Splitting Schemes as Preconditioners

A numerical study of splitting schemes and implicit solution methods for Biot's equation

SIAM GS13 — Recent advances in inexact solvers

J. B. Haga ▪ Kalkulo AS
J. M. Nordbotten ▪ Dept. of Mathematics, University of Bergen
K. A. Mardal ▪ Center for Biomedical Computations, Simula Research Lab.

Outline

- Motivation

  — *What is Biot's equation; what is it good for; and why is it hard to solve*

- Splitting Methods and Preconditioners

  — *Outline of common splitting methods; and how may be formulated as block preconditioners*

- Experiments and results

  — *Test cases; convergence results*

- Summary

What is Biot's equation

- Couples *solid deformation* and *fluid pressure* in a fluid-filled porous material

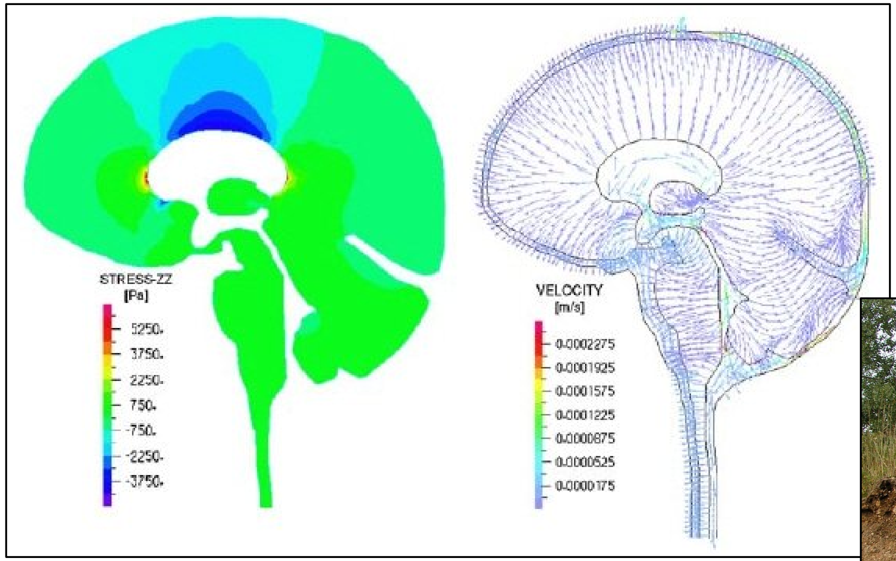
- (Linear case:)
  - Linear elasticity for the solid matrix (*A*)
  - Poisson's equation for the fluid pressure (*C*)
  - Coupled by Darcy's law and the concept of effective stress (*B*)

- Block structured saddle point problem:
  - *A* is positive
  - *C* is negative (or zero)

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

Why is Biot's equation hard to solve?

Very large range of material parameters, and large parameter contrasts



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[image: K. Van Rees]

[image: materialproject.org]

Splitting methods and Preconditioners

Solution methods for the coupled system

- Explicit coupling (or loose coupling)
  - Solve pressure at every time step
  - Update mechanics only occasionally
- Iterative coupling
  - Solve pressure and mechanics (separately) at every time step
  - Uses a splitting scheme
- Implicit coupling (or full coupling)
  - “Full iterations“ (of which iterative coupling is a special case)
  - OR other iterations at each time step (preconditioned Krylov, ...)
  - OR direct solver

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Notation

$$\underbrace{\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{pmatrix} d \\ p \end{pmatrix}}_{\mathrm{x}} = \underbrace{\begin{pmatrix} f_d \\ f_p \end{pmatrix}}_{\mathrm{f}}$$

Splitting schemes

The split-variable (d^*, p^*) is constrained in different ways in the split-step.

$$\begin{bmatrix} d_k \\ p_k \end{bmatrix} \rightarrow \begin{bmatrix} d_{k+1} \\ p^* \end{bmatrix} \rightarrow \begin{bmatrix} d_{k+1} \\ p_{k+1} \end{bmatrix}$$

Pressure split
(Drained, Undrained split)

$$\begin{bmatrix} d_k \\ p_k \end{bmatrix} \rightarrow \begin{bmatrix} d^* \\ p_{k+1} \end{bmatrix} \rightarrow \begin{bmatrix} d_{k+1} \\ p_{k+1} \end{bmatrix}$$

Displacement split
(Fixed strain, stress)

Drained split

1. The displacement is solved with constant pressure,
2. The pressure is then solved with fixed displacement:

$$\begin{bmatrix} d_k \\ p_k \end{bmatrix} \rightarrow \begin{bmatrix} d_{k+1} \\ p^* = p_k \end{bmatrix} \rightarrow \begin{bmatrix} d_{k+1} \\ p_{k+1} \end{bmatrix}$$

The drained split can be written as

$$\mathrm{x}_{k+1} = \mathrm{x}_k - \mathcal{P}_{\mathrm{DS}}^{-1}(\mathcal{A}\mathrm{x}_k - \mathrm{f}),$$

with

$$\mathcal{P}_{\mathrm{DS}}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -B^T A^{-1} & I \end{bmatrix}.$$

Drained split

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Preconditioned Richardson iteration

The drained split can be written as

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with

$$\mathcal{P}_{\mathrm{DS}}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -B^T A^{-1} & I \end{bmatrix}.$$

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with

$$\boxed{\mathcal{P}_{\mathrm{DS}} = \begin{bmatrix} A & 0 \\ B^T & C \end{bmatrix}}$$

Block triangular preconditioner

Drained split

$$\mathrm{x}_{k+1} = \mathrm{x}_k - \mathcal{P}_{\mathrm{DS}}^{-1}(\mathcal{A}\mathrm{x}_k - \mathrm{f}),$$

Relationship between iterative coupling and full iterations

- Iterative coupling is a *single iteration* of full iterations at each time step
- k is then the *time step*, rather than the *iteration number*
- It follows that for constant f, iterative coupling (in time) is the same as full iterations (within a time step).

$$\mathrm{x}_{k+1} = \mathrm{x}_k - \mathcal{P}_{\mathrm{DS}}^{-1}(\mathcal{A}\mathrm{x}_k - \mathrm{f}),$$

Common splitting schemes

Conditionally stable

$$\mathcal{P}_{\mathrm{DS}} = \begin{bmatrix} A & 0 \\ B^T & C \end{bmatrix}$$

$$\mathcal{P}_{\mathrm{FSa}} = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

Drained split | **Fixed strain**

Undrained split | **Fixed stress**

Stable

$$\mathcal{P}_{\mathrm{US}} = \begin{bmatrix} \hat{S}_A & 0 \\ B^T & C \end{bmatrix}$$

$$\mathcal{P}_{\mathrm{FSe}} = \begin{bmatrix} A & B \\ 0 & \hat{S}_C \end{bmatrix}$$

$$\hat{S}_A = A - Bb^{-1}B^T$$

$$\hat{S}_C = C + N_d\beta^{-1}I$$

approximate
displacement Schur complement

approximate
pressure Schur complement

Splitting schemes as general preconditioners

- The preconditioners derived from splitting schemes can be used with other, more efficient iterative methods (i.e., preconditioned Krylov methods).

- Two questions then arise:
  - Is the *block triangular structure* advantageous?
  - Are the *Schur complement approximations* advantageous?

Experiments and results

Implementation overview

- For the main experiments
  - Finite Element discretisation using **FENiCS**
  - Linear algebra and algebraic preconditioners/solvers from **Trilinos** (and the **MUMPS** sparse direct solver, via Trilinos)
  - Blockwise formulation and iterative solvers from **cbc.block**
  - Random initial iterate (to remove influence of load vector / Neumann BC)

- For supplementary experiments
  - Control Volume discretisation

cbc.block

$$\begin{bmatrix} A & 0 \\ B^T & S_C \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ 0 & S_C^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -B^T & I \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & I \end{bmatrix}$$

```
def pressure_schur(A, B, C):
    Ainv = MumpsSolver(A)
    Spre = MumpsSolver(collapse(C – B.T*InvDiag(A)*B))
    Sinv = BiCGStab(C – B.T*Ainv*B, precond=Spre)

    P1 = block_mat([[1,   0  ],
                    [0,  Sinv]])
    P2 = block_mat([[ 1,   0],
                    [-B.T, 1]])
    P2 = block_mat([[Ainv, 0],
                    [ 0,   1]])
    return P1*P2*P3
```

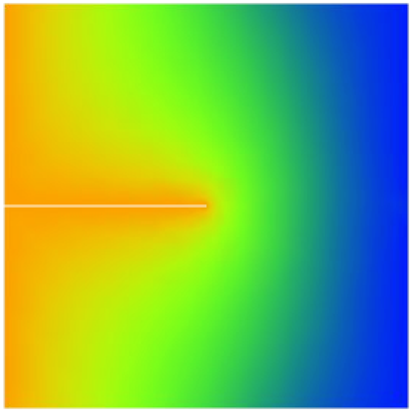
Test case 1: Hydraulic fracturing

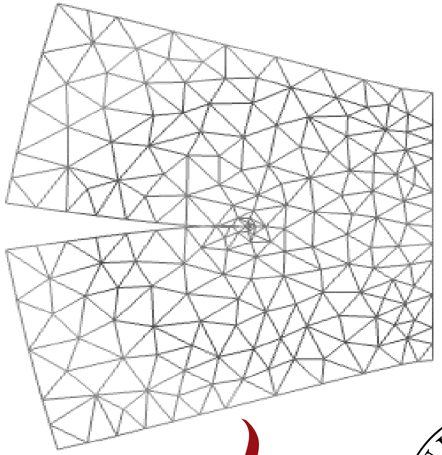
Problem statement

Domain: Unit square

Dirichlet boundary:

