Stat 212b: Topics in Deep Learning
Lecture 5

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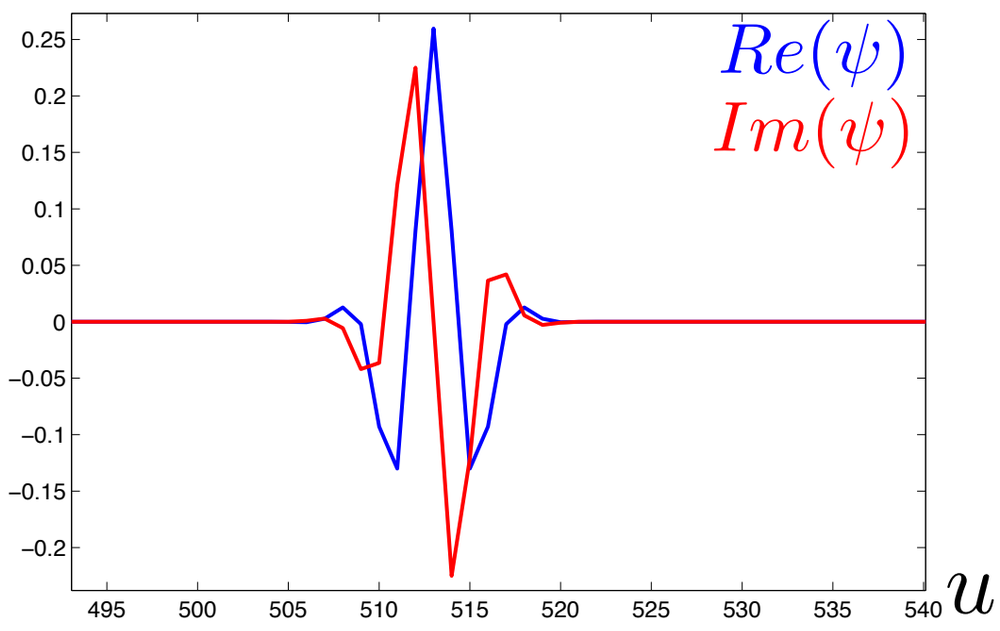
Review: Wavelets

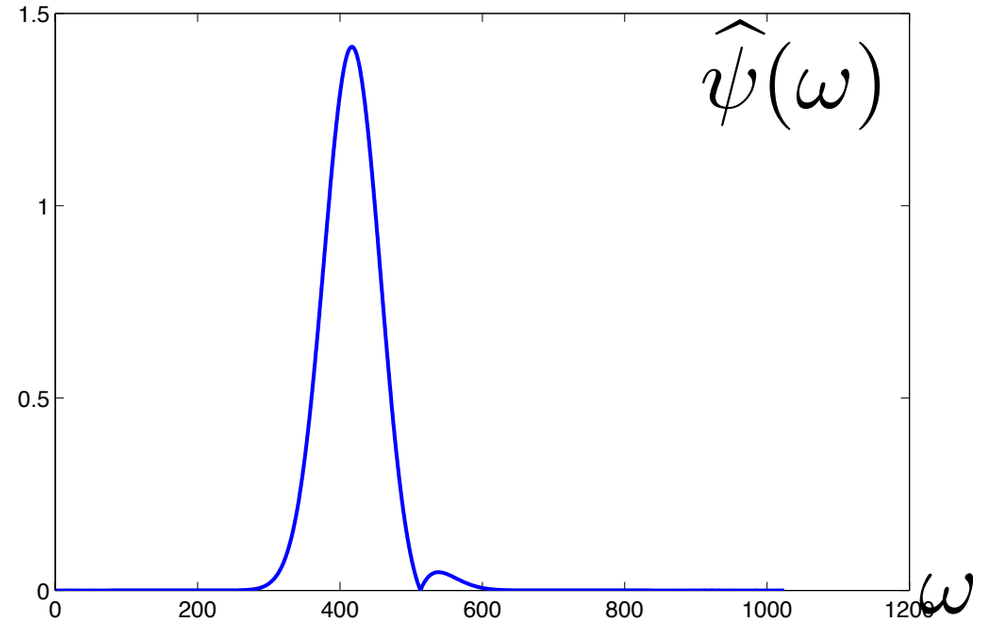
- ψ : bandpass (ie oscillating) signal, well localized in space and frequency.
- At least one vanishing moment: $\int \psi(u)du = 0$

$(\text{we say that } \psi \text{ has } k \text{ vanishing moments if } \int \psi(u)u^l du = 0 \text{ for } l < k)$

- Can be real or complex. $\psi = \psi_r + i\psi_i$

Ex: Morlet wavelet

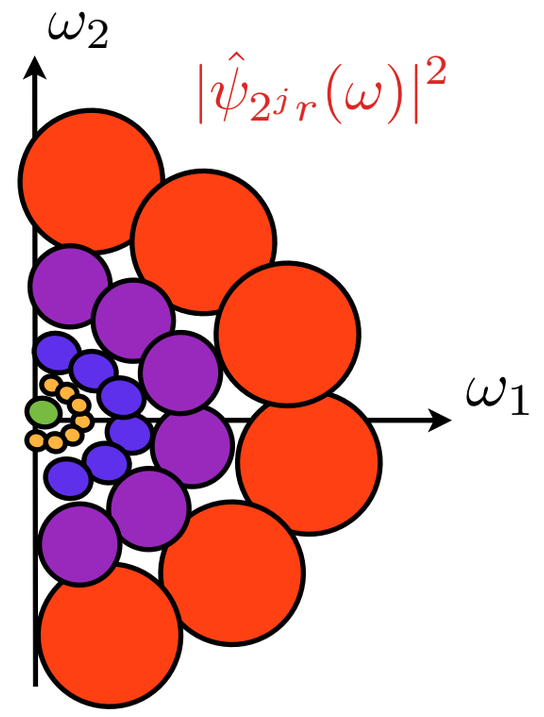




Review: Littlewood-Paley Wavelet Filter Banks

- For images, dilated and rotated wavelets:

$$\psi_\lambda(u) = 2^{-j/2}\psi(2^{-j}ru) \ , \ \text{with } \lambda = 2^j r$$



- Wavelet transform convolutional filter bank:

$$Wx = \{x \star \phi(u) \ , \ x \star \psi_\lambda(u)\}_{\lambda \in \Lambda} \qquad x \star \psi(u) = \int x(v)\psi(u-v)dv \ .$$

Theorem (Littlewood-Paley): If there exists $\delta > 0$ such that

$$\forall \omega > 0 \ , \ 1 - \delta \leq |\hat{\phi}(\omega)|^2 + \frac{1}{2}\sum_\lambda |\hat{\psi}(\lambda^{-1}\omega)|^2 \leq 1 \ ,$$

$$\text{then } \forall\, x \in L^2 \ , \ (1-\delta)\|x\|^2 \leq \|Wx\|^2 \leq \|x\|^2 \ .$$

Review: Wavelets and Deformations

- We saw before that a blurring kernel is nearly invariant to deformations:

  Proposition: The local averaging $\Phi(x) = x * \phi_J$ satisfies $\forall\ \|x\| = 1 \in L^2\ ,\ \tau\ ,\ \|\Phi(x) - \Phi(\varphi_\tau x)\| \leq C\|\tau\|\ .$

- What about the wavelet operator $\Phi(x) = \{x * \psi_\lambda\}_\lambda$?
  - We don't have local invariance, but we have a form of local covariance:

Proposition [Mallat]: For each $\delta > 0$ there exists $C > 0$ such that for all J and all $\tau \in C^2$ with $\|\nabla\tau\|_\infty \leq 1 - \delta$ we have

$$\|W_J\varphi_\tau - \varphi_\tau W_J\| \leq C(J\|\nabla\tau\|_\infty + \|H\tau\|_\infty)\ .$$

($H\tau$: Hessian of τ)

Review: Characterization of stable non-linearities

- Preserve additive stability:

$$\|Mx - Mx'\| \leq \|x - x'\| \ . \qquad M \text{ non-expansive} \ .$$

- Preserve geometric stability: It is sufficient to commute with diffeomorphisms.

