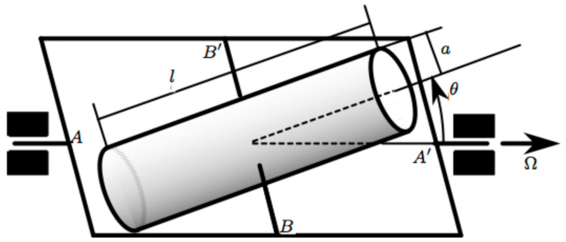
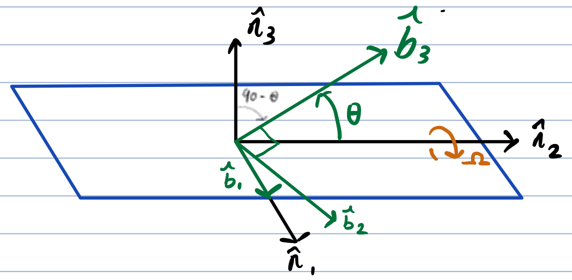


Fig. 1

A solid cylinder of mass m, radius a, and length l is pivoted about a transverse axis (B-B') through its center of mass as shown in Fig. 1. The axis (A-A') rotates with a constant angular velocity Ω. Assume $l > \sqrt{3}a$.

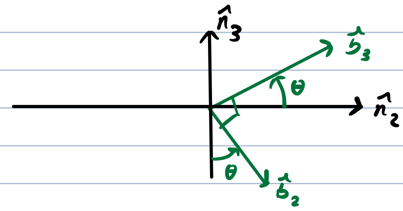
Q5) Find the EOM of the cylinder in terms of θ

- The external frame is rotating abt A-A' at Ω rad/s.
- The cylinder is free to pitch abt B-B' => 1 DOF



n-frame: $\{ O, \hat{n}_1, \hat{n}_2, \hat{n}_3 \}$
B-frame: $\{ O, \hat{b}_1, \hat{b}_2, \hat{b}_3 \}$

- Observe that the B-frame has to be rotated by $(90-\theta)^\circ$ abt $\hat{b}_1 = \hat{n}_1$, for both frames to align.
- So,



$$\hat{b}_1 = \hat{n}_1$$
$$\hat{b}_2 = \sin\theta\, \hat{n}_2 - \cos\theta\, \hat{n}_3$$
$$\hat{b}_3 = \cos\theta\, \hat{n}_2 + \sin\theta\, \hat{n}_3$$

$$\begin{Bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & S\theta & -C\theta \\ 0 & C\theta & S\theta \end{bmatrix} \begin{Bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{Bmatrix}$$

- Observe that the 3x3 matrix is a DCM mapping vectors in n-frame to B-frame.

So,

$${}^{B}\omega = [BN]\, {}^{n}\omega$$

$${}^{B}\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & S\theta & -C\theta \\ 0 & C\theta & S\theta \end{bmatrix} \begin{Bmatrix} \dot{\theta} \\ \Omega \\ 0 \end{Bmatrix}$$

$${}^{B}\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{Bmatrix} \dot{\theta} \\ \Omega \sin\theta \\ \Omega \cos\theta \end{Bmatrix}$$

- In the above method, we assumed no prior knowledge of Euler Angles and have derived the rotation matrix mapping the inertial to body frame, vice versa.

↳ Can we use 1 of 12 Euler Angle sets to get the angular velocity vector in the body frame of the cylinder?

- If we are to use the kinematic Differential Eqn for this eqn, realize that the B- & n-frame are cascaded.
  Ω acts on the n-frame.
  $\dot{\theta}$ is the rate at which the cylinder rotates in B-frame.
  So, mathematically,

$$^{B}\{\omega\} = \underbrace{[B^{-1}]\{\dot{\theta}\}}_{\text{Euler Rate in body mapped to angular vel. in body}} + \underbrace{[BN]\,^{n}\{\omega\}}_{\text{mapping angular vel. of n-frame to body}}$$

If we considered the 3-2-1 Euler Angle set,

$$^{B}\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{bmatrix} -S\theta_2 & 0 & 1 \\ C\theta_2 S\theta_3 & C\theta_3 & 0 \\ C\theta_2 C\theta_3 & -S\theta_3 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_3 \\ \dot{\theta}_2 \\ \dot{\theta}_1 \end{Bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & S\theta & -C\theta \\ 0 & C\theta & S\theta \end{bmatrix} {}^{n}\begin{Bmatrix} 0 \\ \Omega \\ 0 \end{Bmatrix}$$

