Fréchet derivatives under various parameterizations using chain rule

for SeisJIMU module m_parameterization.f90

joeyartech
joeywzhou1986@gmail.com

The University of Texas at Dallas

June 17, 2022

Overview

Physical meaning of symbols uniformly used in this code:

vp	P-wave velocity
vs	S-wave velocity
sp =1/vp	P-wave slowness
sps=vs/vp	inv Vp-Vs ratio
ip =vp*rho	P-wave (acoustic) impedance
rho	density
kpa=rho*vp^2	bulk modulus
lda=rho*(vp^2-2vs^2)	1st Lamé parameter
mu =rho*vs^2	2nd Lamé parameter, shear modulus

Overview

Considered parameterizations

SITUATION	PARAMETERIZATION	ALLOWED PARAMETERS
WaveEquation	moduli-density	kpa (or lda mu) rho
canonical models	velocities-density	vp vs rho
seismic	velocities-impedance	vp vs ip
tomography	slowness-density	sp sps rho
optimization	can be any of above	–

Source-Destination trilogy:

$$\boxed{\text{model m} \xrightarrow{\textit{FWD}} \text{WaveEq lm} \xrightarrow{\textit{PARAMETERIZATION}} \text{optim x,g} \xrightarrow{\textit{FWD}} \text{model m}}$$

- model m $\rightarrow$ WaveEq lm depending on field.
- WaveEq produces gkpa (or glda, gmu) and grho under moduli-density parameterization.
- then m_parameterization.f90 converts them to another parameterization for optim x,g, which can be any of parameterization

ACTIVE versus PASSIVE

- Active parameters will be converted to optim x via feature scaling (e.g. (par-par_min)/(par_max-par_min)), and will be updated by optimization methods.
  - Mono-parameter inversion: 1 active parameter.
  - Multi-parameter inversion: >1 active parameters.
    - Scaling in m_linesearch.f90 depends only on the 1st active parameters (no matter of it's velocity or not).
    - Output of inversion results has same sequence of active parameters listed in setup.in
- Passive parameters will NOT be converted to optim x, and will NOT be updated by optimization methods.
- However, they may still be updated according to user-specified empirical law (e.g. Gardner), which serves as hard constraints between parameters.
  - User has to modified the code where necessary to insert such a law as symbolic computation is not straightforward in fortran
  - Such a law may change the optim g for active parameters.

velocities-density

velocities-impedance

slowness-density

PARAMETERIZATION velocities-density

isotropic ACoustic

$$\left\{\begin{array}{l}\kappa = \rho V_P^2\\ \rho_0 = \rho\end{array}\right.$$

$$\begin{aligned}
\nabla_{V_P} &= \nabla_\kappa \frac{\partial \kappa}{\partial V_P} + \nabla_{\rho_0}\frac{\partial \rho_0}{\partial V_P}\\
&= \nabla_\kappa 2\rho V_P\\
\nabla_\rho &= \nabla_\kappa \frac{\partial \kappa}{\partial \rho} + \nabla_{\rho_0}\frac{\partial \rho_0}{\partial \rho}\\
&= \nabla_\kappa V_P^2 + \nabla_{\rho_0}
\end{aligned}$$

$$\left\{\begin{array}{l}V_P = \sqrt{\frac{\kappa}{\rho_0}}\\ \rho = \rho_0\end{array}\right.$$

$$\begin{aligned}
\nabla_\kappa &= \nabla_{V_P}\frac{\partial V_P}{\partial \kappa} + \nabla_\rho \frac{\partial \rho}{\partial \kappa}\\
&= \nabla_{V_P}\frac{1}{2\rho_0}\sqrt{\frac{\rho_0}{\kappa}}\\
&= \nabla_{V_P} 0.5\rho_0^{-0.5}\kappa^{-0.5}\\
\nabla_{\rho_0} &= \nabla_{V_P}\frac{\partial V_P}{\partial \rho_0} + \nabla_\rho \frac{\partial \rho}{\partial \rho_0}\\
&= \nabla_{V_P}\left(-\frac{\kappa}{2\rho_0^2}\sqrt{\frac{\rho_0}{\kappa}}\right) + \nabla_\rho\\
&= \nabla_{V_P}\left(-0.5\rho_0^{-1.5}\kappa^{0.5}\right) + \nabla_\rho
\end{aligned}$$

