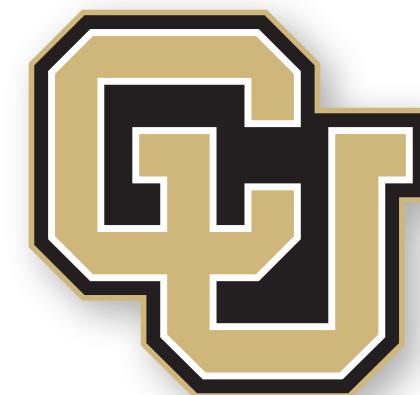


Momentum Exchange Devices

ASEN 5010

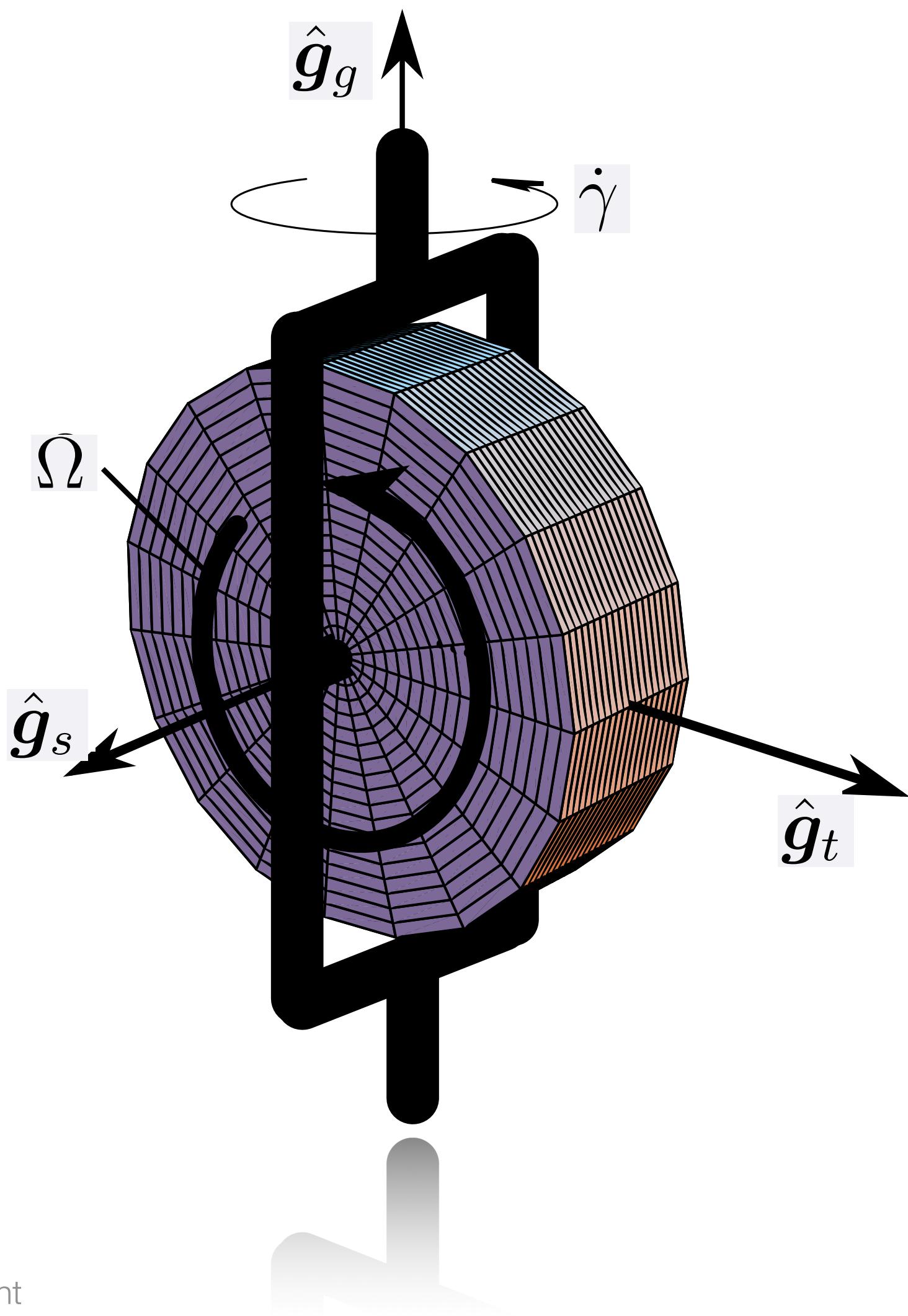
Prof. H. Schaub
hanspeter.schaub@colorado.edu



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Boulder

Outline

- Momentum Control Devices
- Equations of motion of VSCMG
 - single VSCMG
 - motor torque calculation
 - cluster of VSCMG
- Momentum Device Control
 - Overview of RW control solution



Momentum Control Devices

Spinning hardware “thingies” to rotate the spacecraft...

