Proofs for "Expertise and knowledge: a modal logic perspective"

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Appendix A Proofs

Proof of Lemma 7 We first show that R is reflexive and has the *Euclidean property* (xRy and xRz implies yRz). For reflexivity, let $\Gamma \in X_{\mathsf{L}}$. Suppose $\mathsf{A}\varphi \in \Gamma$. By ($\mathrm{T}_{\mathsf{A}}$) and closure of maximally consistent sets under modus ponens, $\varphi \in \Gamma$. Hence $\Gamma R \Gamma$.

For the Euclidean property, suppose $\Gamma R \Delta$ and $\Gamma R \Lambda$. We show $\Delta R \Lambda$ by contraposition. Suppose $\varphi \notin \Lambda$. Since $\Gamma R \Lambda$, this means $\mathsf{A}\varphi \notin \Gamma$. Hence $\neg\mathsf{A}\varphi \in \Gamma$, and by ($5_{\mathsf{A}}$) we get $\mathsf{A}\neg\mathsf{A}\varphi \in \Gamma$. Now $\Gamma R \Delta$ gives $\neg\mathsf{A}\varphi \in \Delta$, so $\mathsf{A}\varphi \notin \Delta$.

To conclude we need to show R is symmetric and transitive. For symmetry, suppose $\Gamma R \Delta$. By reflexivity, $\Gamma R \Gamma$. The Euclidean property therefore gives $\Delta R \Gamma$. For transitivity, suppose $\Gamma R \Delta$ and $\Delta R \Lambda$. By symmetry, $\Delta R \Gamma$. The Euclidean property again gives $\Gamma R \Lambda$. □

Proof of Lemma 8 Part (1) was shown in the main text. For (2), note that by (1) we have

$$\begin{aligned} \mathsf{A}(\varphi \to \psi) \in \Sigma &\iff |\varphi \to \psi|_\Sigma = X_\Sigma \\ &\iff \forall \Gamma \in X_\Sigma : \varphi \to \psi \in \Gamma \end{aligned}$$

Suppose $\mathsf{A}(\varphi \to \psi) \in \Sigma$. Take $\Gamma \in |\varphi|_\Sigma$. Then we have $\varphi, \varphi \to \psi \in \Gamma$, so $\psi \in \Gamma$. This shows $|\varphi|_\Sigma \subseteq |\psi|_\Sigma$. Conversely, suppose $|\varphi|_\Sigma \subseteq |\psi|_\Sigma$. Take $\Gamma \in X_\Sigma$. If $\varphi \notin \Gamma$ then $\neg\varphi \in \Gamma$, so $\neg\varphi \vee \psi \in \Gamma$ and thus $\varphi \to \psi \in \Gamma$. If $\varphi \in \Gamma$ then $\Gamma \in |\varphi|_\Sigma \subseteq |\psi|_\Sigma$, so $\psi \in \Gamma$. Thus $\varphi \to \psi \in \Gamma$ in this case too. Hence $\mathsf{A}(\varphi \to \psi) \in \Sigma$.

For (3) first note that $\mathsf{A}(\alpha \wedge \beta) \in \Sigma$ iff both $\mathsf{A}\alpha \in \Sigma$ and $\mathsf{A}\beta \in \Sigma$. This can be shown using (K_{A}), (MP) and instances of the propositional tautologies $(p \wedge q) \to p$ (for the left-to-right implication) and $p \to q \to (p \wedge q))$ (for the right-to-left implication). Recalling that $\varphi \leftrightarrow \psi$ is an abbreviation for $(\varphi \to \psi) \wedge (\psi \to \varphi)$, we get

$$\begin{aligned} \mathsf{A}(\varphi \leftrightarrow \psi) \in \Sigma &\iff \mathsf{A}(\varphi \to \psi) \in \Sigma \text{ and } \mathsf{A}(\psi \to \varphi) \in \Sigma \\ &\iff |\varphi|_\Sigma \subseteq |\psi|_\Sigma \text{ and } |\psi|_\Sigma \subseteq |\varphi|_\Sigma \\ &\iff |\varphi|_\Sigma = |\psi|_\Sigma \end{aligned}$$

as required.