Theorem: If M is non-expansive operator in L^2 such that $\varphi_\tau M = M\varphi_\tau$ for all τ, then M is point-wise:

$$Mx(u) = \rho(x(u)) \ .$$

- Since we want to smooth orbits, we may choose a point-wise nonlinearity that reduces oscillations:

$$\rho(z) = |z| \text{ or } \rho(z) = \max(0, z)$$

Objectives

- Scattering Representations
  - Main Properties
  - Main Limitations
  - Extensions: Joint rigid scattering.

- Convolutional Neural Networks
  - From fixed groups to adaptive templates

Separable Scattering Operators

- Local averaging kernel: $x \star \phi_J$
  - locally translation invariant
  - stable to additive and geometric deformations
  - loss of high-frequency information.

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- Recover lost information: $\mathcal{U}_J(x) = \{x \star \phi_J \, , \, |x \star \psi_\lambda|\}_{\lambda \in \Lambda_J}$.
  - Point-wise, non-expansive non-linearities: maintain stability.
  - Complex modulus maps energy towards low-frequencies.

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- Cascade the "recovery" operator:

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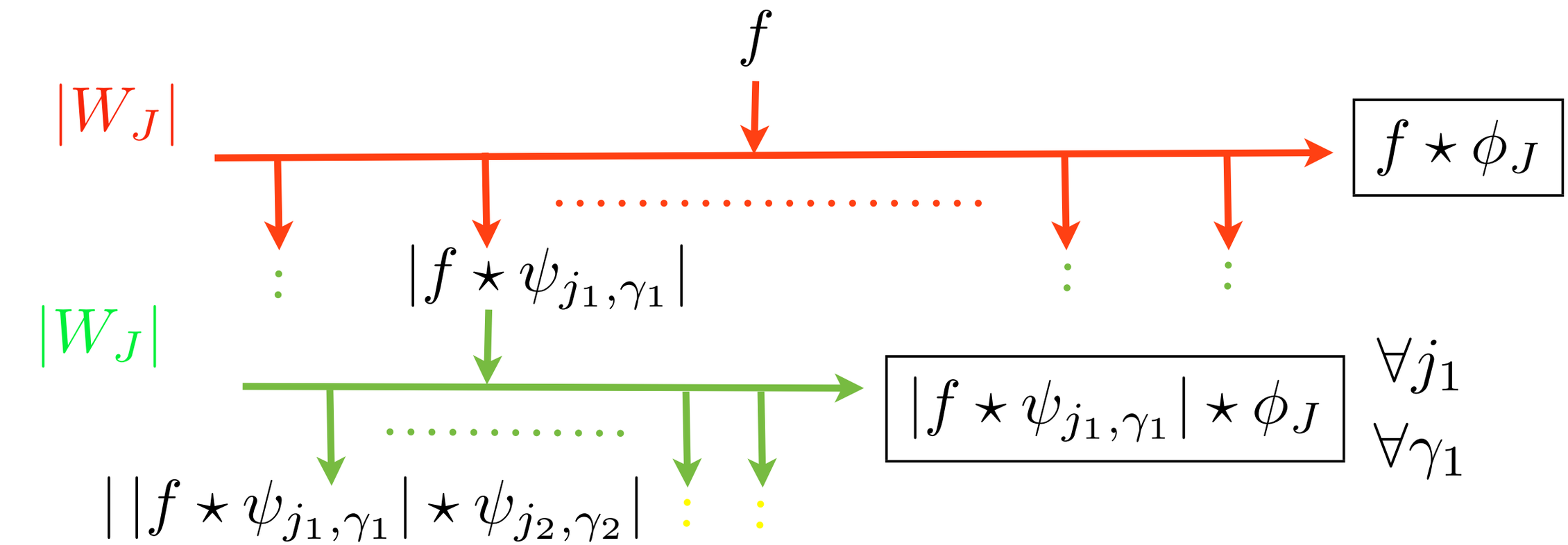
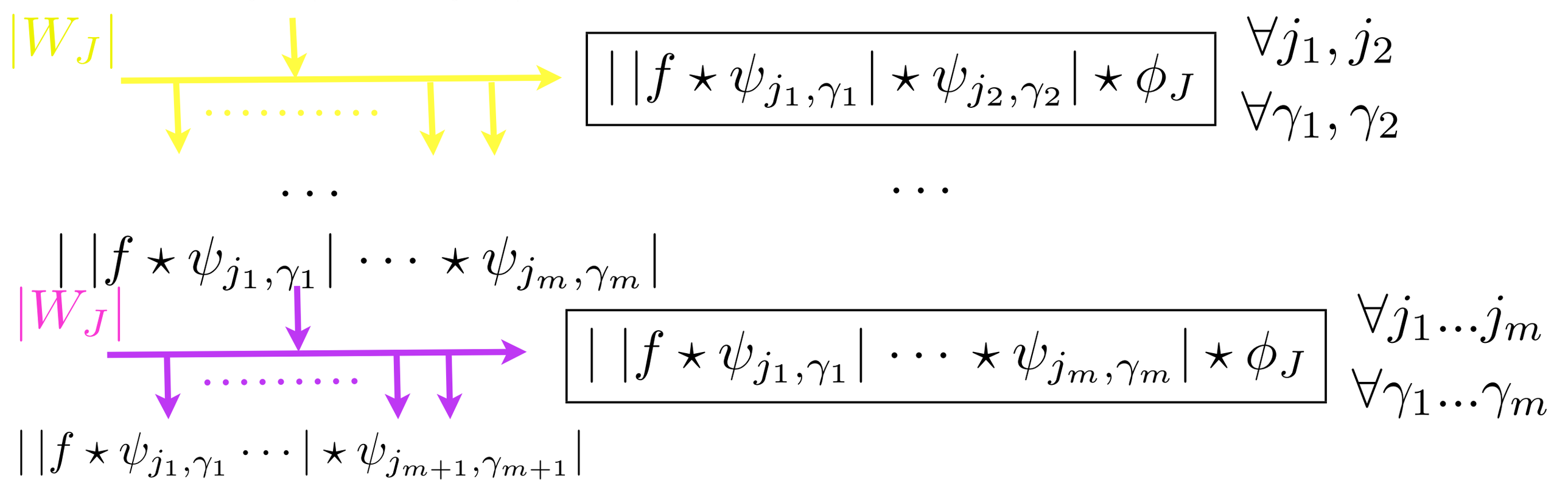
- Scattering coefficient along a path $p = (\lambda_1, \ldots, \lambda_m)$:

$$S_J[p]x(u) = |||x \star \psi_{\lambda_1}| \star \psi_{\lambda_2}| \star \ldots | \star \psi_{\lambda_m}| \star \phi_J(u) \, .$$

