Physics 426 - Instability Notes

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1 Introduction

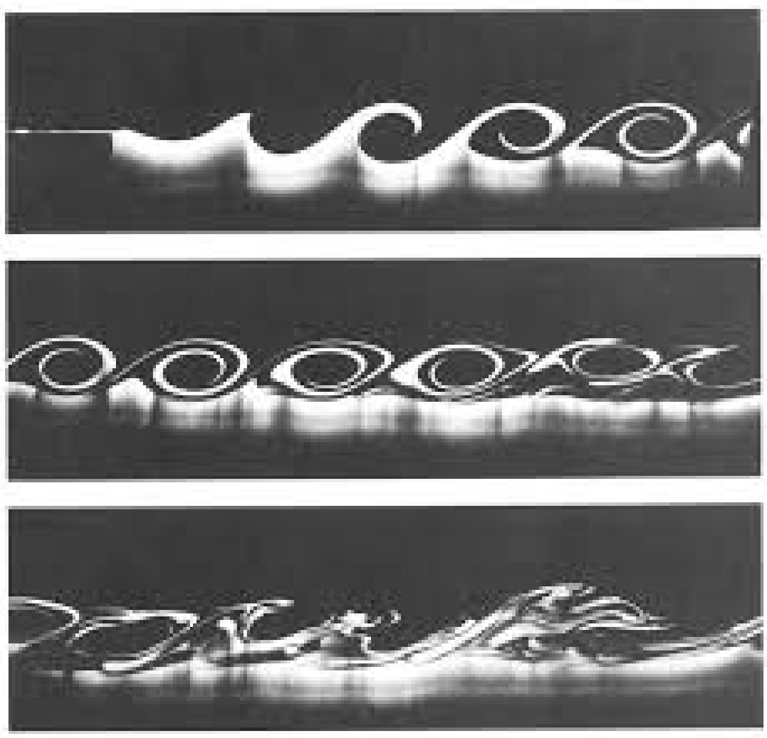

Figure 1: Shear instability in a fluid; the flow in the upper half is moving to the right faster than the fluid in the lower half. The three panels are distance downstream.

We have seen in the lab demos that fluid flow is very susceptible to becoming unstable and breaking down into turbulence. An example is shown in figure 2 on the following page where the flow in the upper layer is moving faster than flow in the lower layer and this leads to a "shear instability". A few things to note:

- There is clearly a preferred horizontal scale to this instability. The billows that grow are one size until they self-interact and become turbulent.
- This horizontal scale must be present in the upstream conditions, but it need not be large.
- The instability takes time to grow.

This leads to a few questions we commonly ask ourselves about a flow

1. Is a flow unstable?
2. If a flow is unstable can we say what scale has the fastest growth rate?
3. How fast is that growth rate?

2 Rayleigh Shear Instability

As a simple situation that often leads to instability lets consider a homogeneous steady background flow $u = U(y)$. The idea of an instability is that there are always small imperfections to this flow. Physically, its easiest to think of these as small displacements of the streamlines around the mean and then consider what happens if there are small perturbations to the flow. If the flow is stable, then all perturbations will be damped, whereas if it is unstable some perturbations will grow.

2.1 Linear perturbation analysis

All of these methods decompose the flow into a background component of the flow and a 'perturbation', and then assume that the perturbation is small so that only linear terms in the perturbation are retained. i.e. if $u = U(y) + u'(x, y, y)$ then terms that are quadratic or higher in u' are assumed small. So, similarly $v = v'$, and $p = p'$ because neither of these two variables have a background component.

So for the flow above the x-, and y-momentum equations, and continuity equations become:

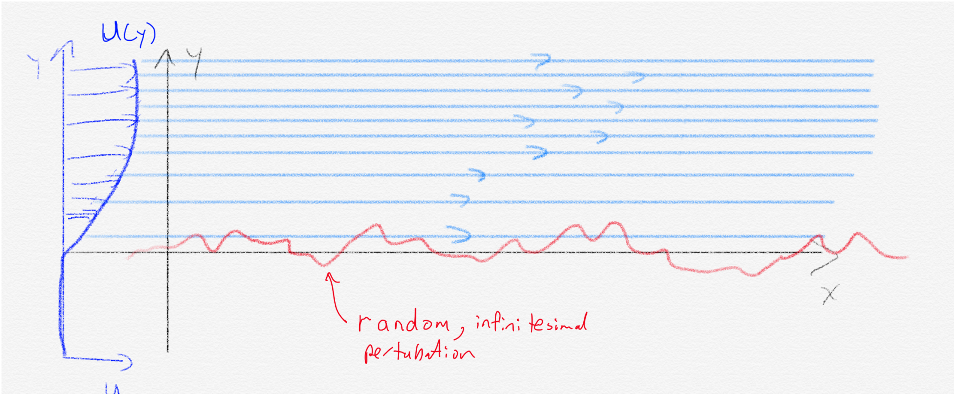

Figure 2: Schematic of shear instability. Background flow is steady and has a dependence in y. The idea of a flow instability is to perturb the flow and see what scales in the flow grow.