□

Proof of Theorem 5 Soundness was shown in the main text. For completeness, we adopt a roughly similar approach to the general case. Let consistency, maximal consistency and other standard notions and notation be defined as before, but now for $\mathsf{L_{int}}$ instead of L. Let $X_{\mathsf{L_{int}}}$ be the set of maximally $\mathsf{L_{int}}$-consistent sets. Define the relation R on $X_{\mathsf{L_{int}}}$ in exactly the same way. Since $\mathsf{L_{int}}$ extends L, R is again an equivalence relation, and we have the analogues of Lemma 8 and Corollary 1.

This time, however, the construction of the canonical model for a given $\Sigma \in X_{\mathsf{L_{int}}}$ is much more straightforward. The set of states is simply X_Σ, i.e. the equivalence class of Σ in R. Overriding earlier terminology, say $A \subseteq X_\Sigma$ is *S-closed* iff $|\varphi|_\Sigma \subseteq A$ implies $|\mathsf{S}\varphi|_\Sigma \subseteq A$ for all $\varphi \in \mathcal{L}$. Then set

$$P_\Sigma = \{A \subseteq X_\Sigma \mid A \text{ is S-closed}\}.$$

Finally, set $V_\Sigma(p) = |p|_\Sigma$, and write $M_\Sigma = (X_\Sigma, P_\Sigma, V_\Sigma)$.

First, we have $M_\Sigma \in \mathbb{M}_{\mathsf{int}}$, i.e. intersections of S-closed sets are S-closed. Indeed, suppose $\{A_i\}_{i\in I}$ is a collection of S-closed sets, and suppose $|\varphi|_\Sigma \subseteq \bigcap_{i\in I} A_i$. Then $|\varphi|_\Sigma \subseteq A_i$ for each i, so S-closure gives $|\mathsf{S}\varphi|_\Sigma \subseteq A_i$. Hence $|\mathsf{S}\varphi|_\Sigma \subseteq \bigcap_{i\in I} A_i$.

Importantly, we have the truth lemma for M_Σ: for all $\Gamma \in X_\Sigma$ and $\varphi \in \mathcal{L}$,

$$M_\Sigma, \Gamma \models \varphi \iff \varphi \in \Gamma,$$

i.e. $\|\varphi\|_{M_\Sigma} = |\varphi|_\Sigma$.

As usual, the proof is by induction on formulas. The case for atomic propositions follows from the definition of V_Σ, the cases for conjunctions and negations hold by properties of maximally consistent sets, and the case for $\mathsf{A}\varphi$ holds by an argument identical to the one used in the general case (Lemma 9). The only interesting cases are therefore for $\mathsf{E}\varphi$ and $\mathsf{S}\varphi$ formulas:

(E): First suppose $\mathsf{E}\varphi \in \Gamma$. We claim $|\varphi|_\Sigma$ is S-closed. This will give $\|\varphi\|_{M_\Sigma} \in P_\Sigma$ by the induction hypothesis and definition of P_Σ, and therefore $M_\Sigma, \Gamma \models \mathsf{E}\varphi$.

So, suppose $|\psi|_\Sigma \subseteq |\varphi|_\Sigma$. Then $\mathsf{A}(\psi \to \varphi) \in \Sigma$. Let $\Delta \in |\mathsf{S}\psi|_\Sigma$. Since $\Delta, \Gamma, \Sigma \in X_\Sigma$, we have $\mathsf{E}\varphi \in \Delta$ and $\mathsf{A}(\psi \to \varphi) \in \Delta$ too. By (W_E), $\mathsf{S}\psi \wedge \mathsf{E}\varphi \to \varphi \in \Delta$. But $\mathsf{S}\psi \in \Delta$, so $\mathsf{S}\psi \wedge \mathsf{E}\varphi \in \Delta$ and thus $\varphi \in \Delta$, i.e. $\Delta \in |\varphi|_\Sigma$. This shows $|\mathsf{S}\psi|_\Sigma \subseteq |\varphi|_\Sigma$, so $|\varphi|_\Sigma$ is S-closed as required.