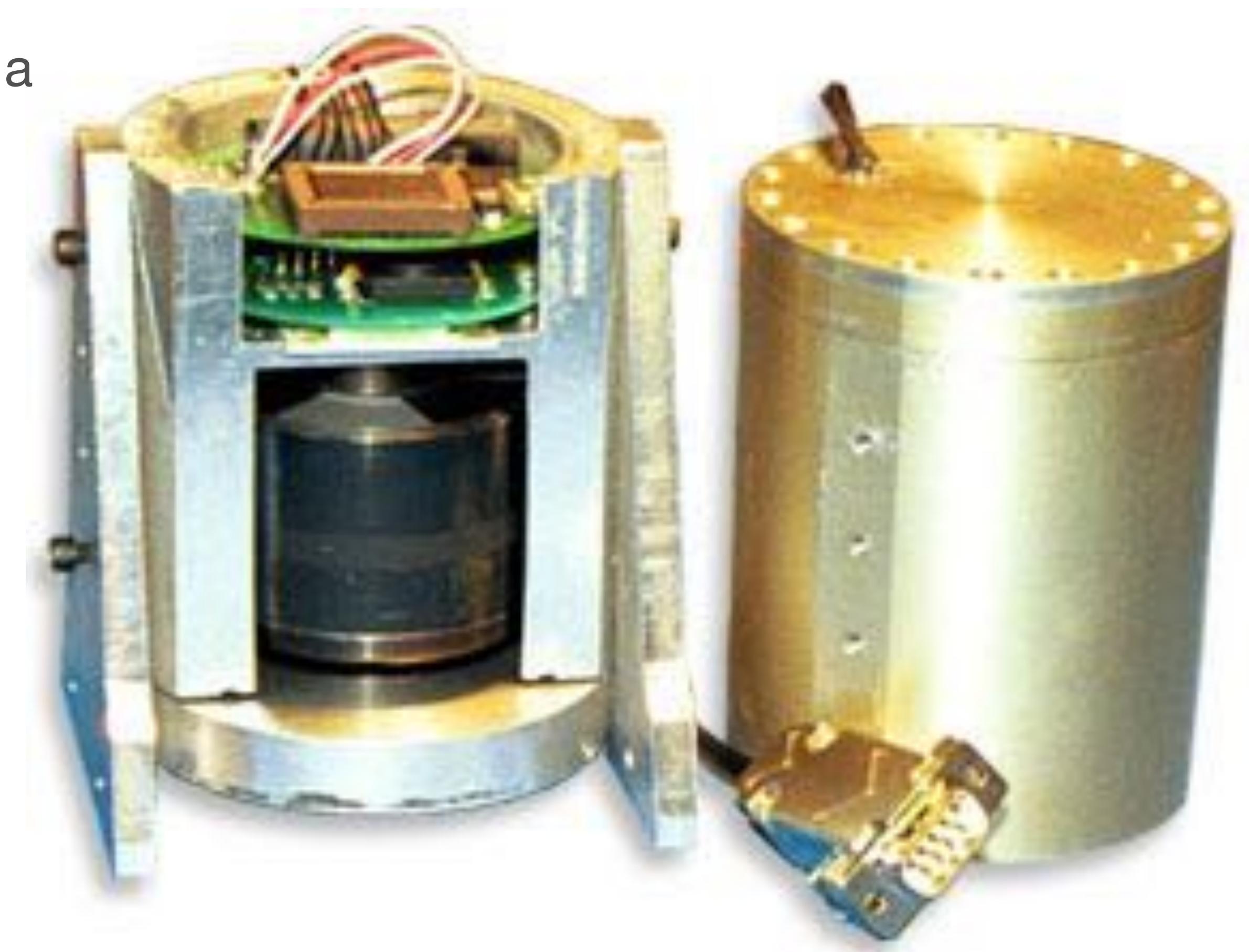


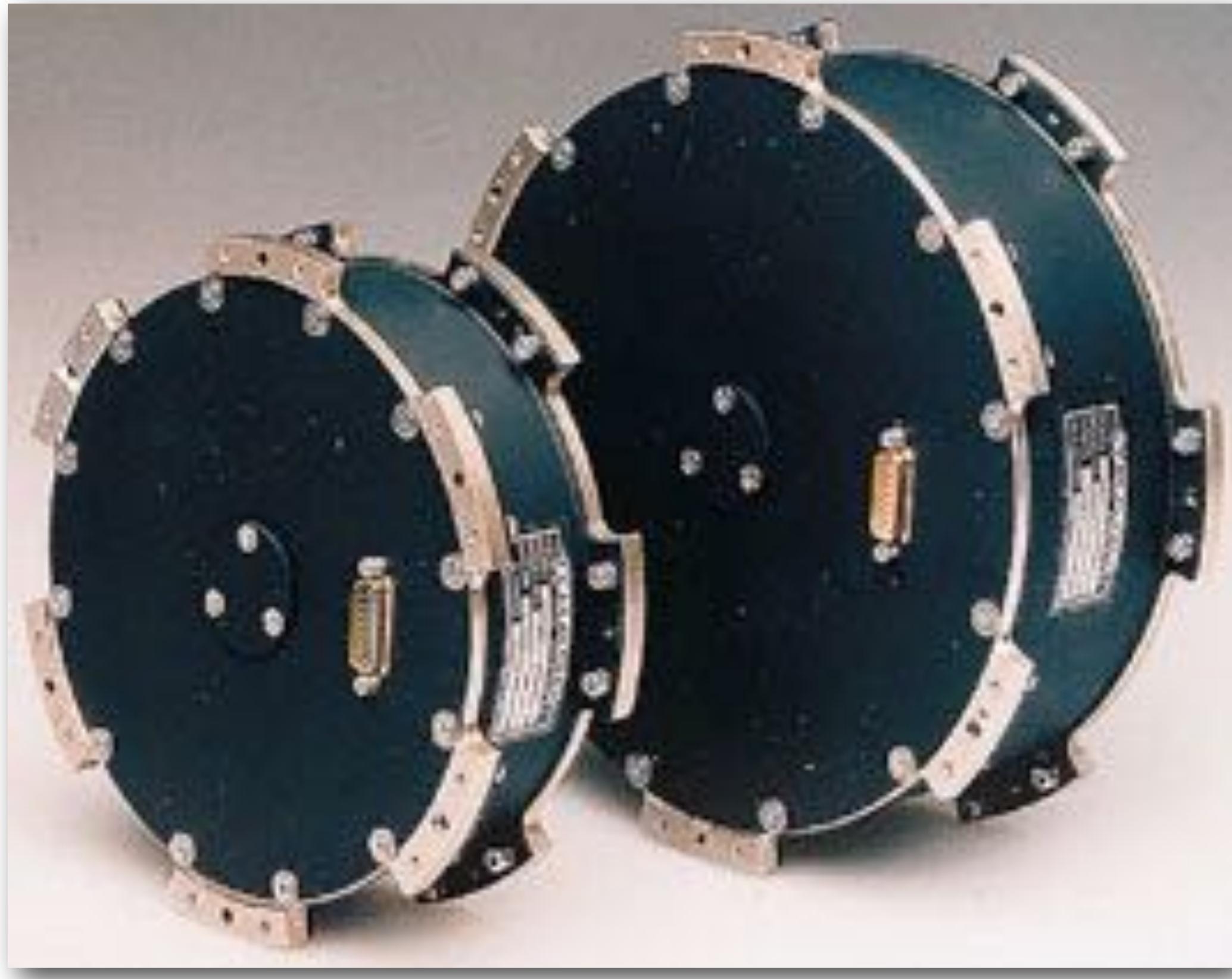
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Reaction Wheels (RW)

- By increasing or decreasing the spin of a disk, a torque is applied onto the spacecraft.
- The torque is parallel to the disk spin axis.
- Simple mechanical device.
- Multiple disks can generate arbitrary torque.
- Wheels can saturate (reach a maximum spin speed).





ITHACO's low-cost but highly reliable reaction wheel designs keep spacecraft correctly oriented as they spin through space. (company description of device)
<http://www.sti.nasa.gov/tto/spinoff1997/t2.html>





Inside CTA Space System's High Torque Reaction/Momentum Wheel is an innovative flywheel/bearing arrangement that allows the entire rotating system to be balanced after it is assembled. (company description)
<http://www.sti.nasa.gov/tto/spinoff1997/t3.html>





Honeywell's reaction wheel assemblies (RWA) and momentum wheel assemblies (MWA) are reliable, lightweight solutions to a variety of momentum control needs, providing stability and attitude-control for small to very large, heavy spacecraft. Earth-pointing satellites and multiple-satellite communication networks are examples of applications that require the fine attitude control that Honeywell RWAs provide. RWAs and MWAs from Honeywell have accumulated more than 9 million hours, or more than a thousand years, in space and have never caused a mission to end prematurely.

Control Moment Gyroscope (CMG)

- Another popular attitude control device.
- A disk is spinning at a constant rate.
- By rotating this disk (called gimbaling), a torque is applied through the gyroscopic effect.
- For a small torque to gimbal the disk, a large torque is produced onto the spacecraft.



Control Moment Gyroscope (CMG)

- Mechanically more complex device than RWs
- Control laws are much more complicated.
- Very large torques can be produced (good for rapid reorientation or large spacecraft such as space station)
- Singular configurations exist where the required torque cannot be produced.





A CMG contains two torque motors.
One to keep the disk spinning at a
constant rate, the other to gimbal the
spinning disk.



A typical CMG setup has 4 devices
aligned in a pyramid configuration.



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Equations of Motion

Let's learn to be one with the truth of gyroscopics...



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Spacecraft with 1 VSCMG

- A Variable-Speed CMG is a classical CMG device where the disk speed is left to be variable.
- Think of a VSCMG device as a hybrid CMG/RW.
- Convenient when developing the equations of motion, since we get both the CMG and RW equations of motion by doing the work only once!!
- Researchers have started to look into actually building and flying a VSCMG devices.
 - Avoids classical CMG singularities
 - Highly redundant system (more robust to component failure)
 - Can be used as a combined power storage/attitude control device.



Battle Plan...

- To derive the equations of motion of a spacecraft with a single VSCMG, we recall Euler's equation

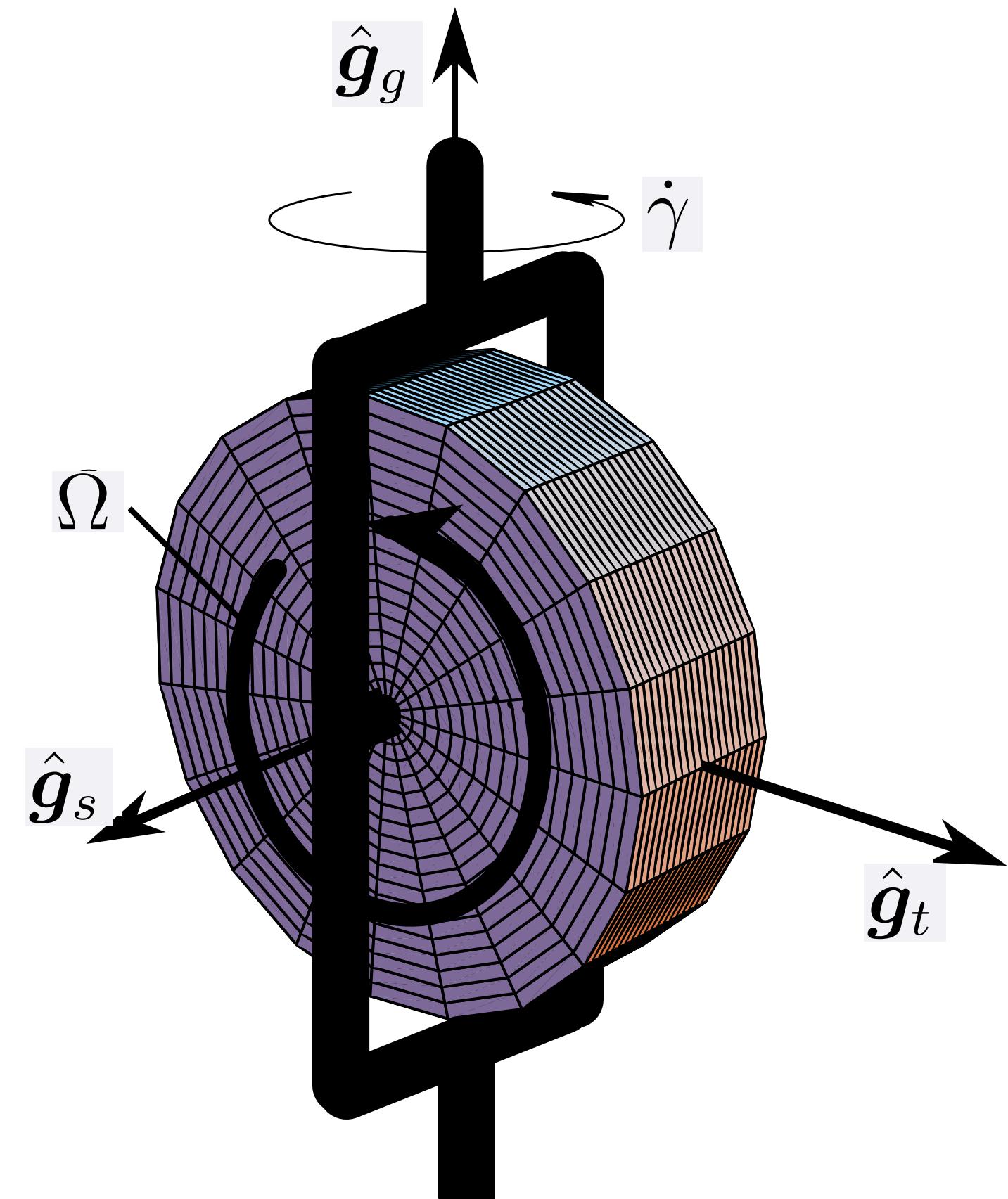
$$\dot{\mathbf{H}} = \mathbf{L}$$

- We will need to find the total angular momentum vector \mathbf{H} for the combined spacecraft/VSCMG system. Once we have this expression, we can then differentiate it to get the desired equations of motion.
- To manage all this algebra, we will break up the whole system into the spacecraft part, the CMG momentum and the RW momentum.



