DOCUMENT FILE FOR CONTENT IN SUBFOLDER: AXIOM–CIRCLE

1. Content in file Basic.lean

In this file, we define the class of circles, and state some basic properties. A more precise plan is the following:

(1) We define the class of circles.
(2) We give different ways to make a circle.
(3) We discuss the position between a point and a circle.
(4) We discuss the position between different points on the same circle.
(5) We define the concepts of antipodes, arc and chord in a circle.

1.1. **Definitions.**

- Structure `Circle`: A *Circle* consists of a center O and a radius r which is positive, i.e. $r > 0$; it is the circle whose center and radius are given.

1.2. **Make of concepts.**

- Definition `Circle.mk_pt_pt`: Given two distinct points O and A, this function returns a circle whose center is O and radius is $\|OA\|$.
- Definition `Circle.mk_pt_pt_pt`: Given three points A, B, C that are not collinear, this function returns a circle which is the circumcircle of triangle ABC. This `def` will be moved into the construction of circumcenter.
- Definition `CIR` as `Circle.mk_pt_pt`: This is to abbreviate the function `Circle.mk_pt_pt` into `CIR`.
- Definition $\odot$ as `Circle.mk_pt_pt`: This is to abbreviate the function `Circle.mk_pt_pt` into $\odot$.
- Definition `Circle.mk_pt_radius`: Given a point O and a positive real number r, this function returns a circle whose center is O and radius is r.
- Definition `Circle.mk_pt_pt_diam`: Given two distinct points A and B, this function returns a circle with AB as its diameter, i.e. the circle's center is the midpoint of AB, and its radius is $\frac{1}{2}\|AB\|$.

1.3. **Position between a point and a circle.**

- Definition `Circle.IsInside`: Given a point A and a circle ω, this function returns whether A lies inside ω; here saying that A lies inside ω means that the distance between A and the center of ω is not greater than the radius of ω.
- Definition `Circle.IsOn`: Given a point A and a circle ω, this function returns whether A lies on ω; here saying that A lies on ω means that the distance between A and the center of ω is equal to the radius of ω.
- Definition `Circle.IsInt`: Given a point A and a circle ω, this function returns whether A lies in the interior of ω; here saying that A lies in the interior of ω means that the distance between A and the center of ω is smaller than the radius of ω.

- Definition `Circle.IsOutside`: Given a point A and a circle ω, this function returns whether A lies outside ω; here saying that A lies outside ω means that the distance between A and the center of ω is greater than the radius of ω.
- Definition `Circle.carrier`: Given a circle, its underlying set is the set of points that lie on this circle.
- Definition `Circle.interior`: Given a circle, its interior is the set of points that lie in the interior of this circle.
- Definition `LiesIn` as `Circle.IsInside`: This is to abbreviate the function `Circle.IsInside` into `LiesIn`.
- Definition `LiesOut` as `Circle.IsOutside`: This is to abbreviate the function `Circle.IsOutside` into `LiesOut`.
- Instance `Circle.pt_liesout_ne_center`: Given a circle ω and a point A that lies outside ω, then A is distinct to the center of ω.
- Instance `Circle.pt_lieson_ne_center`: Given a circle ω and a point A that lies on ω, then A is distinct to the center of ω.
- Instance `Circle.pt_liesout_ne_pt_lieson`: Given a circle ω, a point A that lies outside ω and a point B that lies on ω, then A and B are distinct, i.e. $A \neq B$.
- Instance `Circle.pt_liesint_ne_pt_lieson`: Given a circle ω, a point A that lies in the interior of ω and a point B that lies on ω, then A and B are distinct, i.e. $A \neq B$.
- Instance `Circle.pt_liesout_ne_pt_liesint`: Given a circle ω, a point A that lies outside ω and a point B that lies in the interior of ω, then A and B are distinct, i.e. $A \neq B$.
- Theorem `Circle.liesint_iff_liesin_and_not_lieson`: Given a circle ω and a point A, then A lies in the interior of ω if and only if A lies inside ω and A doesn't lie on ω.
- Theorem `Circle.liesin_iff_liesint_or_lieson`: Given a circle ω and a point A, then A lies inside ω if and only if A lies in the interior of ω or A lies on ω.
- Theorem `Circle.mk_pt_pt_lieson`: Given two distinct points O and A, then A lies on `CIR O A`, i.e. the center of the circle is O and the radius is $\|OA\|$.
- Theorem `Circle.mk_pt_pt_diam_fst_lieson`: Given two distinct points A and B, then the first point A lies on the circle with AB as its diameter.
- Theorem `Circle.mk_pt_pt_diam_snd_lieson`: Given two distinct points A and B, then the second point B lies on the circle with AB as its diameter.
- Definition `Circle.seg_lies_inside_circle`: Given a segment l and a circle ω, this function returns whether l lies inside ω; here saying that l lies inside ω means that the two endpoints of l both lie inside ω.
- Definition `SegInCir` as `Circle.seg_lies_inside_circle`: This is to abbreviate the function `Circle.seg_lies_inside_circle` into `SegInCir`.
- Theorem `Circle.pt_lies_inside_circle_of_seg_inside_circle`: Given a circle ω, a segment l that lies in the interior of ω and a point A that lies in the interior of l, then A lies in the interior of ω. still `sorry`, need a lemma of *Seg*

1.4. **Position between different points.**

- Lemma `Circle.pts_lieson_circle_vec_eq`: Given a circle ω and two distinct points A, B that both lie on ω, if we denote the perpendicular foot of the center of ω to the line AB as P, then $\overrightarrow{AP} = \overrightarrow{PB}$.

