Baez Exercises: Connections

83. For $A = \sum_i T_i \otimes \omega_i$, where T_i are sections of $\mathrm{End}(E)$ and ω_i are 1-forms, show that A is well-defined, i.e. independent of how we write A as a sum $\sum_i T_i \otimes \omega_i$

Can't we just express everything in terms of a basis of sections times a basis of one-forms?

84. Prove that any connection D can be written as $D^0 + A$, for some choice of standard flat connection D^0 stemming from the choice of local trivialization, and A an $\mathrm{End}(E)$-valued 1-form.

If we choose a local trivialization, this sets D^0 as follows. From the LT we get a basis of sections e_j. Then $D^0_v(e_j) = 0$ so that

$$D^0_v s = D^0_v s^j e_j = v(s^j)\, e_j + s^j D^0_v e_j = v(s^j) e_j$$

Now let's check that $D^0 + A$ is a connection.

1. $D_v(\alpha s) = D^0_v(\alpha s) + A(v)(\alpha s) = \alpha D^0_v(s) + \alpha A(v)(s) = \alpha D_v(s)$
2. $D_v(s+t) = D^0_v(s+t) + A(v)(s+t) = D^0_v(s) + D^0_v(t) + A(v)(s) + A(v)(t)$
   $= D_v(s) + D_v(t)$
3. $D_v(fs) = D^0_v(fs) + A(v)(fs) = v(f)s + fD^0_v(s) + f A(v)(s)$
   $= v(f) s + f D_v(s)$
4. $D_{v+w}(s) = D^0_{v+w}(s) + A(v+w)(s) = D^0_v(s) + A(v)(s) + D^0_w(s) + A(w)(s)$
   $= D_v(s) + D_w(s)$

5. $D_{fv}(s) = D^0_{fv}(s) + A(fv)(s) = f D^0_v(s) + f A(v)(s)$

$= f D_v(s)$

Then we must check that if D is a connection, we can write $D = D^0 + A$ for some $\mathrm{End}(E)$-valued 1-form A. So, is $D - D^0$ an $\mathrm{End}(E)$-valued 1-form?

1. It is linear in v, since D and D^0 are

2. It is linear in s:

$$D_v(fs) - D^0_v(fs) = \cancel{v(f)}s + f D_v s - \cancel{v(f)}s - f D^0_v s = f(D_v - D^0_v)s$$

And given D, D^0, how can we define A? Choosing a basis e_j of sections and local coordinates x^μ (actually the latter specify the former) we define

$$A = A^j_{\mu i}\, e_j \otimes e^i \otimes dx^\mu \quad \text{and set } A^j_{\mu i} e_j = (D - D^0)(\partial_\mu) e_i$$

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85. Show that $D'_v(s) = g D_v(g^{-1}s)$ is a connection

1. $D'_{\alpha v}(s) = g D_{\alpha v}(g^{-1}s) = \alpha g D_v(g^{-1}s) = \alpha D'_v(s)$

2. $D'_{v+w}(s) = g D_{v+w}(g^{-1}s) = g D_v(g^{-1}s) + g D_w(g^{-1}s) = D'_v(s) + D'_w(s)$

3. $D'_v(fs) = g D_v(g^{-1}fs) = g D_v(f g^{-1}s) = g\, v(f)\, g^{-1}s + f g D_v(g^{-1}s)$

$= v(f) + f D'_v(s)$ $\quad$ $\hookrightarrow$ f is a function, $g \in \mathrm{End}(E)$

4. $D'_v(s+t) = g D_v(g^{-1}(s+t)) = g D_v(g^{-1}s + g^{-1}t) = D'_v(s) + D'_v(t)$

5. $D'_{fv}(s) = g D_{fv}(g^{-1}s) = f g D_v(g^{-1}s) = f D'_v(s)$