A Riemann Zeta magnitude growth function based on the Riemann Siegel Theta function closely mirrors the $|t^{-0.1548}\zeta(0.5+it)|$ behaviour for t < 10000.

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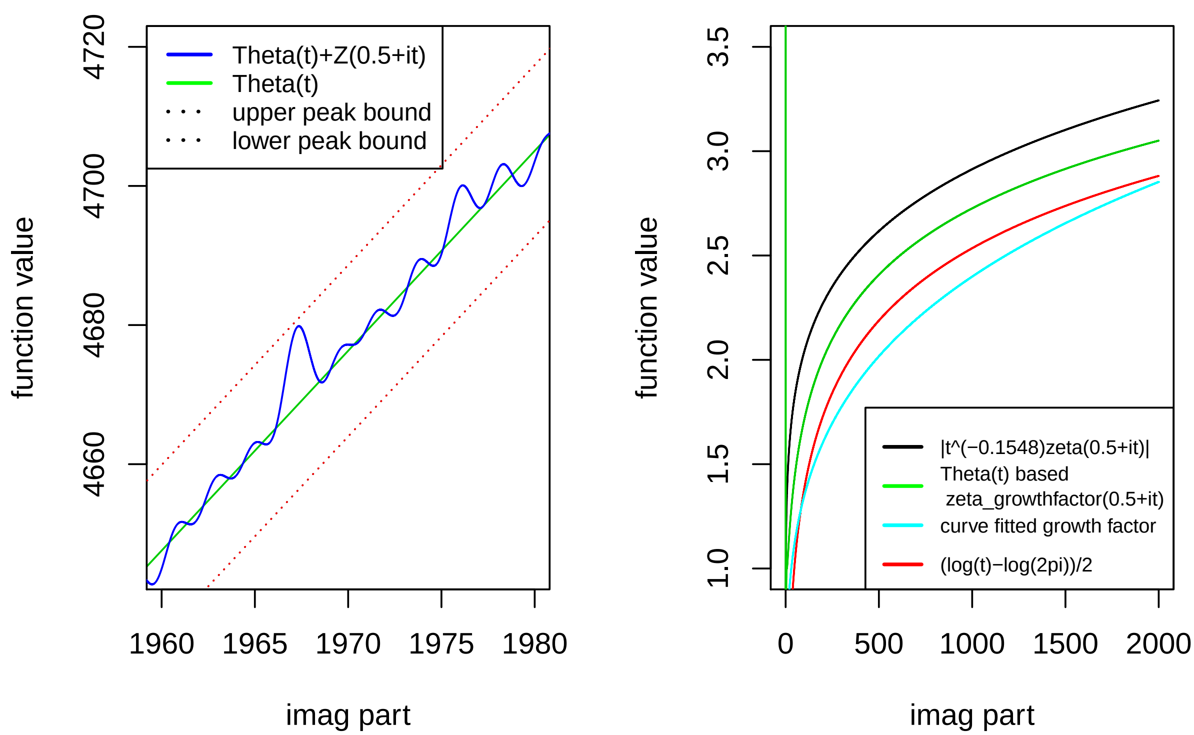
Executive Summary

Using the continuous composite function $\theta(t) + e^{i\theta(t)}\zeta(0.5 + it)$ and imposing the condition that $max(|\zeta_{norm}(0.5 + it)|) = \pi(1 + \gamma/2)$, a growth factor expression for $\zeta(0.5 + it)$ can be derived, $\zeta_{growthfactor}(0.5 + it) = \frac{\pi(1+\frac{\gamma}{2})\frac{1}{\sqrt{\theta'(t)^2+1}}+\theta(t+\pi(1+\frac{\gamma}{2}))\frac{\theta'(t)}{\sqrt{\theta'(t)^2+1}}-\theta(t)}{\pi(1+\frac{\gamma}{2})} \approx \frac{\theta(t+\pi(1+\frac{\gamma}{2}))-\theta(t)}{\pi(1+\frac{\gamma}{2})} = \theta'(t) + \frac{\theta''(t)}{2} + ...$ The growth factor closely mirrors, for low $t < 10000$ the current known Lindelof Hypothesis bound $t^{-0.1548}|\zeta(0.5 + it)|$. The deviation for large t compared to $max(|\zeta_{norm}(0.5 + it)|) = \pi(1 + \gamma/2)$ for both growth estimates suggests extra functional dependence is required to improve both estimates or $max(|\zeta_{norm}(0.5 + it)|) = \pi(1 + \gamma/2) + f(t)$.