- $u_x = 0 \text{ on } \Gamma_{\text{right}}$
- $p = -x \text{ on } \Gamma,\ p = 1 \text{ on } \Gamma_{\text{frac}}$





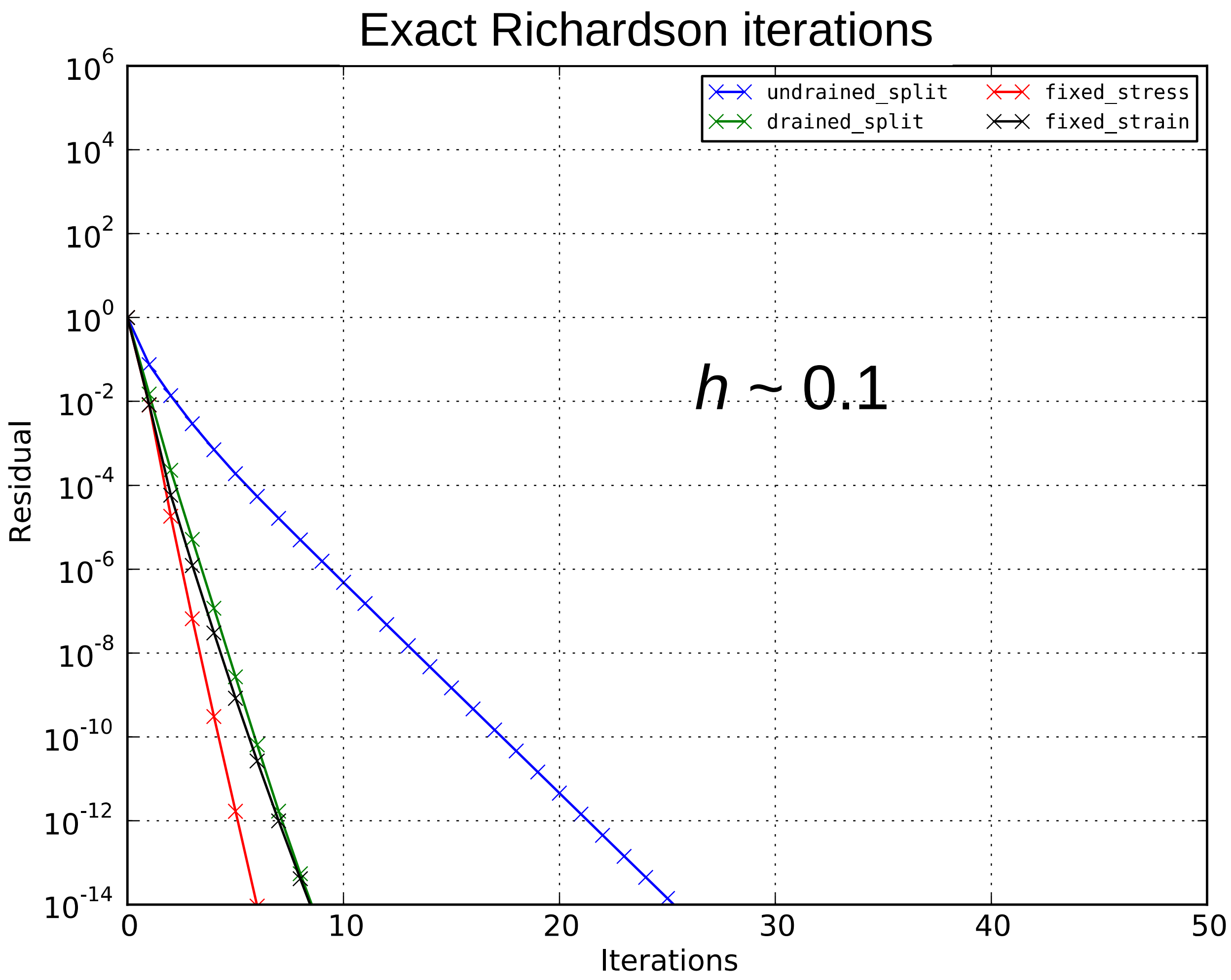
Parameters:

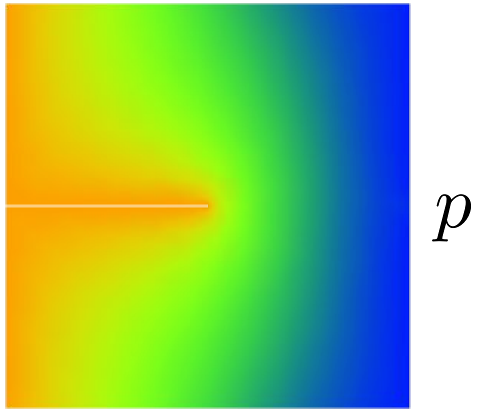
- Everything set to unity (in Lamé formulation)

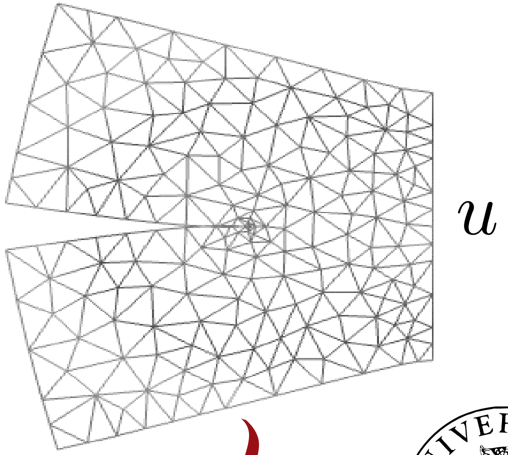
E	Young's modulus	ν	Poisson's ratio	Λ	Fluid mobility	b	Fluid storage coefficient	τ	Coupling strength
2.5		0.25		1		1		0.5	

Test case 1: Hydraulic fracturing

Full iterations converges fast

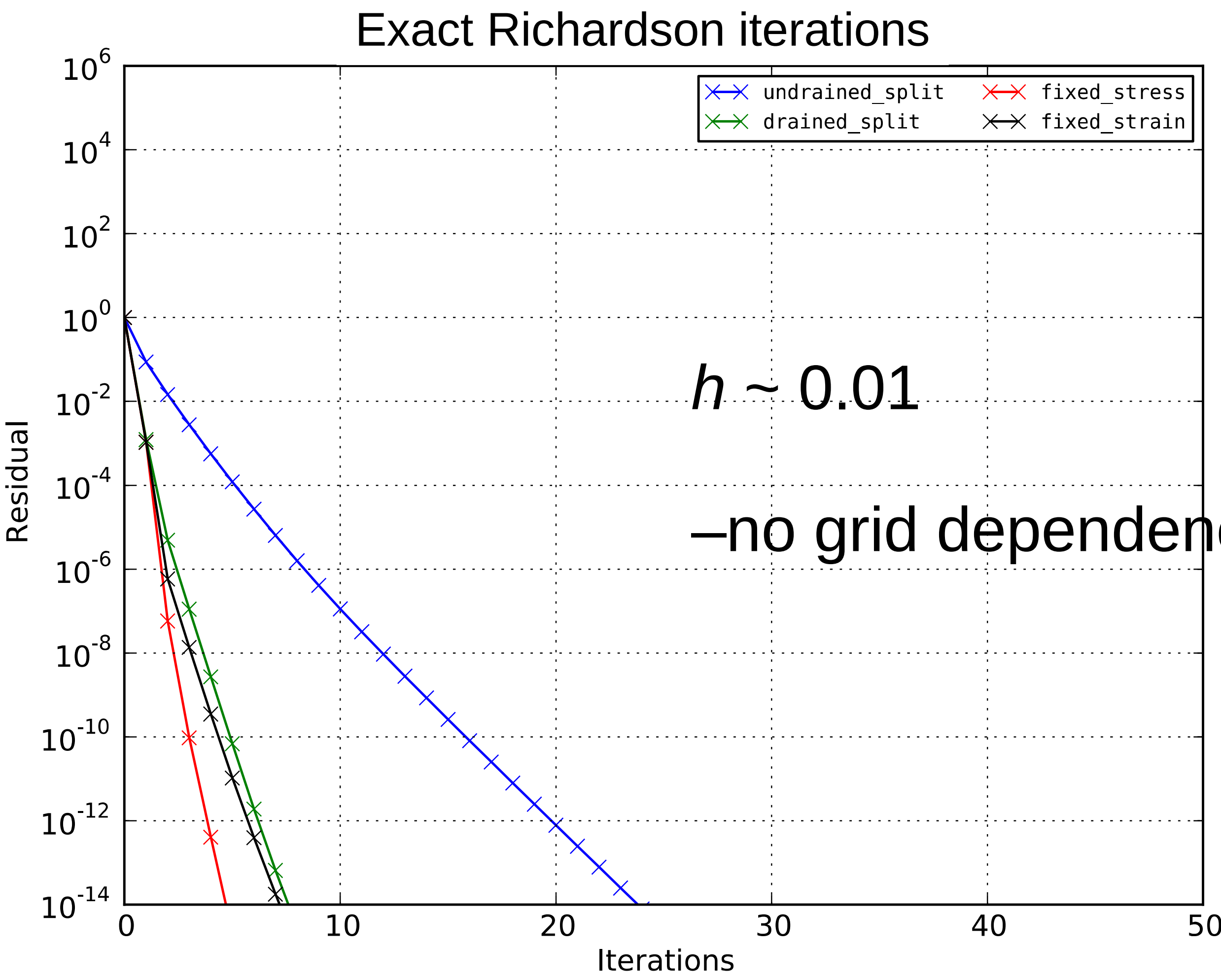






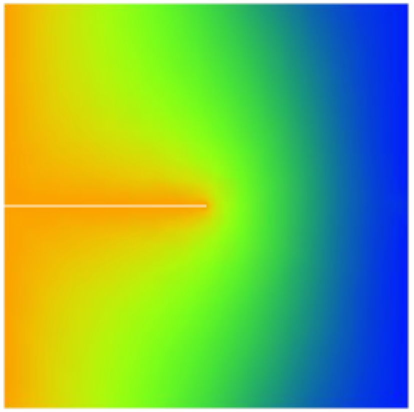
Test case 1: Hydraulic fracturing

Convergence is independent of grid resolution

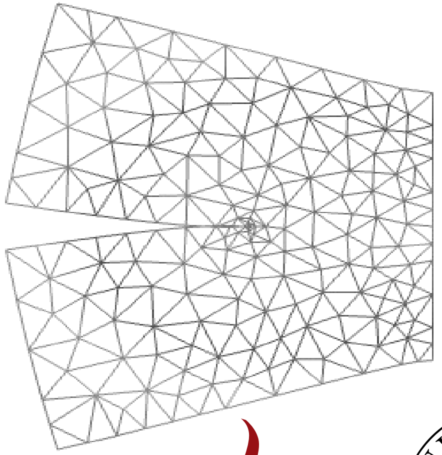


h ~ 0.01

–no grid dependence!



