Pricing Barrier Options using PDEs in C++

A Preprint

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October 5, 2020

Abstract

In this project, we have implemented an exact pricer using a continuous analytic formula and a Partial Differential Equation (PDE) pricer for barrier options in C++ and compared the results obtained from both functions. For the PDE pricer, we have also added a setAlignment method that allows for the grid to be aligned to the barrier level. Furthermore, we have investigated the convergence properties of the PDE pricer. We find that the PDE pricer exhibits a linear order of convergence, and that the fully implicit scheme has smoother convergence (fewer fluctuations) compared to the Crank-Nicholson scheme. Moreover, we also find that the Broadie-Glasserman-Kou adjustment formula for the continuous analytic formula provides a close approximation to the discrete PDE pricer.

Keywords Option Pricing · Partial Differential Equations · Barrier Options · Computational Finance

1 Introduction

Barrier options differ from traditional options in that their payoffs depend on whether the underlying asset price has reached a certain price level before expiration. Thus, the barrier level has a sizeable impact on the present value of the barrier option, and it has varying impacts on calls and puts. We implemented a code that prices European barrier options via closed-form formulae (continuous monitoring) and PDE (discrete-time monitoring) across different initial conditions and different payoff combinations (16 payoff conditions in total: call/put, up/down, in/out, barrier $\leq$ strike / strike $\leq$ barrier). The results indicate that the closed-form price is different from the PDE price, and the daily monitoring provides the closest approximation to the closed-form price. This is because the closed-form price is derived assuming there is monitoring in continuous time, and hence the daily monitoring frequency is the closest approximation to continuous monitoring. We also experimented with the Broadie-Glasserman-Kou (BGK) [1] adjustment formula for the closed-form formulae, and our results indicate that the BGK adjustment provides a good adjustment from a continuous monitoring scheme to a discrete monitoring scheme.

Furthermore, we examined the convergence properties of PDE prices and determined that the PDE prices for barrier options exhibit a linear order of convergence, and that the fully implicit scheme allows for smoother convergence than the Crank-Nicholson scheme. Additionally, the size of the barrier level affects convergence, with convergence improving when the barrier is further away from the spot price.

The next section (section 2) describes our code to price barrier options as well as the 3 research areas we explored in this project. Section 3 describes the results under the 3 research areas as well as our discussion of these results. Section 4 concludes.

2 Code and Approach

We added 2 excel functions to the orflib library. The first function is called barrierOptionBS, and it calculates closed-form formulae (continuous monitoring) prices for barrier options. The closed-form formulae are based on the results from Hull's *Options, Futures and Other Derivatives* [2]. We examined 16 different payoff possibilities in total to describe a barrier option: call/put, up/down, in/out, barrier $\leq$ strike / strike $\leq$ barrier.

2.1 Exact Pricer for Barrier Options

barrierOptionBS includes the following parameter entries and validations:

- Payoff Type: This is an integer input. 1 indicates Call Option, and -1 indicates Put option. We validated the input to be an integer, and an error is thrown when the input is not -1 or 1.
- Barrier Type: This is a string input comprising 2 characters out of a possible 4 combinations: 'UO', 'UI', 'DO', 'DI'. 'U' represents up, 'D' represents down, 'I' represents in, and 'O' represents out. Both upper-case and lower-case characters are acceptable. We validated the input to only accept these 4 possibilities.
- Spot: This is a numerical (double) entry. This indicates the present spot value of the underlying asset. We validated the input to be non-negative, because under the conditions of no-arbitrage, spot prices cannot be negative.
- Strike: This is a numerical (double) entry that represents the strike price of the call/put option. We validated the input to be non-negative, because under the conditions of no arbitrage, strike price cannot be negative.
- Barrier level: This is a numerical (double) entry that represents the barrier level. Just as how strike and spot must be non-negative, we also validated the barrier level entry to ensure that it is non-negative. In addition, we made sure that when the user is pricing a 'down' option, the barrier level must be $\leq$ initial spot price, and when the user is pricing an 'up' option, the barrier level must be $\geq$ initial spot price, because if this is the case, the 'in' barrier option is the same as an ordinary option, and the 'out' option will be worth 0.
- Time to Expiry: This is a numerical (double) entry that represents the time to expiry of the option, measured in years. We validated the time to expiry to make sure that time to expiry is non-negative, because if it is negative, then the option has already expired and will not be traded.
- Interest Rate: This is a numerical (double) entry that represents current interest rate.
- Dividend Yield: This is a numerical (double) entry that represents the dividend yield.
- Volatility: This is a numerical (double) entry that represents the volatility (standard deviation) of returns. We validated the volatility entry to be non-negative, because standard deviations are non-negative by definition.

