Outline for estimating HMC model with Gibbs sampling

Model Parameters:

- d: Number of states
- n: number of observations

Notation:

- $y_{1:k}$: sample of observables from period 1 to k
- $a_{i:}$: row i of matrix A

Variables:

- Y_k: observable in period k. e.g. Inflation rate
- X_k: State in period k. e.g. {good, bad}
- $A = \{a_{ij}\}$: Transition matrix. $P(X_{k+1} = j | X_k = i)$
- $\rho = \{\rho_i\}$: $\{P(X_0 = i)\}$. Probabiliy of initial states
- $\pi_k(s) : P(X_k = s)$. State probabilties in the forward and backward recursions

Distribution for observables:

- $f(Y_k | X_k = i) = \mathcal{N}(\mu_i, \sigma_i^2)$

Priors:

- a_i: Dir(1,1,...,1)

- ρ: Dir(1,1,...,1)
- $\mu_i|\sigma_i^2$: $\mathcal{N}(\xi_i, \sigma_i^2/\nu_i)$
- σ_i^2: $\Gamma^{-1}(\alpha_i, \beta_i)$
- β_i: $\Gamma^{-1}(g, h)$

Hyper Parameter Values:

- R = max(Y) - min(Y)
- M = median(Y)
- ξ = [M-0.25*R, M, M+0.25*R]
- $\alpha = 1_d$
- $g = 0.2\ 1_d$
- $h = 10/R^2\ 1_d$
- $\nu = 0.1\ 1_d$

Conditional Distributions:

$$\rho|\ldots \sim Dir\big(1, 1, \ldots, 1\big) \tag{1}$$

$$a_{i:}|\ldots \sim Dir\big(1 + n_{i1}, 1 + n_{i2}, \ldots, 1 + n_{id}\big) \tag{2}$$

$$n_{ij} = \#\{1 < k \leq n : X_{k-1} = i, X_k = j\}$$

$$\text{i.e. number of transitions from state i to j}$$

$$(\mu_i, \sigma_i^2)|\ldots \sim \text{Normal-Inverse Gamma}(\xi, \nu, \alpha, \beta) \tag{3}$$

$$\sigma_i^2|\ldots \sim \Gamma^{-1}\left(\alpha + \frac{1}{2}n, \beta + \frac{1}{2}\sum_{k=1}^{n}(y_k - \overline{y}_k)^2 + \frac{n_i\nu}{2(n_i + \nu)}(\overline{y}_i - \xi_i)^2\right) \tag{4}$$

$$\mu_i|\sigma_i^2, \ldots \sim \mathcal{N}\left(\frac{n_i\overline{y}_k + \nu\xi_i}{n_i + \nu}, \frac{\sigma_i^2}{n_i + \nu}\right) \tag{5}$$

$$n_i = \#\{1 \leq k \leq n : X_k = i\}$$

$$\beta|\ldots \sim \Gamma^{-1}\left(g + \alpha_i, h + \sigma^2\right) \tag{6}$$

Forward Recursion:

Instead of sampling the states one at a time conditional on the parameters, observables and all other states, we calculate the joint distribution of all the state variables and then sample from the states. This increases the mixing speed of the chain. We calculate this joint probability recursively with a forwrad-backward algorithm.

The forward recusion generates state probabilities based on data only up to period k and is used mostly to calculate the terminal state probabilities. The forward recursion is calulated as follows.

$$p_{krs} \propto p(X_{k-1} = r, X_k = s|Y_{1:k}, \theta)$$

$$= \pi_{k-1}(r|\theta)a_{rs}P(Y_k|X_k = s, \theta)$$

$$\sum_r \sum_s p_{krs} = 1 \text{ Normalization}$$

$$\pi_k(s|\theta) = \sum_r p_{krs}$$

$$p_{1rs} = \rho_r a_{rs} P(Y|X_k = s, \theta)$$

Stochastic Backward Recursion:

The stochastic backward recustion calculates the state probabilties for each observation based on the entire sample.

$$
\begin{aligned}
P'_k &= (p'_{kij}) \\
p'_{krs} &= p(X_{k-1} = r, X_k = s | Y_{1:n}, \theta) \\
p'_{krs} &= p_{krs} \frac{\pi'_k(s|\theta)}{\pi_k(s|\theta)} \\
\pi'_{k-1}(r|\theta) &= \sum_s p'_{krs} \\
\pi'_N(r|\theta) &= \pi_N(r|\theta)
\end{aligned}
$$

Algorithm for drawing X_k:

This nonstochastic backward alogrithm gives us a sequence of states for a given Gibbs sweep instead of a sequence of state probabilities.

