# All The Calculus You Need

## POLS 7012: Introduction to Political Methodology

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# Warm Up

- What is a **function**?
    - A **function** takes one or more inputs and produces an output.

- If $f(x) = 2x^2 + 2x + 3$, what is $f(2)$?

$$f(2) = 2(2)^2 + 2(2) + 3 = 8 + 4 + 3 = 15$$

- $y = 2x + 1$ and $y = 10 - x$. Solve for $x$ and $y$.

$$10 - x = 2x + 1$$

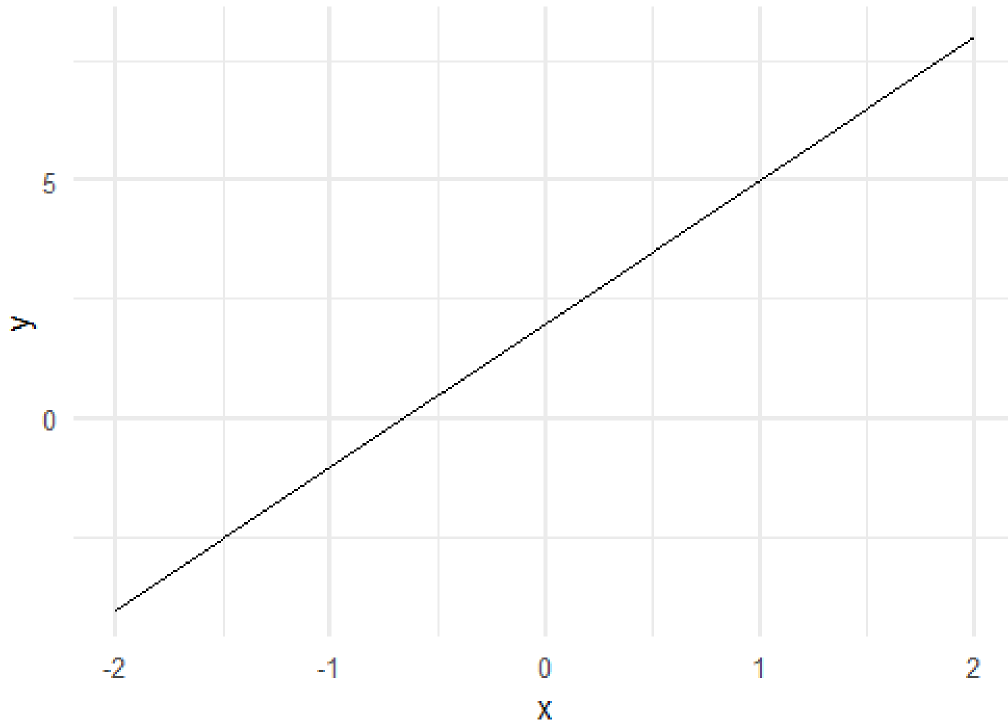
$$3x = 9$$

$$x = 3$$

$$y = 7$$

# Warm Up

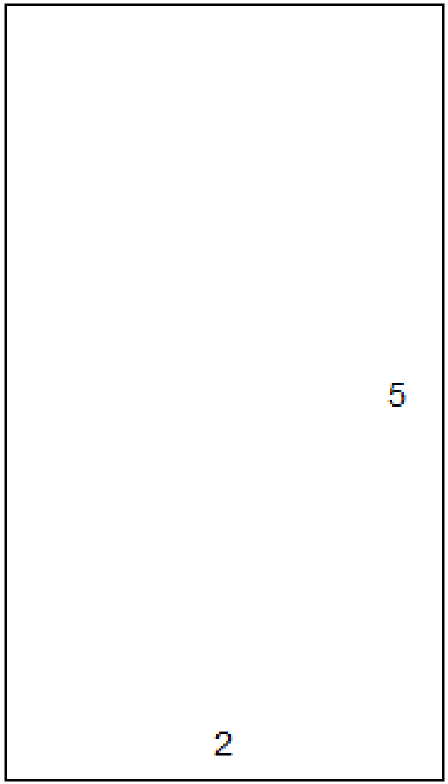
What is the **slope** of $f(x) = 3x + 2$?




The slope of a linear function (a straight line) is measured by how much $y$ increases when you increase $x$ by $1$. In this case, $3$.

# Warm Up

What is the **area** of this rectangle?




$\text{Area} = bh = 2 \times 5 = 10$

**Good news!** You're super close to knowing all the calculus you need.

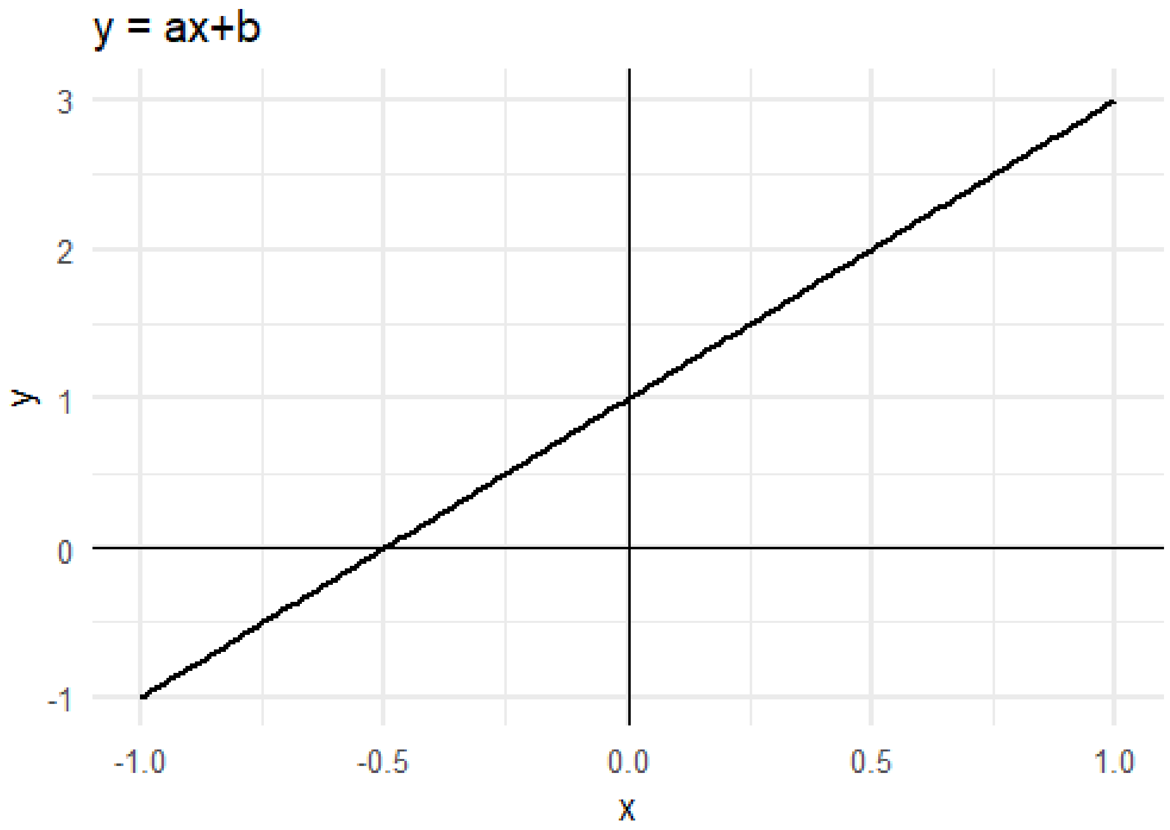
# Learning Objectives

By the end of this week, you will be able to:

- Find the slope of a function at any point
- Identify the *minimum* or *maximum* value of a function
- Compute the area under a curve
- Explain the Fundamental Theorem of Calculus

# Slopes

# Linear Functions



# Linear Functions

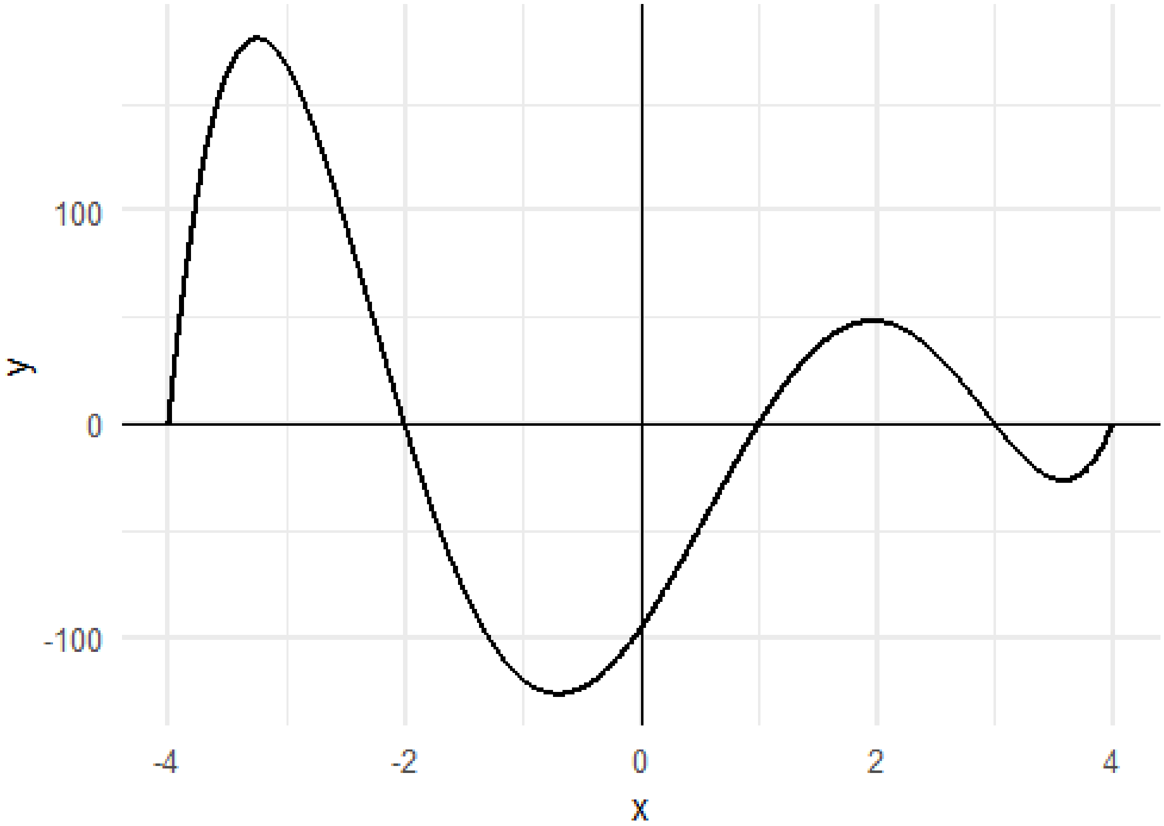
Find the slope of each function:

- $y = 2x + 4$
- $f(x) = \frac{1}{2}x - 2$
- life expectancy (years) = 18.09359 + 5.737335 $\times$ log(GDP per capita)

## Remember:

Slope of a line $= \frac{rise}{run} = \frac{\Delta Y}{\Delta X} = \frac{f(x+h)-f(x)}{h}$

# Nonlinear Functions



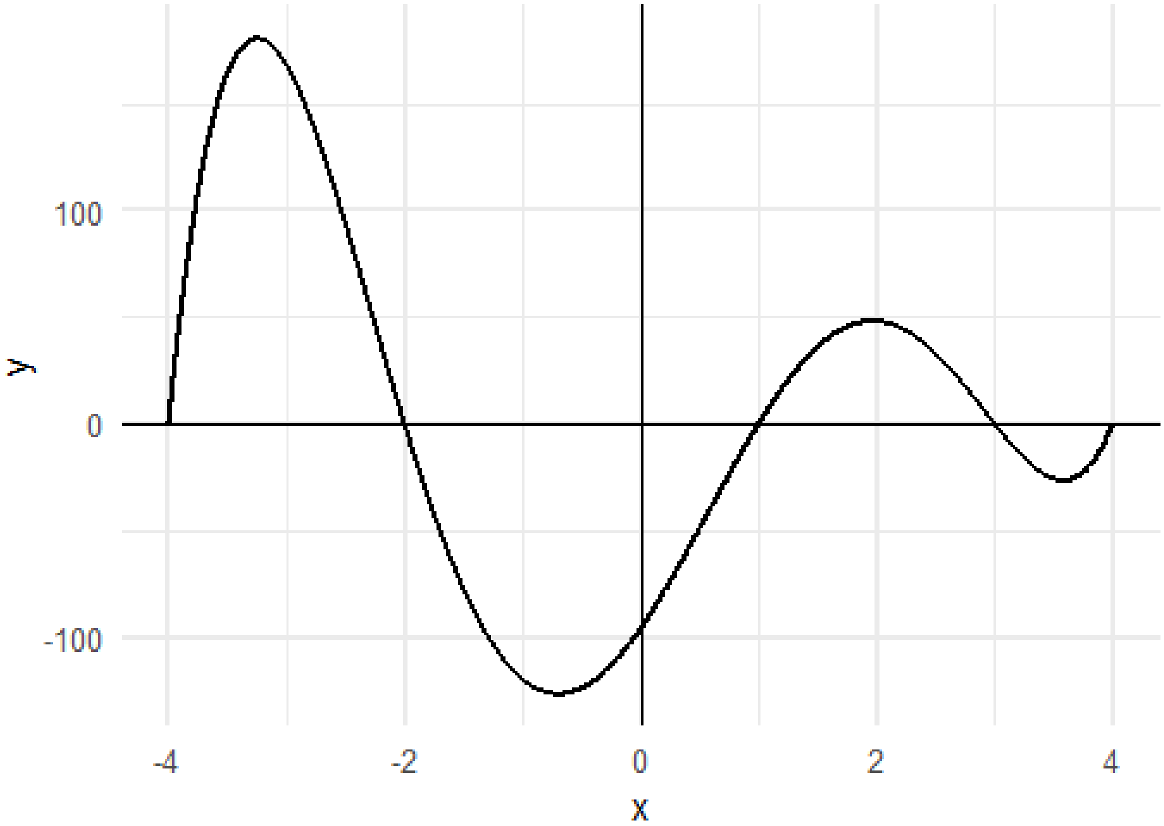
# Isaac Newton

# Gottfried Wilhelm Leibniz

# Newton and Leibniz's Insight

Any curve becomes a straight line if you "zoom in" far enough.

# Zoom and Enhance...



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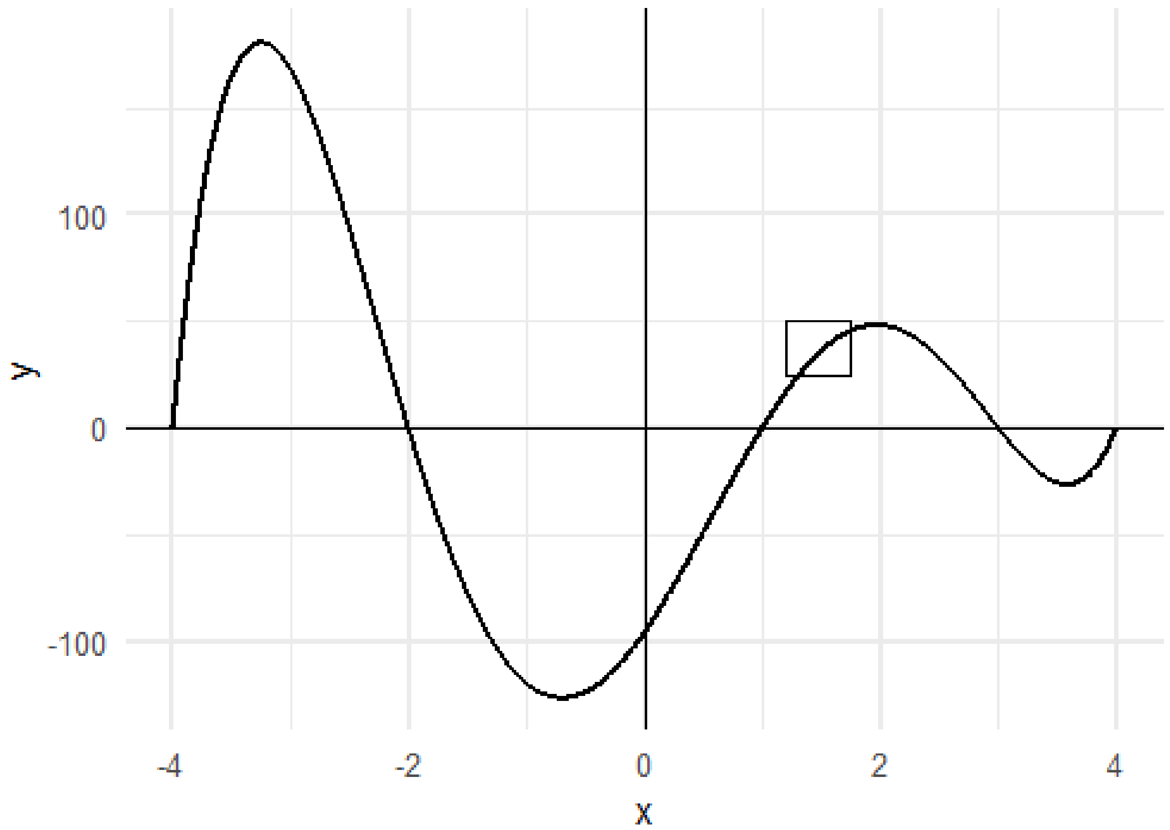

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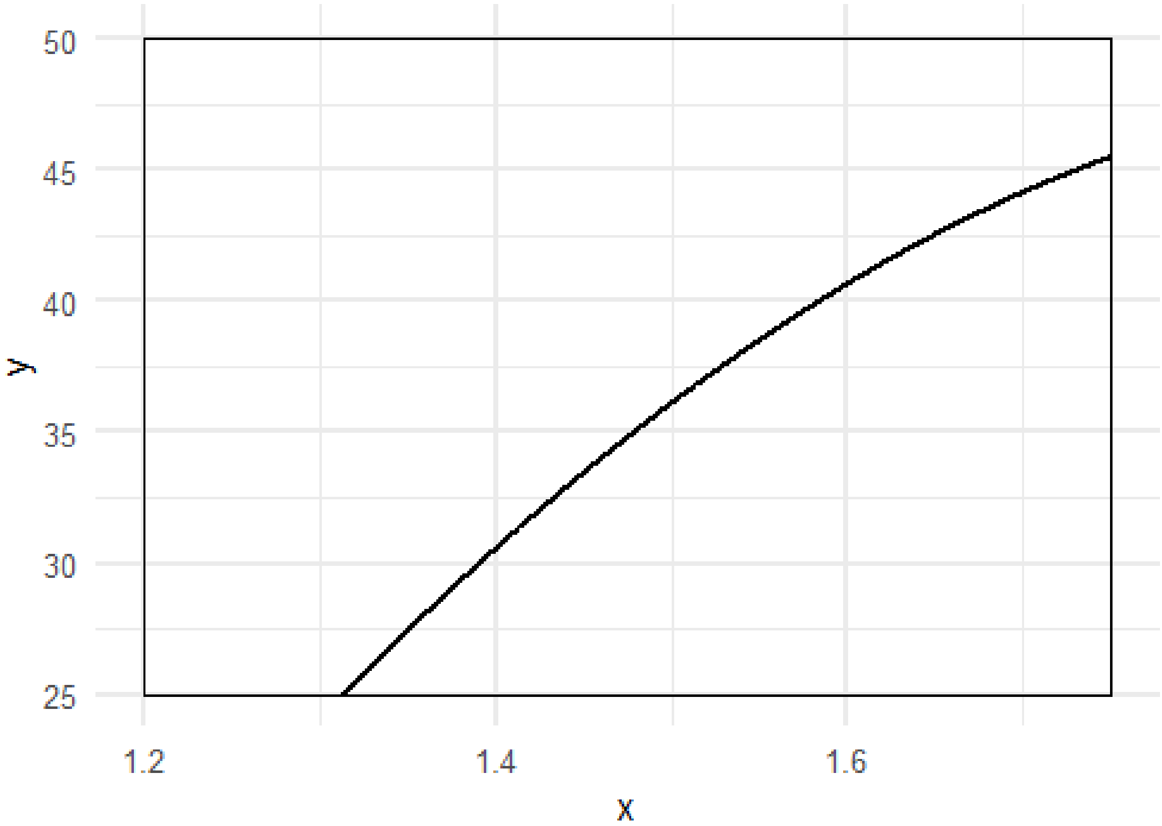