Behaviour of $\theta(t) + e^{i\theta(t)}\zeta(0.5 + it)$ function highlighted by bounds on peak size and the current Lindelof Hypothesis bound $t^{-0.1548}$ and $\theta(t)$ based $\zeta_{growthfactor}(0.5 + it)$ magnitude for low t.

Introduction

On the critical line, in the limit $\Im(0.5+it) \to \infty$, the magnitude of the Riemann Zeta $\zeta(0.5+it) \to \infty$ (1-3). The Riemann Zeta function is defined (1), in the complex plane by the integral

$$\zeta(s) = \frac{\prod(-s)}{2\pi i} \int_{C_{\epsilon,\delta}} \frac{(-x)^s}{(e^x-1)x} dx \tag{1}$$

where $s \, \epsilon \, \mathbb{C}$ and $C_{\epsilon,\delta}$ is the contour about the imaginary poles.

The Riemann Zeta function has been shown to obey the functional equation (2)

$$\zeta(s) = 2^s \pi^{s-1} sin(\frac{\pi s}{2}) \Gamma(1-s) \zeta(1-s) \tag{2}$$

Following directly from the form of the functional equation and the properties of the coefficients on the RHS of eqn (2) it has been shown that any zeroes off the critical line would be paired, ie. if $\zeta(s) = 0$ was true then $\zeta(1-s) = 0$.

Along the critical line (0.5+it), the Riemann Siegel function is an exact function (3) for the magnitude of the Riemann Zeta function with two components Z(t) & $\theta(t)$

$$Z(t) = \zeta(0.5+it) e^{i\theta(t)} \tag{3}$$

and

$$\begin{aligned} \theta(t) &= \Im(log(\Gamma(\frac{1}{4} + \frac{1}{2}it))) - \frac{t}{2} log(\pi) \\ &= -\frac{t}{2} log(\frac{2\pi}{t}) - \frac{t}{2} - \frac{\pi}{8} + \frac{1}{48t} + \frac{7}{7560t} + ... \qquad \text{for } t \to \infty \end{aligned} \tag{4}$$

In Martin (4) the normalised Riemann Zeta function $\zeta_{norm}(s)$ within the critical strip was investigated using an exact curve fitted expression. This expression was shown numerically only to be valid for low $\Im(s)$, within the critical strip, however the following relationship and bounds were identified that could be applicable to all $\Im(s)$

$$abs(\zeta_{norm}(s)) \approx abs(\zeta_{norm}(1-s)) \quad \text{for } 0 \leq \Re(s) \leq 1,\ \Im(s) > 2\pi \tag{5}$$

$$\begin{aligned} max|\zeta_{norm}(s)| &= \pi(1+\frac{\gamma}{2}) \qquad \text{for } s = 0.5+it & (6) \\ &\approx \pi \qquad \text{for } s = \pm 1 + it & (7) \end{aligned}$$

$$min|\zeta_{norm}(s)| \approx \frac{\gamma}{2} \qquad \text{for } s = \pm 1 + it \tag{8}$$

Given the bounded nature of $\zeta_{norm}(s)$ for $\Im(s) > 2\pi$ and sharing the same non-trivial zeroes, the growth factor of the Reimann Zeta function magnitude, within the critical strip, can in principle be obtained as the ratio of the two functions

$$\zeta_{growthfactor}(s) = \frac{maxbound|\zeta(s)|}{maxbound|\zeta_{norm}(s)|} \quad \text{for } 0 \le \Re(s) \le 1 \tag{9}$$