Scattering Convolutional Network



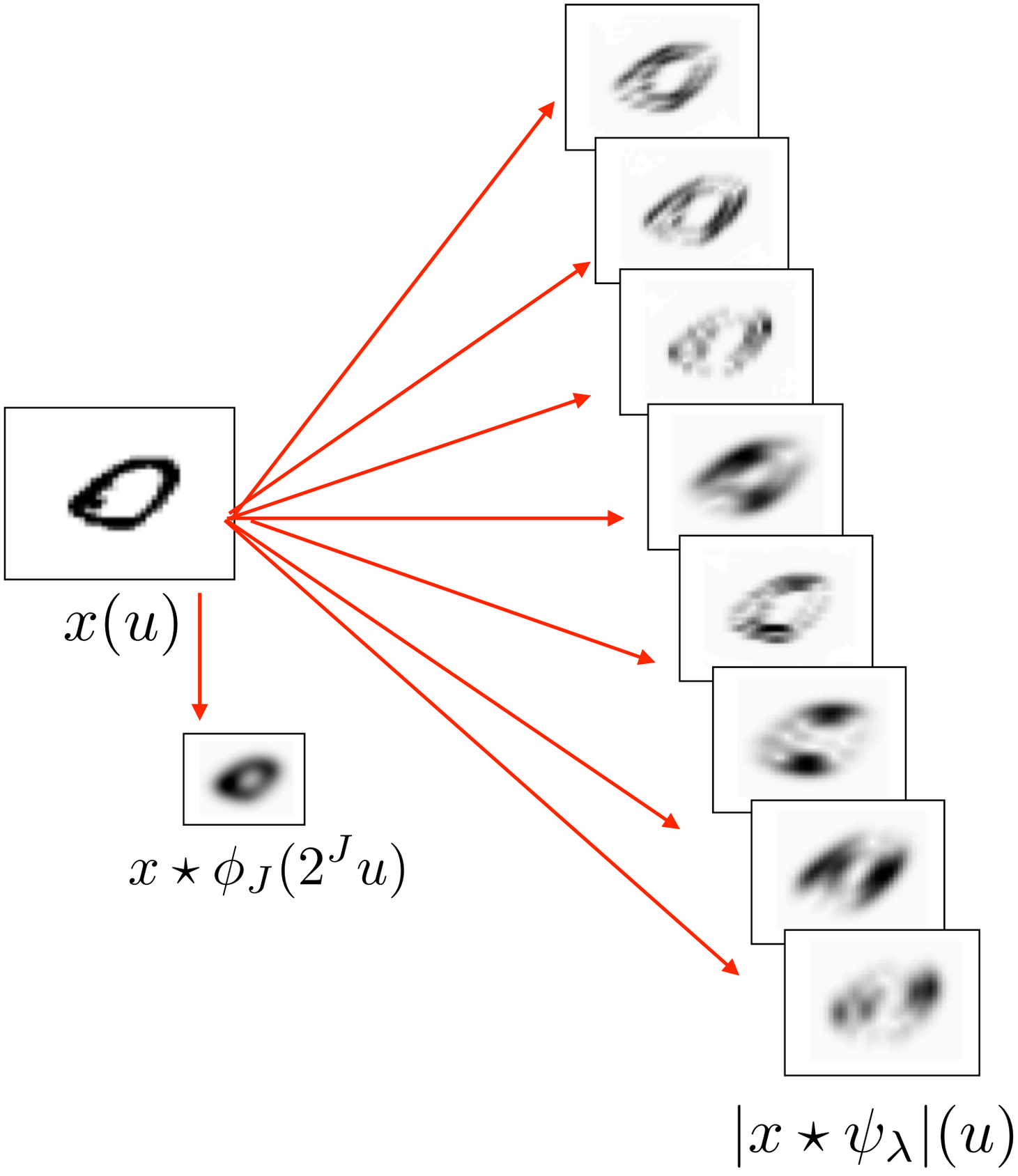


Cascade of contractive operators.