2.2 PDE Pricer for Barrier Options

The second function is called BarrBSPDE, and it makes use of the PDE solver functions in the orflib library to price barrier options. As PDE pricing involves discrete-time monitoring of barrier option prices, we coded 3 monitoring frequencies: MONTHLY (12 times a year), WEEKLY (52 times a year), and DAILY (365 times a year). To achieve this, we coded a new BarrierCallPut class within a new barriercallput.hpp file.

Fundamentally, the design of the PDE European barrier option pricer is based on that of the PDE pricer for the American option. Two key modifications are made, however. First, before the option hits the expiry date, the intrinsic value of the option is set to be 0. This would mean that the option cannot be exercised early. We checked that such a change would make the price of the American option equal to the barrier option. Next, at each time step, instead of checking the exercise condition, we check for whether the spot has hit the barrier level. If it does, then the continuation value *contValue* of the option would be set as 0, which will simulate a barrier option losing its value. This approach is based on the idea of imposing a constraint at each time step by Zvan et al (2000) [3].

To address the existence of a 'stub interval'/remainder in the case that the time-to-expiry of the barrier option is not an integer multiple of the monitoring frequency, we included the stub interval at time 0 instead of at expiry.

Moreover, BarrBSPDE includes an option for the user to align the PDE grid with the barrier level instead of the spot price (which is the default option in orflib), thus allowing users further means to price barrier options under different pricing methods. On a practical level, this is implemented by changing the Pde1Solver to include an additional setAlignment method (function) that can allow the user to setAlignment to the barrier level.

In addition to the 9 parameter entries and validations for barrierOptionBS, BarrBSPDE also includes these parameters and validations:

- Monitoring Frequency: This is an integer entry. '0' indicates monthly monitoring frequency, '1' indicates weekly monitoring frequency, and '2' indicates daily monitoring frequency. We validated the input to only accept these 3 possible entries.
- Discount Curve: This is an entry to input the Yield Curve that is created by the user in excel. An error is thrown when a Yield Curve is not found.
- Volatility: We extended the volatility entry to allow the user to either input a volatility number (double) entry or a Volatility Term Structure name. An error is thrown when neither is found.
- PDE Parameters: PDE parameters is a list of parameters that include: number of time steps, number of spot nodes, and theta (0.5 under Crank-Nicholson scheme, and 1 under fully-implicit scheme).
- Set Alignment: This is a boolean variable. 'TRUE'/1 allows the user to align the PDE grid with the barrier level instead of the spot price, and 'FALSE'/0/'missing' (default) aligns the PDE grid with the spot price.
- All Results: This is a boolean variable. 'TRUE'/1 returns all the results of the PDE calculations, which includes both price and time/spot, whereas 'FALSE'/0/'missing' (default) returns just the present value price of the barrier option.

As described earlier, a key validation that we include for both the exact pricer and the PDE pricer is for the barrier option to not have been knocked-out or knocked-in at the initial time $T = 0$. This would imply that if it was an up barrier, the initial spot must be set lower than the barrier level, and if it was a down barrier, the initial spot must be set higher than the barrier level.

We then used our code to investigate the following areas:

- We compared the prices of knock-out barrier options using both barrierOptionsBS (closed-form formulae) and BarrBSPDE (PDE method).
- We created convergence graphs for one-year ATM up-and-out barrier call with strike $\leq$ barrier and daily frequency, and we compared the results obtained under the Crank-Nicholson scheme and the fully-implicit scheme.
- We compared the converged PDE values (both knock-in and knock-out barrier options) with the closed-form values (modified under the Broadie-Glasserman-Kou adjustment formula).