- Theorem `Circle.pts_lieson_circle_perpfoot_eq_midpoint`: Given a circle ω and two distinct points A, B that both lie on ω, then the perpendicular foot of the center of ω to the line AB is equal to the midpoint of AB.
- Theorem `Circle.three_pts_lieson_circle_not_collinear`: Given a circle ω and three points A, B, C that is distinct to each other, and they all lie on ω, then A, B, C are not collinear.

1.5. **Antipode.**

- Definition `Circle.IsAntipode`: Given a circle ω and two points A, B that both lie on ω, this function returns whether B is A's antipode; here saying that B is A's antipode means that B is the point reflection of A respect to the center of ω.
- Theorem `Circle.antipode_symm`: Given a circle ω and two points A, B that both lie on ω, if B is A's antipode, then A is B's antipode.
- Theorem `Circle.antipode_center_is_midpoint`: Given a circle ω and two points A, B that both lie on ω, if B is A's antipode, then the center of ω is the midpoint of segment AB.
- Theorem `Circle.antipode_iff_collinear`: Given a circle ω and two distinct points A, B that both lie on ω, then B is A's antipode if and only if A, O, B are collinear, where O is the center of ω.
- Theorem `Circle.mk_pt_pt_diam_is_antipode`: Given two distinct points A, B, then B is A's antipode respect to the circle with segment AB as its diameter.

1.6. **Arc.**

- Structure `Arc`: Given a circle ω, an *Arc* consists of two points named `source` and `target`, and properties that these two points both lies on the circle and they are distinct; it is an arc from `source` to `target` respect to ω.
- Definition `Arc.mk_pt_pt_circle`: Given a circle ω and two distinct points A, B that lie on ω, this function returns the arc from A to B respect to ω.
- Definition `ARC` as `Arc.mk_pt_pt_circle`: This is to abbreviate the function `Arc.mk_pt_pt_circle` into `ARC`.
- Definition `Arc.IsOn`: Given a circle ω, a point A and an arc β on ω, this function returns whether A lies on β; here saying that A lies on β means that A lies on ω and A doesn't lie on the left side of the directed line from β's source to target.
- Definition `Arc.ne_endpts`: Given a circle ω, a point A and an arc β on ω, this function returns whether A is not equal to β's endpoints; here saying that A is not equal to β's endpoints means that A is not equal to β's source or target.
- Instance `Arc.pt_ne_source`: Given a circle ω, an arc β on ω and a point A that is not equal to β's endpoints, then A is not equal to β's source.
- Instance `Arc.pt_ne_target`: Given a circle ω, an arc β on ω and a point A that is not equal to β's endpoints, then A is not equal to β's target.
- Definition `Arc.IsInt`: Given a circle ω, a point A and an arc β on ω, this function returns whether A lies in the interior of β; here saying that A lies in the interior of β means that A lies on β and A is not β's endpoints.
- Definition `Arc.carrier`: Given an arc, its underlying set is the set of points that lie on this arc.

- Definition `Arc.interior`: Given an arc, its interior is the set of points that lie in the interior of this arc.
- Theorem `Arc.center_ne_endpts`: Given a circle ω and an arc β on ω, then ω's center is not equal to β's endpoints.
- Instance `Arc.source_ne_center`: Given a circle ω and an arc β on ω, then β's source is not equal to ω's center.
- Instance `Arc.target_ne_center`: Given a circle ω and an arc β on ω, then β's target is not equal to ω's center.
- Definition `Arc.complement`: Given a circle ω and an arc β on ω, this function returns the complement of β; here saying that the complement of β starts from β's target and ends at β's source.
- Lemma `Arc.pt_liesint_not_lieson_dlin`: Given a circle ω, an arc β on ω and a point A that lies in the interior of β, then A doesn't lie on the directed line from β's source to target.
- Theorem `Arc.pt_liesint_liesonright_dlin`: Given a circle ω, an arc β on ω and a point A that lies in the interior of β, then A lies on the right side of the directed line from β's source to target.
- Theorem `Arc.pt_liesint_complementary_liesonleft_dlin`: Given a circle ω, an arc β on ω and a point A that lies in the interior of β's complement, then A lies on the left side of the directed line from β's source to target.
- Is it necessary to define the sum of arcs which are connected?