VSCMG Frames

- The VSCMG spin axis is \hat{g}_s
- The gimbal axis is \hat{g}_g
- The disk spin rate is $\Omega(t)$
- The gimbal rate is $\dot{\gamma}(t)$
- The gimbal coordinate frame G is
$$\mathcal{G} : \{\hat{g}_s, \hat{g}_t, \hat{g}_g\}$$



VSCMG Frames

- Note that the gimbal axis is fixed with respect to the spacecraft body frame B .
- The gimbal frame G angular velocity is

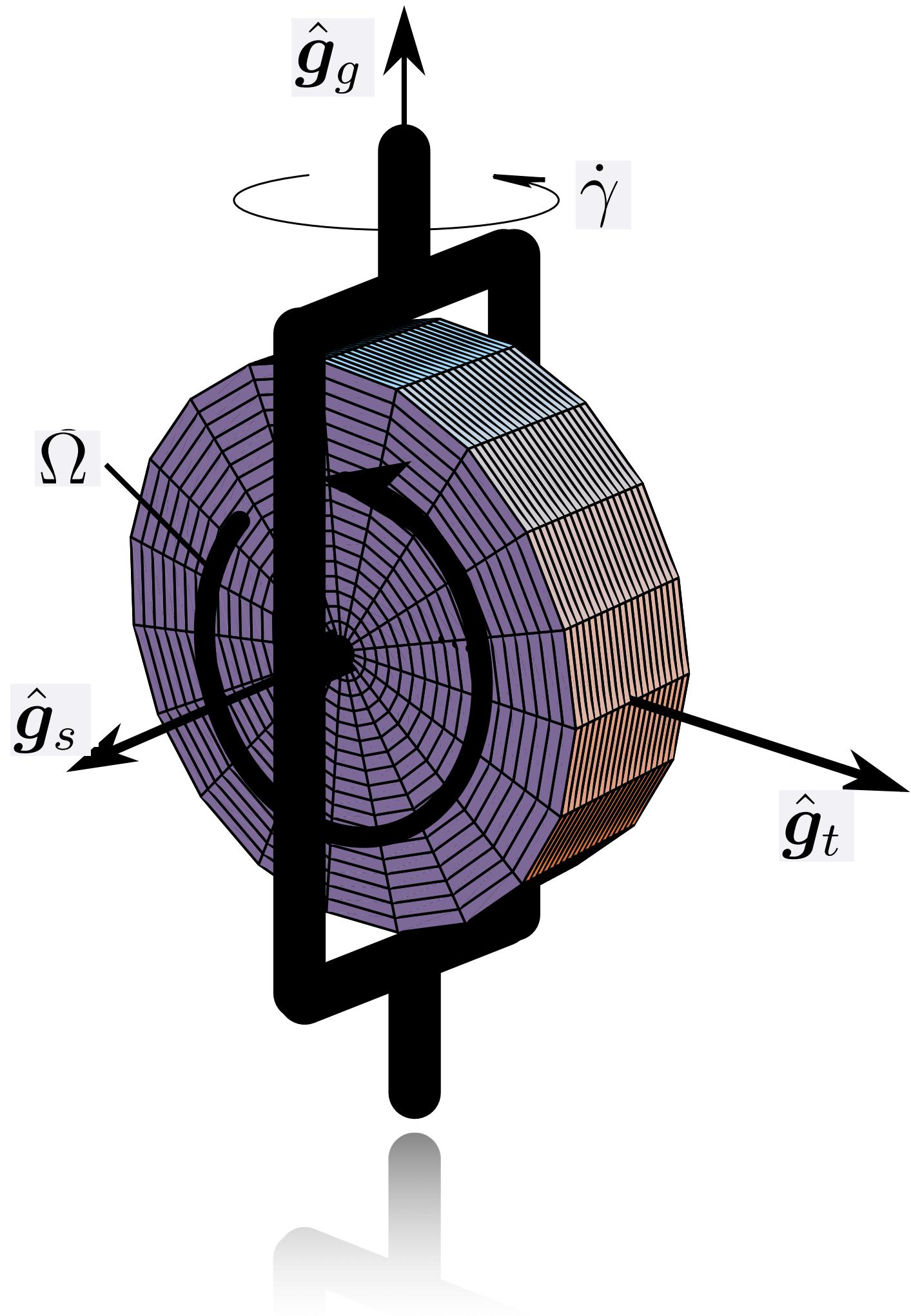
$$\omega_{G/B} = \dot{\gamma} \hat{g}_g$$

- Let W be a frame that tracks the motion of the reaction wheel.

$$W : \{\hat{g}_s, \hat{w}_t, \hat{w}_g\}$$

- It's angular velocity is

$$\omega_{W/G} = \Omega \hat{g}_s$$



VSCMG Inertias

- Let the gimbal frame inertia be

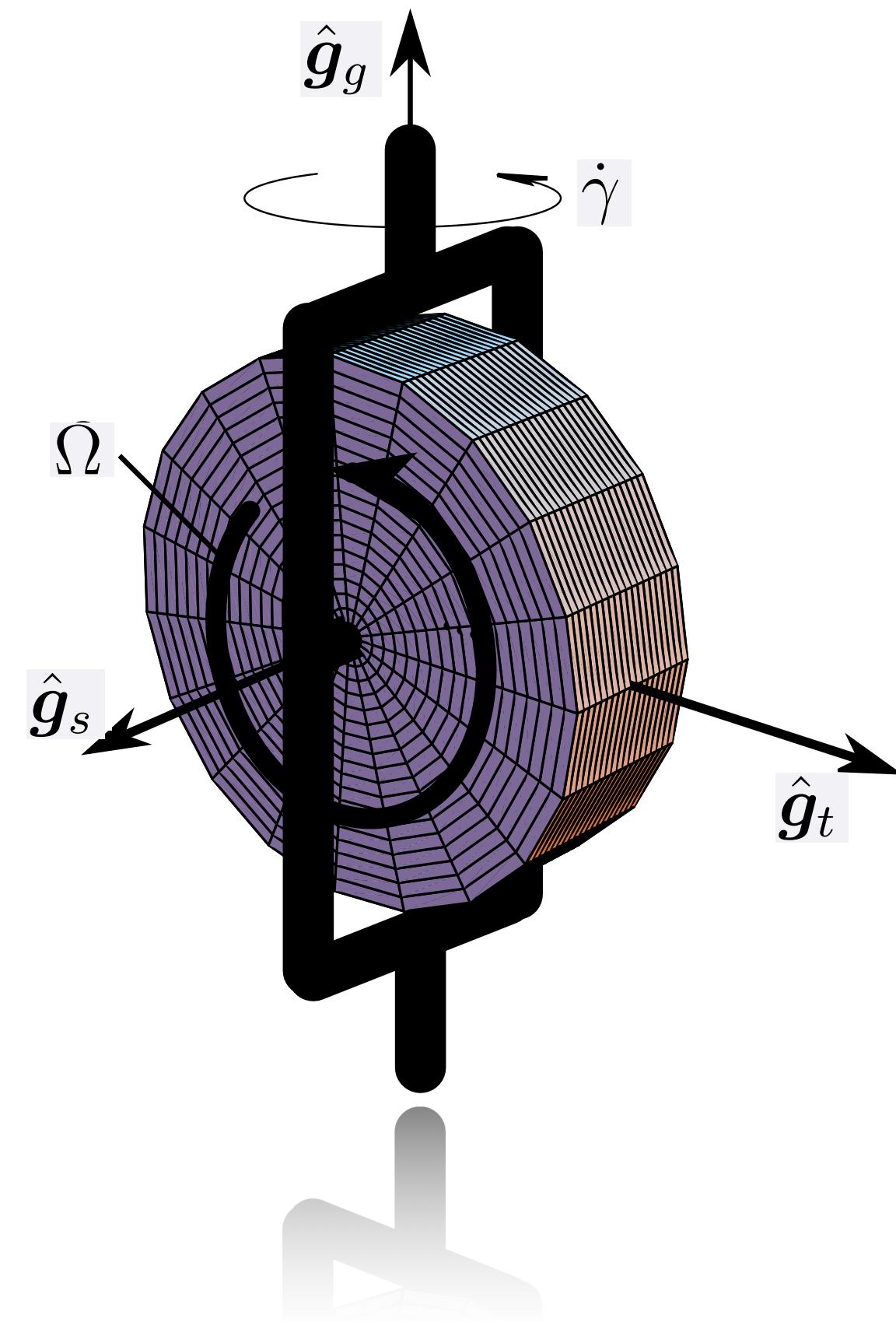
$$[I_G] = {}^G[I_G] = \begin{bmatrix} I_{G_s} & 0 & 0 \\ 0 & I_{G_t} & 0 \\ 0 & 0 & I_{G_g} \end{bmatrix}$$

- The wheel (disk) inertia is

$$[I_W] = {}^W[I_W] = \begin{bmatrix} I_{W_s} & 0 & 0 \\ 0 & I_{W_t} & 0 \\ 0 & 0 & I_{W_t} \end{bmatrix}$$

- Due to symmetry of the disk, we find that

$${}^W[I_W] = {}^G[I_W]$$



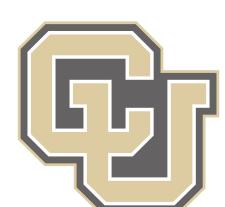
- Assuming the gimbal frame unit vectors are expressed in body frame vector components, then the rotation matrix $[BG]$ can be expressed through

$$[BG] = [\hat{\mathbf{g}}_s \ \hat{\mathbf{g}}_t \ \hat{\mathbf{g}}_g]$$

- The gimbal frame and disk inertias (which were given in gimbal frame components), can be written in body frame components using

$${}^B[I_G] = [BG]^G[I_G][BG]^T = I_{G_s}\hat{\mathbf{g}}_s\hat{\mathbf{g}}_s^T + I_{G_t}\hat{\mathbf{g}}_t\hat{\mathbf{g}}_t^T + I_{G_g}\hat{\mathbf{g}}_g\hat{\mathbf{g}}_g^T$$

$${}^B[I_W] = [BG]^G[I_W][BG]^T = I_{W_s}\hat{\mathbf{g}}_s\hat{\mathbf{g}}_s^T + I_{W_t}\hat{\mathbf{g}}_t\hat{\mathbf{g}}_t^T + I_{W_g}\hat{\mathbf{g}}_g\hat{\mathbf{g}}_g^T$$



Angular Momentum...