Now suppose $M_\Sigma, \Gamma \models \mathsf{E}\varphi$. Then, by the induction hypothesis, $|\varphi|_\Sigma$ is S-closed. Since $|\varphi|_\Sigma \subseteq |\varphi|_\Sigma$ clearly holds, we get $|\mathsf{S}\varphi|_\Sigma \subseteq |\varphi|_\Sigma$. This implies $\mathsf{A}(\mathsf{S}\varphi \to \varphi) \in \Sigma$, and ($\mathrm{Red}_\mathsf{E}$) gives $\mathsf{E}\varphi \in \Sigma$. Since $\Gamma \in X_\Sigma$, we get $\mathsf{E}\varphi \in \Gamma$ as required.

(S): Suppose $\mathsf{S}\varphi \in \Gamma$. Take any $A \in P_\Sigma$ such that $\|\varphi\|_{M_\Sigma} \subseteq A$. By the induction hypothesis, $|\varphi|_\Sigma \subseteq A$. By S-closure of A, $|\mathsf{S}\varphi|_\Sigma \subseteq A$. Hence $\Gamma \in |\mathsf{S}\varphi|_\Sigma \subseteq A$. This shows $M_\Sigma, \Gamma \models \mathsf{S}\varphi$.

For the other direction we show the contrapositive. Suppose $\mathsf{S}\varphi \notin \Gamma$. First, we claim $|\mathsf{S}\varphi|_\Sigma$ is S-closed. Indeed, suppose $|\psi|_\Sigma \subseteq |\mathsf{S}\varphi|_\Sigma$. Then $\mathsf{A}(\psi \to \mathsf{S}\varphi) \in \Sigma$. Take any $\Delta \in |\mathsf{S}\psi|_\Sigma$. Since $\Delta \in X_\Sigma$, $\mathsf{A}(\psi \to \mathsf{S}\varphi) \in \Delta$ also. By (W_S), $\mathsf{S}\psi \to \mathsf{SS}\varphi \in \Delta$. Now $\mathsf{S}\psi \in \Delta$ implies $\mathsf{SS}\varphi \in \Delta$, and ($4_\mathsf{S}$) gives $\mathsf{S}\varphi \in \Delta$, i.e. $\Delta \in |\mathsf{S}\varphi|_\Sigma$. This shows $|\mathsf{S}\psi|_\Sigma \subseteq |\mathsf{S}\varphi|_\Sigma$, and thus $|\mathsf{S}\varphi|_\Sigma$ is S-closed.

Hence $|\mathsf{S}\varphi|_\Sigma$ is a set in P_Σ not containing Γ. Moreover, $\|\varphi\|_{M_\Sigma} \subseteq |\mathsf{S}\varphi|_\Sigma$ by the induction hypothesis and (T_S). Hence $M_\Sigma, \Gamma \not\models \mathsf{S}\varphi$.

Strong completeness now follows. If $\Gamma \not\vdash_{\mathsf{L_{int}}} \varphi$, then $\Gamma \cup \{\neg\varphi\}$ is consistent, so by Lindenbaum's Lemma there is $\Sigma \in X_{\mathsf{L_{int}}}$ with $\Sigma \supseteq \Gamma \cup \{\neg\varphi\}$. Considering the model $M_\Sigma \in \mathbb{M}_{\mathsf{int}}$, we have $M_\Sigma, \Sigma \models \Gamma$ and $M_\Sigma, \Sigma \not\models \varphi$ by the truth lemma. Hence $\Gamma \not\models_{\mathbb{M}_{\mathsf{int}}} \varphi$.

$\square$

Proof of Theorem 6 Soundness was shown in the main text. For completeness, we go via relational semantics using the translation $t : \mathcal{L} \to \mathcal{L}_{\mathsf{KA}}$ and Theorem 3. First, let $\mathsf{L}_{\mathsf{S4A}}$ be the logic of $\mathcal{L}_{\mathsf{KA}}$ formulas formed by the axioms and inference rules shown in Table A1. It is well known that $\mathsf{L}_{\mathsf{S4A}}$ is strongly complete with respect to $\mathbb{M}^*_{\mathsf{S4}}$ (Blackburn, De Rijke, & Venema, 2002, Theorem 7.2).

Table A1 Axioms and inference rules for $\mathsf{L}_{\mathsf{S4A}}$.

