Homework 1

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(10 points) Let $G = (V, E)$ be an undirected graph. Suggest a propositional formula that is satisfiable if and only if G contains a Hamiltonian cycle.

We will describe a formula φ such that the assignment $\alpha \models \varphi$ will give us the Hamiltonian cycle. This formula will be satisfiable iff G has a Hamiltonian cycle.

Let a variable $x_{u,v}$ denote the fact the cycle takes edge $\{u, v\} \in E$ (with a direction from u to v) iff $\alpha(x_{u,v}) = 1$. The formula then has to describe the fact we visit each vertex exactly once and that we exit the vertex exactly once and that there is only one path:

- At least once:

$$\varphi_1 \equiv \bigwedge_{u \in V} \left(\bigvee_{\{u,v\} \in E} x_{u,v} \wedge \bigvee_{\{u,v\} \in E} x_{v,u} \wedge \bigwedge_{\{u,v\} \in E} (\neg x_{u,v} \vee \neg x_{v,u}) \right)$$

- At most once:

$$\varphi_2 \equiv \bigwedge_{(u,v),\{u,w\} \in E \wedge w \neq v} ([\neg x_{u,v} \vee \neg x_{u,w}] \wedge [\neg x_{v,u} \vee \neg x_{w,u}])$$

- One path: Let a variable $y_{u,v,w}$ denote that v is accessible by a path from u through w that is a part of the Hamiltonian cycle iff $\alpha(y_{u,v,w}) = 1$, where $u, v, w \in V$. Then:
  - Extensibility:

$$\varphi_3 \equiv \bigwedge_{u,v,w,x \in V \wedge \{v,x\} \in E \wedge |\{u,v,x\}|=3} ([y_{u,v,w} \wedge x_{v,x}] \to y_{u,x,w})$$

  - Asymmetry (non-triviality):

$$\varphi_4 \equiv \bigwedge_{u,v,w \in V \wedge |\{u,v,w\}|=3} ([y_{u,v,w} \to \neg y_{v,u,w}] \wedge [y_{u,v,w} \to \neg y_{u,w,v}])$$

  - Universality:

$$\varphi_5 \equiv \bigwedge_{u,v \in V \wedge u \neq v} (y_{u,v,u} \wedge y_{u,v,v})$$

The formula has the form: $\varphi \equiv \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4 \wedge \varphi_5$.

φ_1, φ_2 ensure that every vertex will be visited and exited exactly once.

$\varphi_3, \varphi_4, \varphi_5$ ensure that the result will not be broken into separate cycles.