

Examining the convergence behaviour of the finite Dirichlet Series and zeroth order Riemann Siegel formula calculated via the sum of products of finite geometric series of primes and powers of primes near the first Rosser rule violation

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DRAFT Executive summary

A linear combination of products of finite geometric series of primes and power of primes exhibits excellent performance in reproducing the finite Riemann Zeta Dirichlet Series and zeroth order Riemann Siegel formula results for $N=1041$ corresponding to the interval containing the first Rosser rule violation. It can be observed that the first \sqrt{N} of the primes (and their cross products and powers) contribute the majority of the Riemann Zeta large peak at $s=0.5+I*6820051$. However, all primes up to the first quiescent region $N = \sqrt{\frac{t}{2\pi}}$ are required to obtain accurate zeroth order approximations of the zero locations for closely spaced zeroes.

Introduction

For $\Re(s) > 1$, the infinite Euler Product of the primes absolutely converges to the infinite Riemann Zeta Dirichlet Series sum [1,2]

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} = \prod_{\rho=2}^{\infty} \frac{1}{(1 - 1/\rho^s)} \quad \text{for } \Re(s) > 1 \quad (1)$$

Importantly, using the $\log(1-x)$ expansion of $\log(\zeta(s))$ [3-5] the Euler product also has the form

$$\prod_{\rho=2}^{\infty} \frac{1}{(1 - 1/\rho^s)} = \exp\left(\sum_{\rho=2}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n \cdot \rho^{ns}}\right) \quad (2)$$

For $\Re(s) \leq 1$, the partial Euler Product diverges, however, using the above equations for finite sums (products) of integers (primes) the following relationship holds

$$\begin{aligned}
\sum_{k=1}^N \frac{1}{k^s} &= 1 + \left(\sum_{\rho=2}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n \cdot \rho^{ns}} \cdot \delta(\rho^n \leq N) \right) \\
&+ \frac{1}{2!} \left(\sum_{\rho_1=2}^{\infty} \sum_{n=1}^{\infty} \sum_{\rho_2=2}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n \cdot \rho_1^{ns}} \cdot \frac{1}{m \cdot \rho_2^{ms}} \cdot \delta(\rho_1^n \cdot \rho_2^m \leq N) \right) \\
&+ \frac{1}{3!} \left(\sum_{\rho_1=2}^{\infty} \sum_{n=1}^{\infty} \sum_{\rho_2=2}^{\infty} \sum_{m=1}^{\infty} \sum_{\rho_3=2}^{\infty} \sum_{o=1}^{\infty} \frac{1}{n \cdot \rho_1^{ns}} \cdot \frac{1}{m \cdot \rho_2^{ms}} \cdot \frac{1}{o \cdot \rho_3^{os}} \cdot \delta(\rho_1^n \cdot \rho_2^m \cdot \rho_3^o \leq N) \right) \\
&+ \dots
\end{aligned} \tag{3}$$

where the delta functions play a crucial role in appropriately truncating the Euler Product terms. Hence the above expression can be used with the $N \sim \lfloor \frac{t}{\pi} \rfloor$ and $(N \sim \lfloor \sqrt{\frac{t}{2\pi}} \rfloor)$ quiescent regions of the oscillatory divergence of the Riemann Zeta function to obtain useful partial Euler Product based approximations of the Riemann Zeta function in the critical strip (and below) [6].

In [6] empirical calculations showed that the truncated exponential series version of the finite Euler product is a slower running algorithm at the complex plane points presented compared to the simple Dirichlet Series. This is due to the extra multiplication operations and truncation checks that are required at each higher order term of the power series calculation.

In [7], an alternative series expansion for the truncated euler product was given in terms of finite geometric series of primes and powers of primes in the hope of identifying a faster way to calculate a finite Dirichlet Series using only primes. The early terms of the alternative series expansion are shown in the appendix.

In this paper the alternate prime based expansion [7], is used to calculate the finite Dirichlet Series and zeroth order Riemann Siegel formula for the first quiescent region $N = \sqrt{\frac{t}{2\pi}}$ using only a linear combination of finite geometric series of primes and powers of primes near the first Rosser rule violation (gram point 13999525, near by the Riemann Zeta large peak $t=6820051$).

