



Machine Learning

Linear Algebra
review (optional)

Matrices and
vectors

Matrix: Rectangular array of numbers:



$$\rightarrow \boxed{\mathbb{R}^{4 \times 2}}$$



$$\boxed{\mathbb{R}^{2 \times 3}}$$

Dimension of matrix: number of rows x number of columns

Matrix Elements (entries of matrix)

$$A = \begin{bmatrix} 1402 & 191 \\ 1371 & 821 \\ 949 & 1437 \\ 147 & 1448 \end{bmatrix}$$

A_{ij} = " i, j entry" in the i^{th} row, j^{th} column.

$$A_{11} = 1402$$

$$A_{12} = 191$$

$$A_{32} = 1437$$

$$A_{41} = 147$$

~~A_{43}~~ = Undefined (error)

Vector: An $n \times 1$ matrix.

$$\textcircled{y} = \begin{bmatrix} \textcircled{460} \\ \textcircled{232} \\ \textcircled{315} \\ 178 \end{bmatrix}$$

$n = 4$

← 4-dimensional vector.

~~$\mathbb{R}^{3 \times 2}$~~

\mathbb{R}^4

$y_i = i^{th}$ element

$$y_1 = 460$$

$$y_2 = 232$$

$$y_3 = 315$$

→ A, B, C, X

a, b, x, y

1-indexed vs 0-indexed:

$y[1]$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

1-indexed

$y[0]$

$$y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

0-indexed



Machine Learning

Linear Algebra review (optional)

Addition and scalar multiplication

Matrix Addition

$$\begin{array}{c} \downarrow \quad \downarrow \\ \rightarrow \begin{bmatrix} \textcircled{1} & 0 \\ \textcircled{2} & 5 \\ \textcircled{3} & 1 \end{bmatrix} + \begin{bmatrix} \textcircled{4} & 0.5 \\ \textcircled{2} & 5 \\ \textcircled{0} & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0.5 \\ 4 & 10 \\ 3 & 2 \end{bmatrix} \\ \text{3} \times \text{2} \quad \text{3} \times \text{2} \quad \text{3} \times \text{2} \\ \text{matrix} \end{array}$$

$$\begin{array}{c} \rightarrow \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0.5 \\ 2 & 5 \end{bmatrix} = \\ \text{3} \times \text{2} \quad \text{2} \times \text{2} \end{array} \quad \text{error}$$

Scalar Multiplication

← real number

$$3 \times \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 6 & 15 \\ 9 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} \times 3$$

3x2 3x2

$$\begin{bmatrix} 4 & 0 \\ 6 & 3 \end{bmatrix} / 4 = \frac{1}{4} \begin{bmatrix} 4 & 0 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & \frac{3}{4} \end{bmatrix}$$

Combination of Operands

Scalar multiplication

$$3 \times \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} / 3$$

Scalar division

matrix subtraction / vector subtraction

matrix addition / vector addition

$$= \begin{bmatrix} 3 \\ 12 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ \frac{2}{3} \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ 12 \\ 10\frac{1}{3} \end{bmatrix}$$

3x1 matrix
3-dimensional vector



Machine Learning

Linear Algebra review (optional)

Matrix-vector multiplication

Example

$$\begin{bmatrix} 1 & 3 \\ 4 & 0 \\ 2 & 1 \end{bmatrix}_{3 \times 2} \begin{bmatrix} 1 \\ 5 \end{bmatrix}_{2 \times 1} = \begin{bmatrix} 16 \\ 4 \\ 7 \end{bmatrix}_{3 \times 1} \text{ matrix}$$

$$1 \times 1 + 3 \times 5 = 16$$

$$4 \times 1 + 0 \times 5 = 4$$

$$2 \times 1 + 1 \times 5 = 7$$

Details:

$$\underline{A} \times \underline{x} = \underline{y}$$

\underline{A} is an $m \times n$ matrix (m rows, n columns).
 \underline{x} is an $n \times 1$ matrix (n-dimensional vector).
 \underline{y} is an m -dimensional vector.

