Take-home Final
Phy 426, 2018
J. Klymak

DUE: Fri 13 Apr, 2017, 17:00 Exam to be completed independently. Open book, open notes are fine. Show all work, define any constants you need that I don't provide, check your units, etc. Except as noted, the density of the fluid is ρ, gravity is g, the kinematic viscosity ν, and the fluid can be assumed Bousinesque and incompressible.

Please try to make it readable. I will deduct up to 10% for illegible chicken scratches, so please take the time to recopy your work.

The value of each question is indicated in square brackets, the total is out of 75.

Question 1. Laminar boundary layer under a shallow-water wave

Consider a shallow-water wave over a flat-bottom sea-floor of depth H. Assume the wave is specified by the sea-surface displacement $\eta(x,t) = \eta_0 \cos(kx - \omega t)$, where ω is the frequency of the wave (rad/s) and k is the wavenumber (rad/m). Assume that the flow remains laminar throughout the water column, has a viscosity ν, and obeys a no-slip boundary condition at the sea floor.

1. [4] Describe an appropriate Reynolds number for the flow, and show for a 0.1-m amplitude wave in 10-m of water with a 16-s period the Reynolds number is very large.
2. [4] Derive a scaling for the thickness of the bottom boundary layer δ and state how large it might be given the parameters above.
3. [12] Derive an expression for the velocity above the sea-floor at $x = 0$, $u(0, z, t)$. (HINT: think about the scaling here - from the very thin boundary layer, the flow above looks infinite in both the vertical and the horizontal). Sketch the solution at a couple of times.
4. [12] What is the rate of energy dissipation due to the boundary layer viscosity, averaged over a wave period? What is an estimate of the decay time-scale of the wave (i.e. *do not* solve the full decaying time-dependent equation; assume the decay timescale is much greater than the period of the waves, so that you can assume the wave forcing is steady for the calculation).

Question 2. Lossy standing waves

Consider the quasi-steady response in a rectangular basin H deep, W wide, with a vertical wall at one end ("the head") to forcing at a tidal frequency ω. Assume that ω is small enough that the waves are shallow-water waves.

1. [4] A distance L from "the head" of the channel, the pressure is measured with a gauge to vary as $p = p_o \cos \omega t$. Assuming no energy losses in the basin, what is the water height as a function of x, and t?
2. [4] Again, assume that the flow loses no energy, what would the functional dependence of $u(x,t)$ be?

3. [12] Suppose we measure u at $x = L$ from the head, and it is found to be given by $u = u_o \sin(\omega t + \phi)$. what is the average rate of energy loss inside the embayment in terms of p_o, u_o and ϕ?

4. [7] If $\phi << 1$, $p_o << \rho g H$, and $L << \frac{2\pi\sqrt{gH}}{\omega}$, approximate the energy loss just in terms of p_o and ϕ.

Question 3. Flow around a Headland

Consider a headland protruding into a rectangular channel of free-stream depth H. The headland is approximated by a circle with radius R. Assume that R is much less than the width of the channel, and that the free-stream speed far from the headland is U along the channel. The flow separates from the headland at the tip.

1. [4] What is the fastest speed along the headland?

2. [4] Assume that the pressure in the separation bubble behind the tip is the same as at the tip. What is the total drag on the flow exerted by the headland?

3. [4] Assuming that the separation vortex is a Rankine vortex with a radius similar to the headland (i.e. approximately R) and outer speed given by the speed of the flow past the headland, derive an expression for the sea-surface height in the eddy?

4. [4] Suppose the mean flow suddenly turns off. What direction, and how fast will the headland eddy start to move?

Answer Key for Exam $\boxed{\text{A}}$

Question 1. Laminar boundary layer under a shallow-water wave

Consider a shallow-water wave over a flat-bottom sea-floor of depth H. Assume the wave is specified by the sea-surface displacement $\eta(x,t) = \eta_0 \cos(kx - \omega t)$, where ω is the frequency of the wave (rad/s) and k is the wavenumber (rad/m). Assume that the flow remains laminar throughout the water column, has a viscosity ν, and obeys a no-slip boundary condition at the sea floor.