Convergence of the cumulative sum of the alternate prime based expansion [7] compared to the first principle Dirichlet Series (at the first quiescent region)

The cumulative series sum behaviour of the real part of (i) the finite Riemann Zeta Dirichlet Series sum (green) and (ii) the alternate finite geometric series based linear combination sum (red). For $t=6820051$, the first quiescent region is at $N = \lfloor \sqrt{\frac{6820051}{2\pi}} \rfloor = 1041$. Shown in grey is the Dirichlet Series sum $\sum_{n=1}^{N=1041} \frac{1}{n^s}$ used in the zeroth order term of the Riemann Siegel formula.

It can be seen that the contribution from the early terms of the alternate finite geometric series based linear combination sum is dominant and the series sum much more rapidly approaches the expected finite sum value. This is because the geometric series approach accelerates the series sum by incorporating later terms (in the first principles series sum) into the early series terms.

Explicitly, the real part of the early terms in the alternate finite geometric series linear combination

$$\begin{aligned}
\Re \left(\sum_{n=1}^{1041} \frac{1}{n^s} \right) &= \Re \left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots + \frac{1}{1041^s} \right) \\
&= \Re(a_2(s, 1041) + a_3(s, 1041) + a_5(s, 1041) + a_7(s, 1041) + a_9(s, 1041) + \dots + a_{p \leq 1041}(s, 1041))
\end{aligned} \tag{4}$$

