1 Notation

1. $i \in \{1, 2, 3\}$ denotes the state of the Markov process
2. Y_t is an observation of inflation at time t
3. $X_t \in \{1, 2, 3\}$ is the state of the Markov process at time t
4. Θ is a vector of all parameters; state means, state variances, transistion probabilties, and a $T \times 3$ matrix of state probabilties for each period (μ, σ^2, A, π)
5. $\Phi(\cdot)$ is the CDF of the standard Normal
6. F_t is the belief of the agent at time t for the distribution of the parmaters

We have the following by definition of the Markov process

$$Y|i \sim \mathcal{N}(\mu_i, \sigma_i^2)$$

$$\begin{aligned}
Pr_t(Y_t \leq y) &= \int_{\theta \in \Theta} Pr(Y_t \leq y|\theta) \, \mathrm{d}\, F_t(\theta) \\
Pr_t(Y_t \leq y) &= \int_{\theta \in \Theta} \Phi\left(\frac{y - \mu}{\sigma}\right) \mathrm{d}\, F_t(\theta) \\
Pr_t(Y_t \leq y) &= \mathbb{E}_{\theta \sim F_t}\left[\Phi\left(\frac{y - \mu}{\sigma}\right)\right]
\end{aligned}$$

When $t \leq T$, we have

$$\mathbb{E}_{\theta \sim F_t}\left[\Phi\left(\frac{y - \mu}{\sigma}\right)\right] = \mathbb{E}_{\theta \sim F_t} \sum_{i=1}^{3} \Phi\left(\frac{y - \mu_i}{\sigma_i}\right) \pi_t(i)$$

Now we want a function $f : \mathcal{R} \times \Delta(\Theta) \to \mathcal{R}$ such that for a given $F \in \Delta(\Theta)$, f maps a standard normal to Z. Let $Z \sim \mathcal{N}(0, 1)$, then we want a function f such that

$$Y_t = f(Z; F_t)$$

Note that

$$\begin{aligned}
Pr_t(Y_t \leq y) &= Pr(f(Z; F_t) \leq y) \\
Pr_t(Y_t \leq y) &= Pr(Z \leq f^{-1}(y; F_t)) \\
Pr_t(Y_t \leq y) &= \Phi\left(f^{-1}(y; F_t)\right)
\end{aligned}$$

Combining results we have

$$\Phi\left(f^{-1}(y;F_t)\right) = \mathbb{E}_{\theta\sim F_t}\left[\Phi\left(\frac{y-\mu_i}{\sigma_i}\right)\right]$$

Implying

$$f^{-1}(y;F_t) = \Phi^{-1}\left(\mathbb{E}_{\theta\sim F_t}\left[\Phi\left(\frac{y-\mu}{\sigma}\right)\right]\right)$$
$$f^{-1}(y;F_t) = \Phi^{-1}\left(\mathbb{E}_{\theta\sim F_t}\sum_{i=1}^{3}\Phi\left(\frac{y-\mu_i}{\sigma_i}\right)\pi_t(i)\right)$$

$$\mathbb{E}_{\theta\sim F_t}\sum_{i=1}^{3}\Phi\left(\frac{y-\mu_i}{\sigma_i}\right)\pi_t(i) \tag{1}$$

can be estimated via gibbs sampling. For each gibb's sweep calculate the term

$$\sum_{i=1}^{3}\Phi\left(\frac{y-\mu_i}{\sigma_i}\right)\pi_t(i)$$

and average over the sample of these draws. The function f^{-1} can then be estimated by calculating equation 1 over a grid of y.