal{L}_{K^*}^{-1}C\|\|\Sigma_{K^*}\|\mathrm{Tr}\big(E_K^\top(R+B^\top P_K B)^{-1}E_K\big)\\ &\leq\|\Sigma_{K^*}\|\mathrm{Tr}\big(E_K^\top(R+B^\top P_K B)^{-1}E_K\big)\\ &\leq\frac{\|\Sigma_{K^*}\|\mathrm{Tr}\big(E_K^\top E_K\big)}{\sigma_{\min}(R)}.\end{aligned}\tag{S2}$$

From (15), it follows that

$$\begin{aligned}\|\nabla J(K))\|_F^2 &= 4\mathrm{Tr}(C\Sigma_K E_K^\top E_K\Sigma_K C^\top)\\ &\geq 4\mu^2\sigma_{\min}(C)^2\mathrm{Tr}(E_K^\top E_K),\ \forall C\in\mathbb{C}.\end{aligned}\tag{S3}$$

By (S2) and (S3), it holds that

$$J(K) - J(K^*) \leq \frac{\|\Sigma_{K^*}\| \|\nabla J(K))\|_F^2}{4\mu^2 \sigma_{\min}(C)^2 \sigma_{\min}(R)}, \ \forall C \in \mathbb{C}. \quad \text{(S4)}$$

Suppose K' satisfies that $K' = K - X$. According to (S1), we get

$$\begin{aligned} & J(K) - J(K^*) \\ & \quad \geq J(K) - J(K') \\ & \quad = \mathrm{Tr}\big(\mathcal{L}_{K'}^{-1} C \Sigma_{K'} E_K^\top (R + B^\top P_K B)^{-1} E_K \Sigma_{K'} C^\top\big) \\ & \quad \geq \frac{\mu \mathrm{Tr}\big(E_K^\top E_K\big)}{\|R + B^\top P_K B\|}, \ \forall C \in \mathbb{C}. \end{aligned} \quad \text{(S5)}$$

In addition, when $C \in \mathbb{C}$, since we can always identity the state x by $x = (C^\top C)^{-1} C y$, it is clear that $J(K^*) = J_s^*$ for every $C \in \mathbb{C}$. By replacing $J(K^*)$ in (S4) and (S5) with J_s^*, we finally complete the proof. □

S.II. Example: Unstable System with Randomly Initial Controller

For the internally unstable system mentioned in Section VI-C, the stability of the controller can be judged by the spectral radius of the closed-loop system matrix, where model matrices are required. However, stabilizing controllers are relatively difficult to find through model-free methods. Trial and error method can be a feasible scheme to obtain the initial stabilizing controller in model-free case. When applying a controller to the dynamic system, the convergence or divergence of the observation output provides a criterion for the stability of the closed-loop system.

Through these methods, the set of stabilizing controllers $\mathbb{K} = \{K = k : k \in (2.1, 22.05)\}$ for the internally unstable system can be determined. Run all policy gradient methods 10 times with randomly generated initial controllers. The relative errors of control gains are shown in Fig. S1. It can be seen that all policy gradient methods converge within 100 iterations under different initial controllers. Our theoretical convergence results are demonstrated in the internally unstable case with randomly initial controllers.

