Take-home Final
Phy 426, 2019
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DUE: Fri 26 Apr, 2019, 23:59 by email. Exam to be completed independently. Open book, open notes are fine. Show all work, define any constants you need that I don't provide, check your units, etc. Except as noted, the density of the fluid is ρ, gravity is g, the kinematic viscosity ν, and the fluid can be assumed Bousinesque and incompressible.

Please try to make it readable. I will deduct up to 10% for illegible chicken scratches, so please take the time to recopy your work.

The value of each question is indicated in square brackets, the total is out of 22.

Question 1. Laminar flow out a pipe

A barrel with a viscous oil in it has a thin pipe with radius of 0.01 m leading out of it 1 m below the surface of the oil. Assume that the flow is laminar and viscous, with viscosity $\nu = 10^{-4}\ \mathrm{m^2\,s^{-1}}$. Assume that surface tension is not important (though in such a thin pipe it might be).

1. [4] If the pipe is 4-m long and lays flat what is the flow rate out the end of the pipe (in $\mathrm{m^3 s^{-1}}$)? (HINT: from the point of view of deriving the velocity in the pipe, assume that the pipe is infinite - i.e. don't worry about the end-effects.)
2. [2] Given the flow speed at the centre line of the pipe as a velocity scales, what is a horizontal length scale over which the boundary layer would become fully developed? Is the assumption suggested above OK? Is the assumption that the flow is laminar likely OK?
3. [5] Show that the rate of pressure work done by the fluid in the barrel on the fluid in the hose is equal to the rate of turbulent dissipation in the pipe.

Question 2. Waves reflecting from a step

Consider shallow water waves in a narrow channel where the water depth (at rest) shoals from 10 m to 3 m at a step. Assume no viscous losses. Measurements of along-channel velocity and water depth are taken 50m from the step in the deep part of the channel. Both signals are found to have a period of 8 s, and the peak of the positive sea surface height is found to occur 1 s before the peak of positive down-channel velocity. The sea surface height *in the shallow water* is measured to have an amplitude of 0.1 m.

1. [8] The signal at $x = -50$ m consists of a wave moving towards the step, and one partially reflected from the step. Give the information above, derive the amplitudes of these two waves, expressing your answer as two equations in two unknowns. Note that the incoming and reflected waves are in phase at $x = 0$.
2. [3] Numerically solve for the amplitude of the incoming are reflected waves given the information above. (I used `scipy.optimize.fmin`).

Answer Key for Exam $\boxed{\text{A}}$

Question 1. Laminar flow out a pipe

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   Answer: If the pipe is 1 m from the surface, then the pressure at the pipe is $P' = \rho g z + P_{atm}$. At the outlet of the pipe, the pressure is P_{atm}, so the pressure difference if given by $P/\rho = gz$. If the edge effects don't matter, we can infer a pressure gradient $\frac{1}{\rho}\frac{dP}{dx} = -gz/L \equiv p_x = 2.45$, which will be a constant for the problem.

   The flow in a pipe is given in the text, and is easily derivable as well:

   $$u(r) = \frac{r^2 - a^2}{4\nu} p_x \tag{1}$$

   and integrating over the cross-sectional area gives:

   $$Q = -\frac{\pi a^4}{8\nu} p_x \tag{2}$$

   or $Q = 9.6 \times 10^{-5}\ \mathrm{m^3\,s^{-1}}$.

2. [2] Given the flow speed at the centre line of the pipe as a velocity scales, what is a horizontal length scale over which the boundary layer would become fully developed? Is the assumption suggested above OK? Is the assumption that the flow is laminar likely OK?

   Answer: The flow speed at the center of the pipe is given by the above: $= 0.625\ \mathrm{ms^{-1}}$. The time scale to diffuse 1 cm is $T \approx R^2/\nu = 1s$, so the steady flow will be close to set up by 62 cm into the pipe, which is not minor compared to the length of the pipe.

   In terms of the Reynolds number, the length scale is likely something on the order of that 62 cm, and U 62.5 cm/s. so the Reynolds number is something like ~ 3900, which is pretty high, and turbulence is likely to be important.