1.7. **Chord.**

- Structure `Chord`: Given a circle ω, a Chord consists of a non-degenerate segment AB and condition that both A and B lie on ω.
- Instance `Chord.IsND`: Given a circle ω and a chord s in ω, then the source and target of s are distinct.
- Definition `Chord.mk_pt_pt_circle`: Given a circle ω and two distinct points A, B that both lie on ω, this function returns the chord AB in ω.
- Definition `Chord.IsOn`: Given a circle ω, a point A and a chord s in ω, this function returns whether A lies on s; here saying that A lies on s means that A lies on the non-degenerate segment respect to s.
- Definition `Chord.IsInt`: Given a circle ω, a point A and a chord s in ω, this function returns whether A lies in the interior of s; here saying that A lies in the interior of s means that A lies in the interior of the non-degenerate segment respect to s.
- Definition `Chord.carrier`: Given a chord, its underlying set is the set of points that lie on this chord.
- Definition `Chord.interior`: Given a chord, its interior is the set of points that lie in the interior of this chord.
- Definition `Chord.ne_endpts`: Given a circle ω, a point A and a chord s in ω, this function returns whether A is not equal to the endpoints of s; here saying that A is not equal to s's endpoints means that A is not equal to the source or target of s.
- Theorem `Chord.center_ne_endpts`: Given a circle ω and a chord s in ω, then ω's center is not equal to the endpoints of s.
- Instance `Chord.source_ne_center`: Given a circle ω and a chord s in ω, then the source of s is not equal to ω's center.

- Instance `Chord.target_ne_center`: Given a circle ω and a chord s in ω, then the target of s is not equal to ω's center.
- Definition `Chord.reverse`: Given a circle ω and a chord s in ω, then this function returns the reverse chord of s, which starts from the target of s and ends at the source of s.
- Theorem `Chord.pt_liesint_liesint_circle`: Given a circle ω, a chord s in ω and a point A that lies in the interior of s, then A lies in the interior of ω.
- Definition `Arc.toChord`: Given a circle ω and an arc β on ω, this function returns the chord respect to β.
- Definition `Chord.toArc`: Given a circle ω and a chord s in ω, this function returns the arc respect to s.
- Theorem `Circle.complementary_arc_toChord_eq_reverse`: Given a circle ω and an arc β on ω, then the chord respect to the complement of β is equal to the reverse chord respect to β.
- Theorem `Circle.reverse_chord_toArc_eq_complement`: Given a circle ω and a chord s in ω, then the arc respect to the reverse chord of s is equal to the complement arc respect to s.
- Definition `Chord.length`: Given a circle ω and a chord s in ω, this function returns the length of s.
- Definition `Chord.IsDiameter`: Given a circle ω and a chord s in ω, this function returns whether s is a diameter; here saying that s is a diameter means that the center of ω lies on s.
- Theorem `Chord.diameter_iff_antipide`: Given a circle ω and a chord s in ω, then s is a diameter if and only if the source and target of s are antipodes.
- Theorem `Chord.diameter_length_eq_twice_radius`: Given a circle ω and a chord s in ω, if s is a diameter, then the length of s is twice as large as ω's radius, i.e. $|s| = 2r$.

2. Content in file LCPosition.lean

In this file, we define the position between a line and a circle, and there intersected points if intersected.

2.1. Position between a directed line and a circle.

- Definition `Circle.DirLine.IsDisjoint`: Given a directed line l and a circle ω, this function returns whether l is disjoint to ω; here saying that l is disjoint to ω means that the distance from the circle of ω to l is greater than the radius of ω.
- Definition `Circle.DirLine.IsTangent`: Given a directed line l and a circle ω, this function returns whether l is tangent to ω; here saying that l is tangent to ω means that the distance from the circle of ω to l is equal to the radius of ω.
- Definition `Circle.DirLine.IsSecant`: Given a directed line l and a circle ω, this function returns whether l is secant to ω; here saying that l is secant to ω means that the distance from the circle of ω to l is smaller than the radius of ω.
- Definition `Circle.DirLine.IsIntersected`: Given a directed line l and a circle ω, this function returns whether l is intersected with ω; here saying that l is intersected with ω means that the distance from the circle of ω to l is not greater than the radius of ω.

- Definition `Secant` as `Circle.DirLine.IsSecant`: This is to abbreviate the function `Circle.DirLine.IsSecant` into `Secant`.
- Definition `Tangent` as `Circle.DirLine.IsTangent`: This is to abbreviate the function `Circle.DirLine.IsTangent` into `Tangent`.
- Definition `Disjoint` as `Circle.DirLine.IsDisjoint`: This is to abbreviate the function `Circle.DirLine.IsDisjoint` into `Disjoint`.
- Theorem `DirLC.disjoint_pt_liesout_circle`: Given a circle ω, a directed line l which is disjoint to ω, and a point A that lies on l, then A lies outside ω.
- Theorem `DirLC.intersect_iff_tangent_or_secant`: Given a directed line l and a circle ω, then l is intersected with ω if and only if l is tangent to ω or l is secant to ω.
- Theorem `DirLC.pt_liesint_secant`: Given a circle ω, a point A in the interior of ω and a directed line l such that A lies on l, then l is secant to ω.
- Theorem `DirLC.pt_liesint_intersect`: Given a circle ω, a point A in the interior of ω and a directed line l such that A lies on l, then l is intersected with ω.