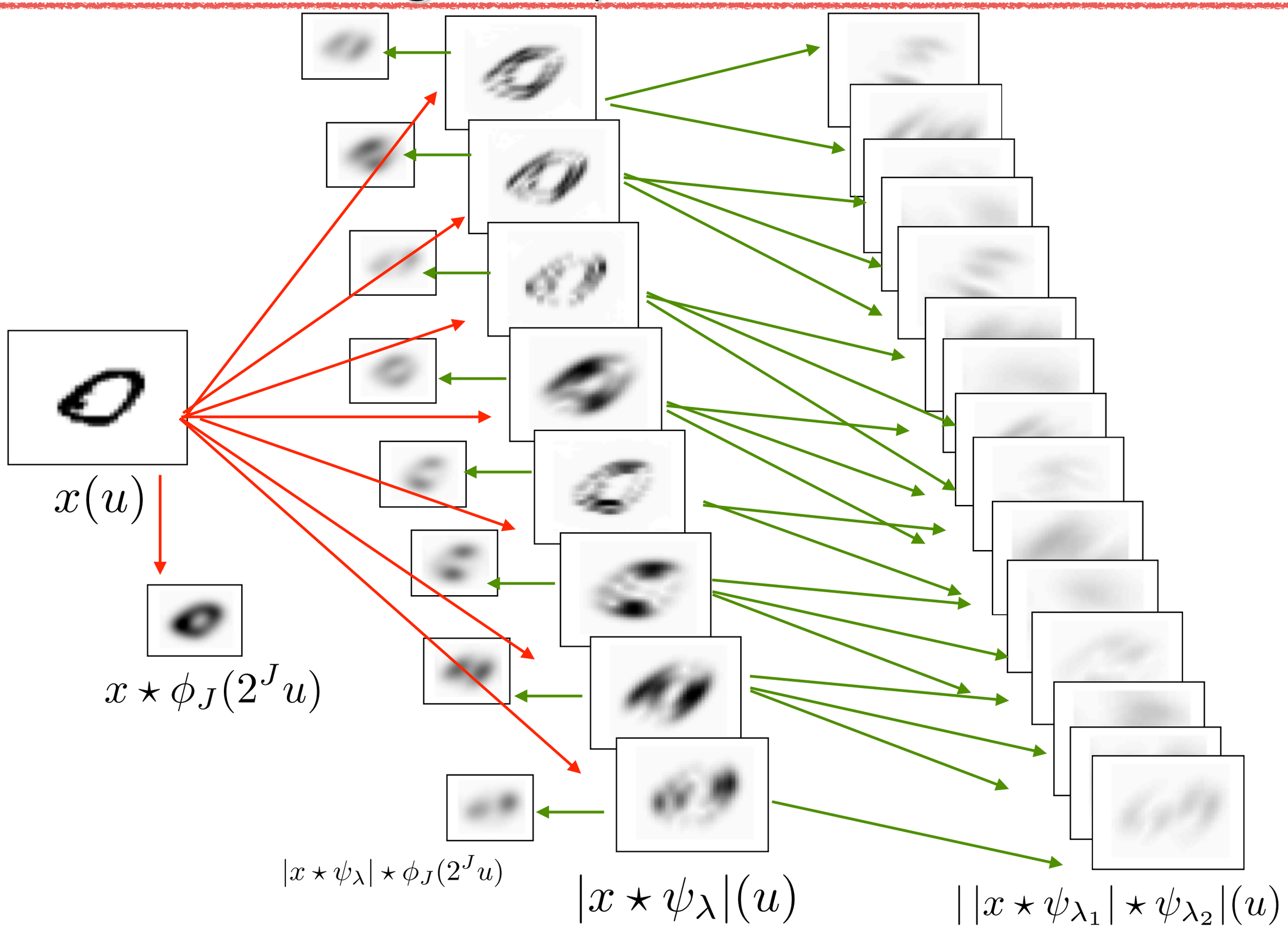
Scattering Example

$x(u)$

Scattering Example

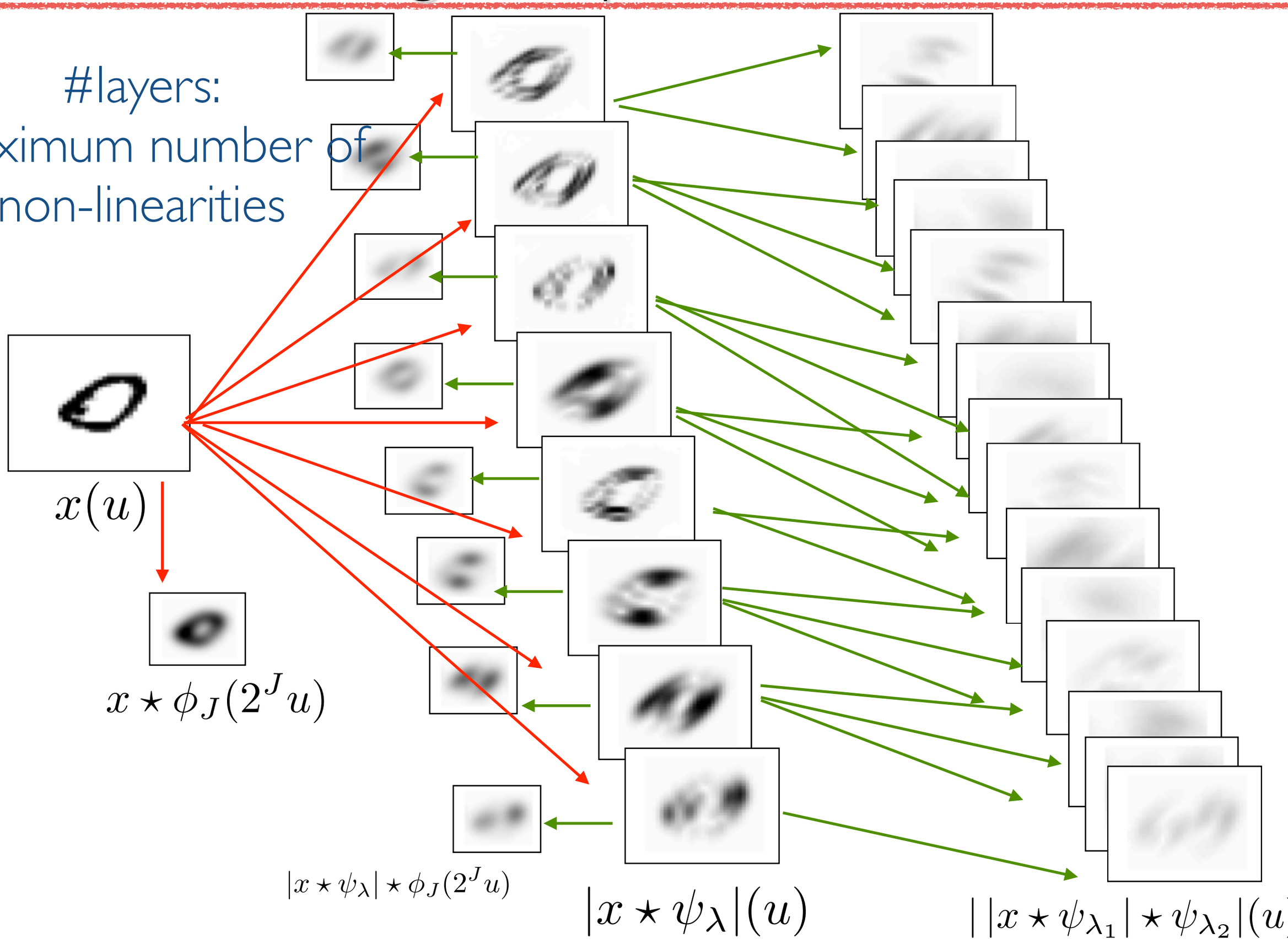


Scattering Example



Scattering Example

#layers:
maximum number of
non-linearities



Scattering with Multi-Resolution Wavelets

- We have considered a collection $\psi_{j,\theta}$ of oriented and dilated wavelets, and a translation co-variant wavelet decomposition operator:

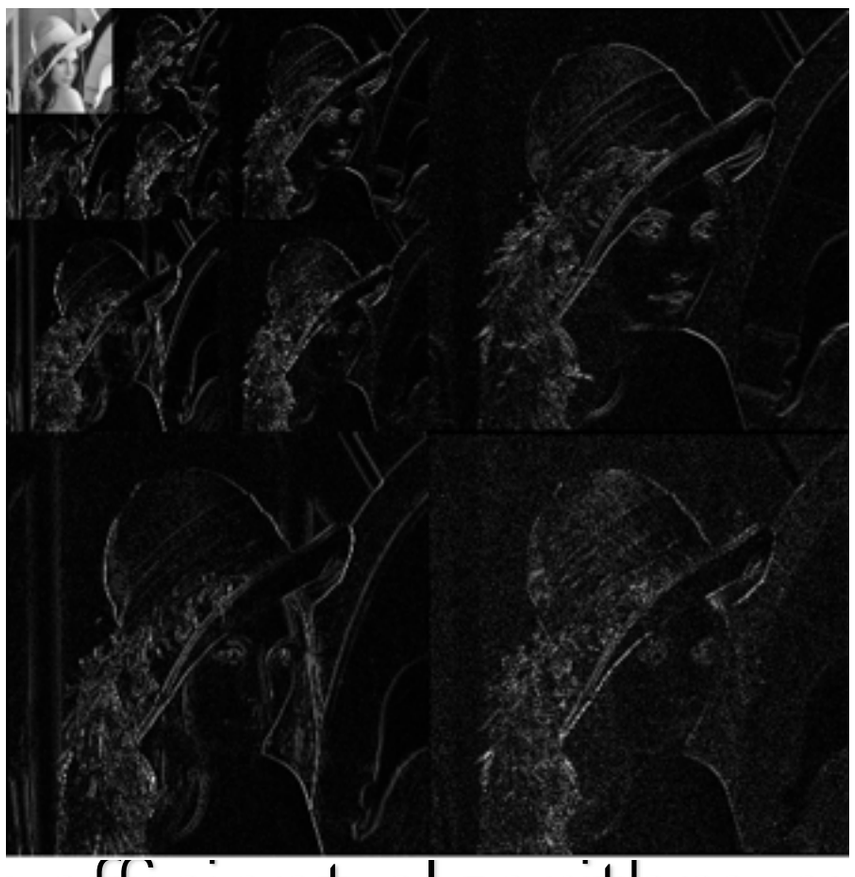
$$Wx = \{x \star \phi_J, x \star \psi_{j,\theta}\}$$

- With *J* scales and *L* orientations, the redundancy is (1+*JL*).

Scattering with Multi-Resolution Wavelets

- With *J* scales and *L* orientations, the redundancy is (1+*JL*).
- This is in contrast with *orthogonal* wavelet transforms, used for compression and (suboptimally) for denoising.

$$x \in \mathbb{R}^N \rightarrow Wx \in \mathbb{R}^N \ . \qquad W^T W = Id$$



example of orthogonal wavelet decomposition

- A very efficient algorithm exists using filter cascades with MultiResolution Analysis.

Multi-Resolution Wavelets

- At each scale j, we consider a low-pass *scaling filter* h and band-pass filters g_θ, $\theta \in [1, \dots, L]$.