- We are now ready to express the total angular momentum of the system using

$$\mathbf{H} = \mathbf{H}_B + \mathbf{H}_G + \mathbf{H}_W$$

- \mathbf{H}_B is the angular momentum of the spacecraft itself, \mathbf{H}_G is the angular momentum of the gimbal frame, while \mathbf{H}_W is the angular momentum of the spinning disk.
- The spacecraft angular momentum is simply that of a rigid body:

$$\mathbf{H}_B = [I_s]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$$



- The inertial angular momentum of the rigid gimbal frame is

$$\mathbf{H}_G = [I_G] \boldsymbol{\omega}_{\mathcal{G}/\mathcal{N}}$$

- where $\boldsymbol{\omega}_{\mathcal{G}/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{G}/\mathcal{B}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$. This can now be rewritten as

$$\mathbf{H}_G = (I_{G_s} \hat{\mathbf{g}}_s \hat{\mathbf{g}}_s^T + I_{G_t} \hat{\mathbf{g}}_t \hat{\mathbf{g}}_t^T + I_{G_g} \hat{\mathbf{g}}_g \hat{\mathbf{g}}_g^T) \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + I_{G_g} \dot{\gamma} \hat{\mathbf{g}}_g$$

- Let us introduce the angular velocity components taken along the gimbal frame axis directions:

$$\omega_s = \hat{\mathbf{g}}_s^T \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \quad \omega_t = \hat{\mathbf{g}}_t^T \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \quad \omega_g = \hat{\mathbf{g}}_g^T \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$$

$${}^{\mathcal{G}}\boldsymbol{\omega} = \omega_s \hat{\mathbf{g}}_s + \omega_t \hat{\mathbf{g}}_t + \omega_g \hat{\mathbf{g}}_g$$

- This allows us to write the gimbal frame angular momentum expression as

$$\mathbf{H}_G = I_{G_s} \omega_s \hat{\mathbf{g}}_s + I_{G_t} \omega_t \hat{\mathbf{g}}_t + I_{G_g} (\omega_g + \dot{\gamma}) \hat{\mathbf{g}}_g$$

- The inertial angular momentum of the disk is

$$\boldsymbol{H}_W = [I_W] \boldsymbol{\omega}_{\mathcal{W}/\mathcal{N}}$$

- where $\boldsymbol{\omega}_{\mathcal{W}/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{W}/\mathcal{G}} + \boldsymbol{\omega}_{\mathcal{G}/\mathcal{B}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$

- The momentum expression can be expanded using

$$\boldsymbol{H}_W = [I_W] \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_W] \boldsymbol{\omega}_{\mathcal{G}/\mathcal{B}} + [I_W] \boldsymbol{\omega}_{\mathcal{W}/\mathcal{G}}$$

It is implied that all vectors are added with components in the same frame.

- The first term can be written as

$$\begin{aligned} [I_W] \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} &= (I_{W_s} \hat{\mathbf{g}}_s \hat{\mathbf{g}}_s^T + I_{W_t} \hat{\mathbf{g}}_t \hat{\mathbf{g}}_t^T + I_{W_g} \hat{\mathbf{g}}_g \hat{\mathbf{g}}_g^T) \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ &= I_{W_s} \omega_s \hat{\mathbf{g}}_s + I_{W_t} \omega_t \hat{\mathbf{g}}_t + I_{W_g} \omega_g \hat{\mathbf{g}}_g \end{aligned}$$



- The second two terms can be written as

$$[I_W]\boldsymbol{\omega}_{\mathcal{G}/\mathcal{B}} = \begin{bmatrix} I_{W_s} & 0 & 0 \\ 0 & I_{W_t} & 0 \\ 0 & 0 & I_{W_t} \end{bmatrix}^{\mathcal{G}} \begin{pmatrix} 0 \\ 0 \\ \dot{\gamma} \end{pmatrix} = I_{W_t} \dot{\gamma} \hat{\mathbf{g}}_g$$

$$[I_W]\boldsymbol{\omega}_{\mathcal{W}/\mathcal{G}} = \begin{bmatrix} I_{W_s} & 0 & 0 \\ 0 & I_{W_t} & 0 \\ 0 & 0 & I_{W_t} \end{bmatrix}^{\mathcal{W}} \begin{pmatrix} \Omega \\ 0 \\ 0 \end{pmatrix} = I_{W_s} \Omega \hat{\mathbf{g}}_s$$

- Combining all these results, the spinning wheel inertial angular momentum is written as

$$\mathbf{H}_W = I_{W_s} (\omega_s + \Omega) \hat{\mathbf{g}}_s + I_{W_t} \omega_t \hat{\mathbf{g}}_t + I_{W_t} (\omega_g + \dot{\gamma}) \hat{\mathbf{g}}_g$$

Some final preparation...

- Before we begin to differentiate the system angular momentum vectors, we need to establish some useful relationships.
- The gimbal frame direction vectors can be written in terms of their initial orientations as

$$\hat{\mathbf{g}}_s(t) = \cos(\gamma(t) - \gamma_0) \hat{\mathbf{g}}_s(t_0) + \sin(\gamma(t) - \gamma_0) \hat{\mathbf{g}}_t(t_0)$$

$$\hat{\mathbf{g}}_t(t) = -\sin(\gamma(t) - \gamma_0) \hat{\mathbf{g}}_s(t_0) + \cos(\gamma(t) - \gamma_0) \hat{\mathbf{g}}_t(t_0)$$

$$\hat{\mathbf{g}}_g(t) = \hat{\mathbf{g}}_g(t_0)$$



- Note that the B frame derivatives of the gimbal frame unit vectors are

$$\frac{\mathcal{B}_d}{dt}(\hat{\mathbf{g}}_s) = \dot{\gamma}\hat{\mathbf{g}}_t \quad \frac{\mathcal{B}_d}{dt}(\hat{\mathbf{g}}_t) = -\dot{\gamma}\hat{\mathbf{g}}_s \quad \frac{\mathcal{B}_d}{dt}(\hat{\mathbf{g}}_g) = 0$$

- The inertial derivatives of these vectors are

$$\begin{aligned}\dot{\hat{\mathbf{g}}}_s &= \frac{\mathcal{B}_d}{dt}(\hat{\mathbf{g}}_s) + \boldsymbol{\omega} \times \hat{\mathbf{g}}_s = (\dot{\gamma} + \omega_g)\hat{\mathbf{g}}_t - \omega_t\hat{\mathbf{g}}_g \\ \dot{\hat{\mathbf{g}}}_t &= \frac{\mathcal{B}_d}{dt}(\hat{\mathbf{g}}_t) + \boldsymbol{\omega} \times \hat{\mathbf{g}}_t = -(\dot{\gamma} + \omega_g)\hat{\mathbf{g}}_s + \omega_s\hat{\mathbf{g}}_g \\ \dot{\hat{\mathbf{g}}}_g &= \frac{\mathcal{B}_d}{dt}(\hat{\mathbf{g}}_g) + \boldsymbol{\omega} \times \hat{\mathbf{g}}_g = \omega_t\hat{\mathbf{g}}_s - \omega_s\hat{\mathbf{g}}_t\end{aligned}$$

use ${}^G\boldsymbol{\omega} = \omega_s\hat{\mathbf{g}}_s + \omega_t\hat{\mathbf{g}}_t + \omega_g\hat{\mathbf{g}}_g$ to derive this result.



- Finally, the following expressions are derived:

$$\begin{aligned}\dot{\omega}_s &= \dot{\hat{\mathbf{g}}}_s^T \boldsymbol{\omega} + \hat{\mathbf{g}}_s^T \dot{\boldsymbol{\omega}} = \dot{\gamma} \boldsymbol{\omega}_t + \hat{\mathbf{g}}_s^T \dot{\boldsymbol{\omega}} \\ \dot{\omega}_t &= \dot{\hat{\mathbf{g}}}_t^T \boldsymbol{\omega} + \hat{\mathbf{g}}_t^T \dot{\boldsymbol{\omega}} = -\dot{\gamma} \boldsymbol{\omega}_s + \hat{\mathbf{g}}_t^T \dot{\boldsymbol{\omega}} \\ \dot{\omega}_g &= \dot{\hat{\mathbf{g}}}_g^T \boldsymbol{\omega} + \hat{\mathbf{g}}_g^T \dot{\boldsymbol{\omega}} = \hat{\mathbf{g}}_g^T \dot{\boldsymbol{\omega}}\end{aligned}$$

- The following combined gimbal and spinning disk inertia matrix will be useful to simplify some results:

$$[J] = [I_G] + [I_W] = \begin{bmatrix} J_s & 0 & 0 \\ 0 & J_t & 0 \\ 0 & 0 & J_g \end{bmatrix}^G$$



And now, the fun...