$$
\begin{aligned}
\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + v'\frac{\partial U}{\partial y} &= -\frac{1}{\rho}\frac{\partial p'}{\partial x} \\
\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} &= -\frac{1}{\rho}\frac{\partial p'}{\partial x} \\
\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} &= 0
\end{aligned}
$$

To see how this flow behaves in the presence of perturbations, we assume the flow is a periodic travelling wave in the x-direction and has a y-direction structure (to be determined):

$$
\begin{aligned}
u' &= \hat{u}(y)\ e^{ik(x-ct)} \\
v' &= \hat{v}(y)\ e^{ik(x-ct)} \\
p' &= \hat{p}(y)\ e^{ik(x-ct)}
\end{aligned}
$$

This gives three coupled PDEs for the vertical structure of the response that depends on k and c:

$$
-ikc\ \hat{u} + ikU\ \hat{u} + U'\ \hat{v} = -ik\ \hat{p}
$$

$$
\begin{aligned}
-ikc\ \hat{v} + ikU\ \hat{v} &= -ik\ \frac{\partial \hat{p}}{\partial y} \\
ik\ \hat{u} + \frac{\partial \hat{v}}{\partial y} &= 0
\end{aligned}
$$

where here we have defined $U' \equiv \frac{\partial U}{\partial y}$.

This is three coupled equations in three unknowns, so as usual we combine, choosing to do so for $\hat{v}(y)$, and after some algebra get

$$
\hat{v}'' + \hat{v}\left(\frac{U''}{c - U} - k^2\right) = 0 \tag{1}
$$

How does this help us? Equation (1) cannot be solved analytically, but as we will see below some useful criteria for stability can be derived analytically using it. However, first what is the character of the numerical solutions? We note that k is always real (its the wavelength of the disturbance we are interested in), but that c may be imaginary in general. If it is imaginary, then we have growing solutions if $\mathcal{I}(c) < 0$ (assuming $k > 0$) and the growth rate will be given by $k\mathcal{I}(c)$.

Now all we need to do is solve for c and $\hat{v}(y)$ for a given $U(y)$ and k. Equation (1) is a classic eigenvalue problem, where c can have a number of distinct values for a given background profile of $U(y)$. When we solve this numerically, the numerical solution ends up being a matrix eigenvalue problem.

2.2 Numerical Solution

Numerically equation (1) is straightforward to set up using first differencing. First we rewrite in an eigen-vector/eigen value form, where $\hat{v}(y)$ are the eigen vectors and c are the eigenvalues:

$$
\left[\left(\frac{\partial^2}{\partial y^2} - k^2\right) U - U''\right]\hat{v} = c\left[\frac{\partial^2}{\partial y^2} - k^2\right]\hat{v} \tag{2}
$$

We write it this way because that allows us to write the discretized form as matrix equation of the form

$$
A\mathbf{v} = cB\mathbf{v} \tag{3}
$$

where A and B are matrices, and $\mathbf{v}$ is an eigenvector we are solving for and c is a *generalized* eigenvalue problem for which there are numerical solvers in most linear algebra packages (i.e. `scipy.linalg.eigh`).

Setting up the matrices A and B is straight forward based on our previous discretization attempts. If divide the y-domain into N grid points, indexed from 0 to $N-1$, then we end up with two tri-diagonal matrices:

$$
\begin{aligned}
A_{i,i-1} = A_{i,i+1} &= U_i/\Delta y^2 \\
A_{i,i} &= \left(-2/\Delta y^2 - k^2\right) U_i - (U_{i+1} + U_{i-1} - 2U_i)/\Delta y^2
\end{aligned}
$$

and for B:

$$
\begin{aligned}
B_{i,i-1} = B_{i,i+1} &= 1/\Delta y^2 \\
B_{i,i} &= -2/\Delta y^2 - k^2
\end{aligned}
$$

These relationships are for interior points where $i > 0$ and $i < N-1$. For the first and last row we need to assume boundary conditions, which in this case is the disturbance goes to zero at the boundaries: $\hat{v}_{-1} = \hat{v}_N = 0$, and assume that U has not gradient there: $U_{-1} = U_0$ and $U_N = U_{N-1}$. If we do that then we just plug those values into the above relations to get

$$
\begin{aligned}
A_{0,0} &= \left(-2/\Delta y^2 - k^2\right) U_0 - (U_1 - U_0)/\Delta y^2 \\
A_{0,1} &= U_0/\Delta y^2 \\
B_{0,0} &= -2/\Delta y^2 - k^2 \\
B_{0,1} &= 1/\Delta y^2
\end{aligned}
$$

and similarly for $i = N-1$.