p



u

Test case 2: Two-dimensional spinal cord

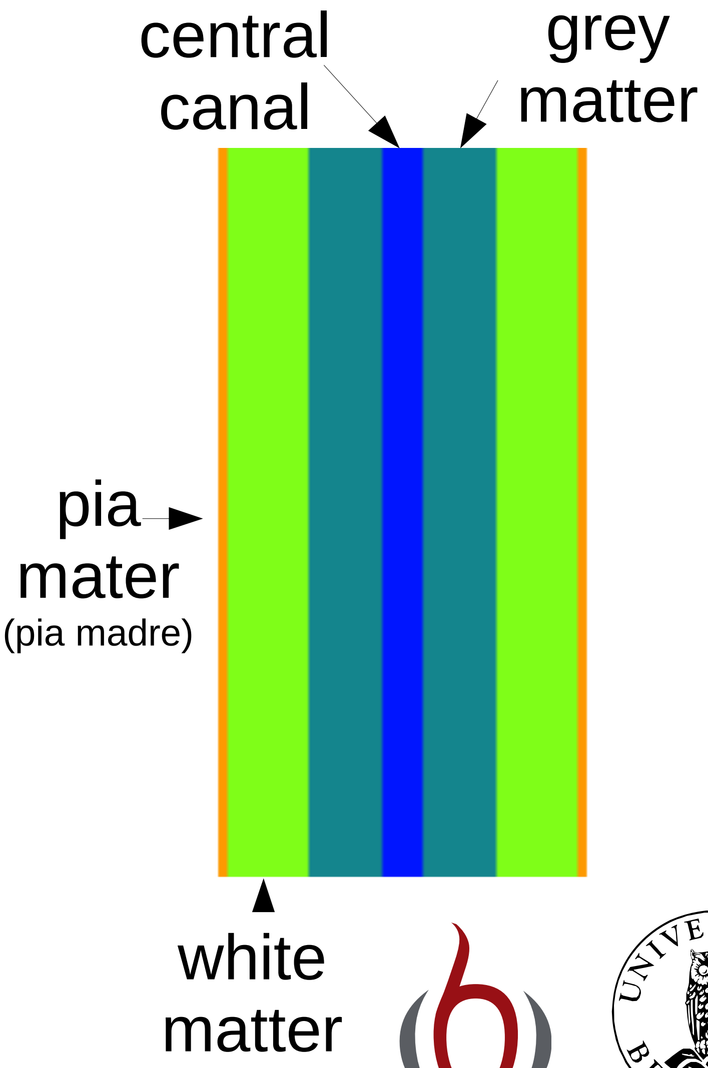
Problem statement

Dirichlet boundary:

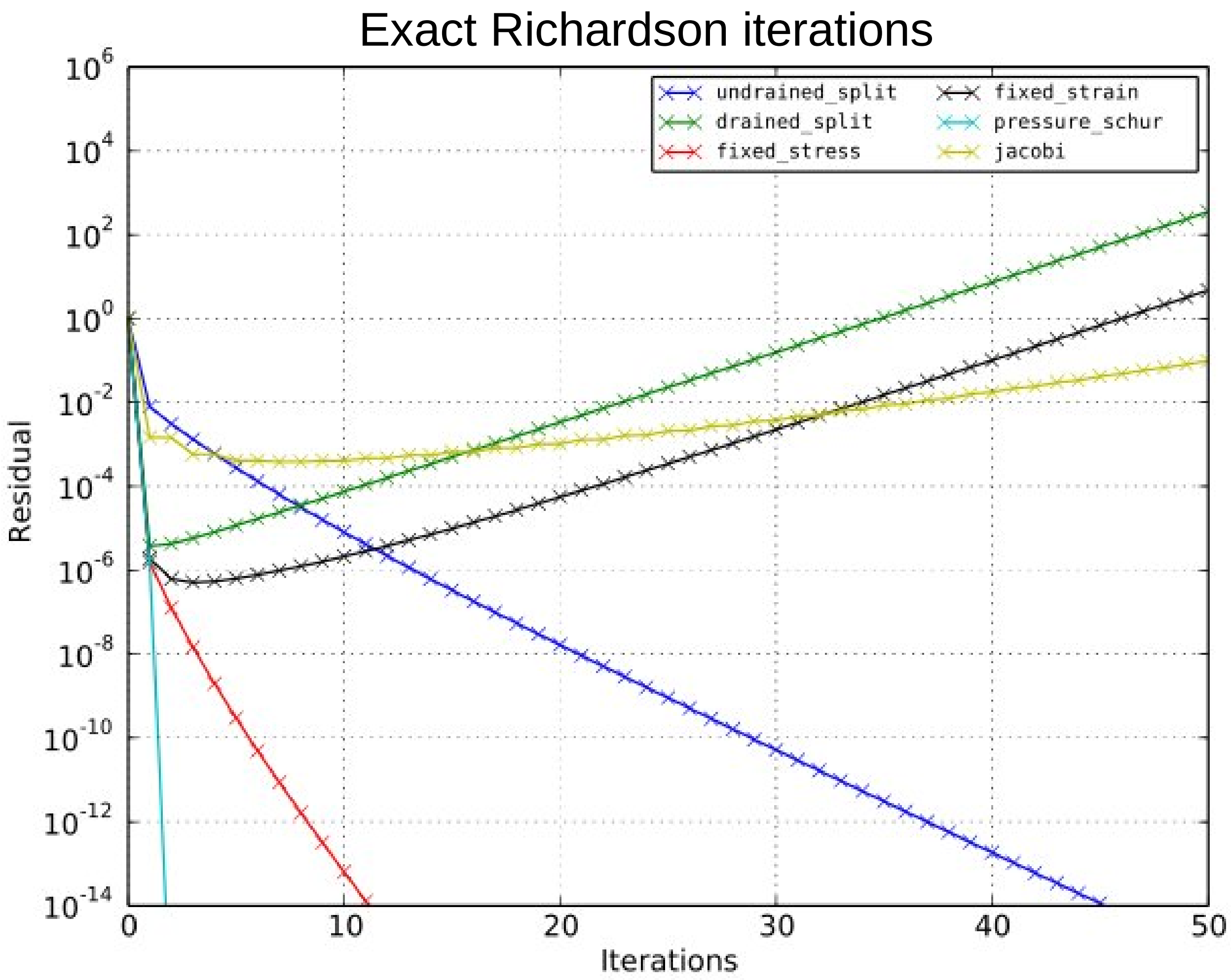
- $u = 0 \text{ on } \Gamma_{\text{pia}}, \; u_y = 0 \text{ on } \Gamma_{\text{top,bot}}$
- $p = p_{\text{applied}} \text{ on } \Gamma$

Parameters:

	E Young's modulus	ν Poisson's ratio	Λ Fluid mobility	b Fluid storage coefficient	τ Coupling strength
Pia mater	$2 \cdot 10^{4}$	0.35	$4 \cdot 10^{-6}$	$1 \cdot 10^{-5}$	4.5
White matter	$5 \cdot 10^{1}$	0.35	$2 \cdot 10^{-5}$	$1 \cdot 10^{-2}$	1.8
Grey matter	$5 \cdot 10^{1}$	0.35	$2 \cdot 10^{-6}$	$1 \cdot 10^{-2}$	1.8
Central canal	$5 \cdot 10^{1}$	0.35	$1 \cdot 10^{-3}$	$1 \cdot 10^{-2}$	1.8