- At this point we are ready to compute the terms in Euler's equation $\dot{\mathbf{H}} = \mathbf{L}$. We have all the required expressions and need to simply carry out the required algebra.
- Taking the inertial derivative of the spinning wheel angular momentum expression \mathbf{H}_W , we find

$$\begin{aligned}\dot{\mathbf{H}}_W &= \hat{\mathbf{g}}_s \left[I_{W_s} \left(\dot{\Omega} + \hat{\mathbf{g}}_s^T \dot{\boldsymbol{\omega}} + \dot{\gamma} \omega_t \right) \right] + \hat{\mathbf{g}}_t \left[I_{W_s} (\dot{\gamma}(\omega_s + \Omega) + \Omega \omega_g) \right. \\ &\quad \left. + I_{W_t} \hat{\mathbf{g}}_t^T \dot{\boldsymbol{\omega}} + (I_{W_s} - I_{W_t}) \omega_s \omega_g - 2I_{W_t} \omega_s \dot{\gamma} \right] \\ &\quad + \hat{\mathbf{g}}_g \left[I_{W_t} \left(\hat{\mathbf{g}}_g^T \dot{\boldsymbol{\omega}} + \ddot{\gamma} \right) + (I_{W_t} - I_{W_s}) \omega_s \omega_t - I_{W_s} \Omega \omega_t \right]\end{aligned}$$



- Taking the derivative of the gimbal frame angular momentum expression \mathbf{H}_G , we find

$$\begin{aligned}\dot{\mathbf{H}}_G &= \hat{\mathbf{g}}_s \left((I_{G_s} - I_{G_t} + I_{G_g}) \dot{\gamma} \omega_t + I_{G_s} \hat{\mathbf{g}}_s^T \dot{\boldsymbol{\omega}} + (I_{G_g} - I_{G_t}) \omega_t \omega_g \right) \\ &\quad + \hat{\mathbf{g}}_t \left((I_{G_s} - I_{G_t} - I_{G_g}) \dot{\gamma} \omega_s + I_{G_t} \hat{\mathbf{g}}_t^T \dot{\boldsymbol{\omega}} + (I_{G_s} - I_{G_g}) \omega_s \omega_g \right) \\ &\quad + \hat{\mathbf{g}}_g \left(I_{G_g} (\hat{\mathbf{g}}_g^T \dot{\boldsymbol{\omega}} + \ddot{\gamma}) + (I_{G_t} - I_{G_s}) \omega_s \omega_t \right)\end{aligned}$$

- Finally, the spacecraft angular momentum inertial derivative is

$$\dot{\mathbf{H}}_B = [I_s] \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times [I_s] \boldsymbol{\omega}$$



- Let us define the time-varying total spacecraft inertia matrix $[I]$:

$$[I] = [I_s] + [J]$$

- Adding up all the terms, and substituting them into Euler's equation $\dot{H} = \mathbf{L}$, we finally arrive at the desired equations of motion of a spacecraft with a single VSCMG.

$$\begin{aligned}[I]\dot{\omega} = & -\omega \times [I]\omega - \hat{g}_s \left(J_s \dot{\gamma} \omega_t + I_{W_s} \dot{\Omega} - (J_t - J_g) \omega_t \dot{\gamma} \right) \\ & - \hat{g}_t ((J_s \omega_s + I_{W_s} \Omega) \dot{\gamma} - (J_t + J_g) \omega_s \dot{\gamma} + I_{W_s} \Omega \omega_g) \\ & - \hat{g}_g (J_g \ddot{\gamma} - I_{W_s} \Omega \omega_t) + \mathbf{L}\end{aligned}$$

These equations of motion are valid for both a RW or CMG device!



Comments...

- By changing the wheel speed or by gimbaling the CMG devices, a torque is applied to the spacecraft and the corresponding attitude is changed.
- RW devices are simpler, but have limits on how large the spin speed Ω can grow.
- Adding the gimbaling mode clearly makes the mathematics much more fun and interesting :-)
- To generally control a spacecraft attitude, three or more of these devices would have to be attached to the spacecraft.



RW Motor Torque

- The equations of motion of only the spinning disk could be found by solving Euler's equations for this disk

$$\dot{\mathbf{H}}_W = \mathbf{L}_w$$

- Note that this is the inertial derivative of the inertial disk angular momentum. We have already found this to be

$$\begin{aligned}\dot{\mathbf{H}}_W = & \hat{\mathbf{g}}_s \left[I_{W_s} \left(\dot{\Omega} + \hat{\mathbf{g}}_s^T \dot{\boldsymbol{\omega}} + \dot{\gamma} \omega_t \right) \right] + \hat{\mathbf{g}}_t [I_{W_s} (\dot{\gamma}(\omega_s + \Omega) + \Omega \omega_g) \\ & + I_{W_t} \hat{\mathbf{g}}_t^T \dot{\boldsymbol{\omega}} + (I_{W_s} - I_{W_t}) \omega_s \omega_g - 2I_{W_t} \omega_s \dot{\gamma}] \\ & + \hat{\mathbf{g}}_g [I_{W_t} (\hat{\mathbf{g}}_g^T \dot{\boldsymbol{\omega}} + \ddot{\gamma}) + (I_{W_t} - I_{W_s}) \omega_s \omega_t - I_{W_s} \Omega \omega_t]\end{aligned}$$



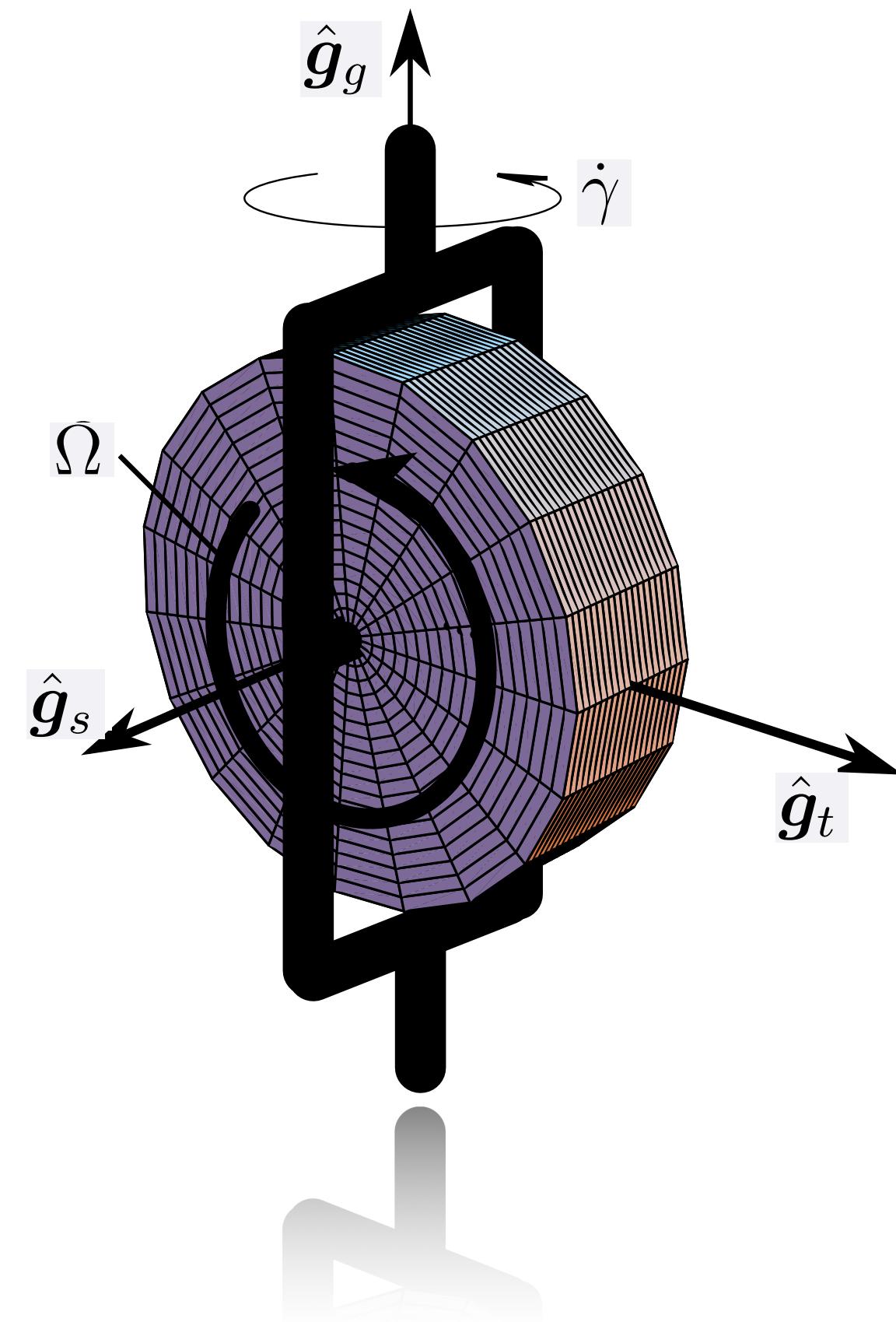
- The only external torque being applied to the spinning disk is through the RW motor.

$$\dot{\mathbf{H}}_W = \mathbf{L}_W = u_s \hat{\mathbf{g}}_s + \tau_{w_t} \hat{\mathbf{g}}_t + \tau_{w_g} \hat{\mathbf{g}}_g$$

- Thus, equating the $\hat{\mathbf{g}}_s$ directions yields:

$$u_s = I_{W_s} \left(\dot{\Omega} + \hat{\mathbf{g}}_s^T \dot{\boldsymbol{\omega}} + \dot{\gamma} \omega_t \right)$$

Given the current disk angular acceleration, spacecraft angular acceleration, or the current gimbal rate, this formula shows how hard the RW motor has to work.