using curve fitting for low $\Im(s)$ (4), the growth factor estimate was

$$\zeta_{growthfactor}(s) = abs\Big(e^{(-abs(\Re(s)-\frac{1}{2})+abs(\Im(s))\frac{\pi}{4}+\frac{\gamma}{4\pi})}\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\Big) \quad \text{for } 0 \le \Re(s) \le 1 \tag{10}$$

which had similar $\Im(s)$ behaviour to the current lowest bound $t^{-0.158}|\zeta(0.5+i\Im(s))|$ for the Lindelof hypothesis (5), for low $\Im(s)$ but eventually the $\zeta_{growthfactor}(s)$ exhibited much higher estimates for growth at higher $\Im(s)$. That is, when comparing the calculated $max|\zeta_{norm}(s)|$ values at larger $\Im(s)$ for known large peaks (6) $max(|\zeta_{norm}(s)|) \ll \pi, \Im(s) \to \infty$. Indeed, for very large $\Im(s)$ using data from (7), it is also the case that $t^{-0.158}|\zeta(0.5+i\Im(s))|$ exhibits diminishing values for large peaks as $\Im(s) \to \infty$.

In this paper, an alternate growth factor estimate based on the composite function $\theta(t) + \zeta(0.5+it)$ has been presented using eqn (7), the previous observation that because the maximum peak height of $\zeta(0.5+it)$ tended to increase in increments of π as $\Im(s)$ increased (8)

$$\Delta(max(\theta_{ext}(0.5+it) + |\zeta(0.5+it)|) - \theta_{ext}(0.5+it))) \approx \pi log(t) \tag{11}$$

and a taylor series for the bounds of $\theta(t) + \zeta(0.5+it)$.

Deriving the $\theta(t)$ based growth factor on the critical line

As shown in (8), the imaginary logarithm of the Riemann Siegel Z function $\Im(log(Z_{ext}(s)))$ can be turned into a continuous function by careful mapping of the branch points which add π around the position of each non-trivial zero. On the critical line, a similar function can easily be constructed by simply adding $\theta_{ext}(0.5+it)+|\zeta(0.5+it)|$ since $Z_{ext}(0.5+it)$ and $\theta_{ext}(0.5+it)+|\zeta(0.5+it)|$ share a $\theta_{ext}(0.5+it)$ dependence.

The other relevant idea from (8) is that the highest peaks of the Riemann Zeta function are bounded when expressed as $\theta_{ext}(0.5+it) + |\zeta(0.5+it)|$.

Using calculations of the slope and normal of $\theta_{ext}(0.5+it)$ the upper and lower bounds of $\theta_{ext}(0.5+it) + |\zeta(0.5+it)|$ based on $\zeta_{norm}(0.5+it)$ are simply translated versions of $\theta_{ext}(0.5+it)$ separated by a constant distance of $\pi(1+\gamma/2)$ along the normal vector from $\theta_{ext}(0.5+it)$.

This means that vertical and horizontal displacement of the upper and lower bounds along the normal vector are

$$verticaldisplacement(\theta(t) + |\zeta(0.5+it)|)_{UB} = \Delta(y) = \pi(1+\frac{\gamma}{2})sin(\frac{\pi}{2} - atan(\theta'(t))) \tag{12}$$

$$verticaldisplacement(\theta(t) + |\zeta(0.5+it)|)_{LB} = -\Delta(y) = -\pi(1+\frac{\gamma}{2})sin(\frac{\pi}{2} - atan(\theta'(t))) \tag{13}$$

$$horizontaldisplacement(\theta(t) + |\zeta(0.5+it)|)_{UB} = -\Delta(t) = -\pi(1+\frac{\gamma}{2})cos(\frac{\pi}{2} - atan(\theta'(t))) \tag{14}$$

$$horizontaldisplacement(\theta(t) + |\zeta(0.5+it)|)_{LB} = \Delta(t) = +\pi(1+\frac{\gamma}{2})cos(\frac{\pi}{2} - atan(\theta'(t))) \tag{15}$$

This mixed spacial displacement means that the function value of the upper and lower bounds vertically above any particular function point will be complex since $\theta(t)$ has curvature.

