Lie Derivatives

To bad Baez and Munian don't give a treatment of Lie derivatives...
Let's just do it ourselves!

The basic idea behind the Lie derivative is that it is the derivative (of whatever) along a flow defined by a vector field v.

Start with the vector field v and let $\phi_t(p)$ be the integral curve through the point p, parameterized by t.
The maps $\phi_t : M \to M$ define the flow, whose defining equation is

$$\frac{d}{dt'}\phi_{t'}(p)\Big|_{t'=t} = v_{\phi_t(p)} \leftarrow \text{vector } v \text{ at point } \phi_t(p)$$

integral curve
differentiated wrt t'
and then evaluated at t

(Note: Baez leaves out the $|_{t'=t}$ part; being more explicit will help later.)

Now consider a function $f : M \to \mathbb{R}$. We can compare the function value at a point p and one a little farther along the flow, at $\phi_t(p)$:

$$\Delta f_t(p) = f(\phi_t(p)) - f(p)$$

This is nothing other than the pullback of f: $f(\phi_t(p)) = \phi_t^* f(p)$, so

$$\Delta f_t(p) = \phi_t^* f(p) - f(p)$$

Using the pullback, we can formulate the change in the function along the flow starting from p in terms of functions evaluated only at p.
Then we can define the derivative $\mathcal{L}_v f$ of f along ϕ_t in the usual manner:

$$\mathcal{L}_v f\Big|_p = \lim_{t\to 0}\frac{\Delta f_t(p)}{t} = \lim_{t\to 0}\frac{f(\phi_t(p)) - f(\phi_0(p))}{t} = \frac{d}{dt} f(\phi_t(p))\Big|_{t=0}$$

which is nothing other than the directional derivative:

$$\mathcal{L}_v f\Big|_p = \left[\frac{d}{dt}\phi_t(p)\Big|_{t=0}\right] f = v_p f.$$

What about the derivative of a vector field or one-form?

For these we follow the same procedure. The flow gives us a way to push vectors forward, ϕ_{t*}, in accordance with

$$(\phi_{t*}v)f = v(\phi_t^* f)$$

Under the flow the tangent vector v_p gets mapped to a new tangent vector $\phi_{t*}v$, now located at $\phi_t(p)$.

Sometimes this is a little confusing, but if we think of $\phi : M \to N$ and f as a function on N, then when evaluating the above definition at some point $p \in M$, we must have

$$[(\phi_* v)f](\phi(p)) = [v(\phi^* f)](p)$$

The moral is: if we pull f back, we have to evaluate it at the pulled-back point.

Keeping this in mind, we can show that the tangent vectors themselves are similarly transformed; (This is Baez I.18)

$$(\phi_* v)_q = \phi_*(v_p) \quad \text{for } q = \phi(p)$$

$$\begin{aligned}(\phi_* v)_q f &= [\phi_* v(f)](q) = [\phi_* v(f)](\phi(p)) \\ &= [v(\phi^* f)](p) = v_p(\phi^* f) \\ &= (\phi_* v_p) f\end{aligned}$$

here we go back to the original point, since we go back to the original field v

Observe that in the definition of the Lie derivative of a function, the pullback was used to bring the function value at some point $q = \phi_t(p)$ further along the flow back to the point p.

However, we cannot pull vectors back under the flow ϕ_t, only push them forward. To achieve the same ends, we can simply push the vector field forward under the inverse flow!

Thus, before we can define the Lie derivative of a vector field, we need to consider inverse flows.

What is the vector field associated with ϕ_{-t}? Intuitively, it's just $-v$ but we should prove this, and further test our skills with the notation. The proof technique is the same as ever — apply it to a function and look at the result.