CMG Motor Torque

- To compute the motor torque of the CMG gimbal mode, we need to look at both the disk and the gimbal frame as one unit.

$$\dot{\mathbf{H}}_G + \dot{\mathbf{H}}_W = \mathbf{L}_G$$

- Again, we have already computed these inertial angular momentum derivatives. The gimbal momentum rate is:

$$\begin{aligned}\dot{\mathbf{H}}_G &= \hat{\mathbf{g}}_s \left((I_{G_s} - I_{G_t} + I_{G_g}) \dot{\gamma} \omega_t + I_{G_s} \hat{\mathbf{g}}_s^T \dot{\boldsymbol{\omega}} + (I_{G_g} - I_{G_t}) \omega_t \omega_g \right) \\ &\quad + \hat{\mathbf{g}}_t \left((I_{G_s} - I_{G_t} - I_{G_g}) \dot{\gamma} \omega_s + I_{G_t} \hat{\mathbf{g}}_t^T \dot{\boldsymbol{\omega}} + (I_{G_s} - I_{G_g}) \omega_s \omega_g \right) \\ &\quad + \hat{\mathbf{g}}_g \left(I_{G_g} (\hat{\mathbf{g}}_g^T \dot{\boldsymbol{\omega}} + \ddot{\gamma}) + (I_{G_t} - I_{G_s}) \omega_s \omega_t \right)\end{aligned}$$



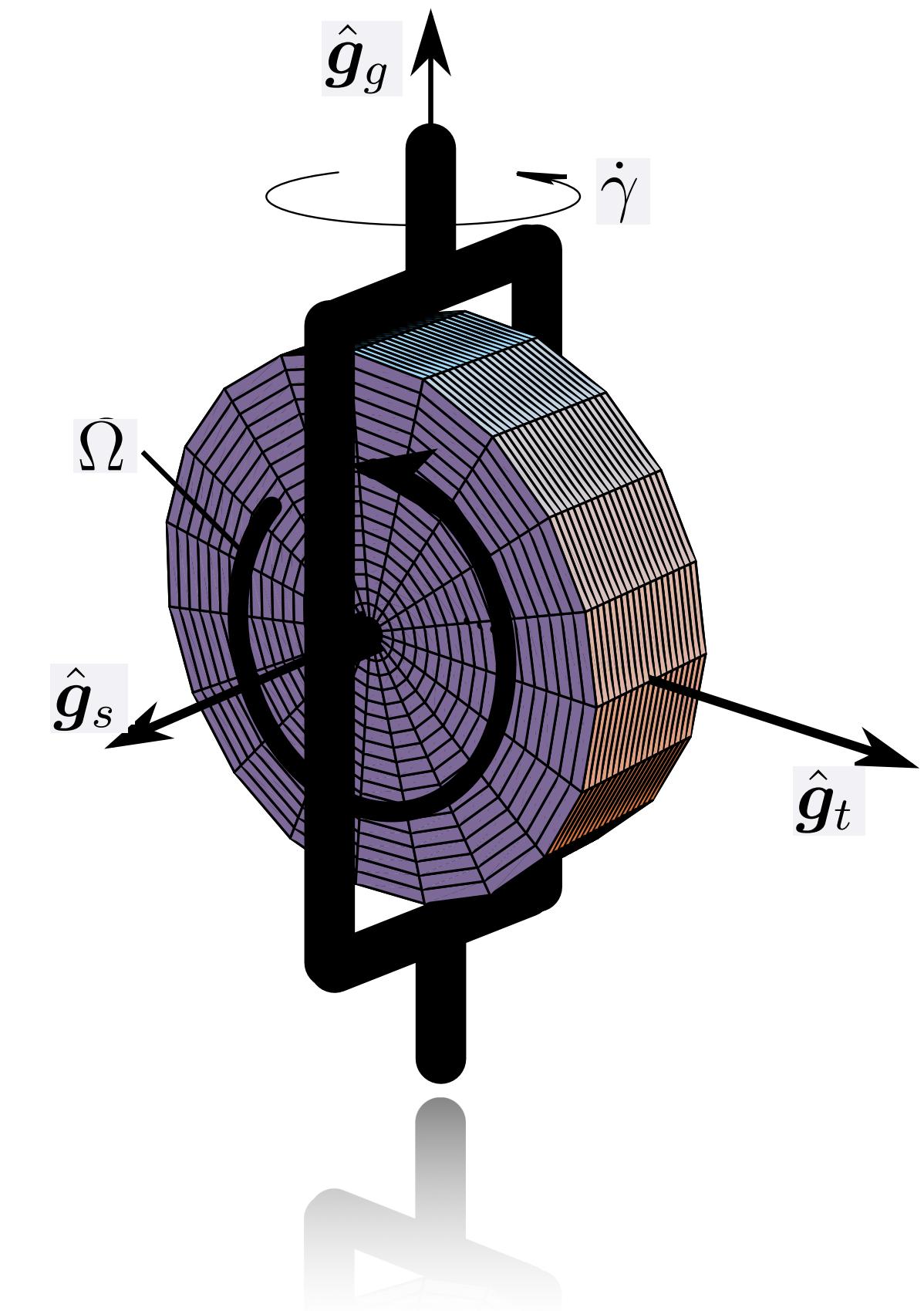
- The only external torque being applied to this two-body system is through the gimbal axis motor.

$$\dot{\mathbf{H}}_G + \dot{\mathbf{H}}_W = \mathbf{L}_G = \tau_{G_s} \hat{\mathbf{g}}_s + \tau_{G_t} \hat{\mathbf{g}}_t + u_g \hat{\mathbf{g}}_g$$

- Thus, equating the $\hat{\mathbf{g}}_g$ directions yields:

$$u_g = J_g (\hat{\mathbf{g}}_g^T \dot{\boldsymbol{\omega}} + \ddot{\gamma}) - (J_s - J_t) \omega_s \omega_t - I_{W_s} \Omega \omega_t$$

Given a commanded gimbal time history $\gamma(t)$, this equation shows us how to compute the actual torque that the gimbal motor must apply.



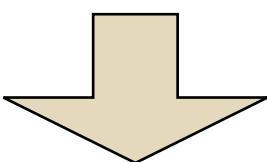
Example: single RW device

$$\dot{\gamma} = 0 \quad \ddot{\gamma} = 0$$

$$[I]\dot{\omega} = -\omega \times [I]\omega - \hat{g}_s J_s \dot{\Omega} \\ - J_s \Omega (\omega_g \hat{g}_t - \omega_t \hat{g}_g) + L$$

using:

$$\omega \times \hat{g}_s = (\omega_s \hat{g}_s + \omega_t \hat{g}_t + \omega_g \hat{g}_g) \times \hat{g}_s = -\omega_t \hat{g}_g + \omega_g \hat{g}_t$$



$$[I]\dot{\omega} = -\omega \times [I]\omega - \hat{g}_s J_s \dot{\Omega} - \omega \times J_s \Omega \hat{g}_s + L$$

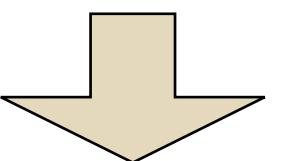
Motor torque:

$$u_s = J_s (\dot{\Omega} + \hat{g}_s^T \dot{\omega})$$

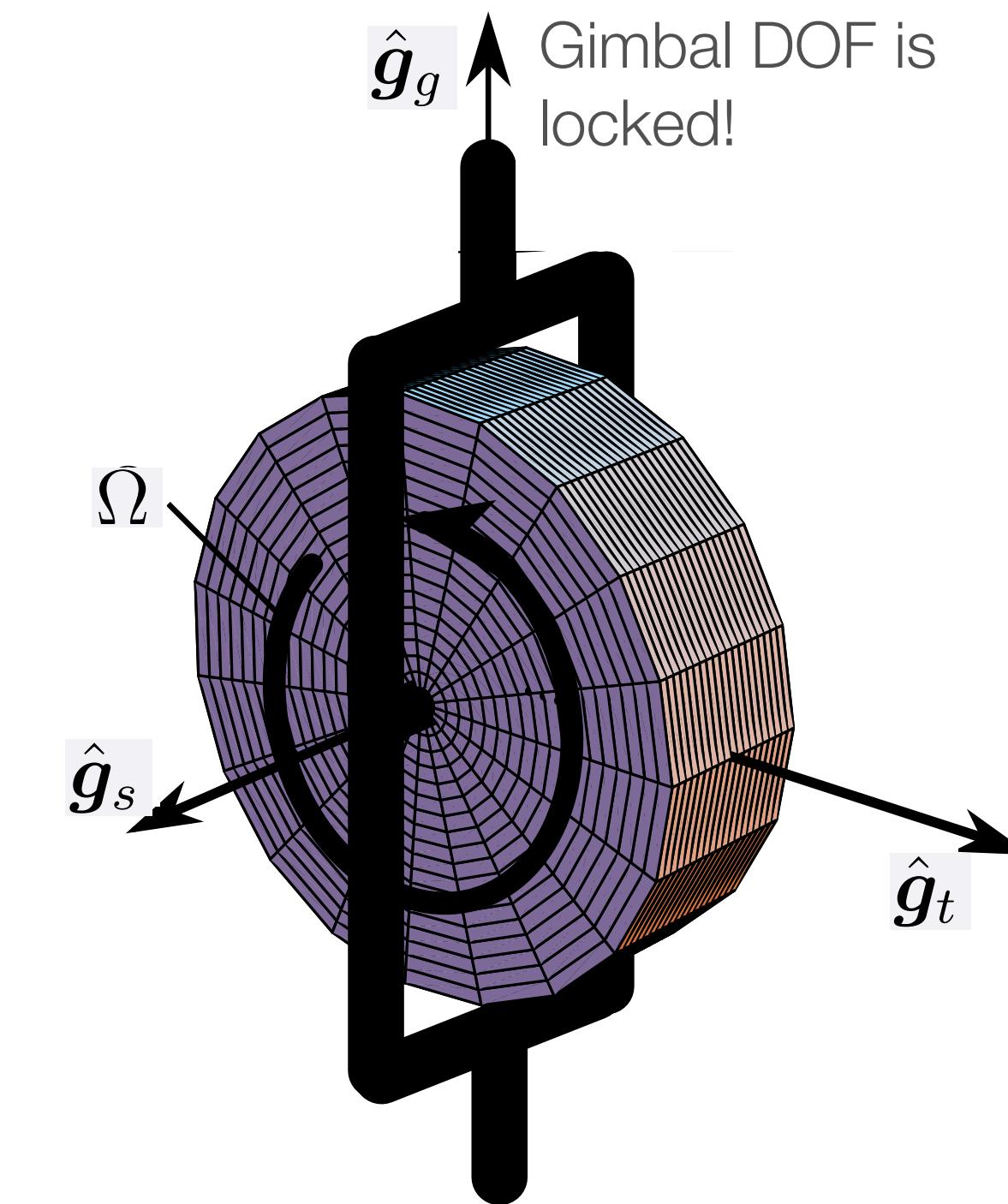
Inertia of spacecraft and

non-spin RW axis:

