Take-home Final Assignment
Phy 426, 2020
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DUE: 21 Apr, 2020, 1200

Exam to be completed independently, though questions to the instructor are welcome/encouraged. Open-book, open-notes are fine. Show all work, define any constants you need that I don't provide, check your units, etc. Except as noted, the density of the fluid is ρ, gravity is g, the kinematic viscosity ν, and the fluid can be assumed Bousinesque and incompressible.

Please try to make it readable. I will deduct up to 10% for illegible chicken scratches, so please take the time to recopy your work. Typeset is preferred.

Question 1. Shear production of turbulent kinetic energy

Consider a slowly evolving flow in the x-direction with a slowly evolving shear in the z-direction given by $\partial U/\partial z \neq 0$. On top of this slowly evolving flow is a flow that is turbulent with velocity fluctuations u', v', w'

1. [10] Using Reynolds averaging form an energy equation for the mean flow and show that the mean flow has a sink of energy given by $+\overline{u'w'}\frac{\partial U}{\partial z}$
2. [4] Explain why this is a sink instead of a source by considering the likely sign of the terms.
3. [8] Again using Reynolds averaging on the energy equation, show that the *turbulent* kinetic energy has a *source* due to this term.

Question 2. Viscous flow down an incline

Consider a viscous fluid spilling down an infinitely long inclined slope with angle α with respect to the horizontal. Assume the flow is in steady-state, and has a two-dimensional flux $q[\mathrm{m}^2\,\mathrm{s}^{-1}]$ down the slope.

1. [3] Before doing any math, what is the dependence of the fluid thickness on the distance down the infinite slope, and why?
2. [12] How thick is the layer of fluid (H) measured perpendicular to the slope? (HINT: determine the velocity profile, by assuming no stress at the fluid/air interface, and no-slip at the incline/fluid interface.)
3. [10] Demonstrate that the rate of viscous dissipation in the fluid is equal to the loss of energy as the fluid flows down the slope.
4. [4] Discuss the solution at the top of the incline, where the boundary layer has not yet had time to develop. What is an appropriate distance downstream where you might expect the boundary layer to be "well developed"?

Question 3. Lossy standing waves

Consider long waves in a basin H deep, with a vertical wall at one end, each having a tidal frequency ω.

1. [4] A distance L from the far end, the pressure is measured with a gauge to vary as $p = p_o \cos \omega t$. Assuming no energy losses in the basin, what is the water height as a function of x, and t?
2. [4] What would the functional dependence of $u(x,t)$ be?
3. [10] Now, suppose the waves have dissipation, and we also measure u at $x = L$ from the far end, and it is given by $u = u_o \sin(\omega t + \phi)$. what is the average rate of energy loss inside the embayment (in terms of p_o and u_o, and ϕ; HINT: you don't need to think about the form of the dissipation or where it happens at all, you just need to consider that the net energy into the inlet is equal to the dissipation – if there were no dissipation, there would be no net energy flux)?
4. [7] If $\phi << 1$, $p_o << \rho g H$, and $L << \frac{2\pi\sqrt{gH}}{\omega}$, approximate the above just in terms of p_o and ϕ.

Question 4. Minimum energy of irrotational flow

Mathematically (as opposed to dynamically), there are infinitely many flow fields $\mathbf{u}(\mathbf{x}, t)$ that can satisfy i) $\nabla \cdot \mathbf{u} = 0$ and ii) $\mathbf{u} \cdot \mathbf{n} = 0$ along any boundary S enclosing a fluid.

1. [15] Show that the unique *irrotational* flow has the minimum kinetic energy of any such flow. (Hint, any flow can be split into an irrotational component, and a residual).

Question 5. Boundary layer in a teacup

Consider the flow in a "teacup", in this case a steady, solid-body flow, at rate Ω [$\mathrm{rad\,s^{-1}}$] above a flat plate with a no-slip bottom boundary layer.

1. [10] Derive the (steady) equations for the boundary layer, assuming that the tangential velocity can be written as: $V_\theta = \Omega r + v_\theta$, that the boundary layer is thin, and that the Reynolds number is low enough that the non-linear boundary layer terms are negligible. (Hint don't forget that there is an "outer flow" and that this *will* have pressure gradients).
2. [10] Solve this equation for the tangential velocity perturbation v_θ and the radial u_r, with appropriate boundary conditions for v_θ and u_r assuming an exponential dependence with height from the bottom. (Hint, the PDE is separable in r and z)
3. [5] Derive an expression for the total transport of water [$m^2\,s^{-1}$] in radial direction across any radius R centered on the axis of rotation of the flow.
4. [5] Given this flow rate, what is the vertical velocity as a function of r out the top of the boundary layer?