Let w give rise to a flow ψ_t, which happens to equal ϕ_{-t}, where ϕ_t is the flow given by v. We want to show that $w = -v$. Apply w_p to an arbitrary function f:

$$wf(p) = \frac{d}{ds} f(\psi_s(p)) = \frac{d}{ds} f(\phi_{-s}(p)) = \frac{d}{dt} f(\phi_t(p)) \frac{ds}{dt} \quad \text{where } t = -s$$

by the usual chain rule. Thus

$$wf(p) = -\frac{d}{dt} f(\phi_t(p)) = -vf(p) \quad \text{so} \quad w = -v$$

An entirely similar argument shows that if $\psi_t = \phi_{g(t)}$, then

$$w_p = g'(p)\, v_p$$

Now we are ready to define $\mathcal{L}_v(u)$, the derivative of v along some vector field u. We want to compare u_p with $u_{\phi_t(p)}$, a little further along the flow. But these vectors live in different tangent spaces, so this isn't immediately possible. By pushing $u_{\phi_t(p)}$ forward along the inverse flow ϕ_{-t}, it ends up in the tangent space at p, and we can then sensibly take their difference. This reasoning leads to the definition

$$\mathcal{L}_v(u)\Big|_p = \lim_{t \to 0} \frac{\phi_{-t*}(u_{\phi_t(p)}) - u_p}{t}$$

Slightly more generally:

$$\mathcal{L}_V(u)\Big|_{\phi_{t_1}(p)} = \mathcal{L}_V(u)\Big|_{\phi_{t_2}(p)} = \lim_{t_2\to t_1} \frac{\phi_{\overset{\Delta t = t_2 - t_1}{-\Delta t *}}\left(u_{\phi_{t_2}(p)}\right) - u_{\phi_{t_1}(p)}}{t_2 - t_1}$$

Choosing $t_1 = 0$, $t_2 = t$ gives the above definition, but we will shortly have occasion to use this more general form.

As a definition, this form is all well and good. But is there a more useful form? Does the Lie derivative of a vector field equal something we can calculate? Yes! In fact, it's just the Lie bracket:

$$\mathcal{L}_V(u)\Big|_p = [V, u]_p$$

following Carroll

There are a couple of ways to prove this. The simplest is to use the initial definition and make a judicious choice of coordinates. Since the integral curves of V form a congruence and "fill" the manifold, at least locally near p, in the sense that through any point q is a unique integral curve of V, we can use them to define a coordinate system. Let x^1 be the parameter along the integral curves and the other x^k be chosen arbitrarily. Then $V = \partial_1$ and moreover, the pushforward takes on a simple form. Under the flow ϕ_t the point p with coordinates $x^\mu(p)$ goes to $q = \phi_t(p)$, with coordinates $x^\mu(q)$. Clearly $x^1(q) = x^1(p) + t$, while $x^j(q) = x^j(p)$ for $j \neq 1$

Then $\phi_{t*}\partial_k = \partial_k$ since $(\phi_{t*}\partial_k)f = \partial_k \phi_t^* f = \partial_k f\circ\phi_t = \frac{\partial f}{\partial y^j}\frac{\partial y^j}{\partial x^k} = \frac{\partial f}{\partial x^k}$

for $y^j = (\phi_t)^j$, and $\frac{\partial y^j}{\partial x^k} = \delta^j{}_k$ $\qquad = \partial_k f$

(see Baez I.19) should imagine flow maps $M \to N$ and use different symbols for coords on M and N

Thus, for any $u = u^\mu \partial_\mu$, $\phi_{t*} u = u^\mu \partial_\mu$

↳ vector field, not tangent vector

In the definition of $\pounds_v(u)|_p$ we therefore have

$$\pounds_v(u)\big|_p = \lim_{t\to 0} \frac{1}{t}\left(\phi_{-t*} u_{\phi_t(p)} - u_p\right)$$

$$= \lim_{t\to 0} \frac{1}{t}\left(u^\mu(\phi_t(p))\, \phi_{-t*}\left(\partial_\mu\big|_{\phi_t(p)}\right) - u^\mu(p)\,\partial_\mu\big|_p\right)$$

$$= \lim_{t\to 0} \frac{1}{t}\left(u^\mu(\phi_t(p))\underbrace{\left(\phi_{-t*}\partial_\mu\right)_p}_{=\partial_\mu|_p} - u^\mu(p)\,\partial_\mu\big|_p\right)$$