S.III. Example: Four-dimensional Stable System

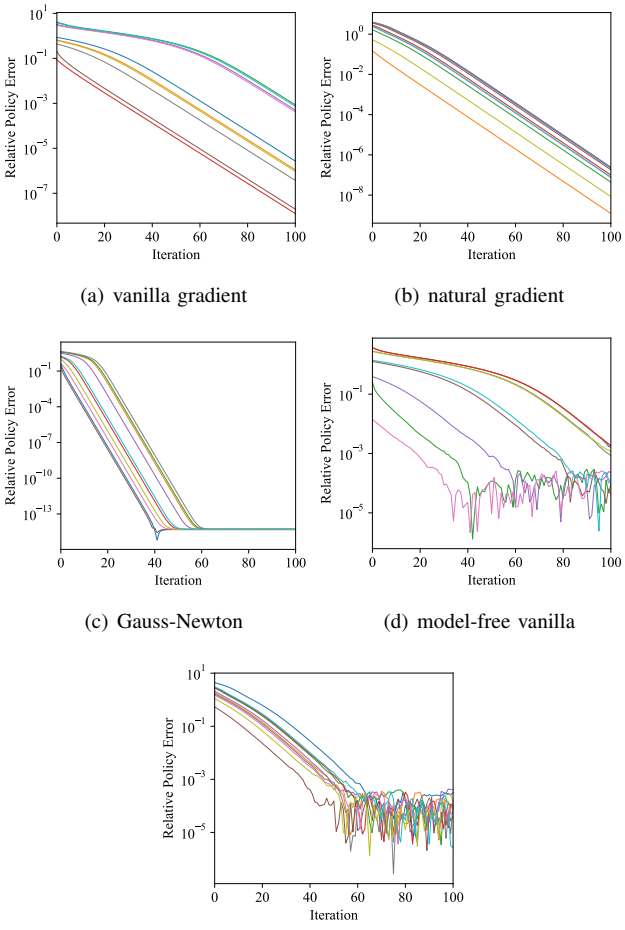
Consider a circuit system given in [*S1*] with

$$A = \begin{bmatrix} 0.90031 & -0.00015 & 0.09048 & -0.00452 \\ -0.00015 & 0.90031 & 0.00452 & -0.09048 \\ -0.09048 & -0.00452 & 0.90483 & -0.09033 \\ 0.00452 & 0.09048 & -0.09033 & 0.90483 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.00468 & -0.00015 \\ 0.00015 & -0.00468 \\ 0.09516 & -0.00467 \\ -0.00467 & 0.09516 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

where $Q = \mathrm{diag}([0.1, 0.2, 0, 0])$, $R = \mathrm{diag}([10^{-6}, 10^{-4}])$, and $X_0 = I_4$. According to [*S2*, Theorem 1] , the optimal gain is

$$K^* = \begin{bmatrix} 2.9738 & -7.2907 \\ 2.1067 & -12.5384 \end{bmatrix}.$$

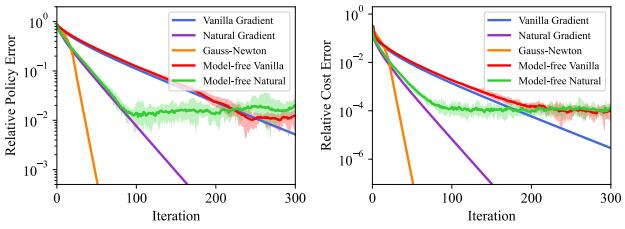


(a) vanilla gradient (b) natural gradient

(c) Gauss-Newton (d) model-free vanilla

(e) model-free natural

Fig. S1. Learning curves of different methods with randomly initial controllers. Each solid line corresponds to one simulation result, and each method is run 10 times.



(a) policy error (b) cost error

Fig. S2. Learning curves of different methods. The solid lines correspond to the mean and the shaded regions correspond to interval between maximum and minimum values over 10 runs.

We set $K_0 = \begin{bmatrix} 0 & -1 \\ 0 & -2 \end{bmatrix}$ for all methods and adopt the same hyperparameters as in Section VI-B. The relative errors of the control gain and the cost function of different methods are shown in Fig. S2. The observed trend of this example is quite similar to the example given in Section VI-B. Overall, the numerical results are consistent with our convergence analysis.

REFERENCES

[S1] F. Lewis, *Applied Optimal Control & Estimation: Digital Design & Implementation*. Prentice Hall, 1992.

[S2] J. te Yu, "An equivalent discrete-time output feedback linear quadratic regulator theory," *2020 7th International Conference on Control, Decision and Information Technologies (CoDIT)*, vol. 1, pp. 868–873, 2020.