Final Practice Problems

These problems represent material covered after the midterm. However the final will be cumulative. For practice problems covering material before the midterm, see 'Midterm Practice Problems'

1. Describe the difference between bubble sort and insertion sort

2. Use a greedy algorithm to make change using quarters, dimes, nickels, and pennies for

   a) 87 cents
   b) 99 cents
   c) 49 cents
   d) 33 cents

3. Determine whether each of these functions is $\mathcal{O}(x^2)$

   a) $f(x) = 17x + 11$
   b) $f(x) = x \log x$
   c) $f(x) = x^2 + 1000$
   d) $f(x) = x^4/2$

4. Arrange the functions $\sqrt{n}, 1000 \log n, n \log n, 2n!, 2^n, 3^n$, and $n^2/1000000$ in a list so that each function is big-O of the next function.

5. Show that if $f_1(x)$ and $f_2(x)$ are functions from the set of positive integers to the set of real numbers and $f_1(x)$ is $\Theta(g_1(x))$ and $f_2(x)$ is $\Theta(g_2(x))$, then $(f_1 f_2)(x)$ is $\Theta((g_1 g_2)(x))$

6. Show that $\lceil xy \rceil$ is $\Omega(xy)$

7. What is the best order to form the product ABC if A, B, and C are matrices with dimensions $3 \times 9, 9 \times 4$, and 4×2, respectively?

8. Prove or disprove that if $a \mid bc$, where $a, b, c \in \mathbb{Z}^+$ and $a \neq 0$, then $a \mid b$ or $a \mid c$

9. Evaluate these quantities

   a) $-17 \bmod 2$
   b) $-101 \bmod 13$
   c) $144 \bmod 7$
   d) $199 \bmod 19$

10. Use fast modular exponentiation to find

    a) $7^{644} \bmod 645$
    b) $3^{2003} \bmod 99$
    c) $11^{644} \bmod 645$

11. Use Euclid's algorithm to find

    a) $\gcd(123, 277)$
    b) $\gcd(54, 12)$
    c) $\gcd(1000, 5040)$

12. Find an inverse of a modulo m for each of these pairs of relatively prime integers.

a) $a = 19, m = 141$

b) $a = 55, m = 89$

c) $a = 89, m = 232$

13. Using the inverses you just found, solve each of these congruences.

    a) $19x \equiv 4 \pmod{141}$

    b) $55x \equiv 34 \pmod{89}$

    c) $89x \equiv 2 \pmod{232}$

14. Use Fermat's little theorem to find

    a) $7^{121} \bmod 13$

    b) $23^{1002} \bmod 41$

    c) $2^{340} \bmod 11$

15. Find all solutions, if any, to the system of congruences

$$\begin{aligned} x &\equiv 7 \pmod{9} \\ x &\equiv 4 \pmod{12} \\ x &\equiv 16 \pmod{21} \end{aligned}$$

16. Give a recursive algorithm for computing the gcd of two nonnegative integers a and b with $a < b$ using the fact that $\gcd(a, b) = \gcd(a, b - a)$

17. Give a recursive definition for the set of integers that are congruent to 3 modulo 7.

18. How many bit strings with length n both begin and end with a 1?

19. How many one-to-one functions are there from a set with five elements to sets with the following number of elements?

    a) 4

    b) 5

    c) 7

20. Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

21. Show that among any group of five (non necessarily consecutive) integers, there are two with the same remainder when divided by 4.

22. Use the binomial theorem to find the expansion of $(x + y)^6$

23. Find the coefficient of x^5y^8 in $(x + y)^{13}$

24. What is the probability that a five-card poker hand contains at least one ace?

25. What is the probability that a fair die never comes up an even number when it is rolled six times?

26. Suppose that E and F are events such that $p(E) = 0.8$ and $p(F) = 0.6$. Show that $p(E \cup F) \geq 0.8$ and $p(E \cap F) \geq 0.4$

27. Suppose that E and F are events in a sample space and $p(E) = 1/3$, $p(F) = 1/2$ and $p(E|F) = 2/5$. Find $p(F|E)$.

28. Consider the nonhomogeneous linear recurrence relation $a_n = 2a_{n-1} + 2^n$

   a) Find all solutions of this recurrence relation (general form)
   b) Find the solution with $a_0 = 2$

29. The Lucas numbers satisfy the recurrence relation $L_n = L_{n-1} + L_{n-2}$ with initial conditions $L_0 = 2$ and $L_1 = 1$. Find an explicit formula for the Lucas numbers.

30. Find the solution to $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$ with $a_0 = 7$, $a_1 = -4$ and $a_2 = 8$

31. List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$ where $(a, b) \in R$ if and only if

   a) $a = b$
   b) $a > b$
   c) $a \mid b$

32. Determine whether the relation R on the set of all people is reflexive, symmetric, and/or transitive where $(a, b) \in R$ if and only if

   a) a is taller than b
   b) a and b were born on the same day
   c) a and b have a common grandparent

33. Which of the following relations on $\{0, 1, 2, 3\}$ are equivalence relations?

   a) $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$
   b) $\{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$
   c) $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

34. Determine whether the relations represented by these adjacency matrices are equivalence relations

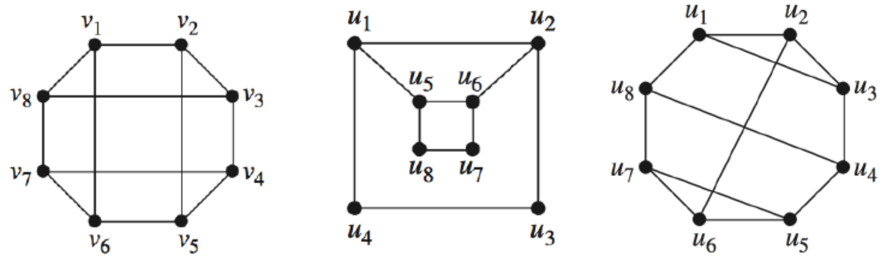
   a)
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

   b)
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

35. For each of the degree sequences shown below, determine whether they represent a valid undirected graph with no self-loops. If they do, draw the graph. If they do not, explain why.

   a) 4, 3, 2, 1, 0
   b) 2, 2, 2, 2, 2
   c) 1, 1, 1, 1, 1
   d) 4, 4, 3, 2, 1

36. For each of the graphs below, determine if the graph has an Eulerian Tour. If it does, give one such tour. If it does not, explain why. For graphs that do not contain Eulerian tours, can you add a small number of edges so that they do contain one?



37. For the same graphs, determine whether each one is bipartite. If it is, specify a two-coloring of the vertices. If it is not, explain why. For graphs that are not bipartite, can you remove a small number of edges so that they become bipartite?

38. Use the greedy coloring algorithm to find a coloring of the vertices of the graphs below. To start out, consider the vertices in alphabetical order. Then choose your own vertex order and repeat the coloring process.