Baez Exercises: Vector Bundle Constructions

78. Prove that if $g_{\alpha\alpha} = 1$ and $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$ where defined, $\pi : E \to M$ is a vector bundle.

We start with the disjoint union $\bigcup_\alpha U_\alpha \times V$, where the U_α are open sets on M.
Points $(p, v) \in U_\alpha \times V$ and $(p, v') \in U_\beta \times V$ are to be identified when

$$v = \rho(g_{\alpha\beta}(p)) v'.$$

Now we need to be able to add points in the fiber E_p above p.
For a given open set U_α we have

$$(p, v_1)_\alpha + (p, v_2)_\alpha = (p, v_1 + v_2)_\alpha$$

Seen from the point of view of open set U_β the rule holds,

$$(p, g_{\beta\alpha} v_1)_\beta + (p, g_{\beta\alpha} v_2)_\beta = (p, g_{\beta\alpha} v_1 + g_{\beta\alpha} v_2)_\beta$$

$$= (p, g_{\beta\alpha}(v_1 + v_2))_\beta,$$

Since the transition function $g_{\alpha\beta}$ is linear.

Similarly, we could use an open set U_γ and mix things up:

$$(p, v_1)_\alpha + (p, g_{\beta\alpha} v_2)_\beta \overset{?}{=} (p, g_{\gamma\alpha}(v_1 + v_2))_\gamma$$

LHS: $(p, v_1)_\alpha = (p, g_{\gamma\alpha} v_1)_\gamma$

$$(p, g_{\beta\alpha} v_2)_\beta = (p, g_{\gamma\beta} g_{\beta\alpha} v_2)_\gamma$$

$= g_{\gamma\alpha} g_{\alpha\gamma} g_{\gamma\beta} g_{\beta\alpha} = g_{\gamma\alpha}$ by the cocycle condition

$$(p, v_1)_\alpha + (p, g_{\beta\alpha} v_2)_\beta = (p, g_{\gamma\alpha} v_1)_\gamma + (p, g_{\gamma\alpha} v_2)_\gamma = \text{RHS}$$

which is what we wanted to prove. So the cocycle condition ensures that if we add vectors belonging to a given open set U_α, this sum can be translated into a sum of vectors corresponding to a different open set U_β in a unique way.

To define the projection operator π, just use π_α for any α:

$$\pi(p, v)_\beta = \pi_\alpha(p, g_{\alpha\beta} v)_\alpha = p$$

Locally it's trivial by construction (~~any~~ take open set U_β and we get $E \simeq U_\beta \times V$)

Finally, it's a vector bundle since under a local trivialization induced by choice of U_α, each fiber $E_p = p \times V$ can be mapped to $p \times \mathbb{R}^n$ by choosing a basis for V.

---

79. Show that if $T: E_p \to E_p$ is of the form $[p, v]_\alpha \mapsto [p, d\rho(x) v]_\alpha$ for some $x \in \mathfrak{g}$ and $p \in U_\alpha \cap U_\beta$, then T is also of the form

$$[p, v']_\beta \mapsto [p, d\rho(x') v']_\beta$$

for some $x' \in \mathfrak{g}$ (typo?)

Choose $v' = g_{\beta\alpha} v$, so $[p, v]_\alpha = [p, g_{\beta\alpha} v]_\beta$

This is mapped to $[p, d\rho(x) v]_\alpha = [p, g_{\beta\alpha} d\rho(x) v]_\beta$

Picking $x' = g_{\beta\alpha} \, x \, g_{\alpha\beta}$ so $d\rho(x') = g_{\beta\alpha} d\rho(x) g_{\alpha\beta}$

Then $d\rho(x') v' = g_{\beta\alpha} d\rho(x) v$, whence the T map has the desired property.