SCF analysis of the Milky Way - Large Magellanic Cloud interaction

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1 Motivation:

Recent measurements have suggest that the Large Magellanic Cloud (LMC) is more massive than previously thought (abundance matching suggests the total halo mass is of order $\sim 10^{11} M_{\odot}$) and that the Clouds are likely approaching the Milky Way (MW) for the first time. We are interested in studying how the MW's Dark Matter (DM) halo is responding to this recent passage of the LMC. In particular we want to capture how the shape of the DM halo is changing in time and in radius. Studying this effect would directly impact models for the dynamics of all halo tracers (stellar streams clusters, halo stars, globular cluster, satellite galaxies).

2 Simulations & SCF method:

We set up N-body simulations representing the interaction of the Milky Way (MW) and the Large Magellanic Cloud (LMC), the basic properties of the simulation are described in Table 1. We consider both a low resolution (LR) and a high resolution (HR) simulation (more resolution tests are planned). The results shown here are for the simulations with the higher mass LMC ($2.5 \times 10^{11} M_{\odot}$). The LMC's orbit is followed for the past $2Gyr$ until the final position and velocity is within 2σ of the Katllivayalill (2013) measurements.

At each snapshot of the simulation we apply the SCF method using only the Dark Matter (DM) particles of the MW halo with the center of mass at cartesian coordinates $[0, 0, 0]$ (The effect of the LMC DM particles on the MW DM halo shape will be accounted for in later analysis, but is excluded here for easier testing of the analysis code). We compute the coefficients S_{nlm} (see §3 for the definitions of these coefficients) and the potential of the MW DM halo using the SCF method following Hernquist and Ostriker (1990) with the notation of Lowing et al (2011).

Figure 1 illustrates equipotential contours of the DM halo of the MW at different times during the simulation, the potential was computed using the coefficients of the basis function expansion explained later in this document. Plotted in red is the orbit of the LMC, but as noted earlier, LMC dark matter particles are not accounted for in this analysis. In the bottom row of the panels in Fig.1 you can see that the equipotential contours are distorted in the inner regions (*the attached movie in the email corresponds to this figure*).

In order to understand the response of the DM halo we would like to identify which coefficients S_{nlm}, T_{nlm} (multipoles in the expansion) contribute more to the time evolving shape of the DM halo. Fig. 2 shows the values of the coefficients when an expansion up to $n_{max} = 5, l_{max} = 5$ was performed, the $y - axis$ is the amplitude of the coefficient and the $x - axis$ shows the number of the coefficient in the following order:

Simulation	Density Profile	Mass ($1 \times 10^{10} M_{\odot}$)	Npart in LR sim	Npart in HR sim
MW disk	Exponential	5	5E5	5E6
MW bulge	Hernquist	1.4	1E5	1E6
MW DM halo	Hernquist	100 (M_{vir})	1E6	3E7
LMC	Hernquist	[3, 5, 8, 10, 25]	5E5	5E6

Table 1: Simulation main parameters. The disk and the bulge are live systems.

(n=1, l=1, m=1, m=2, l=2, m=1, ...) the terms with $m > l$ are included in the plot but they are zero. We found that the values of the coefficients depend highly on accurateley choosing the center of mass of the halo and on the resolution of the simulation, we believe the latter creates noise owing to the discrete representation of the halo.

1. **High order terms like those at $nlm = 100$ in Fig.2 appear in both the low resolution simulation and in the high resolution simulation. In the high resolution simulation these terms have an overall smaller amplitude. This makes me think that these terms might correspond to the noise due to the discrete representantion of the halo. Therefore to really know which coefficients are describing the shape of the halo we should use high resolution simulations and apply the smoothing and the principal component analysis described in your paper Weinberg 1996, do you have any suggestion regarding an optimal resolution for the simulation?**