→ To get \underline{y}_i , multiply \underline{A} 's i^{th} row with elements of vector \underline{x} , and add them up.

Example

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 3 & 0 & 4 \\ -1 & -2 & 0 & 0 \end{bmatrix}_{3 \times 4} \begin{matrix} \downarrow \\ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}_{4 \times 1} \end{matrix} = \begin{bmatrix} 14 \\ 13 \\ -7 \end{bmatrix}_{3 \times 1} = \begin{bmatrix} 14 \\ 13 \\ -7 \end{bmatrix}$$

$$\left. \begin{array}{l} 1 \times 1 + 2 \times 3 + 1 \times 2 + 5 \times 1 = 14 \\ 0 \times 1 + 3 \times 3 + 0 \times 2 + 4 \times 1 = 13 \\ -1 \times 1 + (-2) \times 3 + 0 \times 2 + 0 \times 1 = -7 \end{array} \right\}$$

House sizes:

- 2104
- 1416
- 1534
- 852

Matrix

4x2

1	2104
1	1416
1	1534
1	852

X

$h_0(x)$

2x1

Vector

-40
0.25

=

4x1 matrix

$-40 \times 1 + 0.25 \times 2104$
$-40 \times 1 + 0.25 \times 1416$

$h_0(1416)$

Prediction = Data Matrix * Parameters

4x1

for $i = 1:1000$,
prediction(i) = ...

$$h_{\theta}(x) = -40 + 0.25x$$



Machine Learning

Linear Algebra review (optional)

Matrix-matrix multiplication

Example

$$\begin{array}{l} \begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 0 & 1 \\ \hline 5 & 2 \\ \hline \end{array} = \begin{bmatrix} 11 & 10 \\ 9 & 14 \end{bmatrix} \\ \textcircled{2 \times 3} \quad \textcircled{3 \times 2} \\ \begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \times \begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline 5 \\ \hline \end{array} = \begin{bmatrix} 11 \\ 9 \end{bmatrix} \\ \begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \times \begin{array}{|c|} \hline 3 \\ \hline 1 \\ \hline 2 \\ \hline \end{array} = \begin{bmatrix} 10 \\ 14 \end{bmatrix} \end{array}$$

Handwritten green annotations: The first row of the first matrix is underlined. In the second equation, the first matrix is underlined. In the third equation, the first matrix is underlined. Green arrows point from the 2x2 result matrix to the 2x1 and 1x2 result matrices, with a '2x2' label and arrows pointing to the corresponding dimensions.

Details:

$$\underline{A} \times \underline{B} = \underline{C}$$

$m \times n$ matrix
(m rows,
 n columns)

$n \times o$ matrix
(n rows,
 o columns)

$m \times o$
matrix

The i^{th} column of the matrix C is obtained by multiplying A with the i^{th} column of B . (for $i = 1, 2, \dots, o$)

Example

$$\overset{2 \times 2}{\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}} \overset{2 \times 2}{\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}} =$$

$$\overset{2 \times 2}{\begin{bmatrix} 9 & 4 \\ 15 & 12 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} =$$

$$\begin{bmatrix} 1 \times 0 + 3 \times 3 \\ 2 \times 0 + 5 \times 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} =$$

$$\begin{bmatrix} 1 \times 1 + 3 \times 2 \\ 2 \times 1 + 5 \times 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \end{bmatrix}$$

House sizes:

$$\begin{Bmatrix} \frac{2104}{1416} \\ \frac{1534}{852} \end{Bmatrix}$$

Matrix

$$\begin{bmatrix} 1 & \frac{2104}{1416} \\ 1 & \frac{1534}{852} \\ 1 & \frac{1534}{852} \\ 1 & \frac{1534}{852} \end{bmatrix} \times$$

