CSCI 2824 - CU Boulder, 2019 Summer

Lecture 2: Arguments in Logic

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Content is borrowed from Susanna Epp's Discrete Mathematics with Applications and Andrew Altomare's notes.

1 Review

Last time we got acquainted with the class. We then began talking about logic. We covered the following topics:

- Propositions/Statements
- Connectives: negation, and, or, xor, if-then, if-and-only-if
- Truth tables

I also made the question on satisfiability on the quiz worth zero points since we didn't get a chance yesterday to cover it.

2 Recap on Mindsets

I asked you take a survey about mindsets. I have emailed each of you individually with a response to your "get to know me" and included your mindset score. The class average was 43/60 (0 being a pure static mindset and 60 being a pure growth mindset) with a standard deviation of 8.9, a low of 30, and a high of 56. I think there are differences between we really act and think and how we answer on the survey. The survey might bias us to answer more growth than we really are. The main purpose is to start thinking about your unconscious tendencies in problem solving. Who would be interested in more readings and discussion on mindsets?

2.1 Satsifiability

Definition: *Satisfiabile*
A compound proposition is *satisfiable* if there is an assignment of truth values to its constituent propositions that makes it true. If there is no such case, then the compound proposition is unsatisfiable.

For example, $p \wedge \neg p$ is unsatisfiable. Do you see how in a truth table we have no way to get true for the statement? That's what unsatisfiable means. Satisfiable means we have at least one way.

Exercise
Show that $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$ is satisfiable.

Exercise
Now, show that $(p \to q) \wedge (p \to \neg q) \wedge (\neg p \to q) \wedge (\neg p \to \neg q)$ is unsatisfiable.

2.2 Necessary and sufficient conditions

Let n be a natural number $(0, 1, 2, 3, \ldots)$. It is *sufficient* that n be divisible by 12 for n to be divisible by 6. How can we represent this claim using a conditional?

Let r="n is divisible by 12" and s="n is divisible by 6." This statement is telling us that *under the condition that* n is divisible by 12, it must be divisible by 6. For a sufficient condition, the condition goes at the front of the conditional: $r \to s$.

It is *necessary* for warm surface air to start up convection in order for a severe summer thunderstorm to occur. How can we represent this claim using a conditional? Let t="severe summer thunderstorm occurs" and w="warm surface air spurs convection." This statement tells us that *under the condition that* a thunderstorm occurs, it must be the case that warm surface air has spurred convection. For a *necessary condition*, the condition goes at the end of the conditional: $t \to w$.

Exercise
Consider the following statement: "If it snows, then I crash my bicycle riding home." Is snowing a necessary or sufficient (or neither) condition? Similarly, "I crash my bicycle only if it snows?"

Note that we could do these example using truth tables, but they after are too large to be of any practical use. For n propositions, the truth table will have 2^n rows.

Definition: *Necessary and Sufficient Conditions*
If $P \to Q$, we say P is a *sufficient condition* for Q and Q is a *necessary condition* for P. If R is both a *necessary and sufficient condition* for S then $R \leftrightarrow S$.

2.3 Sudoku satisfiability

Sudoku puzzles can be written (and solved) as satisfiability problems. It turns out that Sudoku puzzles would require 2^{729} rows, which is more than the number of atoms in the universe (between 2^{259} and 2^{272})

Let $p(i, j, n)$ represent the proposition that n occurs at row i and column j. There are 9 rows, 9 columns, and 9 numbers. Thus there are $9 \times 9 \times 9 = 729$ propositions. Hence, 2^{729} rows in your truth table... yikes!

Notation:

$$\bigwedge_{j=1}^{4} p_j = p_1 \wedge p_2 \wedge p_3 \wedge p_4$$

$$\bigvee_{j=1}^{4} p_j = p_1 \vee p_2 \vee p_3 \vee p_4$$

Exercise
How can we represent Sudoku logically?