PARAMETERIZATION velocities-density + PASSIVE Gardner

isotropic ACoustic

$$\boxed{\rho = a V_P^b}$$

$$\begin{cases} \kappa = \rho V_P^2 = a V_P^{b+2} \\ \rho_0 = \rho = a V_P^b \end{cases}$$

$$V_P = \left(\frac{\kappa}{a}\right)^{\frac{1}{b+2}}$$

$$\begin{aligned} \nabla_{V_P} &= \nabla_\kappa \frac{\partial \kappa}{\partial V_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_P} \\ &= \nabla_\kappa a(b+2) V_P^{b+1} + \nabla_{\rho_0} ab V_P^{b-1} \\ &= \left(\nabla_\kappa \frac{b+2}{b} V_P^2 + \nabla_{\rho_0}\right) \frac{b\rho}{V_P} \end{aligned}$$

$$\begin{aligned} \nabla_\kappa &= \nabla_{V_P} \frac{\partial V_P}{\partial \kappa} \\ &= \nabla_{V_P} \frac{1}{a(b+2)} \left(\frac{\kappa}{a}\right)^{-\frac{b+1}{b+2}} \end{aligned}$$

PARAMETERIZATION velocities-density

isotropic P-SV

$$\begin{cases} \lambda = \rho(V_P^2 - 2V_S^2) \\ \mu = \rho V_S^2 \\ \rho_0 = \rho \end{cases}$$

$$\begin{aligned}
\nabla_{V_P} &= \nabla_\lambda \frac{\partial \lambda}{\partial V_P} + \nabla_\mu \frac{\partial \mu}{\partial V_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_P} \\
&= \nabla_\lambda 2\rho V_P \\
\nabla_{V_S} &= \nabla_\lambda \frac{\partial \lambda}{\partial V_S} + \nabla_\mu \frac{\partial \mu}{\partial V_S} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_S} \\
&= \nabla_\lambda (-4)\rho V_S + \nabla_\mu 2\rho V_S \\
&= (-2\nabla_\lambda + \nabla_\mu)\, 2\rho V_S \\
\nabla_\rho &= \nabla_\lambda \frac{\partial \lambda}{\partial \rho} + \nabla_\mu \frac{\partial \mu}{\partial \rho} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial \rho} \\
&= \nabla_\lambda (V_P^2 - 2V_S^2) + \nabla_\mu V_S^2 + \nabla_{\rho_0} \\
&= \nabla_\lambda V_P^2 + (-2\nabla_\lambda + \nabla_\mu) V_S^2 + \nabla_{\rho_0}
\end{aligned}$$

$$\begin{cases} V_P = \sqrt{\frac{\lambda + 2\mu}{\rho_0}} \\ V_S = \sqrt{\frac{\mu}{\rho_0}} \\ \rho = \rho_0 \end{cases}$$

$$\begin{aligned}
\nabla_\lambda &= \nabla_{V_P} \frac{\partial V_P}{\partial \lambda} + \nabla_{V_S} \frac{\partial V_S}{\partial \lambda} + \nabla_\rho \frac{\partial \rho}{\partial \lambda} \\
&= \ldots \\
\nabla_\mu &= \nabla_{V_P} \frac{\partial V_P}{\partial \mu} + \nabla_{V_S} \frac{\partial V_S}{\partial \mu} + \nabla_\rho \frac{\partial \rho}{\partial \mu} \\
&= \ldots \\
\nabla_{\rho_0} &= \nabla_{V_P} \frac{\partial V_P}{\partial \rho_0} + \nabla_{V_S} \frac{\partial V_S}{\partial \rho_0} + \nabla_\rho \frac{\partial \rho}{\partial \rho_0} \\
&= \ldots
\end{aligned}$$