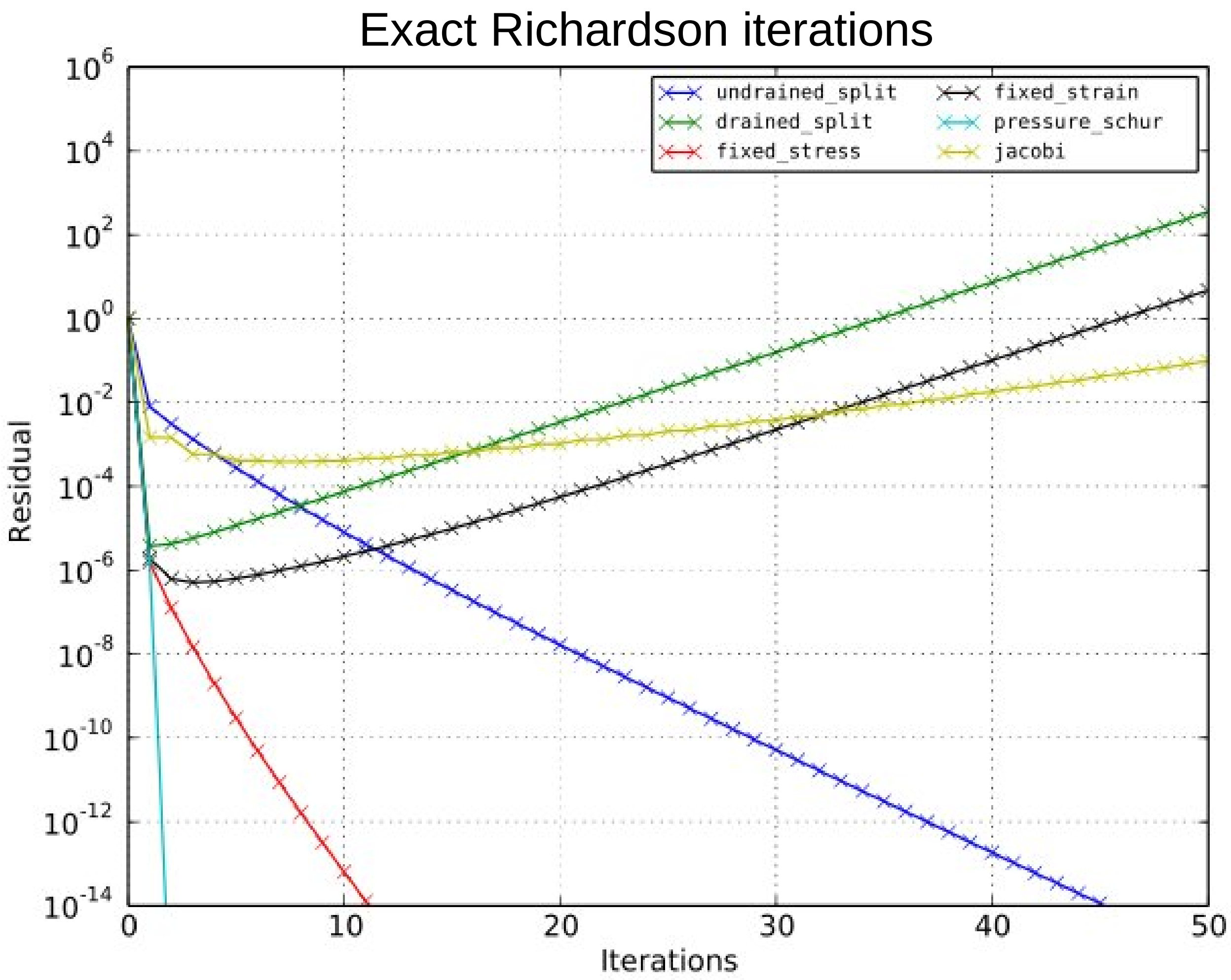


Test case 2: Two-dimensional spinal cord



- The two unstable splittings diverge – as expected.
- Two new preconditioners are introduced as benchmark:
  - Block Jacobi
  - Block triangular press. Schur

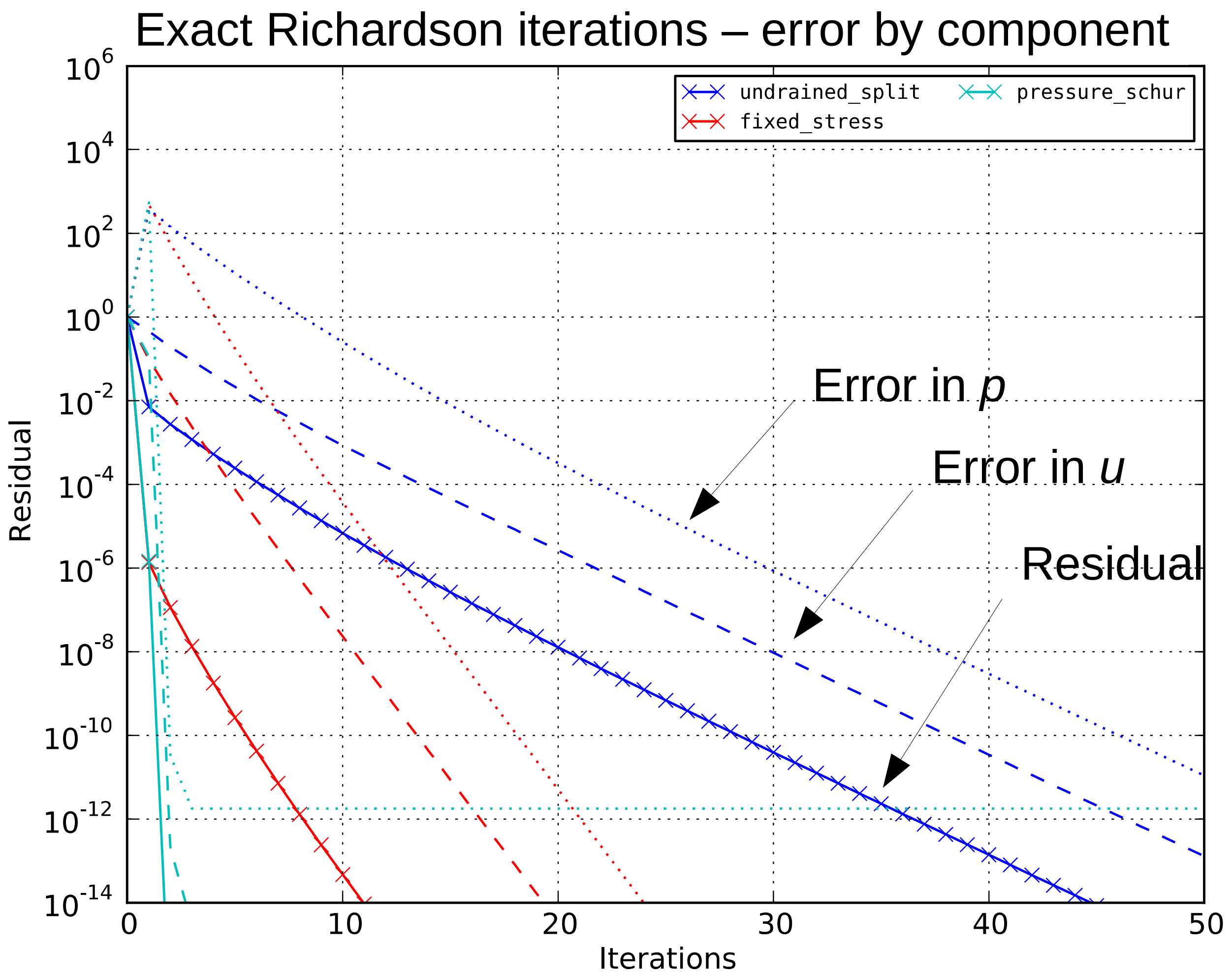
Test case 2: Two-dimensional spinal cord



- The two unstable splittings diverge – as expected.
- Two new preconditioners are introduced as benchmark:
  - Block Jacobi
  - Block triangular press. Schur
- But a digression:

Test case 2: Two-dimensional spinal cord

Real error may be much larger than the residual! (digression)



- Relative reduction of error in p is up to 9 orders of magnitude smaller than reduction of residual!
- The size of this effect is due to the random initial iterate, but something to be aware of.

Test case 2: Two-dimensional spinal cord

A few words about preconditioned Krylov-space solvers

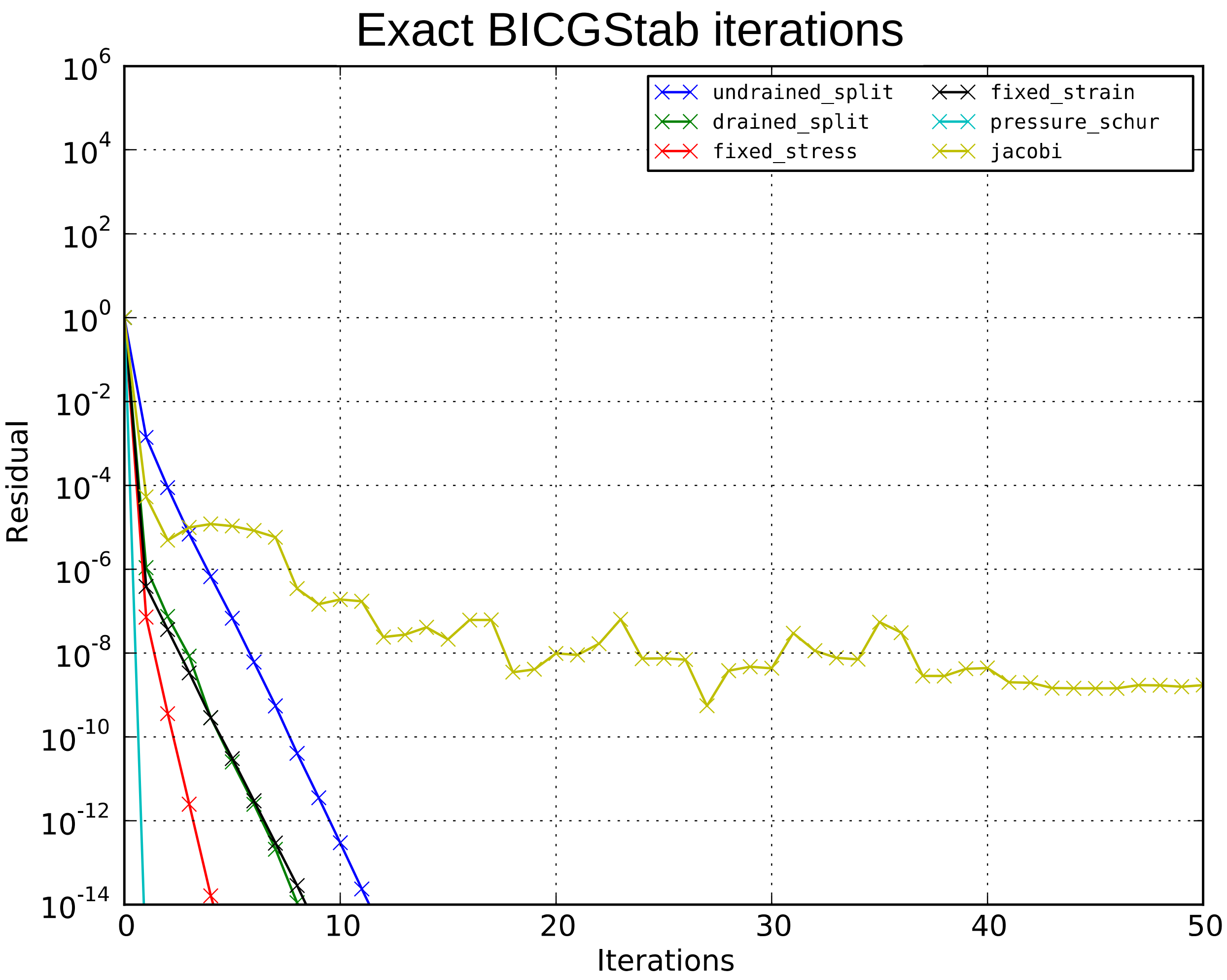
- $\mathcal{A}$ is indefinite symmetric,
- $\mathcal{P}$ is non-symmetric.

