Stat 212b: Topics in Deep Learning
Lecture 2

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Objectives

- Classification, Kernels and metrics
- Representations for recognition
  - curse of dimensionality
  - invariance/covariance
  - discriminability
- Variability models
  - transformation groups and symmetries
  - deformations
  - stationarity
  - clutter and class-specific
- Examples

High-dimensional Recognition Setup

- Input data x lives in a high-dimensional space:

$$x \in \Omega,\ \Omega \subset \mathbb{R}^d \qquad \text{finite-dimensional (but large } d\text{!)}$$

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- We observe $(x_i, y_i)\ ,\ i = 1 \ldots n\ ,$ where

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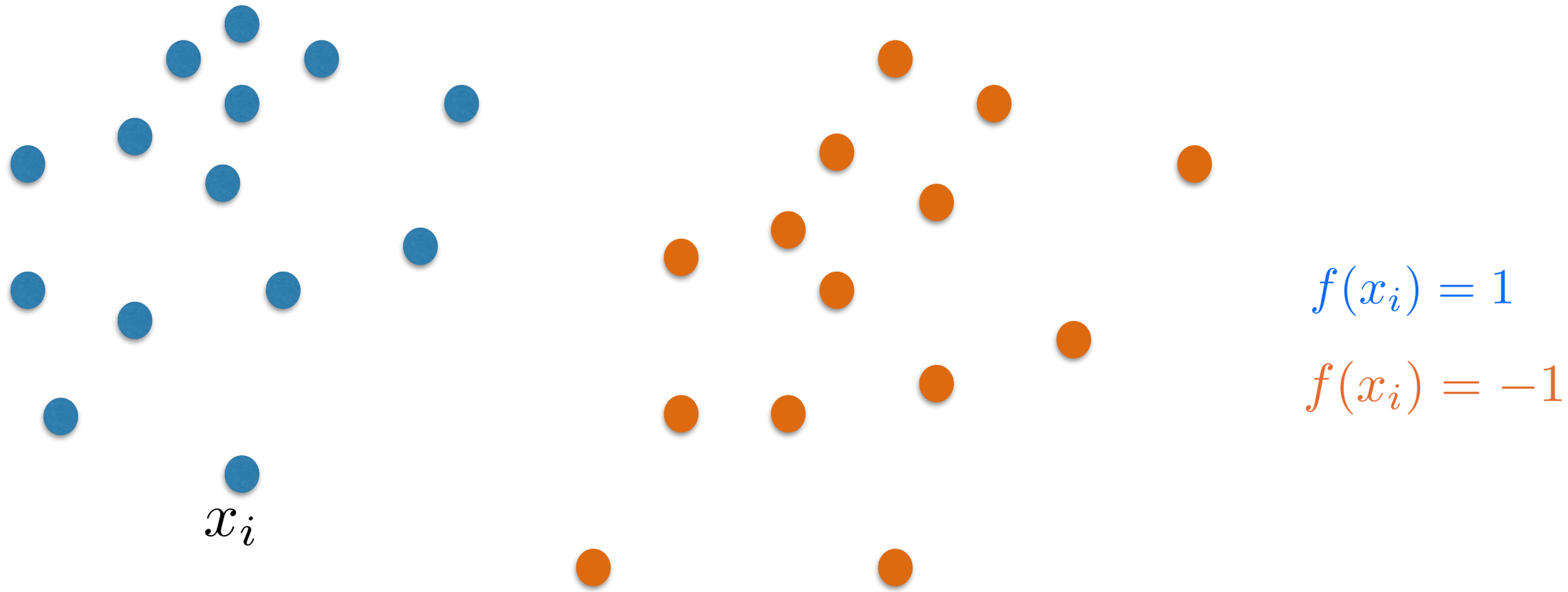
$$y_i \in \{1, K\}\ . \quad \text{(classification)}$$

- We can reduce the former to "interpolating" a function

$$f\ :\ \Omega \to \mathbb{R}^K \quad (f(x) = p(y \mid x) \text{ in the classification case})$$

How to "interpolate" in high-dimensions?

- Let's start with a (very) simple low-dimensional setting:



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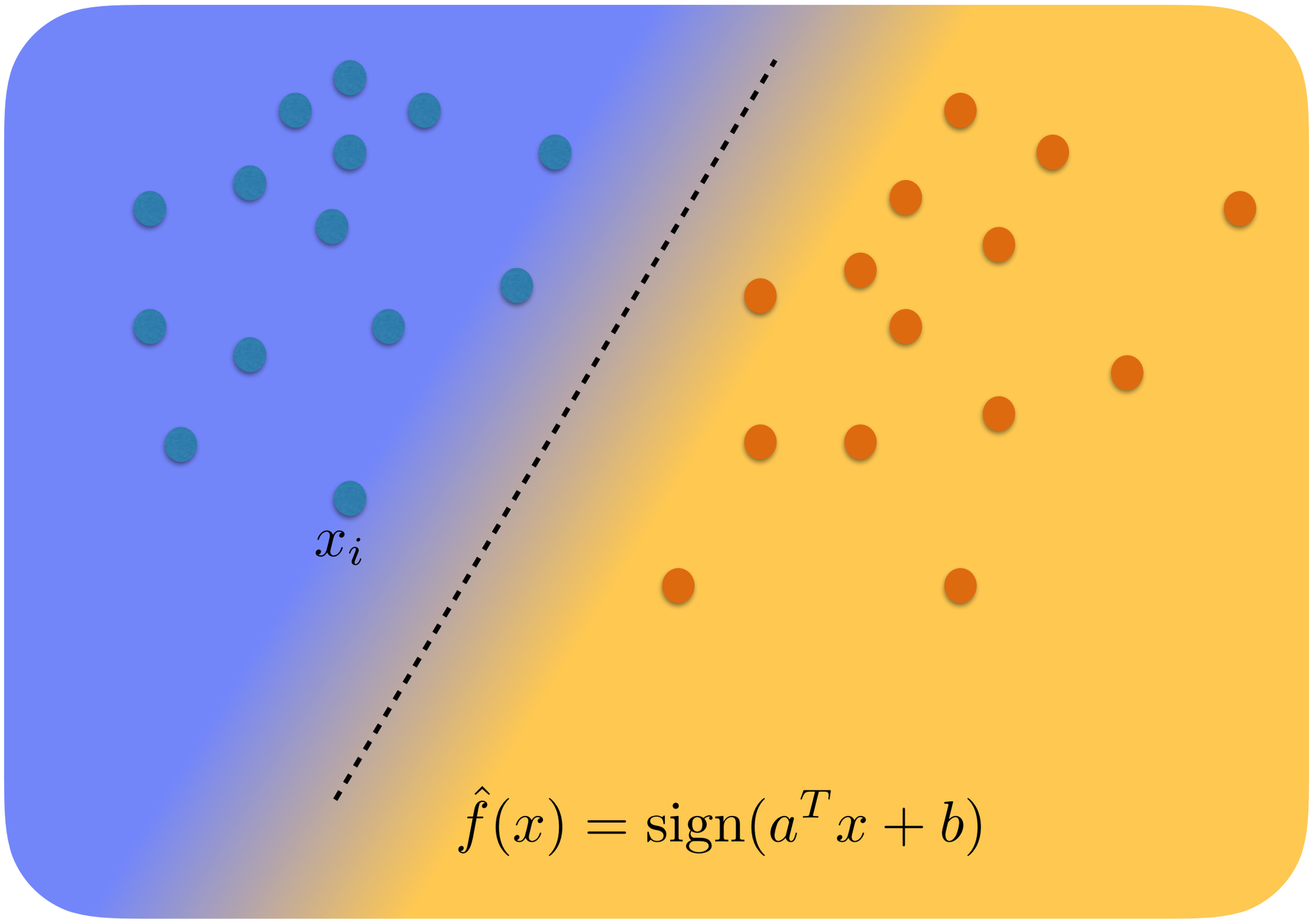