↳ since the body, the cylinder, is constrained abt $\hat{b}_2$ & $\hat{b}_3$,

$$\begin{matrix} \theta_2 = 0 \\ \dot{\theta}_2 = 0 \end{matrix} \quad \& \quad \begin{matrix} \theta_3 = 0 \\ \dot{\theta}_3 = 0 \end{matrix} \quad \& \quad \dot{\theta}_1 = \dot{\theta}$$

↳ $^{n}\{\omega\}$ is strictly only considering angular vel. abt the outer frame.
  There is only Ω abt $\hat{n}_2$ or $A-A'$.

Then,

$$^{B}\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta} \end{Bmatrix} + \begin{Bmatrix} 0 \\ S\theta\,\Omega \\ C\theta\,\Omega \end{Bmatrix}$$

$$^{B}\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{Bmatrix} \dot{\theta} \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \Omega\, S\theta \\ \Omega\, C\theta \end{Bmatrix}$$

$$^{B}\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{Bmatrix} \dot{\theta} \\ \Omega \sin(\theta) \\ \Omega \cos(\theta) \end{Bmatrix} \quad \#$$

- Having found $^{B}\{\omega\}$, we can then find EOM of the cylinder in terms of θ.

Euler's Rotational EOM,

$$I_1 \dot{\omega}_1 + (I_3 - I_2)\,\omega_2 \omega_3 = L_1 \quad -①$$
$$I_2 \dot{\omega}_2 + (I_1 - I_3)\,\omega_3 \omega_1 = L_2 \quad -②$$
$$I_3 \dot{\omega}_3 + (I_2 - I_1)\,\omega_1 \omega_2 = L_3 \quad -③$$

- Assuming no external Torques ; $L_1 = L_2 = L_3 = 0$
- Cylinder is an axisymmetric body ; $\sim I_1 = I_2 = I_t = \frac{1}{12} m (3a^2 + \ell^2)$

$$\sim I_3 = I_a = \tfrac{1}{2} m a^2$$

- Eqn ③ goes to zero as, $L_3 = 0$ & $(I_2 - I_1) = 0$.
we can either use ② or ③.
Since we want the EOM in terms of θ, we use ①,

$$I_1 \dot{\omega}_1 + (I_3 - I_2)\, \omega_2 \omega_3 = 0$$

$$I_t \ddot{\theta} + (I_a - I_t)(\Omega \sin\theta)(\Omega \cos\theta) = 0$$

$$\boxed{\ddot{\theta} + \frac{I_a - I_t}{I_t} \Omega^2 \sin\theta \cos\theta = 0}$$

$$\boxed{\ddot{\theta} - \frac{(I_t - I_a)}{I_t} \frac{\Omega^2}{2} \sin(2\theta) = 0}$$

Q6) Finding the natural freq., ω_n, of small oscillations abt $\theta_0 = \pi/2$.

$$\ddot{\theta} - \frac{\left(\frac{1}{12} m(3a^2 + l^2) - \frac{1}{2} m a^2\right)}{\frac{1}{12} m (3a^2 + l^2)} \frac{\Omega^2}{2} \sin(2\theta) = 0$$

$$\ddot{\theta} - \frac{\cancel{\frac{1}{12} m}(3a^2 + l^2 - 6a^2)}{\cancel{\frac{1}{12} m}(3a^2 + l^2)} \frac{\Omega^2}{2} \sin(2\theta) = 0$$

$$\ddot{\theta} - \frac{(-3a^2 + l^2)}{3a^2 + l^2} \frac{\Omega^2}{2} \sin(2\theta) = 0$$

$$\ddot{\theta} + \frac{(3a^2 - l^2)}{3a^2 + l^2} \frac{\Omega^2}{2} \sin(2\theta) = 0$$

For small perturbations abt $\theta_0 = \pi/2$, $\theta = \theta_0 + \delta\theta$
$\dot{\theta} = \delta\dot{\theta}$
$\ddot{\theta} = \delta\ddot{\theta}$