This eigenvalue problem can then be solved for each value of k to be explored, and the fastest growing unstable mode determined for each k. An example of this calculation is shown for the profile in figure 3 on the following page. There is a clear minimum in the growth time for this profile at $k = 2.3 \times 10^{-4}$ rad m^{-1} indicating that this wavelength will have the fastest growing instability. The shape of the eigenvectors (figure 5 on page 8) can also be determined if that is of interest.

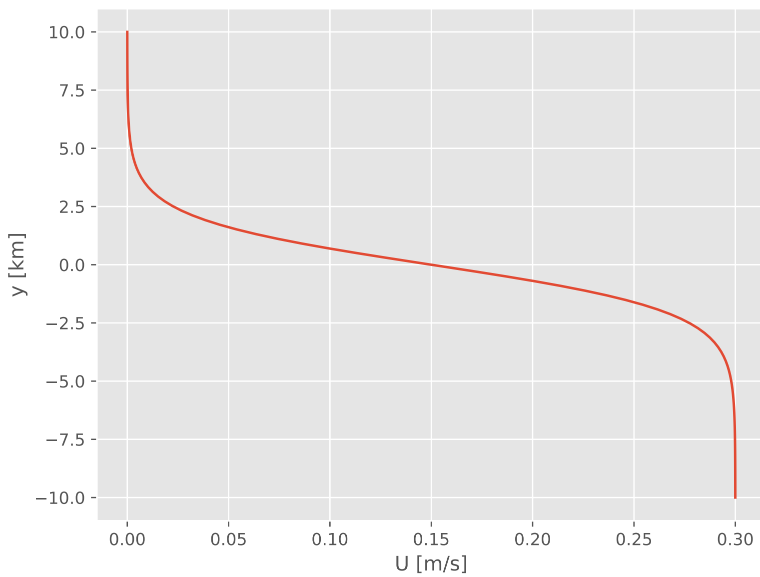

Figure 3: Background velocity profile for stability analysis.

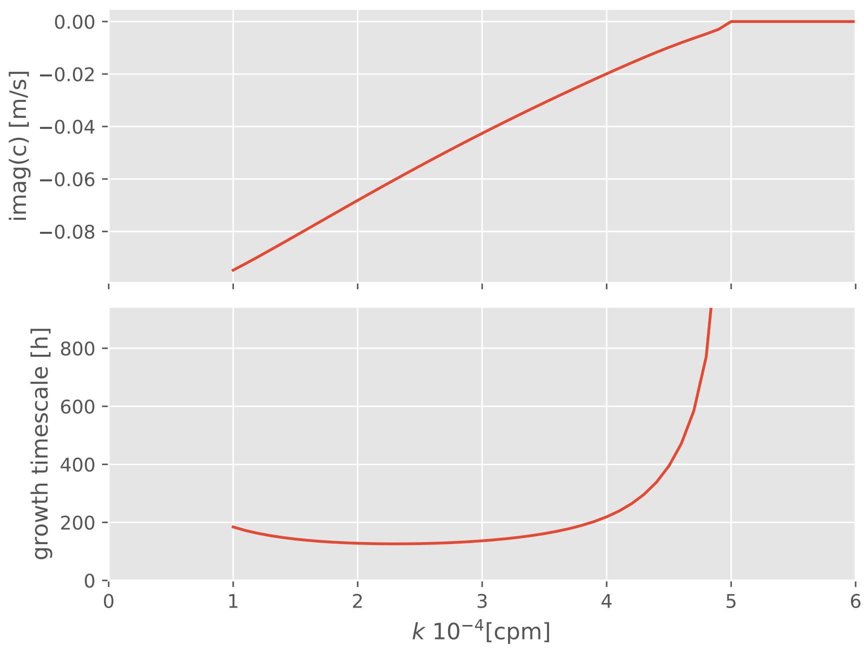

Figure 4: a) Fastest-growing eigenvalue and b) growth timescale as a function of wavenumber for the velocity in figure 3 on the previous page. The fastest growing wavenumber is broadly around 25-km scale.

