CSCI 2824 - CU Boulder, 2019 Summer

Lecture 11: Functions

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Content is borrowed from Susanna Epp's Discrete Mathematics with Applications,
Rosens's Discrete Mathematics and its Applications,
Bettina and Thomas Richmond's A Discrete Transition to Advanced Mathematics, and Andrew Altomare's notes.

1 Recap/Review

Exercise
Prove that $|A \cup B| = |A| + |B| - |A \cap B|$

Let A and B be any sets. Notice from Figure 1 that if we wish to count an element $x \in A \cup B$ that there are three possibilities:

- $x \in A$ but $x \notin B$
- $x \in B$ but $x \notin A$
- $x \in A$ and $x \in B$

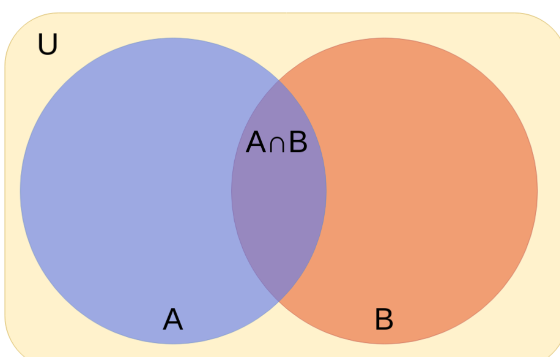

Figure 1: A schematic of two sets

Thus, $|A| + |B|$ double counts the overlapping purple $A \cap B$ and we get the final result.

Exercise
For any prime numbers a, b, and c, $a^2 + b^2 \neq c^2$.

Consider prime numbers a, b, and c. Suppose that $a^2 + b^2 = c^2$. Then, $a^2 = c^2 - b^2 = (c+b)(c-b)$. Since a is prime then the only factors for a^2 are a, a^2, and 1. Thus, we have two scenarios:

1. $c + b = a$ and $c - b = a$. Then, $c + b = c - b$ implies that $b = -b$, which is only possible when $b = 0$, which is not a prime number and is a contradiction.

2. $c + b = 1$ and $c - b = a^2$. $c - b > c + b$. This is only possible if $b < 0$, which is not the case since b must be prime. Thus, we have a contradiction.

3. $c + b = a^2$ and $c - b = 1$. To get a difference of 1, you must subtract an even number from an odd number. The only even prime is 2 implying $b = 2$ and $c = 3$. Then, $c + b = 2 + 3 = 5 = a^2$ implying $a = \sqrt{5}$, which is not an integer and not a prime number.

In all scenarios, we arrive at a contradiction. Therefore, $a^2 + b^2 \neq c^2$.

2 What is a function?

You spent so long in high school talking about functions, but can you define them formally? I dare you to try before you check the definition.

Let's first define a relation.

Definition: *relation*
Let A and B be sets. A relation R from A to B is a subest of $A \times B$. Given an ordered pair $(x, y) \in A \times B$, x is related to y by R, written $x\,R\,y$, if, and only if, $(x, y) \in R$. The set A is called the domain of R and the set B is called its co-domain.

Now, we are prepared!

Definition: *function*
A function f from a set X to a set Y, denoted $f : X \longrightarrow Y$, is a relation from X, the domain of f, to Y, the co-domain of f that satisfies two properties:

- Every element in X is related to some element in Y
- No element in X is related to more than one element in Y.

Thus, given $x \in X$, there is a unique element in Y that is related to x by f.

The set of all values of f taken together is called the range of f or the image of X under f, $\{y \in Y | y = f(x) \text{ for some } x \in X\}$.

Given an element y in Y, there may exist elements in X with y as their image. When x is an element such that $f(x) = y$, then x is the preimage or inverse image of y.

The simplest example of a function might be the identity function.

Definition: *identity function*
Given a set X, the identity function on X is defined as $I_X(x) = x$ for all $x \in X$.

It's just a copier!

We also have already seen sequences which were functions from the natural numbers to some set.

3 Properties of functions

See the book for more examples here. I don't feel comfortable reproducing the figures since I copied them from the book.

Definition: *injective*
Let $F : X \longrightarrow Y$ be a function. F is injective if an only if, $\forall x, x' \in X$ if $F(x) = F(x')$ then $x = x'$ or equivalently if $x \neq x'$ then $F(x) \neq F(x')$, i.e. $\forall x, x' \in X\ F(x) = F(x') \rightarrow x = x'$. Some people call this property one-to-one.

Definition: *surjective*
Let $F : X \longrightarrow Y$ be a function. F is surjective if, and only if, given $y \in Y$, it is possible to find $x \in X$ such that $y = F(x)$, i.e. $\forall y \in Y, \exists x \in X$ such that $F(x) = y$. Some people call this property onto.

Definition: *bijective*
A function $F : X \longrightarrow Y$ is bijective if it is both injective and surjective. Some people call this function F a one-to-one correspondence.

Theorem 3.1. *Suppose $F : X \longrightarrow Y$ is a bijection. Then, there exists a function $F^{-1} : Y \longrightarrow X$ defined such that $F^{-1}(y)$ is the unique element $x \in X$ where $F(x) = y$. That is, $F^{-1}(y) = x \leftrightarrow F(x) = y$. We call this the inverse function for F.*

Exercise
Prove that if X and Y are sets and $F : X \longrightarrow Y$ is a bijection then F^{-1} is also a bijection.