# Zoom and Enhance…Again…

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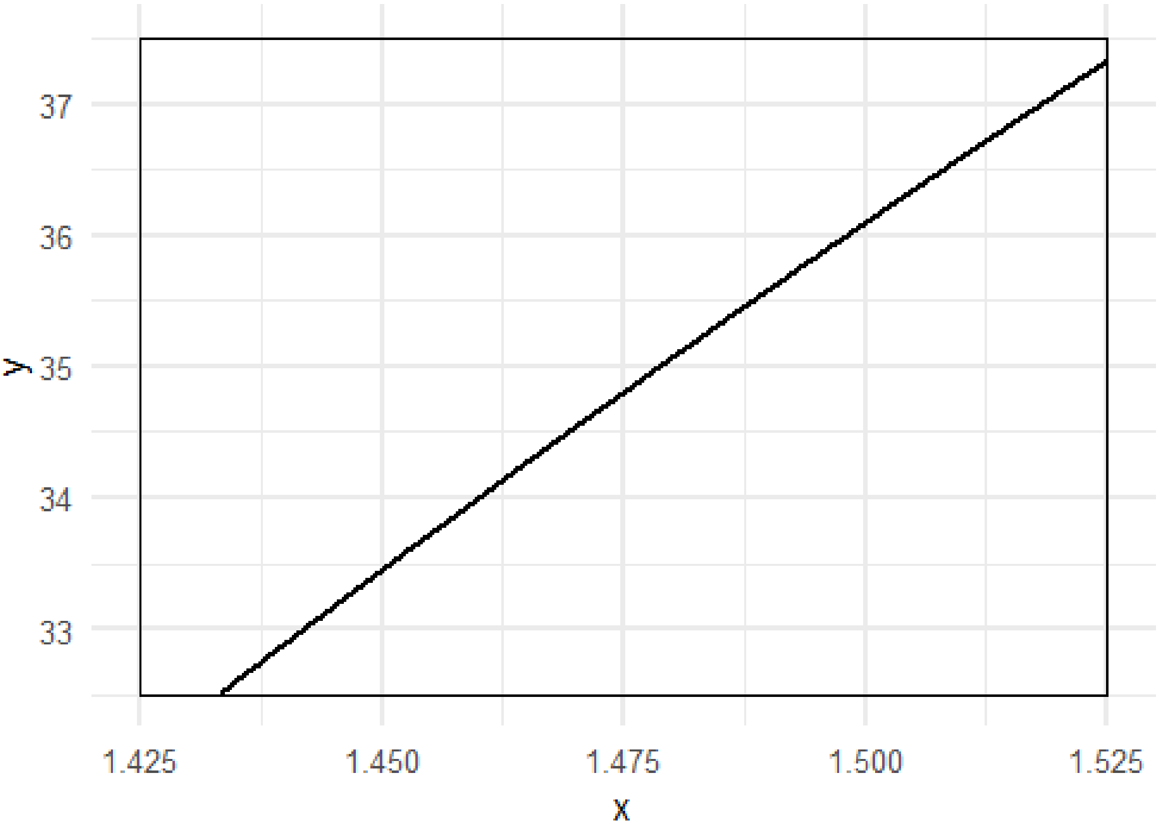
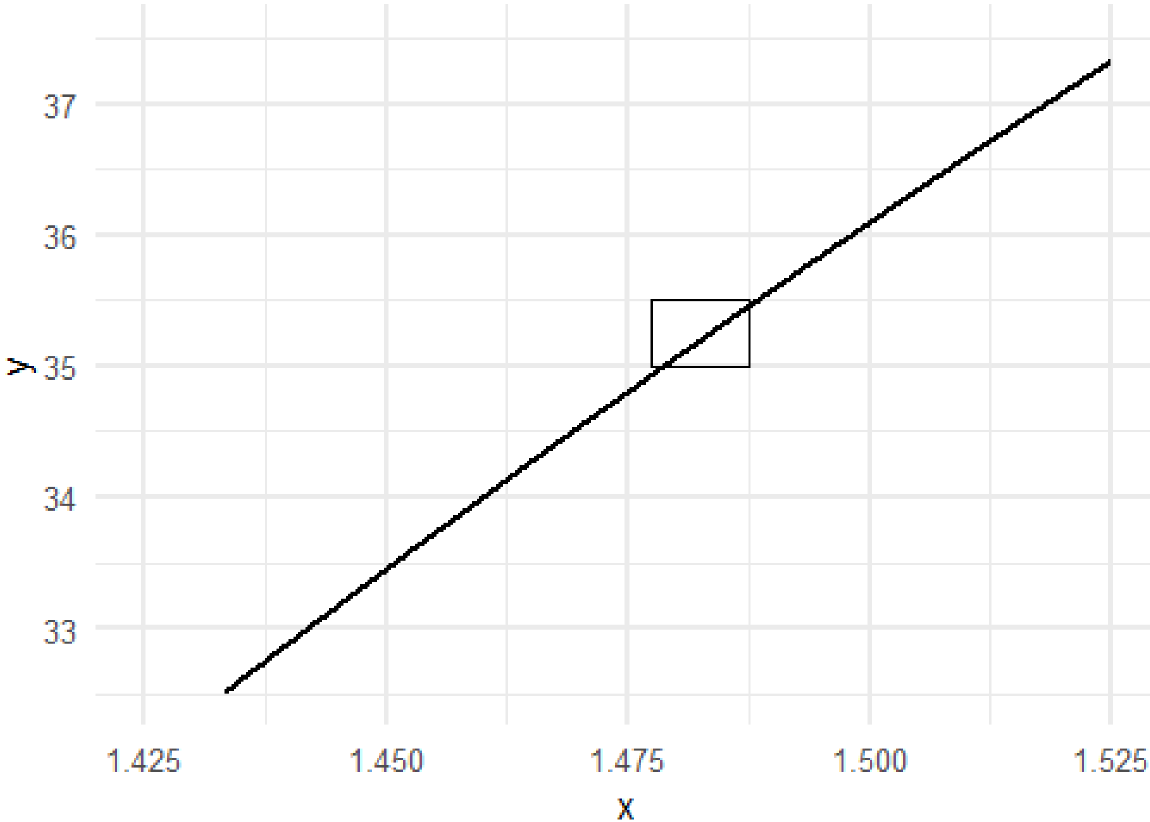


# Zoom and Enhance...Again...And Again...



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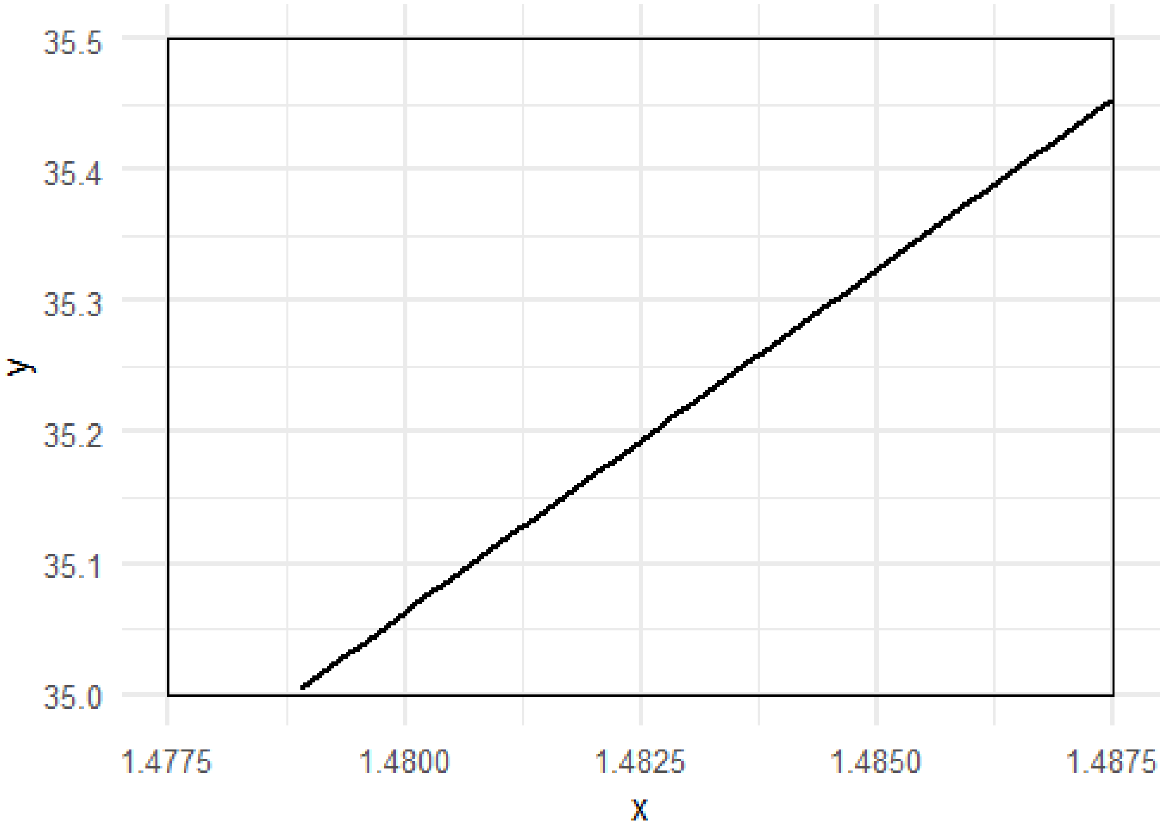



It's basically a straight line! And finding the slope of a straight line is easy...

# Putting All That Into Math...

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

# Putting All That Into Math…

$$f'(x) = \underbrace{\lim_{h \to 0}}_{\text{shrink h really small}} \frac{\overbrace{f(x+h) - f(x)}^{\text{the change in y}}}{\underbrace{h}_{\text{the change in x}}}$$

This is called the **derivative** of a function.

# Example

Let $f(x) = 2x + 3$. What is $f'(x)$?