PARAMETERIZATION velocities-density + PASSIVE Gardner

isotropic P-SV

$$\boxed{\rho = aV_P^b}$$

$$\begin{cases} \lambda = \rho(V_P^2 - 2V_S^2) = aV_P^{b+2} - 2aV_P^b V_S^2 \\ \mu = \rho V_S^2 = aV_P^b V_S^2 \\ \rho_0 = \rho = aV_P^b \end{cases}$$

$$\begin{aligned}
\nabla_{V_P} &= \nabla_\lambda \frac{\partial \lambda}{\partial V_P} + \nabla_\mu \frac{\partial \mu}{\partial V_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_P} \\
&= \nabla_\lambda \left(a(b+2)V_P^{b+1} - 2abV_P^{b-1}V_S^2 \right) + \nabla_\mu abV_P^{b-1}V_S^2 + \nabla_{\rho_0} abV_P^{b-1} \\
&= \left(\nabla_\lambda \left(\frac{b+2}{b} V_P^2 - 2V_S^2 \right) + \nabla_\mu V_S^2 + \nabla_{\rho_0} \right) abV_P^{b-1} \\
&= \left(\nabla_\lambda \frac{b+2}{b} V_P^2 + (-2\nabla_\lambda + \nabla_\mu) V_S^2 + \nabla_{\rho_0} \right) \frac{b\rho}{V_P} \\
\nabla_{V_S} &= \nabla_\lambda \frac{\partial \lambda}{\partial V_S} + \nabla_\mu \frac{\partial \mu}{\partial V_S} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_S} \\
&= \nabla_\lambda (-4aV_P^b V_S) + \nabla_\mu 2aV_P^b V_S \\
&= (-2\nabla_\lambda + \nabla_\mu) 2\rho V_S
\end{aligned}$$

PARAMETERIZATION velocities-density + PASSIVE Castagna

isotropic P-SV

$$\boxed{V_S = aV_P + b}$$

$$\begin{cases} \lambda = \rho(V_P^2 - 2V_S^2) = \rho\left((1-2a^2)V_P^2 - 4abV_P - 2b^2\right) \\ \mu = \rho V_S^2 = \rho(a^2V_P^2 + 2abV_P + b^2) \\ \rho_0 = \rho \end{cases}$$

$$\begin{aligned}
\nabla_{V_P} &= \nabla_\lambda \frac{\partial \lambda}{\partial V_P} + \nabla_\mu \frac{\partial \mu}{\partial V_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_P} \\
&= \nabla_\lambda\ \rho\left(2(1-2a^2)V_P - 4ab\right) + \nabla_\mu\ \rho(2a^2V_P + 2ab) \\
&= 2\rho\left(\nabla_\lambda\left(V_P - 2a^2V_P - 2ab\right) + \nabla_\mu(a^2V_P + ab)\right) \\
&= 2\rho\left(\nabla_\lambda(V_P - 2aV_S) + \nabla_\mu aV_S\right) \\
\nabla_\rho &= \nabla_\lambda \frac{\partial \lambda}{\partial \rho} + \nabla_\mu \frac{\partial \mu}{\partial \rho} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial \rho} \\
&= \nabla_\lambda(V_P^2 - 2V_S^2) + \nabla_\mu V_S^2 + \nabla_{\rho_0}
\end{aligned}$$

velocities-density

velocities-impedance

slowness-density

PARAMETERIZATION velocities-impedance

isotropic ACoustic

$$\boxed{I_P = V_P \rho}$$

$$\begin{cases} \kappa = \rho V_P^2 = V_P I_P \\ \rho_0 = \rho = \frac{I_P}{V_P} \end{cases}$$

$$\begin{aligned}
\nabla_{V_P} &= \nabla_\kappa \frac{\partial \kappa}{\partial V_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_P} \\
&= \nabla_\kappa I_P + \nabla_{\rho_0} (-I_P V_P^{-2}) \\
&= \left(\nabla_\kappa - \nabla_{\rho_0} V_P^{-2} \right) V_P \rho \\
\nabla_{I_P} &= \nabla_\kappa \frac{\partial \kappa}{\partial I_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial I_P} \\
&= \nabla_\kappa V_P + \nabla_{\rho_0} V_P^{-1}
\end{aligned}$$

$$\begin{cases} V_P = \sqrt{\frac{\kappa}{\rho_0}} \\ I_P = \sqrt{\kappa \rho_0} \end{cases}$$

