A gaussian example

$$\rho(\cdot) \sim \mathcal{N}^{\otimes m}(0,1)$$

$$q(\cdot|\theta) \sim \mathcal{N}^{\otimes m}(\theta,1)$$

$$\boxed{\theta_0 = 0}$$

$$\mathbb{E}_{y\sim\rho(\cdot)}\left[\log\left(q(y|\theta_0)\right)\right] = \alpha$$

$$\alpha = -\frac{m}{2}\log(2\pi) - \frac{m}{2} \quad \Big| \quad \hat{\alpha} = \log\left(q(x|\bar{x}_m)\right)$$

$$\hat{\alpha} = -\frac{m}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{m}(x_i - \bar{x}_m)^2$$

$$\mathbb{E}_{x\sim\rho(\cdot)}(\hat{\alpha}) = -\frac{m}{2}\log(2\pi) - \frac{1}{2} \times \mathbb{E}_{x\sim\rho(\cdot)}\left[\sum_{i=1}^{m}(x_i - \bar{x}_m)^2\right]$$

(2)

$$\mathbb{E}_{x\sim p(\cdot)}(\hat{\alpha}) = -\frac{m}{2}\log(2\pi) - \frac{1}{2}(m-1)$$

$$\Rightarrow \mathbb{E}_{x\sim p(\cdot)}(\hat{\alpha}) - \alpha = \frac{m}{2} - \frac{m-1}{2} = \left[\frac{1}{2}\right]$$

The bias is the number of parameters divided by 2

Confidence interval on α

$$\alpha = -\frac{m}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{m}\mathbb{E}_{N(0,1)}\left((y_i-\theta_0)^2\right)$$

$$= -\frac{m}{2}\log(2\pi) - \frac{m}{2}\left(1+\theta_0^2\right)$$

$$= -\frac{m}{2}\left[\log(2\pi) - 1 - \theta_0^2\right]$$

$$\alpha(\theta_0) = -\frac{n}{2}\left[\log(2\pi) - 1 - \theta_0^2\right]$$

$$\alpha^{\#} = \hat{\alpha} - \frac{1}{2} = \log\left(q(x \mid \bar{x}_n)\right) - \frac{1}{2}$$

$$\alpha^{\#} = \hat{\alpha} - \frac{1}{2} = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n}(x_i - \bar{x}_n)^2 - \frac{1}{2}$$

$$\alpha^{\#} - \alpha(\theta_0) = -\frac{1}{2}\sum_{i=1}^{n}(x_i - \bar{x}_n)^2 + \left(\frac{n-1}{2}\right) + \frac{n\theta_0^2}{2}$$

Bootstrap $\alpha^{\#} - \alpha(\theta_0)$

$$-\frac{1}{2}\sum_{i=1}^{n}(x_i^* - \bar{x}_n^*)^2 + \left(\frac{n-1}{2}\right) + \frac{n\bar{x}_n^2}{2}$$

$(x_1^*, ----, x_n^*)$ a bootstrap sample from $(x_1, ----, x_n)$

$$\alpha^{\#} - \alpha = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum(x_i - \bar{x}_n)^2 - \frac{1}{2} - \alpha$$

By construction

Bootstrap $\alpha^{\#} - \alpha(\theta_0)$
is equivalent to
bootstrap $\hat{\alpha} - \alpha(\theta_0)$
We need to evaluate
the bias to get
a bootstrap confidence
interval on α!