

Solutions to Homework #1

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1 (HW1). Rank the following functions by the order of growth. That is, make a list of the functions such that if $f(n)$ is on a higher line than $g(n)$, then $f(n)$ is $\Omega(g(n))$. If $f(n)$ is $\Theta(g(n))$, then $f(n)$ and $g(n)$ should be listed on the same line.

Solution. We can use L'Hopital's rule to find what function grows faster, then rank them by order of growth.

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|-----------------------------------|----------------|----------------------------------|
| 1. $1 = n^{\frac{1}{\ln(n)}}$; 2 | 7. $2^{lg(n)}$ | 13. $\left(\frac{3}{2}\right)^n$ |
| 2. $\ln(\ln(n))$ | 8. $4^{lg(n)}$ | 14. 2^n |
| 3. $\sqrt{lg(n)}$ | 9. n | 15. $n!$ |
| 4. $\ln(n)$; $lg(n)$ | 10. $nlg(n)$ | 16. $(n+1)!$ |
| 5. $lg^2(n)$ | 11. n^2 | 17. 2^{2^n} |
| 6. $\sqrt{2^{lg(n)}}$ | 12. n^3 | 18. $2^{2^{n+1}}$ |

□

2 (HW1). Express the solution to each of the following recurrences using Θ -notation. Show your reasoning.

- We can solve the recurrences below using the Master Theorem, given that:
 $T(n) = aT(n/b) + f(n)$, $a \geq 1, b > 1$. Following three cases:

1. If $\exists \epsilon > 0$ such that $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$, then $T(n) = \mathcal{O}(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a} lg^k n)$, with $k \geq 0$, then $T(n) = \Theta(n^{\log_b a} lg^{k+1} n)$.
3. If $\exists \epsilon > 0$ such that $f(n) = \Omega(n^{\log_b(a+\epsilon)})$, then $T(n) = \Theta(f(n))$.

(a) $T(n) = 2T(n/2) + nlg^2n$.

Solution. Following a more general form of the masters theorem where $T(n) = aT(n/b) + \Theta(n^k \log^\phi(n))$, $a \geq 1$, $b > 1$, and $\phi \in \mathbb{R}$. We have that $a = 2, b = 2, k = 1$ and $f(n) = n^{\log_2 2} lg^2 n$. For this case if $a = b^k$ and $\phi > -1$ then the solution to the recurrence follows case#2 of the general form, such that: $T(n) = \Theta(n^{\log_b a} lg^{\phi+1} n) = \Theta(n^1 lg^3 n)$. \square

(b) $T(n) = 9T(n/2) + n^3$.

Solution. Given that $a = 9$, $b = 2$, $f(n) = n^3$. Since $n^{\log_2 9} \approx n^{3.1699...}$ is polynomially larger than $f(n) = n^3$ this recurrence follows case# 1 of the masters theorem. Such that $f(n) = \mathcal{O}(n^{\log_2(9-\epsilon)})$. Therefore the solution to the recurrence is given by: $T(n) = \Theta(n^{\log_b a}) = \Theta(n^3)$. \square

(c) $T(n) = 8T(\lceil n/4 \rceil) + n^2$.

Solution. When solving recurrences usually we can disregard floor/ceil as we are interested in the growth of the recurrence (running time). In this case we have that $a = 8$, $b = 4$, $f(n) = n^2$ and $n^{\log_4 8} = n^{\frac{3}{2}}$. Since $n^{\log_4 8} \leq n^2$ the recurrence follows case# 3 of the master theorem. Such that $f(n) = \Omega(n^{\log(a+\epsilon)_b}) = \Omega(n^2)$ \square