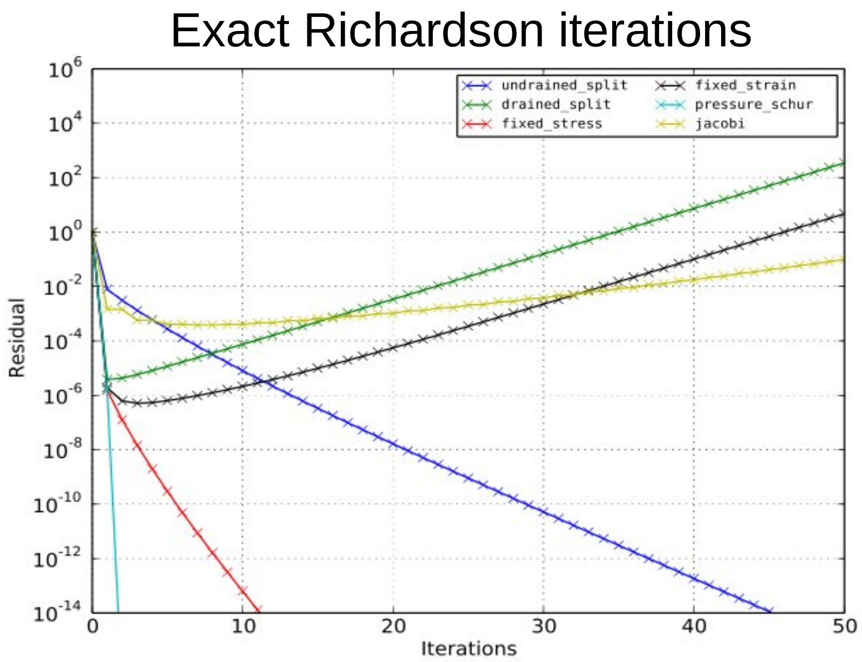
Options include general methods like BiCGStab or GMRES,
or symmetric methods like MinRes or even PCG in a non-standard norm.

- The latter usually places conditions on the initial iterate as well as the RHS.

The results presented here are with BiCGStab, but GMRES behaves similarly.

Test case 2: Two-dimensional spinal cord





Notice:

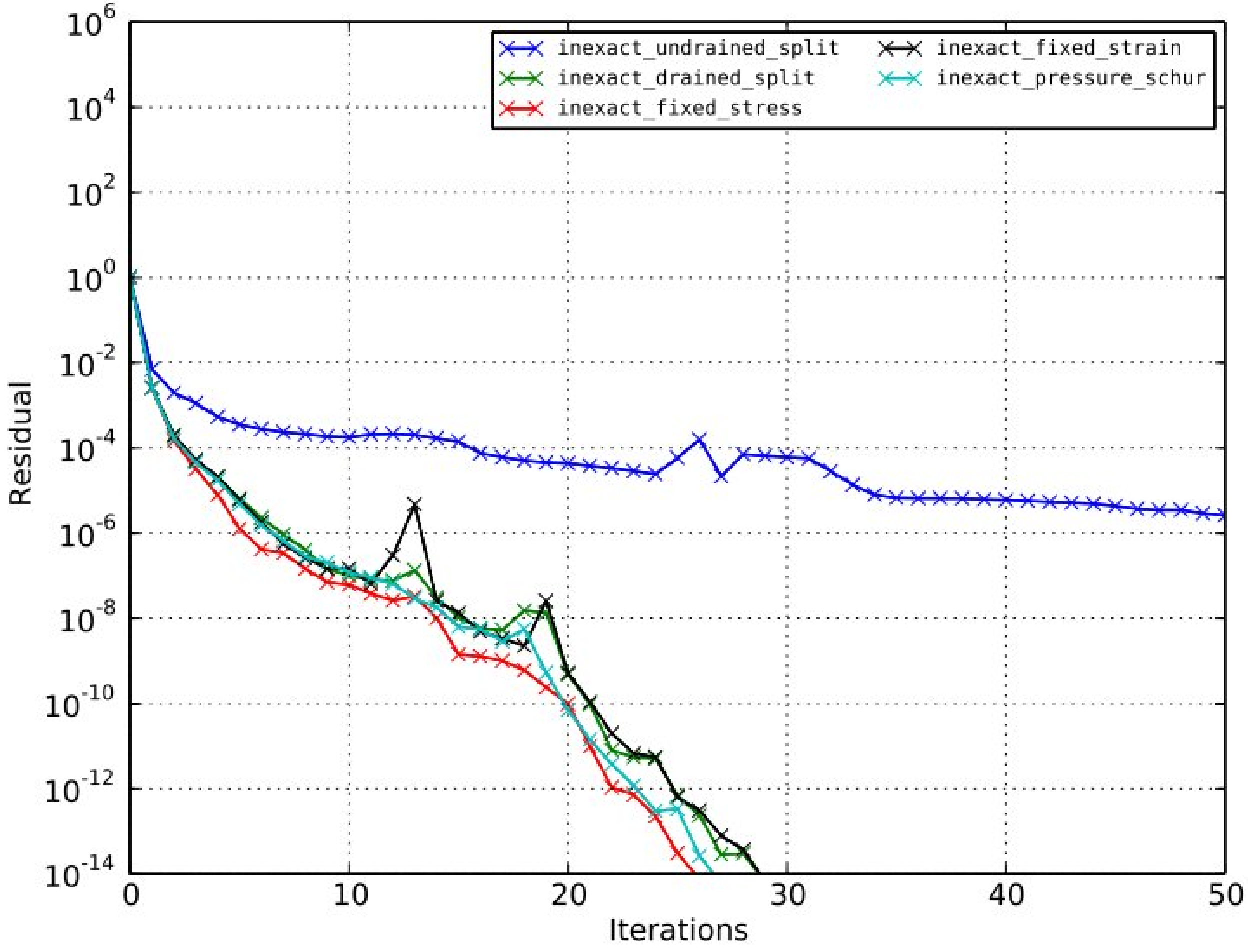
- Unstable splittings work well,
- Jacobi not so much.

Approximate inverses

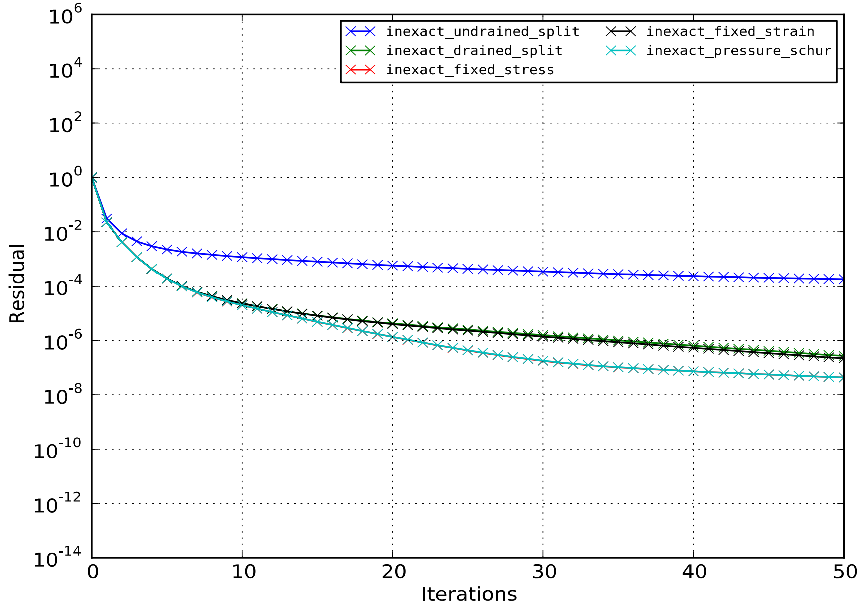
- $\hat{A}^{-1}, \hat{S}_A^{-1}$: ML (from Trilinos)
- $\hat{C}^{-1}, \hat{S}_C^{-1}$: ILUT (from Trilinos/IFPACK)
- The benchmark pressure Schur is: $\hat{S}_C = C - B(\mathrm{diag}A)^{-1}B^T$