$$= \lim_{t\to 0} \frac{1}{t}\left(u^\mu(\phi_t(p)) - u^\mu(p)\right)\partial_\mu\big|_p$$

$$= \frac{d}{dt'} u^\mu(\phi_{t'}(p))\Big|_{t'=t}\,\partial_\mu\big|_p$$

$$= v\, u^\mu(p)\,\partial_\mu\big|_p = vu\big|_p$$

Now consider $uv|_p = u^\mu(p)\underbrace{\partial_\mu\big|_p v^\nu(p)}_{=0}\partial_\nu\big|_p$ since $v^\nu(p) = \delta^{\nu 1}$

$$= 0$$

So $\pounds_v(u)|_p = [v,u]_p$, at least in this coordinate system

But because $[v,u]_p$ is a tangent vector, as is the Lie derivative, this equality holds irrespective of coordinate system. (vu is not a tangent vector)

We can also prove this in a coordinate-free manner by using the slightly more general definition. Consider

$$\pounds_v(u)\Big|_{\phi_{t_2}(p)} = \lim_{t_2 \to t_1} \frac{\phi_{-\Delta t *}\left(u_{\phi_{t_2}(p)}\right) - u_{\phi_{t_1}(p)}}{t_2 - t_1}$$

and take $t_2(p) = r$, $t_2 - t_1 = \Delta t = t$ so that we have

$$\pounds_v(u)\Big|_r = \lim_{t \to 0} \frac{1}{t}\left(\phi_{-t*} u_r - u_{\phi_{-t}(r)}\right)$$

This is akin to $\left.\frac{df}{dx}\right|_{x=a} = \lim_{h \to 0} \frac{f(a) - f(a-h)}{h}$

Now apply to an arbitrary function:

$$\pounds_v(u)\Big|_r f = \left[\pounds_v(u) f\right](r) = \lim_{t \to 0} \frac{1}{t}\left(\left(\phi_{-t*} u_r\right) f - u_{\phi_{-t}(r)} f\right)$$

$$= \lim_{t \to 0} \frac{1}{t}\left(\left(\phi_{-t*} u\right)_{\phi_{-t}(r)} f - u_{\phi_{-t}(r)} f\right)$$

$$= \lim_{t \to 0} \frac{1}{t}\left(\left[\left(\phi_{-t*} u\right) f\right]\left(\phi_{-t}(r)\right) - uf\left(\phi_{-t}(r)\right)\right)$$

$$= \lim_{t \to 0} \frac{1}{t}\left(\left[u\left(\phi_{-t}^{*} f\right)\right](r) - uf\left(\phi_{-t}(r)\right)\right)$$

The vector field u has its own flow, which we call ψ_s. From the defining equation applied to a function, one gets

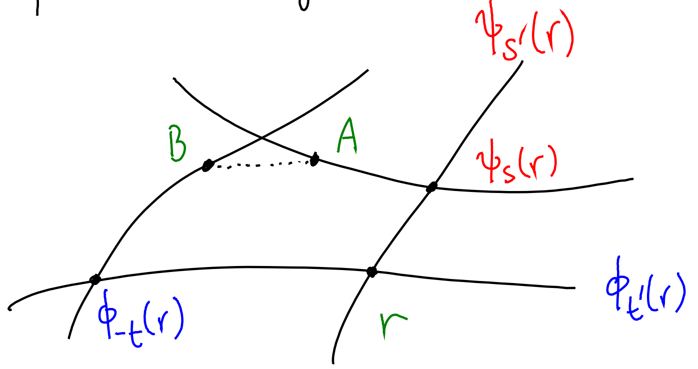
$$u_{\psi_s(p)} f = [uf]\left(\psi_s(p)\right) = \left[\frac{d}{ds'} \psi_{s'}(p)\right]_{s'=s} f = \left.\frac{d}{ds'} f\left(\psi_{s'}(p)\right)\right|_{s'=s}$$