Let's begin by showing F^{-1} is injective. Consider $y, y' \in Y$ such that $F^{-1}(y) = F^{-1}(y')$. Let $x = F^{-1}(y) = F^{-1}(y')$. Then, $x \in X$ and by definition of F^{-1}, $F(x) = y$ and $F(x) = y'$. Therefore, $y = y'$.

Now we will show that F^{-1} is surjective. Consider $x \in X$. Let $y = F(x)$. Then, $y \in Y$ and by definition of F^{-1}, $F^{-1}(y) = x$.

Therefore, F^{-1} is a bijection.

4 Cardinality

We started this aside to figure out the cardinality of infinite sets. We're finally ready!

Definition: *cardinality*
Let A and B be any sets. A has the same cardinality as B if and only if there is a bijection from A to B.

Let's first observe a few properties about cardinality:

Exercise
Show that A has the same cardinality as A.

Suppose A is any set. Consider the identity function I_A from A to A. This is injection because if x and x' are elements in A with $I_A(x) = I_A(x')$ then $x = x'$. It's also surjective because if $y \in A$ then $y = I_A(y)$ by definition. So I_A is a bijection.

Exercise
Show that cardinality is symmetric, i.e. if A has the same cardinality as B then B has the same cardinality as A.

Suppose A and B are any sets and A has the same cardinality as B. Theren there exists a function f from A to B that is a bijection. Therefore, there must exist a function f^{-1} from B to A that is also a bijection.

Exercise
Show that cardinality if transitive, i.e. if A has the same cardinality as B and B has the same cardinality as C then A has the same cardinality as C.

Suppose A, B, and C are any sets and that A has the same cardinality as B and B has the same cardinality as C. Then, there exists a function f from A to B that is a bijection and a function g from B co C that is a bijection. Therefore, $g \circ f$ is a function from A to C that is a bijection.

We then talk about countability of sets. That tells us about the different size of infinite sets.

Exercise
Show that the set of integers is countable.

It's not finite. But notice that the following function is a bijection that maps from $\mathbb{N}$ to $\mathbb{Z}^+$ so they have the same size and thus are both countable.

$$F(n) = \begin{cases} \frac{n}{2} & n \text{ is an even positive integer} \\ \frac{-(n-1)}{2} & n \text{ is an odd positive integer} \end{cases}$$

We can easily pair up positive and negative integers too. (This is hand wavey sounding. Be more rigorous.)

Can we come up with an infinity that isn't countable? Let's try the positive rational numbers... Oh no though Cantor's diagonal argument ruins us.

Exercise
The set of all real numbers between 0 and 1 is uncountable.

We proceed by proof by contradiction. Suppose that the set of all real numbers between 0 and 1 is countable. Then we can write them as a list:

$$\begin{aligned}
&0.a_{11}a_{12}a_{13}\dots a_{1n}\dots\\
&0.a_{21}a_{22}a_{23}\dots a_{2n}\dots\\
&0.a_{31}a_{32}a_{33}\dots a_{3n}\dots\\
&\quad\vdots\\
&0.a_{n1}a_{n2}a_{n3}\dots a_{nn}\dots\\
&\quad\vdots
\end{aligned}$$

We now construct a new decimal number $d = 0.d_1d_2d_3\dots d_n\dots$ as follows:

$$d_n = \begin{cases} 1 & \text{if } a_{nm} \neq 1 \\ 2 & \text{if } a_{nm} = 1 \end{cases}$$

. But, notice that d differs in the nth decimal position from the nth number on the list. Therefore, d is not on the list despite being between 0 and 1 and a real number. This is a contradiction. Hence, the set of all real numbers between 0 and 1 is uncountable!

Exercise
Any subset of any countable set is countable.

Let A be a particular but arbitrarily chosen countable set and let B be any subset of A. Either B is finite or it is infinite. If it is finite, then B is countable by definition. Suppose B is infinite. Since A is countable its elements can be represented as a sequence $a_1, a_2, a_3, \dots$. We now define a function $g : \mathbb{Z}^+ \longrightarrow B$ as follows:

1. Search sequentially through $a_1, a_2, a_3, \dots$ until an element of B is found. Call that element $g(1)$.

2. For each integer $k \geq 2$ suppose $g(k-1)$ has been defined, i.e. $g(k-1) = a_i$ for some a_i in the sequence. Starting with a_{i+1} search sequentially through $a_{i+1}, a_{i+2}, a_{i+3}, \dots$ trying to find an element of B. We must find one eventually because B is infinite and we have only defined finitely many elements in g so far. Call the one that that is found $g(k)$.

Thus, g is defined for each positive integer. Since all $a_1, a_2, a_3, \dots$ is distinct then g is injective. Further, the searches for elements of B are sequential. Thus, every element of A is reached during the search and all elements of B is somewhere in the sequence. So, every element of B is eventually found and made the image of some integer. So, g is surjective. Thus, we have a bijection between $\mathbb{Z}^+$ and B proving it is countably infinite.

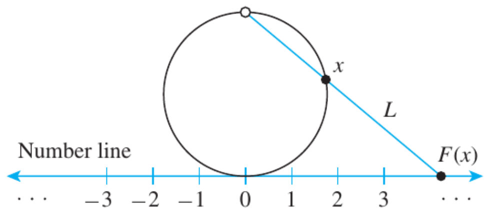

Figure 2: Definition of f

> **Exercise**
> Prove that any set with a countable subset is uncountable.

This is just the contrapositive of the previous theorem.

> **Exercise**
> Show that $\mathbb{R}$ is uncountable.

Define the function $f : S \longrightarrow \mathbb{R}$ as shown in Figure 2. So the real numbers between 0 and 1 have the same cardinality as all the reals.