Using, the known displacements and that the bound curves are parallel to $\theta(t)$ gives the formal functional behaviour

$$(\theta(t) + e^{i\theta(t)}\zeta(0.5+it))_{UB,LB} = \pm\Delta(y) + \theta(t \pm \Delta(t)) \tag{16}$$

The growth factor is then derived from the ratio

$$\begin{aligned}\zeta_{growthfactor}(0.5+it) &= \frac{(\theta(t) + e^{i\theta(t)}\zeta(0.5+it))_{UB} - \theta(t)}{(\theta(t) + \pi(1+\frac{\gamma}{2})) - \theta(t)} \\ &= \frac{(\theta(t) + e^{i\theta(t)}\zeta(0.5+it))_{UB} - \theta(t)}{\pi(1+\frac{\gamma}{2})} \end{aligned}\tag{17}$$

using eqn (16) and geometrical relationships

$$\begin{aligned}\zeta_{growthfactor}(0.5+it) &= \frac{\Delta(y) + \theta(t+\Delta(t)) - \theta(t)}{\pi(1+\frac{\gamma}{2})} && (18)\\ &= \frac{\pi(1+\frac{\gamma}{2})sin(\frac{\pi}{2} - atan(\theta'(t))) + \theta(t + \pi(1+\frac{\gamma}{2})cos(\frac{\pi}{2} - atan(\theta'(t)))) - \theta(t)}{\pi(1+\frac{\gamma}{2})} && (19)\\ &= \frac{\pi(1+\frac{\gamma}{2})cos(atan(\theta'(t))) + \theta(t + \pi(1+\frac{\gamma}{2})sin(atan(\theta'(t)))) - \theta(t)}{\pi(1+\frac{\gamma}{2})} && (20)\\ &= \frac{\pi(1+\frac{\gamma}{2})\frac{1}{\sqrt{\theta'(t)^2+1}} + \theta(t+\pi(1+\frac{\gamma}{2}))\frac{\theta'(t)}{\sqrt{\theta'(t)^2+1}} - \theta(t)}{\pi(1+\frac{\gamma}{2})} && (21)\\ &\approx \frac{\theta(t+\pi(1+\frac{\gamma}{2})) - \theta(t)}{\pi(1+\frac{\gamma}{2})} = \theta'(t) \qquad \text{for } t \to \infty && (22)\end{aligned}$$

The ratio can also be expressed using second order taylor series to reveal additional series terms for low t

$$\begin{aligned}(\theta(t) + |\zeta(0.5+it)|)_{UB,LB} &= \theta(t) \pm \pi * (1+\frac{\gamma}{2})\Big(\theta'(t)cos(\frac{\pi}{2} - atan(\theta'(t))) + ... \\ &+ \frac{\theta''(t)}{2}cos(\frac{\pi}{2} - atan(\theta'(t)))^2 + sin(\frac{\pi}{2} - atan(\theta'(t)))\Big) && (23)\\ &\approx \theta(t) \pm \pi * (1+\frac{\gamma}{2})\Big(\theta'(t)sin(atan(\theta'(t))) + \frac{\theta''(t)}{2}sin(atan(\theta'(t)))^2 + cos(atan(\theta'(t)))\Big) && (24)\\ &\approx \theta(t) \pm \pi * (1+\frac{\gamma}{2})\Big(\theta'(t)\frac{\theta'(t)}{\sqrt{\theta'(t)^2+1}} + \frac{\theta''(t)}{2}(\frac{\theta'(t)}{\sqrt{\theta'(t)^2+1}})^2 + \frac{1}{\sqrt{\theta'(t)^2+1}}\Big) && (25)\\ &\approx \theta(t) \pm \pi * (1+\frac{\gamma}{2})\Big(\theta'(t) + \frac{\theta''(t)}{2} + ...\Big) && (26)\end{aligned}$$

The resultant growth factor is then the ratio estimate

$$\begin{aligned}\zeta_{growthfactor}(0.5+it) &= \frac{\big((\theta(t) + |\zeta(0.5+it)|)_{UB} - \theta(t)\big)}{max|\zeta_{norm}(0.5+it)|} \\ &\approx \theta'(t) + \frac{\theta''(t)}{2} + ... \end{aligned}\tag{27}$$