2.2. Definition of intersected points.

- Structure `DirLCInxpts`: A *DirLCInxpts* consists of two points named `front` and `back`; they are the intersected points of a directed line and a circle, distinguished by the direction of the directed line.
- Lemma `DirLC.dist_pt_line_ineq`: Given a circle ω and a directed line l that is intersected with ω, then we have an inequality $r^2 - d^2 \geq 0$, where r is the radius of ω and d is the distance from the center of ω to l. This lemma makes sure that the definition of intersected points is well defined.
- Definition `DirLC.Inxpts`: Given a circle ω and a directed line l that is intersected with ω, this function returns the intersected points of l and ω.

2.3. Basic properties of intersected points.

- Lemma `DirLC.inx_pts_lieson_dlin`: Given a circle ω and a directed line l that is intersected with ω, then both of the intersected points of l and ω lie on l.
- Theorem `DirLC.inx_pts_lieson_circle`: Given a circle ω and a directed line l that is intersected with ω, then both of the intersected points of l and ω lie on ω.
- Theorem `DirLC.inx_pts_same_iff_tangent`: Given a circle ω and a directed line l that is intersected with ω, then two intersected points of l and ω coincide if and only if l is tangent to ω.
- Lemma `DirLC.inx_pts_ne_center`: Given a circle ω and a directed line l that is intersected with ω, then both of the intersected points of l and ω are distinct with the center of ω.
- Theorem `DirLC.inx_pts_antipode_iff_center_lieson`: Given a circle ω and a directed line l that is intersected with ω, then one of the intersected points of l and ω is the antipode of another if and only if the center of ω lies on l. still `sorry`
- Theorem `DirLC.inxwith_iff_intersect`: Given a circle ω and a directed line l, then the images of l and ω have intersection if and only if l is intersected with ω.
- Theorem `DirLC.inxwith_iff_tangent_or_secant`: Given a circle ω and a directed line l, then the images of l and ω have intersection if and only if l is tangent to ω or

l is secant to ω. Do we need to change the statement of `IsIntersected` to `InxWith` in the above theorems?

2.4. **Tangent point.**

- Definition `DirLC.Tangentpt`: Given a circle ω and a directed line l that is tangent to ω, this function returns the tangent point of l and ω.
- Lemma `DirLC.tangent_pt_ne_center`: Given a circle ω and a directed line l that is tangent to ω, then the tangent point of l and ω is distinct with the center of ω.
- Theorem `DirLC.tangent_pt_center_perp_line`: Given a circle ω and a directed line l that is tangent to ω, then the line between the center of ω and the tangent point is perpendicular to l.
- Theorem `DirLC.tangent_pt_eq_perp_foot`: Given a circle ω and a directed line l that is tangent to ω, then the tangent point is the perpendicular foot from the center of ω to l.

2.5. **The uniqueness of intersected points.**

- Theorem `Circle.DirLC_intersection_eq_inxpts`: Given a circle ω, a directed line l that is intersected with ω and a point A that both lie on l and ω, then A is equal to one of the intersected points of l and ω.
- Theorem `Circle.pt_pt_tangent_eq_tangent_pt`: Given a circle ω and two points A, B that A lies outside ω and B lies on ω, if directed line AB is tangent to ω, then B is the tangent point.
- Theorem `Circle.chord_toDirLine_intersected`: Given a circle ω and a chord s in ω, then the directed line respect to s, which starts from the source of s and ends at the target of s, is intersected with ω.
- Theorem `Circle.chord_toDirLine_inx_front_pt_eq_target`: Given a circle ω and a chord s in ω, then the front intersected point of the directed line respect to s and ω is equal to the target of s. still `sorry`
- Theorem `Circle.chord_toDirLine_inx_back_pt_eq_source`: Given a circle ω and a chord s in ω, then the back intersected point of the directed line respect to s and ω is equal to the source of s. still `sorry`

2.6. **Equivalent condition for tangency.**

- Theorem `Circle.pt_pt_tangent_perp`: Given a circle ω and two points A, B that A lies outside ω and B lies on ω, if directed line AB is tangent to ω, then the directed line from the center of ω to B is perpendicular to directed line AB.
- Theorem `Circle.pt_pt_perp_tangent`: Given a circle ω and two points A, B that A lies outside ω and B lies on ω, if directed line AB is perpendicular to the directed line from the center of ω to B, then directed line AB is tangent to ω.
- Theorem `Circle.pt_pt_perp_eq_tangent_pt`: Given a circle ω and two points A, B that A lies outside ω and B lies on ω, if directed line AB is perpendicular to the directed line from the center of ω to B, then B is the tangent point of directed line AB and ω.

2.7. **Position between a line and a circle.**

- Definition `Circle.Line.IsDisjoint`: Given a line l and a circle ω, this function returns whether l is disjoint to ω; here saying that l is disjoint to ω means that the distance from the circle of ω to l is greater than the radius of ω.
- Definition `Circle.Line.IsTangent`: Given a line l and a circle ω, this function returns whether l is tangent to ω; here saying that l is tangent to ω means that the distance from the circle of ω to l is equal to the radius of ω.
- Definition `Circle.Line.IsSecant`: Given a line l and a circle ω, this function returns whether l is secant to ω; here saying that l is secant to ω means that the distance from the circle of ω to l is smaller than the radius of ω.
- Definition `Circle.Line.IsIntersected`: Given a line l and a circle ω, this function returns whether l is intersected with ω; here saying that l is intersected with ω means that the distance from the circle of ω to l is not greater than the radius of ω.