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{2(x+h) + 3 - (2x+3)}{h}$$

$$= \lim_{h \to 0} \frac{2x + 2h + 3 - (2x+3)}{h}$$

$$= \lim_{h \to 0} \frac{2h}{h}$$

$$= 2$$

# Now A Nonlinear One

Let $f(x) = 3x^2 + 2x + 3$. What is $f'(x)$?

$$= \lim_{h \to 0} \frac{3(x+h)^2 + 2(x+h) + 3 - (3x^2 + 2x + 3)}{h}$$

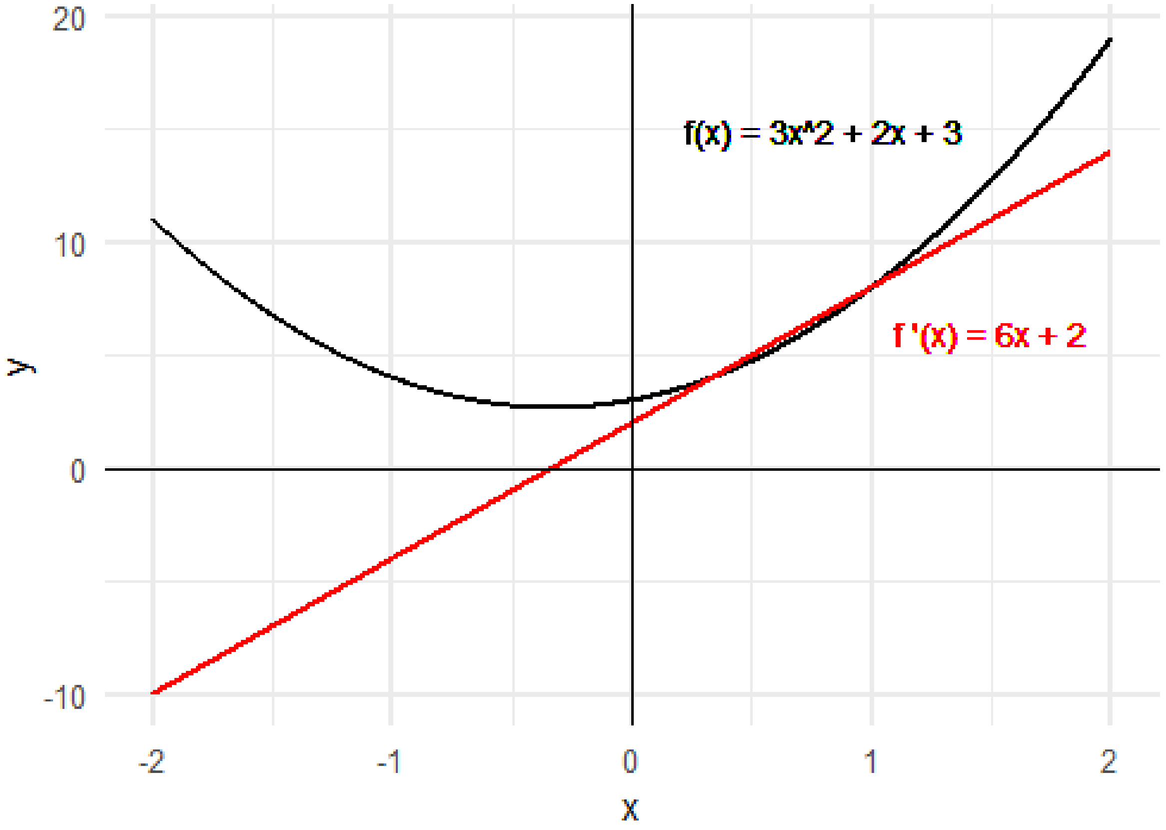
$$= \lim_{h \to 0} \frac{3x^2 + 3h^2 + 6xh + 2x + 2h + 3 - (3x^2 + 2x + 3)}{h}$$

$$= \lim_{h \to 0} \frac{3h^2 + 6xh + 2h}{h}$$

$$= \lim_{h \to 0} 3h + 6x + 2$$

$$= 6x + 2$$

# Solution



More **good news!** You don't have to go through that process every time. Mathematicians have done it for you, and have discovered a whole bunch of useful shortcuts.

# Shortcut 1: The Power Rule

If $f(x) = ax^k$, then $f'(x) = kax^{k-1}$

## Practice Problem:

Let $f(x) = 2x^3$. What is $f'(x)$?

$$f'(x) = 6x^2$$

# Shortcut 2: The Sum Rule

The derivative of a sum is equal to the sum of derivatives.

If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$

## Practice Problem:

If $f(x) = 2x^3 + x^2$, what is $f'(x)$?

$$f'(x) = 6x^2 + 2x$$

# Shortcut 3: The Constant Rule

The derivative of a constant is zero

If $f(x) = c$, then $f'(x) = 0$

## Practice Problem:

If $f(x) = 4x^2 + 3x + 5$, what is $f'(x)$?

$$f'(x) = 8x + 3$$

# Shortcut 4: The Product Rule

The derivative of a product is a bit trickier…

If $f(x) = g(x) \cdot h(x)$, then $f'(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x)$

## Practice Problem

$f(x) = (3x^2 + 6x)(x + 2)$, what is $f'(x)$?

$$f'(x) = (3x^2 + 6x)(1) + (6x + 6)(x + 2)$$

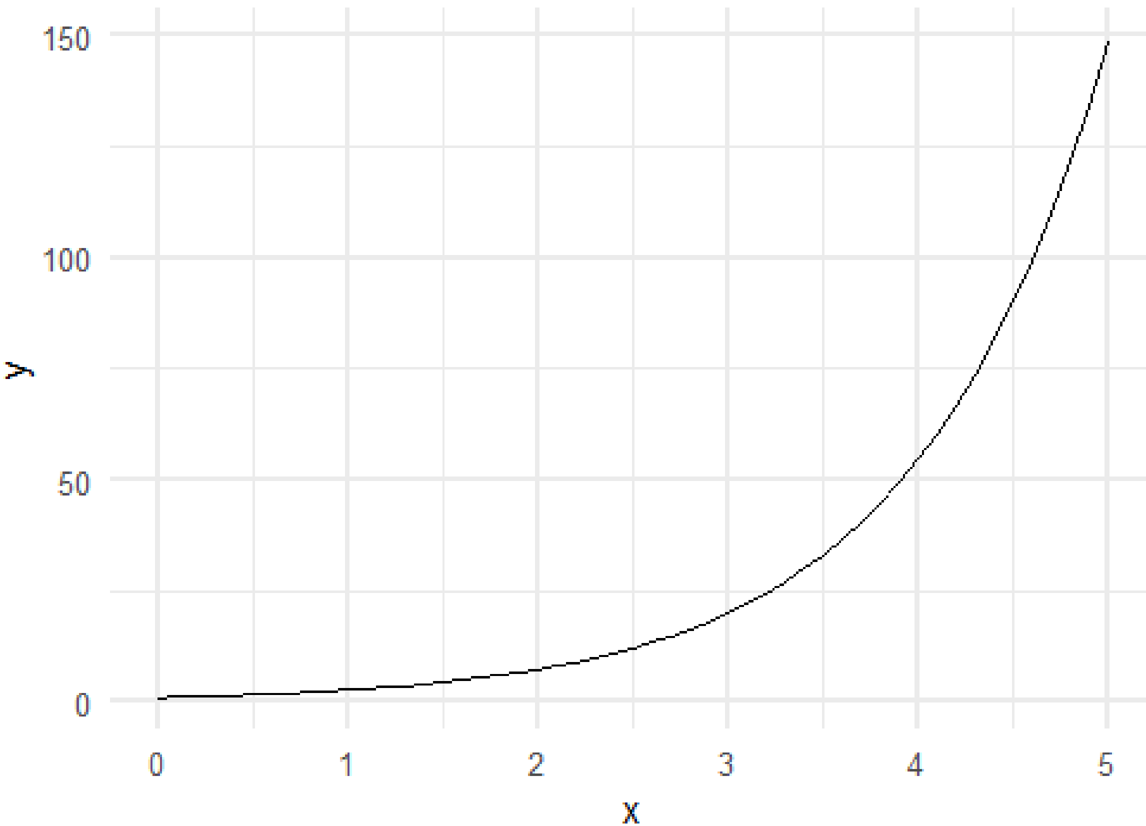
$$f'(x) = 3x^2 + 6x + 6x^2 + 6x + 12x + 12$$

$$f'(x) = 9x^2 + 24x + 12$$

# Shortcut 5: The Exponential Rule

Remember Euler's number? $e = 2.7182818...$

If $f(x) = e^x$, then $f'(x) = e^x$



# Other Derivative Rules

Table 6.1: List of Rules of Differentiation

| | |
|---|---|
| Sum rule | $(f(x)+g(x))' = f'(x)+g'(x)$ |
| Difference rule | $(f(x)-g(x))' = f'(x)-g'(x)$ |
| Multiply by constant rule | $f'(ax) = af'(x)$ |
| Product rule | $(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$ |
| Quotient rule | $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$ |
| Chain rule | $(g(f(x))' = g'(f(x))f'(x)$ |
| Inverse function rule | $(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$ |
| Constant rule | $(a)' = 0$ |
| Power rule | $(x^n)' = nx^{n-1}$ |
| Exponential rule 1 | $(e^x)' = e^x$ |
| Exponential rule 2 | $(a^x)' = a^x(\ln(a))$ |
| Logarithm rule 1 | $(\ln(x))' = \frac{1}{x}$ |
| Logarithm rule 2 | $(\log_a(x))' = \frac{1}{x(\ln(a))}$ |


If you haven't seen these before, it's a lot to absorb. But practice helps.

# More Practice

## Problem 1: Sum of Powers (Polynomial)

Let $f(x) = 2x^3 + 4x + 79$. What is $f'(x)$?

## Problem 2: Multiply By A Constant

Let $f(x) = 3(x^2 + x + 42)$. What is $f'(x)$?

## Problem 3: Product Rule

Let $f(x) = (x^2 + 1)(x + 3)$. What is $f'(x)$?

I taught you that...

...so you could do *this*.

# Optimization

Let $f(x) = 2x^2 + 8x - 32$. At what value of $x$ is the function minimized?


**Key Insight**: Function is minimized when the slope "switches" from decreasing to increasing. Exactly at the point where the slope equals zero.

