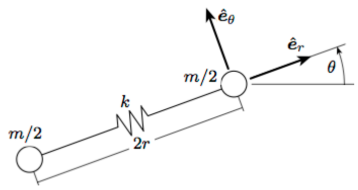

Fig. 1: 2 Balls

Two particles with mass $m/2$ are attached by a linear spring with a spring constant k as shown in Fig. 1. Consider arbitrary the initial position and velocity of each mass on the plane. For simplicity, however, assume that the initial separation $2r_0$ is the unstretched length of the spring, and that the mass center has zero inertial velocity initially. Determine the differential equations of motion whose solution would give $r(t)$ and $\theta(t)$ as functions of time and initial conditions; it is not necessary to solve these differential equations.

- use Polar coord. (alonge $\hat{e}_r$ & $\hat{e}_\theta$)
- Assume there exists some motion along $\hat{e}_r$ & $\hat{e}_\theta$
- Solving using Newtonian mechanics;

  - Force on masses due to spring,

$$\left.\begin{aligned} F_{spring} &= -k\,\Delta r \\ &= -k\,(2r - 2r_0) \\ &= -2k\,(r - r_0) \end{aligned}\right\}$$

This force term will be along $\hat{e}_r$

- Since the mass moves along $\hat{e}_r$ & $\vec{e}_\theta$ only, there exises a forcing term radially.

  So, for planar motion, the position vector can be written as,

$$\vec{r} = r\,\vec{e}_r$$

Then, the velocity vector,

$$\dot{\vec{r}} = \frac{d}{dt}(r\,\vec{e}_r)$$
$$= \frac{d}{dt}(r)\,\vec{e}_r + r\,\frac{d}{dt}(\vec{e}_r)$$
$$= \dot{r}\,\vec{e}_r + r\,\dot{\theta}\,\vec{e}_\theta$$

Then, the accel. vector,

$$\ddot{\vec{r}} = \frac{d}{dt}(\dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta)$$
$$= \frac{d}{dt}(\dot{r})\,\vec{e}_r + \dot{r}\,\frac{d}{dt}(\vec{e}_r)$$
$$+ \frac{d}{dt}(r\dot{\theta})\,\vec{e}_\theta + r\dot{\theta}\,\frac{d}{dt}(\vec{e}_\theta)$$
$$= \ddot{r}\,\vec{e}_r + \dot{r}\,\dot{\theta}\,\vec{e}_\theta + (\dot{r}\dot{\theta} + r\ddot{\theta})\,\vec{e}_\theta - r\dot{\theta}^2\,\vec{e}_r$$
$$= (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta$$

- Since the frame is body-attached and rotating (changing w.r.t. time), the unit vectors must be differentiated.
- Recall the kinematic transport theorem,

$$\frac{{}^N d}{dt}(\vec{r}) = \frac{{}^B d}{dt}(\vec{r}) + \omega \times \vec{r}$$

- so; $\frac{d}{dt}(\vec{e}_r) = 0 + \omega \times \vec{e}_r, \quad \omega = \dot{\theta}\,\vec{e}_z$

$$\boxed{\frac{d}{dt}(\vec{e}_r) = \dot{\theta}\,\hat{e}_\theta}$$

- and, $\frac{d}{dt}(\vec{e}_\theta) = 0 + \omega \times \vec{e}_\theta$

$$\boxed{\frac{d}{dt}(\vec{e}_\theta) = -\dot{\theta}\,\vec{e}_r}$$

- So from the accel. vector, we have the acceleration terms along $\hat{e}_r$ & $\hat{e}_\theta$

- Radially (along $\hat{e}_r$),

$$F_{spring} = ma \quad , \quad 1 \text{ ball is } m/2$$

$$-2k(r-r_0) = \frac{m}{2}(\ddot{r} - r\dot{\theta}^2)$$

$$-4k(r-r_0) = m(\ddot{r} - r\dot{\theta}^2)$$

$$m\ddot{r} - mr\dot{\theta}^2 + 4k(r-r_0) = 0$$

- Tangentially (along $\hat{e}_\theta$),

$$F_{tangential} = ma$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

Using the setup of the prior problem, determine an expression that relates the radial velocity $\dot{r}$ and the angular velocity $\dot{\theta}$ as functions of r, θ and initial conditions. (Hint: use conservation of energy)

- In a conserved system,

$$E = T + V, \qquad T - \text{kinetic energy}, \quad V - \text{potential energy}$$

- In the context of the question,
  - There will be K.E. due to motion of the system
  - there will be P.E. due to spring

- Kinetic energy,

$$T = \frac{1}{2} m (\vec{v} \cdot \vec{v})$$
$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) \Rightarrow \text{K.E. of both masses}$$

- Potential Energy,

$$V = \frac{1}{2} k (\Delta r)^2$$
$$= \frac{1}{2} k (2r - 2r_0)^2$$
$$= \frac{1}{\cancel{2}} k (2^{\cancel{2}})(r - r_0)^2$$
$$= 2k (r - r_0)^2$$

- Then,

$$E = T + V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + 2k (r - r_0)^2$$

- So at time $t = t_0$, and some $t > t_0$, $E = \text{const.}$

$$E_{final} = E_{initial}$$

$$\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + 2k (r - r_0)^2 = \frac{1}{2} m (r_0^2 + r_0^2 \dot{\theta}_0^2) + 2k \cancelto{0}{(r_0 - r_0)^2}$$

Thus,

$$\frac{m}{2} (r_0^2 + r_0^2 \dot{\theta}_0^2) = 2k (r - r_0)^2 + \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2)$$

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