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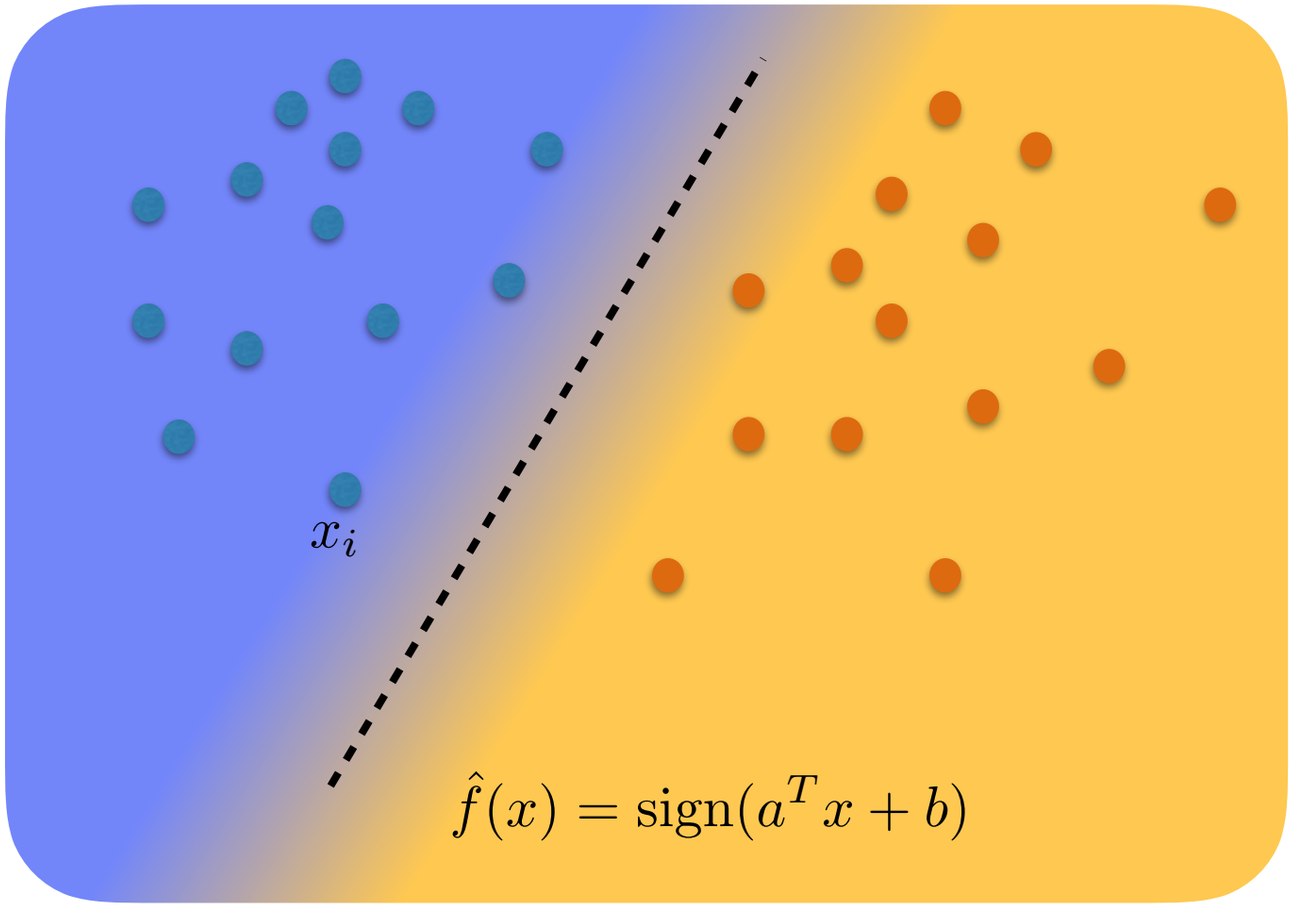
How to interpolate in high dimensions?



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- We have found (linear) features $\Phi(x) = a^T x$ such that

$$|f(x) - f(x')| \leq C\|\Phi(x) - \Phi(x')\|$$

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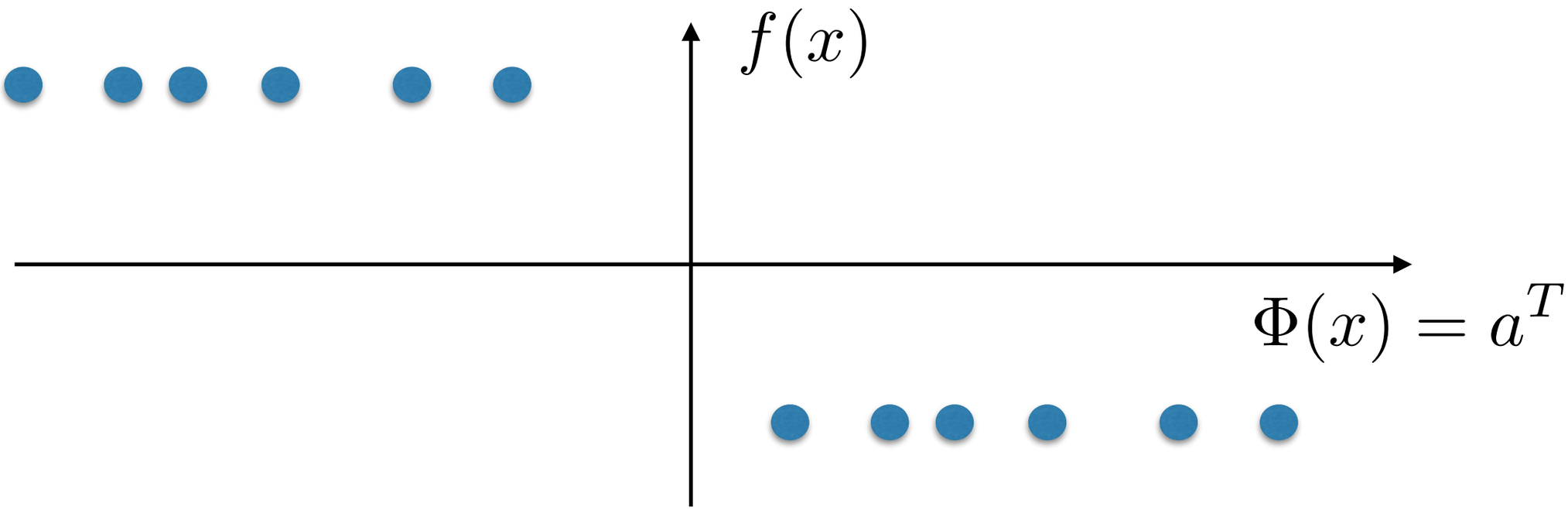

Support Vector Machines

- The previous example is formalized by Support Vector Machines [Vapnik et al, '90s]: given a binary classification problem with data (x_i, y_i), we consider an estimator for f(x) of the form

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- Empirical Risk Minimization:

$$\min_{a,b} \ \frac{1}{n}\sum_{i=1}^{n} \ell(y_i, \hat{f}(x_i)) + \lambda \|a\|^2 \ ,$$

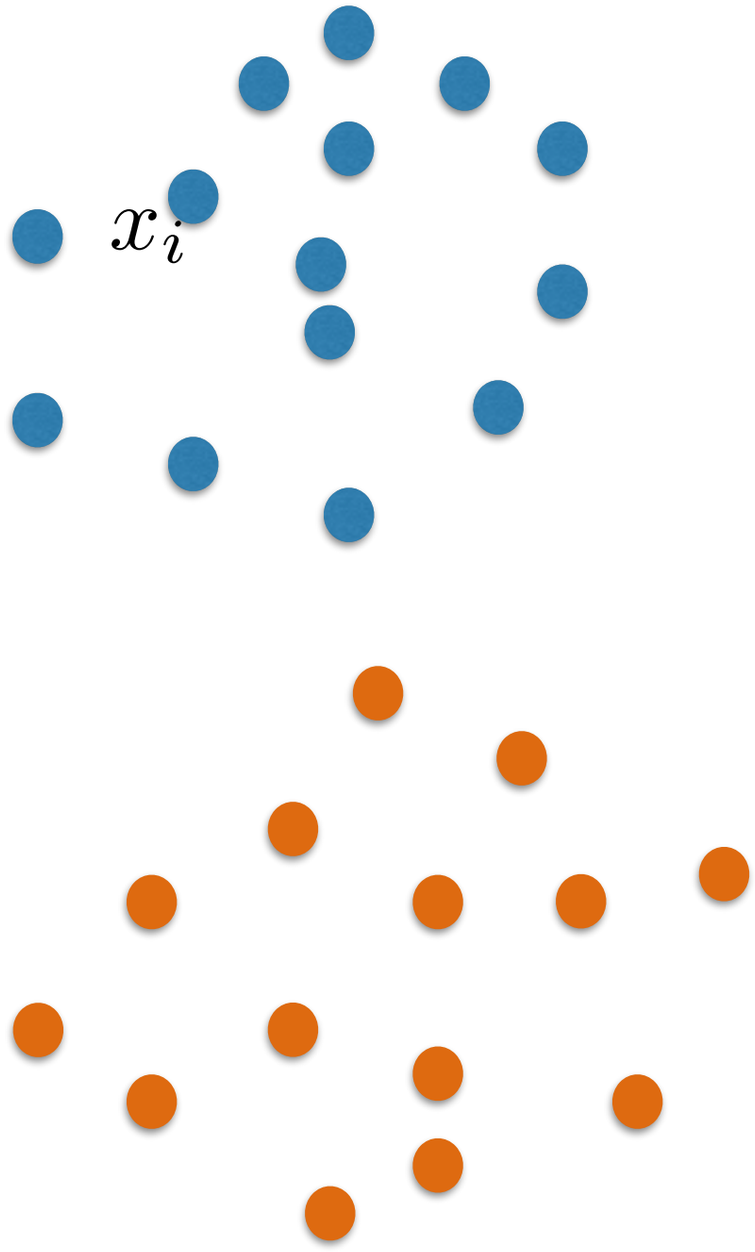
enforces training examples to fall in the right side of the hyperplane

