CSCI 2824 - CU Boulder, 2019 Summer

Lecture 11: Functions

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Content is borrowed from Susanna Epp's Discrete Mathematics with Applications,
Rosens's Discrete Mathematics and its Applications,
Bettina and Thomas Richmond's A Discrete Transition to Advanced Mathematics, and Andrew Altomare's notes.

1 What is a function?

You spent so long in high school talking about functions, but can you define them formally? I dare you to try before you check the definition.

Let's first define a relation.

Definition: *relation*
Let A and B be sets. A relation R from A to B is a subest of $A \times B$. Given an ordered pair $(x, y) \in A \times B$, x is related to y by R, written $x\, R\, y$, if, and only if, $(x, y) \in R$. The set A is called the domain of R and the set B is called its co-domain.

Now, we are prepared!

Definition: *function*
A function f from a set X to a set Y, denoted $f : X \longrightarrow Y$, is a relation from X, the domain of f, to Y, the co-domain of f that satisfies two properties:

- Every element in X is related to some element in Y
- No element in X is related to more than one element in Y.

Thus, given $x \in X$, there is a unique element in Y that is related to x by f.

The set of all values of f taken together is called the range of f or the image of X under f, $\{y \in Y | y = f(x) \text{ for some } x \in X\}$.

Given an element y in Y, there may exist elements in X with y as their image. When x is an element such that $f(x) = y$, then x is the preimage or inverse image of y.

The simplest example of a function might be the identity function.

Definition: *identity function*
Given a set X, the identity function on X is defined as $I_X(x) = x$ for all $x \in X$.

It's just a copier!

We also have already seen sequences which were functions from the natural numbers to some set.

2 Properties of functions

Definition: *injective*
Let $F : X \longrightarrow Y$ be a function. F is injective if an only if, $\forall x, x' \in X$ if $F(x) = F(x')$ then $x = x'$ or equivalently if $x \neq x'$ then $F(x) \neq F(x')$, i.e. $\forall x, x' \in X\ F(x) = F(x') \rightarrow x = x'$. Some people call this property one-to-one.

Definition: *surjective*
Let $F : X \longrightarrow Y$ be a function. F is surjective if, and only if, given $y \in Y$, it is possible to find $x \in X$ such that $y = F(x)$, i.e. $\forall y \in Y, \exists x \in X$ such that $F(x) = y$. Some people call this property onto.

Definition: *bijective*
A function $F : X \longrightarrow Y$ is bijective if it is both injective and surjective. Some people call this function F a one-to-one correspondence.

Theorem 2.1. *Suppose $F : X \longrightarrow Y$ is a bijection. Then, there exists a function $F^{-1} : Y \longrightarrow X$ defined such that $F^{-1}(y)$ is the unique element $x \in X$ where $F(x) = y$. That is, $F^{-1}(y) = x \leftrightarrow F(x) = y$. We call this the inverse function for F.*

Exercise
Prove that if X and Y are sets and $F : X \longrightarrow Y$ is a bijection then F^{-1} is also a bijection.

3 Hash functions

4 Cardinality

We started this aside to figure out the cardinality of infinite sets. We're finally ready!

Definition: *cardinality*
Let A and B be any sets. A has the same cardinality as B if and only if there is a bijection from A to B.

Let's first observe a few properties about cardinality:

Exercise
Show that A has the same cardinality as A.

Exercise
Show that cardinality is symmetric, i.e. if A has the same cardinality as B then B has the same cardinality as A.

Exercise
Show that cardinality if transitive, i.e. if A has the same cardinality as B and B has the same cardinality as C then A has the same cardinality as C.

We then talk about countability of sets. That tells us about the different size of infinite sets.

Exercise
Show that the set of integers is countable.

Can we come up with an infinity that isn't countable? Let's try the positive rational numbers... Oh no though Cantor's diagonal argument ruins us.

Exercise
The set of all real numbers between 0 and 1 is uncountable.

Exercise
Any subset of any countable set is countable.

Exercise
Prove that any set with a countable subset is uncountable.

Exercise
Show that $\mathbb{R}$ is uncountable.