3 Logical equivalence

Recall the conditional: If p, then q which we write as $p \to q$. Consider the statements, $\neg p \vee q$ and $\neg p \wedge q$.

p	q	$\neg p$	$p \to q$	$\neg p \vee q$	$\neg p \wedge q$
T	T	F	T	T	F
T	F	F	F	F	F
F	T	T	T	T	T
F	F	T	T	T	F

Notice how $p \to q$ has the same truth table as $\neg p \vee q$? These statements are said to be logically equivalent!

Definition: *Logical equivalence*
Two propositions are *logically equivalent* if they have the same truth values for all combinations of their constituents.

There are three other conditionals closely related to $\boldsymbol{p} \to q$:

1. The *converse*: $q \to p$
2. The *inverse*: $\neg p \to \neg q$
3. The *contrapositive*: $\neg q \to \neg p$

For example: If I am a math teacher then I'm not a banana. Let p="I am a math teacher", q="I'm not a banana"

- Converse: If I'm not a banana then I am a math teacher
- Inverse: If I'm not a math teacher, then I am a banana
- Contrapositive: If I am a banana, then I'm not a math teacher

On your homework, you will be asked to consider the logical equivalences in these statements. Understanding the relationships will be helpful later when you're writing proofs. Sometimes it's easier to prove one logically equivalent form instead of proving the original statement.

3.1 Tautologies, contradictions, and contingencies

Consider the statement “Marcus is wearing shoes or he is not wearing shoes."

- Let p=Marcus is wearing shoes
- Then we have $p \lor \neg p$

p	$\neg p$	$p \lor \neg p$
T	F	T
F	T	T

But wait! This statement is always true! That’s special.

Definition: *Tautology*
A compound proposition that is always true is called a *tautology*.

There are many tautologies, for example: $p \rightarrow p$ and $((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)$. If p is a statement that is logically equivalent to q then $p \Leftrightarrow q$ is a tautology. For your own good, verify these with a truth table!

What do we call the opposite, when the statement is always false?

Definition: *Contradiction*
A *contradiction* is a compound proposition that is **F** for all possible combinations of constituent proposition truth values.

For example, there is no possible way for the statement “Today it will rain and today it will not rain $(p \land \neg p)$" to be true. You can take any tautology, negate it, and you have a contradiction.

Definition: *Contingency*
A compound proposition that is neither a tautology or a contradiction is a contingency.

You can probably imagine lots of statements that are contingencies. For example: “It is not the case that Gary is boring and rides a motorcycle." Another way to express this is, “Either Gary is not boring or Gary does not ride a motorcycle." Let p= Gary is boring, q=Gary rides a motorcycle. Then the original proposition is: $\neg(p \land q)$. And the revised more normal-sounding version is $\neg p \lor \neg q$. Are they logically equivalent?

p	q	$\neg p$	$\neg q$	$p \land q$	$\neg(p \land q)$	$\neg p \lor \neg q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

These two examples combine to give us a powerful pair of logical manipulations that we can use to rearrange or simplify compound propositions.

3.2 De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

3.3 The power of logical equivalency

There are lots of these kinds of identities. Example: Consider the conditional $p \to q$ If it snows then Andrew crashes his bike. Or rephrased: Either it isn't snowing, or Andrew crashes his bike. $\neg p \vee q$. This is known as **relation by implication**:

$$p \to q \equiv \neg p \vee q$$

. We've already seen this example.

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1. *Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. *Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. *Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. *Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5. *Negation laws:*	$p \vee {\sim}p \equiv \mathbf{t}$	$p \wedge {\sim}p \equiv \mathbf{c}$
6. *Double negative law:*	${\sim}({\sim}p) \equiv p$	
7. *Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. *Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9. *De Morgan's laws:*	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
10. *Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. *Negations of* **t** *and* **c**:	${\sim}\mathbf{t} \equiv \mathbf{c}$	${\sim}\mathbf{c} \equiv \mathbf{t}$

Figure 1: A list of logical equivalencies from Epp (pg 35 in edition 4)

Logical equivalences provide an elegant, and potentially much simpler, alternative to truth tables. We can construct a chain of logical equivalences starting from the first compound proposition and leading to the second one This is exactly how we construct mathematically sound arguments.

