# Group Problem 1

Due Mar. 2, 2018

A direct numerical simulation of turbulent flow between two parallel walls is undertaken to determine the average velocity in the center of the channel $U_c$ (among other things). There are two sources of error in this calculation: the statistical error due to the fact that the averages are taken over a finite sample, and the discretization error due to the numerical approximations in the calculation. The goal of the current problem is to estimate the discretization error given data for the center line velocity calculated with several different mesh resolutions. Following is the data you have:

| Name | $h$ | $\tilde{U}_{ch}$ | $\sigma_{U_{ch}}$ |
|---|---|---|---|
| Coarsest | 2.0 | 1.16828362427 | 0.0001283 |
| Coarse | $\sqrt{2}$ | 1.16429173392 | 0.0003982 |
| Nominal | 1.0 | 1.16367827195 | 0.0001282 |
| Fine | 0.5 | 1.16389876649 | 0.0002336 |

Here $h$ is the resolution parameter (normalized grid spacing), $\tilde{U}_{ch}$ is the observed value of the averaged center line velocity calculated for resolution parameter $h$, and $\sigma_{U_{ch}}$ is the standard deviation of the statistical noise in $\tilde{U}_{ch}$ due to finite averaging. Further, generalizations of the central limit theorem imply that the statistical noise is approximately normally distributed (i.e. Gaussian). The value of $U_{ch}$ in the limit of an arbitrarily large averaging sample, i.e. when the statistical error is zero, is expected to have the form:

$$U_{ch} = U_c - Ch^p \tag{1}$$

where $U_c$ is the "exact" center line velocity, $C$ a coefficient and $p$ is the order of convergence. Note that $Ch^p$ is the discretization error at resolution $h$, and that $C$ is the discretization error for $h = 1$.

Other researchers have studied this flow and report various values of $U_c$, but none has done a detailed error analysis. The best available information is that $U_c \approx 1.1627$, with an error expected to be less than 5% with 95% confidence. There is no *a priori* reason to think that the discretization error and thus $C$ is either positive or negative. Further, on the nominal mesh ($h = 1$), we have 95% confidence that the discretization error has magnitude less than 0.5% of $U_c$. Finally, $p$ is positive and the various numerical schemes used in the computations have convergence rates consistent with $p = 2$, $p = 3$, $p = 6$ and $p = 7$. Because in the range of $h$ considered, it is not clear which numerical approximations dominate the error, the value of $p$ is presumed to be between 1 and 10.

Use the information provided above and the data in the table to develop estimates of the exact center line velocity $U_c$, the discretization error for the nominal mesh, and the convergence order $p$. In particular:

1. Develop probabilistic representations for the information detailed above for use as a prior.
2. Develop a likelihood function describing the likelihood of obtaining the observed values of $\tilde{U}_{ch}$ in the table for specific values of $U_c$, $C$ and $p$.

3. Use the results of (1) and (2) to obtain and plot posterior distributions of the $U_c$, $C$ and $p$.

4. What conclusions can be drawn from these results.

Note that because the relationship in (1) is non-linear, you will not be able to determine the posteriors analytically as we did in class. Instead, you can do the computations numerically using a Markov Chain Monte Carlo (MCMC) algorithm. Software and documentation will be available on canvas. Alternatively, since the posterior is a probability density in only three dimensions, you can evaluate the posterior on a grid in three-dimensions. Note that for normalization and to determine marginal distributions of the paperameters, you will need to be able to compute numerical quadratures on the grid.