3 Results and Discussion

3.1 Comparison of PDE prices with closed-form prices

We compared the PDE prices with the closed-form prices calculated for 4 knock-out barrier options. Our payoff conditions are stipulated below:

Spot	100.00
Strike	100.00
Barrier Level	108 (UO) / 92 (DO)
Time to expiry	1.00 (yrs)
Discount/Yield Curve	5.00 % flat yield curve
Volatility	30.00%
Dividend Yield	2.00%

PDE Parameters	
TimeSteps	500
SpotNodes	500
StdDevs	4.0
Theta	0.5

Table 1: Payoff conditions for comparison

Up and Out Call	Continuous	0.01594
	Daily	0.021441573
	Weekly	0.040428461
	Monthly	0.085056991
Down and Out Call	Continuous	7.25011
	Daily	7.889442061
	Weekly	8.747544996
	Monthly	9.937667844
Up and Out Put	Continuous	5.11904
	Daily	5.410399997
	Weekly	6.148905547
	Monthly	7.219364387
Down and Out Put	Continuous	0.02122
	Daily	0.034140838
	Weekly	0.060182025
	Monthly	0.119350327

Table 2: Comparison of closed-form and PDE prices for knock-out options

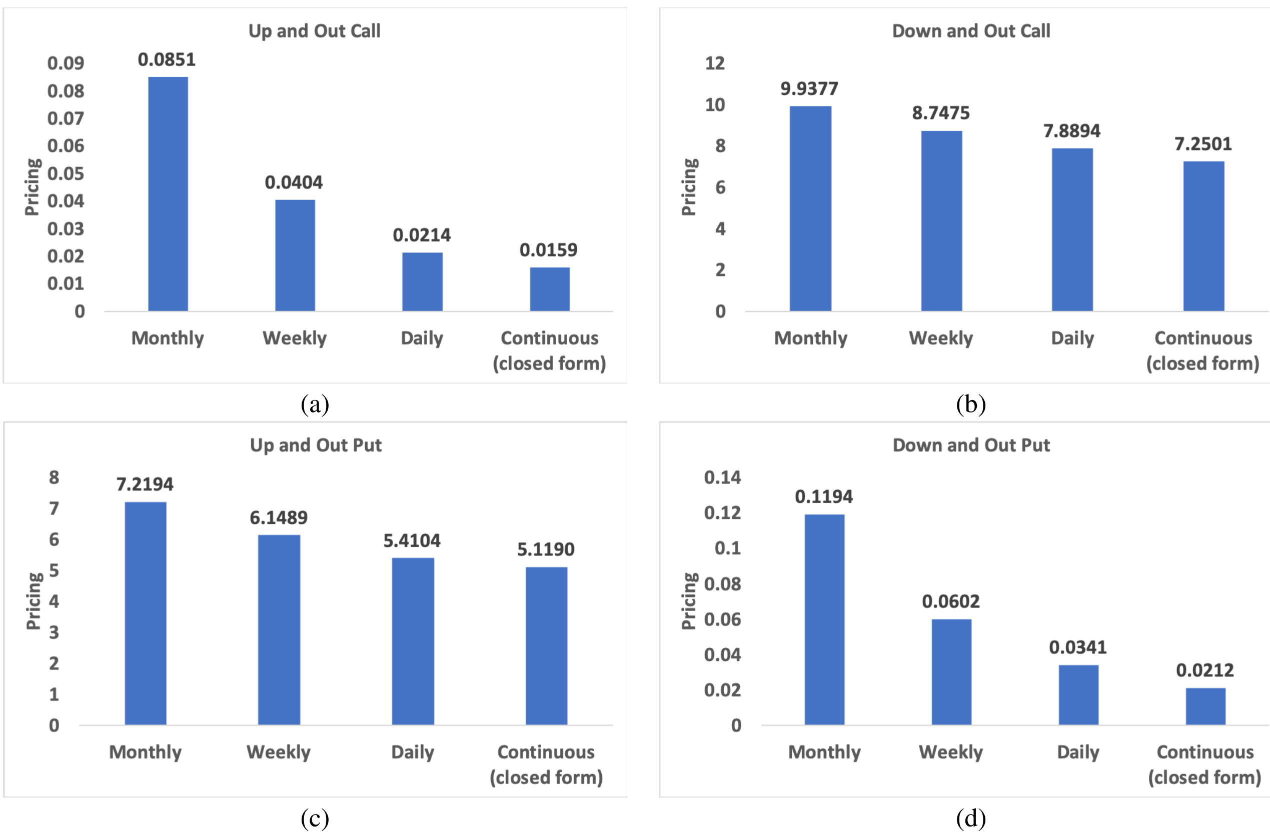


Figure 1: The pricing at different monitoring frequencies for (a) up and out call; (b) down and out call; (c) up and out put and (d) down and out put. The price decreases as monitoring frequency increases and converge to the price of obtained from closed form continuous formula.