Answer Key for Exam $\boxed{\text{A}}$

Question 1. Shear production of turbulent kinetic energy

Consider a slowly evolving flow in the x-direction with a slowly evolving shear in the z-direction given by $\partial U/\partial z \neq 0$. On top of this slowly evolving flow is a flow that is turbulent with velocity fluctuations u', v', w'

1. [10] Using Reynolds averaging form an energy equation for the mean flow and show that the mean flow has a sink of energy given by $+\overline{u'w'}\frac{\partial U}{\partial z}$

   Answer: This is covered explicitly in the text in the chapter on turbulence: "Kinetic Energy Budget of Mean Flow".

2. [4] Explain why this is a sink instead of a source by considering the likely sign of the terms.

   Answer: This is again covered explicitly in the text: if $dU/dz > 0$ then $w' > 0$ will, on average bring water upwards so that it is moving slower then U, and hence $u' < 0$. Similarly if the converse is true, and hence $\overline{u'w'} < 0$, and $\overline{u'w'}\frac{\partial U}{\partial z} < 0$. The converse all applies if $dU/dz < 0$.

3. [8] Again using Reynolds averaging on the energy equation, show that the *turbulent* kinetic energy has a *source* due to this term.

   Answer: This is covered in the *next* section, entitled "Kinetic Energy Budget of Turbulent Flow" where after all is done, the term $-\overline{u'w'}\frac{\partial U}{\partial z}$ appears, and is hence usually a source.

Question 2. Viscous flow down an incline

Consider a viscous fluid spilling down an infinitely long inclined slope with angle α with respect to the horizontal. Assume the flow is in steady-state, and has a two-dimensional flux $q[\mathrm{m^2\,s^{-1}}]$ down the slope.

1. [3] Before doing any math, what is the dependence of the fluid thickness on the distance down the infinite slope, and why?

   Answer: The fluid thickness cannot change as a function of distance down the slope. If it did, then the thickness would get infinitely thin somewhere, with infinite velocities, and infinite shear stresses.

2. [12] How thick is the layer of fluid (H) measured perpendicular to the slope? (HINT: determine the velocity profile, by assuming no stress at the fluid/air interface, and no-slip at the incline/fluid interface.)

   Answer: First, we adopt a reference frame where x is along the slope, and z is perpendicular to the slope. In this frame we have

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + g_S + \nu\frac{\partial^2 u}{\partial z^2} \qquad (1)$$

where $g_S = g \sin\alpha$ and $w = 0$. From the logic above, there can't be any variations in x, and $w = 0$, so steady-state implies:

$$0 = g_S + \nu \frac{\partial^2 u}{\partial z^2}. \tag{2}$$

The boundary condition on u is $u(z = 0) = 0$, and no stress at the surface, i.e. $\partial u(z = H)/\partial z = 0$, so it is straightforward to integrate to get

$$u = \frac{g_S}{\nu}\left(Hz - \frac{z^2}{2}\right) \tag{3}$$

To get H, we integrate and relate to q:

$$\begin{aligned} q &= \int_o^H u \mathrm{d}z && (4) \\ &= \frac{g_S}{3\nu} H^3 && (5) \end{aligned}$$

so $H = \left(\frac{3\nu q}{g_S}\right)^{\frac{1}{3}}$.

3. [10] Demonstrate that the rate of viscous dissipation in the fluid is equal to the loss of energy as the fluid flows down the slope.

