CSCI 2824 - CU Boulder, 2019 Summer

Lecture 4: Multiply Quantified Logic

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Content is borrowed from Susanna Epp's Discrete Mathematics with Applications and Andrew Altomare's notes.

1 Review

- Universal quantifier $\forall x P(x)$ means "for all x in my **domain**, $P(x)$
- Existential quantifier $\exists x P(x)$ means "there exists an x in my domain such that $P(x)$"
- Distribution rules

$$\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$$
$$\exists x(P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$$

- And recall that distribution of $\forall$ over $\vee$ and $\exists$ over $\wedge$ did **not** work

1.1 Nested Quantifiers

We can include multiple quantifiers for a propositional function. Consider the propositional function $C(x, y)$="y is the favorite color of x." What about the domains? Consider x: all people and y: all colors.

We could write: $\forall x \exists y C(x, y)$="for all people, there exists a color such that that color is their favorite."

Consider the domain of all real numbers. What does the following statement mean? $\forall x \exists y(x + y = 0)$. In English, we can write it "For all x, there exists a y such that $x + y = 0$. This is **true**. This is expressing the fact that all real numbers have an additive inverse.

1.1.1 Swapping order

How can we express the law of *commutation of addition* (That is $x + y = y + x$.) Let the domain be the real numbers. Then we could use $\forall x \forall y(x+y = y+x)$. What happens if you swap the order of $\forall x$ and $\forall y$? Then we would have instead: $\forall y \forall x(x+y = y+x)$ Turns out nothing changes. You still loop over all the combinations of x's and y's

Returning to the previous example: $\forall x \exists y(x + y = 0) \stackrel{?}{\equiv} \exists y \forall x(x + y = 0)$. i.e. what happens when we swap the order of $\forall$ and $\exists$? The original statement was "For every x there exists some y

such that $x + y = 0$." The new one is "There exists some y such that for every x, $x + y = 0$." This is **false**. After switching the order, we have completely changed the meaning of our statement.

Rules for switching quantifiers:

- Okay to swap $\forall x$ and $\forall y$
- Okay to swap $\exists x$ and $\exists y$ (verify this for yourself)
- Generally, not okay to sway $\forall x$ and $\exists y$

1.1.2 Domain Caution

Consider the domain of all real numbers. How can we express the fact that all numbers have a *multiplicative inverse*? (A number we can multiply the original by to get 1.)

First off, is this even true? Do *all* real numbers have a multiplicative inverse? No, but all do! How do we say this with quantifiers? "For all x that aren't 0, there exists some number y such that $xy = 1$. Note that "that aren't 0" is a *condition* that we need to satisfy in order to move on to the second part of this statement. (We will need to use a conditional.) So maybe: $\forall x((x \neq 0) \Rightarrow \exists y(xy = 1))$

1.1.3 Tricky examples

Example: Translate the statement "You can fool some of the people all of the time." Let $F(p, t)$ be the statement "you can fool person p at time t." Let the domain for p be all people, the domain for t be time. Then we have $\exists p \forall t F(p, t)$

Example: Translate the statement "You can't fool all of the people all of the time." First, "It is not the case that for every person, for all times, they can be fooled." So, $\neg(\forall p \forall t F(p, t))$. What if we push the negation through?

$$\begin{aligned} \neg(\forall p \forall t F(p, t) &\equiv \exists p \neg(\forall t F(p, t)) \\ &\equiv \exists p \exists t \neg F(p, t) \end{aligned}$$

So "there exists some person for some time that can't be fooled"

References