It is also noted that $\theta(t)$ is very suitable for second order taylor series expansion away from real axis given

$$\theta(t) = -\frac{t}{2}log(\frac{2\pi}{t}) - \frac{t}{2} - \frac{\pi}{8} + \frac{1}{48t} + \frac{7}{7560t} + ... \qquad \text{for } t \to \infty \tag{28}$$

$$\theta'(t) = \frac{log(t)}{2} - \frac{log(2\pi)}{2} + ... \qquad \text{for } t \to \infty \tag{29}$$

$$\theta''(t) = \frac{1}{2t} + ... \qquad \text{for } t \to \infty \tag{30}$$

Collecting the leading terms

$$\zeta_{growthfactor}(0.5+it) \approx \frac{(log(t) - log(2\pi))}{2} + \frac{1}{4t} + ... \tag{31}$$

Results

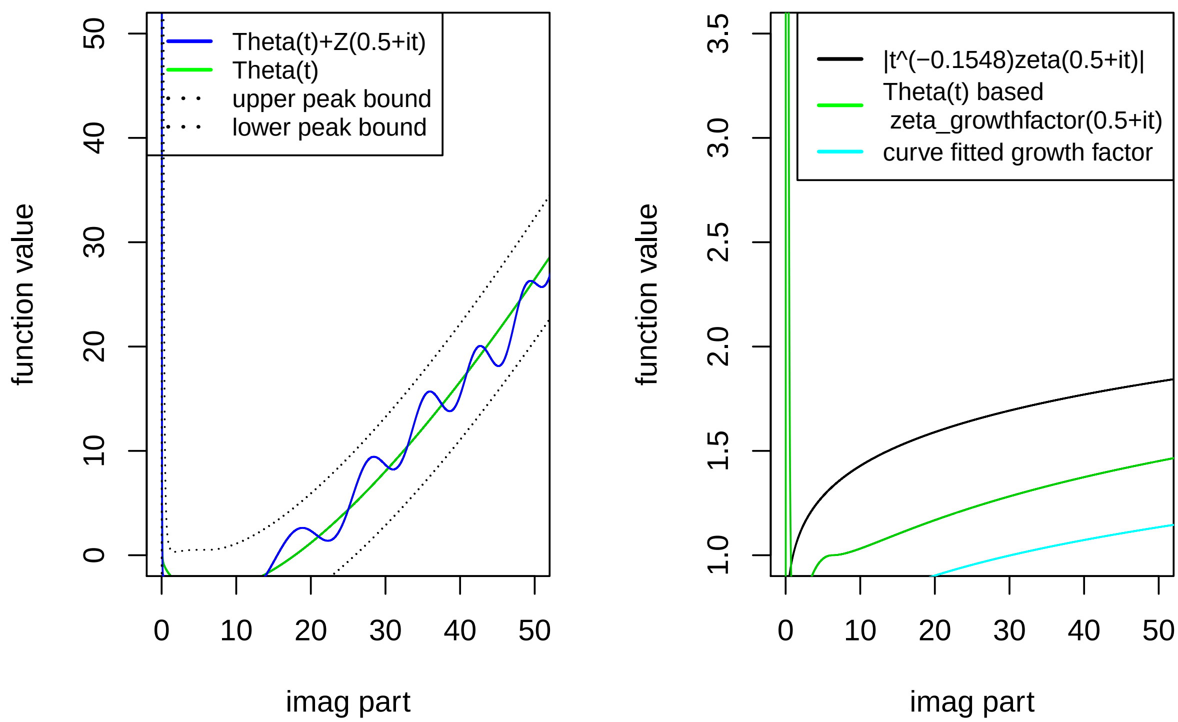
The following two graphs plus the executive summary graph show the close correspondence of the $\theta(t)$ growth factor of the Riemann Zeta magnitude along the critical line to the current Lindelof Hypothesis bound $t^{-0.1548}|\zeta(0.5+it)|$ for low t and the curved fitted growth factor from (4)