$\mathsf{K}(\varphi \to \psi) \to (\mathsf{K}\varphi \to \mathsf{K}\psi)$	$(\mathrm{K}_{\mathsf{K}})$
$\mathsf{K}\varphi \to \varphi$	$(\mathrm{T}_{\mathsf{K}})$
$\mathsf{K}\varphi \to \mathsf{K}\mathsf{K}\varphi$	(4_{K})
$\mathsf{A}(\varphi \to \psi) \to (\mathsf{A}\varphi \to \mathsf{A}\psi)$	$(\mathrm{K}_{\mathsf{A}})$
$\mathsf{A}\varphi \to \varphi$	$(\mathrm{T}_{\mathsf{A}})$
$\neg\mathsf{A}\varphi \to \mathsf{A}\neg\mathsf{A}\varphi$	(5_{A})
$\mathsf{A}\varphi \to \mathsf{K}\varphi$	$(\mathrm{Inc}_{\mathsf{K}})$
From φ infer $\mathsf{A}\varphi$	$(\mathrm{Nec}_{\mathsf{A}})$
From $\varphi \to \psi$ and φ infer ψ	(MP)

Now, define a translation $u : \mathcal{L}_{\mathsf{KA}} \to \mathcal{L}$ as follows:

$$\begin{aligned} u(p) &= p \\ u(\varphi \wedge \psi) &= u(\varphi) \wedge u(\psi) \\ u(\neg\varphi) &= \neg u(\varphi) \\ u(\mathsf{K}\varphi) &= \neg\mathsf{S}\neg u(\varphi) \\ u(\mathsf{A}\varphi) &= \mathsf{A}u(\varphi). \end{aligned}$$

Recall the translation $t : \mathcal{L} \to \mathcal{L}_{\mathsf{KA}}$ from Section 4. While u is not the inverse of t (for instance, there is no $\psi \in \mathcal{L}_{\mathsf{KA}}$ with $u(\psi) = \mathsf{E}p$), for any $\varphi \in \mathcal{L}$ we have that φ is $\mathsf{L}_{\mathsf{top}}$-provably equivalent to $u(t(\varphi))$.

Claim 1 *Let $\varphi \in \mathcal{L}$. Then $\vdash_{\mathsf{L}_{\mathsf{top}}} \varphi \leftrightarrow u(t(\varphi))$.*

Proof By induction on $\mathcal{L}$ formulas. The cases of atomic propositions and propositional connectives are straightforward. For the other cases, first note that the "replacement of equivalents" rule is derivable in L (and thus in $\mathsf{L}_{\mathsf{top}}$) for S, E and A:

$$\text{From } \varphi \leftrightarrow \psi \text{ infer } \bigcirc\varphi \leftrightarrow \bigcirc\psi \quad (\bigcirc \in \{\mathsf{S}, \mathsf{E}, \mathsf{A}\}).$$

For S this follows from $(\mathrm{Nec}_{\mathsf{A}})$ and $(\mathrm{W}_{\mathsf{S}})$; for E from $(\mathrm{Nec}_{\mathsf{A}})$ and $(\mathrm{RE}_{\mathsf{E}})$, and for A from $(\mathrm{Nec}_{\mathsf{A}})$ and $(\mathrm{K}_{\mathsf{A}})$. Now for the inductive step, suppose $\vdash_{\mathsf{L}_{\mathsf{top}}} \varphi \leftrightarrow u(t(\varphi))$.