1. [4] Describe an appropriate Reynolds number for the flow, and show for a 0.1-m amplitude wave in 10-m of water with a 16-s period the Reynolds number is very large.

   Answer: A Reynolds number is the ratio of the advection term to the viscous term. Here, advection is horizontal, so we want to compare $\frac{UL}{\nu}$, where U us a velocity scale and L is a *horizontal* advection scale. In order to get the velocity scale, we turn to the x-momentum equation for hydrostatic waves:

   $$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \tag{1}$$

   and find that $U = \frac{gk}{\omega}\eta_0$, which using the dispersion relation for surface waves gives the scale: $U = \sqrt{\frac{g}{H}}\eta_0$
   The horizontal advective scale is just $L = U/\omega$, so we have a Reynolds number:

   $$Re = \frac{g\ \eta_0^2}{H\omega\nu} \approx 25,000 \tag{2}$$

2. [4] Derive a scaling for the thickness of the bottom boundary layer δ and state how large it might be given the parameters above.

   Answer: The boundary layer thickness is $\sqrt{\nu T}$ where T is some time scale which in this case is simply ω^{-1}, so $\delta \approx \sqrt{\nu/\omega} \approx 1.5$ mm

3. [12] Derive an expression for the velocity above the sea-floor at $x = 0$, $u(0,z,t)$. (HINT: think about the scaling here - from the very thin boundary layer, the flow above looks infinite in both the vertical and the horizontal). Sketch the solution at a couple of times.

   Answer: We assume that $\frac{\partial}{\partial x} << \frac{\partial}{\partial z}$, then the boundary layer equations are:

   $$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}$$

   The boundary condition is that $u(0,t) = 0$ at the sea floor and that $u(z >>\delta$, t) = $u_\infty(0,z,t)$. u_∞ is just the velocity of shallow-water gravity waves:

   $$u_\infty = \eta_0\sqrt{\frac{g}{H}}\cos(\omega t) = u_0\cos(\omega t) \tag{3}$$

As we usually do for boundary layer problems, we assume that the horizontal pressure gradient is independent of the boundary layer, so

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\frac{\partial \eta}{\partial x} = gk\eta_0 \sin(\omega t) = \sqrt{\frac{g}{H}}\omega\eta_0 \sin(\omega t) \tag{4}$$

But, perhaps more useful is to note that

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial u_\infty}{\partial t} \tag{5}$$

so we can rewrite the differential equation as

$$\begin{aligned}
\frac{\partial(u-u_\infty)}{\partial t} &= \nu\frac{\partial^2(u-u_\infty)}{\partial z^2} \\
\frac{\partial(u')}{\partial t} &= \nu\frac{\partial^2(u')}{\partial z^2}
\end{aligned}$$

where now u' has boudnary conditions:

$$\begin{aligned}
u'(0,t) &= -u_\infty(t) \\
u'(z >> \delta, t) &= 0
\end{aligned}$$

To solve, we assume assume that u' is separable and oscillatory with the same frequency as the forcing: $u' = F(z)e^{i\omega t}$, yielding

$$i\omega F = \nu F'' \tag{6}$$

which has an exponentially decaying solution:

$$F = Ae^{-(i+1)(z/\delta)} \tag{7}$$

where $\delta = \sqrt{2\nu/\omega}$ (as above except an extra 2).
Note that the solution oscillates in z as well as time:

$$u' = u_0 e^{-z/\delta}\cos(\omega t - z/\delta) \tag{8}$$

of course, $u = u' + u_\infty$.

4. [12] What is the rate of energy dissipation due to the boundary layer viscosity, averaged over a wave period? What is an estimate of the decay time-scale of the wave (i.e. *do not* solve the full decaying time-dependent equation; assume the decay timescale is much greater than the period of the waves, so that you can assume the wave forcing is steady for the calculation).