Comparing the three figures,

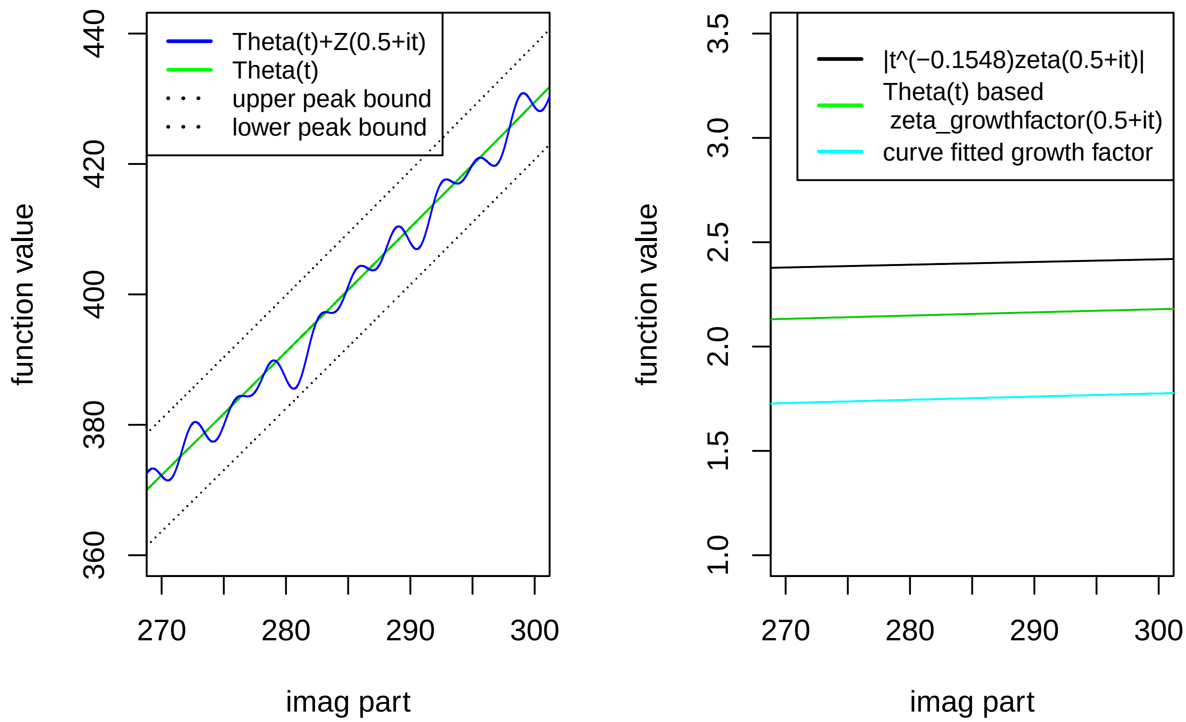
(i) the change of frame of reference from the unscaled presentation of $|\zeta(0.5+it)|$ to the scaled sloping function $\theta(t) + e^{i\theta(t)}\zeta(0.5+it)$ with integral scaling by π allows the highest peaks of the Riemann Zeta function to be recognised as bounded features. The gradual growth of large peaks by π can now be understood arising from $\theta(t)$ influencing the bounds of the function amplitude as well as the periodicity of the zeroes.

(ii) the $\theta(t) + e^{i\theta(t)}\zeta(0.5+it)$ based growth factor is closer to the Lindelof Hypothesis upper bound result than the curve fitted growth factor from (4) reflecting the deeper concept that $\theta(t) + e^{i\theta(t)}\zeta(0.5+it)$ represents compared to curve fitting at low t

(iii) the executive summary figure, in particular shows that the $\theta(t) + e^{i\theta(t)}\zeta(0.5+it)$ growth factor is actually very similar to the Lindelof Hypothesis upper bound and may represent a natural limiting value for the t^{-0.1458} estimate methodology.



Behaviour of $\theta(t)+e^{i\theta(t)}\zeta(0.5+it)$ function for very low t, highlighted by bounds on peak size and the current Lindelof Hypothesis bound $t^{-0.1548}$, $\theta(t)$ based $\zeta_{growthfactor}(0.5+it)$ magnitude for low t and a curve fitted growth factor.



