Homework 3

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(10 points) One of the preprocessing steps can be to eliminate some of the variables using so-called DP-elimination (or DP-resolution). In particular, assume we have a CNF φ and a variable x which we want to eliminate. Denote

$$\varphi_0 = \{C \in \varphi | \neg x \in C\}$$

$$\varphi_1 = \{C \in \varphi | x \in C\}$$

$$\varphi_r = \{C \in \varphi | C \cap \{x, \neg x\} = \emptyset\}$$

Namely, φ_0 consists of the clauses containing negative literal $\neg x$, φ_1 consists of the clauses containing positive literal x, φ_r contains the rest of the clauses. Let us now define

$$\varphi_{dp} = \{Res(C_0, C_1) | C_0 \in \varphi_0, C_1 \in \varphi_1\}$$

where $Res(C_0, C_1)$ denotes the clauses originating from C_0 and C_1 by resolution. Show that φ is equisatisfiable with $\varphi' = \varphi_r \wedge \varphi_{dp}$.

First, we define: $\varphi'_0 = \{C \setminus \{\neg x\} | C \in \varphi_0\}, \varphi'_1 = \{C \setminus \{x\} | C \in \varphi_1\}$
It follows that: $\varphi_0 \equiv \{x \to C | C \in \varphi'_0\}, \varphi_1 \equiv \{\neg x \to C | C \in \varphi'_1\}$ (1)
and from that: $\varphi_0 \equiv x \to \{C | C \in \varphi'_0\}, \varphi_1 \equiv \neg x \to \{C | C \in \varphi'_1\}$
and $\varphi_0 \wedge \varphi_1 \equiv x \to \{C | C \in \varphi'_0\} \wedge \neg x \to \{C | C \in \varphi'_1\}$ and (using existence quantifier to denote capturing of variable x):

$$\exists_x(\varphi_0 \wedge \varphi_1) \equiv \{C | C \in \varphi'_0\} \vee \{C | C \in \varphi'_1\}$$

From distributivity of $\wedge$ and $\vee$ and the previous: $\exists_x(\varphi_0 \wedge \varphi_1) \equiv \{C_0 \vee C_1 | C_0 \in \varphi'_0, C_1 \in \varphi'_1\}$

We can finalize this as follows:

$$\exists_x(\varphi_0 \wedge \varphi_1) \equiv \{C_0 \vee C_1 | C_0 \in \varphi'_0, C_1 \in \varphi'_1\} \stackrel{Def}{\equiv} \exists_x(\{Res(\{\neg x\} \cup C_0, \{x\} \cup C_1) | C_0 \in \varphi'_0, C_1 \in \varphi'_1\})$$

$$\stackrel{(1)}{=} \exists_x(\{Res(C_0, C_1) | C_0 \in \varphi_0, C_1 \in \varphi_1\}) = \exists_x(\varphi_{dp})$$

And, since: $\exists_x(\varphi_0 \wedge \varphi_1) \equiv \exists_x(\varphi_{dp})$, it holds that: $\exists_x(\varphi) = \exists_x(\varphi_0 \wedge \varphi_1 \wedge \varphi_r) \stackrel{x \notin var(\varphi_r)}{\equiv} \exists_x(\varphi_0 \wedge \varphi_1) \wedge \varphi_r \equiv \exists_x(\varphi_{dp}) \wedge \varphi_r \stackrel{x \notin var(\varphi_r)}{\equiv} \exists_x(\varphi_{dp} \wedge \varphi_r) = \exists_x(\varphi')$
And thus:

$$\varphi \equiv_{SAT} \varphi'$$