Answer: Dissipation is given by $\epsilon = \nu\left(\partial u/\partial z\right)^2$, and to get the total per are of the wave, we integrate vertically: $D = \int_0^H \epsilon \, \mathrm{d}z$. Also note that u_∞ has no vertical dependence, so

$$\begin{aligned}
\frac{\partial u'}{\partial z} &= -\frac{1}{\delta}u_0 e^{-z/\delta}\left[\cos(\omega t - z/\delta) + \sin(\omega t - z/\delta)\right] \\
\epsilon &= \nu\frac{u_0^2}{\delta^2}e^{-2z/\delta}\left[\cos^2(\omega t - z/\delta) + \sin^2(\omega t - z/\delta) + 2\cos(\omega t - z/\delta)\sin(\omega t - z/\delta)\right] \\
&= \nu\frac{u_0^2}{\delta^2}e^{-2z/\delta}
\end{aligned}$$

where the last expression comes because we average the last term over a wave period so the last term averages to zero. Integrating vertically:

$$D = \nu \frac{1}{2} \frac{u_0^2}{\delta} \tag{9}$$

To compare to the energy in the wave, we note that the energy density in the wave, averaged over a tidal period is $E = \frac{1}{2} g \eta_0^2$, so the decay timescale is simply $\tau = E/D = g\eta_0^2 \delta/(\nu u_0^2) = H\delta/\nu$. For our case, $\tau = 22,600s$, which isn't a super long time!

Question 2. Lossy standing waves

Consider the quasi-steady response in a rectangular basin H deep, W wide, with a vertical wall at one end ("the head") to forcing at a tidal frequency ω. Assume that ω is small enough that the waves are shallow-water waves.

1. [4] A distance L from "the head" of the channel, the pressure is measured with a gauge to vary as $p = p_o \cos \omega t$. Assuming no energy losses in the basin, what is the water height as a function of x, and t?

   Answer: This is just a standing wave so we have

$$\begin{aligned} p &= (A/2)\left(\cos\left(kx - \omega t\right) + \cos\left(kx + \omega t\right)\right) && (10) \\ &= A\cos(kx)\cos(\omega t) && (11) \\ &= p_o \frac{\cos(kx)}{\cos(kL)} \cos(\omega t) && (12) \end{aligned}$$

   and the waves are long, so the surface height is given by:

$$\eta(x,t) = \frac{p_0}{\rho g} \frac{\cos(kx)}{\cos(kL)} \cos(\omega t) \tag{13}$$

2. [4] Again, assume that the flow loses no energy, what would the functional dependence of $u(x,t)$ be?

   Answer: For long waves

$$\frac{\partial u}{\partial t} = -g \frac{\partial \eta}{\partial x} \tag{14}$$

   so,

$$\begin{aligned} u(x,t) &= \frac{p_0}{2\rho} \frac{k}{\omega} \left(-\cos\left(kx - \omega t\right) + \cos\left(kx + \omega t\right)\right) && (15) \\ &= \frac{p_0}{\rho} \frac{k}{\omega} \frac{\sin(kx)}{\cos(kL)} \sin(\omega t) && (16) \end{aligned}$$

   using the same angle identities we used for the standing wave above. Note that u and eta are out of phase and that $u(x = 0) = 0$.

3. [12] Suppose we measure u at $x = L$ from the head, and it is found to be given by $u = u_o \sin\left(\omega t + \phi\right)$. what is the average rate of energy loss inside the embayment in terms of p_o, u_o and ϕ?

Answer: The instantaneous energy flux (per inlet width) into the inlet is given by:

$$F = \int_o^H up \, \mathrm{d}z \tag{17}$$
$$= Hup. \tag{18}$$

If no energy is lost in the inlet, there will be no net energy flux into the inlet and this integral will be zero. And since u and p are in quadrature in the non-lossy situation, that would work out mathematically. Here there is a slight phase shift, so they are not in quadrature, leading to a net energy flux that must be balanced by dissipation. To get the mean, we average over the wave period $T = 2\pi/\omega$:

$$\langle F \rangle = \frac{H}{T}\int_o^T up\mathrm{d}t \tag{19}$$
$$= \frac{H}{T}p_o u_o \int_o^T \sin(\omega t + \phi)\cos(\omega t)\mathrm{d}t \tag{20}$$
$$= \frac{H}{T}p_o u_o \int_o^T \left(\sin\omega t \cos\omega t \cos\phi + \cos^2\omega t \sin\phi\right)\mathrm{d}t \tag{21}$$

The first integrand vanishes, and the second is easy to evaluate:

$$< F >= \frac{Hp_o u_o}{2}\sin\phi \tag{22}$$

The rate of energy loss is simply this net flux into the bay. Note that if $\phi = 0$, there is not energy loss, and we are back to a standing wave.

4. [7] If $\phi << 1$, $p_o << \rho g H$, and $L << \frac{2\pi\sqrt{gH}}{\omega}$, approximate the energy loss just in terms of p_o and ϕ.

Answer: In this limit, the small phase change means that the bay largely rises and falls in qudrature. Also, the seasurface slope is small, so the flux of water into the bay is equal to the seasurface height change:

$$Hu = L\frac{\partial\eta}{\partial t} \tag{23}$$

so, given our variables:

$$u_o = (L/H)\omega\frac{p_o}{\rho g} \tag{24}$$

so, we have

$$< F >= \frac{p_o^2}{2\rho g}\cos\phi \tag{25}$$

Question 3. Flow around a Headland

Consider a headland protruding into a rectangular channel of free-stream depth H. The headland is approximated by a circle with radius R. Assume that R is much less than the width of the channel, and that the free-stream speed far from the headland is U along the channel. The flow separates from the headland at the tip.

1. [4] What is the fastest speed along the headland?

   Answer: The flow in the channel is just the same as flow around a cylinder, if the channel is wide enough. Therefore, assuming the flow is irrotational:

   $$u_\theta = -2U\sin\theta \tag{26}$$

   which has a maximum at $\theta = \pi/2$, and $max(u) = 2U$ in the downstream direction.

2. [4] Assume that the pressure in the separation bubble behind the tip is the same as at the tip. What is the total drag on the flow exerted by the headland?

   Answer: The drag on the headland is simply the pressure force exerted in the x-direction:

   $$D = R\int_0^\pi p(\theta)\cos\theta \mathrm{d}\theta. \tag{27}$$

   The pressure on the surface is given by $p(\theta) = \frac{1}{2}\rho U^2\left(1 - 4\sin^2\theta\right)$ for $\pi/2 < \theta < \pi$, and $-3/2\rho U^2$, for $0 < \theta < \pi/2$. For the back half:

   $$D_2 = -3/2\rho R U^2. \tag{28}$$

   For the front half

   $$\begin{aligned} D_1 &= R\int_{\pi/2}^{\pi}\frac{1}{2}\rho U^2\left(1 - 4\sin^2\theta\right)\cos\theta \mathrm{d}\theta. && (29)\\ &= R\frac{1}{2}\rho U^2(-1 + \frac{4}{3}). && (30)\\ &= \frac{1}{6}\rho R U^2 && (31)\end{aligned}$$

   So, the total drag is $-\frac{4}{3}\rho R U^2$.

3. [4] Assuming that the separation vortex is a Rankine vortex with a radius similar to the headland (i.e. approximately R) and outer speed given by the speed of the flow past the headland, derive an expression for the sea-surface height in the eddy?

   Answer: The azimuthal velocity in a Rankine vortex is given by $u_\theta = 2Ur/R$. The water surface provides the pressure gradient that keeps the water moving in a circle:

   $$\begin{aligned} -g\frac{\partial\eta}{\partial r} &= -\frac{u_\theta^2}{r} && (32)\\ &= -\frac{4U^2}{R^2}r && (33)\end{aligned}$$

   so, $\eta(r) = \frac{2U^2}{g}(\frac{r^2}{R^2} - 1)$ for $r < R$, assuming $\eta = 0$ at $r = R$. Outside the vortex, the surface can also be determined, but it will likely be swamped by other flow factors.

4. [4] Suppose the mean flow suddenly turns off. What direction, and how fast will the headland eddy start to move?

   Answer: The method of images tells us that the eddy will be self-advected. Because the eddy is in solid body rotation, the actual interaction will be complicated, but lets approximate by thinking about where the center of the eddy will move. In that case, if the eddy is R from the wall, it is $2R$ from its image eddy. The velocity outside the core is just $u_\theta(r) = 2UR/r$, so the self-advection is at speed U towards the headland. Note that this is consistent with the eddy being held in place by the oncoming flow.