Homework 2

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(10 points) Let φ be a CNF and $C \in \varphi$ a clause in φ and $l \in C$ a literal in C. We say that C is blocked by literal l if for every other clause $D \in \varphi$ which contains $\neg l$ we have that $Res_l(C, D)$ is a tautology (i.e. there is another literal $l_D \in D$ such that $\neg l_D \in C$). Show that φ is equisatisfiable with $\varphi \setminus \{C\}$, i.e. φ is satisfiable if and only if $\varphi \setminus \{C\}$ is satisfiable.

We can rewrite φ into $C \wedge \varphi_l \wedge \varphi_{\neg l} \wedge \varphi_r$ where φ_l contains the clauses other than C that contain l; $\varphi_{\neg l}$ contains the clauses that contain $\neg l$; and φ_r contains the clauses of φ that don't contain neither l nor $\neg l$.

It holds that: $C \wedge \varphi_l \wedge \varphi_{\neg l} \wedge \varphi_r \equiv (\neg l \to (C' \wedge \varphi'_l)) \wedge (l \to \varphi'_{\neg l}) \wedge \varphi_r$ where C' is the clause C after removing the literal l; and φ'_l consist of the clauses in φ_l after removing the literal l; and, similarly, $\varphi'_{\neg l}$ contains the clauses from $\varphi_{\neg l}$ after removing $\neg l$ from each.

And then, since neither of C', φ'_l, $\varphi'_{\neg l}$, nor φ_r contain either of the literals l and $\neg l$, it holds that: $(\neg l \to (C' \wedge \varphi'_l)) \wedge (l \to \varphi'_{\neg l}) \wedge \varphi_r \equiv_{SAT} ((C' \wedge \varphi'_l) \vee \varphi'_{\neg l}) \wedge \varphi_r$.

Then, from distributivity of $\vee$ and $\wedge$, we get: $((C' \wedge \varphi'_l) \vee \varphi'_{\neg l}) \wedge \varphi_r \equiv \bigwedge_{D' \in \varphi_{\neg l}} (C' \vee D') \wedge \bigwedge_{D' \in \varphi'_{\neg l}} (\varphi'_l \vee D') \wedge \varphi_r = \bigwedge_{D \in \varphi_{\neg l}} Res_l(C, D) \wedge \bigwedge_{D' \in \varphi'_{\neg l}} (\varphi'_l \vee D') \wedge \varphi_r \equiv \bigwedge_{D \in \varphi_{\neg l}} Res_l(C, D) \wedge (\varphi'_l \vee \varphi'_{\neg l}) \wedge \varphi_r$; the equality follows the definition of C' and $\varphi'_{\neg l}$ from C and $\varphi_{\neg l}$; and the second equivalence, again, from distributivity of $\vee$ and $\wedge$.

The assignment states that $Res_l(C, D)$ is a tautology for every $D \in \varphi_{\neg l}$, and thus: $\bigwedge_{D \in \varphi_{\neg l}} Res_l(C, D) \wedge (\varphi'_l \vee \varphi'_{\neg l}) \wedge \varphi_r \equiv_{SAT} (\varphi'_l \vee \varphi'_{\neg l}) \wedge \varphi_r \equiv_{SAT} (\neg l \to \varphi'_l) \wedge (l \to \varphi'_{\neg l}) \wedge \varphi_r \equiv \varphi_l \wedge \varphi_{\neg l} \wedge \varphi_r = \varphi \setminus \{C\}$.

And thus:

$$\varphi \equiv_{SAT} \varphi \setminus \{C\}$$