- Wavelets and the bluring kernel are obtained at each j by cascading these filters:

$$\phi_j = \phi_{j-1} \star h_j \quad \psi_{j,\theta} = \phi_{j-1} \star g_{j,\theta} \ .$$

- Decompositions are obtained by cascading fine-to-coarse:

$$x \star \phi_j(u) = (x \star \phi_{j-1}) \star h_j(u) \quad , \quad x \star \psi_{j,\theta}(u) = (x \star \phi_{j-1}) \star g_{j,\theta}(u) \ .$$

- Downsampling (or "*stride*") adaptive to signal smoothness:

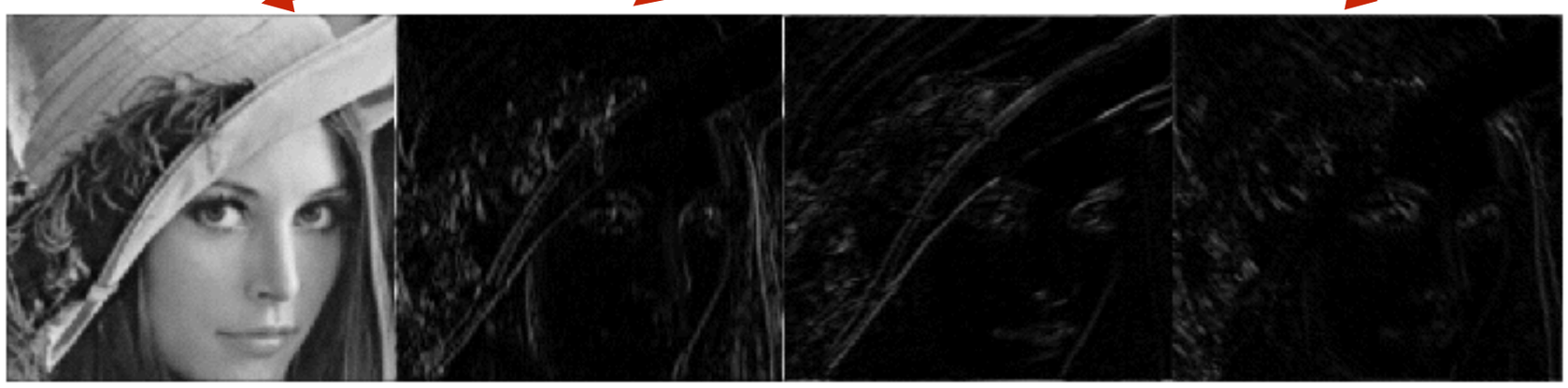
$$x \star \phi_j(u) = (x \star \phi_{j-1}) \star h(2u) \quad , \quad x \star \psi_{j,\theta}(u) = (x \star \phi_{j-1}) \star g_\theta(2u) \ .$$

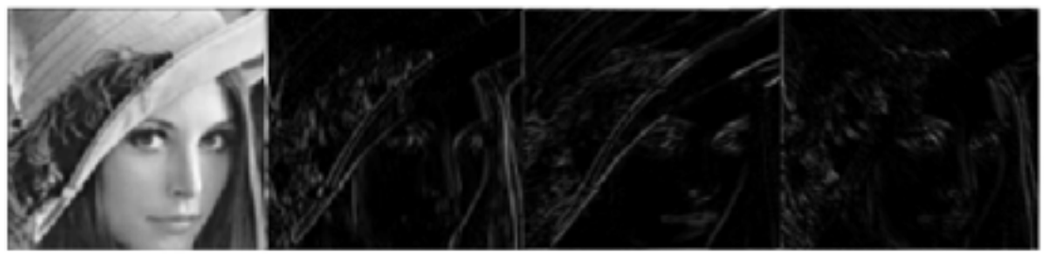
Scattering with Multi-Resolution Wavelets

x

$x \star h$

$|x \star g_\theta|$





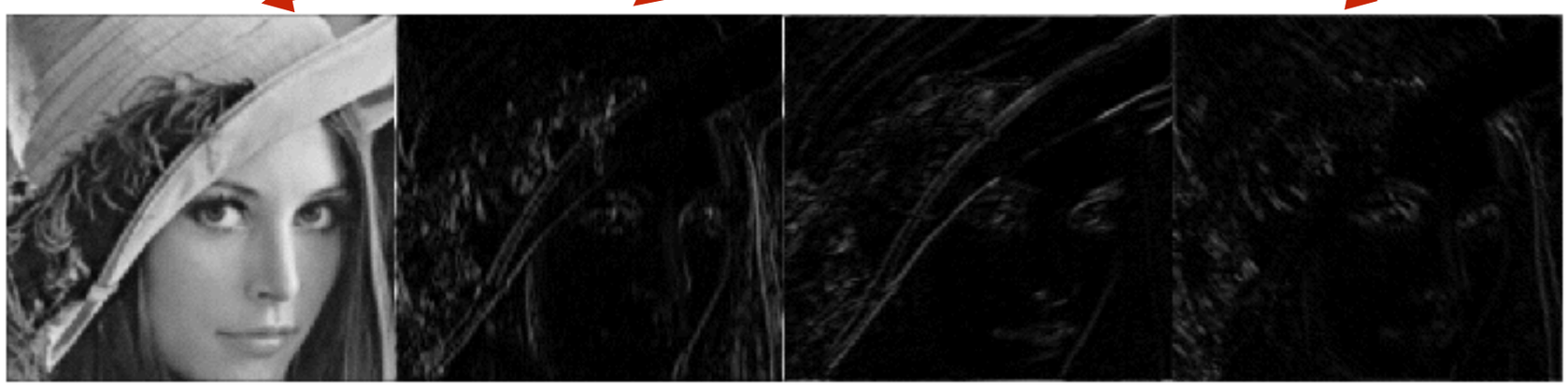
Scattering with Multi-Resolution Wavelets

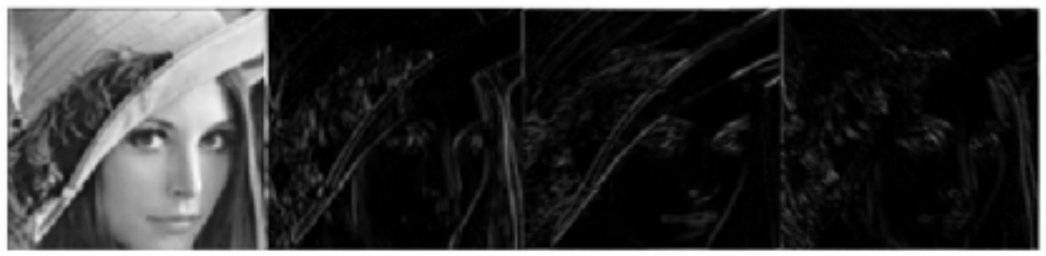
#layers:
maximum scale

x

$x \star h$

$|x \star g_\theta|$





Scattering Conservation of Energy

Theorem *(Mallat)*: For appropriate wavelets, the scattering representation is contractive, $\|S_J x - S_J x'\| \leq \|x - x'\|$,

and unitary, $\|S_J x\| = \|x\|$.

$$\|S_J x\|^2 = \sum_{p \in \mathcal{P}_J} \|S_J[p]x\|^2$$

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- In practice, the transform is limited to a finite number of layers m_{max}. This result shows residual error converges to 0.
- The result requires complex wavelets (ie, not real).

Interpretation

- Unitary Wavelet decomposition preserves energy:

$$\|x\|^2 = \|x \star \phi_J\|^2 + \sum_{j \leq J, \theta} \|x \star \psi_{j,\theta}\|^2 \ .$$

- Repeat formula on each output $|x \star \psi_{j,\theta}|$:

$$\| |x \star \psi_{j,\theta}| \|^2 = \| |x \star \psi_{j,\theta}| \star \phi_J \|^2 + \sum_{j_2 \leq J, \theta_2} \| |x \star \psi_{j,\theta}| \star \psi_{j_2,\theta_2} \|^2 \ .$$

$$\|x\|^2 = \|S_J[0]x\|^2 + \sum_{|p|=1} \|S_J[p]x\|^2 + \sum_{|p|=2} \| |x \star \psi_{\lambda_1}| \star \psi_{\lambda_2} \|^2$$

$\forall\ m$

$$\|x\|^2 = \sum_{|p|<m} \|S_J[p]x\|^2 + \sum_{|p|=m} \| \ldots |x \star \psi_{\lambda_1}| \star \psi_{\lambda_2}| \ldots \psi_{\lambda_m} \|^2$$

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- Result amounts to proving that

$$\lim_{m\to\infty} \sum_{|p|=m, j_i \leq J} \| |x \star \psi_{\lambda_1}| \star \ldots | \star \psi_{\lambda_m} | \|^2 = 0 \ .$$

- Fact: Every time we apply the (complex) wavelet modulus, we push energy towards the low frequencies.
- Result is obtained by formally showing this fact.

