

Outline for estimating HMC model with Gibbs sampling

Model Parameters:

- d : Number of states
- n : number of observations

Notation:

- $y_{1:k}$: sample of observables from period 1 to k
- a_i : row i of matrix A

Variables:

- Y_k : observable in period k . e.g. Inflation rate
- X_k : State in period k . e.g. {good, bad}
- $A = \{a_{ij}\}$: Transition matrix. $P(X_{k+1} = j | X_k = i)$
- $\rho = \{\rho_i\}$: $\{P(X_0 = i)\}$. Probabiliy of initial states
- $\pi_k(s) : P(X_k = s)$. State probabilties in the forward and backward recursions

Distribution for observables:

- $f(Y_k | X_k = i) = \mathcal{N}(\mu_i, \sigma_i^2)$

Priors:

- a_i : $\text{Dir}(1, 1, \dots, 1)$

- ρ : $\text{Dir}(1,1,\dots,1)$
- $\mu_i|\sigma_i^2$: $\mathcal{N}(\xi_i, \sigma_i^2/\nu_i)$
- σ_i^2 : $\Gamma^{-1}(\alpha_i, \beta_i)$
- β_i : $\Gamma^{-1}(g, h)$

Hyper Parameter Values:

- $R = \max(Y) - \min(Y)$
- $M = \text{median}(Y)$
- $\xi = [M-0.25*R, M, M+0.25*R]$
- $\alpha = 1_d$
- $g = 0.2 \ 1_d$
- $h = 10/R^2 \ 1_d$
- $\nu = 0.1 \ 1_d$

Conditional Distributions:

$$\rho | \dots \sim \text{Dir}(1, 1, \dots, 1) \quad (1)$$

$$a_{i:} | \dots \sim \text{Dir}(1 + n_{i1}, 1 + n_{i2}, \dots, 1 + n_{id}) \quad (2)$$

$$n_{ij} = \#\{1 < k \leq n : X_{k-1} = i, X_k = j\}$$

i.e. number of transitions from state i to j

$$(\mu_i, \sigma_i^2) | \dots \sim \text{Normal-Inverse Gamma}(\xi, \nu, \alpha, \beta) \quad (3)$$

$$\sigma_i^2 | \dots \sim \Gamma^{-1} \left(\alpha + \frac{1}{2}n, \beta + \frac{1}{2} \sum_{k=1}^n (y_k - \bar{y}_k)^2 + \frac{n_i \nu}{2(n_i + \nu)} (\bar{y}_i - \xi_i)^2 \right) \quad (4)$$

$$\mu_i | \sigma_i^2, \dots \sim \mathcal{N} \left(\frac{n_i \bar{y}_k + \nu \xi_i}{n_i + \nu}, \frac{\sigma_i^2}{n_i + \nu} \right) \quad (5)$$

$$n_i = \#\{1 \leq k \leq n : X_k = i\}$$

$$\beta | \dots \sim \Gamma^{-1} (g + \alpha_i, h + \sigma^2) \quad (6)$$

Forward Recursion:

Instead of sampling the states one at a time conditional on the parameters, observables and all other states, we calculate the joint distribution of all the state variables and then sample from the states. This increases the mixing speed of the chain. We calculate this joint probability recursively with a forward-backward algorithm.

The forward recursion generates state probabilities based on data only up to period k and is used mostly to calculate the terminal state probabilities. The forward recursion is calculated as follows.

$$\begin{aligned} p_{krs} &\propto p(X_{k-1} = r, X_k = s | Y_{1:k}, \theta) \\ &= \pi_{k-1}(r | \theta) a_{rs} P(Y_k | X_k = s, \theta) \\ \sum_r \sum_s p_{krs} &= 1 \text{ Normalization} \\ \pi_k(s | \theta) &= \sum_r p_{krs} \\ p_{1rs} &= \rho_r a_{rs} P(Y | X_k = s, \theta) \end{aligned}$$

Stochastic Backward Recursion:

The stochastic backward recursion calculates the state probabilities for each observation based on the entire sample.

$$\begin{aligned}P'_k &= (p'_{kij}) \\p'_{krs} &= p(X_{k-1} = r, X_k = s | Y_{1:n}, \theta) \\p'_{krs} &= p_{krs} \frac{\pi'_k(s|\theta)}{\pi_k(s|\theta)} \\\pi'_{k-1}(r|\theta) &= \sum_s p'_{krs} \\\pi'_N(r|\theta) &= \pi_N(r|\theta)\end{aligned}$$

Algorithm for drawing X_k :

This nonstochastic backward algorithm gives us a sequence of states for a given Gibbs sweep instead of a sequence of state probabilities.