3. Content in file CCPosition.lean

In this file, we define the position between two circles, and there intersected points if intersected.

3.1. Position between two circles.

- Definition `Circle.CC.IsSeparated`: Given two circles ω_1, ω_2, this function returns whether ω_1 is separated from ω_2; here saying that ω_1 is separated from ω_2 means that the distance between their centers is greater than the sum of their radius, i.e. $d > r_1 + r_2$.
- Definition `Circle.CC.IsIntersected`: Given two circles ω_1, ω_2, this function returns whether ω_1 is intersected with ω_2; here saying that ω_1 is intersected with ω_2 means that the distance between their centers is smaller than the sum of their radius and greater than the absolute value of the difference between their radius, i.e, $|r_1 - r_2| < d < r_1 + r_2$.
- Definition `Circle.CC.IsExtangent`: Given two circles ω_1, ω_2, this function returns whether ω_1 is external tangent to ω_2; here saying that ω_1 is external tangent to ω_2 means that the distance between their centers is equal to the sum of their radius, i.e. $d = r_1 + r_2$.
- Definition `Circle.CC.IsIntangent`: Given two circles ω_1, ω_2, this function returns whether ω_1 is internal tangent to ω_2; here saying that ω_1 is internal tangent to ω_2 means that the distance between their centers is equal to ω_2's radius minus ω_1's radius, i.e. $d = r_2 - r_1$, and their centers are distinct. Here we put the smaller circle in the first position.
- Definition `Circle.CC.IsIncluded`: Given two circles ω_1, ω_2, this function returns whether ω_1 is included in ω_2; here saying that ω_1 is included in ω_2 means that the distance between their centers is smaller than ω_2's radius minus ω_1's radius, i.e. $d < r_2 - r_1$. Here we put the smaller circle in the first position.
- Definition `Separate` as `Circle.CC.IsSeparated`: This is to abbreviate the function `Circle.CC.IsSeparated` into `Separate`.
- Definition `Intersect` as `Circle.CC.IsIntersected`: This is to abbreviate the function `Circle.CC.IsIntersected` into `Intersect`.
- Definition `Extangent` as `Circle.CC.IsExtangent`: This is to abbreviate the function `Circle.CC.IsExtangent` into `Extangent`.

- Definition `Intangent` as `Circle.CC.IsIntangent`: This is to abbreviate the function `Circle.CC.IsIntangent` into `Intangent`.
- Definition `IncludeIn` as `Circle.CC.IsIncluded`: This is to abbreviate the function `Circle.CC.IsIncluded` into `IncludeIn`.

3.2. **Properties of separated.**

- Theorem `CC.separate_symm`: Given two circles ω_1, ω_2, then ω_1 is separated from ω_2 if and only if ω_2 is separated from ω_1.
- Theorem `CC.separated_pt_liesout_second_circle`: Given two circles ω_1, ω_2 that are separated, and a point A lies on ω_1, then A lies outside ω_2.
- Theorem `CC.separated_pt_liesout_first_circle`: Given two circles ω_1, ω_2 that are separated, and a point A lies on ω_2, then A lies outside ω_1.

3.3. **Properties of external tangent.**

- Theorem `CC.extangent_symm`: Given two circles ω_1, ω_2, then ω_1 is external tangent to ω_2 if and only if ω_2 is external tangent to ω_1.
- Lemma `CC.extangent_centers_distinct`: Given two circles ω_1, ω_2 that are external tangent, then their centers are distinct.
- Definition `CC.Extangentpt`: Given two circles ω_1, ω_2 that are external tangent, this function returns the external tangent point of ω_1 and ω_2. Is it necessary to state the coercion when the external tangent condition is flipped?
- Theorem `CC.extangent_pt_lieson_circles`: Given two circles ω_1, ω_2 that are external tangent, then their external tangent point lies on both ω_1 and ω_2.
- Theorem `CC.extangent_pt_centers_collinear`: Given two circles ω_1, ω_2 that are external tangent, then their centers and the external tangent point are collinear.

3.4. **Properties of internal tangent.**

- Theorem `CC.intangency_pt_liesin_second_circle`: Given two circles ω_1, ω_2 that ω_1 is inscribed in ω_2, and a point A that lies on ω_1, then A lies inside ω_2.
- Definition `CC.Intangentpt`: Given two circles ω_1, ω_2 that ω_1 is inscribed in ω_2, this function returns the inscribed point of ω_1 and ω_2.
- Theorem `CC.intangent_pt_lieson_circles`: Given two circles ω_1, ω_2 that ω_1 is inscribed in ω_2, then their inscribed point lies on both ω_1 and ω_2.
- Theorem `CC.intangent_pt_centers_collinear`: Given two circles ω_1, ω_2 that ω_1 is inscribed in ω_2, then their centers and the inscribed point are collinear.

3.5. **Properties of included.**

- Theorem `CC.included_pt_liesint_second_circle`: Given two circles ω_1, ω_2 that ω_1 is included in ω_2, and a point A that lies on ω_1, then A lies in the interior of ω_2.
- Theorem `CC.included_pt_liesout_first_circle`: Given two circles ω_1, ω_2 that ω_1 is included in ω_2, and a point A that lies on ω_2, then A lies outside ω_1.