We can make several observations with regard to the results above. Firstly, for all 4 knock-out barrier options, the price of both call and put options increase as the monitoring frequency decreases (Monthly > Weekly > Daily > Continuous). Intuitively, as the monitoring frequency increases, it is more likely for an option to be knocked out at any point in time before expiry, which causes the prices to decrease. This is also the reason that PDE prices > closed-form prices. And as expected, the Daily monitoring frequency yields the closest approximation to the closed-form price.

Additionally, given the conditions stipulated above, it is easy to see that the Down-and-Out call has a higher price than the Up-and-Out call, whereas the Down-and-Out put has a lower price than the Up-and-Out put. The Up-and-Out call has a low price because the upside earnings for the investor is capped at 8 (108 - 100): once spot prices reach above 108, the call option is knocked out. As for the Down-and-Out call option, the investor is entitled to unlimited upside as long as spot price does not reach below 92.

Conversely, for the Down-and-Out put, the investor's earnings are also capped at 8 (100-92), because once the spot price reaches below 92, the put option is knocked out. As for the Up-and-Out put, the investor is able to enjoy the full payoffs of a put option provided that the spot does not reach above 108.

This result above confirms that our barrier option pricing functions yield results that make intuitive sense, thus confirming our approach and code.

Furthermore, we experimented with different Up-and-Out and Down-and-Out barrier levels for both knock-out calls and puts. The results indicate that as the barrier gets further from the spot price, the prices of the knock-out options get closer and closer to the prices of the ordinary options. This is to be expected, because as the barrier gets further away from the spot price, the probability of the spot price hitting the barrier before expiry (and thereby knocking out the barrier option) decreases, and therefore in expectation, the payoffs of the barrier options more strongly resemble the payoffs of ordinary options.

3.2 Convergence graphs for Crank-Nicholson and Fully-Implicit schemes

We also compared the convergence graphs for a one-year ATM Up-and-Out barrier call using fully implicit and Crank-Nicholson schemes.

Spot	100.00
Strike	100.00
Barrier Level	110
Time to expiry	1.00 (yrs)
Discount/Yield Curve	0.00 % flat yield curve
Volatility	25.00%
Dividend Yield	0.00%

Table 3: Payoff conditions for comparison

To compute the convergence graphs, we created 6 refinements in total. We increased TimeSteps and SpotNodes exponentially: 100, 200, 400, 800, 1600, 3200.

As shown in Fig 2, the convergence is in general closer to linear. This is in line with the results from Zvan et al (2002) [3]. Compared to the fully implicit scheme, the Crank-Nicholson scheme exhibits fluctuations, especially in cases of H = 110 and 125, when the barrier is closer to the strike. This is consistent with the conditions required to prevent oscillations provided by Zvan et al.(2000) as the following [3]:

$$\Delta S_{i-1/2} < \frac{\sigma^2 S_i}{r} \tag{1}$$

$$\frac{1}{(1-\theta)\Delta t^*} > \frac{\sigma^2 S_i^2}{2}\left(\frac{1}{\Delta S_{i-1/2}\Delta S_i} + \frac{1}{\Delta S_{i+1/2}\Delta S_i}\right) + r \tag{2}$$

where $\Delta S_i = 1/2(S_{i+1} - S_{i-1})$, $\Delta S_{i+1/2} = S_{i+1} - S_i$ and $\Delta S_{i-1/2} = S_{i1} - S_{i-1}$. σ is the volatility. r is the risk-free interest rate. Δt^* is the time step. In our problem, Eqn 1 is satisfied. Eqn 2 is always satisfied for a fully implicit scheme (θ=1), but is only satisfied at small time step and small S_i grid size as Eqn 2 restricts the time step as a function of S_i grid size [3]. Thus, in this problem, a fully implicit scheme is more stable than a Crank-Nicholson scheme.