$$\begin{aligned}
\nabla_\kappa &= \nabla_{V_P} \frac{\partial V_P}{\partial \kappa} + \nabla_{I_P} \frac{\partial I_P}{\partial \kappa} \\
&= \nabla_{V_P} \frac{1}{2\rho_0} \sqrt{\frac{\rho_0}{\kappa}} + \nabla_{I_P} \frac{\rho_0}{2\sqrt{\kappa \rho_0}} \\
&= \nabla_{V_P} 0.5 \rho_0^{-0.5} \kappa^{-0.5} + \nabla_{I_P} 0.5 \rho_0^{0.5} \kappa^{-0.5} \\
&= (\nabla_{V_P} + \nabla_{I_P} \rho_0)\, 0.5 \rho_0^{-0.5} \kappa^{-0.5} \\
\nabla_{\rho_0} &= \nabla_{V_P} \frac{\partial V_P}{\partial \rho_0} + \nabla_{I_P} \frac{\partial I_P}{\partial \rho_0} \\
&= \nabla_{V_P} \frac{-\kappa}{2\rho_0^2} \sqrt{\frac{\rho_0}{\kappa}} + \nabla_{I_P} \frac{\kappa}{2\sqrt{\kappa \rho_0}} \\
&= \nabla_{V_P} (-0.5) \rho_0^{-1.5} \kappa^{0.5} + \nabla_{I_P} 0.5 \rho_0^{-0.5} \kappa^{0.5} \\
&= \left(-\nabla_{V_P} \rho_0^{-1} + \nabla_{I_P} \right) 0.5 \rho_0^{-0.5} \kappa^{0.5}
\end{aligned}$$

PARAMETERIZATION velocities-impedance

isotropic P-SV

$$\begin{cases} \lambda = \rho(V_P^2 - 2V_S^2) = V_P I_P - \frac{2V_S^2 I_P}{V_P} \\ \mu = \rho V_S^2 = \frac{V_S^2 I_P}{V_P} \\ \rho_0 = \rho = \frac{I_P}{V_P} \end{cases}$$

$$\begin{aligned}
\nabla_{V_P} &= \nabla_\lambda \frac{\partial \lambda}{\partial V_P} + \nabla_\mu \frac{\partial \mu}{\partial V_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_P} \\
&= \nabla_\lambda \left(I_P + \frac{2V_S^2 I_P}{V_P^2} \right) + \nabla_\mu \left(-\frac{V_S^2 I_P}{V_P^2} \right) + \nabla_{\rho_0} \left(-\frac{I_P}{V_P^2} \right) \\
&= \left(\nabla_\lambda (V_P^2 + 2V_S^2) - \nabla_\mu V_S^2 - \nabla_{\rho_0} \right) \frac{I_P}{V_P^2} \\
&= \left(\nabla_\lambda V_P^2 + (2\nabla_\lambda - \nabla_\mu) V_S^2 - \nabla_{\rho_0} \right) \frac{\rho}{V_P} \\
\nabla_{V_S} &= \nabla_\lambda \frac{\partial \lambda}{\partial V_S} + \nabla_\mu \frac{\partial \mu}{\partial V_S} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial V_S} \\
&= \nabla_\lambda \frac{-4V_S I_P}{V_P} + \nabla_\mu \frac{2V_S I_P}{V_P} \\
&= (-2\nabla_\lambda + \nabla_\mu) \frac{2V_S I_P}{V_P} \\
&= (-2\nabla_\lambda + \nabla_\mu) 2V_S \rho
\end{aligned}$$

PARAMETERIZATION velocities-impedance

isotropic P-SV *continued*

$$\begin{cases} \lambda = \rho(V_P^2 - 2V_S^2) = V_P I_P - \frac{2V_S^2 I_P}{V_P} \\ \mu = \rho V_S^2 = \frac{V_S^2 I_P}{V_P} \\ \rho_0 = \rho = \frac{I_P}{V_P} \end{cases}$$

$$\begin{aligned}
\nabla_{I_P} &= \nabla_\lambda \frac{\partial \lambda}{\partial I_P} + \nabla_\mu \frac{\partial \mu}{\partial I_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial I_P} \\
&= \nabla_\lambda \left(V_P - \frac{2V_S^2}{V_P} \right) + \nabla_\mu \frac{V_S^2}{V_P} + \nabla_{\rho_0} \frac{1}{V_P} \\
&= \left(\nabla_\lambda (V_P^2 - 2V_S^2) + \nabla_\mu V_S^2 + \nabla_{\rho_0} \right) \frac{1}{V_P} \\
&= \left(\nabla_\lambda V_P^2 + (-2\nabla_\lambda + \nabla_\mu) V_S^2 + \nabla_{\rho_0} \right) \frac{1}{V_P}
\end{aligned}$$

