PROBLEM SET #5

For the problems below, you need to write programs into a single Jupyter notebook document. Use `Markdown` cells and the hash(`#`) symbol to indicate the problem numbers, explanation, and comments. Name your notebook document using your student `ID` number as `HW5_ID.ipynb`, and email it to your teaching assistant at `hangyeol@snu.ac.kr` before the deadline. No homework will be accepted after the deadline.

1. An isothermal self-gravitating sphere in hydrostatic equilibrium ought to satisfy

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{du}{d\xi}\right) = -e^{-u}, \tag{1}$$

where ξ is the the dimensionless radius and $u(\xi)$ is related to the density ρ via $\rho = \rho_c e^{-u}$, with ρ_c being the central density (at $\xi = 0$). The proper boundary conditions are $u(0) = du/d\xi|_{\xi=0} = 0$.

(a) For small ξ, one can seek for a power series solution $u(\xi) = \sum_{m=0}^{\infty} a_m \xi^m$ of Equation (1), with a_m's being coefficients. Using Taylor expansions, express the first five coefficients a_0, a_1, a_2, a_3, and a_4.

(b) Write a program to solve Equation (1), and plot ρ/ρ_c as a function of $\xi \in [10^{-1}, 10^2]$.

(c) The dimensionless mass within ξ of the sphere is given by

$$m(\xi) \equiv \frac{1}{\sqrt{4\pi}}\int_0^{\xi} \xi'^2 e^{-u(\xi')} d\xi'. \tag{2}$$

Plot $p \equiv m^2 e^{-u}$ as a function of $r = (\xi du/d\xi)^{-1}$. Find the values of ξ and r where p is maximized. (*Hint: You can integrate Equation* (2) *analytically by using Equation* (1).)

2. In cosmology, the cosmic expansion is described by the scale factor $a(t)$. By definition, $a = 0$ at the time of the Big Bang (i.e., $t = 0$), and $a = 1$ at the present time t_0. The scale factor is related to the redshift z through

$$a = \frac{1}{1+z}, \tag{3}$$

and also to the Hubble parameter

$$H = \frac{1}{a}\frac{da}{dt}. \tag{4}$$

The Friedmann equation that governs the expansion of our Universe is given by

$$H^2(z) = H_0^2[\Omega_R(1+z)^4 + \Omega_M(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda], \tag{5}$$

where $H_0 = 71$ km s^{-1} Mpc^{-1} is the Hubble constant at the present time, Ω_R, Ω_M, Ω_k, and Ω_Λ denote the Ω parameters for radiation, matter, curvature, and dark energy, respectively, with the condition that $\Omega_R + \Omega_M + \Omega_k + \Omega_\Lambda = 1$.

(a) Suppose a fictitious, mass-dominated universe with $\Omega_R = \Omega_k = \Omega_\Lambda = 0$ and $\Omega_M = 1$. Solve Equation (5) numerically, and compare your results with the analytic solution

$$a(t) = \left(\frac{3H_0}{2}t\right)^{2/3}. \tag{6}$$

Note that the age of the universe is $t_0 = 2/(3H_0)$.

(b) The current estimates of the Ω parameters in our Universe are $\Omega_R = 3 \times 10^{-5}$, $\Omega_k = 0$ (flat universe), $\Omega_M = 0.27$ and $\Omega_\Lambda = 0.73$. Solve Equation (5) numerically, and plot a as a function of time t. What is the current age of the Universe in Gyr?

(c) When did the Universe switch from decelerating to accelerating expansion? What were the values of the the redshift and Hubble parameter at that time?

3. Consider a binary system consisting of two stars with masses $m_1 = 1$ and $m_2 = 0.5$ placed in the x-y plane. Initially ($t = 0$), the two stars are located at $\mathbf{r}_1 = (x_1, y_1) = (-0.5, 0)$ and $\mathbf{r}_2 = (1, 0)$, and have velocities of $\mathbf{v}_1 = (0.01, 0.05)$ and $\mathbf{v}_2 = (0.02, 0.2)$. The two stars orbit with each other due to the mutual gravity. The relevant equation of motion is

$$\ddot{\mathbf{r}}_i = -m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}, \quad (i \neq j), \tag{7}$$

for $i, j = 1$ or 2.

(a) Integrate Equation (7) from $t = 0$ to $t = 50$ using the Leap-frog integrator, and plot the orbits of the two stars in the x-y plane. You need to choose a small enough dt for accurate orbit calculations.

(b) The center of mass is defined as $\mathbf{r} = \sum m_i \mathbf{r}_i / \sum m_i$. Indicate the motion of the center of mass in the figure you draw in Part (a).

(c) Plot the total energy defined by

$$E = \frac{1}{2}m_1 \mathbf{v}_1^2 + \frac{1}{2}m_2 \mathbf{v}_2^2 - \frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}. \tag{8}$$

over $t = 0 - 1000$. Comment on the accuracy of the calculated orbits in terms of the energy conservation.

4. Find smallest five values of $\lambda\,(> 0)$ that satisfies

$$y'' + (\lambda - |x|)y = 0, \quad \text{for} \quad -5 \leq x \leq 5, \tag{9}$$

subject to $y(-5) = 0$, $y'(-5) = 0.1$, and $y(0) = 0$. Plot the corresponding eigenfunctions over $x \in [-5, 5]$.