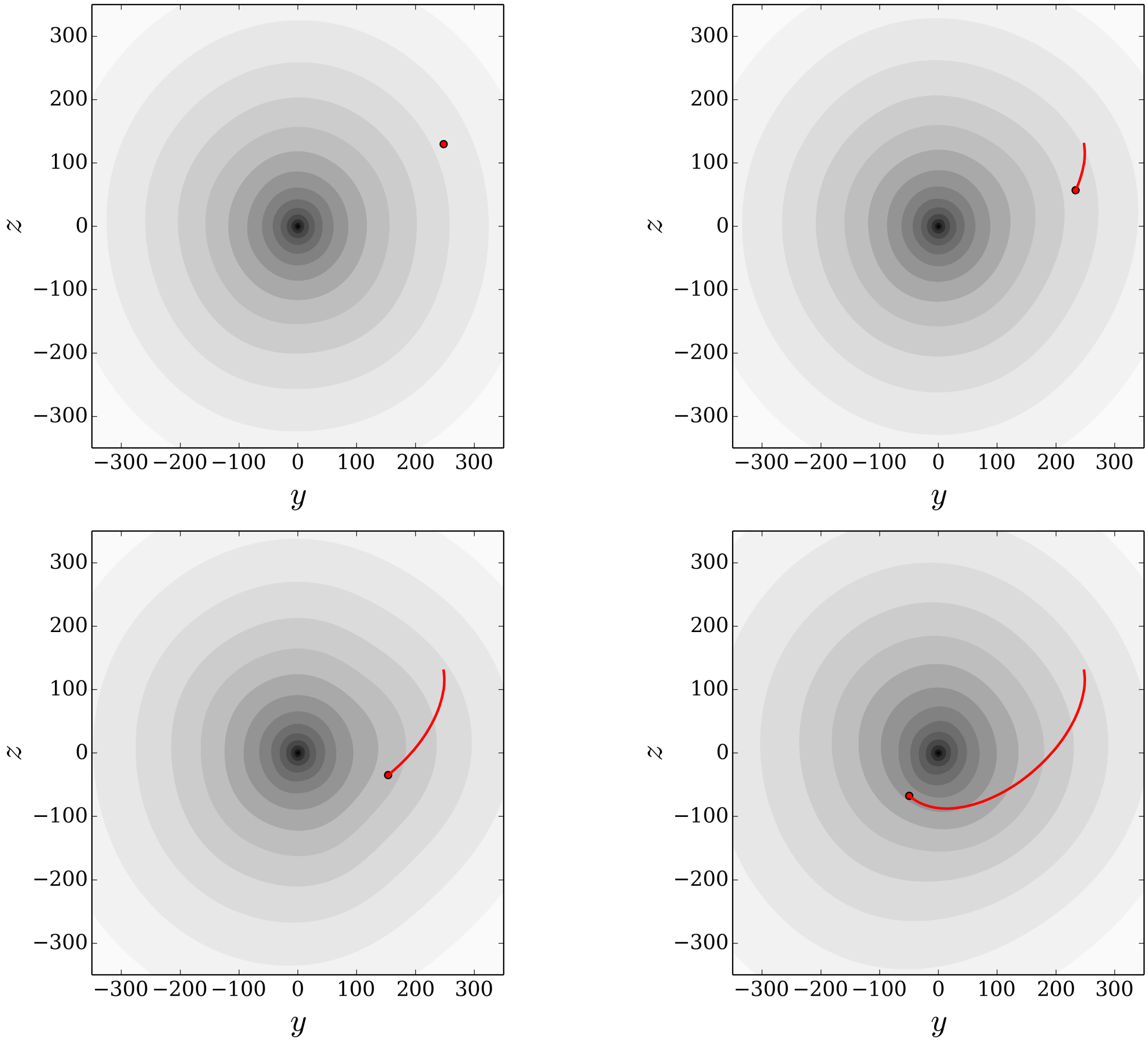

Figure 1: MW Dark Matter halo equipotential contours at three different times during the simulation $t = 0.0, 0.4, 0.8, 1.2$ Gyrs corresponding to the top left, top right, bottom left and bottom right panels respectively. Red line shows the orbit of LMC. The potential were reconstructed using the SCF method from the DM particles in the MW halo of the LR simulation. The coefficient used to compute the potential are shown in Fig.2.