$$[I_{RW}] = [I_s] + J_t \hat{g}_t \hat{g}_t^T + J_g \hat{g}_g \hat{g}_g^T$$



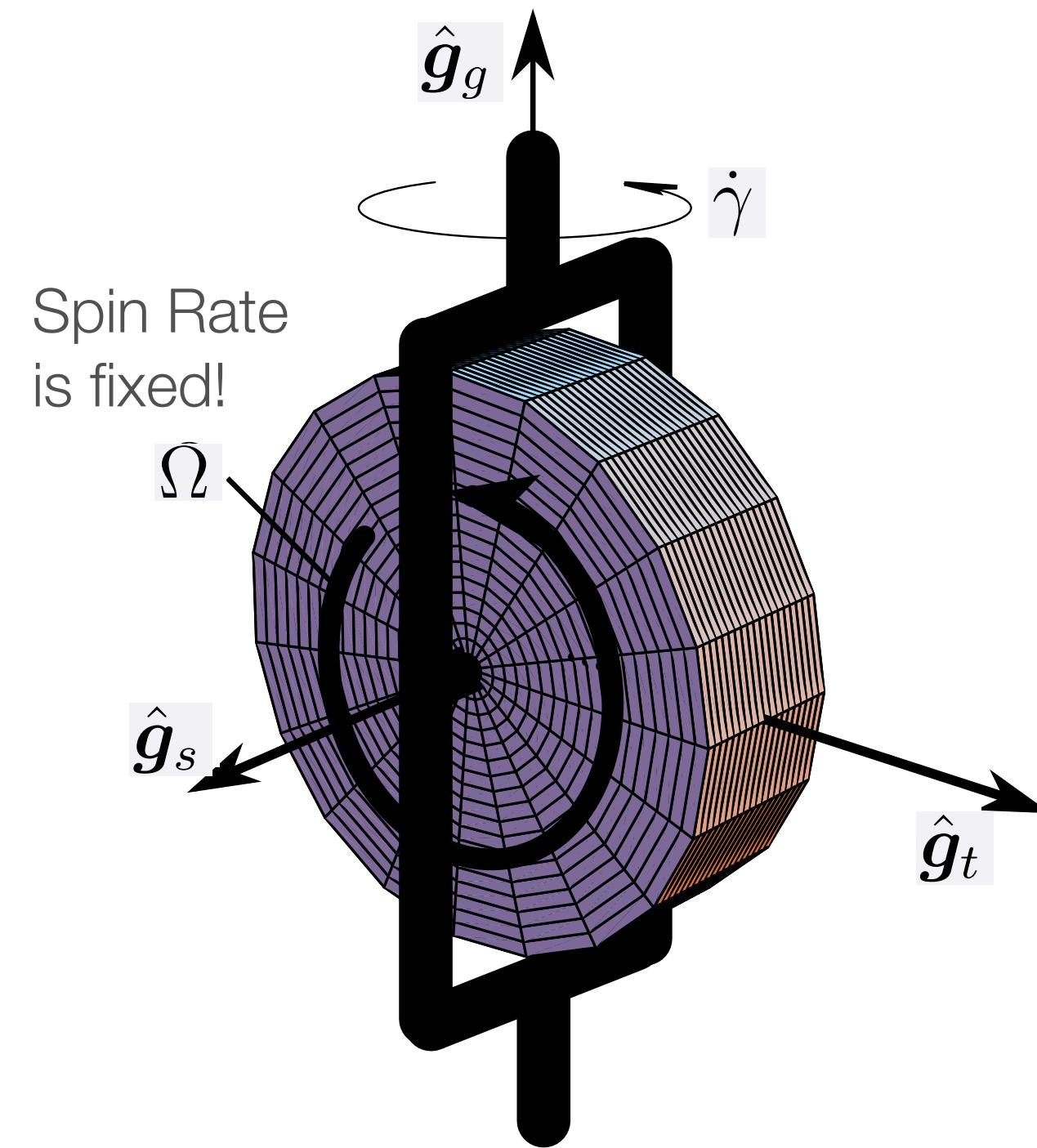
$$[I_{RW}]\dot{\omega} = -\omega \times [I_{RW}]\omega - \omega \times J_s \hat{g}_s (\omega_s + \Omega) - u_s \hat{g}_s + L$$



Example: single CMG device

An inner-servo loop is holding the wheel spin rate fixed:

$$\dot{\Omega} = 0$$



$$\begin{aligned}[I]\dot{\omega} = & -\omega \times [I]\omega - \hat{g}_s (J_s \dot{\gamma} \omega_t - (J_t - J_g) \omega_t \dot{\gamma}) \\ & - \hat{g}_t (J_s (\omega_s + \Omega) \dot{\gamma} - (J_t + J_g) \omega_s \dot{\gamma} + J_s \Omega \omega_g) \\ & - \hat{g}_g (J_g \ddot{\gamma} - J_s \Omega \omega_t) + L\end{aligned}$$

CMG controls are discussed shortly...



Multiple VSCMGs

- To accommodate a spacecraft with N VSCMG devices, we need to employ a little “book-keeping” to account for the various momentum contributions:

We define the $3 \times N$ matrices:

$$[G_s] = [\hat{\mathbf{g}}_{s_1} \cdots \hat{\mathbf{g}}_{s_N}] \quad [G_t] = [\hat{\mathbf{g}}_{t_1} \cdots \hat{\mathbf{g}}_{t_N}] \quad [G_g] = [\hat{\mathbf{g}}_{g_1} \cdots \hat{\mathbf{g}}_{g_N}]$$

New inertia matrix definition:

$$[I] = [I_s] + \sum_{i=1}^N [J_i] = [I_s] + \sum_{i=1}^N J_{s_i} \hat{\mathbf{g}}_{s_i} \hat{\mathbf{g}}_{s_i}^T + J_{t_i} \hat{\mathbf{g}}_{t_i} \hat{\mathbf{g}}_{t_i}^T + J_{g_i} \hat{\mathbf{g}}_{g_i} \hat{\mathbf{g}}_{g_i}^T$$



$$\boldsymbol{\tau}_s = \begin{bmatrix} J_{s_1} \left(\dot{\Omega}_1 + \dot{\gamma}_1 \omega_{t_1} \right) - (J_{t_1} - J_{g_1}) \omega_{t_1} \dot{\gamma}_1 \\ \vdots \\ J_{s_N} \left(\dot{\Omega}_N + \dot{\gamma}_N \omega_{t_N} \right) - (J_{t_N} - J_{g_N}) \omega_{t_N} \dot{\gamma}_N \end{bmatrix}$$

Torque-like vectors:

$$\boldsymbol{\tau}_t = \begin{bmatrix} J_{s_1} (\Omega_1 + \omega_{s_1}) \dot{\gamma}_1 - (J_{t_1} + J_{g_1}) \omega_{s_1} \dot{\gamma}_1 + J_{s_1} \Omega_1 \omega_{g_1} \\ \vdots \\ J_{s_N} (\Omega_N + \omega_{s_N}) \dot{\gamma}_N - (J_{t_N} + J_{g_N}) \omega_{s_N} \dot{\gamma}_N + J_{s_N} \Omega_N \omega_{g_N} \end{bmatrix}$$

$$\boldsymbol{\tau}_g = \begin{bmatrix} J_{g_1} \ddot{\gamma}_1 - J_{s_1} \Omega_1 \omega_{t_1} \\ \vdots \\ J_{g_N} \ddot{\gamma}_N - J_{s_N} \Omega_N \omega_{t_N} \end{bmatrix}$$

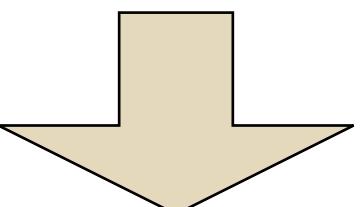


EOM of spacecraft with N VSCMGs:

$$[I]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I]\boldsymbol{\omega} - [G_s]\boldsymbol{\tau}_s - [G_t]\boldsymbol{\tau}_t - [G_g]\boldsymbol{\tau}_g + \mathbf{L}$$

Energy expression:

$$T = \frac{1}{2}\boldsymbol{\omega}^T [I_s]\boldsymbol{\omega} + \frac{1}{2} \sum_{i=1}^N J_{s_i} (\Omega_i + \omega_{s_i})^2 + J_{t_i} \omega_{t_i}^2 + J_{g_i} (\omega_{g_i} + \dot{\gamma}_i)^2$$



After much algebra, or by using
the work-energy-principle...

$$\dot{T} = \boldsymbol{\omega}^T \mathbf{L} + \sum_{i=1}^N \dot{\gamma}_i u_{g_i} + \Omega_i u_{s_i}$$



Example: multiple RW devices

$$\dot{\gamma} = 0$$

$$\ddot{\gamma} = 0$$

\hat{g}_g Gimbal DOF is locked!

