Solutions to Homework #1

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1 (HW1). Rank the following functions by the order of growth. That is, make a list of the functions such that if $f(n)$ is on a higher line that $g(n)$, then $f(n)$ is $\Omega(g(n))$. If $f(n)$ is $\Theta(g(n))$, then $f(n)$ and $g(n)$ should be listed on the same line.

Solution. We can use L'Hopital's rule to find what function growths faster, then rank them by order of growth.

1. $1 = n^{\frac{1}{ln(n)}}$; 2
2. $ln(ln(n))$
3. $\sqrt(lg(n))$
4. $ln(n)$; $lg(n)$
5. $lg^2(n)$
6. $\sqrt(2^{lg(n)})$
7. $2^{lg(n)}$; n
8. $nlg(n)$
9. n^2; $4^{lg(n)}$
10. n^3
11. $\left(\frac{3}{2}\right)^n$
12. 2^n
13. $n!$
14. $(n+1)!$
15. 2^{2^n}
16. $2^{2^{n+1}}$

□

2 (HW1). Express the solution to each of the following recurrences using Θ-notation. Show your reasoning.

- We can solve the recurrences below using the Master Theorem, given that:
$T(n) = aT(n/b) + f(n)$, $a \geq 1$,$b > 1$. Following three cases:

1. If $\exists\epsilon > 0$ such that $f(n) = \mathcal{O}(n^{log_b a-\epsilon})$, then $T(n) = (n^{log_b a})$.
2. If $f(n) = \Theta(n^{log_b a}lg^k n)$,with $k \geq 0$, then $T(n) = \Theta(n^{log_b a}lg^{k+1}n)$.
3. If $\exists\epsilon > 0$ such that $f(n) = \Omega(n^{log_b(a+\epsilon)})$,then $T(n) = \Theta(f(n))$.

(a) $T(n) = 2T(n/2) + nlg^2n$.

Solution. Following a more general form of the masters theorem where $T(n) = aT(n/b) + \Theta(n^k log^{\phi}(n))$, $a \geq 1$, $b > 1$, and $\phi \in \mathbb{R}$. We have that $a = 2$,$b = 2$,$k = 1$ and $f(n) = n^{log_2 2} lg^2 n$. For this case if $a = b^k$ and $\phi > -1$ then the solution to the recurrence follows case#2 of the general form, such that: $T(n) = \Theta(n^{log_b a} lg^{\phi+1} n) = \Theta(n^1 lg^3 n)$. □

(b) $T(n) = 9T(n/2) + n^3$.

Solution. Given that $a = 9$, $b = 2$, $f(n) = n^3$. Since $n^{log_2 9} \approx n^{3.1699...}$ is polynomially larger than $f(n) = n^3$ this recurrance follows case# 1 of the masters theorem. Such that $f(n) = \mathcal{O}(n^{log_2(9-\epsilon)})$. Therefore the solution to the recurrance is given by:$T(n) = \Theta(n^{log_b a}) = \Theta(n^3)$. □

(c) $T(n) = 8T(\lceil n/4 \rceil) + n^2$.

Solution. When solving recurrances usually we can disregard floor/ceil as we are interested in the growth of the recurrance (running time). In this case we have that $a = 8$, $b = 4$, $f(n) = n^2$ and $n^{log 8_4} = n^{\frac{3}{2}}$. Since $n^{log 8_4} \leq n^2$ the recurrance follows case# 3 of the master theorem. Such that $f(n) = \Omega(n^{log_b(a+\epsilon)}) = \Omega(n^2)$ □