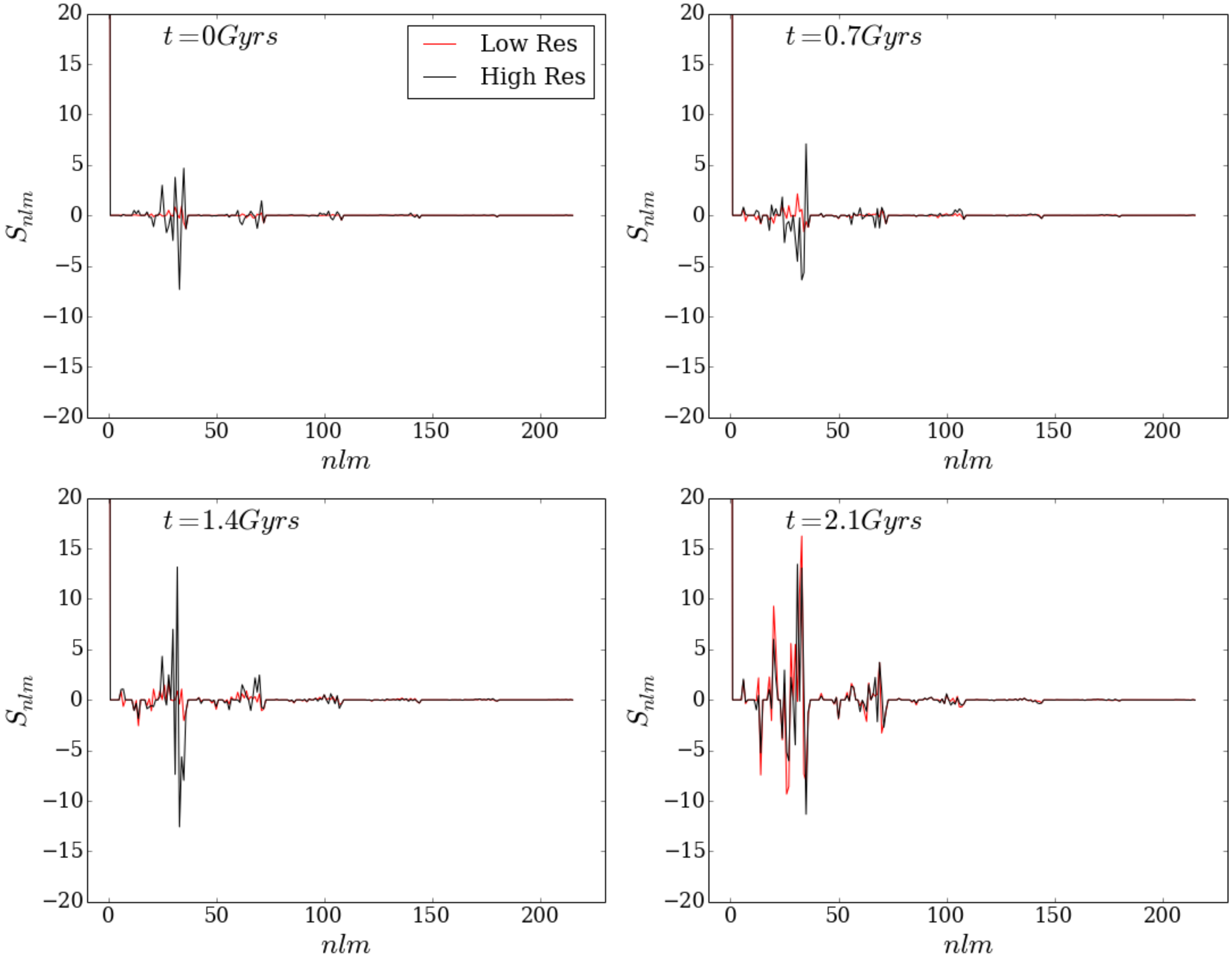

Figure 2: Coefficients S_{nlm} of the Dark Matter halo particles of the Milky Way for three different times. These coefficients were used to recover the potential shown in Fig.1. Black solid lines represent the low resolution simulation coefficients, while the red solid lines show the high resolution coefficients. The different panels are the coefficients computed at different times during the simulation. The T_{nlm} coefficients show a similar behave.

In order to get a deeper understanding of the response of the DM halo to the LMC passage we need to understand what are the 'true' multipoles that contribute to the shape of the DM halo. This is the main subject of the next section.

3 Variance computation and smoothing:

This section follows the computations presented in Weinberg (1996) Appendix 2. With the notation of Lowing (2011) (This is for my own sanity check in order to be sure that I am computing the right thing! There is nothing new here that is not in Weinberg's appendix). Bellow I follow the steps of how to compute the variance of the coefficients S_{nlm} in the basis expansion **due to the particles x_i that contribute to the coefficient (This is not the variance of the coefficients with respect to the other coefficients).**