1. Draw X_N from π_N
2. Draw X_{N-1} from the Categorical distribution with probabilities propotional to the X_N column of P_N
3. Iterate backwards until full sample of X's have been sampled

Gibbs Sweep Steps

1. draw μ_i's and σ_i's from Normal-Inverse Gamma distribution via gibbs sampling.

2. Update β_i's

3. update ρ

4. update A

5. Forward update P and π

6. Backward update P' and π'

7. Reorder the states so that the μ_i's are in increasing order

8. Update X

Forecasts

To generate the "real time", h-period ahead foreacst for date k with no signals about the future, we estimate the model based on data from from the begining of the sample and ending on date k. After the Gibbs sampler has been run a sufficient number of times we calculate the state probabilites by averaging over the state probabilties calculated in the stochastic backward recursion during each Gibbs sweep. In particular, this gives us the estiamted state probabiity for date k, $\pi_k(s)$. We then calculate the probabilty of being in each state h periods ahead as $\pi_{k+h}(s) = \pi_k(s)A^h$ where $\pi_k(s)$ is a row vector of state probabilties. The point forecast for expected infaltion in period $k + h$ is calculated as $\mathbb{E}_k(Y_{k+h}) = \sum_{s\in S} \mathbb{E}_k(Y_{k+h}|X_{k+h} = s)\pi_{k+h}(s) = \sum_{s\in S} \mu_s \pi_{k+h}(s)$

Adding 1 Period Ahead Signals

In the model where the forecaster's information set in period k is the history of inflation and a 1 period ahead noisy signal we would need to make a few changes. We would essentially need to run the model for one extra period and treat the signal as another observation, making standard adjustments for calculating the mean and variance of each state's inflation for the fact that the last observation is a noisy signal. Tbe forward recursion calculation would also need to be adjusted for the fact that in period k we have a noisy signal so the emision probabilty $P(Y_{k+1}|X_{k+1} = s, \theta)$ is calculated differently than the first k emision probabilites.

For the forecasts we now have some information about the state $k+1$ so the model will give us an informed estimate for the state probabilties in period $k+1$, $\pi_{k+1}(s)$. To calculate the forecast for period $k+h$ we would do the same as before except $\pi_{k+h}(s) = \pi_{k+1}(s)A^{h-1}$.

Changes Made

In this section the state indexes i are supressed for clarity. The signal structure is assumed to be

$$
\begin{aligned}
S_k &= Y_k + \epsilon_k \\
Y_k &\sim N(\mu, \sigma^2) \\
\epsilon_k &\sim N(0, \kappa\sigma^2) \\
S_k &\sim N(\mu, (1+\kappa)\sigma^2)
\end{aligned}
$$

There are n direct measurments of y_k and m measurements of the noisy signal s_k

The postior for the mean of the observables is given by

$$
\begin{aligned}
\mu|\sigma^2 ... &\sim \mathcal{N}(\zeta, \omega) \\
\tau_y &= \frac{n}{\sigma^2} \\
\tau_0 &= \frac{\nu}{\sigma^2} \\
\tau_y &= \frac{m}{(1+\kappa)\sigma^2} \\
\zeta &\equiv \frac{\tau_y \overline{y} + \tau_0 \xi + \tau_s \overline{s}}{\tau_y + \tau_0 + \tau_s} \\
\zeta &\equiv \frac{n\overline{y} + \nu\xi + \frac{m}{1+\kappa}\overline{s}}{n + \nu + \frac{m}{1+\kappa}} \\
\omega &= \frac{\sigma^2}{n + \nu + \frac{m}{1+\kappa}}
\end{aligned}
$$

The postior for the variance σ^2 is

$$
\begin{aligned}
\sigma_i^2 | \ldots &\sim \Gamma^{-1}(\tilde{\alpha}, \tilde{\beta}) \\
\tilde{\alpha} &= \alpha + \frac{1}{2}n + \frac{1}{2}m \\
\tilde{\beta} &= \beta + \frac{1}{2}\sum_{k=1}^{n}(y_k - \overline{y}_k)^2 + \frac{1}{2(1+\kappa)}\sum_{k=n+1}^{m}(y_k - \overline{y})^2 + \frac{\left(n + \frac{m}{1+\kappa}\right)\nu}{2(n + \frac{m}{1+\kappa} + \nu)}(\overline{y} - \xi)^2
\end{aligned}
$$

In the forward recursion step we need to adjust the emission probabilites for the signals which is straight forwadly done by useing the modified variance of the signals.

$$
\begin{aligned}
y_k | x_k &= s, \theta \sim \mathcal{N}(\mu, \sigma^2) \\
s_k | x_k &= s, \theta \sim \mathcal{N}(\mu, (1+\kappa)\sigma^2)
\end{aligned}
$$