CSCI 2824 - CU Boulder, 2019 Summer

Lecture 4: Multiple Quantifiers

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Content is borrowed from Susanna Epp's Discrete Mathematics with Applications and Andrew Altomare's notes.

1 Review

- We went over everything we've covered so far, from propositions until now.
- I supplied a list of named logical equivalencies for your convenience below in Section 1.1.
- Universal quantifier $\forall x P(x)$ means "for all x in my **domain**, $P(x)$
- Existential quantifier $\exists x P(x)$ means "there exists an x in my domain such that $P(x)$"
- Distribution rules

$$\forall x(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$$
$$\exists x(P(x) \lor Q(x)) \equiv \exists x P(x) \lor \exists x Q(x)$$

- And recall that distribution of $\forall$ over $\lor$ and $\exists$ over $\land$ did **not** work

1.1 List of Logical Equivalencies

Name	Equivalence
Disjunctive Identity	$p \wedge \mathbf{T} \equiv p$
Conjunctive Identity	$p \vee \mathbf{F} \equiv p$
Disjunctive Domination	$p \vee \mathbf{T} \equiv \mathbf{T}$
Conjunctive Domination	$p \wedge \mathbf{F} \equiv \mathbf{F}$
Disjunctive Idempotent	$p \vee p \equiv p$
Conjunctive Idempotent	$p \wedge p \equiv p$
Double negation	$\neg(\neg p) \equiv p$
Commutative	$p \vee q \equiv q \vee p$
Commutative	$p \wedge q \equiv q \wedge p$
Associative	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Associative	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Distributive	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Distributive	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
DeMorgans	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
DeMorgans	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
Absorption	$p \vee (p \wedge q) \equiv p$
Absorption	$p \wedge (p \vee q) \equiv p$
Negation	$p \vee \neg p \equiv \mathbf{T}$
Negation	$p \wedge \neg p \equiv \mathbf{F}$
	$p \rightarrow q \equiv \neg p \vee q$
	$p \rightarrow q \equiv \neg q \rightarrow \neg p$
	$p \wedge q \equiv \neg(p \rightarrow \neg q)$
	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
DeMorgans	$\neg \forall x P(x) \equiv \exists x \neg P(x)$
DeMorgans	$\neg \exists x P(x) \equiv \forall x \neg P(x)$
	$\forall x \forall y P(x) \equiv \forall y \forall x P(x)$
	$\exists x \exists y P(x) \equiv \exists y \exists x P(x)$
	$\forall x (P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$
	$\exists x (P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$

2 Nested Quantifiers

We can include multiple quantifiers for a propositional function. Consider the propositional function $C(x, y)$="y is the favorite color of x." What about the domains? Consider x: all people and y: all colors.

We could write: $\forall x \exists y C(x, y)$="for all people, there exists a color such that that color is their favorite."

Consider the domain of all real numbers. What does the following statement mean? $\forall x \exists y (x + y = 0)$. In English, we can write it "For all x, there exists a y such that $x + y = 0$. This is **true**.

This is expressing the fact that all real numbers have an additive inverse.

2.0.1 Swapping order

How can we express the law of *commutation of addition* (That is $x + y = y + x$.) Let the domain be the real numbers. Then we could use $\forall x \forall y(x+y = y+x)$. What happens if you swap the order of $\forall x$ and $\forall y$? Then we would have instead: $\forall y \forall x(x+y = y+x)$ Turns out nothing changes. You still loop over all the combinations of x's and y's

Returning to the previous example: $\forall x \exists y(x + y = 0) \stackrel{?}{\equiv} \exists y \forall x(x + y = 0)$. i.e. what happens when we swap the order of $\forall$ and $\exists$? The original statement was "For every x there exists some y such that $x + y = 0$." The new one is "There exists some y such that for every x, $x + y = 0$." This is **false**. After switching the order, we have completely changed the meaning of our statement.

Rules for switching quantifiers:

- Okay to swap $\forall x$ and $\forall y$
- Okay to swap $\exists x$ and $\exists y$ (verify this for yourself)
- Generally, not okay to sway $\forall x$ and $\exists y$

2.0.2 Domain Caution

Consider the domain of all real numbers. How can we express the fact that all numbers have a *multiplicative inverse*? (A number we can multiply the original by to get 1.)

First off, is this even true? Do *all* real numbers have a multiplicative inverse? No, but all do! How do we say this with quantifiers? "For all x that aren't 0, there exists some number y such that $xy = 1$. Note that "that aren't 0" is a *condition* that we need to satisfy in order to move on to the second part of this statement. (We will need to use a conditional.) So maybe: $\forall x((x \neq 0) \Rightarrow \exists y(xy = 1))$

2.0.3 Tricky examples

Example: Translate the statement "You can fool some of the people all of the time." Let $F(p, t)$ be the statement "you can fool person p at time t." Let the domain for p be all people, the domain for t be time. Then we have $\exists p \forall t F(p, t)$

Example: Translate the statement "You can't fool all of the people all of the time." First, "It is not the case that for every person, for all times, they can be fooled." So, $\neg(\forall p \forall t F(p, t))$. What if we push the negation through?

$$\begin{aligned} \neg(\forall p \forall t F(p,t)) &\equiv \exists p \neg(\forall t F(p,t)) \\ &\equiv \exists p \exists t \neg F(p,t) \end{aligned}$$

So "there exists some person for some time that can't be fooled"

3 Time to practice

Exercise
Prove that $\neg(p \vee (\neg p \wedge q))$ is logically equivalent to $\neg p \wedge \neg q$.

