PROBLEM SET #4

For the problems below, you need to write programs into a single Jupyter notebook document. Use `Markdown` cells and the hash(`#`) symbol to indicate the problem numbers, explanation, and comments. Name your notebook document using your student `ID` number as `HW4_ID.ipynb`, and email it to your teaching assistant at `hangyeol@snu.ac.kr` before the deadline. No homework will be accepted after the deadline.

1. On a windy night, a drunkard begins walking at the origin of a two-dimensional coordinate system (x, y). His steps are 1 unit in length and are random in the following way: with probability $\frac{1}{6}$, he takes a step in the $+x$ direction; with probability $\frac{1}{4}$, he takes a step in the $+y$ direction; with probability $\frac{1}{4}$, he takes a step in the $-y$ direction; with probability $\frac{1}{3}$, he takes a step in the $-x$ direction.

(a) What is the probability that after 50 steps, he will be more than 20 units distant from the origin? Your need to give the mean value and the standard deviation.

(b) Find the most probable position and the deviation of the drunkard after 10^3 steps.

2. Suppose that in a room of n people, each of the 365 days of the year is equally likely to be someone's birthday. Perform Monte Carlo simulations for $n = 10, 30$, and 50 to calculate the probability that at least two of the n people have the same birthday. What is the smallest n for the probability to be greater than fifty-fifty?

3. Suppose a photon incident on the bottom of a plane-parallel slab with optical depth $\tau_{\rm max}$. We assume that the slab is infinite in the x and y directions: z-axis is normal to the slab. The photon can be scattered (with no absorption) at any point within the slab. We begin with $z = 0$ (bottom of the slab) and follow the photon's trajectories up to $z_{\rm max} = 1$ (top of the slab) by taking following steps:

Step 1: We inject a photon from below whose flux is isotropic in any direction. The probability for a certain injection angle at $z = 0$ (with respect to the z-axis), is given by $p(\mu)d\mu = 2\mu d\mu$ with $0 \le \mu \le 1$, where $\mu \equiv \cos\theta$. Use a uniform deviate ξ_1 to sample $\mu = \sqrt{\xi_1}$. Use another uniform deviate ξ_2 to obtain $\phi = 2\pi\xi_2$. Calculate the initial direction of the photon $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Note that the initial position of the photon is $(x, y, z) = (0, 0, 0)$.

Step 2: When the optical depth of a photon is τ, the probability P that a photon interacts with the medium is given by $P = 1 - e^{-\tau}$. Noting that P is random over [0, 1), τ can be sampled from a uniform deviate ξ_3 as $\tau = -\ln(\xi_3)$. Pick up a value for τ.

Step 3: The distance traveled by a photon along a ray is given by $L = \tau z_{\rm max}/\tau_{\rm max}$. Update the photon's new position as $x = x + L\sin\theta\cos\phi$, $y = y + L\sin\theta\sin\phi$, and $z = z + L\cos\theta$.

Step 4: If $z < 0$, skip to Step 7 below. If $z > z_{\rm max}$, add one to the number of transmitted photons, record its emergent direction, $\mu = \cos\theta$, and go to Step 7.

Step 5: Assume that the photons are scattered uniformly into 4π steradians. Generate the new direction by sampling uniformly for ϕ in the range from 0 to 2π and μ in the range from -1 to 1: $\phi = 2\pi\xi_4$ and $\mu = 2\xi_5 - 1$, where ξ_4 and ξ_5 are two independent uniform deviates in [0, 1).

Step 6: Repeat Steps 2–5 until the fate of the photon has been determined in Step 4.

Step 7: Repeat Steps 1–6 with additional incident photons until sufficient data has been obtained.

(a) Calculate the transmission probabilities for $\tau_{\rm max} = 0.01$, 0.1, 1, and 10. Begin with 10^3 incident photons and increase this number until satisfactory statistics are obtained. Give a qualitative explanation of your results.

(b) Draw some typical paths of photons in the x–z plane.

(c) For $\tau_{\rm max} = 10$, place all emergent photons in 10 uniform bins in μ with a bin width $\Delta\mu = 0.1$ in the range of [0,1]. Let n_i denote the number of photons in the i-th bin. Then the fractional emergent *energy* of photons in the angle between μ_i and $\mu_i + \Delta\mu$ is given by $E_i = n_i/n_{\rm tot}$, where $n_{\rm tot} = \sum_{i=0}^{9} n_i$ is the total number of emergent photons. The normalized emergent intensity is given by $I_i = E_i/(2\mu_i\Delta\mu)$. Plot I_i as a function of θ_i and compare it with the prediction $I(\theta) = (2 + 3\cos\theta)/5$ of the gray, LTE atmosphere under the Eddington approximation.

4. *The Bondi Problem*: The Bondi problem involves the spherical accretion of gas to a gravitating mass such as a black hole. In spherical symmetry, the steady flow of an isothermal gas with velocity u obeys the following equations

$$\left(u - \frac{1}{u}\right)\frac{du}{dx} = \frac{2}{x} - \frac{1}{x^2}, \tag{1}$$

where is the dimensionless distance from the gravitating mass.

(a) Solve Equation (1) inward starting from $x = 5$ to $x = 0.1$ for three values of $u = 3$, 0.1, and 0.01 at $x = 5$, and plot the results on the u–x plane. You will see that the case with $u(x = 5) = 0.1$ does not give the solution you want. Why does this happen? How can you circumvent this problem?

(b) Now you want to obtain the transonic solutions (that is, solutions with $u = 1$ at some point). In view of Equation (1), $u = 1$ should occur at $x = 1/2$ for regular solutions. Show that the transonic solutions should satisfy

$$\Delta u = \pm 2\Delta x, \tag{2}$$

where Δu and Δx denote the small changes in u and x around 1 and 1/2, respectively. (That is, $x = 0.5 + \Delta x$ and $u = 1 + \Delta u$ for $|\Delta x|, |\Delta u| \ll 1$.) Draw the transonic solutions in the u–x plane, with x in the range between 0.1 and 5.