- S: Note that
$$u(t(\mathsf{S}\varphi)) = u(\neg\mathsf{K}\neg t(\varphi)) = \neg\neg\mathsf{S}\neg\neg u(t(\varphi)).$$
By the inductive hypothesis, propositional logic and replacement of equivalents, $\vdash_{\mathsf{L_{top}}} \mathsf{S}\varphi \leftrightarrow u(t(\mathsf{S}\varphi))$.
- E: We have
$$\begin{aligned} u(t(\mathsf{E}\varphi)) &= u(\mathsf{A}(\neg t(\varphi) \to \mathsf{K}\neg t(\varphi))) \\ &= \mathsf{A}u(\neg t(\varphi) \to \mathsf{K}\neg t(\varphi)) \\ &= \mathsf{A}(u(\neg t(\varphi)) \to u(\mathsf{K}\neg t(\varphi))) \\ &= \mathsf{A}(\neg u(t(\varphi)) \to \neg\mathsf{S}\neg u(\neg t(\varphi))) \\ &= \mathsf{A}(\neg u(t(\varphi)) \to \neg\mathsf{S}\neg\neg u(t(\varphi))). \end{aligned}$$
Taking the contrapositive of the implication, and using replacement of equivalents together with the inductive hypothesis, we get
$$\vdash_{\mathsf{L_{top}}} u(t(\mathsf{E}\varphi)) \leftrightarrow \mathsf{A}(\mathsf{S}\varphi \to \varphi).$$
But we have already seen that $\vdash_{\mathsf{L_{int}}} \mathsf{E}\varphi \leftrightarrow \mathsf{A}(\mathsf{S}\varphi \to \varphi)$; since $\mathsf{L_{top}}$ extends $\mathsf{L_{int}}$, we get $\vdash_{\mathsf{L_{top}}} \mathsf{E}\varphi \leftrightarrow u(t(\mathsf{E}\varphi))$.
- A: This case is straightforward by the inductive hypothesis and replacement of equivalents, since $u(t(\mathsf{A}\varphi)) = \mathsf{A}u(t(\varphi))$.

□

Next we show that if $\varphi \in \mathcal{L}_{\mathsf{KA}}$ is a theorem of $\mathsf{L_{S4A}}$, then $u(\varphi)$ is a theorem of $\mathsf{L_{top}}$.

Claim 2 *Let $\varphi \in \mathcal{L}_{\mathsf{KA}}$. Then $\vdash_{\mathsf{L_{S4A}}} \varphi$ implies $\vdash_{\mathsf{L_{top}}} u(\varphi)$.*

Proof By induction on the length of $\mathsf{L_{S4A}}$ proofs. The base case consists of showing that if φ is an instance of an $\mathsf{L_{S4A}}$ axiom or a subtitution instance of a propositional tautology, then $\vdash_{\mathsf{L_{top}}} u(\varphi)$. The case for instances of tautologies is straightforward, since u does not affect the structure of a propositional formula. We take the axioms of $\mathsf{L_{S4A}}$ in turn.

- (K_{K}): We have
$$\begin{aligned} &u(\mathsf{K}(\varphi \to \psi) \to (\mathsf{K}\varphi \to \mathsf{K}\psi)) \\ &\quad= \neg\mathsf{S}\neg(u(\varphi) \to u(\psi)) \to (\neg\mathsf{S}\neg u(\varphi) \to \neg\mathsf{S}\neg u(\psi)) \\ &\quad= \hat{\mathsf{S}}(u(\varphi) \to u(\psi)) \to (\hat{\mathsf{S}}u(\varphi) \to \hat{\mathsf{S}}u(\psi)) \end{aligned}$$
which is an instance of (K_{S}).
- (T_{K}): We have
$$u(\mathsf{K}\varphi \to \varphi) = \neg\mathsf{S}\neg u(\varphi) \to u(\varphi)$$
Taking the contrapositive, this is $\mathsf{L_{top}}$-provably equivalent to $\neg u(\varphi) \to \mathsf{S}\neg u(\varphi)$, which is an instance of (T_{S}).
- (4_{K}): We have
$$u(\mathsf{K}\varphi \to \mathsf{K}\mathsf{K}\varphi) = \neg\mathsf{S}\neg u(\varphi) \to \neg\mathsf{S}\neg\neg\mathsf{S}\neg u(\varphi)$$
This is provably equivalent to $\mathsf{S}\mathsf{S}\neg u(\varphi) \to \mathsf{S}\neg u(\varphi)$, which is an instance of (4_{S}).
- (K_{A}): We have
$$u(\mathsf{A}(\varphi \to \psi) \to (\mathsf{A}\varphi \to \mathsf{A}\psi)) = \mathsf{A}(u(\varphi) \to u(\psi)) \to (\mathsf{A}u(\varphi) \to \mathsf{A}u(\psi))$$
which is an instance of (K_{A}) in $\mathsf{L_{top}}$.

