Chapter 3

Baez Exercises: Curvature

94. Check that $[v, fw] = f[v,w] + v(f)\,w$

Apply to a function h:

$$[v,fw]h = v\circ(fw)h - fwvh = v(f(wh)) - fwvh$$
$$= v(f)\,wh + f\,vwh - fwvh$$
$$= \big(v(f)\,w + f[v,w]\big)h \qquad \text{as intended.}$$

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95. Prove that $H(\gamma, D) = 1 - \varepsilon^2 F_{\mu\nu}$ for γ an infinitesimal loop along the coordinates x^μ, x^ν

We use the path-ordered exponential:

$$H(\gamma, D) = P\exp\left[-\int_0^S A(\gamma'(s);\gamma(s))\,ds\right]$$

where by $A(\gamma'(s), \gamma(s))$ we denote the vector potential, as a function of the vector field $\gamma'(s)$ and the position $\gamma(s)$. The latter is implicitly contained in the former, but in this calculation we must be careful to be explicit.

Now consider the curve γ:

$$\gamma(t) = \begin{cases} (4t\varepsilon, 0) & 0 \le t \le 1/4 \\ (\varepsilon, \varepsilon(4t-1)) & 1/4 < t \le 1/2 \\ (\varepsilon(3-4t), \varepsilon) & 1/2 \le t \le 3/4 \\ (0, 4\varepsilon(1-t)) & 3/4 \le t \le 1 \end{cases}$$

where (a,b) means $x^\mu = a$, $x^\nu = b$. It's immediately obvious that

$$\gamma'(t) = \begin{cases} 4\varepsilon\,\partial_\mu & 0 \le t \le 1/4 \\ 4\varepsilon\,\partial_\nu & 1/4 < t \le 1/2 \\ -4\varepsilon\,\partial_\mu & 1/2 \le t \le 3/4 \\ -4\varepsilon\,\partial_\nu & 3/4 \le t \le 1 \end{cases}$$

Now we're ready to start.

$$H(\gamma, D) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} P\left(\int_0^1 ds\ A(\gamma'(s);\gamma(s))\right)^n$$

$$= 1 - \int_0^1 ds\ A(\gamma'(s);\gamma(s)) + \frac{1}{2} P\left(\int_0^1 ds\ A(\gamma'(s);\gamma(s))\right)^2 + \text{terms of higher order}$$

since $A(\gamma'(s);\gamma(s)) \propto \varepsilon$, we don't need to keep any more terms

Let's concentrate on the first integral:

$$\int_0^1 ds\ A(\gamma'(s);\gamma(s)) = \int_0^{1/4} ds\ 4\varepsilon\, A_\mu(4s\varepsilon, 0) + \int_{1/4}^{1/2} ds\ 4\varepsilon\, A_\nu(\varepsilon, \varepsilon(4s-1))$$

$$- \int_{1/2}^{3/4} ds\ 4\varepsilon\, A_\mu(\varepsilon(3-4s), \varepsilon) - \int_{3/4}^{1} ds\ 4\varepsilon\, A_\nu(0, 4\varepsilon(1-s))$$

here $A_\rho(a,b)$ is shorthand for $A(\partial_\rho;(a,b))$

Since ε is small, we can Taylor expand the integrands:

$$A_\mu(4s\varepsilon, 0) = A_\mu(0,0) + \varepsilon \frac{d}{d\varepsilon'} A_\mu(4s\varepsilon', 0)\Big|_{\varepsilon'=0} + O(\varepsilon^2)$$

$$= A_\mu(0,0) + 4\varepsilon s\, \partial_\mu A_\mu\big|_{(0,0)} + O(\varepsilon^2)$$

when there's no point specified, it's $(0,0)$

$$A_\nu(\varepsilon, \varepsilon(4s-1)) = A_\nu(0,0) + \varepsilon\, \partial_\mu A_\nu + \varepsilon(4s-1)\,\partial_\nu A_\nu + O(\varepsilon^2)$$

$$A_\mu(\varepsilon(3-4s), \varepsilon) = A_\mu + \varepsilon(3-4s)\,\partial_\mu A_\mu + \varepsilon\,\partial_\nu A_\mu + O(\varepsilon^2)$$

$$A_\nu(0, 4\varepsilon(1-s)) = A_\nu + 4\varepsilon(1-s)\,\partial_\nu A_\nu + O(\varepsilon^2)$$

$$\int_0^1 ds\, A(\gamma'(s);\gamma(s)) = \int_0^{1/4} ds\left(4\varepsilon A_\mu + 16\varepsilon^2 \partial_\mu A_\mu s\right) + \int_{1/4}^{1/2} ds\left(4\varepsilon A_\nu + 4\varepsilon^2 \partial_\mu A_\nu + 4\varepsilon^2(4s-1)\partial_\nu A_\nu\right)$$