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- Fact: Every time we apply the (complex) wavelet modulus, we push energy towards the low frequencies.
- Result is obtained by formally showing this fact.

- It requires a non-linearity that produces smooth envelopes:
  - complex wavelets OK
  - real wavelets: ??

Scattering Geometric Stability

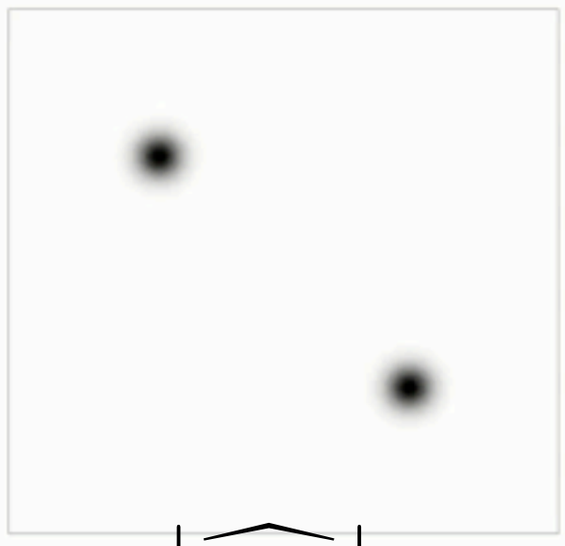
- Geometric Stability:

$$\|S_J x\|^2 = \sum_{p \in \mathcal{P}_J} \|S_J[p]x\|^2$$

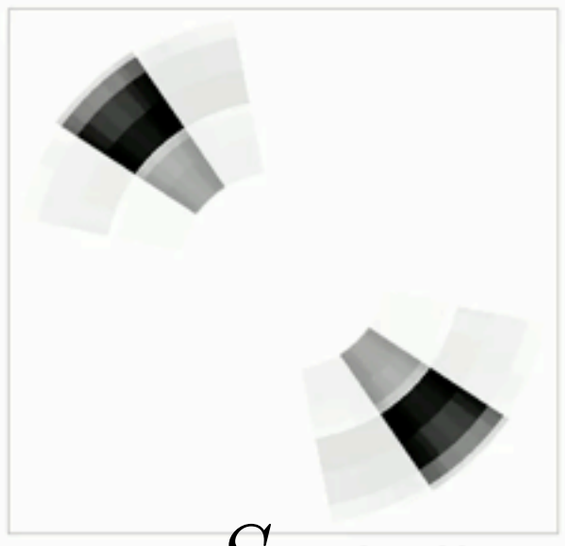
Theorem (Mallat'10): There exists C such that for all $x \in L^2(R^d)$ and all m, the m-th order scattering satisfies

$$\|S_J \varphi_\tau x - S_J x\| \leq Cm\|x\|(2^{-J}|\tau|_\infty + \|\nabla\tau\|_\infty + \|H\tau\|_\infty) \ .$$

$\varphi_\tau x$



$|\widehat{\varphi_\tau x}|$



$S_J \varphi_\tau x$

Interpretation

- Denote

$$A_J x = x \star \phi_J \qquad W_J x = \{x \star \psi_\lambda\}_\lambda \qquad Mx = |x|$$

- We know that

$$\|A_J - A_J \varphi_\tau\| \leq C(2^{-J}|\tau|_\infty + |\nabla \tau|_\infty)$$

$$\|W_J \varphi_\tau - \varphi_\tau W_J\| \leq C(J|\nabla \tau|_\infty + |H\tau|_\infty)$$

$$M\varphi_\tau = \varphi_\tau M \qquad ([A, B] = AB - BA \ : \ \text{Commutator})$$

- $S_J = \{A_J, A_J M W_J, A_J M W_J M W_J, \dots\}$

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Interpretation

- Each order contributes separately:

$$\|S_J - S_J\varphi_\tau\|^2 = \|A_J - A_J\varphi_\tau\|^2 + \|A_J MW_J - A_J MW_J\varphi_\tau\|^2 + \dots$$

- Let us inspect a generic term:

$$\|A_J \underbrace{MW_J MW_J \dots MW_J}_{k \text{ times}} - A_J \underbrace{MW_J MW_J \dots MW_J}_{k \text{ times}} \varphi_\tau\|$$

$$(U_J = MW_J)$$

$$\|A_J U_J^k - A_J U_J^k \varphi_\tau\| \le \|A_J U_J^k - A_J U_J^{k-1}\varphi_\tau U_J\| + \|A_J U_J^{k-1}\varphi_\tau U_J - A_J U_J^k \varphi_\tau\|$$

$$\le \|A_J U_J^{k-1} - A_J U_J^{k-1}\varphi_\tau\| + \|A_J U_J^{k-1}[\varphi_\tau, U_J]\|$$

$$\le \|A_J U_J^{k-1} - A_J U_J^{k-1}\varphi_\tau\| + \|[\varphi_\tau, U_J]\|$$

$$\le k\|[\varphi_\tau, U_J]\| + \|A_J - A_J\varphi_\tau\| \le k\|[\varphi_\tau, W_J]\| + \|A_J - A_J\varphi_\tau\|$$

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Scattering Discriminability

- For appropriate wavelets, the information is preserved at each layer:

Theorem: (Waldspurger) For appropriate wavelets, the operator $Ux = \{x \star \phi_J, |x \star \psi_j|\}_{j \leq J}$ is injective.

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- Very different situation than Fourier modulus (why?)
- The representation is highly redundant.
- However, the inverse is unstable for large J: we might be contracting too much in general.
- How to prevent that?

Discriminability and Sparsity

- Typical non-linearities are contractive:

$$\|\rho(x) - \rho(x')\| \leq \|x - x'\|$$

Discriminability and Sparsity

- Typical non-linearities are contractive:

$$\|\rho(x) - \rho(x')\| \leq \|x - x'\|$$

- However, if x, x' are sparse, this inequality is an equality in most of the signal domain.

- Thus sparsity is a means to control and prevent excessive contraction of different signal classes.