We also experimented with different values of the barrier level vis-a-vis that of the strike price/spot price. Why do barrier options converge better when the barrier is far away from the spot?

It is well-known that pricing barrier options using discretized numerical methods presents its challenges due to the barrier's discontinuity in the option payoff. In our implementation of the Crank-Nicholson scheme, the solution is affected by spurious oscillations derived from "an inaccurate approximation of the very sharp gradient produced by the knock-out clause, generating an error that is damped out very slowly"[4].

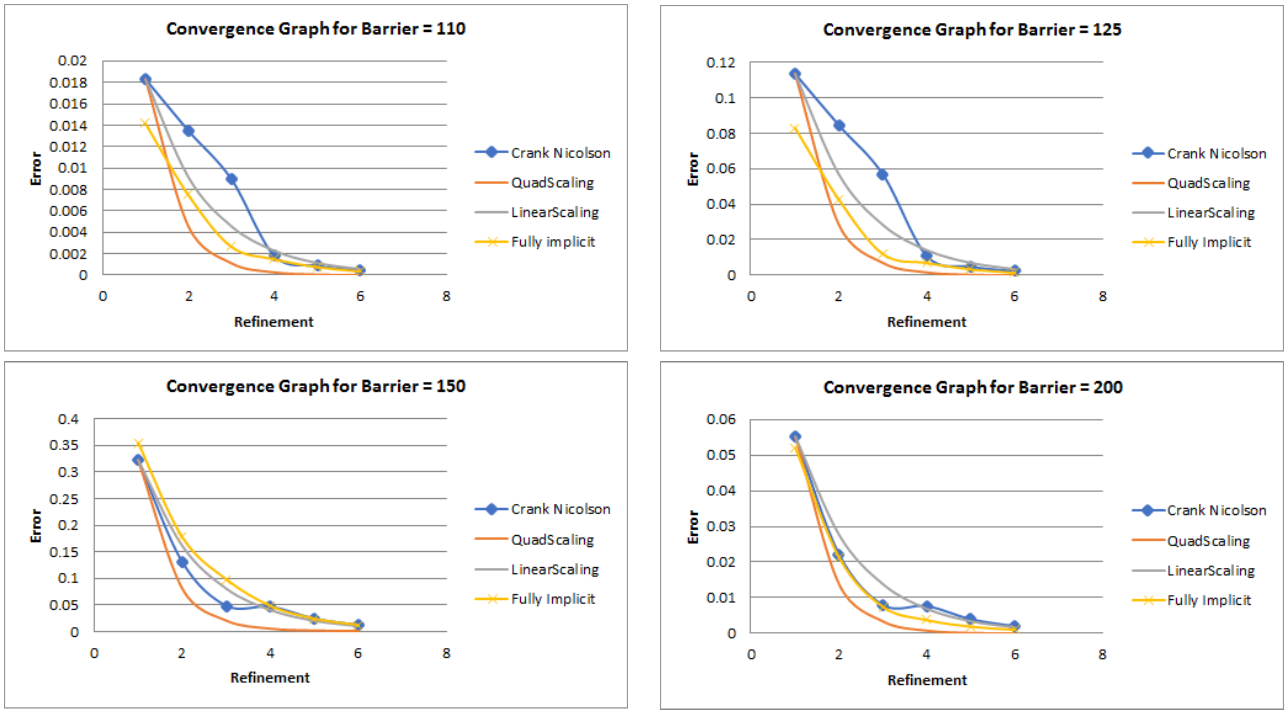


Figure 2: Convergence for one-year ATM up and out barrier call with K = 100, H = 110, 125, 150 and 200 and daily frequency using Crank-Nicolson and fully implicit schemes.

Spurious oscillations can be visualized in both option values and Greeks by comparing continuous and discretely monitored cases. Umeorah and Mashele do just so in their research of rebate barrier options. Figure 3 present Greeks vs stock price for rebate barrier options with a spot price of 60, strike price of 50, and barrier level of 35. The first row displays a continuously monitored case, while the second is a discretely monitored case with 5 monitoring periods.

We can see from Figure 3 that oscillations arise in Delta and Gamma near the barrier level of 35 in the discrete case.