$$- \int_{1/2}^{3/4} ds\left(4\varepsilon A_\mu + 4\varepsilon^2(3-4s)\partial_\mu A_\mu + 4\varepsilon^2 \partial_\nu A_\mu\right) - \int_{3/4}^{1} ds\left(4\varepsilon A_\nu + 16\varepsilon^2(1-s)\partial_\nu A_\nu\right)$$

$$= \varepsilon A_\mu + \frac{1}{2}\varepsilon^2 \partial_\mu A_\mu + \varepsilon A_\nu + \varepsilon^2 \partial_\mu A_\nu + \frac{1}{2}\varepsilon^2 \partial_\nu A_\nu$$

$$- \varepsilon A_\mu - \frac{1}{2}\varepsilon^2 \partial_\mu A_\mu - \varepsilon^2 \partial_\nu A_\mu - \varepsilon A_\nu - \frac{1}{2}\varepsilon^2 \partial_\nu A_\nu$$

$$= \varepsilon^2\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)$$ so far, so good.

Now for the second integral:

$$\frac{1}{2} P\left(\int_0^1 ds\, A(\gamma'(s);\gamma(s))\right)^2 = \int_{1>s_1>s_2>0} ds_1\, ds_2\, A(\gamma'(s_1);\gamma(s_1))\, A(\gamma'(s_2);\gamma(s_2))$$

$$= \int_0^1 \int_0^{s_1} ds_2\, ds_1\, A(s_1)\, A(s_2)$$

We can also Taylor expand the integrand here, but because we only want to keep terms up to ε^2, we'll only need the zeroth-order term. That is, we can treat A as constant over the different pieces of the path.

First look at the inside integral:

$$\int_0^{s_1} ds_2\, A(s_2) = 4\varepsilon \begin{cases} s_1 A_\mu & 0 \le s_1 \le 1/4 \\ \frac{1}{4} A_\mu + (s_1 - \frac{1}{4}) A_\nu & 1/4 \le s_1 \le 1/2 \\ \frac{1}{4} A_\mu + \frac{1}{4} A_\nu - (s_1 - \frac{1}{2}) A_\mu & 1/2 \le s_1 \le 3/4 \\ \frac{1}{4} A_\nu - (s_1 - \frac{3}{4}) A_\nu & 3/4 \le s_1 \le 1 \end{cases}$$

Inserting this in the s_1 integral gives

$$\int_0^1 ds_1 \int_0^{s_1} ds_2 \; A(s_1)\,A(s_2) = 16\varepsilon^2 \Bigg[\int_0^{1/4} ds \; s\, A_\mu A_\mu + \int_{1/4}^{1/2} ds \; A_\nu \Big(\tfrac{1}{4} A_\mu + (s - \tfrac14) A_\nu \Big)$$

$$- \int_{1/2}^{3/4} ds \; A_\mu \Big(\tfrac14 A_\mu + \tfrac14 A_\nu - (s - \tfrac12) A_\mu \Big) - \int_{3/4}^{1} ds \; A_\nu \Big(\tfrac14 A_\nu - (s - \tfrac34) A_\nu \Big) \Bigg]$$

$$= \tfrac12 \varepsilon^2 \cancel{A_\mu^2} + \varepsilon^2 A_\nu A_\mu + \tfrac12 \varepsilon^2 \cancel{A_\nu A_\nu} - \varepsilon^2 \cancel{A_\mu^2} - \varepsilon^2 A_\mu A_\nu + \tfrac12 \cancel{\varepsilon^2 A_\mu^2} - \cancel{\varepsilon^2 A_\nu^2} + \cancel{\tfrac12 \varepsilon^2 A_\nu^2}$$

$$= \varepsilon^2 \left[A_\nu , A_\mu \right]$$

Altogether we have

$$H(\gamma, D) = 1 - \varepsilon^2 \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) - \varepsilon^2 \left[A_\mu , A_\nu \right] = 1 - \varepsilon^2 F_{\mu\nu}$$

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96. Show that the holonomies along any two homotopic paths are the same if the curvature vanishes.

Let's choose a homotopy γ_s between two paths γ_0 and γ_1 and show that $\frac{d}{ds} H(\gamma_s, D) = 0$ if D is flat.

$$\frac{d}{ds} \left(u - \int_0^t dt_1 \, A(\gamma_s'(t_1) ; \gamma_s(t_1))\, u + \int_0^t \int_0^{t_1} dt_1\, dt_2 \; \underbrace{A(\gamma_s'(t_1) ; \gamma_s(t_1))\, A(\gamma_s'(t_2) ; \gamma_s(t_2))} \, u + \dots \right)$$

$$= - \int_0^t dt_1 \, \frac{d}{ds} A(\gamma_s'(t_1) ; \gamma_s(t_1))\, u + \int_0^t \int_0^{t_1} dt_1\, dt_2 \, \frac{d}{ds} \Big[\qquad \Big] u + \dots$$

$$= - \int_0^t dt_1 \, A' u + \int_0^t \int_0^{t_1} dt_1\, dt_2 \Big(A'(t_1) A(t_2) + A(t_1) A'(t_2) \Big) u + \dots$$

$$H(\gamma_s, D) = \sum_{n=0}^{\infty} (-1)^n \int_{t \geq t_1 \geq \cdots \geq t_n \geq 0} A(\gamma_s'(t_1))\, A(\gamma_s'(t_2)) \cdots A(\gamma_s'(t_n))\, dt_n \cdots dt_1$$