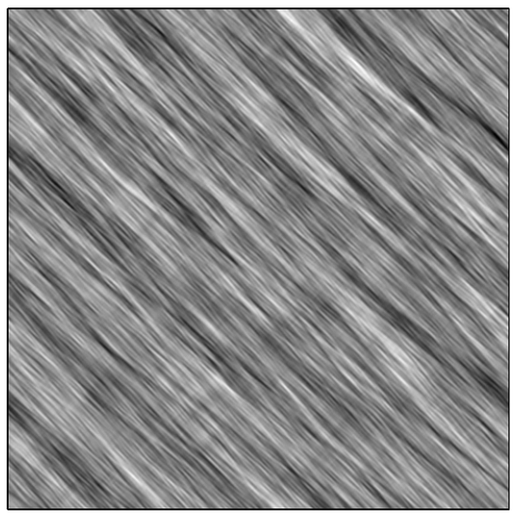
Image Examples

Images

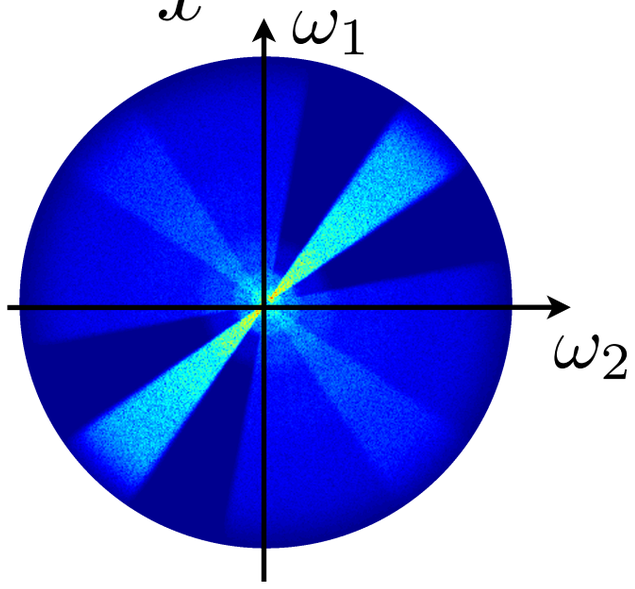
Fourier

Wavelet Scattering

x

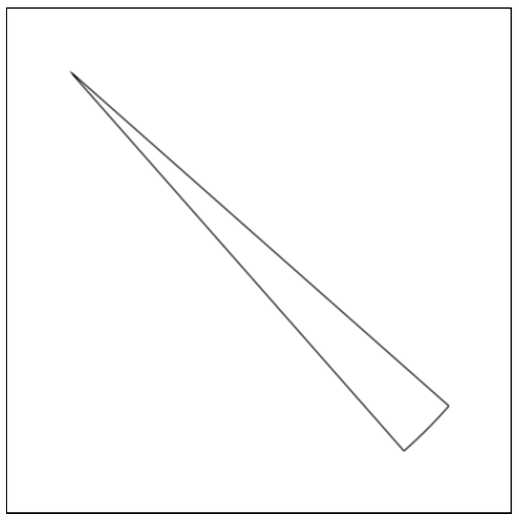


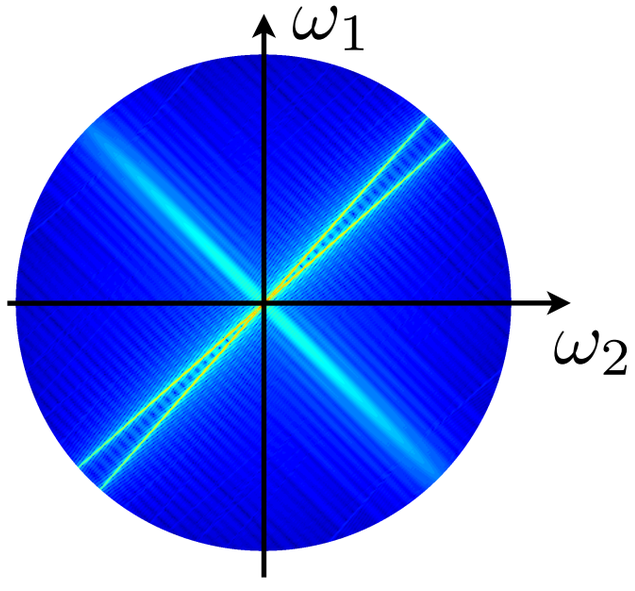
$\hat{x}$

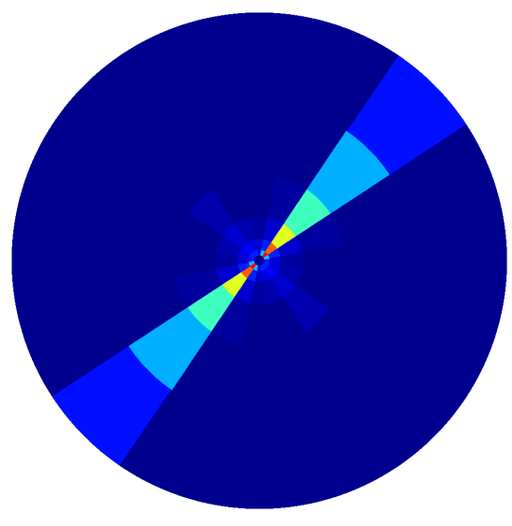


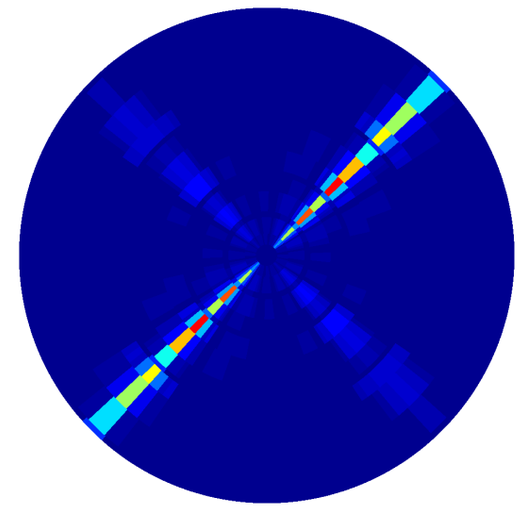
$|x \star \psi_{\lambda_1}| \star \phi_J$

$||x \star \psi_{\lambda_1}| \star \psi_{\lambda_2}| \star \phi_J$









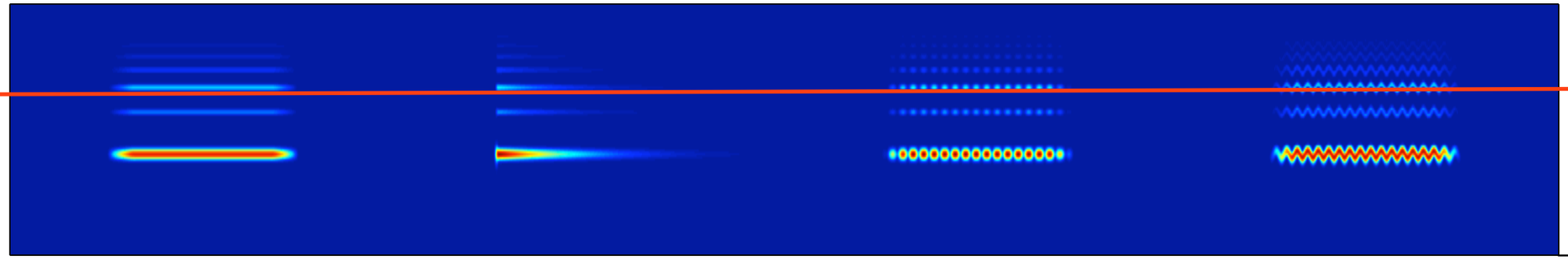
window size = image size

Sound Examples

(courtesy J. Anden)

