POLS 7012: Problem Set 6 (Calculus)

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Problem 1

A dictator is deciding how much tax revenue to steal for himself and how much to give to his generals. He can steal some fraction of revenue x between 0 and 1. If he steals all of it ($x = 1$), then his generals will get mad, overthrow him, and he gets nothing. If he gives everything to his generals ($x = 0$), then his generals are happy, don't overthrow him, but he still gets nothing because he gave away all the loot.

If the probability that the generals leave him in place equals $1 - x$, how much should the dictator steal to maximize his **expected happiness** (probability of staying in power $\times$ the amount of money stolen)?

Solution

$$H(x) = x(1 - x) = x - x^2$$

$$H'(x) = 1 - 2x$$

To find the maximum, set the derivative equal to zero.

$$1 - 2x = 0$$

$$x^* = \frac{1}{2}$$

Problem 2

Suppose you have three data points: x_1, x_2 and x_3. Find the value m that minimizes the sum of squared errors $f(m)$:

$$f(m) = (x_1 - m)^2 + (x_2 - m)^2 + (x_3 - m)^2$$

Solution

$$f(m) = x_1^2 + m^2 - 2x_1m + x_2^2 + m^2 - 2x_2m + x_3^2 + m^2 - 2x_3m$$

$$\frac{\partial f}{\partial m} = 6m - 2x_1 - 2x_2 - 2x_3$$

To find the minimum, set that derivative equal to zero.[1]

$$6m - 2x_1 - 2x_2 - 2x_3 = 0$$

[1] The second derivative is positive, so we know it's a minimum.

$$m = \frac{x_1 + x_2 + x_3}{3}$$

Hey you proved a theorem! The **mean** of a variable minimizes the sum of squared errors.

Problem 3

Consider the following regression model:

$$Y = 2X_1 + 5X_2 - 4X_1X_2 + \varepsilon$$

If you increase X_1 by 1 unit, what happens to Y?

Solution

$$\frac{\partial Y}{\partial X_1} = 2 - 4X_2$$

When you increase X_1 by one unit, the value of Y may increase, or may decrease, depending on the value of X_2. If $X_2 > \frac{1}{2}$, the slope is negative, and otherwise positive.

Remember that when you start working with linear-interaction models! The answer to "what is the slope" is "it depends on the other variables!"