Answer: Consider a control volume in the flow at x and $x + \Delta x$. The potential energy of the fluid at x is clearly larger than that down the slope, so there is an energy loss in the control volume. To calculate it, we consider the energy flowing in at x and subtract from the energy flowing out at $x + \Delta x$:
Flowing in, the water is higher than flowing out:

$$\begin{aligned} Q_{in} &= \int_0^H u(z) PE(z) \mathrm{d}z && (6) \\ &= \int_0^H u(z)(\rho g(z + z_o + \Delta x \sin\alpha)) \mathrm{d}z && (7) \end{aligned}$$

$$Q_{out} = \int_0^H u(z)(\rho g(z + z_o)) \mathrm{d}z \tag{8}$$

where z_o is an arbitrary offset. The net flux into the control volume is simply

$$\begin{aligned} Q_{in} - Q_{out} &= \rho g \Delta x \sin\alpha \int_0^H u(z) \mathrm{d}z && (9) \\ &= \rho g \Delta x \sin\alpha \frac{g_S H^3}{3\nu} && (10) \end{aligned}$$

or taking Δx to zero:

$$dQ/dx = \rho \frac{g_S^2 H^3}{3\nu}. \tag{11}$$

This has to equal the depth-integrated dissipation

$$
\begin{aligned}
D &= \rho\nu \int_0^H \left(\frac{\partial u}{\partial z}\right)^2 \mathrm{d}z && (12)\\
&= \rho\nu \int_0^H \frac{g_s^2}{\nu^2}\left(H^2 - 2Hz + z^2\right) \mathrm{d}z && (13)\\
&= \rho\frac{g_S^2}{\nu}\frac{H^3}{3}. && (14)
\end{aligned}
$$

and so the proposition is true.

4. [4] Discuss the solution at the top of the incline, where the boundary layer has not yet had time to develop. What is an appropriate distance downstream where you might expect the boundary layer to be "well developed"?

Question 3. Lossy standing waves

Consider long waves in a basin H deep, with a vertical wall at one end, each having a tidal frequency ω.

1. [4] A distance L from the far end, the pressure is measured with a gauge to vary as $p = p_o \cos\omega t$. Assuming no energy losses in the basin, what is the water height as a function of x, and t?

   Answer: This is just a standing wave so we have

$$
\begin{aligned}
p &= (A/2)\left(\cos(kx-\omega t) + \cos(kx+\omega t)\right) && (15)\\
&= A\cos(kx)\cos(\omega t) && (16)\\
&= p_o\frac{\cos(kx)}{\cos(kL)}\cos(\omega t) && (17)
\end{aligned}
$$

   and the waves are long, so the surface height is given by:

$$
\eta(x,t) = \frac{p_0}{\rho g}\frac{\cos(kx)}{\cos(kL)}\cos(\omega t) \qquad (18)
$$

2. [4] What would the functional dependence of $u(x,t)$ be?

   Answer: For long waves

$$
\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \qquad (19)
$$

   so,

$$
\begin{aligned}
u(x,t) &= \frac{p_0}{2\rho}\frac{k}{\omega}\left(-\cos(kx-\omega t) + \cos(kx+\omega t)\right) && (20)\\
&= \frac{p_0}{\rho}\frac{k}{\omega}\frac{\sin(kx)}{\cos(kL)}\sin(\omega t) && (21)
\end{aligned}
$$

   using the same angle identities we used for the standing wave above. Note that u and eta are out of phase and that $u(x=0)=0$.

3. [10] Now, suppose the waves have dissipation, and we also measure u at $x = L$ from the far end, and it is given by $u = u_o \sin(\omega t + \phi)$. what is the average rate of energy loss inside the embayment (in terms of p_o and u_o, and ϕ; HINT: you don't need to think about the form of the dissipation or where it happens at all, you just need to consider that the net energy into the inlet is equal to the dissipation – if there were no dissipation, there would be no net energy flux)?

   Answer: The instantaneous energy flux into the inlet is given by

$$F = \int_o^H up\,\mathrm{d}z \tag{22}$$
$$= Hup \tag{23}$$

and then to get the mean, we average over the time period $T = 2\pi/\omega$:

$$\langle F \rangle = \frac{H}{T}\int_o^T up\,\mathrm{d}t \tag{24}$$
$$= \frac{H}{T}p_o u_o \int_o^T \sin(\omega t + \phi)\cos(\omega t)\mathrm{d}t \tag{25}$$
$$= \frac{H}{T}p_o u_o \int_o^T \left(\sin\omega t\cos\omega t\cos\phi + \cos^2\omega t \sin\phi\right)\mathrm{d}t \tag{26}$$

The first integrand vanishes, and the second is easy to evaluate:

$$< F >= \frac{Hp_o u_o}{2}\sin\phi \tag{27}$$

The rate of energy loss is simply this net flux into the bay. Note that if $\phi = 0$, there is no energy loss, and we are back to a standing wave.