Differentiating, we obtain

$$\frac{d}{ds} H(\gamma_s, D) = \sum_{n=0}^{\infty} (-1)^n \int_{t \geq t_1 \geq \cdots \geq t_n \geq 0} \left(A_1' A_2 \cdots A_n + A_1 A_2' \cdots A_n + \cdots + A_1 A_2 \cdots A_n' \right) dt_n \cdots dt_1$$

using the shorthand $A_k' = \frac{d}{ds} A(\gamma'(t_k))$

more to come (at least include the case of Abelian vector potential)

97. Show that every connection on a bundle $\pi: E \to M$ is flat if M is one dimensional.

The connection is flat when the curvature vanishes:

$$F(u,v) = [D_u, D_v] - D_{[u,v]} = 0$$

Since M is one dimensional, in any open set $\mathcal{U}$ we have but one coordinate, say x, and one associated coordinate vector field, ∂_x, which is a basis for vector fields on $\mathcal{U}$ (having functions as components). Thus $u = u'(x)\partial_x$ and $v = v'(x)\partial_x$ so that

$$F(u,v) = u'(x)v'(x)\, F(\partial_x, \partial_x) = 0 \quad \text{since } F \text{ is antisymmetric.}$$

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98. Use the results of exercise 72 to show that any E-valued differential form can be written – not necessarily uniquely – as a sum of those of the form $s \otimes \omega$, where s is a section of E and ω is an ordinary differential form on M.

Isn't the point of these "not nec. uniquely" exercises the same as the HJW theorem of QIT? In other words, for the vector space $V = U \otimes W$, we can choose a basis for V by choosing bases for U and W and multiplying them. Then any element of V can be expressed uniquely. But there's still the freedom to use an overcomplete set of product vectors in the decomposition, and all decompositions will be related by unitaries in the manner of the HJW theorem.

Here, E is a vector bundle, so its got a basis, and the space of forms certainly does, too, so the above arguments apply.

99. Using the previous exercise, show that there is a unique way to define the wedge product of an E-valued form, and an ordinary form such that the wedge of the E-valued form $s \otimes \omega$ and the ordinary form μ is given by

$$(s \otimes \omega) \wedge \mu = s \otimes (\omega \wedge \mu)$$

and such that the wedge product depends $C^\infty(M)$-linearly on each factor.

We shall work in a product basis, define the wedge product by the formula given, and then show that the resulting expression is basis-independent. Start from an E-valued form α:

$$\alpha = \sum_{jk} b_{jk} s_j \otimes \omega_k = \sum_{jk} c_{jk} s'_j \otimes \omega'_k$$

for some bases $\{s_j\}$, $\{\omega_j\}$ and $\{s'_k\}$, $\{\omega'_k\}$. The bases are related by $s_j = T_j^k s'_k$, $\omega_j = R_j^k \omega'_k$. Choosing the first, define

$$\alpha \wedge \mu = \sum_{jk} b_{jk} s_j \otimes (\omega_k \wedge \mu) \qquad \text{for } \mu \text{ an ordinary form.}$$

Now change the basis.

$$\alpha \wedge \mu = \sum_{jk} b_{jk} T_j^l s'_l \otimes (R_k^m \omega'_m \wedge \mu)$$

The wedge product is linear, so

$$\alpha \wedge \mu = \sum_{\substack{jk \\ lm}} \underbrace{b_{jk} T_j^l R_k^m}_{c_{lm}} s'_l \otimes (\omega'_m \wedge \mu)$$

$$= \sum_{lm} c_{lm} s'_l \otimes (\omega'_m \wedge \mu)$$

which is what we would have obtained by defining the product in the other basis. Thus the product is independent of basis, and $C^\infty(M)$-linear by construction.

100. Show that the two definitions of the exterior covariant derivative are equivalent.

On the one hand we define d_D by saying $(d_D s)(v) = D_v s$ for any v. On the other, if we work in a coordinate system x^μ, then we have $d_D s = D_\mu s \otimes dx^\mu$. To see that these are equivalent, act on an arbitrary vector:

$$(d_D s)\, v = (D_\mu s \otimes dx^\mu)\, v = v(x^\mu)\, D_\mu s = v^\nu \partial_\nu (x^\mu)\, D_\mu s = v^\mu D_\mu s$$

$$= D_v s \qquad \text{as intended.}$$

101. Check that the (wedge) product of End(E)-valued forms with E-valued forms is well-defined by its action on products:

$$(T \otimes \omega) \wedge (s \otimes \mu) = T(s) \otimes (\omega \wedge \mu)$$

See 98 and 99. Everything is linear, so the choice of basis doesn't matter.