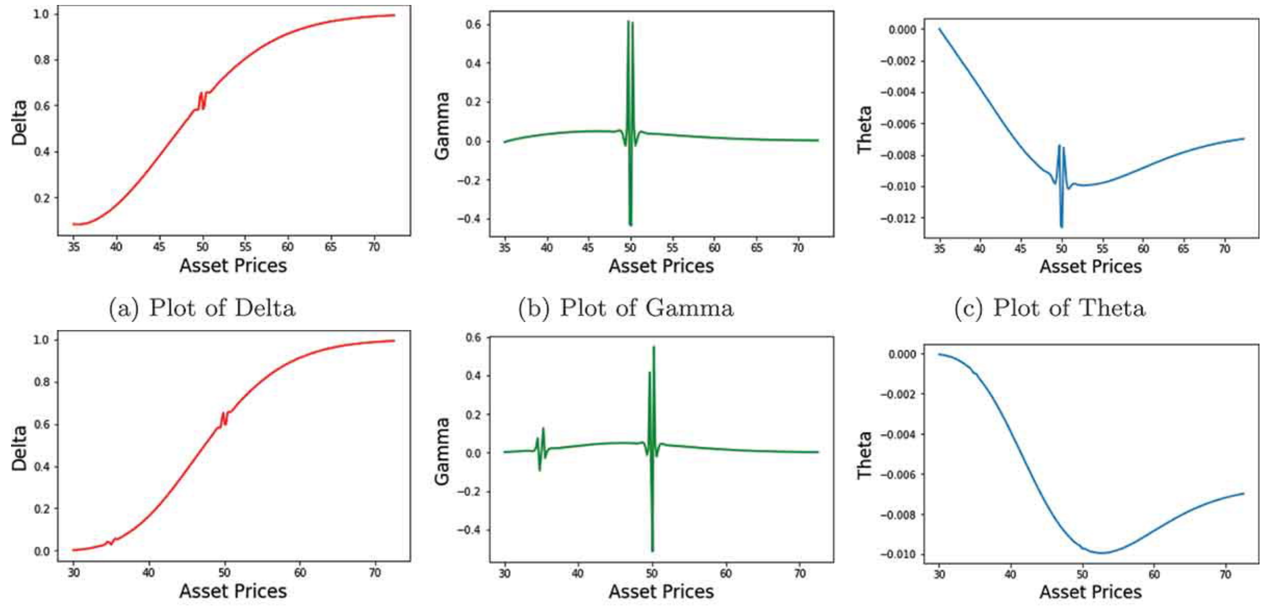
This issue is exacerbated when the barrier level threshold is close to the initial spot price. The calculations of option Greeks may change sharply for small movements in the underlying. Furthermore, the path-dependent nature of barrier options makes the threshold relevant at each point in time until the barrier is hit. When the barrier threshold is near initial spot, further simulated trajectories may be needed, which can be costly[6].

Empirical evidence from research by Broadie and Glasserman provides proof of greater pricing inaccuracies when the barrier threshold is near spot. Across the board in all options priced discretely, the absolute relative error increased as the barrier approached the initial spot level of 100[1].

We can see this effect play out in our project by examining the convergence rates of barrier options closer to the initial spot price. The Crank-Nicholson scheme converges slower relative to Quadratic and Linear Scaling for barrier levels of 110 and 125. When the barrier level is further away at 150 and 200, the scheme improves and converges right in between these benchmarks.

3.3 Comparison of converged PDE prices with closed-form prices

We compared the prices calculated from the closed-form formulae and the PDE methods, and we examined the difference in prices for all 8 possible barrier options: (call,put) x (up, down) x (in, out) and not just for knock-out barriers. Our payoff conditions are detailed below:



(a) Plot of Delta (b) Plot of Gamma (c) Plot of Theta

(a) Plot of Delta (b) Plot of Gamma (c) Plot of Theta

Figure 3: Greeks vs Spot for Continuously Monitored (Top Row) and Discretely Monitored with 5 Monitoring Periods (Bottom Row) Rebate Barrier Options (S0 = 60, K = 50, B = 35) [5]

Spot	100.00
Strike	100.00
Barrier Level	110 (Up) / 90 (Down)
BGK-adjusted Barrier Level	110.842 (Up) / 89.316 (Down)
Time to expiry	1.00 (yrs)
Discount/Yield Curve	0.00 % flat yield curve
Volatility	25.00%
Dividend Yield	0.00%

PDE Parameters	
TimeSteps	10,000
SpotNodes	10,000
StdDevs	4.0
Theta	1

Table 4: Payoff conditions for comparing between closed-form and PDE prices

We applied the BGK-adjustment formula to the Up and Down barrier levels when calculating the closed-form prices. As can be seen, the BGK-adjustment increases the 'Up' barrier and decreases the 'Down' barrier respectively.