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \vee \neg(\neg p \wedge q) && \text{DeMorgans}\\
&\equiv \neg p \wedge (\neg(\neg p) \vee \neg q) && \text{DeMorgans}\\
&\equiv \neg p \wedge (p \vee \neg q) && \text{Double negation}\\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{Distributive law}\\
&\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{since } \neg p \wedge p \equiv \mathbf{F}\\
&\equiv (\neg p \wedge \neg q) \vee \mathbf{F} && \text{commutative law}\\
&\equiv \neg p \wedge \neg q && \text{identity law}
\end{aligned}
$$

Exercise
Show $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

$$
\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{By RBI}\\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by DeMorgans}\\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by associative and commutative disjunction}\\
&\equiv \mathbf{T} \vee \mathbf{T} && \text{known equivalence}\\
&\equiv \mathbf{T} && \text{disjunction definition}
\end{aligned}
$$

Exercise
Let the domain be the real numbers and let $F(x, y, z)$ be the statement "$xy = z$". Determine the truth value of the following statements: (Justify your answer).

1. $\forall x \forall y \exists z F(x, y, z)$
2. $\exists z \forall x \forall y F(x, y, z)$

1. **True**. For any x, for any y there exists a z such that $xy = z$. In other words, multiply any two real numbers and you end up with a real number. (Closure of $\mathbb{R}$ under multiplication.)
2. **False**. There exists a z such that for any x and any y $xy = z$. In other words, there is some magical number z such that multiplying any two real numbers produces z.

Exercise
Use quantifiers to express the following statements:

1. Every computer science student needs a course in discrete mathematics.
2. There is a student in this class who owns a quantum computer.
3. Every student in this class has taken at least one computer science course.
4. There is a student in this class who has taken at least one course in computer science.
5. Every student in this class has been in every building on campus.
6. There is a student in this class who has been in every room of at least one building on campus.
7. Every student in this class has been in at least one room of every building on campus.

1. $\forall x P(x)$ where $P(x)$ is "x needs a course in discrete math" and the universe of discourse is the set of all computer science students.
2. $\exists x P(x)$ where $P(x)$ is "x owns a quantum computer" and the universe is the set of students in the class.
3. $\forall x \exists y P(x, y)$ where $P(x, y)$ is "x has taken y" and the universe of discourse for x is the students in the class and the universe for discourse of y is CS classes
4. $\exists x \exists y P(x, y)$ where $PP(x, y)$ and universe of discourse are same as in (c)
5. $\forall x \forall y P(x, y)$ where $P(x, y)$ is "x has been in y" and x's universe is students in class and y's universe is buildings on campus
6. $\exists x \exists y \forall z (P(z, y) \rightarrow Q(x, z))$ where $P(x, y)$ is "z is in y" and $Q(x, z)$ is "x has been in z". x's universe is students in class, y's universe is buildings on campus and z's universe is set of rooms
7. $\forall x \forall y \exists z (P(z, y) \wedge Q(x, y))$ from previous

Rules of inference	Tautology	Name
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$(p \lor q) \lor r$ $\therefore p \lor (q \lor r)$	$((p \lor q) \lor r) \to (p \lor (q \lor r))$	Associative
$p \land q$ $\therefore q \land p$	$(p \land q) \to (q \land p)$	Commutative
$p \to q$ $q \to p$ $\therefore p \leftrightarrow q$	$((p \to q) \land (q \to p)) \to (p \leftrightarrow q)$	Law of biconditional propositions
$(p \land q) \to r$ $\therefore p \to (q \to r)$	$((p \land q) \to r) \to (p \to (q \to r))$	Exportation
$p \to q$ $\therefore \neg q \to \neg p$	$(p \to q) \to (\neg q \to \neg p)$	Transposition or contraposition law
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \to q$ $\therefore \neg p \lor q$	$(p \to q) \to (\neg p \lor q)$	Material implication
$(p \lor q) \land r$ $\therefore (p \land r) \lor (q \land r)$	$((p \lor q) \land r) \to ((p \land r) \lor (q \land r))$	Distributive
$p \to q$ $\therefore p \to (p \land q)$	$(p \to q) \to (p \to (p \land q))$	Absorption
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
p $\therefore \neg\neg p$	$p \to (\neg\neg p)$	Double negation
$p \lor p$ $\therefore p$	$(p \lor p) \to p$	Disjunctive simplification
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution
$p \to q$ $r \to q$ $p \lor r$ $\therefore q$	$((p \to q) \land (r \to q) \land (p \lor r)) \to q$	Disjunction Elimination

Figure 1: From Wikipedia