3. [5] Show that the rate of pressure work done by the fluid in the barrel on the fluid in the hose is equal to the rate of turbulent dissipation in the pipe.

   Answer: The rate of work is (in m^5/s^3 or W/kg) is

   $$\int_0^a (u(P_{upstream} + P_{atm})/\rho\ 2\pi r\ dr - \int_0^a u P_{atm}/\rho\ \ 2\pi r\ dr = Q P_{upstream}/\rho \tag{3}$$

both of which we calculated above.

The dissipation rate is

$$\begin{aligned} D &= L\int_0^a \nu\left(\frac{\partial u}{\partial r}\right)^2 2\pi r\ dr \\ &= L\int_0^a \nu\left(\frac{r}{2\nu}p_x\right)^2 2\pi r\ dr \\ &= L\int_0^a \frac{r^3}{2\nu}p_x^2\pi\ dr \\ &= L\frac{\pi a^4}{8\nu}p_x \end{aligned}$$

which is equal to the work done.

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1. [8] The signal at $x = -50$ m consists of a wave moving towards the step, and one partially reflected from the step. Give the information above, derive the amplitudes of these two waves, expressing your answer as two equations in two unknowns. Note that the incoming and reflected waves are in phase at $x = 0$.

   Answer: Define the incoming wave as:

$$\begin{aligned} \eta_i(x,t) &= A_i\cos\left(kx-\omega t\right) \\ u_i(x,t) &= A_i\frac{gk}{\omega}\cos\left(kx-\omega t\right) \end{aligned}$$

   with $\omega/k = (gh)^{1/2}$, making k a known. And the refelcted wave by:

$$\begin{aligned} \eta_r(x,t) &= A_r\cos\left(kx+\omega t\right) \\ u_r(x,t) &= -A_r\frac{gk}{\omega}\cos\left(kx+\omega t\right) \end{aligned}$$

   noting the changes of sign of the direction of propagation and if the reflected co-efficient in front of u_r. Using angle identities we can rewrite $\eta = \eta_i + \eta_r$ as:

$$\eta = (A_i + A_r)\cos kx\ \cos\omega t + (A_i - A_r)\cos kx\ \sin\omega t \tag{4}$$

   and then note that we want to write in terms of an amplitude and phase:

$$\begin{aligned} \eta &= A\cos\left(\omega t+\phi\right) \\ &= A\left(\cos\omega t\cos\phi - \sin\omega t\sin\phi\right) \end{aligned}$$

```
In [140]: import numpy as np
import scipy.optimize as sio
h = 10
hs = 3
om = np.pi * 2 / 8
g = 9.81
c = np.sqrt(g*h)
k = om / c
cs = np.sqrt(g*hs)
ks = om / cs
phi = 1 * om
x0 = 50
A = 0.1
def func(x):
    Ai = x[0]
    Ar = x[1]
    hhs = np.sqrt(h / hs)
    first = Ai**2 + Ar**2 + 2 * Ai * Ar * (np.cos(k*x0)**2 - np.sin(k*x0)**2) - A**2
    first = (Ai**2 - Ar**2)*c - cs * A**2
    tankx = np.tan(k*50)
    second = (np.arctan2(-(Ai - Ar) * tankx, Ai + Ar) -
              np.arctan2(-(Ai + Ar) * tankx, Ai - Ar) + phi)
    if np.abs(Ai) < np.abs(Ar):
        second = second*10
    return np.max([np.abs(first), np.abs(second)])

ans = sio.fmin(func, [1, 0.2], ftol=1e-8)
print(ans)
print(np.sqrt(np.sum(ans**2)))
print(ans[0]**2 * c - ans[1]**2 * c - A**2 * cs)
```

```
0.7853981633974483
9.904544411531507 5.424942396007538 43.3995391680603 79.23635529225206
Optimization terminated successfully.
         Current function value: 0.000000
         Iterations: 101
         Function evaluations: 198
[ 0.08134594 -0.03376293]
0.08807438912041984
1.2297164925234583e-08
```

Answer: