COMP 460 Lecture Notes

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1 Order Statistics

Selection problem: Find the ith smallest elt. from a set of n elts. Special cases: *maximum*, *minimum*, *median*.

Can solve in $O(n \lg n)$ time by sorting.
But can do in $O(n)$ time:

- straightforward for max., min.
- in general, practical avg. case alg.
- also worst-case alg. w. bigger const.

} No assumptions about input, unlike linear-time sorting.

1.1 Minimum (or Maximum)

```
    MINIMUM(A)
1   min ← A[1]
2   for i ← 2 to length[A]
3       if min > A[i] then min ← A[i] endif
4   endfor
5   return min
```

Uses $n-1$ comparisons, and $n-1$ are required.

1.2 Simultaneous Min. & Max.

Doing each separately: $(n-1)+(n-1) = 2n-2$ comparisons.

To get $\lceil 3n/2 \rceil - 2$ comparisons, pick up pairs of elts. and compare to each other. Then compare smaller to min. so far and larger to max. so far.

1.3 Selection in Expected Linear Time

Find ith smallest elt. in array A:

```
     RANDOMIZED-SELECT(A, p, r, i)
1    if p = r then return A[p] endif
2    Partition A[p..r] into A[p..q − 1], A[q] = pivot, and A[q + 1..r]
       as in RANDOMIZED-QUICKSORT.
3    k ← q − p + 1
4    if i = k then return A[q] endif
5    if i < k
6    then return RANDOMIZED-SELECT(A, p, q − 1, i)
7     else return RANDOMIZED-SELECT(A, q + 1, r, i − k)
8    endif
```

In line 1.3, we have $k < i$. In that case, the ith smallest in $A[p..r]$ is the $i - k$th smallest in $A[q + 1..r]$.

Worst-case time: $\Theta(n^2)$ as for quicksort.

Avg. time: Worst situation is when ith elt. is always in the larger subarray. Analysis similar to quicksort:

$$\begin{aligned} T(n) &\leq O(n) + \frac{1}{n}\sum_{k=0}^{n-1} T(\max\{k, n-1-k\}) \\ &\leq O(n) + \frac{2}{n}\sum_{k=\lfloor n/2 \rfloor}^{n-1} T(k) \end{aligned}$$

We can solve this recurrence by substitution with the guess $T(n) \leq cn$:

$$\begin{aligned} T(n) &\leq O(n) + \frac{2}{n}\sum_{k=\lfloor n/2 \rfloor}^{n-1} ck \\ &= O(n) + \frac{2c}{n}(n - \lfloor n/2 \rfloor)(\frac{n - 1 + \lfloor n/2 \rfloor}{2}) \\ &\leq O(n) + \frac{c}{n}(n - (\frac{1}{2}n - \frac{1}{2}))(n - 1 + \frac{1}{2}n) \\ &= O(n) + \frac{c}{n}(\frac{1}{2}n + \frac{1}{2})(\frac{3}{2}n - 1) \\ &= O(n) + \frac{c}{n}(\frac{3}{4}n^2 + \frac{1}{4}n - \frac{1}{2}) \\ &\leq O(n) + \frac{3}{4}cn + \frac{1}{4}c \\ &\leq cn \text{ for large enough } c \text{ \& } n \end{aligned}$$

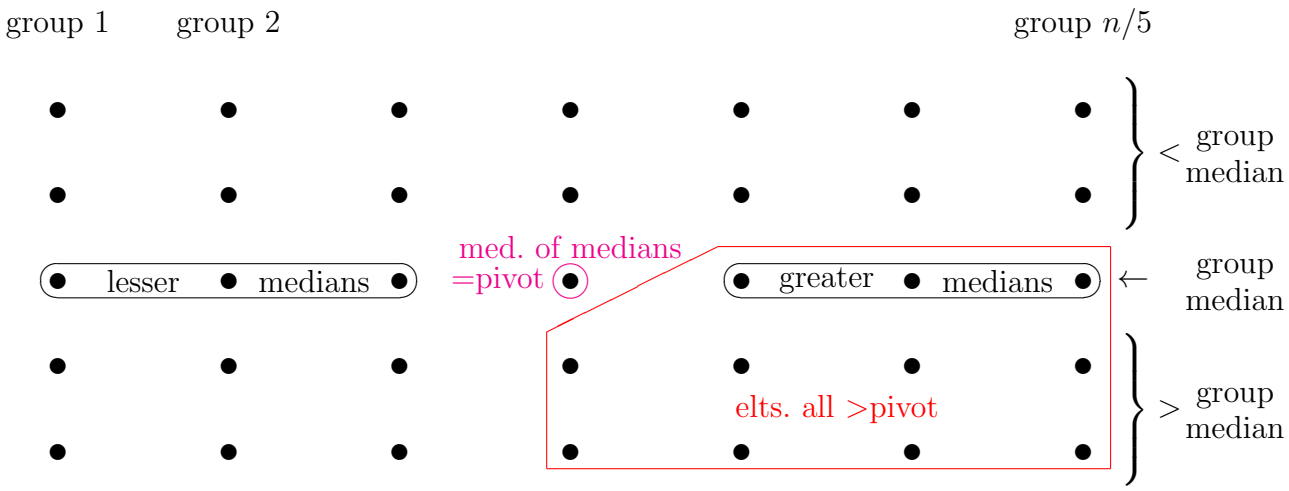
1.4 Selection in Worst-Case Linear Time

SELECT is like RANDOMIZED-SELECT except that we guarantee a reasonably balanced partition by choosing an appropriate pivot:

1. Divide n elts. into $n/5$ groups of 5.
2. Find median of each group of 5 (const. time per group)
3. Recursively call SELECT to find median of medians; that's the pivot.
4. Use pivot to partition into 2 subarrays.
5. If pivot is the ith elt., return it; otherwise, recursively call SELECT on appropriate subarray.

Time: $O(n)$ in steps 1,2, and 4.
$T(n/5)$ in step 3.
$\leq T(x)$ in step 5, where x is largest possible size of larger subarray.

How big can x be?



No. of elts. $\geq$ pivot is at least

$$3 \cdot \frac{1}{2} \cdot \frac{n}{5} = \frac{3}{10}n \text{ . (3 elts. from half the groups)}$$

So the larger subarray has size $\leq \frac{7}{10}n$ (assuming a good balance is arranged when elements are not all distinct). So

$$T(n) \leq T(\frac{1}{5}n) + T(\frac{7}{10}n) + O(n) \text{ .}$$

Can prove $T(n) \leq cn$ by the substitution method.

(Exercise for interested reader:

$$T(n) \leq T(an) + T(bn) + O(n) \text{ w. } a + b < 1 \Rightarrow T(n) = O(n) \text{ .)}$$

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1 Hash Tables

Supports "dictionary" operations:

- Insert: add elt. to set
- Delete: delete elt. from set
- Search: find elt. in set

General notation:

- $U = \{0, 1, 2, \ldots, |U| - 1\}$ is universe of possible keys.
- $K =$ set of n keys actually stored in "dictionary" at a given time.
- $T[0..m-1]$ comprises *hash table* for storing K.

1.1 Hashing w. Chaining (Open Hashing)

Object w. key k goes into slot $h(k)$ of hash table. When keys hashed to same slot (collision), put them in a linked list.

1.1.1 Worst-Case Times

Insert: $O(1)$
Delete: $O(1)$ if *doubly*-linked lists (given ptr. to object to delete).
Same as searching if singly-linked lists.
Search: proportional to length of list of objects w. same *hash value* as key we're searching for.
Worst-case $\Theta(n)$ but better avg. case.

1.1.2 Avg. Case Analysis of Searching

Search for object w. key k assuming keys used are random ($\Rightarrow$ any given key equally likely to go into any of the m slots given a reasonable hash fn.)

Def.: *load factor* $\alpha = n/m$ = avg. no. of elts. in a chain

- Unsuccessful search (usual case if search for a random key):

  Must walk through entire linked list in slot $h(k)$. Expected no. of elts. examined is α.

- Successful search

  If search for an elt. that is in the table, we will examine the cell containing that elt. and, on average, cells for half the other elts. in slot $h(k)$. Expected no. of elts. examined is

$$1 + \frac{1}{2} \cdot \frac{n-1}{m} = 1 + \frac{\alpha}{2} - \frac{\alpha}{2n} \ .$$

- Search for a random key

  Expected no. of elts. examined is

$$\frac{n}{|U|}(1 + \frac{\alpha}{2} - \frac{\alpha}{2n}) + (1 - \frac{n}{|U|})\alpha = \alpha + \frac{n}{|U|}(1 - \frac{\alpha}{2} - \frac{\alpha}{2n}) \ .$$

In all three cases, the total search time is $\Theta(1 + \alpha)$, which equals $\Theta(1)$ if $n = O(m)$.

1.1.3 Universal Hashing

Similar to our move from QUICKSORT to RANDOMIZED-QUICKSORT. Instead of fixing the hash fn. (pivot location) in advance, choose it at random from a collection of good hash functions (all array positions). Then for any input, you will probably get good results.

Def.: A colln. $\mathcal{H}$ of hash functions is universal if

$$\forall \text{ distinct } k, l \in U, |\{\, h \in \mathcal{H} : h(k) = h(l) \,\}| \leq |\mathcal{H}| \,/m \ .$$

Thm: If h is chosen randomly from such a collection $\mathcal{H}$, the expected search time is $\Theta(1 + \alpha)$ regardless of the sequence of hash table operations (as long as the sequence of operations is not chosen by a malicious adversary who knows which member of $\mathcal{H}$ has been selected).

Pf.: Essentially same as avg. case argument above, but now no need to assume random keys.

Corollary: Universal hashing w. chaining in a table w. m slots takes expected time $\Theta(n)$ time for *any* sequence of n Insert, Search, and Delete operations w. $O(m)$ Insert operations (or even an $O(m)$ bound on $|K|$ throughout the sequence).

