To any vector field v one can associate a flow ϕ_t which maps points p along the integral curves of v. Can we think of the pullback ϕ_t^* as an exponential?

Consider an arbitrary function f, evaluated at some point along the flow: $f(\phi_t(p))$. f is smooth, so we could hope that the Taylor series converges, and then

$$f(\phi_t(p)) = \sum_{k=0}^{\infty} \frac{d^k}{ds^k} f(\phi_s(p))\Big|_{s=0} \frac{1}{k!} t^k$$

$$= f(p) + \frac{d}{ds} f(\phi_s(p))\Big|_{s=0} t + \frac{1}{2} \frac{d^2}{ds^2} f(\phi_s(p))\Big|_{s=0} t^2 + \dots$$

The second term is just $t[vf](p)$ since ϕ_s is the flow of v.

We'd like to write the second term as $\frac{1}{2}t^2[vvf](p)$. Is this true?

$$[vvf](p) = \frac{d}{ds}[vf](\phi_s(p))\Big|_{s=0} = \frac{d^2}{ds\,ds'} f(\phi_{s'}(\phi_s(p)))\Big|_{s=s'=0}$$

$$= \frac{d^2}{ds\,ds'} f(\phi_{s+s'}(p))\Big|_{s=s'=0}$$

Letting $t=s+s'$, we have $\frac{d}{dt} = \frac{d}{ds} = \frac{d}{ds'}$, so

$$[vvf](p) = \frac{d^2}{dt^2} f(\phi_t(p))\Big|_{t=0}$$

The same argument will work for $[v^k f](p)$, so we have

$$f(\phi_t(p)) = [\phi_t^* f](p) = \sum_{k=0}^{\infty} \frac{1}{k!} t^k [v^k f](p)$$

Thus $\phi_t^* = \sum_{k=0}^{\infty} \frac{1}{k!} t^k v^k = \exp(tv)$

Does this simplify the Lie derivative def$^{\underline{n}}$? yes!

Return to the original definition of $\mathcal{L}_v(u)$:

$$\mathcal{L}_v(u) = \lim_{t\to 0} \frac{1}{t}\left(\phi_{-t*}u - u\right)$$

We can easily show that $\phi_{-t*}u = \phi_t^* u \phi_{-t}^*$ by applying both sides to a function and evaluating the result at point p:

$$\left[(\phi_{-t*}u)f\right](p) = \left[u\phi_{-t}^* f\right](\phi_t(p)) = \left[\phi_t^*\left(u\phi_{-t}^* f\right)\right](p)$$

Thus, in the Lie derivative we have

$$\mathcal{L}_v(u) = \lim_{t\to 0}\frac{1}{t}\left(\phi_t^* u \phi_{-t}^* - u\right)$$

$$= \lim_{t\to 0}\frac{1}{t}\left(e^{tv} u e^{-tv} - u\right)$$

$$= \lim_{t\to 0}\frac{1}{t}\left(t[v,u] + O(t^2)\right)$$

$$= [v,u]$$

The simplest proof so far.