Test case 2: Two-dimensional spinal cord

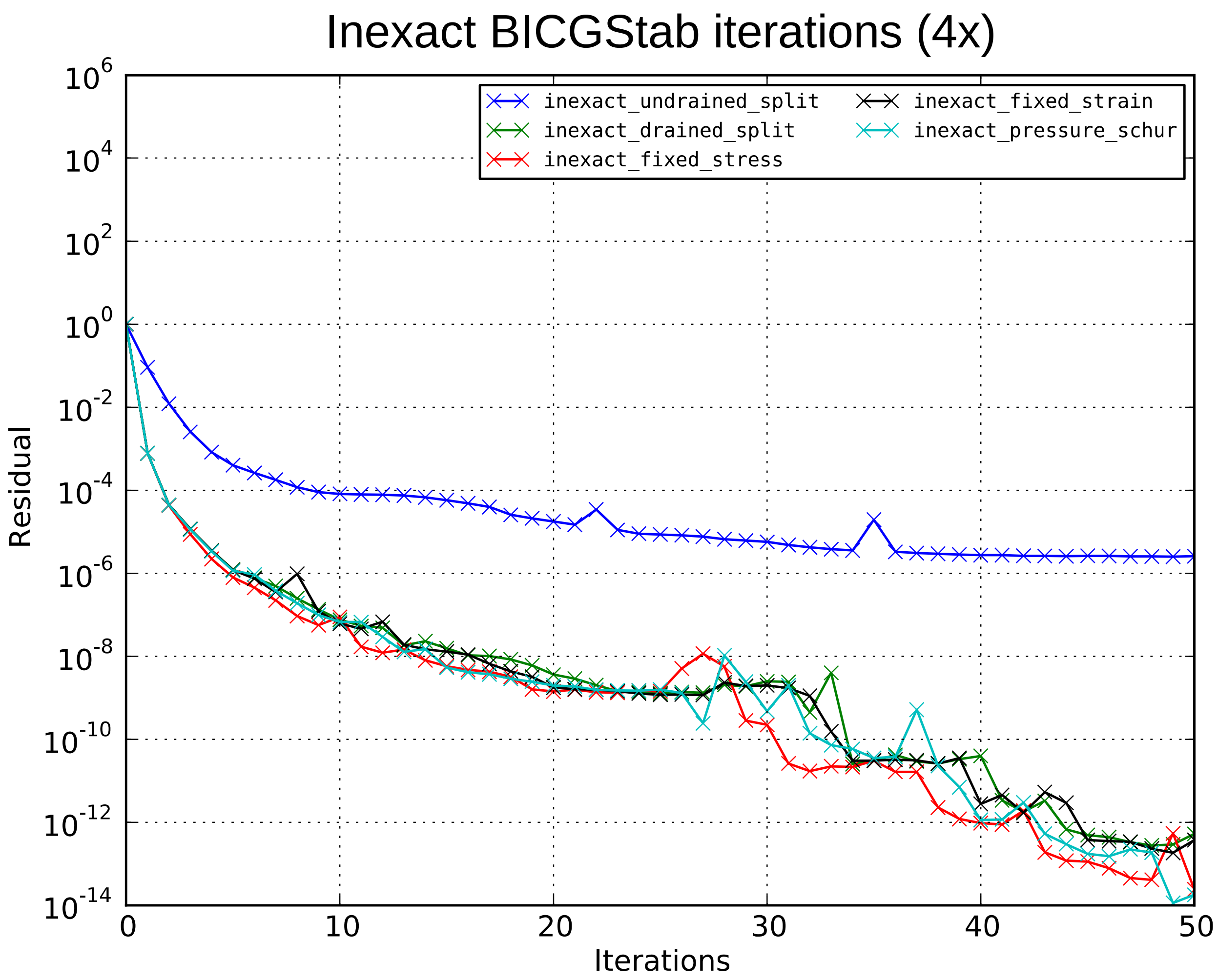
Inexact BICGStab iterations

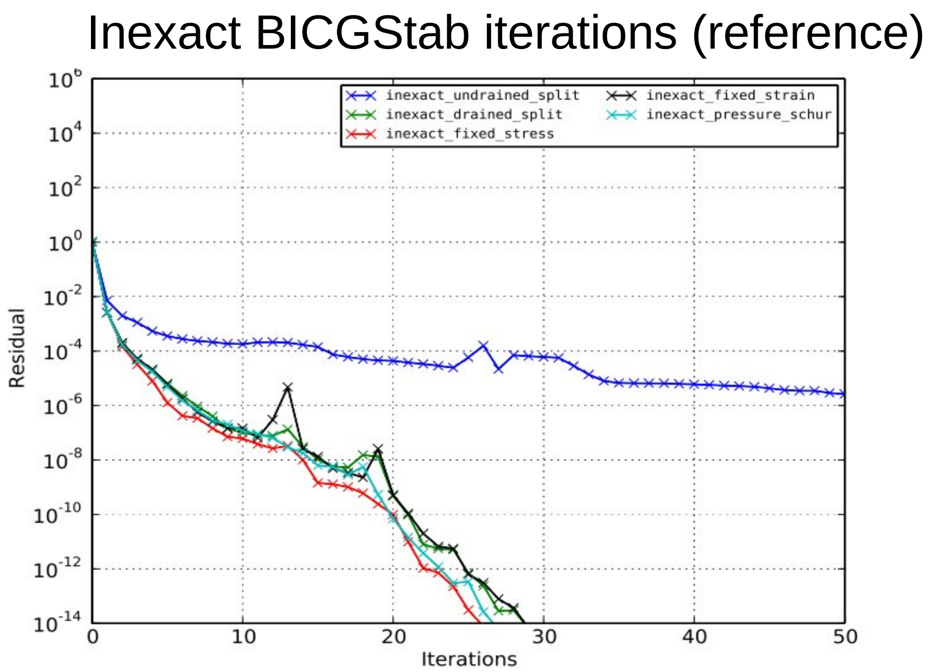


Inexact Richardson iterations



Test case 2: Two-dimensional spinal cord





We have lost grid independence!

- iterations ~ $h^{-1/2}$

Test case 3: Three dimensional spinal cord

Problem statement

Dirichlet boundary:

- $u = 0 \text{ on } \Gamma_{\text{pia}},\ u_y = 0 \text{ on } \Gamma_{\text{top,bot}}$
- $p = p_{\text{applied}} \text{ on } \Gamma$

Parameters (same as 2D):

	E Young's modulus	ν Poisson's ratio	Λ Fluid mobility	b Fluid storage coefficient	τ Coupling strength
Pia mater	$2\cdot10^{4}$	0.35	$4\cdot10^{-6}$	$1\cdot10^{-5}$	4.5
White matter	$5\cdot10^{1}$	0.35	$2\cdot10^{-5}$	$1\cdot10^{-2}$	1.8
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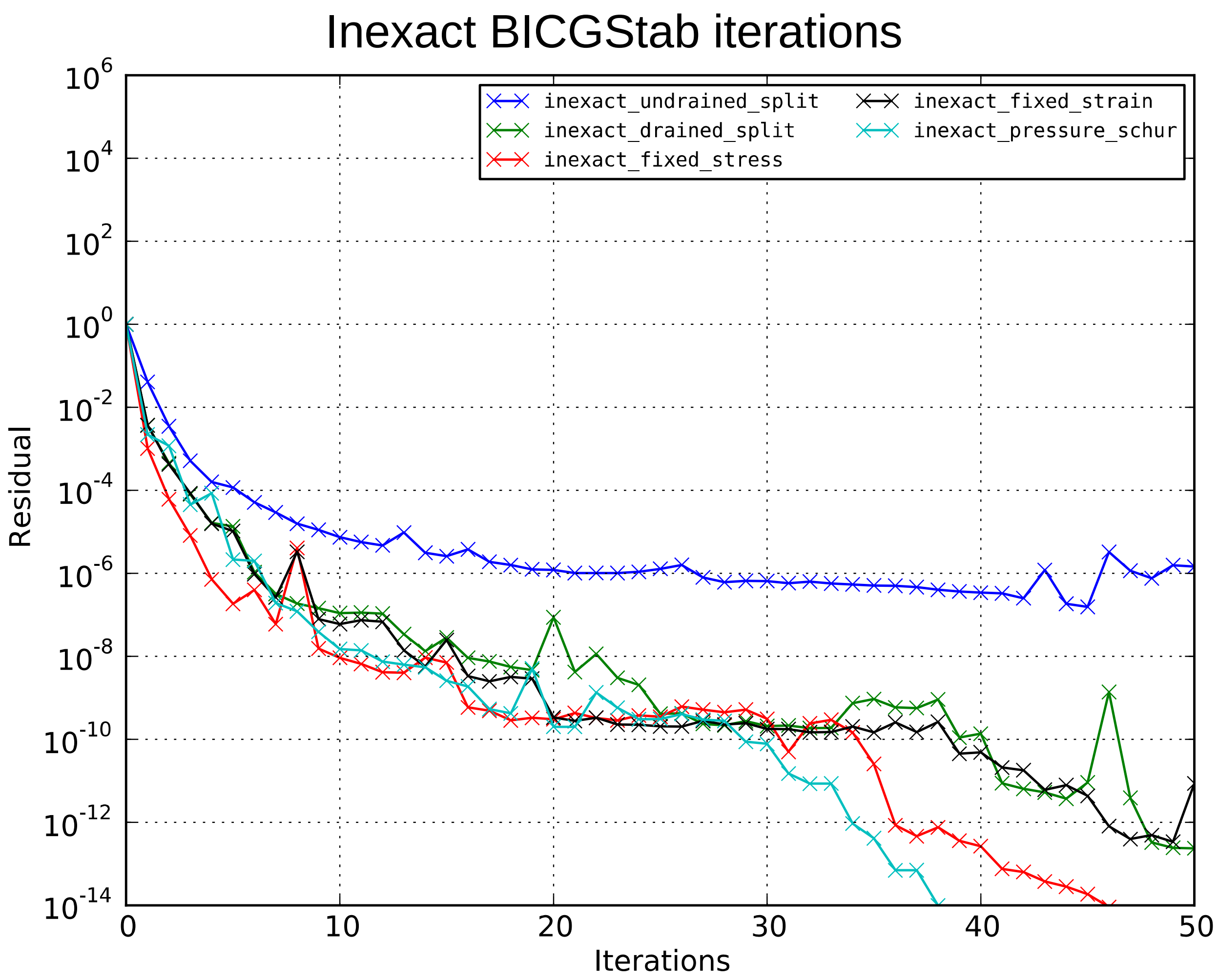
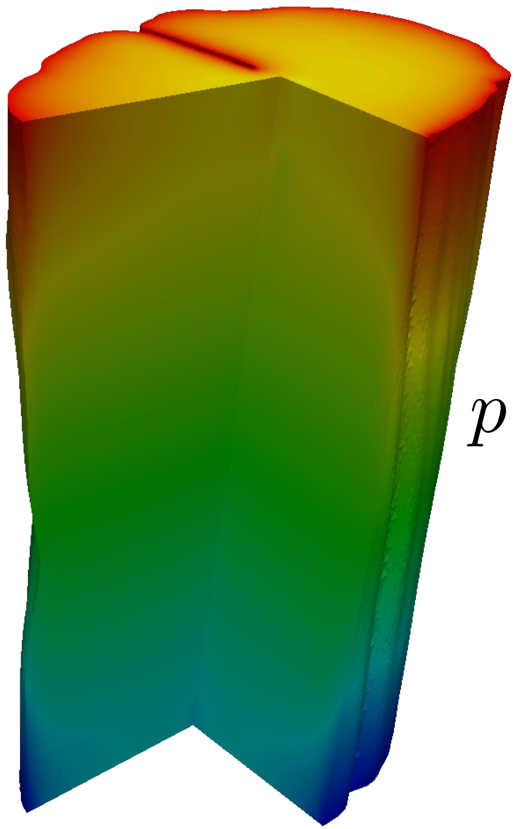
Central canal	$5\cdot10^{1}$	0.35	$1\cdot10^{-3}$	$1\cdot10^{-2}$	1.8