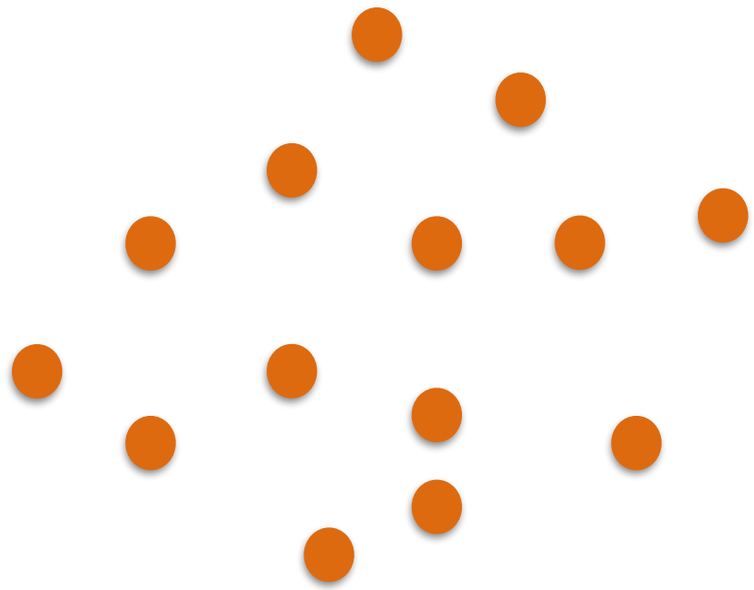
enforces large margin

$$\ell(y, \hat{y}) = \max(0, 1 - y \cdot \hat{y}) \quad : \text{hinge loss} \ .$$

SVMs and Kernels

- Not all problems are linearly separable:





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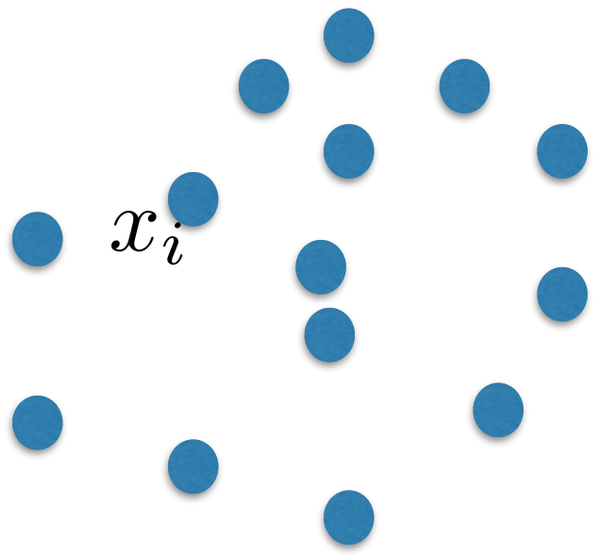

the "XOR" problem

- By using the Lagrangian dual of the previous program, we can rewrite our previous solution as

$$\hat{f}(x) = \text{sign}\left(\sum_i \alpha_i y_i K(x_i, x)\right) ,$$

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- We can replace the linear kernel by a non-linear one, eg
  - polynomial: $K(x, y) = \langle x, y \rangle^d$.
  - Gaussian radial basis function: $K(x, y) = \exp(-\|x - y\|^2/\sigma^2)$.

The Kernel "trick"

- For a wide class of psd kernels (Mercer Kernels), we have a representation in terms of an inner product:

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- But is that all?

Curse of Dimensionality

- It is easy to create discriminative features:
  - Using a Gaussian RBF, it suffices to let $\sigma^2 \to 0$.
  - The estimator converges to the *nearest neighbor* classifier:

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- Underlying question: how to compare signals in high-dim?

Curse of Dimensionality

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$$\text{for each } x \in \Omega, \text{ exists constants } c, C \text{ such that}$$
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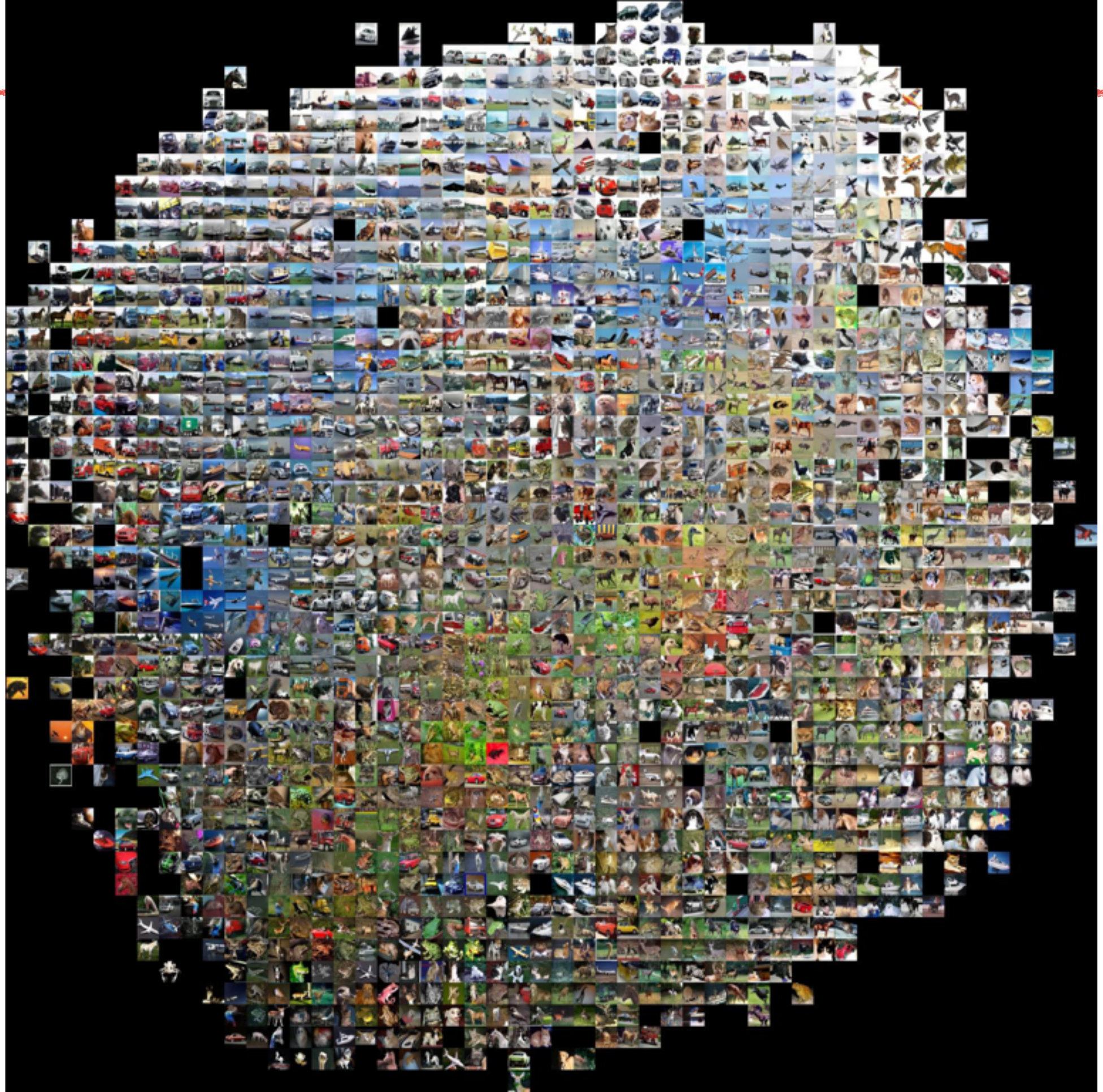
- So, we need a guiding principle that plays well with our data (images, sounds, etc.)



2-dimensional embedding of CIFAR-10 using Euclidean similarity

from A. Karpathy

Linearization

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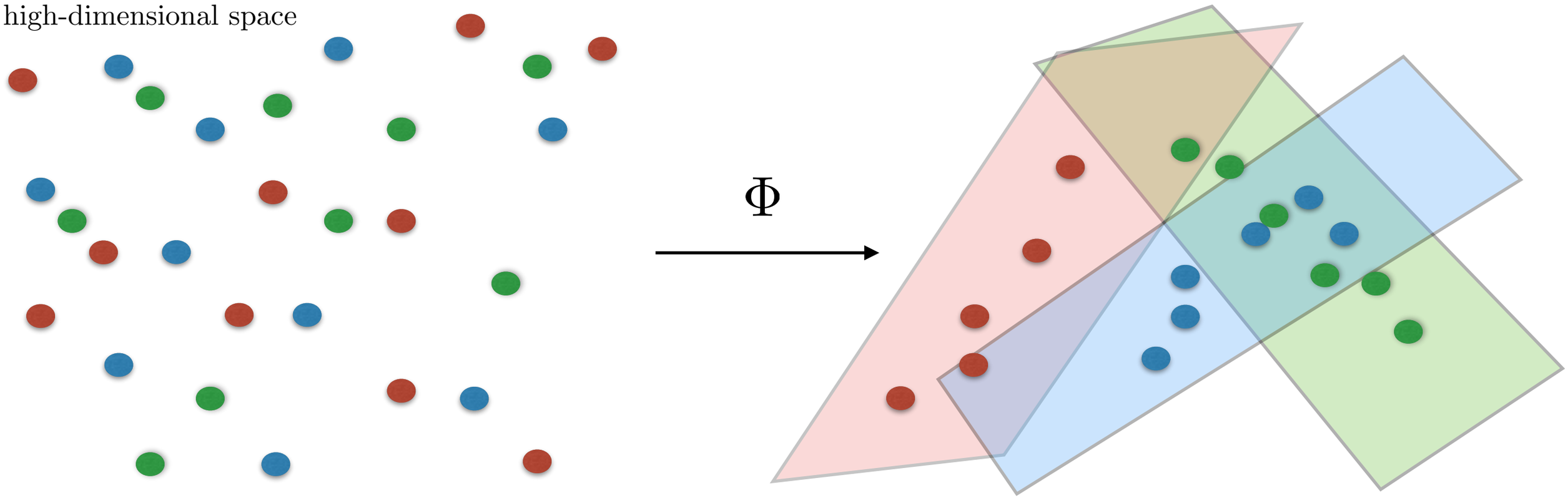
Thus hyperplanes orthogonal to ***a*** *in the feature space should contain only examples of one class.*

Linearization



- In order to beat the curse of dimensionality, we need to capture *invariance* while preserving *discriminative information*.