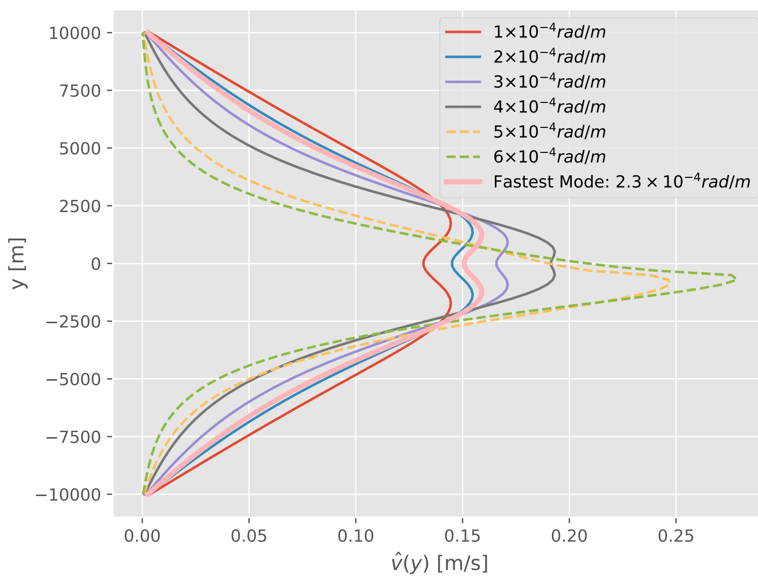

Figure 5: Eigenvectors for discrete wavenumbers k. The eigenvectors with solid colours are unstable, and the dashed ones are stable.

2.3 Fully non-linear simulation

The same velocity profile can be used as the initial conditions of a primitive equation numerical model. Here we set up a channel that has solid boundaries in y and is periodic in x over a distance of 400 km (figure 6) and use the same velocity profile as figure 3 on page 6. The flow is seeded with very small imperfections that then grow or are suppressed. We can see the growth starting in figure 6b.

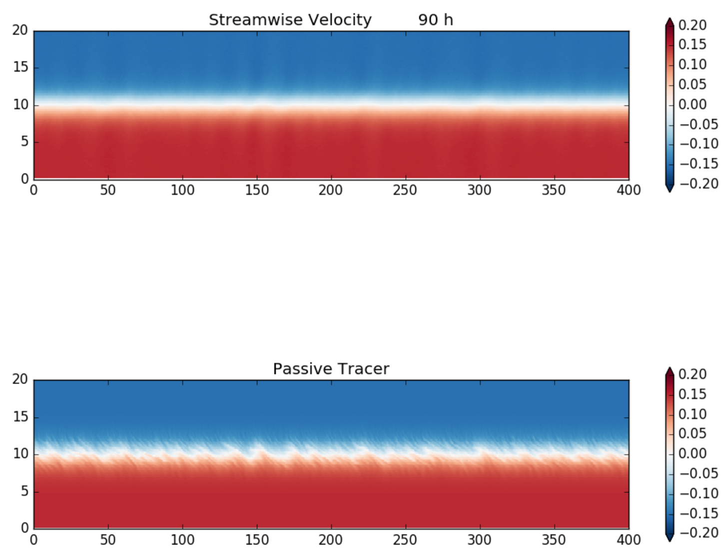

Figure 6: Channel simulation with shear as in figure 3 on page 6, and some added numerical noise after it has been allowed to evolve for 90 h. Upper plot is velocity, and lower plot is a passive tracer with the same initial distribution as the velocity

After the flow has evolved for longer (160h) than the fastest growth scale, it is clear that a scale of about 25 km has been "selected" in the flow instability (figure 7 on the next page). The other scales are still present, but did not grow as fast as this scale and hence it dominates the figure. Its worth noting that the billows in this simulation are all slightly different, and that is the chaotic amplification of slightly different-phased disturbances.

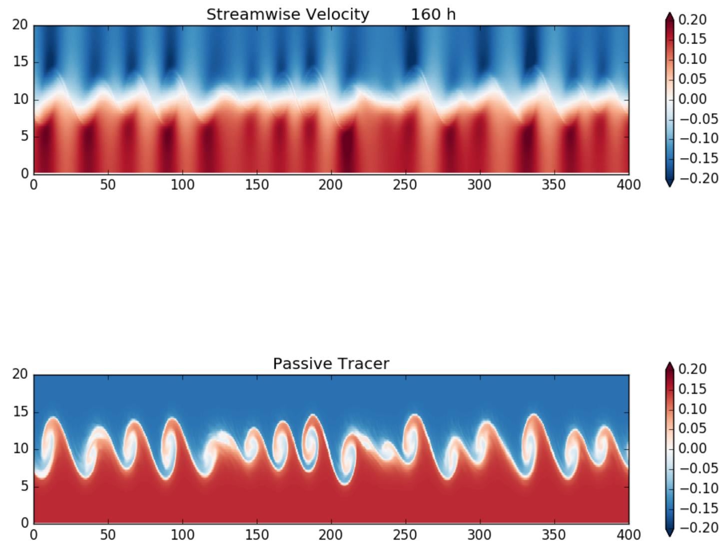

Figure 7: As figure 6 on the previous page except after 160 h, when the instabilites have grown to noticeable size.