$$\rho(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{n=0}^{\infty} Y_{l,m}(\theta)\rho_{nl}(r)(\hat{S}_{nlm}cos\ m\phi + \hat{T}_{nlm}sin\ m\phi) \tag{1}$$

Where the coefficients $\hat{S}_{nlm}$ and $\hat{T}_{nlm}$ are defined as:

$$\hat{S}_{nlm} = (1-\delta_{m0})\tilde{A_{nl}}\sum_k m_k\phi_{nl}(r_k)Y_{lm}(\theta_k)cos\ m\phi_k \tag{2}$$

$$\hat{T}_{nlm} = (1-\delta_{m0})\tilde{A_{nl}}\sum_k m_k\phi_{nl}(r_k)Y_{lm}(\theta_k)sin\ m\phi_k \tag{3}$$

The variance of the coefficients is going to be:

$$var(\hat{S}_{nlm}cos\ m\phi+\hat{T}_{nlm}sin\ m\phi) = cos\ m\phi\ var(\hat{S}_{nlm})+sin\ m\phi\ var(\hat{T}_{nlm})+2cos\ m\phi\ sin\ m\phi\ cov(\hat{S}_{nlm},\hat{T}_{nlm}) \tag{4}$$

Where the variances $var(\hat{S}_{nlm})$ and $var(\hat{T}_{nlm})$ can be computed as follows:

$$var(\hat{S}_{nlm}) = E[(\hat{S}_{nlm}-E[S_{nlm}])^2] = E\left[\sum_{i,j}^{N} m_i m_j \Psi_{nlm}(x_i)\Psi_{nlm}(x_j)\right] - E\left[\sum_i^N m_i\Psi_{nlm}(x_i)\right] E\left[\sum_i^N m_i\Psi_{nlm}(x_i)\right] \tag{5}$$

Where Ψ_{nlm} for the coefficients $\hat{S}_{nlm}$is defined as:

$$\Psi_{nlm}(x_i) = (2-\delta_{m,0})\tilde{A}_{nl}\Phi_{nl}(r_i)Y_{lm}(\theta_i)cos(m\phi_i) \tag{6}$$

And for $\hat{T}_{nlm}$ we defined $\Psi'_{nlm}(x_i)$ as:

$$\Psi'_{nlm}(x_i) = (2-\delta_{m,0})\tilde{A}_{nl}\Phi_{nl}(r_i)Y_{lm}(\theta_i)sin(m\phi_i) \tag{7}$$