velocities-density

velocities-impedance

slowness-density

PARAMETERIZATION slowness-density

isotropic ACoustic

$$\boxed{S_P = 1/V_P}$$

$$\begin{cases} \kappa = \rho V_P^2 = \rho S_P^{-2} \\ \rho_0 = \rho \end{cases}$$

$$\begin{aligned}
\nabla_{S_P} &= \nabla_\kappa \frac{\partial \kappa}{\partial S_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial S_P} \\
&= \nabla_\kappa (-2\rho S_P^{-3}) \\
&= \nabla_\kappa (-2\rho V_P^3) \\
\nabla_\rho &= \nabla_\kappa \frac{\partial \kappa}{\partial \rho} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial \rho} \\
&= \nabla_\kappa S_P^{-2} + \nabla_{\rho_0} \\
&= \nabla_\kappa V_P^2 + \nabla_{\rho_0}
\end{aligned}$$

$$\begin{cases} S_P = \sqrt{\frac{\rho_0}{\kappa}} \\ \rho = \rho_0 \end{cases}$$

$$\begin{aligned}
\nabla_\kappa &= \nabla_{S_P} \frac{\partial S_P}{\partial \kappa} + \nabla_\rho \frac{\partial \rho}{\partial \kappa} \\
&= \nabla_{S_P} \left(\frac{1}{2} \sqrt{\frac{\kappa}{\rho_0}} \frac{-\rho_0}{\kappa^2} \right) \\
&= \nabla_{S_P} (-0.5) \rho_0^{0.5} \kappa^{-1.5} \\
\nabla_{\rho_0} &= \nabla_{S_P} \frac{\partial S_P}{\partial \rho_0} + \nabla_\rho \frac{\partial \rho}{\partial \rho_0} \\
&= \nabla_{S_P} \left(\frac{1}{2} \sqrt{\frac{\kappa}{\rho_0}} \frac{1}{\kappa} \right) + \nabla_\rho \\
&= \nabla_{S_P} 0.5 \rho_0^{-0.5} \kappa^{-0.5} + \nabla_\rho
\end{aligned}$$

PARAMETERIZATION slowness-density + PASSIVE Gardner

isotropic ACoustic

$$\boxed{\rho = a S_P^{-b}}$$

$$\begin{cases} \kappa = \rho V_P^2 = a S_P^{-b-2} \\ \rho_0 = \rho = a S_P^{-b} \end{cases}$$

$$\begin{aligned}
\nabla_{S_P} &= \nabla_\kappa \frac{\partial \kappa}{\partial S_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial S_P} \\
&= \nabla_\kappa a(-b-2) S_P^{-b-3} \\
&\quad + \nabla_{\rho_0} (-ab) S_P^{-b-1} \\
&= \left(\nabla_\kappa \frac{b+2}{b} + \nabla_{\rho_0} S_P^2 \right) (-ab) S_P^{-b-3} \\
&= \left(\nabla_\kappa \frac{b+2}{b} + \nabla_{\rho_0} V_P^{-2} \right) (-ab) V_P^{b+3}
\end{aligned}$$

$$S_P = \left(\frac{\kappa}{a} \right)^{-\frac{1}{b+2}}$$

$$\begin{aligned}
\nabla_\kappa &= \nabla_{S_P} \frac{\partial S_P}{\partial \kappa} \\
&= \nabla_{S_P} \frac{-1}{a(b+2)} \left(\frac{\kappa}{a} \right)^{-\frac{b+3}{b+2}}
\end{aligned}$$

PARAMETERIZATION slowness-density

isotropic P-SV

$$\boxed{S_{PS} = V_S / V_P = V_S S_P}$$

$$\begin{cases} \lambda = \rho(V_P^2 - 2V_S^2) = \rho S_P^{-2}(1 - 2S_{PS}^2) \\ \mu = \rho V_S^2 = \rho(S_{PS}/S_P)^2 \\ \rho_0 = \rho \end{cases}$$