- (T_{A}): We have
$$u(\mathsf{A}\varphi \to \varphi) = \mathsf{A}u(\varphi) \to u(\varphi)$$
which is an instance of (T_{A}) in $\mathsf{L_{top}}$.
- (5_{A}): We have
$$u(\neg\mathsf{A}\varphi \to \mathsf{A}\neg\mathsf{A}\varphi) = \neg\mathsf{A}u(\varphi) \to \mathsf{A}\neg\mathsf{A}u(\varphi)$$
which is an instance of (5_{A}) in $\mathsf{L_{top}}$.
- ($\mathrm{Inc}_{\mathsf{K}}$): We have
$$u(\mathsf{A}\varphi \to \mathsf{K}\varphi) = \mathsf{A}u(\varphi) \to \neg\mathsf{S}\neg u(\varphi) = \mathsf{A}u(\varphi) \to \hat{\mathsf{S}}u(\varphi)$$
which is an instance of (Inc).

For the inductive step, we show that for each inference rule $\frac{\psi_1,\ldots,\psi_n}{\varphi}$, if $\vdash_{\mathsf{L_{top}}} u(\psi_i)$ for each i then $\vdash_{\mathsf{L_{top}}} u(\varphi)$.

- ($\mathrm{Nec}_{\mathsf{A}}$): If $\vdash_{\mathsf{L_{top}}} u(\varphi)$, then from ($\mathrm{Nec}_{\mathsf{A}}$) in $\mathsf{L_{top}}$ we get $\vdash_{\mathsf{L_{top}}} \mathsf{A}u(\varphi)$. But $\mathsf{A}u(\varphi) = u(\mathsf{A}\varphi)$, so we are done.
- (MP): Similarly, this clear from (MP) for $\mathsf{L_{top}}$ and the fact that $u(\varphi \to \psi) = u(\varphi) \to u(\psi)$.

□

Claims 1 and 2 easily imply the following.

Claim 3 *Let $\varphi \in \mathcal{L}$. Then $\vdash_{\mathsf{L_{S4A}}} t(\varphi)$ implies $\vdash_{\mathsf{L_{top}}} \varphi$.*

Proof Suppose $\vdash_{\mathsf{L_{S4A}}} t(\varphi)$. By Claim 2, $\vdash_{\mathsf{L_{top}}} u(t(\varphi))$. By Claim 1, $\vdash_{\mathsf{L_{top}}} \varphi \leftrightarrow u(t(\varphi))$. By (MP), $\vdash_{\mathsf{L_{top}}} \varphi$.

□

We can now show strong completeness. Suppose $\Gamma \subseteq \mathcal{L}$, $\varphi \in \mathcal{L}$ and $\Gamma \models_{\mathbb{M}_{\mathsf{top}}} \varphi$. We claim $t(\Gamma) \models_{\mathbb{M}^*_{\mathsf{S4}}} t(\varphi)$. Indeed, if $M^* \in \mathbb{M}^*_{\mathsf{S4}}$ and x is a state in M^* with $M^*, x \models t(\psi)$ for all $\psi \in \Gamma$, then with f as in Theorem 3 we have $f^{-1}(M^*), x \models \psi$ for all $\psi \in \Gamma$. Since $f^{-1}(M^*) \in \mathbb{M}_{\mathsf{int}} \cap \mathbb{M}_{\mathsf{unions}} \subseteq \mathbb{M}_{\mathsf{top}}$, $\Gamma \models_{\mathbb{M}_{\mathsf{top}}} \varphi$ gives $f^{-1}(M^*), x \models \varphi$, and thus $M^*, x \models t(\varphi)$.

By (strong) completeness of $\mathsf{L_{S4A}}$ for $\mathbb{M}^*_{\mathsf{S4}}$, we get $t(\Gamma) \vdash_{\mathsf{L_{S4A}}} t(\varphi)$. That is, there are $\psi_0, \ldots, \psi_n \in \Gamma$ such that $\vdash_{\mathsf{L_{S4A}}} t(\psi_0) \wedge \cdots \wedge t(\psi_n) \to t(\varphi)$. Since t passes over conjunctions and implications, this means $\vdash_{\mathsf{L_{S4A}}} t(\psi_0 \wedge \cdots \wedge \psi_n \to \varphi)$. By Claim 3, $\vdash_{\mathsf{L_{top}}} \psi_0 \wedge \cdots \wedge \psi_n \to \varphi$. Hence $\Gamma \vdash_{\mathsf{L_{top}}} \varphi$, and we are done.