1.2 Hash Functions

Keys can be non-numeric; easy to convert them to numbers, e.g.,

$\mathtt{a} = 1, \mathtt{b} = 2, \mathtt{c} = 3, \ldots, \mathtt{z} = 26$
$\mathtt{abc} = 1 + 2 \times 26 + 3 \times 26^2$

Then:

- division method: $h(k) = k \bmod m$.

  A good choice for m is a prime not too close to a power of 2.

- multiplication method: $h(k) = \lfloor m(kA \bmod 1) \rfloor$ where $0 < A < 1$.

  ($x \bmod 1$ denotes the fractional part of x.)

  Choosing m as a power of 2 eases computer implementation via bit operations. Knuth: $A \approx (\sqrt{5} - 1)/2$ is a good choice.

- universal hashing (one method):

  Def.: For p prime, $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ and $\mathbb{Z}_p^* = \{1, \ldots, p-1\}$.

  Def.: $h_{a,b}(k) = ((ak + b) \bmod p) \bmod m$

  Consider a family of $p(p-1)$ hash functions based on a prime $p \geq |U| > m$ defined by

  $$\mathcal{H}_{p,m} = \left\{ h_{a,b} : a \in \mathbb{Z}_p^* \wedge b \in \mathbb{Z}_p \right\} .$$

  Thm.: $\mathcal{H}_{p,m}$ is universal. (Recall def.: $\forall$ distinct $k, l \in U, |\{ h \in \mathcal{H} : h(k) = h(l) \}| \leq |\mathcal{H}| / m$)

  Pf.: Consider two distinct keys k and l. ($k \not\equiv l \pmod p$, since $p \geq |U|$.)
  For $h_{a,b} \in \mathcal{H}_{p,m}$, let

  $r = (ak + b) \bmod p$
  $s = (al + b) \bmod p$

  $r \neq s$, since $r - s \equiv a(k-l) \bmod p$, and a and $k-l$ are nonzero mod p. (Number theory in Chapter 31 indicates a product of nonzero nos. is nonzero, even mod a prime.)

  Furthermore, each choice of (a, b) (in $\mathbb{Z}_p^* \times \mathbb{Z}_p$) yields a different (r, s).

  (Same principle as above:
  Suppose

  $$r = (ak + b) \bmod p = (ck + d) \bmod p \quad (1)$$
  $$s = (al + b) \bmod p = (cl + d) \bmod p \quad (2)$$

Then

$$\begin{aligned}(a-c)k+(b-d) &\equiv 0 \pmod p \text{ from (1)}\\ (a-c)l+(b-d) &\equiv 0 \pmod p \text{ from (2)}\end{aligned}$$

which implies $(a-c)(k-l) \equiv 0 \pmod p$. Since $k-l$ is nonzero mod p, and $a, b, c, d \in \mathbb{Z}_p$, we have $a = c \wedge b = d$.)

So there is a 1-1 correspondence between the $p(p-1)$ pairs $(a,b) \in \mathbb{Z}_p^* \times \mathbb{Z}_p$ and the $p(p-1)$ pairs $(r,s) \in \mathbb{Z}_p \times \mathbb{Z}_p$ w. $r \neq s$.

So the no. of fns. in $\mathcal{H}_{p,m}$ that map k and l to the same slot is the no. of $(r,s) \in \mathbb{Z}_p \times \mathbb{Z}_p$ w. $r \neq s$ and $r \equiv s \pmod m$. There are p choices for r. For each one, the no. of $s \in \mathbb{Z}_p$ w. $r \equiv s \pmod m$ is at most $\lceil p/m \rceil$, and the no. of such values $\neq r$ is at most

$$\begin{aligned}\lceil p/m \rceil - 1 &\leq (p+m-1)/m - 1\\ &= (p-1)/m\end{aligned}$$

so the no. of (r,s) pairs meeting our criteria is $\leq p(p-1)/m$. Thus the fraction of the $p(p-1)$ fns. in $\mathcal{H}_{p,m}$ that map k and l to the same slot is at most $1/m$.

Q. E. D.

1.3 Open Addressing (Closed Hashing)

Instead of using chaining to handle collisions, look at a sequence of table positions

$$h(k,0), h(k,1), h(k,2), \ldots, h(k,m-1)$$

until find an empty one to insert into. (Need $n \leq m$ here.)

Similarly, searching looks through seq. of positions until find the key or an empty slot. This makes deletion problematical. Either outlaw deletion, or replace deleted keys w. a special key, DELETED.

- Linear probing: $h(k,i) = (h'(k) + i) \bmod m$.

  Probes consecutive table locations. Tends to have bad clustering (*primary clustering*).

- Quadratic probing: $h(k,i) = (h'(k) + c_1 i + c_2 i^2) \bmod m$.

  Better than linear probing, but probe seq. still depends only on $h'(k)$. *secondary clustering.*

- Double hashing: $h(k,i) = (h_1(k) + i h_2(k)) \bmod m$.

  Comes closest to the theoretical ideal of *uniform hashing*, where each key is equally likely to have any of the $m!$ permutations of $\{0, 1, \ldots, m-1\}$ as its probe seq.

  Book analyzes uniform hashing.