Exercise
Show that $p \to q \equiv \neg q \to \neg p$ without using a truth table.

I do not advise trying to memorize these, especially since different people may use different names. Instead, familiarize yourself with them by using them enough and you naturally will build intuition and essentially memorize them in context. As long as you know how to prove that they're equivalent, you can always double check your intuition.

4 Logical inference

4.1 Rules of inference

We need to learn how to construct valid arguments. Think of an argument as a symbolic template that starts with some assumptions (called premises) and proceeds along a path of logical inferences to reach a *conclusion*.

Example:

If Xerxes can bleed, then Xerxes is a mortal.

Xerxes can bleed.

Therefore, Xerxes is a mortal.

This is an example of a specific valid argument. We need to cast this argument into a symbolic template.

- p="Xerxes can bleed"
- q="Xerxes is mortal"

Then in symbolic logic, our argument becomes

$p \to q$

p

$\therefore q$

Note: the symbol $\therefore$ means *therefore*. Use this to denote the conclusion of the argument.

Definition: *Argument*
An **argument** is a set of **premises** coupled with a **conclusion**. A **valid** argument is an argument such that there is no circumstance in which the premises could be true and the conclusion be false.

Your intuition probably suggests that the previous argument is valid, but let's formalize this. Consider the compound proposition: $((p \to q) \wedge p) \to q$ Note that this is a conditional. Premise 1: $(p \to q)$ and premise 2: p. In general, $[(premise_1) \wedge (premise_2) \wedge \cdots] \to conclusion$. The hypothesis is the conjunction of the premises.

For our argument to be valid, it must be the case that there is no situation (i.e. truth values for p and q) in which the premises of the argument are true but the conclusion is false. In other words, the conditional describing our argument must be a tautology.

p	q	$p \to q$	$(p \to q) \wedge p$	$((p \to q) \wedge p) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

This general form of argument is so useful and common that it has a special name and is designated as a *rule of inference*.

Definition: *Modus Ponens*
Modus Ponens: “the way that affirms by affirming"

1. $p \to q$
2. p

$\therefore q$

I might call this “affirming the antecedent."

Rules of inference are common, valid mini-arguments that we can link together to construct more complex valid arguments. Let’s prove a few more rules of inference.

Definition: *Modus Tollens*
Modus Tollens: “the way that denies by denying"

1. $p \to q$
2. $\neg q$

$\therefore \neg p$

I might call this “denying the consequent."

Exercise
Prove:

If it rains today, then my basement will flood.

My basement did not flood.

$\therefore$ It did not rain today

Definition: *Disjuntive syllogism*
Disjunctive Syllogism (Historically: *Modus Tollendo Ponens*)

1. $p \lor q$
2. $\neg p$

$\therefore q$

Exercise
An example of disjunctive syllogism:

My foot is disfigured or there is a rock in my shoe

My foot is not disfigured

$\therefore$ I have a rock in my shoe

Come up with your own example now.

We can even derive disjunctive syllogism to convince you it's valid!

$p \lor q$ premise

$\neg p$ premise

$\neg p \to q$ relation by implication using (1)

$\therefore q$ modus ponens, using (2) and (3)

Exercise
What can you conclude from the following?

If it is sunny outside then I will go to the park.

