Take-home Final Assignment
Phy 426, 2020
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DUE: 21 Apr, 2020, 1200

Exam to be completed independently, though questions to the instructor are welcome/encouraged. Open-book, open-notes are fine. Show all work, define any constants you need that I don't provide, check your units, etc. Except as noted, the density of the fluid is ρ, gravity is g, the kinematic viscosity ν, and the fluid can be assumed Bousinesque and incompressible.

Please try to make it readable. I will deduct up to 10% for illegible chicken scratches, so please take the time to recopy your work. Typeset is preferred.

Question 1. Shear production of turbulent kinetic energy

Consider a slowly evolving flow in the x-direction with a slowly evolving shear in the z-direction given by $\partial U/\partial z \neq 0$. On top of this slowly evolving flow is a flow that is turbulent with velocity fluctuations u', v', w'

1. [10] Using Reynolds averaging from an energy equation for the mean flow and show that the mean flow has a sink of energy given by $+\overline{u'w'}\frac{\partial U}{\partial z}$
2. [4] Explain why this is a sink instead of a source by considering the likely sign of the terms.
3. [8] Again using Reynolds averaging, argue that the *turbulent* kinetic energy has a *source* due to this term.

Question 2. Viscous flow down an incline

Consider a viscous fluid spilling down an infinitely long inclined slope with angle α with respect to the horizontal. Assume the flow is in steady-state, and has a two-dimensional flux $q[\mathrm{m}^2\,\mathrm{s}^{-1}]$ down the slope.

1. [3] Before doing any math, what is the dependence of the fluid thickness on the distance down the infinite slope, and why?
2. [12] How thick is the layer of fluid (H) measured perpendicular to the slope? (HINT: determine the velocity profile, by assuming no stress at the fluid/air interface, and no-slip at the incline/fluid interface.)
3. [10] Demonstrate that the rate of viscous dissipation in the fluid is equal to the loss of energy as the fluid flows down the slope.
4. [4] Discuss the solution at the top of the incline, where the boundary layer has not yet had time to develop. What is an appropriate distance downstream where you might expect the boundary layer to be "well developed"?

Question 3. Lossy standing waves

Consider long waves in a basin H deep, with a vertical wall at one end, each having a tidal frequency ω.

1. [4] A distance L from the far end, the pressure is measured with a gauge to vary as $p = p_o \cos \omega t$. Assuming no energy losses in the basin, what is the water height as a function of x, and t?
2. [4] What would the functional dependence of $u(x, t)$ be?
3. [10] Now, suppose the waves have dissipation, and we also measure u at $x = L$ from the far end, and it is given by $u = u_o \sin(\omega t + \phi)$. what is the average rate of energy loss inside the embayment (in terms of p_o and u_o, and ϕ)?
4. [7] If $\phi << 1$, $p_o << \rho g H$, and $L << \frac{2\pi\sqrt{gH}}{\omega}$, approximate the above just in terms of p_o and ϕ.

Question 4. Minimum energy of irrotational flow

Mathematically (as opposed to dynamically), there are infinitely many flow fields $\mathbf{u}(\mathbf{x}, t)$ that can satisfy i) $\nabla \cdot \mathbf{u} = 0$ and ii) $\mathbf{u} \cdot \mathbf{n} = 0$ along any boundary S enclosing a fluid.

1. [15] Show that the unique *irrotational* flow has the minimum kinetic energy of any such flow.

Question 5. Boundary layer in a teacup

Consider the flow in a "teacup", in this case a steady, solid-body flow, at rate Ω [$\mathrm{rad\,s^{-1}}$] above a flat plate with a no-slip bottom boundary layer.

1. [10] Derive the (steady) equations for the boundary layer, assuming that the tangential velocity can be written as: $V_\theta = \Omega r + v_\theta$, that the boundary layer is thin, and that the Reynolds number is low enough that the non-linear boundary layer terms are negligible.
2. [10] Solve this equation for the tangential velocity perturbation v_θ and the radial u_r, with appropriate boundary conditions for v_θ and u_r assuming an exponential dependence with height from the bottom. (Hint, the PDE is separable in r and z)
3. [5] Derive an expression for the total transport of water [$m^2\, s^{-1}$] in radial direction across any radius R centered on the axis of rotation of the flow.
4. [5] Given this flow rate, what is the vertical velocity as a function of r out the top of the boundary layer?