Invariance and Symmetry

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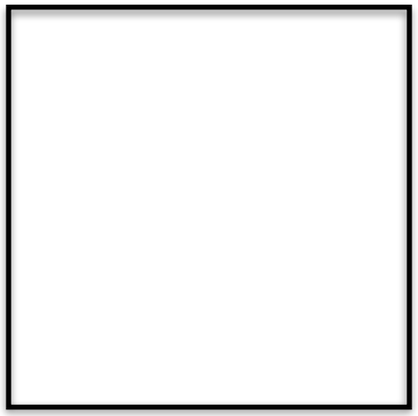
- What are those symmetries? How to impose them on Φ without breaking discriminability?

Discrete symmetries

- Which transformations leave this square unchanged?



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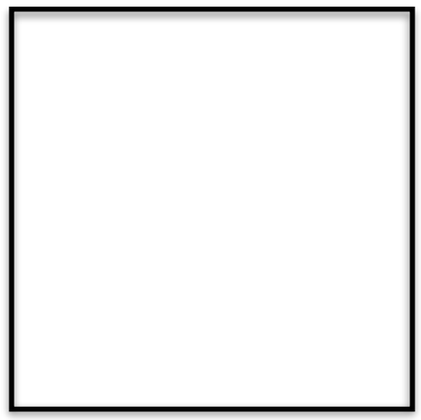

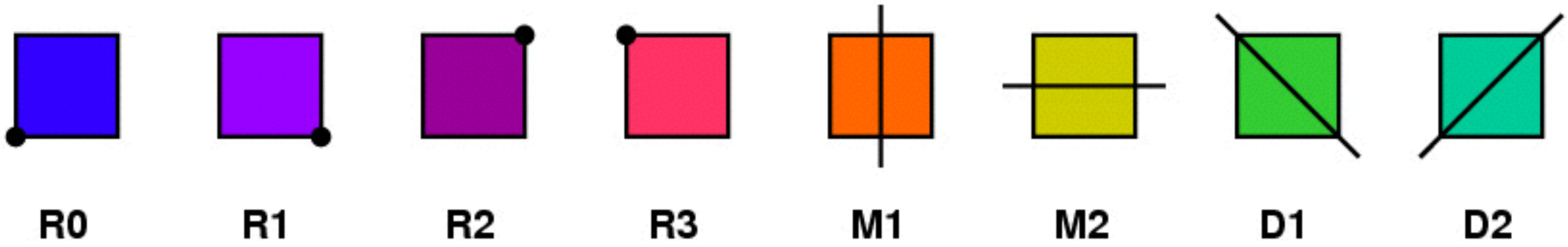

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(from http://www.cs.umb.edu/~eb/)

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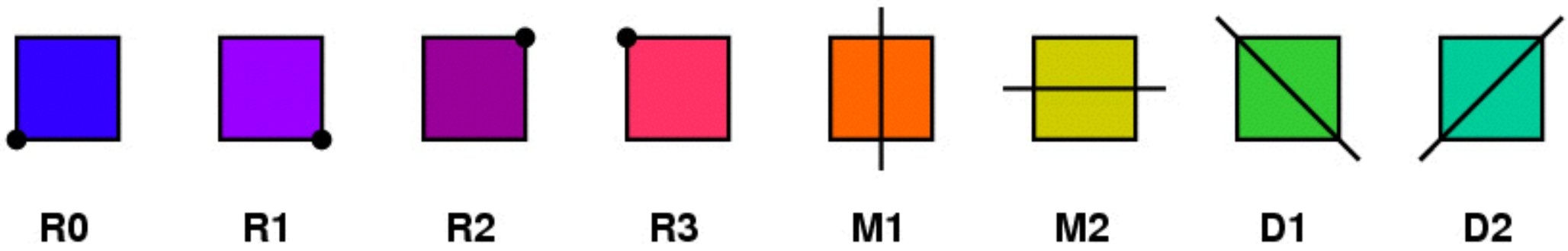

- The set of all symmetries forms a *group* G :

- group operation: $\forall\ g_1, g_2 \in G, \quad g_1 \cdot g_2 \in G\ .$

- identity element: $\exists e \in G\ s.t.\ g \cdot e = e \cdot g = g\ \ \forall\, g \in G\ .$

- inverse: $\forall\ g \in G\ \exists\, g^{-1} \in G\ s.t.\ g \cdot g^{-1} = e\ .$

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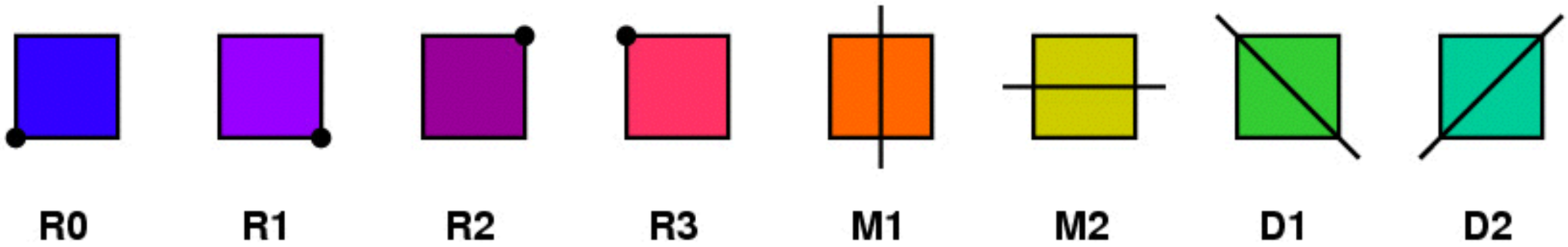
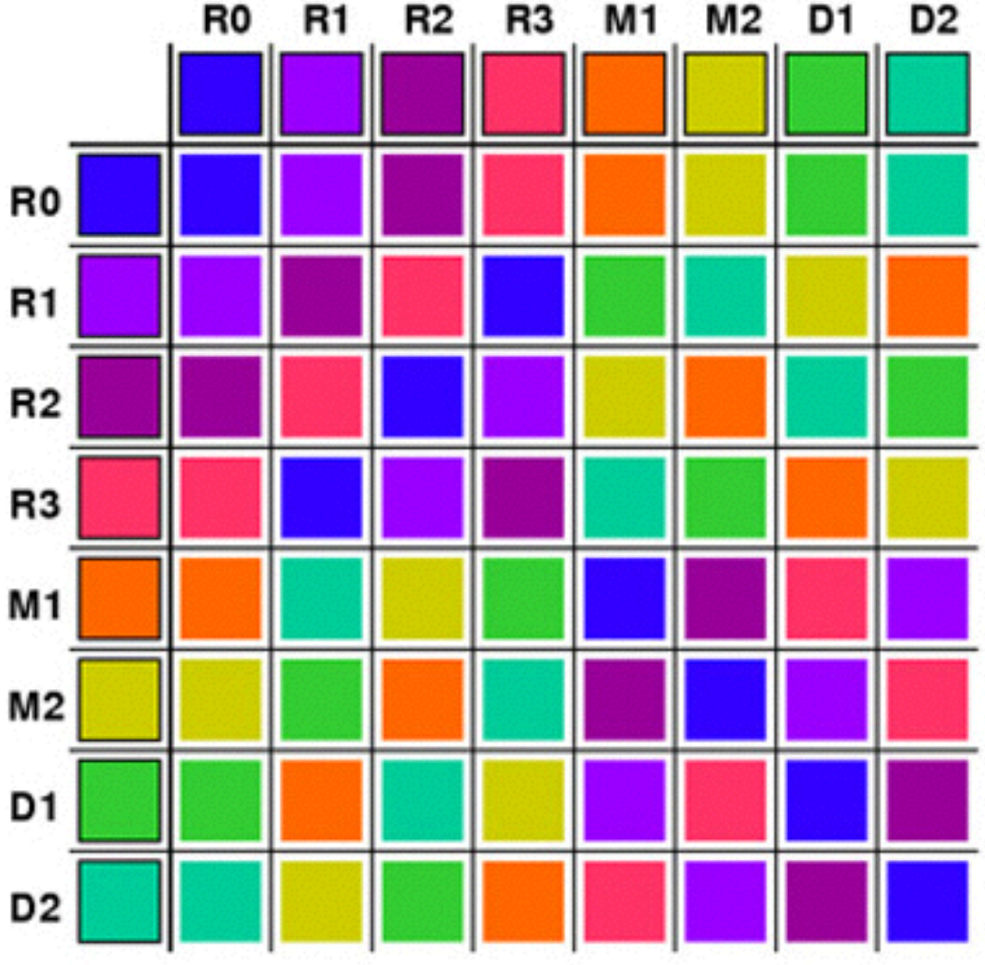


- Discrete groups are completely characterized by their multiplication table:



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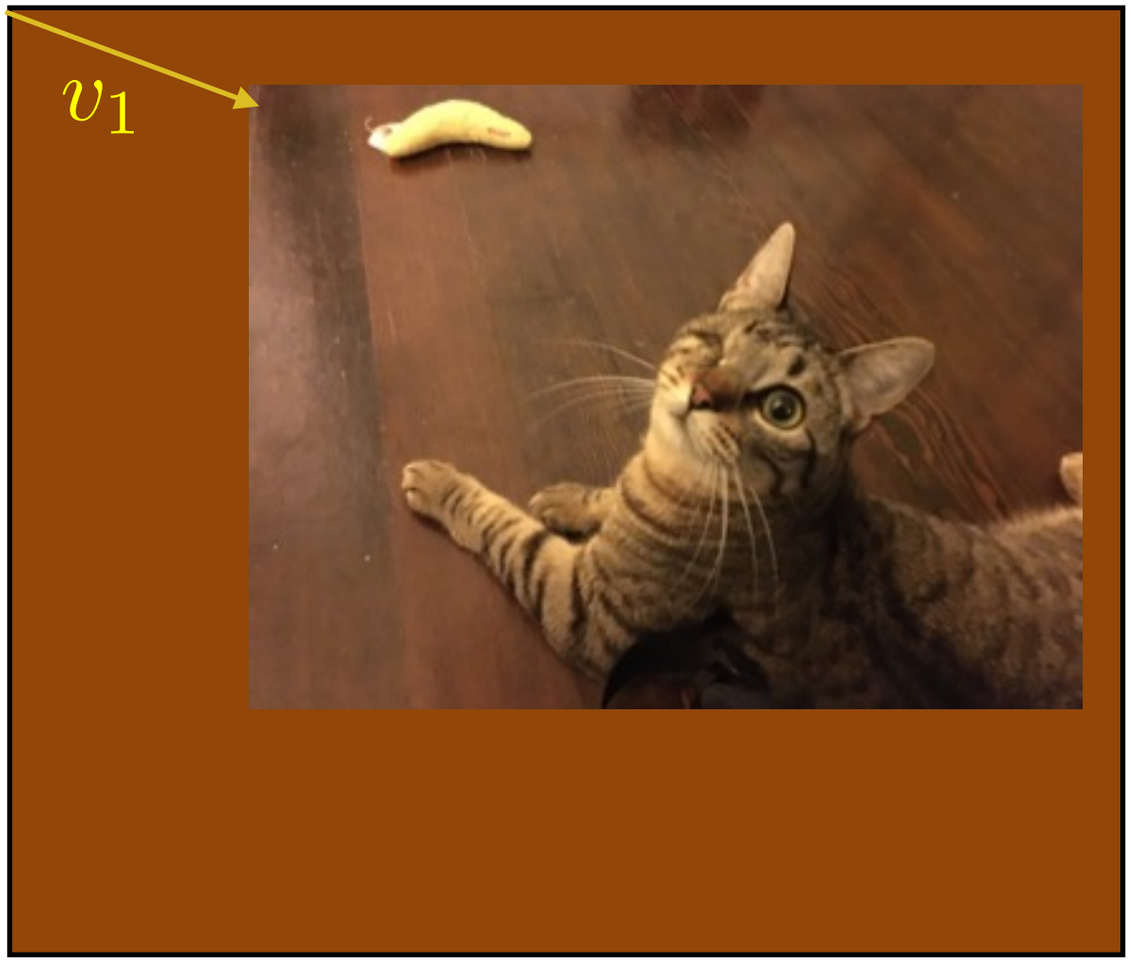
Rigid transformation symmetries

- Which symmetries are we likely to find in image recognition problems?



Translations: $\{\varphi_v\,; v \in \mathbb{R}^2\}$, with $\varphi_v(x)(u) = x(u - v)$.

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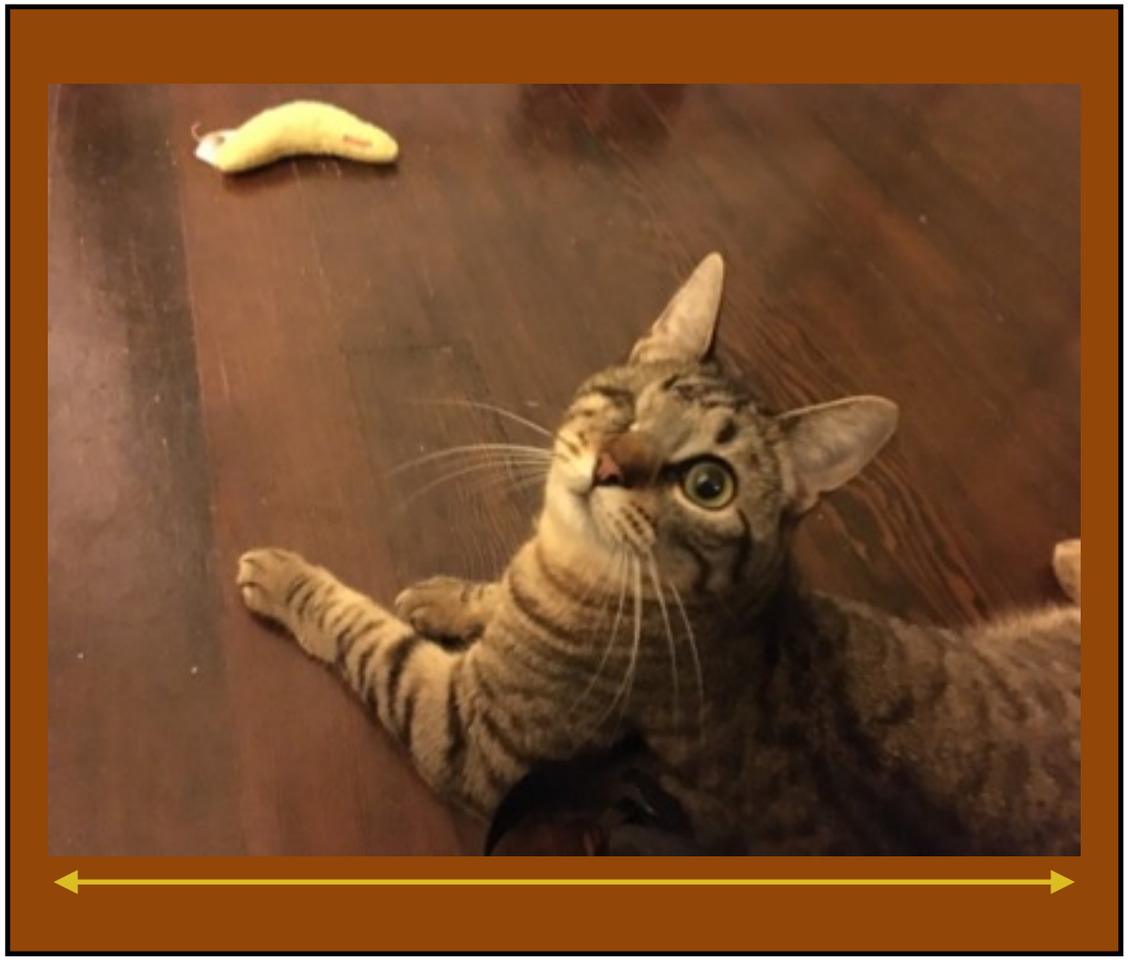