If I go to the park, then I will get ice cream

Modus Ponens	$p \to q$ p $\therefore q$	**Elimination**	**a.** $p \vee q$ $\sim q$ $\therefore p$ **b.** $p \vee q$ $\sim p$ $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$	**Transitivity**	$p \to q$ $q \to r$ $\therefore p \to r$
Generalization	**a.** p $\therefore p \vee q$ **b.** q $\therefore p \vee q$	**Proof by Division into Cases**	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$
Specialization	**a.** $p \wedge q$ $\therefore p$ **b.** $p \wedge q$ $\therefore q$		
Conjunction	p q $\therefore p \wedge q$	**Contradiction Rule**	$\sim p \to c$ $\therefore p$

Figure 2: Some examples of logical inference. Table taken from Epp Edition 4 pg. 61

$\therefore ?$

$\therefore$ If it is sunny outside, then I will get ice cream. This is called **hypothetical syllogism**:

1. $p \to q$
2. $q \to r$

$\therefore p \to r$

As you can imagine there are many more:

Exercise
One tricky form of inference is called **Resolution**:

1. $p \vee q$
2. $\neg q \vee r$

$\therefore p \vee r$

The intuition is that q can either be true or false. If q is true then r must be true. If q is false, then p must be true. Either way, at least one of p or r must be true (or both). **Try to derive resolution formally!**

It's not the easiest thing. Attempting will help you learn. Here is one derivation:

1. $p \vee q$ premise
2. $\neg q \vee r$ premise
3. $\neg p \to q$ RBI (1)
4. $q \to r$ RBI (2)
5. $\neg p \to r$ hypothetical syllogism, (3) and (4)

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6. $\therefore p \vee r$ RBI (5)

Try and find another way?

We use the rules of inference to help us determine whether arguments are valid without having to construct truth tables. Consider the following argument:

$p \vee q \to \neg r$

$\neg r \to s$

p

$\therefore s$

We can use the rules of inference, which we have proven, to show that it follows!

1. $p \vee q \to \neg r$ premise
2. $\neg r \to s$ premise
3. p premise
4. $p \vee q$ addition, using (3)
5. $\neg r$ modus ponens, using (1) and (4)

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6. $\therefore s$ modus ponens, using (2) and (5)

4.2 Translation

One of the trickiest parts can be translating an informal statement into formal logic. What valid argument form is present in the following? You first will need to convert into symbols and then identify the pattern.

- If n is a real number with $n > 3$ then $n^2 > 9$
- Suppose that $n^2 \leq 9$. Then $n \leq 3$

This argument takes the form: (**Modus Tollens**)

1. $p \to q$
2. $\neg q$

$\therefore \neg p$

Exercise
What form is the following argument?

- If $\sqrt{2} > 3/2$ then $(\sqrt{2})^2 > (3/2)^2$.
- We know that $\sqrt{2} > 3/2$
- Consequently $(\sqrt{2}^2 = 2 > (3/2)^2 = 9/4$

Does something look wrong in the previous exercise? A **valid** argument is one where there is no way the conclusion can be false *if the premises are true*. Valid arguments are patterns of logical reasoning. But just because an argument is valid does not mean you can trust the conclusion. In the previous example, the conclusion that $2 > 9/4$ is false. The problem arises because the premise that $\sqrt{2} > 3/2$ is false. We want to be able to tell which arguments are not only valid but “useful" or “nice" also.

Definition: *Sound argument*
When an argument is both **valid** *and* the **premises are true**, we call the argument **sound**.

5 Fallacies

It is not uncommon to see invalid arguments out in the wild. Usually people make invalid arguments when conditionals are involved.

One example is when we affirm the conclusion (assuming the converse).

If you study hard in this class, then you will get an A.

You got an A.

Therefore, you must have studied hard.

But you could have gotten an A for reasons aside from studying hard. So this is **not** a valid argument. We will call this a converse error.

We could do something similar by denying the hypothesis/antecedent (assuming the inverse).

If you choose a strong password, then your email will not be hacked

You did not choose a strong password.

Therefore, your email will be hacked.

Your email might not be hacked even if you have a bad password. So this is **not** a valid argument. We will call this an inverse error.

