

PROBLEM SET #6

For the problems below, you need to write programs into a single Jupyter notebook document. Use **Markdown** cells and the hash(#) symbol to indicate the problem numbers, explanation, and comments. Name your notebook document using your student ID number as `HW6-ID.ipynb`, and email it to your teaching assistant at `hangyeol@snu.ac.kr` before the deadline. **No homework will be accepted after the deadline.**

1. Consider a 5-level model of the O^{+2} (which produces the [O III] lines). The statistical equilibrium between transitions into and out of any level i yields an equation of the form:

$$n_i \left[n_e \sum_{k \neq i} q_{ik} + \sum_{k < i} A_{ik} \right] = n_e \sum_{k \neq i} n_k q_{ki} + \sum_{k > i} n_k A_{ki} \quad (i = 1, \dots, 5). \quad (1)$$

This leads to five equations in the five unknowns n_1, \dots, n_5 , but it can be shown that this system is *degenerate*, i.e., any one of the equations can be constructed as a linear combination of the other four. We thus introduce another equation, which is the equation of normalization: $\sum_{i=1}^5 n_i = 1$. This plus any four of the others may then be solved for the relative populations n_i .

To solve Equation (1), you need to know the atomic parameters such as statistical weights ω_i , differences between the energy levels E_{ij} , Einstein A_{ji} -values, and collisional parameters Ω_{ji} , which are given in Tables 1 and 2. The rate of downward collisions $j \rightarrow i$ is given by

$$q_{ji} = \frac{8.629 \times 10^{-8} \Omega_{ji}(t)}{t^{1/2} \omega_j} \text{ cm}^3 \text{ s}^{-1}, \quad (j > i), \quad (2)$$

while the rate of upward collisions can be found from

$$q_{ij} = \frac{\omega_j}{\omega_i} q_{ji} e^{-1.1605 E_{ij}/t}, \quad (3)$$

where the energy separation E_{ij} is in eV.

index i	ω_i	designation
1	1	3P_0
2	3	3P_1
3	5	3P_2
4	5	1D_2
5	1	1S_0

Table 1: The first five levels of O III

i	j	λ_{ij}	E_{ij} (eV)	A_{ji}	$\Omega_{ji}(1.0)$
1	2	88.356 μ m	0.014	2.7×10^{-5}	0.55
1	3	32.661 μ m	0.038	3.1×10^{-11}	0.27
1	4	4931.1 \AA	2.514	1.7×10^{-6}	0.254
1	5	—	5.300	0.0	0.032
2	3	51.814 μ m	0.024	9.7×10^{-5}	1.29
2	4	4958.9 \AA	2.500	6.8×10^{-3}	0.763
2	5	2321.0 \AA	5.314	2.3×10^{-1}	0.097
3	4	5006.9 \AA	2.476	2.0×10^{-2}	1.272
3	5	2331.4 \AA	5.318	6.1×10^{-4}	0.161
4	5	4363.2 \AA	2.842	1.6	0.58

Table 2: Data for all possible transitions between the levels

- (a) Write a computer program to calculate the populations for any given (n_e, T) pair. Find the populations for $(n_e, T) = (10^2 \text{ cm}^{-3}, 10^4 \text{ K})$, $(10^2 \text{ cm}^{-3}, 2 \times 10^4 \text{ K})$, and $(10 \text{ cm}^{-3}, 10^4 \text{ K})$. Make sure that all n_i 's should be positive.
- (b) The ratio of the $\lambda(5007 + 4959)$ line to the $\lambda 4363$ line of [O III]

$$R = \frac{n_4(A_{43}\lambda_{34} + A_{42}\lambda_{42})}{n_5 A_{54}\lambda_{54}}, \quad (4)$$

can be an important indicator of the electron temperature in ionized nebulae. Plot this ratio over the temperature range $5,000 \text{ K} < T < 20,000 \text{ K}$ for densities $n_e = 10, 10^3$, and 10^5 cm^{-3} .

- (c) The total energy loss (per particle) by collisional excitation of the levels of the O^{+2} ion is given by the sum over all the radiative transitions:

$$L(\text{O}^{+2}) = \sum_{i=2}^5 n_i \left[\sum_{k=1}^{i-1} A_{ik} \cdot E_{ki} \right]. \quad (5)$$

Make a log-log plot of $L(\text{O}^{+2})/n_e$ against the electron temperature for $500 \text{ K} < T < 50,000 \text{ K}$ and $n_e = 10, 10^3$, and 10^5 cm^{-3} .

2. Write a program to find a minimum of

$$f(x, y) = (x^2 - 10y)^2 + 2y^2 - 3x \quad (6)$$

using

- (a) the steepest descent, and
(b) the Powell method.

For both methods, start from $(x, y) = (-5, 2)$ and take $\epsilon = 10^{-6}$ for the tolerance value. For each method, give the minimum point and plot the path taken to it (together with contours of $f(x, y)$).

3. Use the conjugate gradient method to minimize the banana function in three dimensions

$$f(x, y, z) = 100(y - x^2)^2 + (1 - x)^2 + 100(z - y^2)^2 + (1 - z)^2 \quad (7)$$

starting from $(x, y, z) = (0, 2, 1)$. Where is the minimum point? Give the number of iterations to arrive at the minimum point.

4. The ‘sol_vel.dat’ file at <http://mirzam.snu.ac.kr/~wkim/Comp2020/> contains three-column ascii data for line-of-sight velocities of gas in the quiet region of the Sun. The first column is time, t , in units of minutes, while the second and third columns denote the velocities, $v_{\text{H}\alpha}$ and v_{CaII} , measured from H α lines and Ca II lines, respectively, in units of km s^{-1} .

- (a) Plot $v_{\text{H}\alpha}$ and v_{CaII} as functions of t in a single panel.
- (b) Perform Fourier transforms of $v_{\text{H}\alpha}$ and v_{CaII} and plot the power spectra as functions of frequency f , in a single panel. Make sure to indicate the units of f in the plot.
- (c) From the results of (b), what are the oscillation periods of $v_{\text{H}\alpha}$ and v_{CaII} ?
- (d) The cross-correlation of $v_{\text{H}\alpha}$ and v_{CaII} is defined by

$$\text{Corr}(t) = \int_{-\infty}^{\infty} v_{\text{H}\alpha}(\tau + t) v_{\text{CaII}}(\tau) d\tau. \quad (8)$$

Plot $\text{Corr}(t)$ as a function of t .