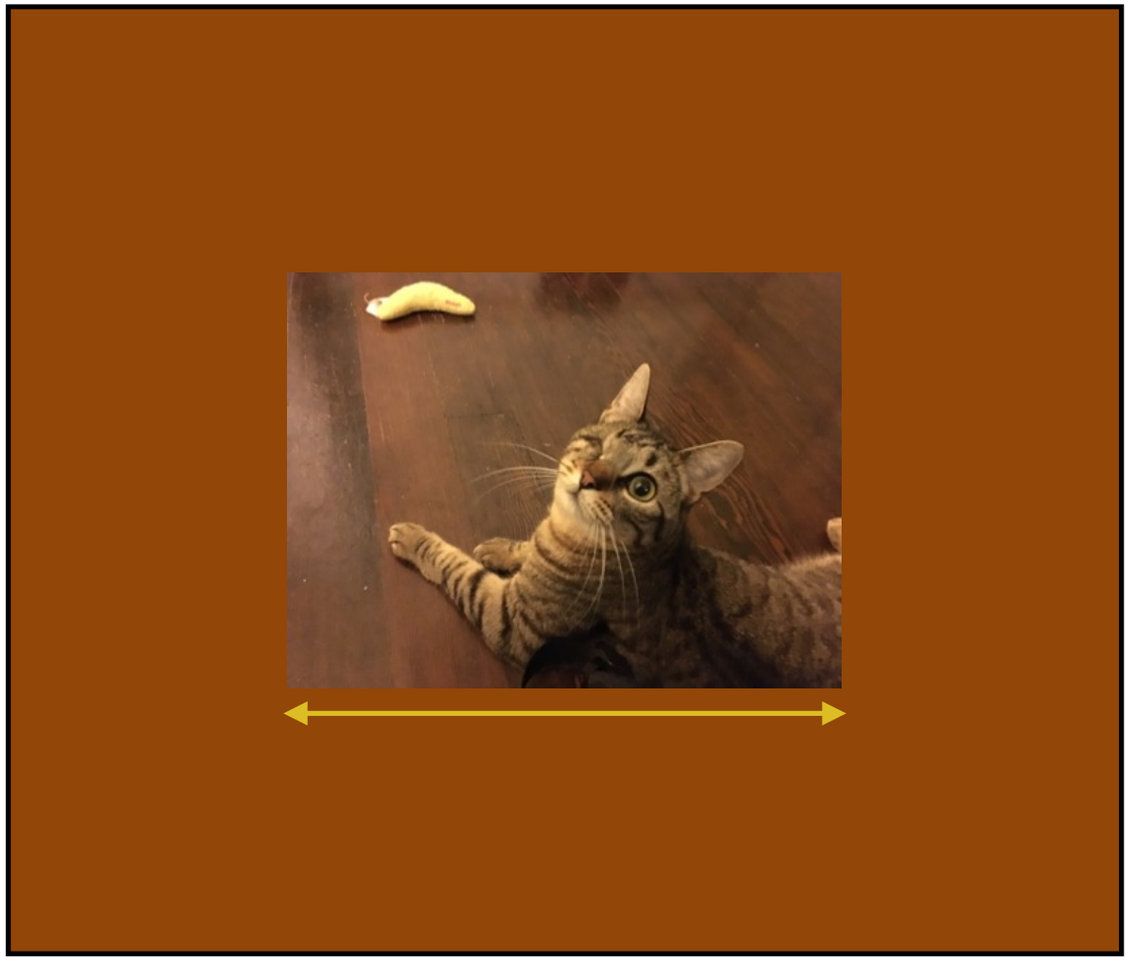

Dilations: $\{\varphi_s\,;s\in\mathbb{R}_+\}$, with $\varphi_s(x)(u)=s^{-1}x(s^{-1}u)$.

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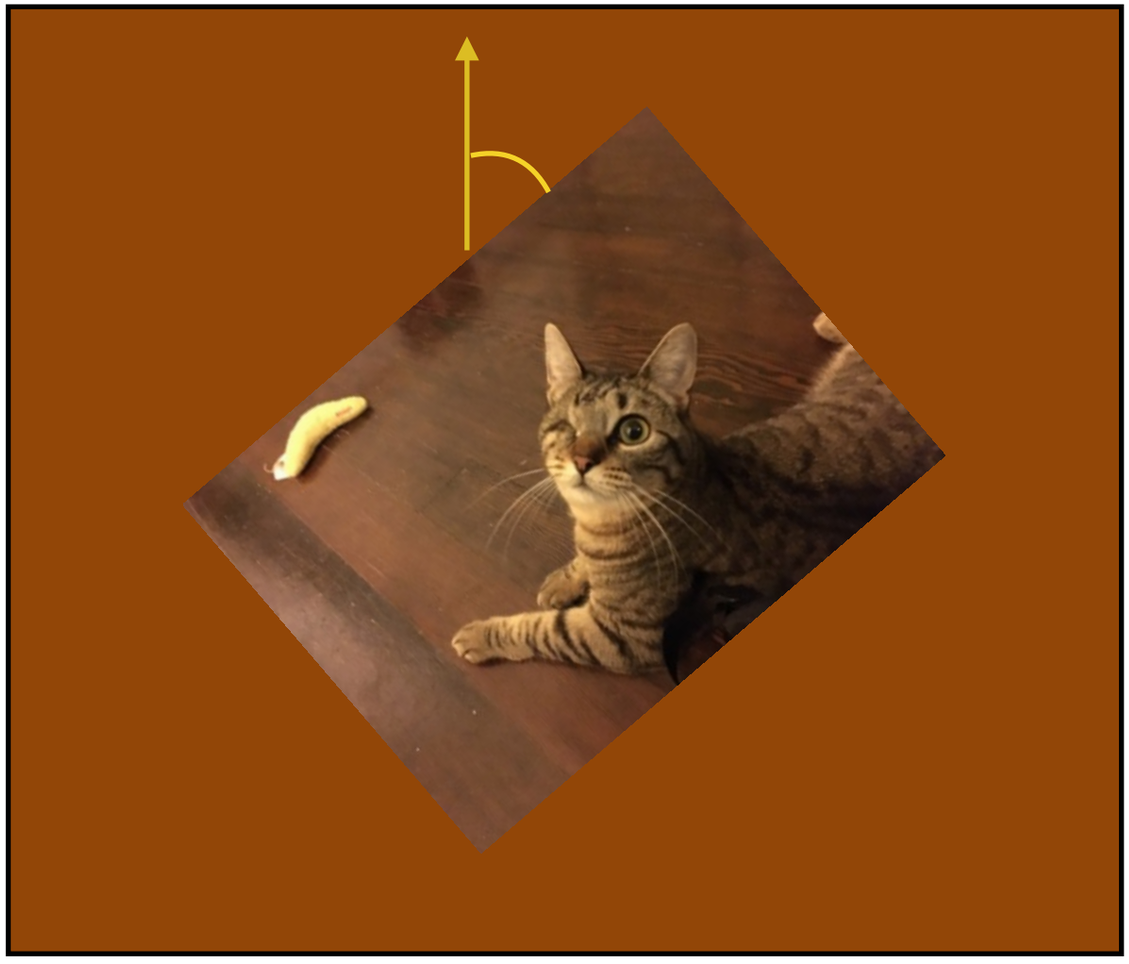

Rotations: $\{\varphi_\theta\,; \theta \in [0, 2\pi)\}$, with $\varphi_\theta(x)(u) = x(R_\theta u)$.

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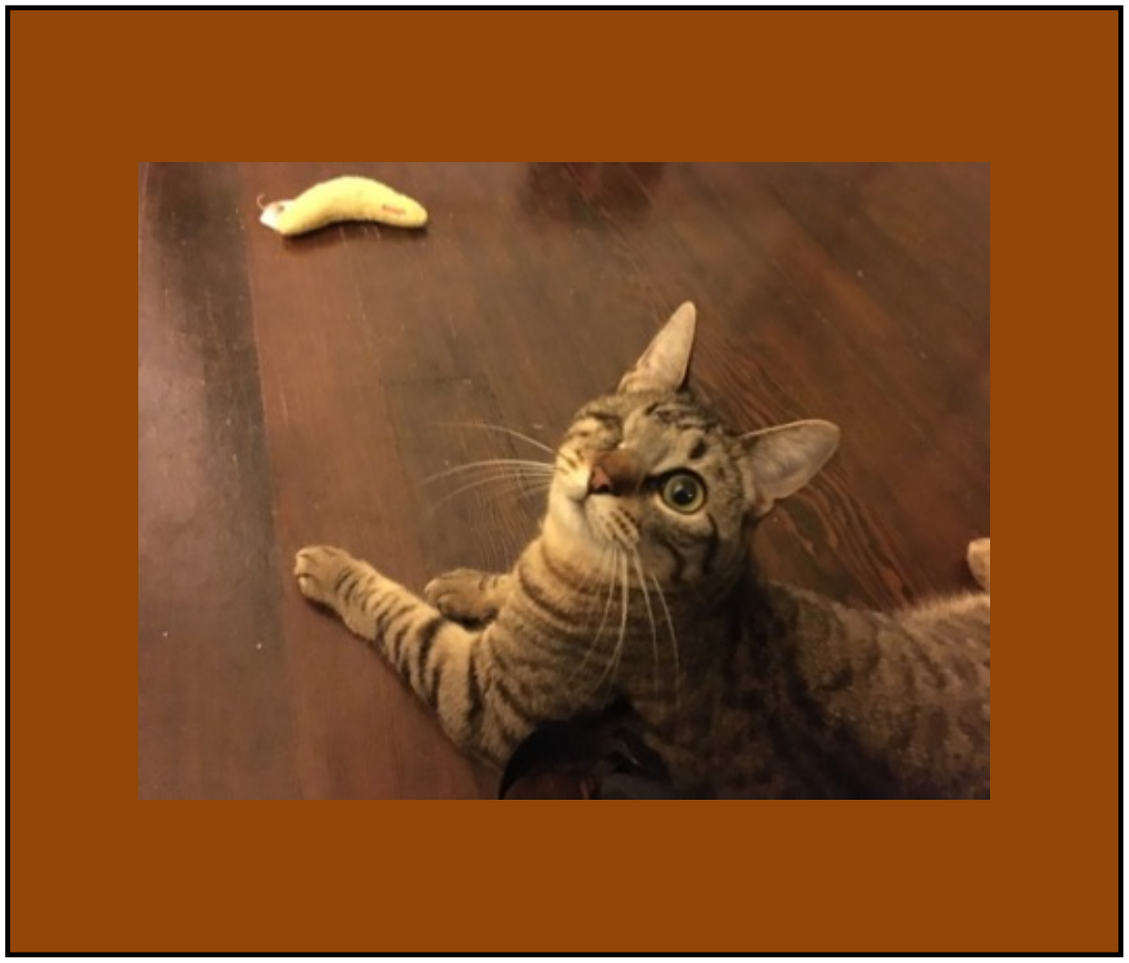