Exercise
What rule of inference or logical fallacy is demonstrated by the following argument? Is the argument sound?

- If earth is flat, then if you fly for a long time in one direction, you will fall of the edge.
- The earth is flat.
- Therefore, if you fly for a long time in one direction for too long, you will fall off the earth.

6 Digital Logic Circuits

Imagine you have a light bulb (which we'll symbolize by the curly circle symbol on the right of the circuits in Figure 3) and a battery (which we'll symbolize by the lines on the left of the circuits in Figure 3). You connect them with switches that you can open, preventing electricity from flowing, or close to allow electricity to flow.

You can then ask what happens to the light depending on the state of the circuit as shown in Figure 4.

Do you notice something cool? If we replace the words *closed* and *one* with *True* and replace *open* and *off* with *False*, we have truth tables for and in the Series Circuit and or with the Parallel Circuit. This observation allows us to reason about electronics using the logic you have learned.

More generally, we can create simple units that represent logical states on electrical inputs and then chain them together to do computation! See Figure 5 for our new *gates*.

We can then create circuits to do computation.

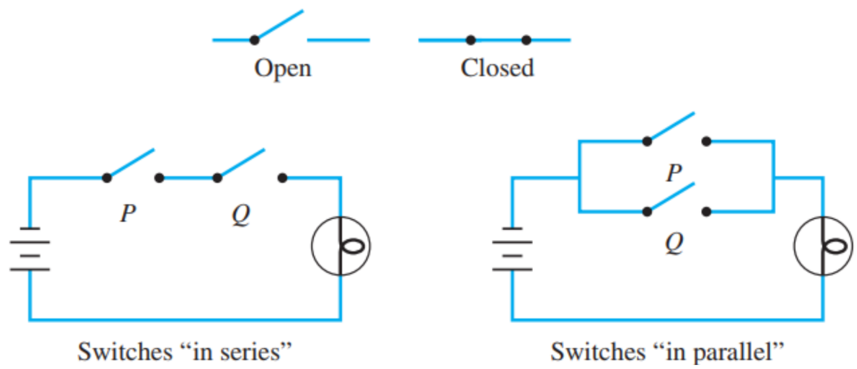


Figure 3: An example of circuits on the bottom with the on versus off states on top. This figure is edited from Epp Edition 5 pg 79.

(a) Switches in Series

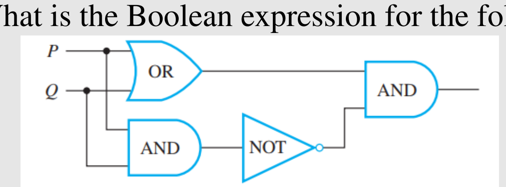
Switches		Light Bulb
P	***Q***	**State**
closed	closed	on
closed	open	off
open	closed	off
open	open	off

(b) Switches in Parallel

Switches		Light Bulb
P	***Q***	**State**
closed	closed	on
closed	open	on
open	closed	on
open	open	off

Figure 4: The state of the light bulb depending on the switches. Taken from Epp Edition 5 pg 80.

Exercise
What is the Boolean expression for the following circuit?



Exercise
What is the circuit for the expression $(\neg P \wedge Q) \vee \neg Q$?

In fact if you check out Section 2.5 in Epp Edition 5, you can see that these circuits can be used for mathematical operations too!

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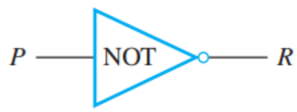
<tr><td>NOT</td><td>
</td><td>
<table>
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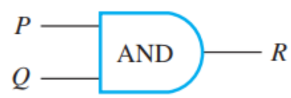
</td></tr>
<tr><td>AND</td><td>
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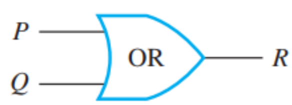
<tr><td>OR</td><td>
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Figure 5: Simple gates that perfom our logical operations. Taken from Epp Edition 4 pg 67.