$$\begin{aligned}
\nabla_{S_P} &= \nabla_\lambda \frac{\partial \lambda}{\partial S_P} + \nabla_\mu \frac{\partial \mu}{\partial S_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial S_P} \\
&= \nabla_\lambda \rho(-2)S_P^{-3}(1 - 2S_{PS}^2) + \nabla_\mu \rho S_{PS}^2(-2)S_P^{-3} \\
&= \left(\nabla_\lambda(1 - 2S_{PS}^2) + \nabla_\mu S_{PS}^2\right)(-2)\rho S_P^{-3} \\
&= \left(\nabla_\lambda V_P^3 + (-2\nabla_\lambda + \nabla_\mu)V_P V_S^2\right)(-2)\rho \\
\nabla_{S_{PS}} &= \nabla_\lambda \frac{\partial \lambda}{\partial S_{PS}} + \nabla_\mu \frac{\partial \mu}{\partial S_{PS}} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial S_{PS}} \\
&= \nabla_\lambda \rho S_P^{-2}(-4)S_{PS} + \nabla_\mu 2\rho S_{PS}/S_P^2 \\
&= (-2\nabla_\lambda + \nabla_\mu)\, 2\rho S_{PS} S_P^{-2} \\
&= (-2\nabla_\lambda + \nabla_\mu)\, 2\rho V_P V_S
\end{aligned}$$

PARAMETERIZATION slowness-density

isotropic P-SV *continued*

$$\boxed{S_{PS} = V_S / V_P = V_S S_P}$$

$$\left\{ \begin{array}{l} \lambda = \rho(V_P^2 - 2V_S^2) = \rho S_P^{-2}(1 - 2S_{PS}^2) \\ \mu = \rho V_S^2 = \rho(S_{PS}/S_P)^2 \\ \rho_0 = \rho \end{array} \right.$$

$$\begin{aligned} \nabla_\rho &= \nabla_\lambda \frac{\partial \lambda}{\partial \rho} + \nabla_\mu \frac{\partial \mu}{\partial \rho} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial \rho} \\ &= \nabla_\lambda S_P^{-2}(1 - 2S_{PS}^2) + \nabla_\mu (S_{PS}/S_P)^2 + \nabla_{\rho_0} \\ &= \left(\nabla_\lambda (1 - 2S_{PS}^2) + \nabla_\mu S_{PS} \right) S_P^{-2} + \nabla_{\rho_0} \\ &= \nabla_\lambda V_P^2 + (-2\nabla_\lambda + \nabla_\mu) V_P V_S + \nabla_{\rho_0} \end{aligned}$$

PARAMETERIZATION slowness-density + PASSIVE Gardner

isotropic P-SV

$$\boxed{S_{PS} = V_S S_P;\, \rho = a S_P^{-b}}$$

$$\left\{ \begin{array}{l} \lambda = \rho S_P^{-2}(1 - 2S_{PS}^2) = a S_P^{-b-2}(1 - 2S_{PS}^2) \\ \mu = \rho (S_{PS}/S_P)^2 = a S_P^{-b-2} S_{PS}^2 \\ \rho_0 = a S_P^{-b} \end{array} \right.$$

$$\begin{aligned}
\nabla_{S_P} &= \nabla_\lambda \frac{\partial \lambda}{\partial S_P} + \nabla_\mu \frac{\partial \mu}{\partial S_P} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial S_P} \\
&= \nabla_\lambda a(-b-2) S_P^{-b-3}(1 - 2S_{PS}^2) \\
&\quad + \nabla_\mu a(-b-2) S_P^{-b-3} S_{PS}^2 + \nabla_{\rho_0} a(-b) S_P^{-b-1} \\
&= \left(\nabla_\lambda (1 - 2S_{PS}^2) + \nabla_\mu S_{PS}^2 + \nabla_{\rho_0} b(b+2) S_P^2 \right) \frac{-a}{b+2} S_P^{-b-3} \\
&= \left(\nabla_\lambda V_P^2 + (-2\nabla_\lambda + \nabla_\mu) V_S^2 + \nabla_{\rho_0} b(b+2) \right) \frac{-a}{b+2} V_P^{b+1} \\
\nabla_{S_{PS}} &= \nabla_\lambda \frac{\partial \lambda}{\partial S_{PS}} + \nabla_\mu \frac{\partial \mu}{\partial S_{PS}} + \nabla_{\rho_0} \frac{\partial \rho_0}{\partial S_{PS}} \\
&= \nabla_\lambda a S_P^{-b-2}(-4) S_{PS} + \nabla_\mu a S_P^{-b-2} 2 S_{PS} \\
&= (-2\nabla_\lambda + \nabla_\mu)\, 2a S_P^{-b-2} S_{PS} \\
&= (-2\nabla_\lambda + \nabla_\mu)\, 2a V_P^{b+1} V_S
\end{aligned}$$