# Homework 3

**Jiří Klepl**

(10 points) One of the preprocessing steps can be to eliminate some of the variables using so-called DP-elimination (or DP-resolution). In particular, assume we have a CNF $\varphi$ and a variable $x$ which we want to eliminate. Denote

$$\varphi_0 = \{C \in \varphi | \neg x \in C\}$$

$$\varphi_1 = \{C \in \varphi | x \in C\}$$

$$\varphi_r = \{C \in \varphi | C \cap \{x, \neg x\} = \emptyset\}$$

Namely, $\varphi_0$ consists of the clauses containing negative literal $\neg x$, $\varphi_1$ consists of the clauses containing positive literal $x$, $\varphi_r$ contains the rest of the clauses. Let us now define

$$\varphi_{dp} = \{Res(C_0, C_1) | C_0 \in \varphi_0, C_1 \in \varphi_1\}$$

where $Res(C_0, C_1)$ denotes the clauses originating from $C_0$ and $C_1$ by resolution. Show that $\varphi$ is equisatisfiable with $\varphi_0 = \varphi_r \wedge \varphi_{dp}$.

First, we define: $\varphi_0' = \{C \setminus \{\neg x\} | C \in \varphi_0\}, \varphi_1' = \{C \setminus \{x\} | C \in \varphi_1\}$
It follows that: $\varphi_0 \equiv \{x \to C | C \in \varphi_0'\}, \varphi_1 \equiv \{\neg x \to C | C \in \varphi_1'\}$ (1)
and from that: $\varphi_0 \equiv x \to \{C | C \in \varphi_0'\}, \varphi_1 \equiv \neg x \to \{C | C \in \varphi_1'\}$
and $\varphi_0 \wedge \varphi_1 \equiv x \to \{C | C \in \varphi_0'\} \wedge \neg x \to \{C | C \in \varphi_1'\}$ and (since those are the only two occurences of variable $x$):

$$\varphi_0 \wedge \varphi_1 \equiv_{SAT} \{C | C \in \varphi_0'\} \vee \{C | C \in \varphi_1'\}$$

From distributivity of $\wedge$ and $\vee$ and the previous: $\varphi_0 \wedge \varphi_1 \equiv_{SAT} \{C_0 \vee C_1 | C_0 \in \varphi_0', C_1 \in \varphi_1'\}$

We can finalize this as follows:

$$\varphi_0 \wedge \varphi_1 \equiv_{SAT} \{C_0 \vee C_1 | C_0 \in \varphi_0', C_1 \in \varphi_1'\} \overset{Def}{=} \{Res(\{\neg x\} \cup C_0, \{x\} \cup C_1) | C_0 \in \varphi_0', C_1 \in \varphi_1'\} \overset{(1)}{=}$$

$$\{Res(C_0, C_1) | C_0 \in \varphi_0, C_1 \in \varphi_1\} = \varphi_{dp}$$