0.5M unknowns

Test case 3: Three dimensional spinal cord

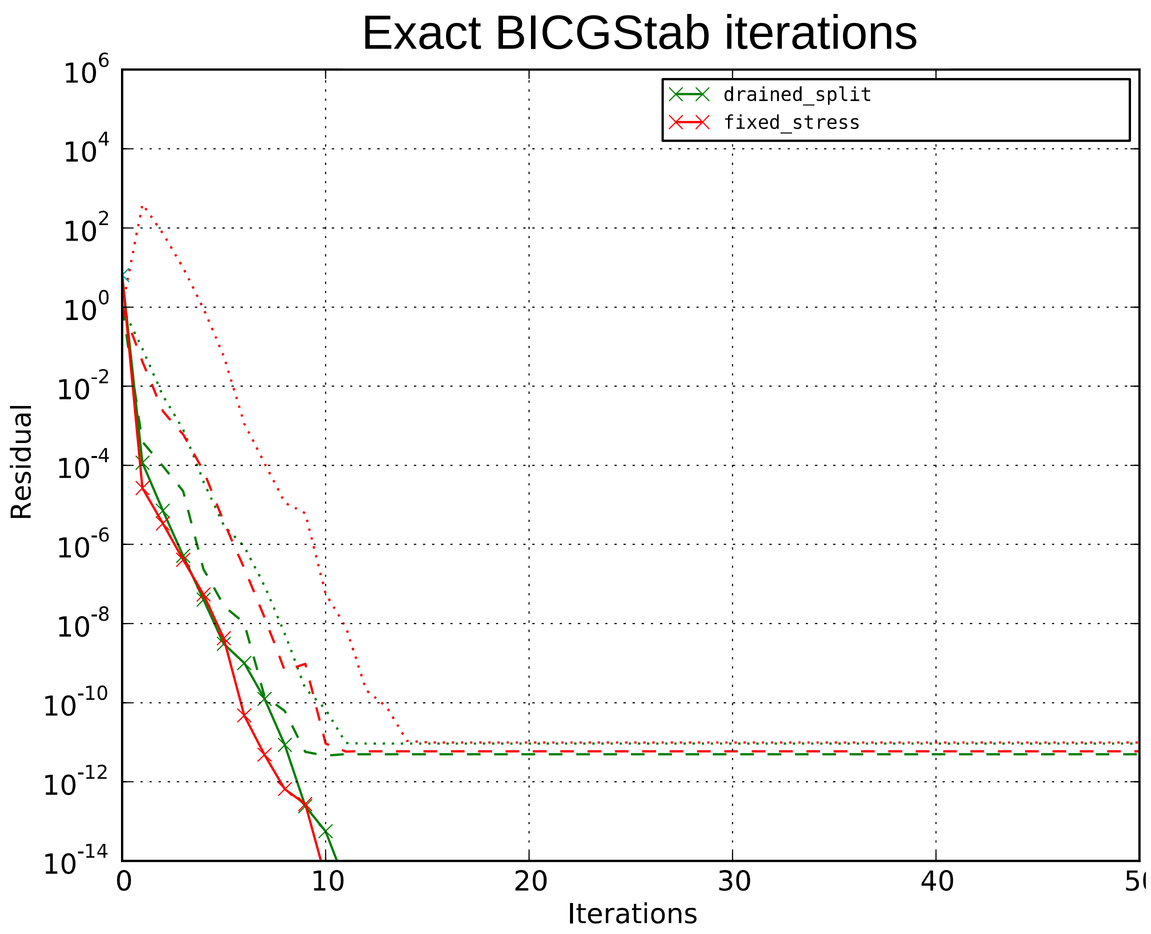


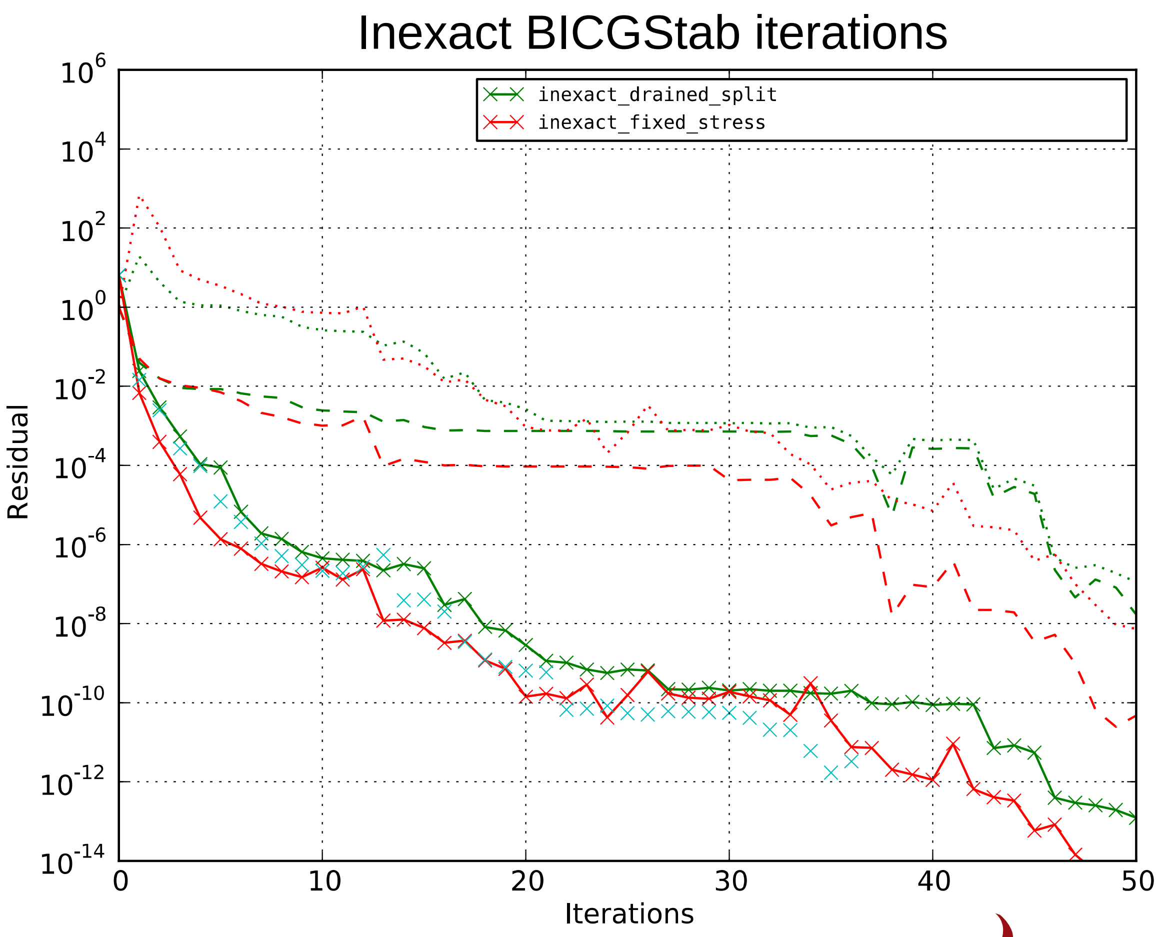
- All work fairly well, except undrained split.
- But another digression...



Test case 3: Three dimensional spinal cord

A final digression...





