In-class questions July 10

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I noticed people really struggled with these problems. We only just covered the concept, but I see the trend that people often don't know where to start, so they write nothing. Even if you don't realize that the binomial theorem is your friend here, you should learn the habit of trying simpler versions of the problem. I suggest thinking about $n = 1$ and then $n = 2$ and then $n = 3$ and so on to see if you can identify a pattern. It really helped some people on the first problem. This is a good practice for any problem and will help you make at least some progress. Thinking in your head only about the problem or telling yourself that you don't know where to start will not make nearly as much progress as just trying things. Take the risk and you'll get the reward.

Theorem 1 (Problem 1)**.**

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n} = 0$$

Proof. Note that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n}$$

can be more succinctly written as $\sum_{k=0}^{n} (-1)^n \binom{n}{k}$. Further, we can add in a power of one: $\sum_{k=0}^{n} 1^{n-k} (-1)^n \binom{n}{k}$. By the binomial theorem this is equivalent to $(1 + (-1))^n$ for any $n \geq 1$. Notice that $0^k = 0$ for $k \geq 1$. Therefore, we have proven the statement. □

Theorem 2 (Problem 2)**.**

$$3^n = \binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \ldots + 2^n\binom{n}{n}$$

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Proof. Consider:

$$\begin{aligned}
3^n &= (1+2)^n \\
&= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 2^k && \text{binomial theorem} \\
&= \sum_{k=0}^{n} \binom{n}{k} 2^k && \text{dropping the multiply by one} \\
&= \binom{n}{k} + 2\binom{n}{1} + 2^2\binom{n}{2} + \ldots + 2^n\binom{n}{n} && \text{expanding}
\end{aligned}$$

Thus, we've proven the statement. □