Behaviour of $\theta(t) + e^{i\theta(t)}\zeta(0.5+it)$ function for slightly higher t, highlighted by bounds on peak size and the current Lindelof Hypothesis bound $t^{-0.1548}$, $\theta(t)$ based $\zeta_{growthfactor}(0.5+it)$ magnitude for low t and a curve fitted growth factor.

High t issues for growth factor estimates, to be continued

Using known lists of $\zeta(s)$ maxima (5,7) for higher t the three growth factors $|t^{-0.1548}\zeta(s)|$, $\zeta_{growthfactor}(0.5+it)$ and curve fitted factor (4) all fail to be consistent with the idea of a flat bound in magnitude for $\zeta_{norm}(s)$ of $(1+\gamma/2)\pi$ on the critical line.

In the results below, if the calculated peak heights represent growth from a baseline normalised Riemann Zeta function, then scaled values of $\sim \pi \approx 3.14$ would have been expected for $\zeta_{norm}(s)$ & $|t^{-0.1548}\zeta(s)|$

for s=1/2+1i*2445745756408.35596, $\zeta(s) = 297.435$, $\zeta_{norm}(s) = 22.22$, $|t^{-0.1548}\zeta(s)| = 3.60$

for s=1/2+1i*310678833629083965667540576593682.058, $\zeta(s) = 16858.119$, $\zeta_{norm}(s) = 461.8294$, $|t^{-0.1548}\zeta(s)|$ $= 0.157398$

Conclusions

The composite function $\theta(t) + e^{i\theta(t)}\zeta(0.5+it)$ in tandem with the concept of a flat bounded normalised Riemann Zeta function can be used to derive a growth factor estimate for the magnitude of the Riemann Zeta function, valid for low $\Im(s)$. The approach also has a useful frame of reference change to understand the rate of change in the highest peaks.

Comparing to calculations using the latest known Lindelof Hypothesis bound results $t^{-0.158}|\zeta(0.5+it)|$, a fully consistent $\zeta_{norm}(s)$ functional form for all large $\Im(s)$ within the critical strip, requires additonal functional dependence in all these growth factor estimates or the assumption of $max(|\zeta_{norm}(0.5+it)|) = \pi(1+\gamma/2)$ is failing.

References

1. Edwards, H.M. (1974). Riemann's zeta function. Pure and Applied Mathematics 58. New York-London: Academic Press. ISBN 0-12-242750-0. Zbl 0315.10035.
2. Riemann, Bernhard (1859). "*Ü*ber die Anzahl der Primzahlen unter einer gegebenen Gr*ö*sse". Monatsberichte der Berliner Akademie.. In Gesammelte Werke, Teubner, Leipzig (1892), Reprinted by Dover, New York (1953).
3. Berry, M. V. "The Riemann-Siegel Expansion for the Zeta Function: High Orders and Remainders." Proc. Roy. Soc. London A 450, 439-462, 1995.
4. Martin, J.P.D. "Exact functional dependence for the growth in the magnitude of the Riemann Zeta function within the critical strip" (2018) http://dx.doi.org/10.6084/m9.figshare.5765796
5. numbers.computation.free.fr/Constants/Miscellaneous/MaxiZAll.txt Copyright © 1999-2010 by Xavier Gourdon & Pascal Sebah
6. Bourgain, Jean (2017), "Decoupling, exponential sums and the Riemann zeta function", Journal of the American Mathematical Society, 30 (1): 205–224, arXiv:1408.5794 doi:10.1090/jams/860
7. Tihanyi, N., Kovács, A. & Kovács, J. "Computing Extremely Large Values of the Riemann Zeta Function" J Grid Computing (2017) 15: 527. https://doi.org/10.1007/s10723-017-9416-0
8. Martin, J.P.D. "Mapping the Extended Riemann Siegel Z Theta Functions about branch points in the complex plane" (2016) http://dx.doi.org/10.6084/m9.figshare.3813999