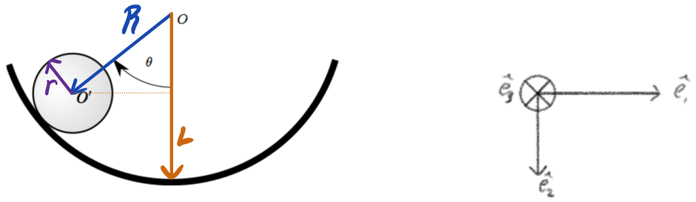


Fig. 1

A solid disk with mass m and radius r is rolling under the influence of a constant gravity field inside a cylinder of radius L, as shown in Fig. 1. Assume $\hat{e}_3$ points into the paper.

- Motion is Planar
- C.O.M. of disk is, $R = L - r$, away from O.

Q4) Finding angular momentum vector, $\vec{H}_o$, relative to cylinder centre O.

- $\vec{r}_{o'/o} = -R\sin\theta\,\hat{e}_1 + R\cos\theta\,\hat{e}_2$

- $\vec{v}_{o'/o} = \frac{d}{dt}(\vec{r}_{o'/o}) = \frac{d}{dt}(-R\sin\theta(t)\,\hat{e}_1 + R\cos\theta(t)\,\hat{e}_2)$

$$= -R\cos\theta\,\dot{\theta}\,\hat{e}_1 - R\sin\theta\,\dot{\theta}\,\hat{e}_2$$

- Angular momentum due to C.O.M. of smaller cylinder 'orbiting' abt O,

$$H_{orbit} = \vec{r}_{o/o'} \times m\vec{v}_{o'/o}$$

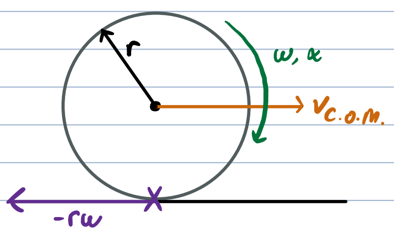
$$= (-R\sin\theta\,\hat{e}_1 + R\cos\theta\,\hat{e}_2) \times (-mR\cos\theta\,\dot{\theta}\,\hat{e}_1 - mR\sin\theta\,\dot{\theta}\,\hat{e}_2)$$

$$= mR^2\sin^2\theta\,\dot{\theta}\,\hat{e}_3 - mR^2\cos^2\theta\,\dot{\theta}\,(-\hat{e}_3)$$

$$= mR^2\dot{\theta}\,(\sin^2\theta + \cos^2\theta)\,\hat{e}_3$$

$$= mR^2\dot{\theta}\,\hat{e}_3$$

- Angular Momentum due to spin of disk abt O',



$$H_{spin} = I_{disk}\,\omega_{spin}\,\hat{e}_3$$

$$= (\tfrac{1}{2} m r^{\cancel{2}})(-R\dot{\theta}/\cancel{r})\,\hat{e}_3$$

$$= -\frac{1}{2} m r R\dot{\theta}\,\hat{e}_3$$

- For a disk without slip, centre of mass velocity and spin velocity is exactly equal,

$$V_{c.o.m.} + V_{spin} = 0$$

$$R\dot{\theta} + (-r\omega_{spin}) = 0$$

$$\omega_{spin} = \frac{-R\dot{\theta}}{r}$$

- Total angular momentum abt O due to disk,

$$H_o = H_{orbit} + H_{spin}$$

$$= (mR^2\dot{\theta} - \tfrac{1}{2} m r R\dot{\theta})\,\hat{e}_3$$

$$= m[(L-r)^2 - \tfrac{1}{2} r(L-r)]\,\dot{\theta}\,\hat{e}_3$$

$$= m\left[\frac{2(L^2 - 2Lr + r^2) - rL + r^2}{2}\right]\dot{\theta}\,\hat{e}_3$$

$$= m\left[\frac{2L^2 - 4Lr + 2r^2 - rL + r^2}{2}\right]\dot{\theta}\,\hat{e}_3$$

$$= m(L^2 - \tfrac{5}{2}Lr + \tfrac{3}{2}r^2)\,\dot{\theta}\,\hat{e}_3$$

Q5) EOM of the disk.

- To find the E.O.M., we find the total energy then differentiate w.r.t. time.
  - ↳ We assume the total energy is conserved in the system. So for $t \in [0, \infty)$, E is const.
  - ↳ K.E. & P.E. contributions 'encode' force & torque constraints
  - ↳ Differentiating E 'captures & retains' linear and angular relations

- $\vec{V}_{o'/o} = -R\cos\theta\,\dot{\theta}\,\hat{e}_1 - R\sin\theta\,\dot{\theta}\,\hat{e}_2$

