Baez Exercises: Holonomy

88. The covariant derivative of a vector field u along a curve $\gamma(t)$ is

$$D_{\gamma'(t)}\, u(t) = \frac{d}{dt} u(t) + A(\gamma'(t))\, u(t)$$

where $u(t)$ is the field at $\gamma(t)$ and $\frac{d}{dt} u(t)$ is really $D^{\circ}_{\gamma'(t)}\, u(t)$. Show that $D_{\gamma'(t)}\, u(t)$ defined in this manner is actually independent of the choice of local trivialization.

Since D itself is basis independent, the expression will be, too. To check that it is basis independent, we'd have to know the transformation rule for $A \to A'$. But this just comes from the invariance of D itself: $D_v = D^{\circ}_v + A(v) = D'^{\circ}_v + A'(v)$; thus A will automatically have the required transformation properties. In other words, there's no point in checking! In any case, the transformation rule is, clearly,

$$A'(v)\, s = A(v) s + (D^{\circ}_v - D'^{\circ}_v)(s)$$

In coordinates we set $[A(v)]^j_k = e^j(A(v) e_k)$, so

$$[A'(v)]^j_k = [A(v)]^j_k + e^j(D^{\circ}_v(e_k) - D'^{\circ}_v(e_k))$$

Supposing $e_k = \Lambda^\ell_k e'_\ell$, the second term is

$$\delta^j_k - e^j D'^{\circ}_v(\Lambda^\ell_k e'_\ell) = \delta^j_k - v(\Lambda^\ell_k)\, e^j(e'_\ell) - \Lambda^\ell_k e^j(e'_\ell)$$

$$= \delta^j_k - v(\Lambda^\ell_k)(\Lambda^{-1})^j_\ell - \Lambda^\ell_k(\Lambda^{-1})^j_\ell = -v(\Lambda^\ell_k)(\Lambda^{-1})^j_\ell$$

$$\Rightarrow [A'(v)]^j_k = [A(v)]^j_k - v(\Lambda^\ell_k)(\Lambda^{-1})^j_\ell$$

89. Check convergence...

90. Let $\alpha:[0,T]\to M$ be a piecewise smooth path and let $f:[0,S]\to[0,T]$ be any piecewise smooth function with $f(0)=0$, $f(S)=T$. Let β be the reparameterized path given by $\beta(t)=\alpha(f(t))$. Show that for any connection D on a vector bundle $\pi:E\to M$, $H(\alpha,D)=H(\beta,D)$

The goal here is to show that the holonomy only depends on the <u>path</u>, not the <u>curve</u>, i.e. not on the particulars of the parameterization. This will hold if the covariant derivative is <u>zero</u> independent of the param. One has to be pretty careful with the notation here. For a given vector field u, let's call $u_\alpha(t)$ the vector in the fiber above $\alpha(t)$ and $u_\beta(t)$ the vector in the fiber above $\beta(t)$. Then we just check if

$$D_{\alpha'(t)}\, u_\alpha(t) \overset{?}{=} D_{\beta'(t)}\, u_{\beta(t)} = 0 \quad \text{for } \beta(t)=\alpha(f(t)).$$

First, $\beta'(t) = \frac{d}{dt}\alpha(f(t)) = \underbrace{f'(t)}_{\text{function}}\ \underbrace{\alpha'(f(t))}_{\text{vector}}$

while $u_\beta(t) = u_\alpha(f(t))$. u at $\alpha(f(t))=\beta(t)$

$$\Rightarrow D_{\beta'(t)}\, u_\beta(t) = D_{\beta'(t)}\, u_\alpha(f(t)) = \frac{d}{dt} u_\alpha(f(t)) + A(\beta'(t))\, u_\alpha(f(t))$$

$$= \frac{d}{ds} u_\alpha(s)\Big|_{s=f(t)} f'(t) + A(f'(t)\,\alpha'(f(t)))\, u_\alpha(f(t))$$

$$= f'(t)\Big[\underbrace{\frac{d}{ds} u_\alpha(s) + A(\alpha'(s))\, u_\alpha(s)}_{=0}\Big]_{s=f(t)} = 0.$$

So we have shown that if u is parallel-transported along $\alpha(t)$, it is also parallel-transported along $\beta(t)$.

There's an even easier way to see this using the path-ordered exponential:

$$H(\gamma, D) = P e^{-\int_0^t dt\, A(\gamma'(t))}$$

Choose $\gamma'(t) = \beta'(t) = f'(t)\,\alpha'(f(t))$. Then $dt\, A(\beta'(t)) = dt\, f'(t)\, A(\alpha'(f(t)))$, which is $ds\, A(\alpha'(s))$ for $s = f(t)$. Thus

$$H(\beta, D) = P e^{-\int_0^T dt\, A(\beta'(t))} = P e^{-\int_0^S ds\, A(\alpha'(s))} = H(\alpha, D)$$

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91. Check the identities $H(1_q\alpha, D) = H(\alpha, D)$, $H(\alpha 1_p, D) = H(\alpha, D)$, and $H(1_p, D) = 1$.

We only need to check the last one, since then the other two follow. Since $1_p(t) = p$ for all t, it follows that $1_p'(t) = 0$ since

$$1_p'(t) f = \frac{d}{dt}(f(1_p(t)) = \frac{d}{dt} f(p) = 0$$

Thus $A(1_p'(t)) = 0$ and therefore the equation for the covariant derivative becomes

$$D_{1_p'(t)}\, u(t) = \frac{d}{dt} u(t) = 0$$

so $u(t) = u(0)$ and therefore $H(1_p(t), D) = 1$.

$H(1_q \alpha, D) = H(1_q, D)\, H(\alpha, D) = H(\alpha, D)$ and similarly for $H(\alpha 1_p, D)$.

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92. Check that the formula for gauge transformed holonomies,

$$H(\gamma, D') = g(\gamma(T))\, H(\gamma, D)\, g(\gamma(0))^{-1}$$

holds even when the path γ does not stay within an open set over which we have trivialized the G-bundle E, by breaking up γ into smaller paths.

Well, the holonomy doesn't depend on the local trivialization, since the covariant derivative doesn't, so we just break up the path in the different local trivializations, and the result will be independent of the choices of local triv:

$$H(\gamma, D') = H(\gamma_2, D') \, H(\gamma_1, D')$$
$$= g(\gamma_2(T)) \, H(\gamma_2, D) \, g(\gamma_2(0))^{-1} \, g(\gamma_1(T)) \, H(\gamma_1, D) \, g(\gamma_1(0))^{-1}$$
$$= g(\gamma_2(T)) \, H(\gamma_2, D) \, H(\gamma_1, D) \, g(\gamma_1(0))^{-1} = g(\gamma(T)) \, H(\gamma, D) \, g(\gamma(0))^{-1}$$

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93. Show that if D is a G-connection on a G-bundle and γ is a loop, the holonomy $H(\gamma, D)$ lives in G.

D is a G-connection when the components A_μ of the vector potential live in $\mathfrak{g}$ (the Lie algebra of G). Then, according to the path integral formula, the holonomy is an operator generated by elements of $\mathfrak{g}$, so it must live in the Lie group G.