Where the first term in the last equality can be expressed in terms of the diagonal and off-diagonal terms as:

$$E\left[\sum_{i,j}^{N} m_i m_j \Psi(x_i)\Psi(x_j)\right] = E\left[\sum_i^N m_i^2\Psi_{nlm}(x_i)\Psi_{n'l'm'}(x_i) + \sum_{i\neq j}^{N} m_i m_j\Psi_{nlm}(x_i)\Psi_{n'l'm'}(x_j)\right] \tag{8}$$

The second term in the right hand (the terms outside the diagonal of the matrix $\Psi_{nlm}\Psi_{n'l'm'}$) can be expressed as:

$$\sum_{i\neq j}^{N} m_i m_j\Psi(x_i)\Psi(x_j) = \sum_i^N m_i\Psi(x_i)\sum_j^N m_j\Psi(x_j) - \frac{1}{N}\sum_i^N m_i\Psi(x_i)\sum_j^N m_j\Psi(x_j) \tag{9}$$

2. This step is not entirely clear to me, Why are the terms in the diagonal equal to the average of all the terms (second term in the right hand of the equation above)?

$$E\left[\sum_{i,j}^{N} m_i m_j\Psi(x_i)\Psi(x_j)\right] = E\left[\sum_i^N m_i^2\Psi_{nlm}(x_i)\Psi_{n'l'm'}(x_i) + \frac{(N-1)}{N}\sum_i^N m_i\Psi(x_i)\sum_j^N m_j\Psi(x_j)\right] \tag{10}$$

Taking into account that $E[a+b] = E[a]+E[b]$:

$$E\left[\sum_{i,j}^{N} m_i m_j \Psi(x_i)\Psi(x_j)\right] = E\left[\sum_{i}^{N} m_i^2 \Psi_{nlm}(x_i)\Psi_{n'l'm'}(x_i)\right] + \frac{(N-1)}{N} E\left[\sum_{i}^{N} m_i \Psi(x_i) \sum_{j}^{N} m_j \Psi(x_j)\right] \quad (11)$$

Combining Eq.5 and Eq.11

$$var(S_{nlm}) = E\left[\sum_{i}^{N} m_i^2 \Psi_{nlm}(x_i)^2\right] - \frac{1}{N} E\left[\sum_{i}^{N} m_i \Psi(x_i) \sum_{j}^{N} m_j \Psi(x_j)\right] \quad (12)$$

Connecting with Weinberg (1996) Eq.14, the first term in the right hand side in **??** is equivalent to:

$$E\left[\sum_{i}^{N} m_i^2 \Psi_{nlm}(x_i)^2\right] = \frac{1}{N} E[\Psi(x_i)] \quad (13)$$

And the second term in the right hand side is:

$$\frac{1}{N} E\left[\sum_{i}^{N} m_i \Psi(x_i) \sum_{j}^{N} m_j \Psi(x_j)\right] = \frac{1}{N} E\left[\Psi(x_i)\right] E\left[\Psi(x_j)\right] \quad (14)$$

Which is consistent with the notation Weinberg (1996) Eq.14:

$$var(\hat{a}_j) = \frac{1}{N}\{E[\Psi_j \Psi_j] - E[\Psi_j]E[\Psi_j]\} \quad (15)$$

On the other hand the covariance of the coefficients $cov(S_{nlm}, T_{nlm})$ is defined as:

$$cov(S_{nlm}, T_{nlm}) = E[S_{nlm}T_{nlm}] - E[S_{nlm}]E[T_{nlm}] \quad (16)$$

$$cov(S_{nlm}, T_{nlm}) = E\left[\sum_{i,j}^{N} m_i m_j \Psi_{nlm}(x_i)\Psi'_{nlm}(x_j)\right] - E\left[\sum_{i} m_i \Psi(x_i)\right] E\left[\sum_{j} m_j \Psi'(x_j)\right] \quad (17)$$