As the flow evolves for a longer period of time, these un-even vortices start to self-interact, and pairs of vortices merge to create larger channel-spanning vortices (figure 8). These eventually completely collapse into near-homogeneous turbulence.

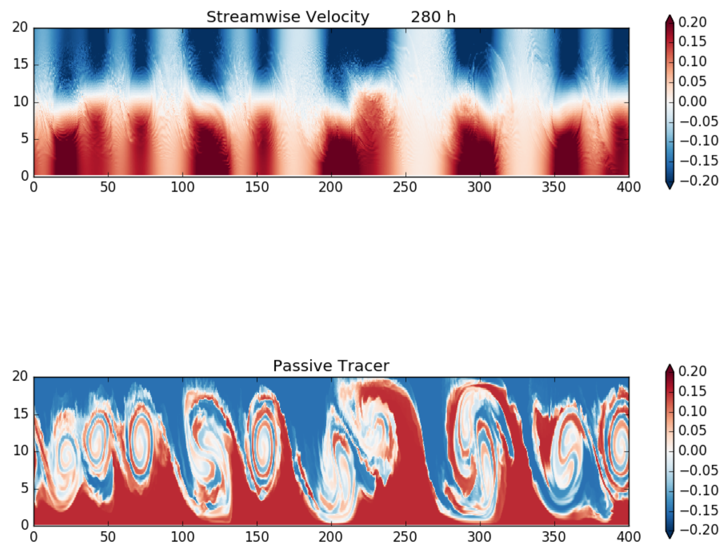

Figure 8: As figure 6 on page 9 except after 160 h, when the instabilites have grown to noticeable size.

The evolving character of the flow is well-visualized by looking at spectra of the tracer variance in the along-stream direction (figure 9). Note that the fastest-growing wavenumbers are near the wavenumber predicted by the linear perturbation theory ($k = 0.036$ cpkm $= 2.3 \times 10^{-4}$ rad/m).

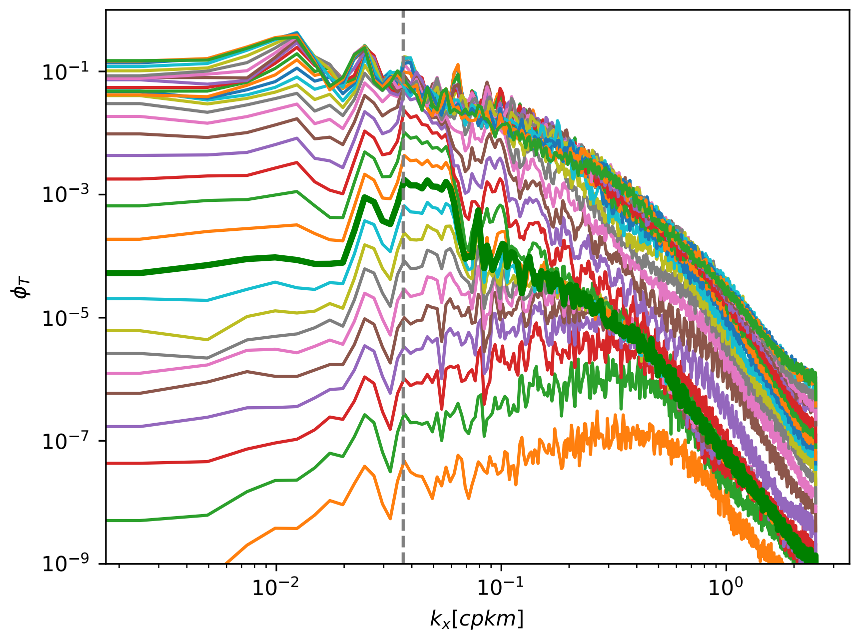

Figure 9: Spectra of tracer variance in the along-flow direction. Spectra with more variance are from later in the simulation. Note the broad peak near the wavenumber of the fastest-growing mode (grey dashed line)

Note that there is a steep roll-off of the tracer spectrum to high wavenumbers. This is a rough approximation of the inertial and convective subranges of turbulence, where I say they are rough because the numerics are such that the Reynolds number of these flows is not very high.