As discussed in the previous section, the fully implicit scheme is more stable than the Crank-Nicholson scheme, and thus we decided to stick with the fully implicit scheme. We increased TimeSteps and SpotNodes to 10,000 to ensure sufficient convergence for the PDE prices.

		Difference with BGK	Difference without BGK
Knock-out Call	Up	8.91E-05	1.06E-03
	down	1.52E-03	9.32E-02
Knock-out Put	Up	1.67E-03	7.79E-02
	Down	9.57E-05	1.37E-03
Knock-in Call	Up	3.73E-05	5.88E-02
	down	1.39E-03	1.51E-01
Knock-in Put	Up	1.55E-03	1.36E-01
	Down	3.08E-05	5.91E-02

Table 5: Difference between BGK-adjusted closed-form and PDE prices (Daily monitoring)

		Difference with BGK	Difference without BGK
Knock-out Call	Up	1.66E-03	2.32E-02
	down	2.92E-03	4.32E-01
Knock-out Put	Up	3.23E-03	4.63E-01
	Down	1.61E-03	3.10E-02
Knock-in Call	Up	1.54E-03	8.09E-02
	down	2.80E-03	4.90E-01
Knock-in Put	Up	3.10E-03	5.20E-01
	Down	1.48E-03	8.88E-02

Table 6: Difference between BGK-adjusted closed-form and PDE prices (Weekly monitoring)

		Difference with BGK	Difference without BGK
Knock-out Call	Up	1.95E-02	7.18E-02
	down	8.20E-03	9.53E-01
Knock-out Put	Up	5.69E-03	1.06E+01
	Down	1.96E-02	9.47E-02
Knock-in Call	Up	1.93E-02	1.29E-01
	down	8.07E-03	1.01E+01
Knock-in Put	Up	5.57E-03	1.12E+01
	Down	1.94E-02	1.52E-01

Table 7: Difference between BGK-adjusted closed-form and PDE prices (Monthly monitoring)

As can be seen, we have verified numerically that the PDE prices are very close to the exact prices given by the BGK-adjusted closed-form formulae, and the difference in price is to the order of 10^{-2} or smaller. We conducted the same exercise for the non-BGK-adjusted closed-form prices, and the differences in prices are higher. This suggest that the BGK-adjustment does allow the continuous-monitoring prices to better approximate discrete-monitoring PDE prices in our case.

Additionally, we can observe that the precision of approximation increases with the frequency of monitoring: the closed-form prices offer the most precise approximations of the daily frequency PDE prices, and they offer the least precise approximations of the monthly frequency PDE prices. As discussed before, this is to be expected, as the daily-monitoring of option prices is the best discrete approximation for continuous-monitoring of option prices.

4 Conclusion

In conclusion, we have extended the orflib library by adding 2 new functions: exact pricer and PDE pricer for barrier options. We investigated the difference in barrier option prices computed via the exact pricer and PDE pricer functions, and we found that monitoring frequency as well as the size of the barrier relative to spot and strike are significant in accounting for these differences. Specifically, the exact prices calculated from closed-form formulae provide the closest approximation to daily-monitored barrier prices, and as the barrier moves further away from the current spot price, the prices of knock-out options increase and converge to the prices of ordinary options, which confirm the predictions of financial theory. We have also verified that the BGK-adjustment allows us to yield closed-form prices that are numerically close to the PDE prices.

We investigated the convergence properties of the PDE pricer and observed that PDE prices exhibit a linear order of convergence. Moreover the fully implicit scheme has smoother convergence (fewer fluctuations) as compared to the Crank-Nicholson scheme, which, after extensive research, is in line with current literature. Furthermore, we observed that the convergence of the PDE prices becomes smoother as the barrier level moves further away from the spot price, which is also in line with present literature.

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