Following the same procedure as for $var(\hat{S}_{nlm})$ I found that:

$$cov(S_{nlm}, T_{nlm}) = E\left[\sum_{i}^{N} m_i^2 \Psi_{nlm} \Psi'_{n'l'm'}\right] - \frac{1}{N} E\left[\sum_{i}^{N} m_i \Psi_{nlm}(x_i) \sum_{j}^{N} m_j \Psi'_{nlm}(x_j)\right] \quad (18)$$

Using the results of Eq.12 and Eq.18 we can express the full variance of the coefficients Eq.4 as:

$$\begin{aligned} var(\hat{S}_{nlm} cos\ m\phi + \hat{T}_{nlm} sin\ m\phi) = cos\ m\phi & \left(E\left[\sum_{i}^{N} m_i^2 \Psi_{nlm}(x_i)^2\right] - \frac{1}{N} E\left[\sum_{i}^{N} m_i \Psi(x_i) \sum_{j}^{N} m_j \Psi(x_j)\right]\right) \\ + sin\ m\phi & \left(E\left[\sum_{i}^{N} m_i^2 \Psi'_{nlm}(x_i)^2\right] - \frac{1}{N} E\left[\sum_{i}^{N} m_i \Psi'(x_i) \sum_{j}^{N} m_j \Psi'(x_j)\right]\right) \\ + 2 cos\ m\phi\ sin\ m\phi & \left(E\left[\sum_{i}^{N} m_i^2 \Psi_{nlm} \Psi'_{n'l'm'}\right] - \frac{1}{N} E\left[\sum_{i}^{N} m_i \Psi_{nlm}(x_i) \sum_{j}^{N} m_j \Psi'_{nlm}(x_j)\right]\right) \end{aligned} \quad (19)$$

Finally the smoothing coefficient b_{nlm} would be:

$$b_{nlm} = \frac{1}{1 + \frac{var(\hat{S}_{nlm}cos\ m\phi + \hat{T}_{nlm}sin\ m\phi)}{(\hat{S}_{nlm}cos\ m\phi + \hat{T}_{nlm}sin\ m\phi)^2}} \tag{20}$$

For the case of $m = 0$

$$b_{nl0} = \frac{1}{1 + \frac{var(\hat{S}_{nl0})}{\hat{S}^2_{nl0}}} \tag{21}$$

Question 3. Do you agree with this derivation, in particular with Eqns. 18 & 19?

4 Principal Component Analysis

As pointed out first in Weinberg (1996) some coefficients might be suppressed with the smoothing method but they might still contribute to a global signal, therefore the having the coefficients would alleviate this over smoothing. We follow the procedure shown in the Appendix 1 of Weinberg (1996), where the outer-product matrix in Lowing (2011) notation is going to be:

$$\tilde{S}_{nlmn'l'm'} = (\hat{S}_{nlm}cos(m\phi) + \hat{T}_{nlm}sin(m\phi)) \cdot (\hat{S}_{n'l'm'}cos(m'\phi) + \hat{T}_{n'l'm'}sin(m'\phi)) \tag{22}$$

For the case of $l = m = 0$ and $l = m' = 0$ this reduces to:

$$\tilde{S}_{n00n'00} = \hat{S}_{n00} \cdot \hat{S_{n00}} \tag{23}$$

.

The eigenvectors of $\tilde{S}_{nlmn'l'm'}$ form the transformation $\tilde{T}$ matrix that converts the coefficients in the principal basi, this is:

$$S^*_{nlm}cos(m\phi) + T^*_{nlm}sin(m\phi) = \sum \tilde{T} \cdot (S^*_{nlm}cos(m\phi) + T^*_{nlm}sin(m\phi)) \tag{24}$$

The rest of the procedure is analogous as explained in Weinberg (1996).

Question 4: The outer-product matrix is computed for every harmonic and its eigenvectors form the tranformation matrix $\tilde{T}$ which is used to find the principal component basis. With these I will get the coefficients in the principal basis for every harmonic, and then I can compute the contribution to the potential of every harmonic. Is this correct?