LaTeXsubmissions are mandatory. Submitting your assignment in another format will result in a **loss of 10 points** on the assignment. The template is here.

Problem 1 *worth 15 points*
Find each of these values.

(a) $\left(99^2 \bmod 32\right)^3 \bmod 15$

(b) $\left(3^4 \bmod 17\right)^2 \bmod 11$

(c) $\left(19^3 \bmod 23\right)^2 \bmod 31$

(d) 9009 div 223

(e) $(177 \bmod 31 + 270 \bmod 31) \bmod 31$

(f) $(177 \bmod 31 \cdot 270 \bmod 31) \bmod 31$

Problem 2 *worth 10 points*
Find counterexamples to each of these statements about congruences.

(a) If $ac \equiv bc (\text{mod } m)$ where a, b, c, and m are integers with $m \geq 2$, then $a \equiv b (\text{mod } m)$.

(b) If $a \equiv b (\text{mod } m)$ and $c \equiv d (\text{mod } m)$, where a, b, c, d, and m are integers with c and d positive and $m \geq 2$, then $a^c \equiv b^d (\text{mod } m)$.

Problem 3 *worth 15 points*
Prove the transitivity of modular congruence. That is, prove that for all integers a, b, c, and n with $n > 1$, if $a \equiv b (\text{mod } n)$ and $b \equiv c (\text{mod } n)$ then $a \equiv c (\text{mod } n)$.

Problem 4 *worth 10 points*
The formal definition for $\Theta-$ notation, written using quanities and variables, is: $f(n)$ is $\Theta(g(n))$ if, and only if, $\exists$ positive real numbers k, A, and B such that $\forall n \geq k$ we have $Ag(n) \leq f(n) \leq Bg(n)$. Write the formal **negation** of the definition using $\forall$ and $\exists$. Then, restate what it means in English without using $\forall, \exists$, or the words "for any," "for every," or "there exists."

Problem 5 *worth 25 points*
Consider the following questions about $7n^3 + 10n^2 + 3$.

(a) Prove that for any integer $n \geq 1$, $0 \leq 7n^3 + 10n^2 + 3 \leq 20n^3$.

(b) Prove that for any integer $n \geq 1$, $7n^3 \leq 7n^3 + 10n^2 + 3$.

(c) Sketch (you can use Desmos or another plotting tool) the graph of all functions in the previous two parts.

(d) Use $O-$ and $\Omega-$ notations to express the results of the first two parts.

(e) What can you deduce about the order of $7n^3 + 10n^2 + 3$?

Problem 6 *worth 10 points*
Show that $\frac{1}{5} + \frac{4}{5^2} + \frac{4^2}{5^3} + \cdots + \frac{4^n}{5^{n+1}}$ is $\Theta(1)$.

Problem 7 *worth 10 points*
Given real-valued functions f and g with the same domain D, the sum of f and g is defined for each real number x as $(f+g)(x) = f(x) + g(x)$. Show that if f and g are both increasing on set S then $f+g$ is also increasing on S.

Problem R *worth 5 points*
Exercise a growth mentality by reflecting on this assignment and your work. Feel free to say whatever you want, but you are required to answer the following. You are graded on whether you complete this, not on what you say.

- How many hours did you spend on this assignment?
- What problem was hardest? Why?
- What problem was easiest? Why?
- Have you done anything differently in your studying or homework since the midterm? Why?