Then,

$$\delta\ddot{\theta} + \frac{(3a^2 - l^2)}{3a^2 + l^2} \frac{\Omega^2}{2} \sin\left(2\left(\frac{\pi}{2} + \delta\theta\right)\right) = 0$$

$$\delta\ddot{\theta} + \frac{3a^2 - l^2}{3a^2 + l^2} \frac{\Omega^2}{2} \sin(\pi + 2\delta\theta) = 0$$

$$\delta\ddot{\theta} + \frac{3a^2 - l^2}{3a^2 + l^2} \frac{\Omega^2}{2} \left(\cancel{\sin(\pi)} \cos(2\delta\theta) + \overset{1}{\cancel{\cos(\pi)}} \sin(2\delta\theta)\right) = 0$$

$$\delta\ddot{\theta} + \frac{3a^2 - l^2}{3a^2 + l^2} \frac{\Omega^2}{2} \sin(2\delta\theta) = 0 \quad , \quad \sin(2\delta\theta) \approx 2\delta\theta$$

$$\delta\ddot{\theta} + \frac{3a^2 - l^2}{3a^2 + l^2} \Omega^2 (\delta\theta) = 0$$

↳ This is in the form of ; $\ddot{x} + \omega_n^2 x = 0$

- Note that $l > \sqrt{3}\, a$
$l^2 > 3a^2$
$l^2 - 3a^2 > 0$

So if we take the square root of the coeff. of $\delta\theta$, ω_n wouldn't be a real value.
So,

$$\delta\ddot{\theta} - \frac{l^2-3a^2}{l^2+3a^2}\Omega^2(\delta\theta) = 0$$

Then,

$$\omega_n = \sqrt{\frac{l^2-3a^2}{l^2+3a^2}\Omega^2}$$

$$\boxed{\omega_n = |\Omega|\sqrt{\frac{l^2-3a^2}{l^2+3a^2}}}$$

#

Q7) What is $\dot{\theta}$ when $\theta = \pi/2$, if the cylinder is released from $\theta=0$ with a small $+\dot{\theta}_0$?

$$\ddot{\theta} - \frac{l^2-3a^2}{l^2+3a^2}\Omega^2 \sin(2\theta) = 0$$

$$\ddot{\theta} = \frac{l^2-3a^2}{l^2+3a^2}\Omega^2 \sin(2\theta)$$

$$\frac{d(\dot{\theta})}{dt} = \frac{l^2-3a^2}{l^2+3a^2}\Omega^2 \sin 2\theta$$

$$d\dot{\theta} = \frac{l^2-3a^2}{l^2+3a^2}\Omega^2 \sin 2\theta \, dt \quad , \quad \dot{\theta} = \frac{d\theta}{dt}$$

$$\dot{\theta}\, d\dot{\theta} = \frac{l^2-3a^2}{l^2+3a^2}\Omega^2 \sin 2\theta \frac{d\theta}{\cancel{dt}} \cancel{dt}$$

$$\dot{\theta}\, d\dot{\theta} = \frac{l^2-3a^2}{l^2+3a^2}\Omega^2 \sin 2\theta \, d\theta$$

$$\int_{\dot{\theta}_0}^{\dot{\theta}} \dot{\theta}\, d\dot{\theta} = \int_{\theta_0}^{\theta} \frac{l^2-3a^2}{l^2+3a^2}\Omega^2 \sin 2\theta \, d\theta$$

$$\left[\frac{\dot{\theta}^2}{2}\right]_{\dot{\theta}_0}^{\dot{\theta}} = \frac{l^2-3a^2}{l^2+3a^2}\Omega^2\left[\frac{-\cos 2\theta}{2}\right]_{\theta_0}^{\theta}$$

$$\frac{\dot{\theta}^2 - \dot{\theta}_0^2}{2} = \frac{l^2-3a^2}{l^2+3a^2}\Omega^2\left[\frac{-\cos 2\theta}{2} + \frac{\cos 2\theta_0}{2}\right]$$

$$\dot{\theta}^2 - 0 = \frac{l^2-3a^2}{l^2+3a^2}\Omega^2\left(-\overset{-1}{\cancel{\cos(\pi)}} + \overset{1}{\cancel{\cos(0)}}\right)$$

$$\boxed{\dot{\theta} = \Omega\sqrt{\frac{l^2-3a^2}{l^2+3a^2}}}$$