Inertia matrix definition:

$$[I_{RW}] = [I_s] + \sum_{i=1}^N (J_{t_i} \hat{g}_{t_i} \hat{g}_{t_i}^T + J_{g_i} \hat{g}_{g_i} \hat{g}_{g_i}^T)$$

Let us define the momentum vector \mathbf{h}_s as:

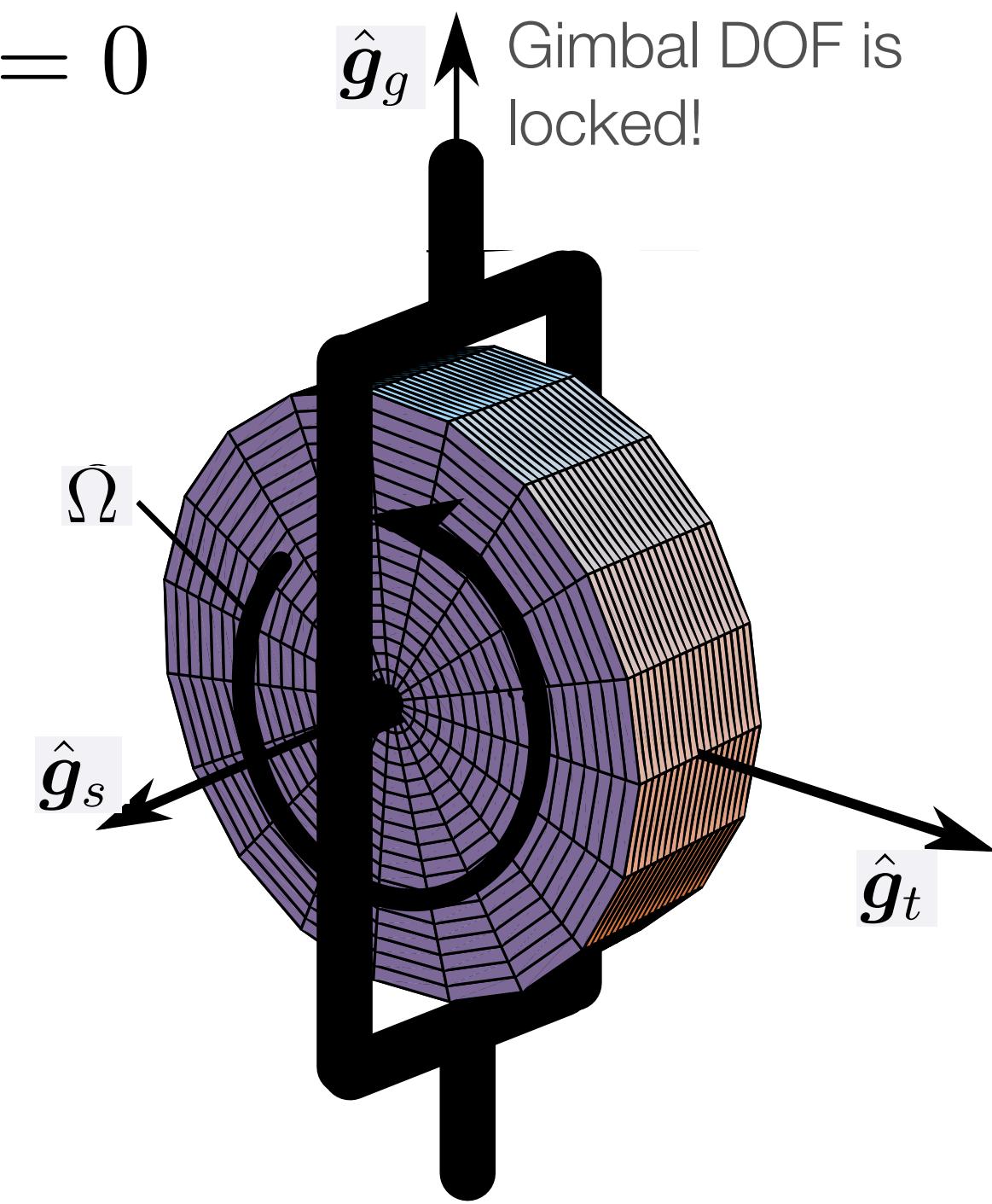
$$\mathbf{h}_s = \begin{pmatrix} \vdots \\ J_{s_i} (\omega_{s_i} + \Omega_i) \\ \vdots \end{pmatrix}$$

The equations of motion then become:

$$[I_{RW}] \dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I_{RW}] \boldsymbol{\omega} - \boldsymbol{\omega} \times [G_s] \mathbf{h}_s - [G_s] \mathbf{u}_s + \mathbf{L}$$

For the special case with 3 RWs aligned with the principal axis, $[G_s]$ becomes an identity matrix and the EOM reduce to

$$[I_{RW}] \dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I_{RW}] \boldsymbol{\omega} - \boldsymbol{\omega} \times \mathbf{h}_s - \mathbf{u}_s + \mathbf{L}$$



Momentum-Device Control Laws

This is where the pudding starts to come together...



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RW Control Devices

- First let us develop a feedback control law for a spacecraft with N reaction wheels with general orientation.

EOM:

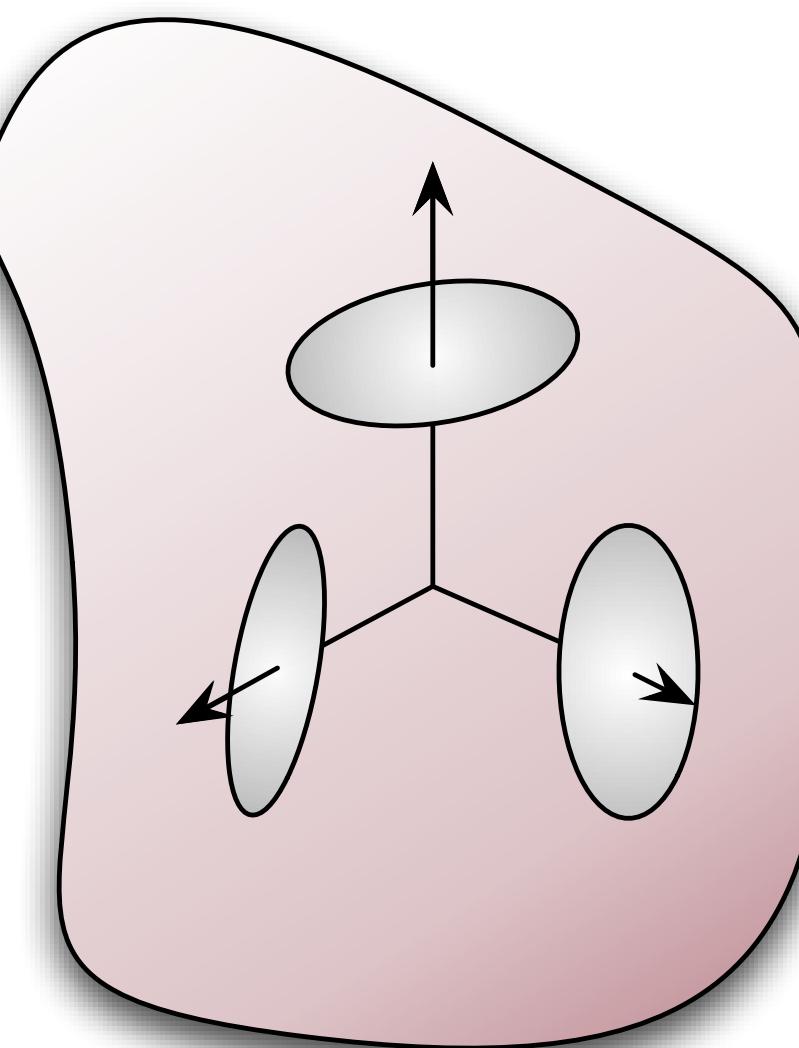
$$[I_{RW}]\dot{\omega} = -\omega \times [I_{RW}]\omega - \omega \times [G_s]\mathbf{h}_s - [G_s]\mathbf{u}_s + \mathbf{L} \quad \text{with} \quad \mathbf{h}_s = \begin{pmatrix} \vdots \\ J_{s_i}(\omega_{s_i} + \Omega_i) \\ \vdots \end{pmatrix}$$

Inertia Matrix:

$$[I_{RW}] = [I_s] + \sum_{i=1}^N (J_{t_i}\hat{\mathbf{g}}_{t_i}\hat{\mathbf{g}}_{t_i}^T + J_{g_i}\hat{\mathbf{g}}_{g_i}\hat{\mathbf{g}}_{g_i}^T)$$

The RW motor control torque vector is:

$$\mathbf{u}_s = \begin{pmatrix} \vdots \\ J_{s_i}(\dot{\Omega}_i + \hat{\mathbf{g}}_{s_i}^T \dot{\omega}) \\ \vdots \end{pmatrix}$$



Spacecraft Tracking Errors:

σ - MRP vector of body frame relative to reference frame

$\delta\omega = \omega - \omega_r$ - body angular velocity tracking error vector

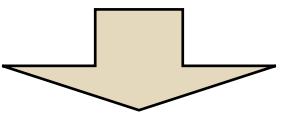
Lyapunov Function:

$$V(\sigma, \delta\omega) = \frac{1}{2} \delta\omega^T [I_{RW}] \delta\omega + 2K \ln(1 + \sigma^T \sigma)$$

components taken in
the B frame

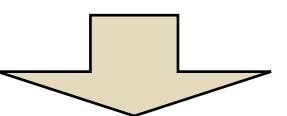
Let's set the Lyapunov Rate to:

$$\dot{V} = -\delta\omega^T [P] \delta\omega \leq 0$$



$$[I_{RW}] \frac{d}{dt} (\delta\omega) + K\sigma + [P]\delta\omega = 0$$

closed-loop dynamics



$$[G_s] \mathbf{u}_s = K\sigma + [P]\delta\omega - [\tilde{\omega}]([I_{RW}]\omega + [G_s]\mathbf{h}_s) - [I_{RW}](\dot{\omega}_r - \omega \times \omega_r) + \mathbf{L}$$

\mathbf{L}_r

Control condition:

$$[G_s] \mathbf{u}_s = \mathbf{L}_r$$

Case 1: 3 RWs aligned with principal axes of spacecraft.

$$\mathbf{u}_s = \mathbf{L}_r$$

Case 2: N RWs aligned generally.

$$\mathbf{u}_s = [G_s]^T ([G_s][G_s]^T)^{-1} \mathbf{L}_r$$

minimum-norm inverse

Energy rate:

$$\dot{T} = \omega^T \mathbf{L} + \sum_{i=1}^N \Omega_i u_{s_i}$$

work/energy principle



The End. . .



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