1. Draw X_N from π_N
2. Draw X_{N-1} from the Categorical distribution with probabilities proportional to the X_N column of P_N
3. Iterate backwards until full sample of X 's have been sampled

Gibbs Sweep Steps

1. draw μ_i 's and σ_i 's from Normal-Inverse Gamma distribution via gibbs sampling.
2. Update β_i 's
3. update ρ
4. update A
5. Forward update P and π
6. Backward update P' and π'
7. Reorder the states so that the μ_i 's are in increasing order
8. Update X

Forecasts

To generate the “real time”, h -period ahead forecast for date k with no signals about the future, we estimate the model based on data from the beginning of the sample and ending on date k . After the Gibbs sampler has been run a sufficient number of times we calculate the state probabilities by averaging over the state probabilities calculated in the stochastic backward recursion during each Gibbs sweep. In particular, this gives us the estimated state probability for date k , $\pi_k(s)$. We then calculate the probability of being in each state h periods ahead as $\pi_{k+h}(s) = \pi_k(s)A^h$ where $\pi_k(s)$ is a row vector of state probabilities. The point forecast for expected inflation in period $k + h$ is calculated as
$$\mathbb{E}_k(Y_{k+h}) = \sum_{s \in S} \mathbb{E}_k(Y_{k+h} | X_{k+h} = s) \pi_{k+h}(s) = \sum_{s \in S} \mu_s \pi_{k+h}(s)$$

Adding 1 Period Ahead Signals

In the model where the forecaster's information set in period k is the history of inflation and a 1 period ahead noisy signal we would need to make a few changes. We would essentially need to run the model for one extra period and treat the signal as another observation,

making standard adjustments for calculating the mean and variance of each state's inflation for the fact that the last observation is a noisy signal. The forward recursion calculation would also need to be adjusted for the fact that in period k we have a noisy signal so the emission probability $P(Y_{k+1}|X_{k+1} = s, \theta)$ is calculated differently than the first k emission probabilities.

For the forecasts we now have some information about the state $k + 1$ so the model will give us an informed estimate for the state probabilities in period $k + 1$, $\pi_{k+1}(s)$. To calculate the forecast for period $k + h$ we would do the same as before except $\pi_{k+h}(s) = \pi_{k+1}(s)A^{h-1}$.

Model Changes Made

In this section the state indexes i are suppressed for clarity. The noiseness of the signal is assumed to be proportional to the variance of inflation in the state the signal is drawn from which allows for closed form solution. The signal structure is assumed to be

$$\begin{aligned} S_k &= Y_k + \epsilon_k \\ Y_k &\sim N(\mu, \sigma^2) \\ \epsilon_k &\sim N(0, \kappa\sigma^2) \\ S_k &\sim N(\mu, (1 + \kappa)\sigma^2) \end{aligned}$$

There are n direct measurements of y_k and m measurements of the noisy signal s_k

The posterior for the mean of the observables is given by

$$\begin{aligned} \mu|\sigma^2 \dots &\sim \mathcal{N}(\zeta, \omega) \\ \tau_y &= \frac{n}{\sigma^2} \\ \tau_0 &= \frac{\nu}{\sigma^2} \\ \tau_y &= \frac{m}{(1 + \kappa)\sigma^2} \\ \zeta &\equiv \frac{\tau_y \bar{y} + \tau_0 \xi + \tau_s \bar{s}}{\tau_y + \tau_0 + \tau_s} \\ \zeta &\equiv \frac{n \bar{y} + \nu \xi + \frac{m}{1+\kappa} \bar{s}}{n + \nu + \frac{m}{1+\kappa}} \\ \omega &= \frac{\sigma^2}{n + \nu + \frac{m}{1+\kappa}} \end{aligned}$$

The postior for the variance σ^2 is

$$\begin{aligned}\sigma_i^2 | \dots &\sim \Gamma^{-1}(\tilde{\alpha}, \tilde{\beta}) \\ \tilde{\alpha} &= \alpha + \frac{1}{2}n + \frac{1}{2}m \\ \tilde{\beta} &= \beta + \frac{1}{2} \sum_{k=1}^n (y_k - \bar{y}_k)^2 + \frac{1}{2(1+\kappa)} \sum_{k=n+1}^m (y_k - \bar{y})^2 + \frac{(n + \frac{m}{1+\kappa}) \nu}{2(n + \frac{m}{1+\kappa} + \nu)} (\bar{y} - \xi)^2\end{aligned}$$

In the forward recursion step we need to adjust the emission probabilities for the signals which is straight forwardly done by using the modified variance of the signals.

$$\begin{aligned}y_k | x_k = s, \theta &\sim \mathcal{N}(\mu, \sigma^2) \\ s_k | x_k = s, \theta &\sim \mathcal{N}(\mu, (1 + \kappa)\sigma^2)\end{aligned}$$

Estimating Models with Noise

To calculate the 12-month ahead forecast for inflation with a signal about inflation in the following month for a sample ending in January 2000 I did the following.

First I estimated a model with data up to February 2000, treating the February data as an observable. I used this sample to estimate the variance of the signals and took the equally weighted mean of the variances across states, i.e.

$$\bar{\sigma}^2 = \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{3}$$

where σ_i^2 is the postior mean of the variances across states.

I then ran a loop 300 times wherein I constructed the sample with noise by drawing a standard normal variable ϵ and then constructed the noisy signal as

$$S_{Feb,2000} = Y_{Feb,2000} + \bar{\sigma}\kappa\epsilon$$

where $\kappa \in \{1, 3, 7\}$ depending on the noise level.

$Y_{Feb,2000}$ was replaced by $S_{Feb,2000}$ and the model was estimated taking into account the noise in $S_{Feb,2000}$. The 12-month ahead forecast was then calculated.