$|f \star \psi_{\lambda_1}|(x)$

$\lambda_1 = \log(\omega_1)$



1977 Hz

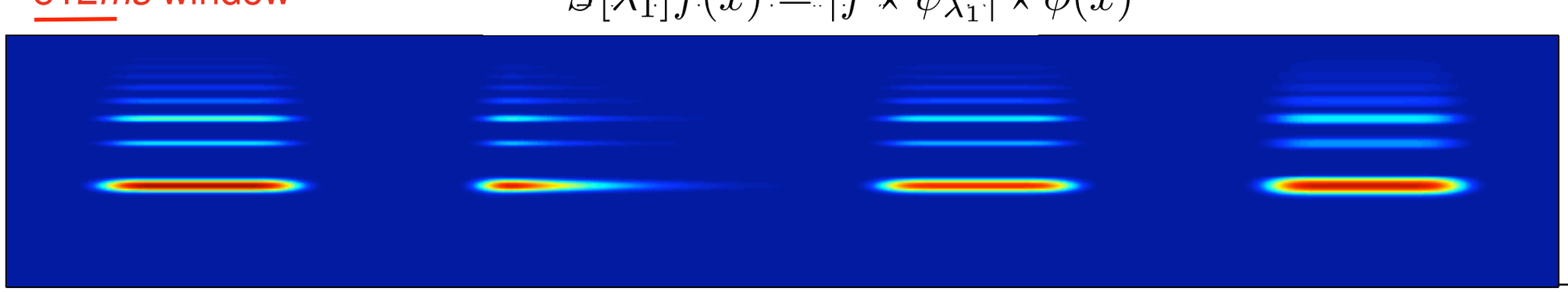
x

t

512*ms* window

$S[\lambda_1]f(x) = |f \star \psi_{\lambda_1}| \star \phi(x)$

$\lambda_1 = \log(\omega_1)$



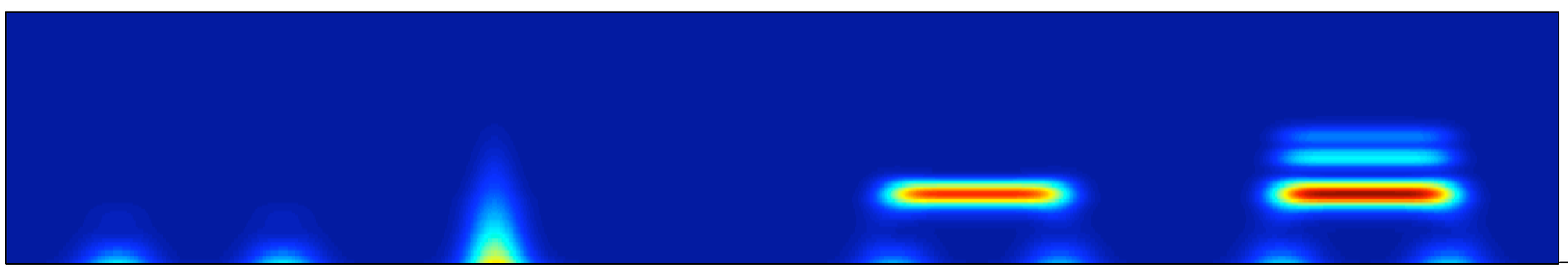
MFCC

x

t

$S[\lambda_1, \lambda_2]f(x) = ||f \star \psi_{\lambda_1}| \star \psi_{\lambda_2}| \star \phi(x) \quad \text{for } \lambda_1 = \log(1977)$

$\lambda_2 = \log(\omega_2)$



18 Hz

x

t

Limitations of Separable Scattering

- No feature dimensionality reduction
  - The number of features increases exponentially with depth and polynomially with scale.

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  – We cannot capture the *joint* deformation structure of feature maps
  – Loss of discriminability.

- The deformation model is rigid and non-adaptive
  – We cannot adapt to each class
  – Wavelets are hard to define *a priori* on high-dimensional domains.

Joint versus Separable Invariance

Wavelet Covariants

$$\rho(x_0 \star \psi_{j,\theta})(u) = x_1(u, j, \theta)$$

Let $\tilde{x}_0 = R_\alpha x_0$ be a rotation of α degrees.

$$\rho(\tilde{x}_0 \psi_{j,\theta})(u) = x_1(R_\alpha u, j, \theta + \alpha)$$

Similarly, roto-translation acts on x_1 by rotating and translating spati
coordinates and translating orientation coordinates

Group Convolutions

Joint Scattering

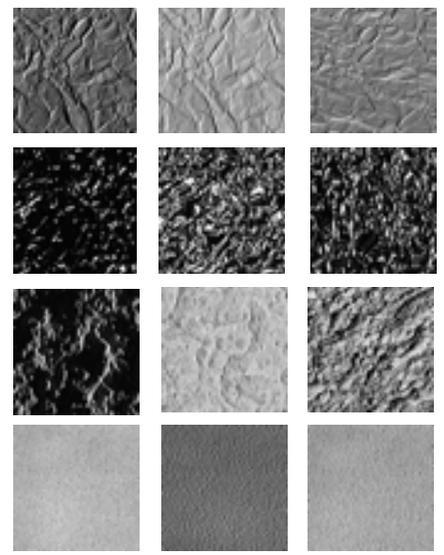
Joint Scattering

Example: Roto-Translation Scattering

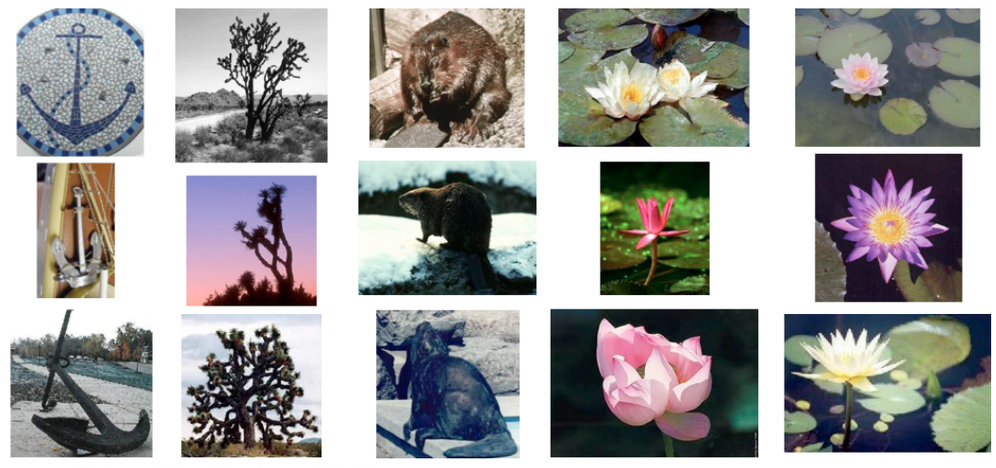
Classification with Scattering

- State-of-the art on pattern and texture recognition:
  - MNIST, USPS [Pami'13]

  - Texture (CUREt, UIUC) [Pami'13]



- Object Recognition:
  - ~17% error on Cifar-10 [Oyallon&Mallat, CVPR'15]



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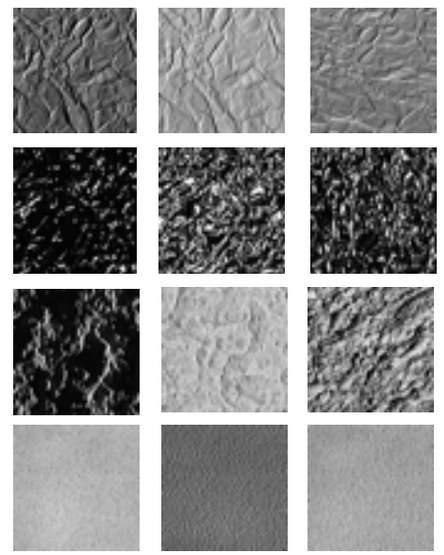

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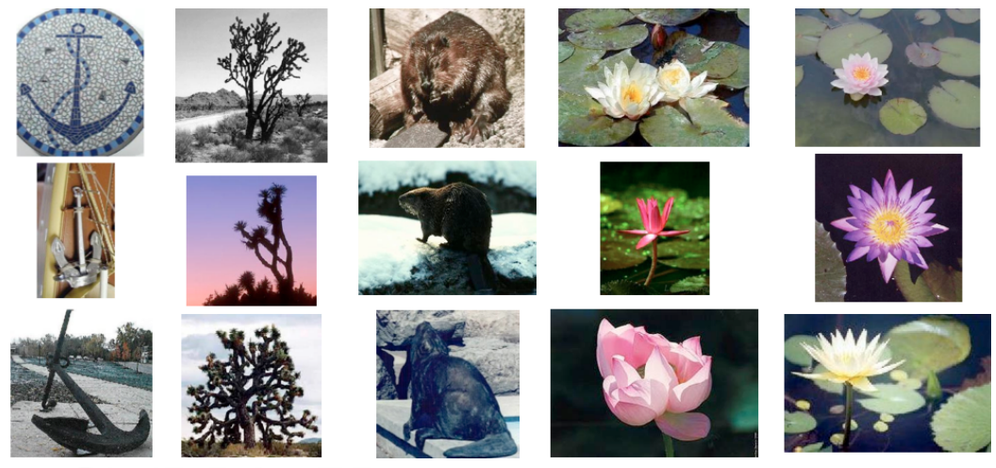

Limitations of Joint Scattering

From Scattering to CNNs

From Scattering to CNNs

Convolutional Neural Networks