Matrix

$$\begin{bmatrix} -40 \\ 0.25 \end{bmatrix} \begin{bmatrix} 200 \\ 0.1 \end{bmatrix} \begin{bmatrix} -150 \\ 0.4 \end{bmatrix} =$$

$$\begin{bmatrix} 486 \\ 314 \\ 344 \\ 173 \end{bmatrix} \begin{bmatrix} 410 \\ 342 \\ 353 \\ 285 \end{bmatrix} \begin{bmatrix} 692 \\ 416 \\ 464 \\ 191 \end{bmatrix}$$

Prediction
of first
 h_θ

Predictions
of 2nd
 h_θ

Have 3 competing hypotheses:

1. $h_\theta(x) = -40 + 0.25x$

2. $h_\theta(x) = 200 + 0.1x$

3. $h_\theta(x) = -150 + 0.4x$



Machine Learning

Linear Algebra review (optional)

Matrix multiplication properties

$$3 \times 5 = 5 \times 3$$


"Commutative"

Let A and B be matrices. Then in general,
 $A \times B \neq B \times A$. (not commutative.)

E.g.


$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$


$$\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$$


$A \times B$
 $m \times n \quad n \times m$

$A \times B$ is $m \times m$

$B \times A$ is $n \times n$



$$\underline{3 \times 5 \times 2}$$

$$3 \times 10 = 30 = 15 \times 2$$

$$3 \times (5 \times 2) = (3 \times 5) \times 2$$

"Associative"

$$\begin{array}{l} A \times (B \times C) \leftarrow \\ \underline{(A \times B)} \times C \leftarrow \end{array}$$



$$A \times B \times C.$$

Let $D = B \times C$. Compute $A \times D$.

Let $E = A \times B$. Compute $E \times C$.

$$\begin{array}{l} A \times (B \times C) \\ (A \times B) \times C \end{array} \rightarrow \text{Some answer.}$$

1 is identity

$$1 \times z = z \times 1 = z$$

for any z

Identity Matrix

Denoted I (or $I_{n \times n}$).

Examples of identity matrices:

$[1]$
 1×1

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2×2

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3×3

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4×4

Informally:

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

For any matrix A ,

$$A \cdot I = I \cdot A = A$$

$m \times n$ $n \times n$ $m \times m$ $m \times n$ $m \times n$

$I_{n \times n}$

Note:

$AB \neq BA$ in general

$$AI = IA \checkmark$$



Machine Learning

Linear Algebra
review (optional)

Inverse and
transpose

$$\underline{1 = \text{"identity"}}$$

$$3 \underbrace{(3^{-1})}_{\frac{1}{3}} = 1$$

$$12 \times \underbrace{(12^{-1})}_{\frac{1}{12}} = 1$$

$$0 \underbrace{(0^{-1})}_{\text{undefined}}$$

Not all numbers have an inverse.

Matrix inverse: \swarrow square matrix
(#rows = #columns) A^{-1}

If A is an $m \times m$ matrix, and if it has an inverse,

$$\rightarrow \underline{A(A^{-1})} = \underline{A^{-1}A} = \underline{I}.$$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \swarrow$$

e.g. $\underbrace{\begin{bmatrix} 3 & 4 \\ 2 & 16 \end{bmatrix}}_A \underbrace{\begin{bmatrix} 0.4 & -0.1 \\ -0.05 & 0.075 \end{bmatrix}}_{A^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A^{-1}A} = I_{2 \times 2}$

Matrices that don't have an inverse are "singular" or "degenerate"

Matrix Transpose

Example:

$$\underline{A} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 5 & 9 \end{bmatrix}_{2 \times 3}$$
$$\underline{B} = \underline{A}^T = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 0 & 9 \end{bmatrix}_{3 \times 2}$$

Let A be an $m \times n$ matrix, and let $B = A^T$.

Then B is an $n \times m$ matrix, and

$$\underline{B}_{ij} = \underline{A}_{ji}.$$

$$B_{12} = A_{21} = 2$$

$$B_{32} = 9 \quad A_{23} = 9.$$