□

Proof of Theorem 7 For soundness, we need to check that (5_{S}) is valid on $\mathbb{M}_{\mathsf{int}} \cap \mathbb{M}_{\mathsf{compl}}$. Let $M = (X, P, V)$ be closed under intersections and complements, and suppose $M, x \models \mathsf{S}\neg\mathsf{S}\varphi$. Note that $\|\mathsf{S}\varphi\|_M = \bigcap\{A \in P \mid \|\varphi\|_M \subseteq A\}$ is an intersection from P, so $\|\mathsf{S}\varphi\|_M \in P$. By closure under complements, $\|\neg\mathsf{S}\varphi\|_M \in P$ too. Hence $M, x \models \mathsf{S}\neg\mathsf{S}\varphi \wedge \mathsf{E}\neg\mathsf{S}\varphi$. By Proposition 1 (4), we get $M, x \models \neg\mathsf{S}\varphi$.

The completeness proof goes in exactly the same way as Theorem 6. Letting $\mathsf{L_{S5A}}$ be the extension of $\mathsf{L_{S4A}}$ with the (5_{K}) axiom $\neg\mathsf{K}\varphi \to \mathsf{K}\neg\mathsf{K}\varphi$, it can be shown that $\mathsf{L_{S5A}}$ is strongly complete with respect to $\mathbb{M}^*_{\mathsf{S5}}$. With u as in the proof of Theorem 6, we have that $\vdash_{\mathsf{L_{S5A}}} \varphi$ implies $\vdash_{\mathsf{L_{int-compl}}} u(\varphi)$, for $\varphi \in \mathcal{L}_{\mathsf{KA}}$ (the only new part to check there is that $u(\neg\mathsf{K}\varphi \to \mathsf{K}\neg\mathsf{K}\varphi)$ is a theorem of $\mathsf{L}_{\mathsf{int}-\mathsf{compl}}$, but this follows from (5_{S})). The remainder of the proof goes through as before, this time appealing to the bijection $g : \mathbb{M}_{\mathsf{int}} \cap \mathbb{M}_{\mathsf{compl}} \to \mathbb{M}^{*}_{\mathsf{S5}}$.

□

Proof of Proposition 5 Let $M^{*} = (X, \{R_j\}_{j \in \mathcal{J}}, V)$ be a multi-source relational model. Since $\mathsf{K}^{\mathsf{com}}_{J}\psi \to \mathsf{K}^{\mathsf{sh}}_{J}\psi$ is valid for any ψ, the left-to-right implication of the above equivalence is straightforward.

For the right-to-left implication, suppose $M^{*}, x \models \mathsf{A}(\neg\varphi \to \mathsf{K}^{\mathsf{sh}}_{J}\neg\varphi)$. We show by induction that $M^{*}, x \models \mathsf{A}(\neg\varphi \to \mathsf{K}^{n}_{J}\neg\varphi)$ for all $n \in \mathbb{N}$, from which the result follows.

The base case $n = 1$ is given, since $\mathsf{K}^{1}_{J}\neg\varphi = \mathsf{K}^{\mathsf{sh}}_{J}\neg\varphi$. For the inductive step, suppose $M^{*}, x \models \mathsf{A}(\neg\varphi \to \mathsf{K}^{n}_{J}\neg\varphi)$. Take $y \in X$ such that $M^{*}, y \models \neg\varphi$. Let $j \in J$. Take $z \in X$ such that yR_jz. From the initial assumption we have $M^{*}, y \models \mathsf{K}^{\mathsf{sh}}_{J}\neg\varphi$, so $M^{*}, y \models \mathsf{K}_j\neg\varphi$ and thus $M^{*}, z \models \neg\varphi$. By the inductive hypothesis, $M^{*}, z \models \mathsf{K}^{n}_{J}\neg\varphi$. This shows that $M^{*}, y \models \mathsf{K}_j\mathsf{K}^{n}_{J}\neg\varphi$ for all $j \in J$, and thus $M^{*}, y \models K^{n+1}_{J}\neg\varphi$. Hence $M^{*}, x \models \mathsf{A}(\neg\varphi \to \mathsf{K}^{n+1}_{J}\neg\varphi)$ as required.

□

References

Blackburn, P., De Rijke, M., Venema, Y. (2002). *Modal logic* (Vol. 53). Cambridge University Press.