3.6. **Properties of intersected.**

- Theorem `CC.intersected_symm`: Given two circles ω_1, ω_2, then ω_1 is intersected with ω_2 if and only if ω_2 is intersected with ω_1.
- Lemma `CC.intersected_centers_distinct`: Given two circles ω_1, ω_2 that are intersected, then their centers are distinct.

- Structure `CCInxpts`: A *CCInxpts* consists of two points named `left` and `right`; they are the intersected points of two circles, distinguished by their position to the directed line between two circles' center.
- Definition `Circle.radical_axis_dist_to_the_first`: Given two circles ω_1, ω_2, denoting their centers as O_1, O_2, this function returns the directed distance from O_1 to their radical axis respect to the direction $\overrightarrow{O_1O_2}$; here saying that this distance is equal to $\frac{r_1^2+d^2-r_2^2}{2d}$.
- Lemma `Circle.radical_axis_dist_lt_radius`: Given two circles ω_1, ω_2 that are intersected, then the absolute value of the directed distance from the center of ω_1 to their radical axis is smaller than the radius of ω_1.
- Definition `CC.Inxpts`: Given two circles ω_1, ω_2 that are intersected, this function returns the two intersected points of ω_1 and ω_2. Is it necessary to state the coercion when the external tangent condition is flipped?
- Theorem `CC.inx_pts_distinct`: Given two circles ω_1, ω_2 that are intersected, then they have two different intersected points.
- Theorem `CC.inx_pts_lieson_circles`: Given two circles ω_1, ω_2 that are intersected, then both of their intersected points lies on both ω_1 and ω_2.
- Lemma `CC.inx_pts_centers_not_collinear`: Given two circles ω_1, ω_2 that are intersected, then both of their intersected points is not collinear with their centers.
- Theorem `CC.inx_pts_tri_acongr`: Given two circles ω_1, ω_2 that are intersected, then ... How to translate `acongr`?
- Theorem `CC.inx_pts_line_perp_center_line`: Given two circles ω_1, ω_2 that are intersected, then the line between their intersected points is perpendicular to the line between their centers. still `sorry`
- Theorem `CC.inx_pts_uniqueness`: Given two circles ω_1, ω_2 that are intersected, and a point A that lies on both ω_1 and ω_2, then A is equal to one of the intersected points of ω_1 and ω_2.

4. Content in file InscribedAngle.lean

In this file, we define concept of central angle and inscribed angle, also state some properties.

4.1. Central angle.

- Definition `Arc.cangle`: Given a circle ω and an arc β on ω, this function returns the central angle of β, which is $\angle AOB$, where A is the source of β and B is the target of β.
- Definition `Arc.IsMajor`: Given a circle ω and an arc β on ω, this function returns whether β is a major arc; here saying that β is a major arc means that the value of the central angle of β is negative.
- Definition `Arc.IsMinor`: Given a circle ω and an arc β on ω, this function returns whether β is a minor arc; here saying that β is a minor arc means that the value of the central angle of β is positive.
- Definition `Chord.cangle`: Given a circle ω and a chord s in ω, this function returns the central angle of s, which is $\angle AOB$, where A is the source of s and B is the target of s.

- Theorem `Circle.cangle_of_arc_eq_cangle_of_toChord`: Given a circle ω and an arc β on ω, then the central angle of β is equal to the central angle of the chord respect to β.
- Theorem `Circle.cangle_of_chord_eq_cangle_of_toArc`: Given a circle ω and a chord s in ω, then the central angle of s is equal to the central angle of the arc respect to s.
- Theorem `Chord.cangle_eq_pi_iff_is_diameter`: Given a circle ω and a chord s in ω, then the value of the central angle of s is equal to π if and only if s is a diameter.
- Theorem `Arc.complement_cangle_reverse`: Given a circle ω and an arc β on ω, then the central angle of the complement of β is equal to the reverse of the central angle of β.
- Theorem `Chord.reverse_cangle_reverse`: Given a circle ω and a chord s in ω, then the central angle of the reverse of s is equal to the reverse of the central angle of s.
- Theorem `Circle.cangle_of_complementary_arc_eq_neg`: Given a circle ω and an arc β on ω, then the value of the central angle of β's complement is equal to negative value of the central angle of β.
- Theorem `Circle.cangle_of_reverse_chord_eq_neg`: Given a circle ω and a chord s in ω, then the value of the central angle of the reverse of s is equal to negative value of the central angle of s.
- Theorem `Chord.cangle_eq_iff_length_eq`: Given a circle ω and two chords s_1, s_2 both in ω, then the value of the central angle of s_1 is equal to the value of the central angle of s_2 if and only if the length of s_1 is equal to the length of s_2. still `sorry`