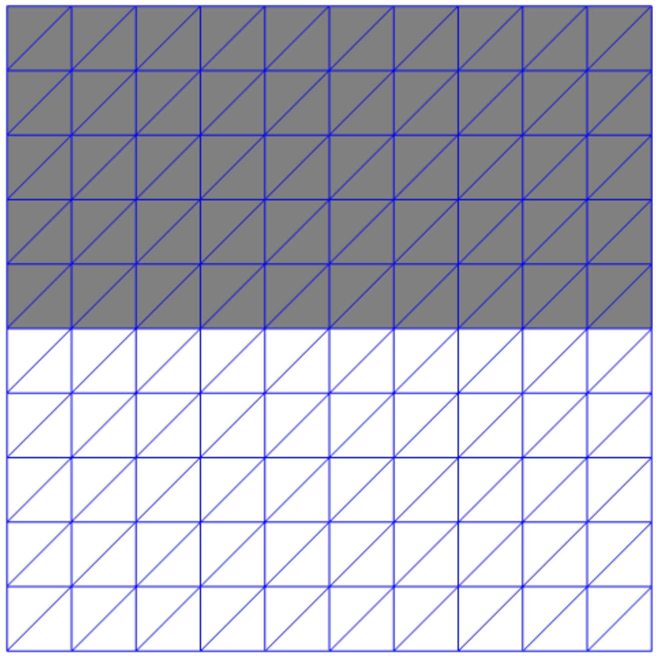
The error is significantly larger for FSe with exact solvers, but the difference disappears with inexact solvers.

Test case 4: Pore seal

Problem statement

Dirichlet boundary:

- $u = 0 \text{ on } \Gamma_{\text{bottom}}$

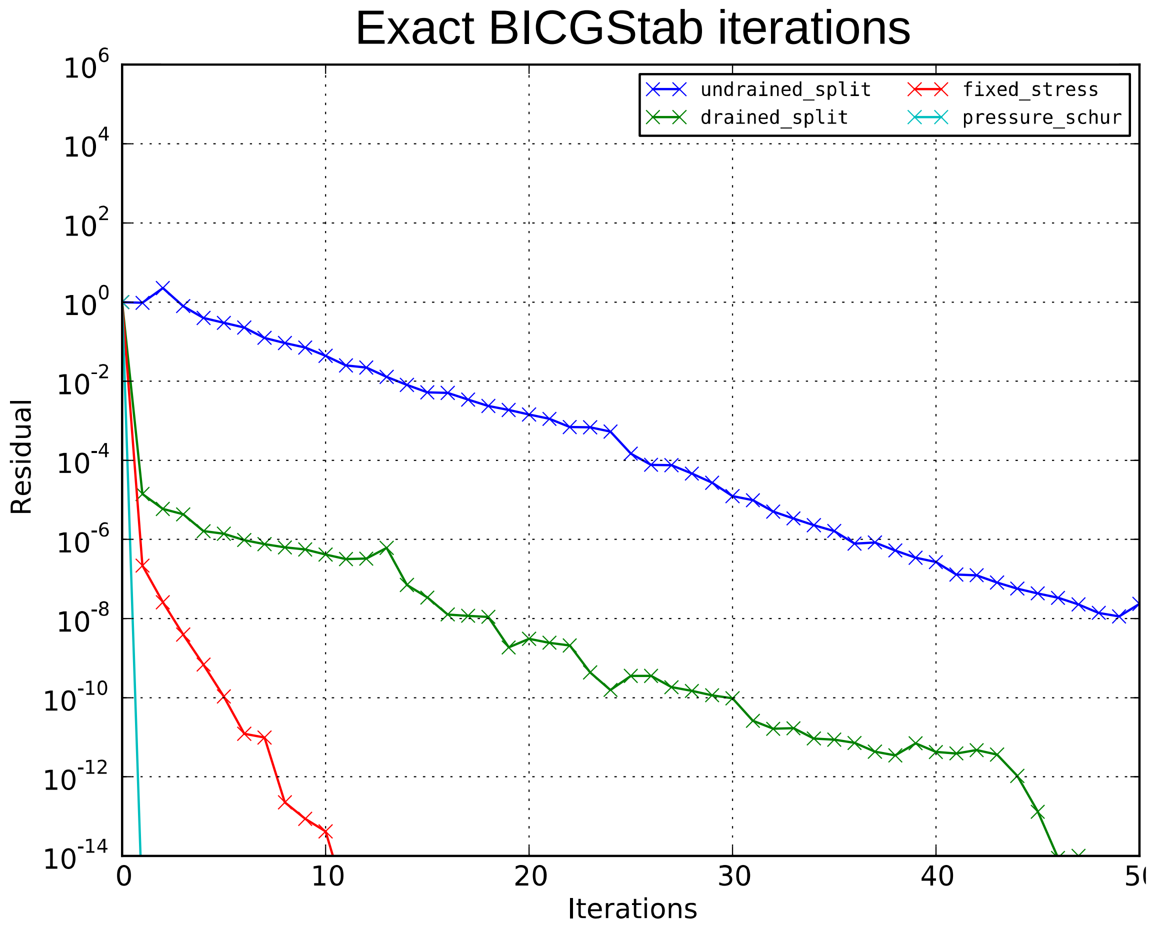


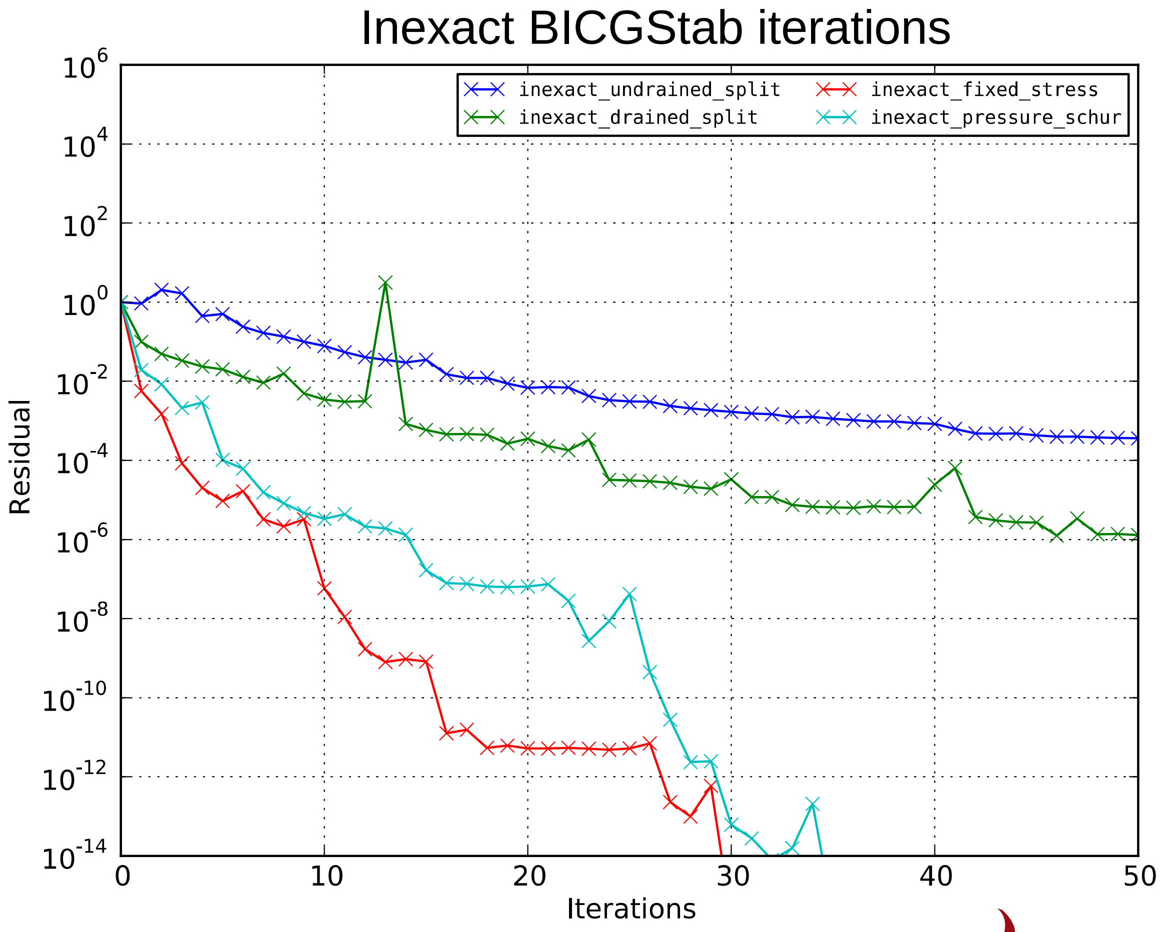
Parameters:

	E Young's modulus	ν Poisson's ratio	Λ Fluid mobility	b Fluid storage coefficient	τ Coupling strength
Top half	$3\cdot10^{3}$	0.45	$1\cdot10^{-8}$	$1\cdot10^{-6}$	91
Bottom half	$3\cdot10^{3}$	0.45	1	$1\cdot10^{-6}$	91

Designed to be hard!

Test case 4: Pore seal





Summary

- All the splitting methods can be rewritten as block triangular preconditioners.
- These work well for the preconditioned BiCGStab method (and for GMRES, results not shown), and will have significantly faster convergence than full iterations in „hard“ cases.
- The *block triangular structure* is advantageous; i.e., superior to the block Jacobi benchmark (as well as Schur-Jacobi, results not shown).
- The *pressure Schur approximation* of the Fixed Stress method seems to have comparable performance to the benchmark approximation when used with inexact solvers. The Schur formulation becomes important with really tough parameters.
- But be aware of the error!

Some references

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  - Pestana and Wathen. Combination preconditioning of saddle point systems for positive definiteness. *Report 11/13, Oxford University Mathematical Institute,* 2011.
- **FEniCS, cbc.block**
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  - http://fenics.org