# Optimization in Three Steps

1. **Take the derivative of the function.**

2. **Set it equal to zero.**

3. **Solve for $x$.**

# Optimization in Three Steps

**1. Take the derivative of the function.**

$$f(x) = 2x^2 + 8x - 32$$

$$f'(x) = 4x + 8$$

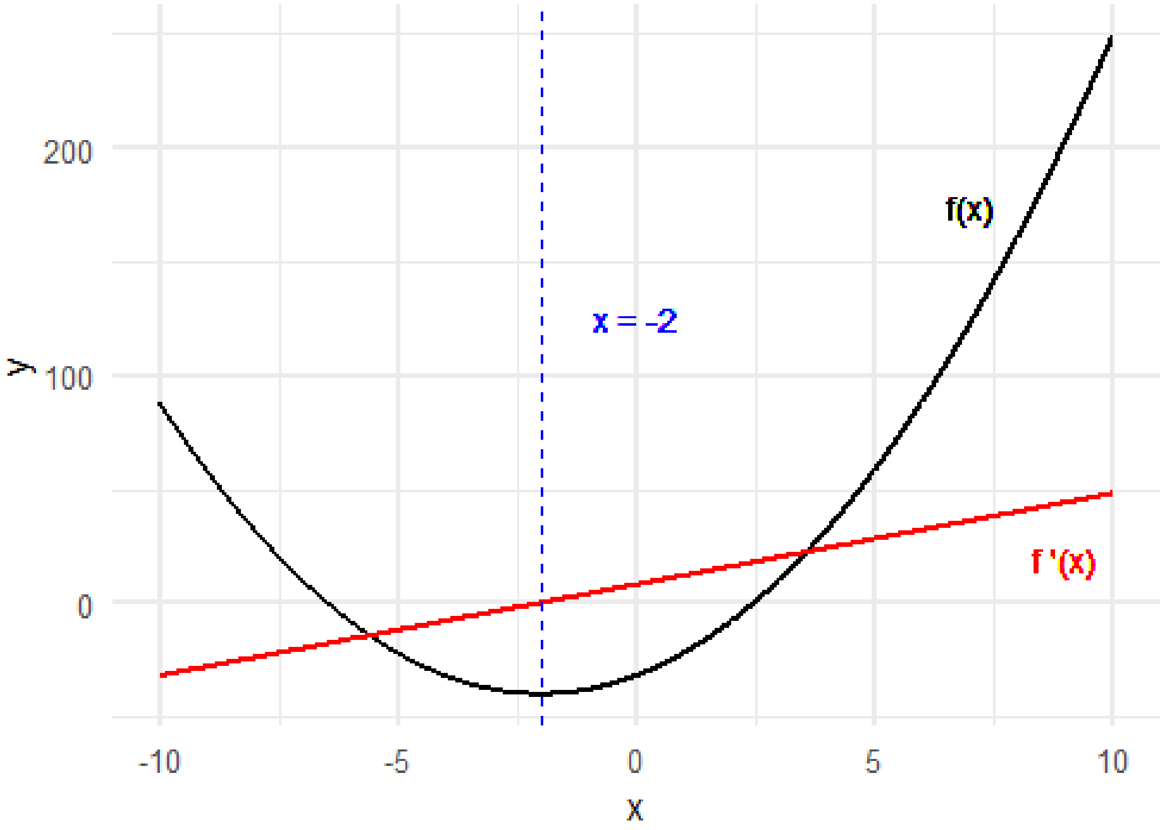
**2. Set it equal to zero**

$$4x + 8 = 0$$

**3. Solve for $x$.**

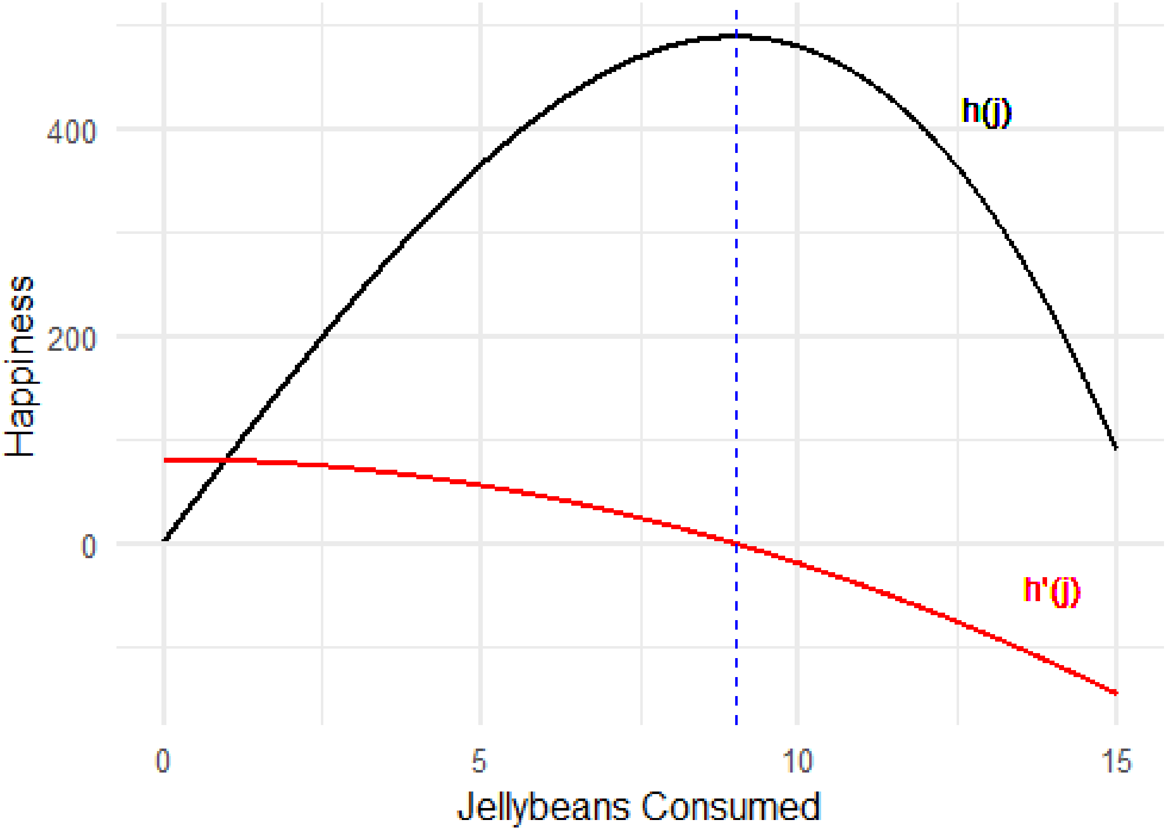
$$x = -2$$

# Optimization in Three Steps



# Now You Try It

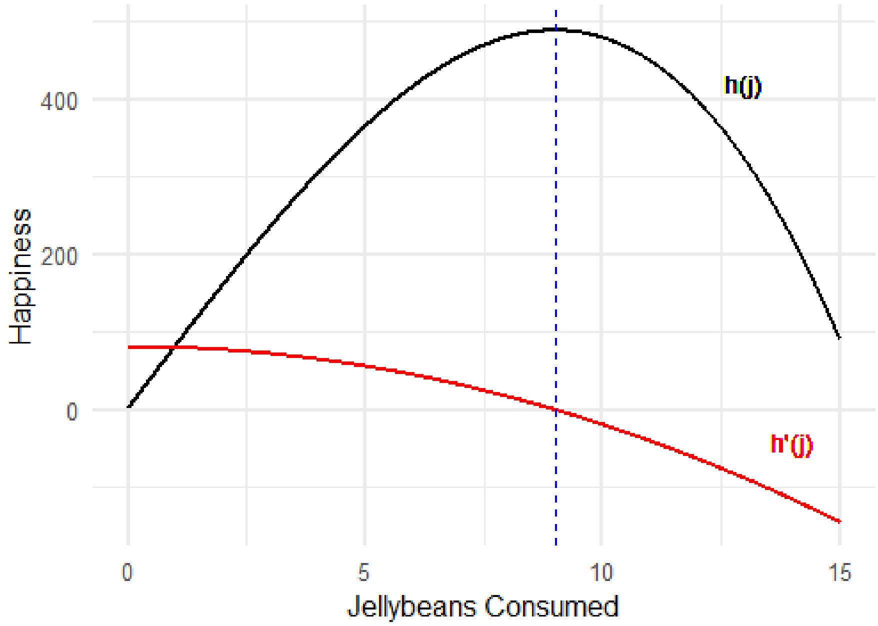
Suppose that happiness as a function of jellybeans consumed is $h(j) = -\frac{1}{3}j^3 + 81j + 2$. How many jellybeans should you eat? (Assume you can only eat a positive number of jellybeans).



Wait, how do you know if it's a maximum or a minimum?

# Jellybeans Again

$h(j) = \frac{1}{3}j^3 + 81j + 2$ and $h'(j) = 81 - j^2$




It's a maximum when the slope is **decreasing**, and a minimum when then slope is **increasing**. How do you figure out if the slope is increasing or decreasing?

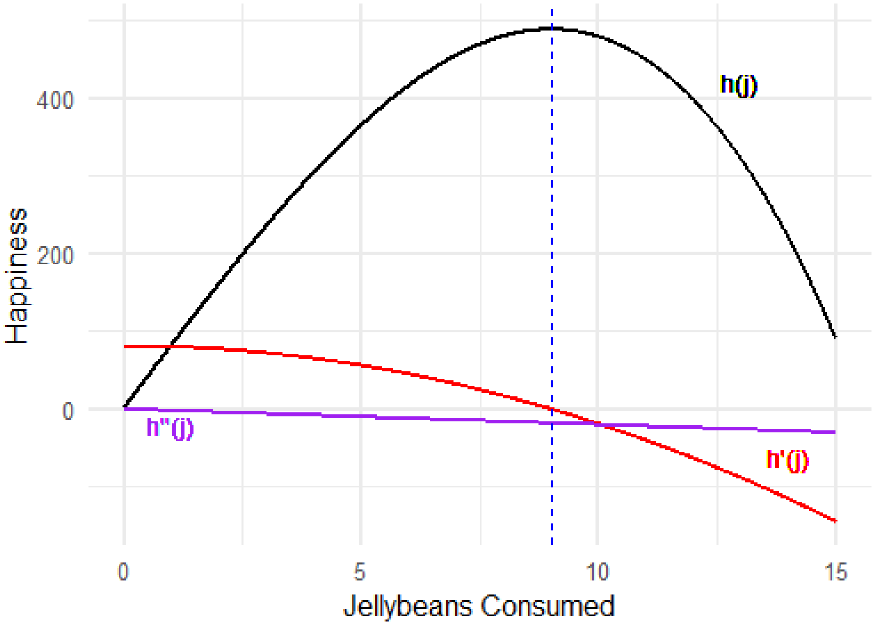
That's right. You find the **slope of the slope** (aka the **second derivative**).

# The Second Derivative Test

$h(j) = \frac{1}{3}j^3 + 81j + 2$ and $h'(j) = 81 - j^2$

What is $h''(j)$? Is it positive or negative when you eat 9 jellybeans?

$$h''(j) = -2j$$



# Partial Derivatives

# Partial Derivatives

What if you have a multivariable function?

$$f(x, y) = 2x^2y + xy - 4x + y - 6$$

Same procedure! To get the derivative of a function *with respect to* $x$ or $y$, treat the other variable as a constant.

$$\frac{\partial f}{\partial x} = 4yx + y - 4$$

$$\frac{\partial f}{\partial y} = 2x^2 + x + 1$$

# Now You Try...

Suppose happiness as a function of jellybeans and Dr. Peppers consumed is

$$h(j, d) = 8j - \frac{1}{2}j^2 + 2d - 3d^2 + jd + 100$$

How many jellybeans should you eat? How many Dr. Peppers should you drink?

Intuitively, the $jd$ term is an **interaction effect**. The effect of jellybeans on happiness increases if you also drink more Dr. Peppers.