4. [7] If $\phi << 1$, $p_o << \rho g H$, and $L << \frac{2\pi\sqrt{gH}}{\omega}$, approximate the above just in terms of p_o and ϕ.

   Answer: In this limit, the small phase change means that the bay largely rises and falls in qudrature. Also, the seasurface slope is small, so the flux of water into the bay is equal to the seasurface height change:

$$Hu = L\frac{\partial\eta}{\partial t} \tag{28}$$

so, given our variables:

$$u_o = (L/H)\omega\frac{p_o}{\rho g} \tag{29}$$

so, we have

$$< F >= \frac{p_o^2\omega L}{2\rho g}\sin\phi \tag{30}$$

Question 4. Minimum energy of irrotational flow

Mathematically (as opposed to dynamically), there are infinitely many flow fields $\mathbf{u}(\mathbf{x}, t)$ that can satisfy i) $\nabla \cdot \mathbf{u} = 0$ and ii) $\mathbf{u} \cdot \mathbf{n} = 0$ along any boundary S enclosing a fluid.

1. [15] Show that the unique *irrotational* flow has the minimum kinetic energy of any such flow. (Hint, any flow can be split into an irrotational component, and a residual).

Answer: An irrotational flow can be represented by $\mathbf{u}_I = \nabla\phi$ where ϕ is the flow potential, so any arbitrary flow field can be written as the unique irrotational flow plus an anomaly: $\mathbf{u} = \nabla\phi + \mathbf{u}'$.

The kinetic energy, T, of such a flow is $2T = \nabla\phi^2 + 2\mathbf{u}'\cdot\nabla\phi + \mathbf{u}'^2$. Integrating over the volume enclosing the fluid we get

$$\int_V T \;\mathrm{d}V = \int_V \nabla\phi^2 \;\mathrm{d}V + \int_V \mathbf{u}'^2 \;\mathrm{d}V + 2\int_V \mathbf{u}'\cdot\nabla\phi \;\mathrm{d}V.$$

(31)

The first term is the kinetic energy of the irrotational flow, the second term is always positive, so only the last term could counterbalance to to make the total kinetic energy smaller than $\int_V \nabla\phi^2 \;\mathrm{d}V$. However, we can easily show that this term is zero because the fact that $\nabla\cdot\mathbf{u}' = 0$ means that $\mathbf{u}'\cdot\nabla\phi = \nabla\cdot\,(\mathbf{u}'\phi)$ allowing us to use the divergence theorem:

$$\begin{aligned}\int_V \mathbf{u}'\cdot\nabla\phi \;\mathrm{d}V &= \int_V \nabla\cdot\,(\mathbf{u}'\phi)\;\mathrm{d}V\\ &= \oint_S \phi\mathbf{u}'\cdot\mathrm{d}\mathbf{S}\\ &= 0\end{aligned}$$

So,

$$\begin{aligned}\int_V T \;\mathrm{d}V &= \int_V \nabla\phi^2 \;\mathrm{d}V + \int_V \mathbf{u}'^2 \;\mathrm{d}V\\ &\geq \int_V \nabla\phi^2 \;\mathrm{d}V\end{aligned}$$

and therefore the irrotational flow has the minimum energy.

Question 5. Boundary layer in a teacup

Consider the flow in a "teacup", in this case a steady, solid-body flow, at rate Ω [$\mathrm{rad\,s^{-1}}$] above a flat plate with a no-slip bottom boundary layer.

1. [10] Derive the (steady) equations for the boundary layer, assuming that the tangential velocity can be written as: $V_\theta = \Omega r + v_\theta$, that the boundary layer is thin, and that the Reynolds number is low enough that the non-linear boundary layer terms are negligible. (Hint don't forget that there is an "outer flow" and that this *will* have pressure gradients).