Setting $s=0$ gives $\quad u_p f = \left.\frac{d}{ds} f\left(\psi_s(p)\right)\right|_{s=0}$. And so

$$\pounds_v(u)|_r f = \lim_{t\to 0} \frac{1}{t} \frac{d}{ds}\left(\phi_{-t}^* f(\psi_s(r)) - f(\psi_s(\phi_{-t}(r)))\right)\Big|_{s=0}$$

$$= \lim_{t\to 0} \frac{1}{t}\frac{d}{ds}\left(f(\phi_{-t}(\psi_s(r))) - f(\psi_s(\phi_{-t}(r)))\right)\Big|_{s=0}$$

Drawing a little diagram reveals what's going on geometrically:



$A = \phi_{-t}(\psi_s(r))$

$B = \psi_s(\phi_{-t}(r))$

Inside the brackets is simply $f(A) - f(B)$, which is precisely what is measured by the Lie bracket. To see this formally, first add and subtract a term $f(\psi_s(r)) = f(\phi_0(\psi_s(r))) = f(\psi_s(\phi_0(r)))$.

Then $\lim_{t\to 0} \frac{1}{t}\left(f(\phi_{-t}(\psi_s(r))) - f(\phi_{-0}(\psi_s(r)))\right) = \frac{d}{dt} f(\phi_{-t}(\psi_s(r)))\Big|_{t=0}$,

and similarly for the other two terms

so we obtain

$$\pounds_v(u)|_r f = \frac{d^2}{dt\,ds}\left[f(\phi_{-t}(\psi_s(r))) - f(\psi_s(\phi_{-t}(r)))\right]_{s=t=0}$$

By the results of Baez I.23 this is nothing other than $[u,-v]_r$, meaning $\pounds_v(u)|_r = [v,u]_r$ or just $\underline{\underline{\pounds_v(u) = [v,u]}}$

Next, what about the Lie derivative of a one-form ω? Following the logic above, we define it as

$$\mathcal{L}_V\omega\Big|_p = \lim_{t\to 0}\frac{1}{t}\left[\phi_t^*\omega_{\phi_t(p)} - \omega_p\right] = \lim_{t\to 0}\frac{1}{t}\left[(\phi_t^*\omega)_p - \omega_p\right]$$

This form is also not very useful by itself. But we can prove a key property in a flash, namely, that the Lie derivative commutes with the exterior derivative d. This follows immediately from the fact that $d\phi^*f = \phi^* df$, i.e. the exterior derivative commutes with pullbacks. (Actually, this is true for any p-form ω)

$$\left[\mathcal{L}_V\, df\right]_p = \lim_{t\to 0}\frac{1}{t}\left[(\phi_t^* df)_p - df_p\right] = d\left[\lim_{t\to 0}\frac{1}{t}\left[(\phi^* f)_p - f_p\right]\right]$$

$$= d\left[\mathcal{L}_V f\right]_p$$

... so this is a proof that $\mathcal{L}_V\, d\omega = d\left[\mathcal{L}_V\omega\right]$

With this in hand we can prove the Leibniz rule:

$$\mathcal{L}_V\, \omega(u) = (\mathcal{L}_V\omega)u + \omega\mathcal{L}_V(u)$$

Consider an exact form df; since these span the space of one-forms, the proof for an arbitrary one-form follows by linearity. The LHS is

$$\mathcal{L}_V\, df(u) = v(df(u)) = vuf$$

Meanwhile, on the RHS we have

$$(\mathcal{L}_V\, df)u + df\,\mathcal{L}_V(u) = \left[d(\mathcal{L}_V f)\right]u + df\left[v,u\right]$$

$$= u\mathcal{L}_V f + [v,u]f = \cancel{uvf} + vuf - \cancel{uvf}$$

So the Leibniz rule holds. (phew!)

This could have been taken as the definition of $\mathcal{L}_V\omega$, so it's good to see that everything is consistent.

Still to prove: $\mathcal{L}_V\omega = d\left[\omega(v)\right] + d\omega(v)$