PROBLEM SET #6

For the problems below, you need to write programs into a single Jupyter notebook document. Use `Markdown` cells and the hash(`#`) symbol to indicate the problem numbers, explanation, and comments. Name your notebook document using your student `ID` number as `HW6_ID.ipynb`, and email it to your teaching assistant at `hangyeol@snu.ac.kr` before the deadline. No homework will be accepted after the deadline.

1. Consider a 5-level model of the O^{+2} (which produces the [O III] lines). The statistical equilibrium between transitions into and out of any level i yields an equation of the form:

$$n_i \left[n_e \sum_{k \neq i} q_{ik} + \sum_{k<i} A_{ik} \right] = n_e \sum_{k \neq i} n_k q_{ki} + \sum_{k>i} n_k A_{ki} \quad (i = 1, \cdots, 5). \tag{1}$$

This leads to five equations in the five unknowns $n_1, \cdots, n_5$, but it can be shown that this system is *degenerate*, i.e., any one of the equations can be constructed as a linear combination of the other four. We thus introduce another equation, which is the equation of normalization: $\sum_{i=1}^{5} n_i = 1$. This plus any four of the others may then be solved for the relative populations n_i.

To solve Equation (1), you need to know the atomic parameters such as statistical weights ω_i, differences between the energy levels E_{ij}, Einstein A_{ji}-values, and collisional parameters $\Omega_{ji} = \Omega_{ij}$, which are given in Tables 1 and 2. The rate of downward collisions $j \to i$ is given by

$$q_{ji} = \frac{8.629 \times 10^{-8}}{t^{1/2}} \frac{\Omega_{ji}}{\omega_j} \text{ cm}^3 \text{ s}^{-1}, \quad (j > i), \tag{2}$$

with temperature $t \equiv T/(10^4\,\text{K})$, while the rate of upward collisions can be found from

$$q_{ij} = \frac{\omega_j}{\omega_i} q_{ji} e^{-1.1605 E_{ij}/t}, \tag{3}$$

where the energy separation E_{ij} is in eV.

index i	ω_i	designation
1	1	$^3P_0$
2	3	$^3P_1$
3	5	$^3P_2$
4	5	$^1D_2$
5	1	$^1S_0$

Table 1: The first five levels of O III

i	j	λ_{ij}	E_{ij} (eV)	A_{ji}	Ω_{ji}
1	2	88.356μm	0.014	2.7×10^{-5}	0.55
1	3	32.661μm	0.038	3.1×10^{-11}	0.27
1	4	4931.1Å	2.514	1.7×10^{-6}	0.254
1	5	$--$	5.300	0.0	0.032
2	3	51.814μm	0.024	9.7×10^{-5}	1.29
2	4	4958.9Å	2.500	6.8×10^{-3}	0.763
2	5	2321.0Å	5.314	2.3×10^{-1}	0.097
3	4	5006.9Å	2.476	2.0×10^{-2}	1.272
3	5	2331.4Å	5.318	6.1×10^{-4}	0.161
4	5	4363.2Å	2.842	1.6	0.58

Table 2: Data for all possible transitions between the levels

(a) Write a computer program to calculate the populations for any given (n_e, T) pair. Find the populations for $(n_e, T) = (10^2\ \mathrm{cm}^{-3}, 10^4\ \mathrm{K})$, $(10^2\ \mathrm{cm}^{-3}, 2\times10^4\ \mathrm{K})$, and $(10\ \mathrm{cm}^{-3}, 10^4\ \mathrm{K})$. Make sure that all n_i's should be positive.

(b) The ratio of the $\lambda(5007+4959)$ line to the $\lambda4363$ line of [O III]

$$R = \frac{n_4(A_{43}\lambda_{34} + A_{42}\lambda_{42})}{n_5 A_{54}\lambda_{54}}, \tag{4}$$

can be an important indicator of the electron temperature in ionized nebulae. Plot this ratio over the temperature range 5,000 K $< T <$ 20,000 K for densities $n_e = 10, 10^3$, and 10^5 cm^{-3}.

(c) The total energy loss (per particle) by collisional excitation of the levels of the O^{+2} ion is given by the sum over all the radiative transitions:

$$L(\mathrm{O}^{+2}) = \sum_{i=2}^{5} n_i \left[\sum_{k=1}^{i-1} A_{ik}\cdot E_{ki}\right]. \tag{5}$$

Make a log-log plot of $L(\mathrm{O}^{+2})/n_e$ against the electron temperature for 500 K $< T <$ 50,000 K and $n_e = 10, 10^3$, and 10^5 cm^{-3}.

2. Write a program to find a minimum of

$$f(x,y) = (x^2 - 10y)^2 + 2y^2 - 3x \tag{6}$$

using

(a) the steepest descent, and

(b) the Powell method.

For both methods, start from $(x, y) = (-5, 2)$ and take $\epsilon = 10^{-6}$ for the tolerance value. For each method, give the minimum point and plot the path taken to it (together with contours of $f(x, y)$).

3. Use the conjugate gradient method to minimize the banana function in three dimensions

$$f(x, y, z) = 100(y - x^2)^2 + (1 - x)^2 + 100(z - y^2)^2 + (1 - z)^2 \tag{7}$$

starting from $(x, y, z) = (0, 2, 1)$. Where is the minimum point? Give the number of iterations to arrive at the minimum point.

4. The '`sol_vel.dat`' file at `http://mirzam.snu.ac.kr/~wkim/Comp2020/` contains three-column ascii data for line-of-sight velocities of gas in the quiet region of the Sun. The first column is time, t, in units of minutes, while the second and third columns denote the velocities, $v_{\mathrm{H}\alpha}$ and v_{CaII}, measured from Hα lines and Ca II lines, respectively, in units of km s^{-1}.

(a) Plot $v_{\mathrm{H}\alpha}$ and v_{CaII} as functions of t in a single panel.

(b) Perform Fourier transforms of $v_{\mathrm{H}\alpha}$ and v_{CaII} and plot the power spectra as functions of frequency f, in a single panel. Make sure to indicate the units of f in the plot.

(c) From the results of (b), what are the oscillation periods of $v_{\mathrm{H}\alpha}$ and v_{CaII}?

(d) The cross-correlation of $v_{\mathrm{H}\alpha}$ and v_{CaII} is defined by

$$\mathrm{Corr}(t) = \int_{-\infty}^{\infty} v_{\mathrm{H}\alpha}(\tau + t) v_{\mathrm{CaII}}(\tau) d\tau. \tag{8}$$

Plot Corr(t) as a function of t.