Answer: First, note that the boundary layer has a pressure gradient in it given by the outer flow: In this case,

$$\begin{aligned}\frac{1}{\rho}\frac{\partial P}{\partial r} &= \frac{v_\theta^2}{r} = \Omega^2 r \\ \frac{1}{\rho}\frac{\partial P}{\partial \theta} &= 0\end{aligned}$$

Unless there is some kind of weird turbulence, the boundary-layer flow will not depend on θ, but it will depend on r and z. Considering the radial momentum equation:

$$u_r\frac{\partial u_r}{\partial r} + w_z\frac{\partial u_r}{\partial z} - \frac{(\Omega r + v_\theta)^2}{r} = -\Omega^2 r + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{\partial^2 u_r}{\partial z^2}\right) \tag{32}$$

or

$$u_r\frac{\partial u_r}{\partial r} + w_z\frac{\partial u_r}{\partial z} - 2\Omega v_\theta + \frac{v_\theta^2}{r} = \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{\partial^2 u_r}{\partial z^2}\right) \tag{33}$$

Similarly for the tangential velocity:

$$u_r\frac{\partial v_\theta}{\partial r} + w_z\frac{\partial v_\theta}{\partial z} + 2\Omega u_r + u_r v_\theta = \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) + \frac{\partial^2 v_\theta}{\partial z^2}\right) \tag{34}$$

Assuming the boundary layer is thin and dropping the non-linear terms:

$$\begin{aligned}-2\Omega v_\theta &= \nu\frac{\partial^2 u_r}{\partial z^2} \\ 2\Omega u_r &= \nu\frac{\partial^2 v_\theta}{\partial z^2}\end{aligned}$$

2. [10] Solve this equation for the tangential velocity perturbation v_θ and the radial u_r, with appropriate boundary conditions for v_θ and u_r assuming an exponential dependence with height from the bottom. (Hint, the PDE is separable in r and z)

Answer: First combine to get a single PDE for v_θ:

$$v_\theta = -\left(\frac{\nu}{2\Omega}\right)^2\frac{\partial^4 v_\theta}{\partial z^4} \tag{35}$$

Assume solutions of form $v_\theta = A(r)e^{kz}$, where k is a complex wavenumber, and we get:

$$-\left(\frac{2\Omega}{\nu}\right)^2 = k^4 \tag{36}$$

or $k^2 = \pm i\frac{2\Omega}{\nu}$ and $k = \pm\sqrt{\frac{\Omega}{\nu}}(1+i)$ Now, the boundary layer does not admit the exponentially growing root, so defining $\delta \equiv \sqrt{\frac{\nu}{\Omega}}$:

$$v_\theta = A(r)\exp\left(-\frac{z}{\delta}\right)\cos\left(\frac{z}{\delta}\right) \tag{37}$$

Now, we know from the boundary conditions that $V_\theta = 0$ at $z = 0$, so $v_\theta = -\Omega r$ at $z = 0$:

$$v_\theta = -\Omega r\ \exp\left(-\frac{z}{\delta}\right)\cos\left(\frac{z}{\delta}\right) \tag{38}$$

We can derive u_r from the equations above, with a bit of care, to show that

$$u_r = -\Omega r\ \exp\left(-\frac{z}{\delta}\right)\sin\left(\frac{z}{\delta}\right) \tag{39}$$

3. [5] Derive an expression for the total transport of water $[m^2\, s^{-1}]$ in radial direction across any radius R centered on the axis of rotation of the flow.

   Answer: For this we simply integrate $\int_0^\infty u_r \; \mathrm{d}z$ and then integrate around the circle:

$$\begin{aligned}
\int_0^\infty u_r \; \mathrm{d}z &= -\Omega R \int_0^\infty \exp\left(-z/\delta\right) \sin\left(z/\delta\right) \\
&= -\Omega R \frac{\delta}{2} \left[\exp\left(-z/\delta\right) \left(\sin(z/\delta) - \cos(z/\delta)\right)\right]_0^\infty \\
&= -\frac{\Omega R \delta}{2}
\end{aligned}$$

   which, integrated around a a circle, gives: $T_r = -\pi \Omega R^2 \delta$, towards the centre.

4. [5] Given this flow rate, what is the vertical velocity as a function of r out the top of the boundary layer?

   Answer: Since the transport is $-\pi\Omega\delta R^2$ then the net transport into the area bounded by two rings radius R_1 and R_2, where $R_1 > R_2$ is simply $-\pi\Omega\delta\,(R_1^2 - R_2^2)$, and hence there is everywhere a positive vertical velocity. The area of such an annulus is $\pi(R_1^2 - R_2^2)$ so the vertical velocity is a constant $w = \pi\Omega\delta$.