Homework 4

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(10 points) Show that every CNF φ can be translated in polynomial time into an equisatisfiable CNF ψ which is a conjunction of a 2-CNF and a Horn formula. How many additional variables are needed?

Let T denote the translation from φ to ψ. First, we do few observations:

- Any variable in φ with no positive or with no negative literal occurrence trivially satisfies the clauses it appears in. The same applies to all clauses which contain a positive and a negative literal occurrence of a certain variable.
- Any clause C in φ with less than two positive literals is already a Horn clause and we can simplify the translation from $T(\varphi) = \psi$ to $T(\varphi') \wedge C = \psi$, where φ' is φ without the clause C.

So, without loss of generality, we can assume φ consists of clauses such that each has at least two positive literals, and that every variable has a positive and a negative occurrence and never in the same clause.

Then we will describe the algorithm for $T(\varphi)$ as follows:

function $T(\varphi)$
    $\varphi' \leftarrow \top$
    while $\varphi \notin$ **Horn do**
        $x \leftarrow$ a variable with most positive occurrences in non-Horn clauses
        $l \leftarrow \text{Lit } x$
        $x' \leftarrow$ fresh variable
        $\varphi' \leftarrow \varphi' \wedge (x \vee x') \wedge (\neg x \vee \neg x')$
        $\varphi \leftarrow \varphi[l := \neg x']$ $\triangleright$ rewrite positive occurrences of x with $\neg x'$
    end while
    return $\varphi \wedge \varphi'$
end function

The needed number of fresh variables is $\leq |\mathbf{x}|$ (if we consider only the variables with positive occurrences in clauses with at least two positive literals). We could achieve a slightly higher number with tseitin transformation of the maximal positive subsets of clauses.