4.2. **Inscribed angle.**

- Definition `Arc.IsIangle`: Given a circle ω, an arc β on ω and an angle ang, this function returns whether ang is an inscribed angle of β; here saying that ang is an inscribed angle of β means that the source of ang lies on ω and is distinct with the endpoints of β, and the source of β lies on the start ray of ang and the target of β lies on the end ray of ang.
- Definition `Chord.IsIangle`: Given a circle ω, a chord s in ω and an angle ang, this function returns whether ang is an inscribed angle of s; here saying that ang is an inscribed angle of s means that the source of ang lies on ω and is distinct with the endpoints of s, and the source of s lies on the start ray of ang and the target of s lies on the end ray of ang.
- Theorem `Arc.iangle_eq`: Given a circle ω, an arc β on ω and an angle ang that is an inscribed angle of β, then $\angle ASB$ is equal to ang where A is the source of β, B is the target of β and S is the source of ang.
- Theorem `Chord.iangle_eq`: Given a circle ω, a chord s in ω and an angle ang that is an inscribed angle of s, then $\angle ASB$ is equal to ang where A is the source of s, B is the target of s and S is the source of ang.
- Theorem `Arc.angle_mk_pt_is_iangle`: Given a circle ω, an arc β on ω and a point C that lies on ω and is distinct with the endpoints of β, then $\angle ACB$ is an inscribed angle of β where A is the source of β and B is the target of β.

- Theorem `Chord.angle_mk_pt_is_iangle`: Given a circle ω, a chord s in ω and a point C that lies on ω and is distinct with the endpoints of s, then $\angle ACB$ is an inscribed angle of s where A is the source of s and B is the target of s.
- Theorem `Circle.iangle_of_arc_is_iangle_of_toChord`: Given a circle ω, an arc β on ω and an angle ang that is an inscribed angle of β, then ang is an inscribed angle of the chord respect to β.
- Theorem `Circle.iangle_of_chord_is_iangle_of_toArc`: Given a circle ω, a chord s in ω and an angle ang that is an inscribed angle of s, then ang is an inscribed angle of the arc respect to s.
- Theorem `Arc.cangle_eq_two_times_inscribed_angle`: Given a circle ω, an arc β on ω and an angle ang that is an inscribed angle of β, then the value of the central angle of β is twice as large as the value of ang.
- Theorem `Chord.cangle_eq_two_times_inscribed_angle`: Given a circle ω, a chord s in ω and an angle ang that is an inscribed angle of s, then the value of the central angle of s is twice as large as the value of ang.
- Theorem `Circle.iangle_of_diameter_eq_mod_pi`: Given a circle ω, a chord s in ω and an angle ang that is an inscribed angle of s, if s is a diameter, then the value of ang is equal to $\frac{\pi}{2}$ in the sense of mod π.
- Theorem `Arc.iangle_invariant_mod_pi`: Given a circle ω, an arc β on ω and two angles ang_1, ang_2 that both are inscribed angles of β, then the value of ang_1 is equal to the value of ang_2 in the sense of mod π.
- Theorem `Chord.iangle_invariant_mod_pi`: Given a circle ω, a chord s in ω and two angles ang_1, ang_2 that both are inscribed angles of s, then the value of ang_1 is equal to the value of ang_2 in the sense of mod π.

4.3. **The value of inscribed angle in the sense of mod π.**

- Definition `Arc.iangdv`: Given a circle ω and an arc β on ω, this function returns the value of any inscribed angle of β in the sense of mod π.
- Definition `Chord.iangdv`: Given a circle ω and a chord s in ω, this function returns the value of any inscribed angle of s in the sense of mod π.
- Theorem `Arc.iangle_dvalue_eq`: Given a circle ω, an arc β on ω and an angle ang that is an inscribed angle of β, then
- Theorem `Chord.iangle_dvalue_eq`: Given a circle ω, a chord s in ω and an angle ang that is an inscribed angle of s, then
- Theorem `Circle.same_chord_same_iangle_dvalue`: Given a circle ω, two chords s_1, s_2 both in ω and two angles ang_1, ang_2 that ang_1 is an inscribed angle of s_1 and ang_2 is an inscribed angle of s_2, then the value of ang_1 is equal to the value of ang_2 in the sense of mod π.

5. Content in file CirclePower.lean

In this file, we define the power of a point respect to a circle, and state the circle power theorem.

5.1. **Definition and basic properties of power.**

- Definition `Circle.power`: Given a circle ω and a point A, this function returns the power of A respect to ω; here saying that the power of A respect to ω is equal to $|OA|^2 - r^2$, where O is ω's center and r is ω's radius.
- Theorem `Circle.pt_liesin_circle_iff_power_npos`: Given a circle ω and a point A, then A lies inside ω if and only if the power of A respect to ω is not positive.
- Theorem `Circle.pt_liesint_circle_iff_power_neg`: Given a circle ω and a point A, then A lies in the interior of ω if and only if the power of A respect to ω is negative.
- Theorem `Circle.pt_lieson_circle_iff_power_zero`: Given a circle ω and a point A, then A lies on ω if and only if the power of A respect to ω is equal to 0.
- Theorem `Circle.pt_liesout_circle_iff_power_pos`: Given a circle ω and a point A, then A lies outside ω if and only if the power of A respect to ω is positive.