# Now You Try...

$$h(j,d) = 8j - \frac{1}{2}j^2 + 2d - 3d^2 + jd + 100$$

$$\frac{\partial h}{\partial j} = 8 - j + d = 0$$

$$\frac{\partial h}{\partial d} = 2 - 6d + j = 0$$

$$j = 8 + d$$

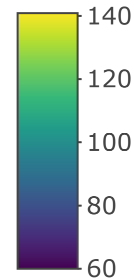
$$j = 6d - 2$$

$$d^* = 2$$

$$j^* = 10$$

# Visualizing That Function

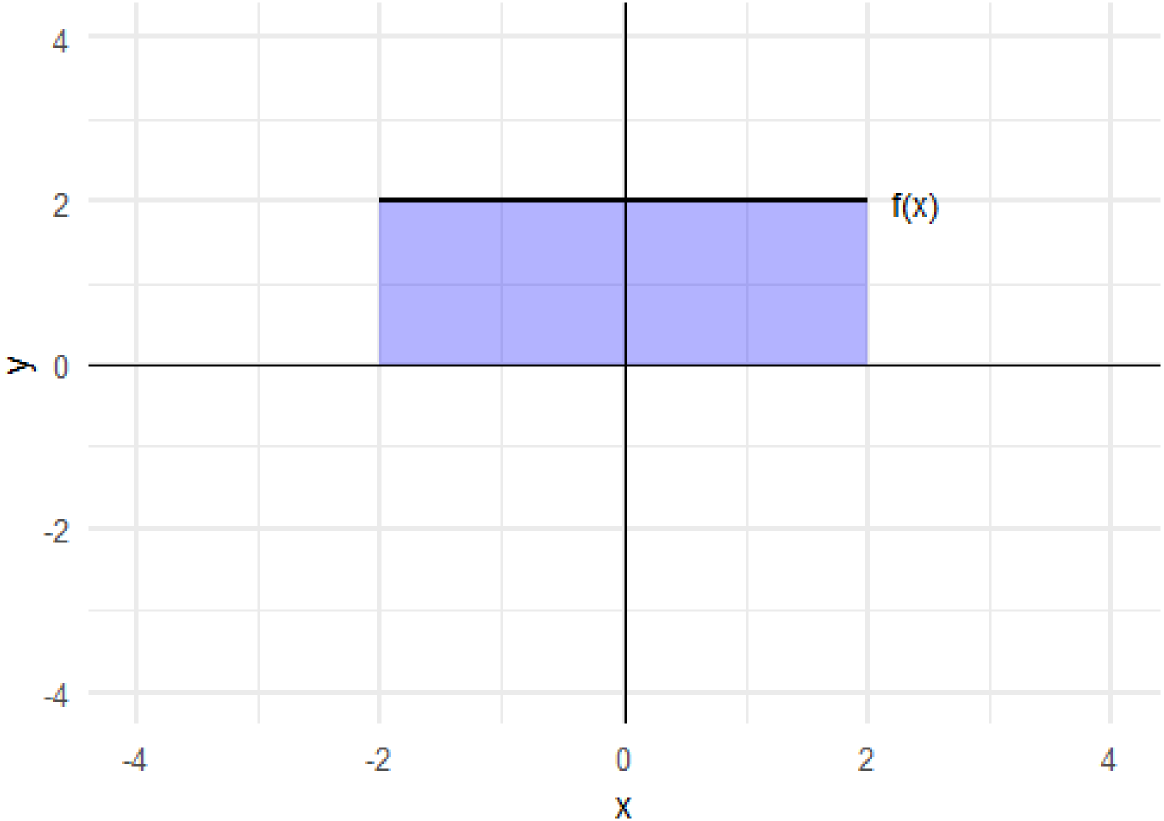
$$h(j, d) = 8j - \frac{1}{2}j^2 + 2d - 3d^2 + jd + 100$$



# Integrals

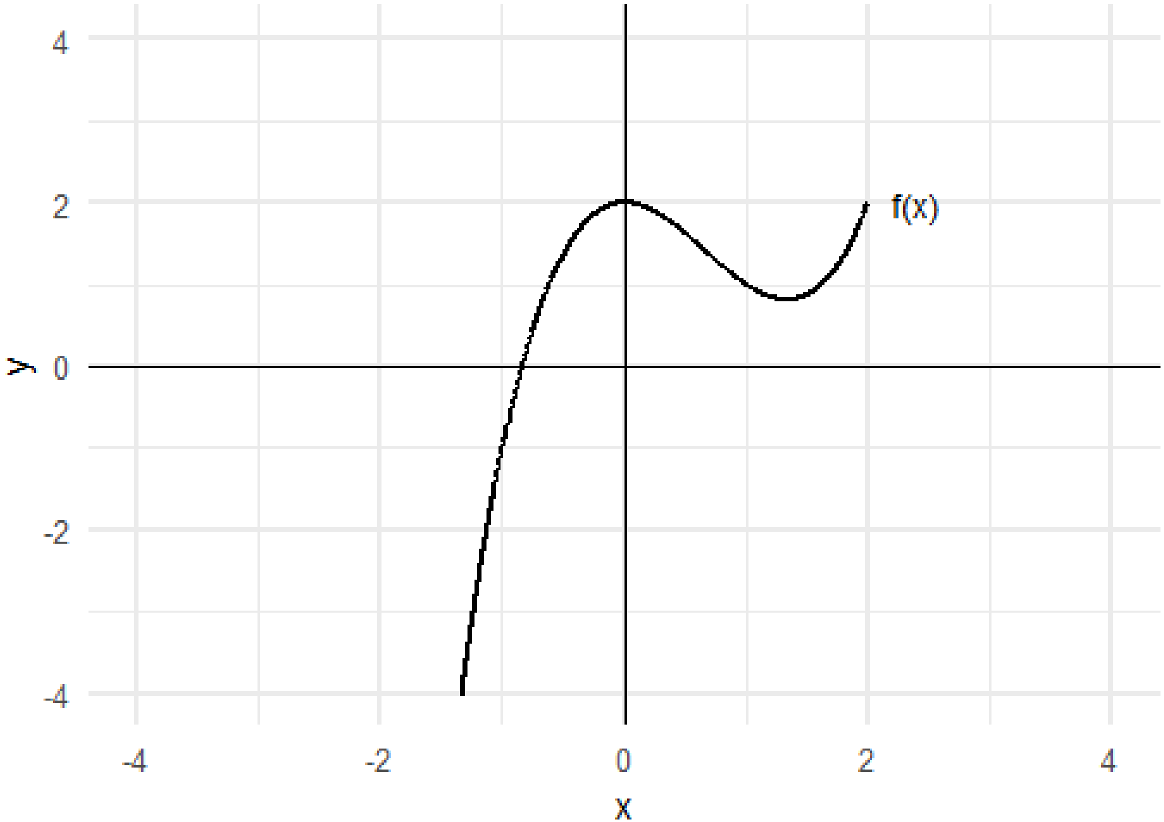
# The Area Under a Curve

Rectangles are easy. $A = bh$



# The Area Under a Curve

Curves are harder. We didn't learn this in geometry!

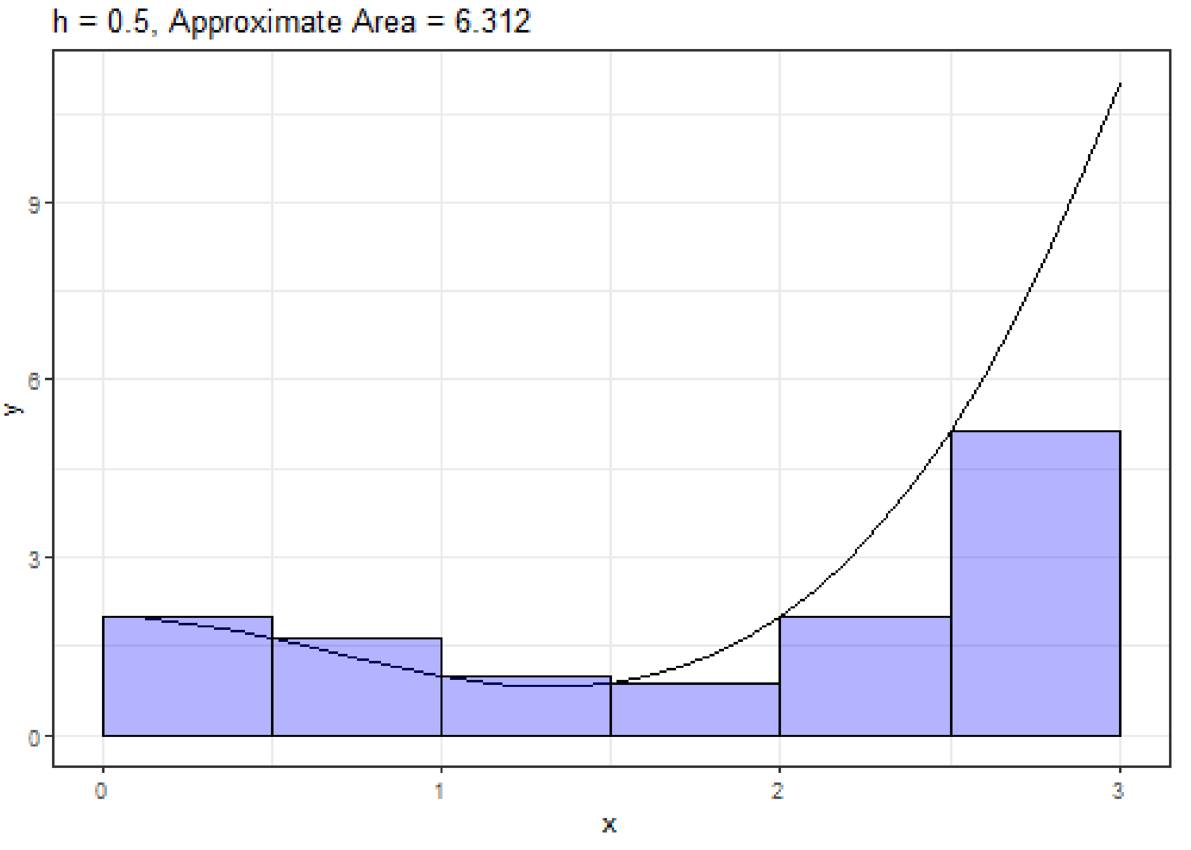


# The Area Under a Curve

- With derivatives, we approximated a hard problem (the slope of a curve) using an easy problem (the slope of a line) by zooming in close enough.
- With integrals, we'll use a similar trick.
- We approximate a hard problem (the area under a curve) using an easy problem (the area of a bunch of rectangles) by making the rectangles *really* thin.

# Riemann Sum

$f(x) = x^3 - 2x^2 + 2$. Find area under curve from $x = 0$ to $x = 3$.



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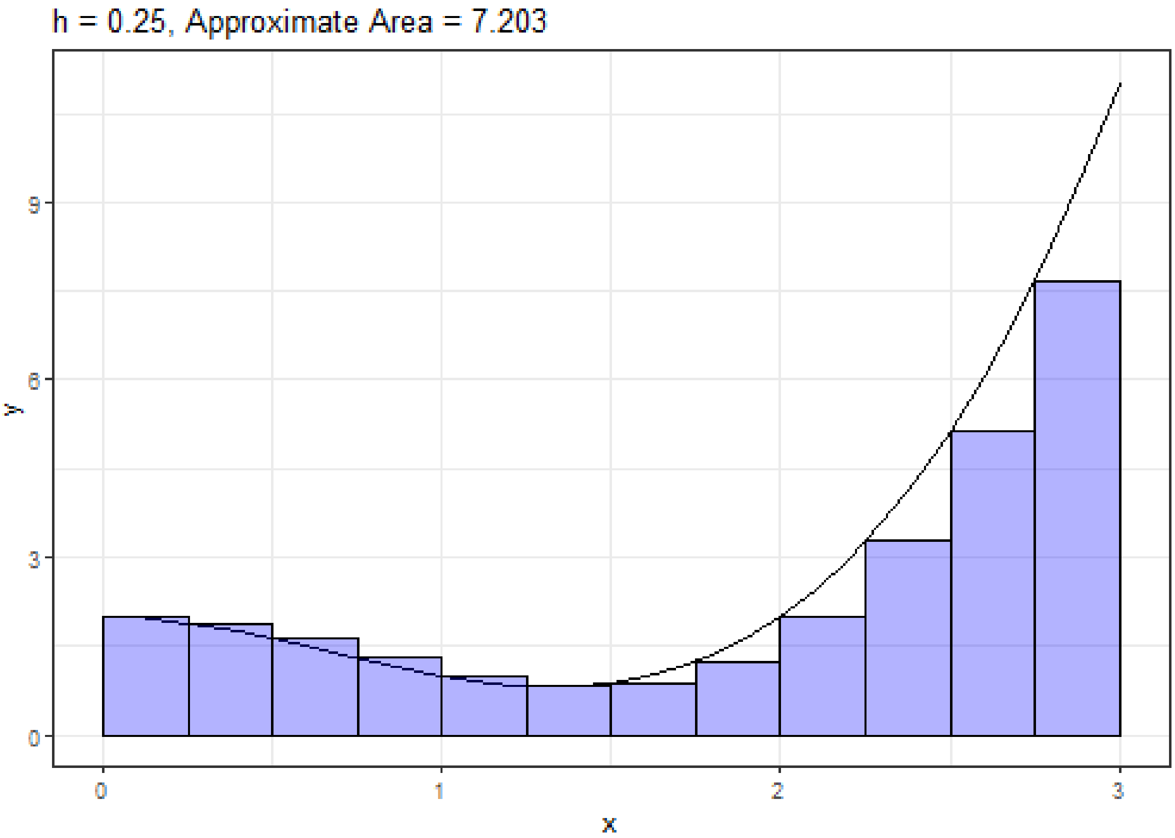

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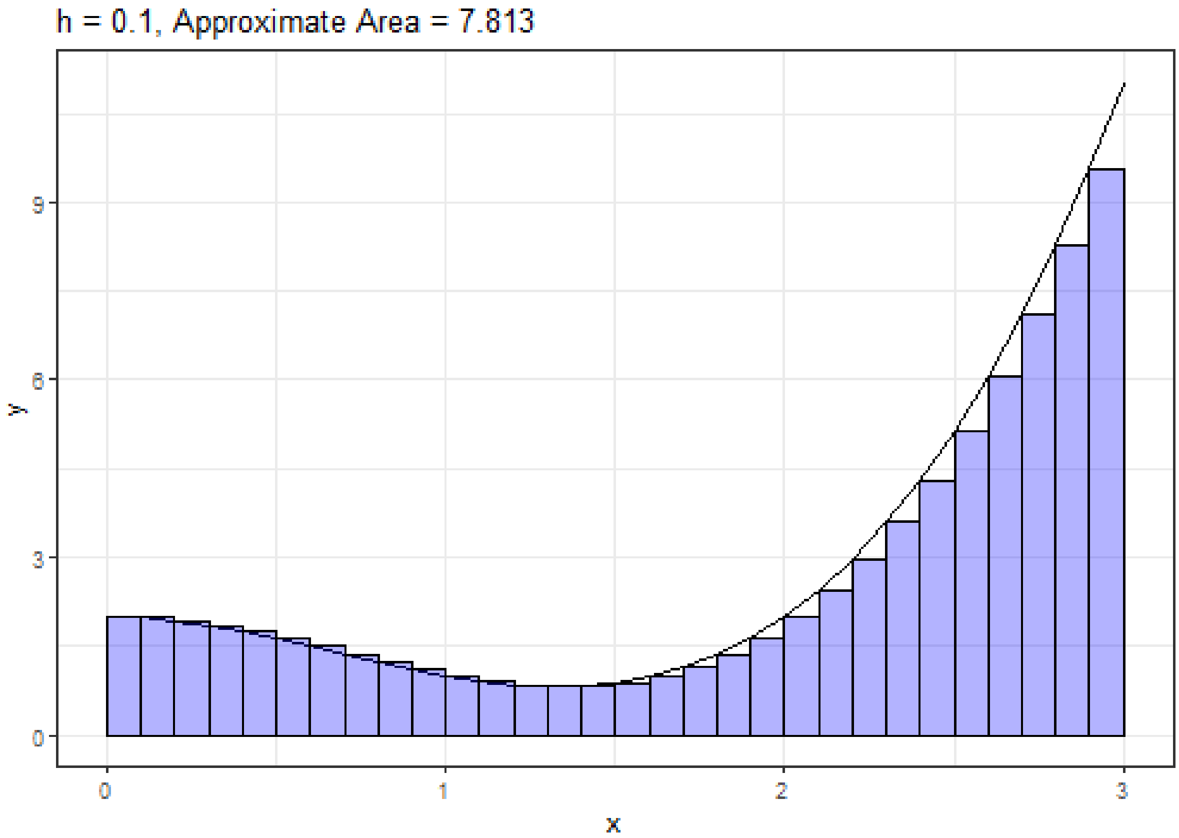

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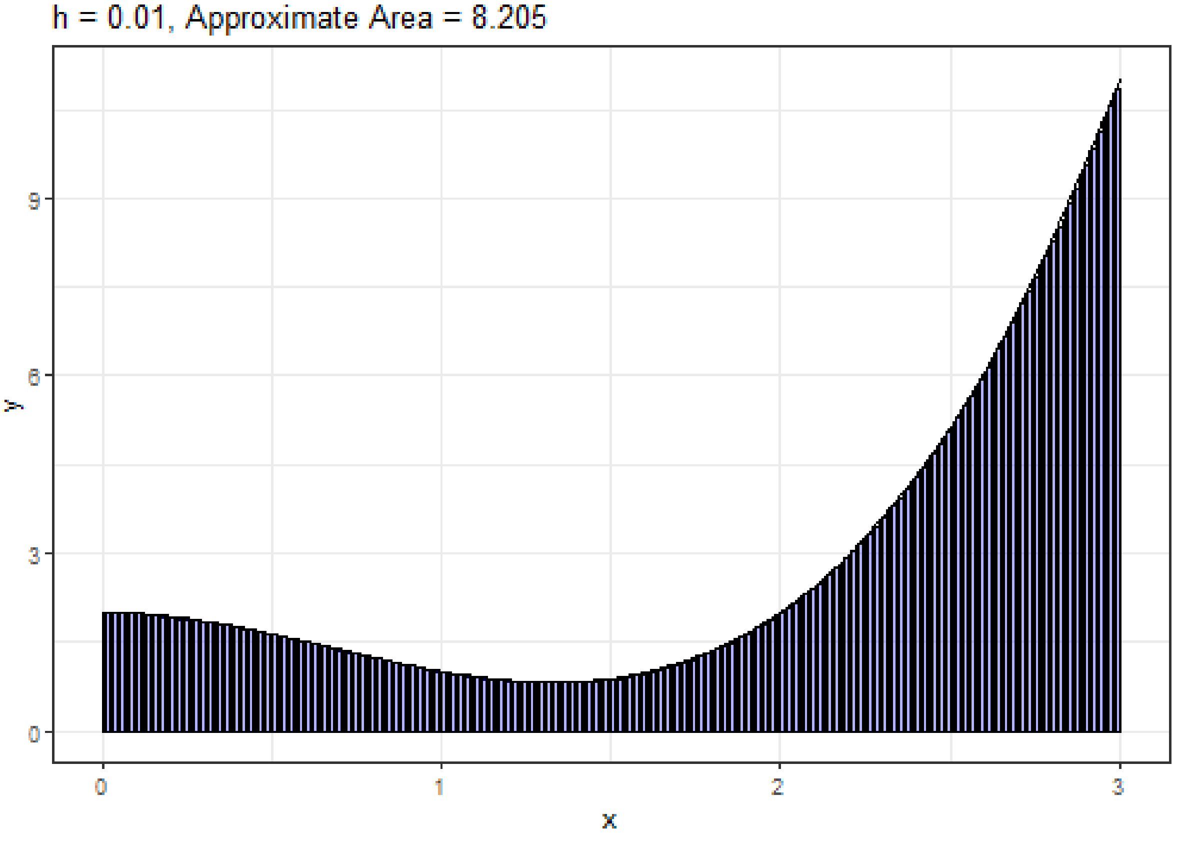