- K.E.,

$$T = T_{orbit} + T_{spin}$$

$$T = \frac{1}{2} m (\vec{V}_{o'/o} \cdot \vec{V}_{o'/o}) + \frac{1}{2} I_{c.o.m.}\, \omega_{spin}^2$$

$$= \frac{1}{2} m \left((-R\cos\theta\,\dot{\theta})^2 + (-R\sin\theta\,\dot{\theta})^2 \right) + \frac{1}{2}\left(\frac{1}{2} m \cancel{r^2} \times \frac{R^2\dot{\theta}^2}{\cancel{r^2}} \right)$$

$$= \frac{1}{2} m \left(R^2\cos^2\theta\,\dot{\theta}^2 + R^2\sin^2\theta\,\dot{\theta}^2 \right) + \frac{1}{4} m R^2 \dot{\theta}^2$$

$$= \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{4} m R^2 \dot{\theta}^2$$

$$= \frac{3}{4} m R^2 \dot{\theta}^2$$

- P.E.,

$$V = m g h_{c.o.m.}$$

⇒ Potential energy of disk initially comes from gravitational potential energy due to its displacement from ground.

$$= mg(L - R\cos\theta)$$

$$= mg(L - (L-r)\cos\theta)$$

- Total Energy,

$$E = T + V$$

$$= \frac{3}{4} m R^2 \dot{\theta}^2 + mg(L - R\cos\theta)$$

- E.O.M.,

$$\frac{dE}{dt} = 0 \quad \Rightarrow \text{Conserved}$$

$$\frac{d}{dt}\left(\frac{3}{4} m R^2 \dot{\theta}^2 + mg(L - R\cos\theta) \right) = 0$$

$$\frac{3}{4} m R^2 \frac{d}{dt}\left(\dot{\theta}(t)^2\right) + mg \frac{d}{dt}(L - R\cos\theta) = 0$$

$$\frac{3}{4} m R^2 \left(2\dot{\theta}(t)\ddot{\theta}(t)\right) + mg\left(0 - R(-\sin\theta)\,\dot{\theta}\right) = 0$$

$$\frac{3}{4} \cancel{m} R^2 (2\dot{\theta}\,\ddot{\theta}) + \cancel{m}g(R\sin\theta\,\dot{\theta}) = 0$$

$$\left(\frac{3}{2} R^2 \ddot{\theta} + gR\sin\theta \right)\dot{\theta} = 0$$

$$\frac{3}{2} R^2 \ddot{\theta} + gR\sin\theta = 0$$

$$\frac{3}{2}(L-r)\ddot{\theta} + g\sin\theta = 0$$

Q6) Natural Freq. of the motion (assuming small θ)

- From Q5, we have the EOM as a 2^{nd} - order ODE,

$$\frac{3}{2}(L-r)\ddot{\theta} + g\sin\theta = 0$$

Assuming small θ, $\sin\theta \approx \theta$. So,

$$\frac{3}{2}(L-r)\ddot{\theta} + g\theta = 0$$

$$\ddot{\theta} + \frac{g}{\frac{3}{2}(L-r)}\theta = 0$$

$$\ddot{\theta} + \frac{2}{3}\frac{g}{L-r}\theta = 0 \Rightarrow$$

similar to 2^{nd} order undamped ODE,

$$\ddot{x} + \omega_n^2 x = 0$$

- Thus,

$$\omega_n^2 = \frac{2}{3} \cdot \frac{g}{L-r}$$

$$\omega_n = \sqrt{\frac{2}{3} \cdot \frac{g}{L-r}}$$

Q7) Given $\theta(0)$ & $\dot{\theta}(0)$, find $\dot{\theta}$ when $\theta = 0°$

- At $t=0$, $E_{initial} = \frac{3}{4}mR^2\dot{\theta}^2 + mg(L-R\cos\theta)$
$= \frac{3}{4}mR^2\dot{\theta}_0^2 + mg(L-R\cos\theta_0)$

- At some t, $E = \frac{3}{4}mR^2\dot{\theta}^2 + mg(L-R\cos\theta)$

- Then by conservation of energy,

$$E = E_{initial}$$

$$\frac{3}{4}mR^2\dot{\theta}^2 + mg(L-R\cos\theta) = \frac{3}{4}mR^2\dot{\theta}_0^2 + mg(L-R\cos\theta_0)$$

$$\frac{3}{4}mR^2\dot{\theta}^2 = \frac{3}{4}mR^2\dot{\theta}_0^2 + mg(\cancel{L}-R\cos\theta_0 - \cancel{L}+R\cos\theta)$$

$$\dot{\theta}^2 = \frac{4}{3mR^2}\left(\frac{3}{4}mR^2\dot{\theta}_0^2 + mgR(\cos\theta-\cos\theta_0)\right)$$

$$\dot{\theta}^2 = \dot{\theta}_0^2 + \frac{4g}{3}\frac{\cos\theta-\cos\theta_0}{R}$$

Assuming small θ, $\cos\theta \approx 1$ $\Rightarrow$ $\dot{\theta}^2 = \dot{\theta}_0^2 + \frac{4g}{3}\frac{1-\cos\theta_0}{(L-r)}$