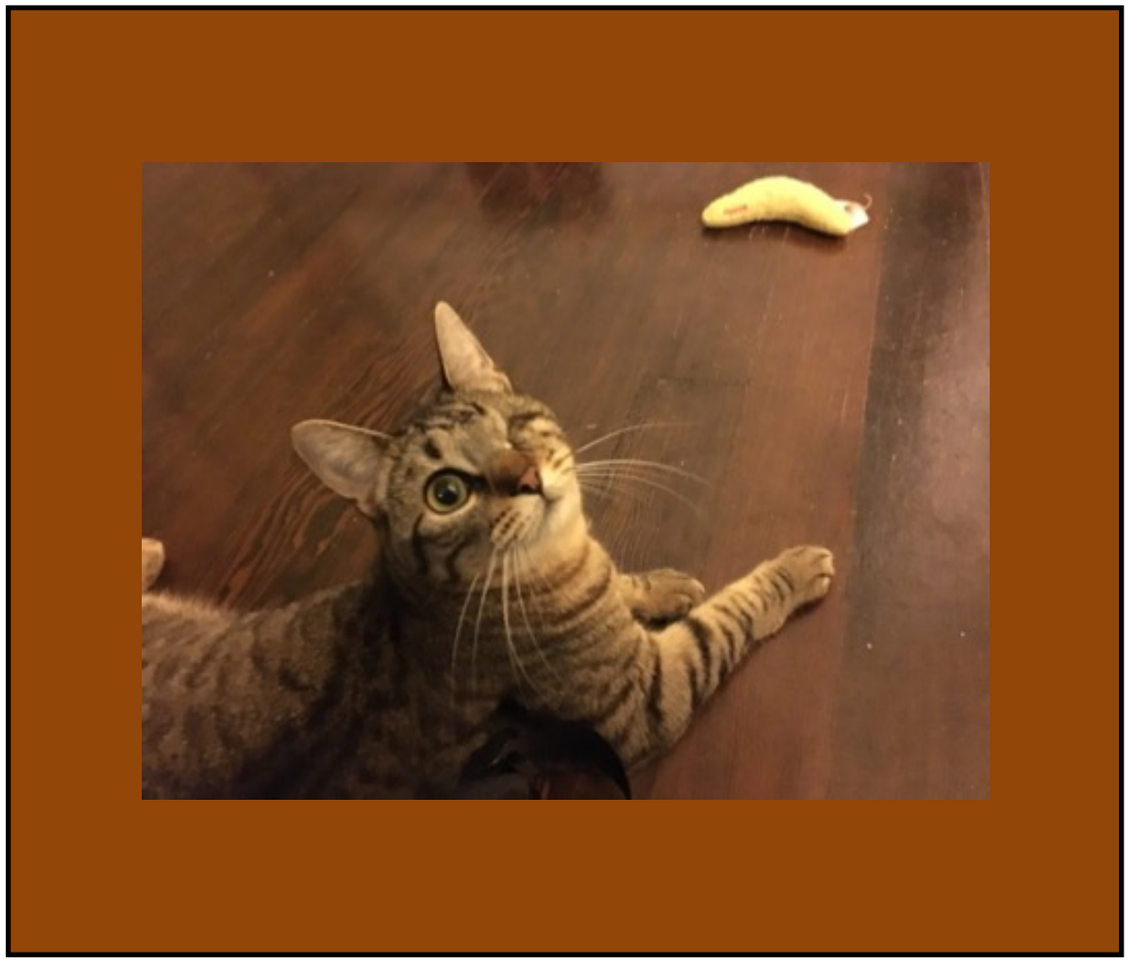

Mirror symmetry: $\{e\,, M\}$, with $Mx(u_1, u_2) = x(-u_1, u_2)$.

Rigid transformation symmetries

- We can combine all these transformations into a single group, the Affine Group $\mathrm{Aff}(\mathbb{R}^2)$.

- It has 6 degrees of freedom; in the representation

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

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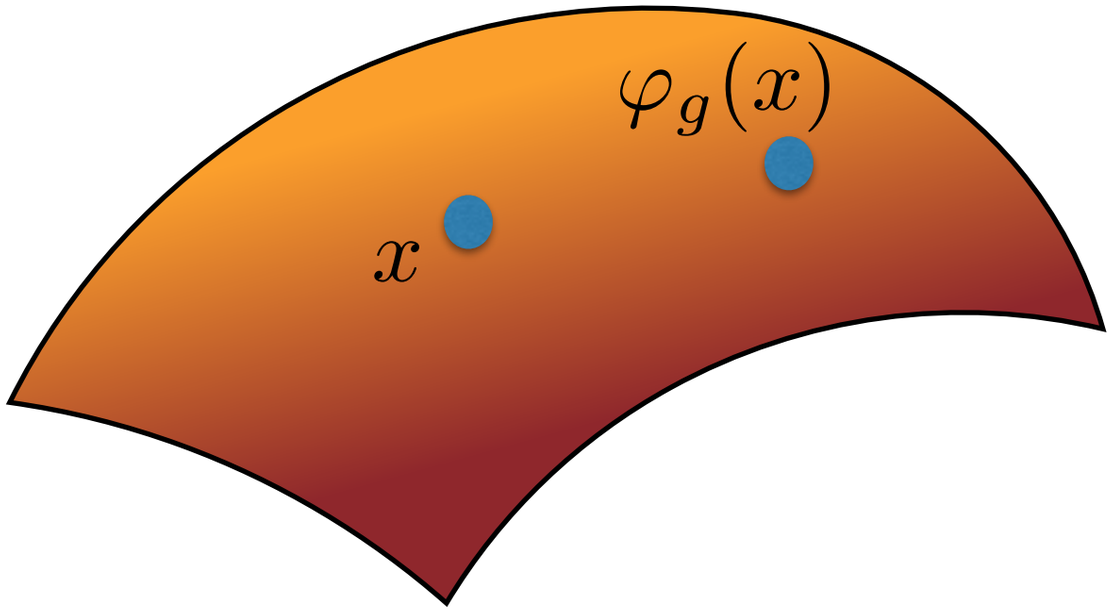
- For some groups, we might only observe partial invariance (e.g. rotation and dilation).

Invariant Representations

- Given a transformation group *G* and an input *x*, the *action* of *G* onto *x* is called an *orbit*:

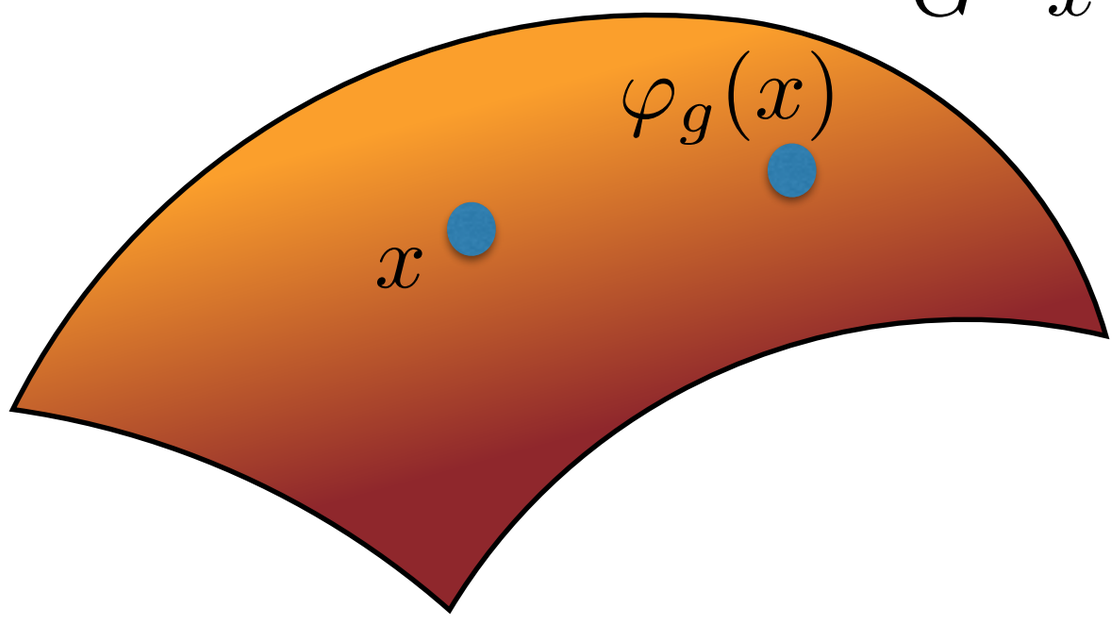
$$G \cdot x = \{\varphi_g(x)\,, g \in G\}$$



- Impact on the learning task?
- Since our estimator is linear in $\Phi(x)$, $\Phi(G \cdot x)$ should be "flat".

Invariant Representations

$$G \cdot x = \{\varphi_g(x)\,, g \in G\}$$



- Problem? A 6-dimensional curvy space looks flat in a high-dimensional space.
- Group symmetries are not sufficient to beat the curse of dimensionality.

From Invariance to Stability

- Symmetry is a very strict criteria. Can we relax it?