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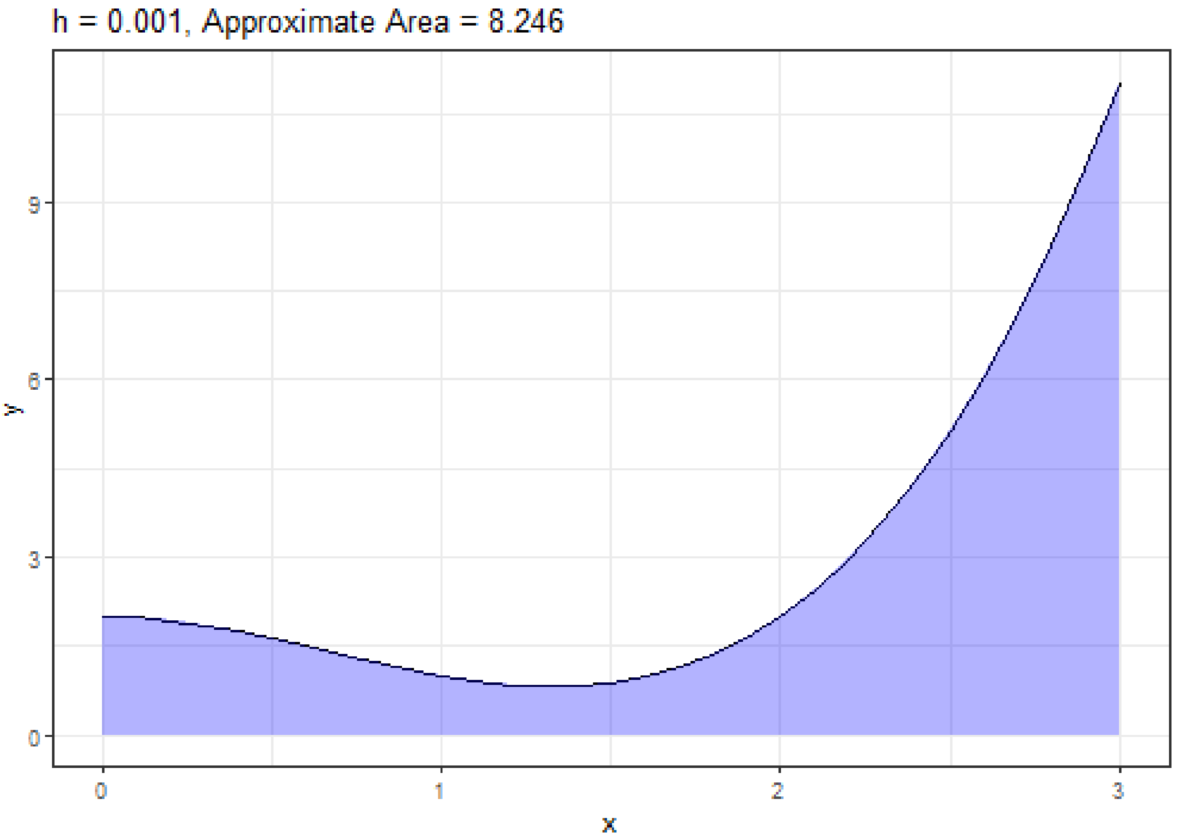


# Integral Notation

$$\lim_{h \to 0} \sum f(x) \cdot h = \int f(x) dx$$

$dx$ is an "infinitesimal" (infinitely small value).

So $\int f(x) dx$ is the area of an infinite number of infinitely skinny rectangles.

If we want the area under a curve between $a$ and $b$, we denote it like so:

$$\int_a^b f(x) dx$$

# There has to be an easier way!

What we want is a function $F(x)$; let's call it the **area function**.

- $F(a)$ gives the area under $f(x)$ between $-\infty$ and $a$.
- $F(b) - F(a)$ gives the area under $f(x)$ between $a$ and $b$.
- As $h$ approaches zero, our skinny rectangles become a better and better approximation of the area function...

$$f(x) \cdot h = \lim_{h \to 0} F(x+h) - F(x)$$

Divide by $h$ on both sides:

$$f(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

Hey. We've seen that before!

$f(x) = F'(x)$. In other words, $F(x)$ is the *antiderivative* of $f(x)$.

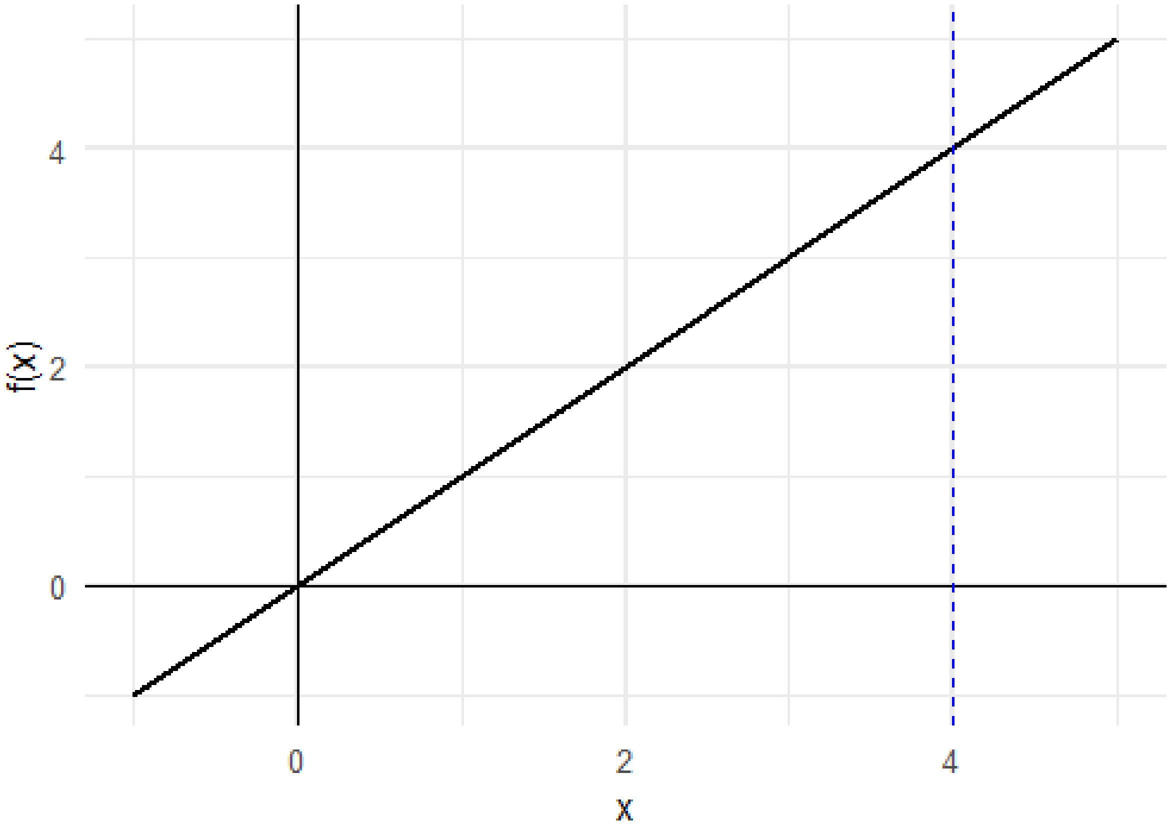
# The Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

Computing the area under the curve and taking the antiderivative are equivalent operations!

# Try It!

If $f(x) = x$, find the area under the curve between $x = 0$ and $x = 4$.




**Hint**: It's a triangle, so the answer should be 8.

# The Area Under a Curve

If $f(x) = x$, find the area under the curve between $x = 0$ and $x = 4$.

Use the Fundamental Theorem of Calculus:

$$\int_0^4 f(x)dx = F(4) - F(0)$$

$F(x) = \frac{1}{2}x^2 + C$

$F(4) - F(0) = \frac{1}{2} \cdot 4^2 = 8$

# Now a Nonlinear Example...

If $f(x) = x^3 - 2x^2 + 2$, find the area under the curve between $x = 0$ and $x = 3$.

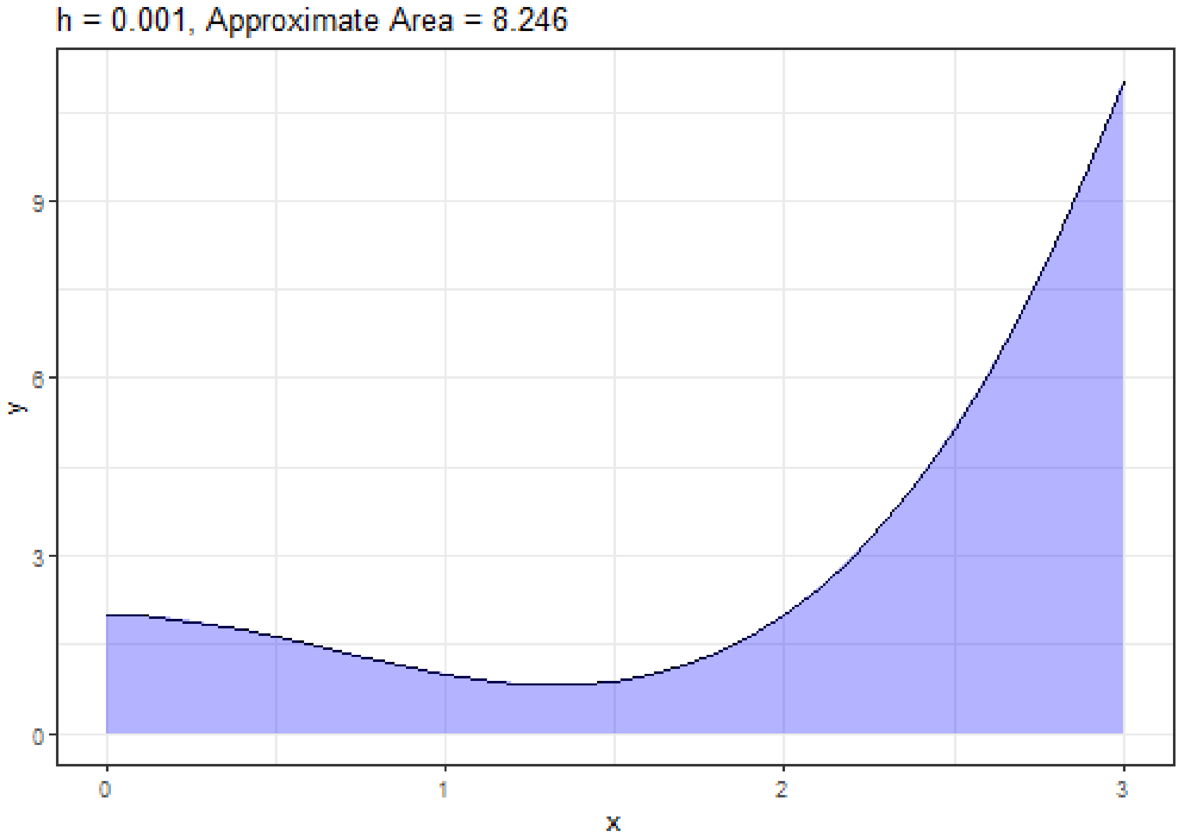
$$\int_0^3 f(x)dx = F(3) - F(0)$$

$F(x) = \frac{1}{4}x^4 - \frac{2}{3}x^3 + 2x + C$

$F(3) - F(0) = \frac{1}{4}3^4 - \frac{2}{3}3^3 + 2(3) - [\frac{1}{4}0^4 - \frac{2}{3}0^3 + 2(0)] = 8.25$

# Now a Nonlinear Example...

That's the same answer that we got from the skinny rectangles!



...and that's All The Calculus You Need