5.2. Tangent lines from a point outside circle.

- Structure `Tangents`: A *Tangents* consists of two points named `left` and `right`, which stores the two tangent points from a point outside a circle.
- Lemma `Circle.tangent_circle_intersected`: Given a circle ω and a point A that lies outside ω, then the circle whose diameter is AO, where O is the center of ω, is tangent with ω.
- Definition `Circle.pt_outside_tangent_pts`: Given a circle ω and a point A outside ω, this function returns the structure *Tangents* that represents the two tangent points of the tangent lines from A to ω, distinguished by the direction $\overrightarrow{AO}$, where O is the center of ω; here we define these two tangent points by the intersected points of circle whose diameter is AO and ω.
- Theorem `Circle.tangents_lieson_circle`: Given a circle ω and a point A outside ω, then the two tangent points respect to A both lie on ω.
- Lemma `Circle.tangents_ne_pt`: Given a circle ω and a point A outside ω, then the two tangent points respect to A are distinct with A.
- Lemma `Circle.tangents_ne_center`: Given a circle ω and a point A outside ω, then the two tangent points respect to A are distinct with the center of ω.
- Lemma `Circle.tangents_perp1`: Given a circle ω and a point A outside ω, then directed line AM is perpendicular to directed line OM, where M is the `left` tangent point respect to A and O is the center of ω.
- Lemma `Circle.tangents_perp2`: Given a circle ω and a point A outside ω, then directed line AN is perpendicular to directed line ON, where N is the `right` tangent point respect to A and O is the center of ω.
- Theorem `Circle.line_tangent_circle`: Given a circle ω and a point A outside ω, then the directed line from A to the tangent point respect to A that we construct before is tangent to ω.
- Theorem `Circle.tangent_pts_eq_tangents`: Given a circle ω and a point A outside ω, then the tangent points respect to A are the tangent point of the tangent lines from A to ω.
- Lemma `Circle.tangent_length_sq_eq_power`: Given a circle ω, a directed line l that is tangent to ω and a point A that lies on l, then the square of distance between A and the tangent point of l and ω is equal to the power of A respect to ω.

- Lemma `Circle.tangent_length_sq_eq_power'`: Given a circle ω and two points A, B that A lies outside ω and B lies on ω, if directed line AB is tangent to ω, then the square of distance between A and B is equal to the power of A respect to ω.
- Theorem `Circle.length_of_tangent_eq`: Given a circle ω and a point A outside ω, then the length of two tangent lines from A to ω are equal, i.e. $|AM| = |AN|$, where M and N are the tangent points respect to A.
- Theorem `Circle.length_of_tangent_eq'`: Given a circle ω and three points A, B, C that A lies outside ω and B, C both lie on ω, if both directed lines AB and AC are tangent to ω, then the distance between A and B is equal to the distance between A and C ,i.e. $|AB| = |AC|$.

5.3. **Circle Power Theorem.**

- Lemma `Circle.pt_liesout_ne_inxpts`: Given a circle ω, a directed line l that is intersected with ω and a point A outside ω that also lies on l, then A is distinct with the two intersected points of l and ω.
- Lemma `Circle.pt_liesint_ne_inxpts`: Given a circle ω, a directed line l that is intersected with ω and a point A in the interior of ω that also lies on l, then A is distinct with the two intersected points of l and ω.
- Theorem `Circle.pt_liesout_back_lieson_ray_pt_front`: Given a circle ω, a directed line l that is intersected with ω and a point A outside ω that also lies on l, then N lies on ray AM, where M, N are respectively the front and back intersected point of l and ω.
- Theorem `Circle.pt_liesint_back_lieson_ray_pt_front_reverse`: Given a circle ω, a directed line l that is intersected with ω and a point A in the interior of ω that also lies on l, then N lies on the reverse ray of ray AM, where M, N are respectively the front and back intersected point of l and ω.
- Theorem `Circle.power_thm`: Given a circle ω, a directed line l that is intersected with ω and a point A that lies on l, then the inner product of vector $\overrightarrow{AM}$ and $\overrightarrow{AN}$ is equal to the power of A respect to ω.
- Theorem `Circle.chord_power_thm`: Given a circle ω, a directed line l that is intersected with ω and a point A in the interior of ω that also lies on l, then the product of distance $|AM|$ and $|AN|$ is equal to negative power of A respect to ω, where M, N are respectively the front and back intersected point of l and ω.
- Theorem `Circle.secant_power_thm`: Given a circle ω, a directed line l that is intersected with ω and a point A outside ω that also lies on l, then the product of distance $|AM|$ and $|AN|$ is equal to the power of A respect to ω, where M, N are respectively the front and back intersected point of l and ω.
- Theorem `Circle.intersecting_chords_thm`: Given a circle ω, a point S in the interior of ω and two chords s_1, s_2 in ω such that S lies on both s_1 and s_2, then we have $|SA| \cdot |SB| = |SC| \cdot |SD|$, where A, B are respectively the source and target of s_1 and C, D are respectively the source and target of s_2. still `sorry`
- Theorem `Circle.intersecting_secants_thm`:Given a circle ω, a point S outside ω and two directed line l_1, l_2 intersected with ω such that S lies on both l_1 and l_2, then we have $|SA| \cdot |SB| = |SC| \cdot |SD|$, where A, B are respectively the front and back intersected point of l_1 and ω, and C, D are respectively the front and back intersected point of l_2 and ω.