- Although image and audio recognition does not have high-dimensional symmetry groups, it is *stable* to local deformations.

$$x \in L^2(\mathbb{R}^m) \ , \ \tau : \mathbb{R}^m \to \mathbb{R}^m \text{ diffeomorphism}$$

$$x_\tau = \varphi_\tau(x) \ , \ x_\tau(u) = x(u - \tau(u))$$

φ_τ is a change of variables: (think of x_τ as adding noise to the pixel *locations* rather than to the pixel values)

From Invariance to Stability

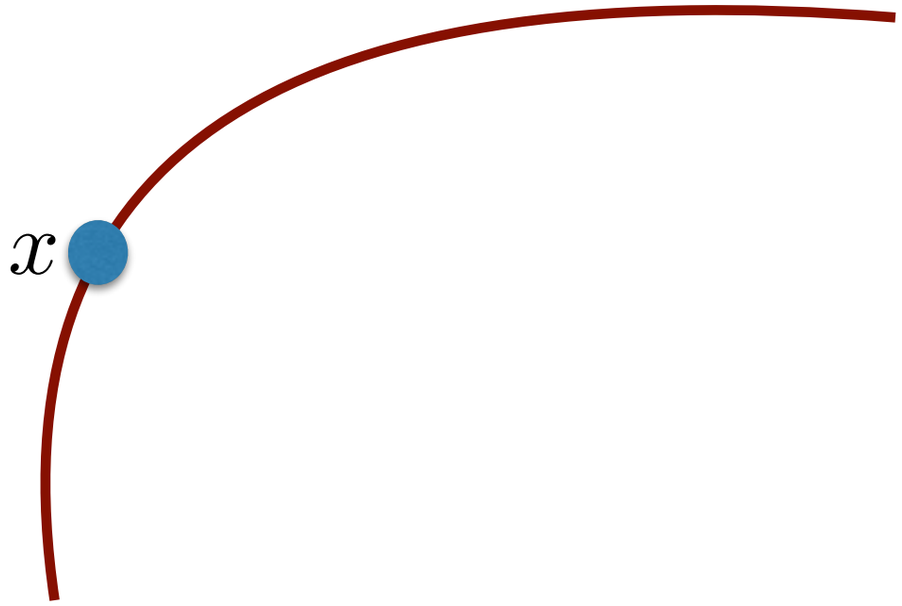
- Informally, if $\|\tau\|$ measures the amount of deformation, many recognition tasks satisfy

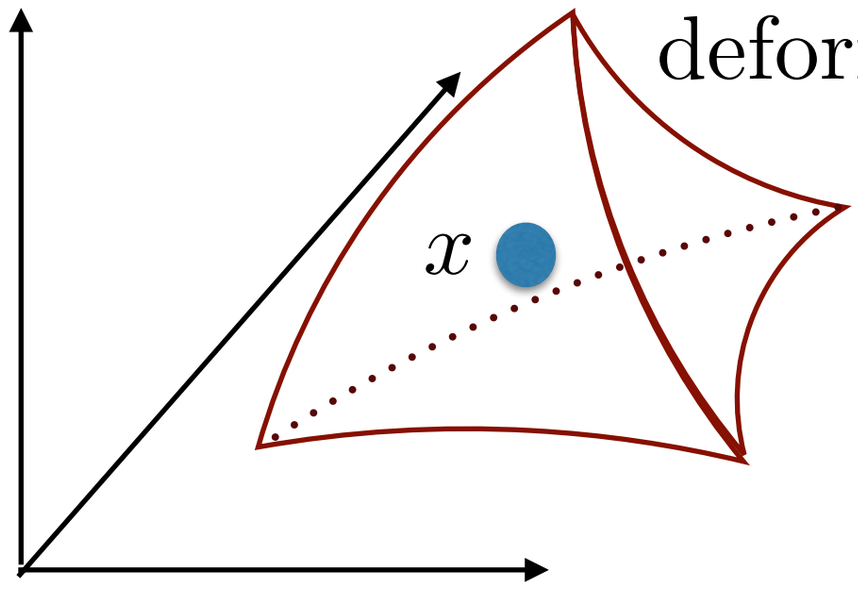
$$\forall\ x, \tau,\ \ |f(x) - f(x_\tau)| \lesssim \|\tau\|$$

- If our representation is stable, then

$$\forall\ x, \tau,\ \ \|\Phi(x) - \Phi(x_\tau)\| \leq C\|\tau\| \Longrightarrow |\hat{f}(x) - \hat{f}(x_\tau)| \leq \tilde{C}\|\tau\|$$

symmetry group: low dimension





deformations fill the space

Deformations in Image/Audio Recognition

- Can model 3D viewpoint changes, changes in pitch/ timbre in speech recognition.

- Deformable parts model [Feltzenszwalb et al, '10]
  - State-of-the-art on object detection pre-CNN.

- Deformable templates [Grenader, Younes, Trouve, Amit]
  - Equip deformable templates with differentiable structure

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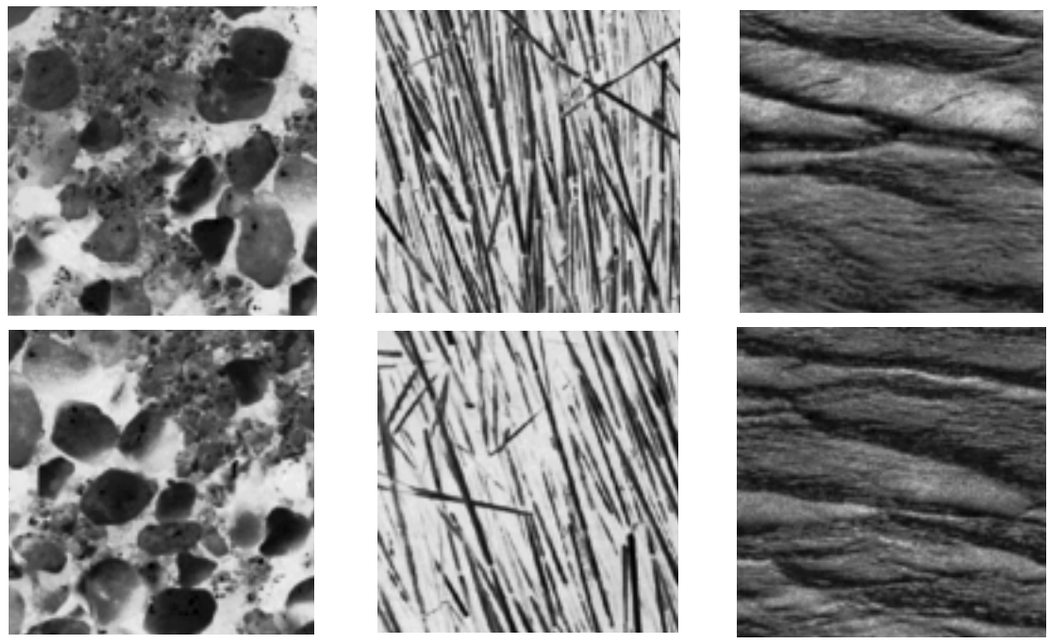
- Deformation cost.

Stability Condition

- Lipschitz

Representation of Stationary Processes

$x(u)$: realizations of a stationary process $X(u)$ (not Gaussian)



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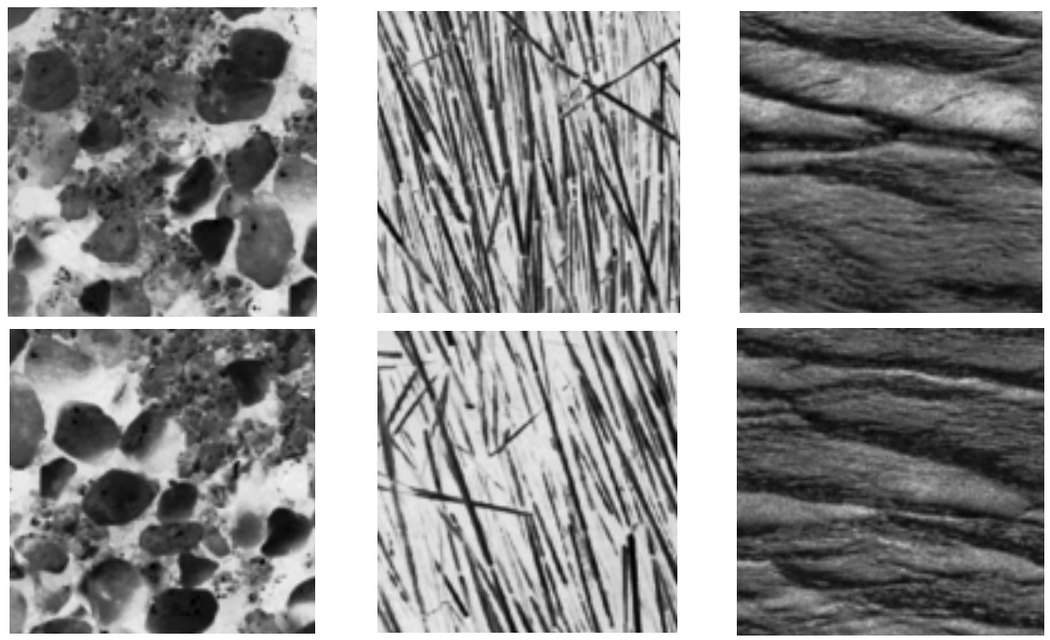

$$\Phi(X) = \{E(f_i(X))\}_i$$

Estimation from samples $x(n)$: $\widehat{\Phi}(X) = \left\{ \frac{1}{N} \sum_n f_i(x)(n) \right\}_i$

Discriminability: need to capture high-order moments

Stability: $E(\|\widehat{\Phi}(X) - \Phi(X)\|^2)$ small

Ergodicity

Stability as Consistency

Examples

Examples

Class-specific variability

- Imagenet Examples

- Definition of clutter

Recognition with Shallow Models

- Sift + Spatial Pyramid
- Random projections
- K-means