$$\tag{5}$$

are for $s = 0.5 + I \cdot 6820051$

$$\Re(a_2(s, 1041)) = 3.017649 \quad (6)$$

$$\Re(a_3(s, 1041)) = 1.620435 \quad (7)$$

$$\Re(a_5(s, 1041)) = 2.045711 \quad (8)$$

$$\Re(a_7(s, 1041)) = 1.352946 \quad (9)$$

$$\Re(a_9(s, 1041)) = 0.834580 \quad (10)$$

$$\Re(a_{11}(s, 1041)) = 2.344479 \quad (11)$$

$$\Re(a_{13}(s, 1041)) = 2.484002 \quad (12)$$

$$\Re(a_{17}(s, 1041)) = 1.474045 \quad (13)$$

$$\Re(a_{19}(s, 1041)) = 1.671007 \quad (14)$$

$$\Re(a_{23}(s, 1041)) = 2.034249 \quad (15)$$

$$\Re(a_{25}(s, 1041)) = 0.535378 \quad (16)$$

$$\Re(a_{27}(s, 1041)) = 0.414098 \quad (17)$$

$$\Re(a_{29}(s, 1041)) = 1.388086 \quad (18)$$

$$\Re(a_{31}(s, 1041)) = -0.461051 \quad (19)$$

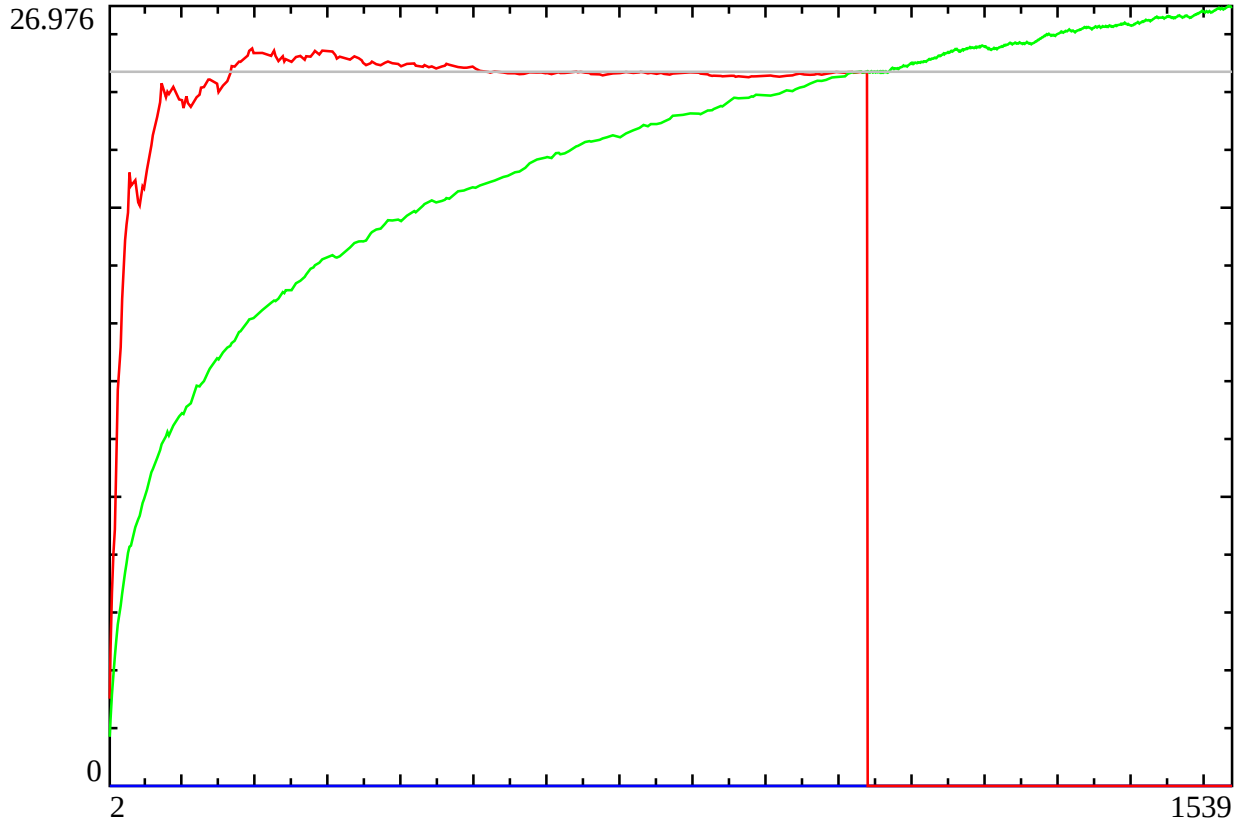


Figure 1. A comparison of the convergence of the cumulative sums (up to and just beyond the first quiescent region of the Riemann Zeta Dirichlet Series sum) for $s = 0.5 + I \cdot 6820051$. At this point on the critical line the Riemann Zeta function has a large peak $|\zeta(s)| \sim 50$ (and the zeroth order finite dirichlet series sum is 24.7 (approximately one half of 50 as expected). The finite geometric series linear combination based sum is only calculated up to the $i=1039$ term, belonging to the nearest prime below $N=1041$.

Using the alternate prime based expansion [7] for generating Riemann Siegel type calculations

Based on the first quiescent region, the interval for N=1041 calculations spans the interval

$$1041^2 \cdot 2 \cdot \pi \leq t < 1042^2 \cdot 2 \cdot \pi \quad (20)$$

$$[6808968.5368 \leq t < 6822056.4118] \quad (21)$$

so the same alternate prime based expansion for N=1041 can be reused over a sizeable interval which offsets the cost of setting up the N=1041 expansion.

An approximation of Riemann Zeta function behaviour can be obtained by the finite dirichlet series based zeroth order Riemann Siegel formula [1,2] at the first quiescent region $N_1 = \sqrt{\frac{t}{2\pi}}$ of the oscillatory divergence of the dirichlet series

$$\zeta_{RS \text{ zeroth order}}(s) = \sum_{n=1}^{\lfloor \sqrt{\frac{t}{2\pi}} \rfloor} \frac{1}{n^s} + \chi(s) \cdot \sum_{n=1}^{\lfloor \sqrt{\frac{t}{2\pi}} \rfloor} \frac{1}{n^{(1-s)}} \quad (22)$$

$$Z_{RS \text{ zeroth order}}(s) = \exp(I \cdot \theta(t)) \cdot \left(\sum_{n=1}^{\lfloor \sqrt{\frac{t}{2\pi}} \rfloor} \frac{1}{n^s} + \chi(s) \cdot \sum_{n=1}^{\lfloor \sqrt{\frac{t}{2\pi}} \rfloor} \frac{1}{n^{(1-s)}} \right) \quad (23)$$

where (i) $\chi(s)$ is the multiplicative factor of the Riemann Zeta functional equation $\zeta(s) = \chi(s)\zeta(1-s)$, (ii) $\theta(t)$ the Riemann Siegel Theta function is a continuous function and (iii) equation (23) is the zeroth order Riemann Siegel Z function [1-3].

Since the alternate finite geometric series based linear combination is equivalent to the finite dirichlet series the following Riemann Siegel prime based formulae can be written

$$\zeta_{\text{primeRS zeroth order}}(s) = \sum_{p=2}^{P \leq \lfloor \sqrt{\frac{t}{2\pi}} \rfloor} a_p(s, \lfloor \sqrt{\frac{t}{2\pi}} \rfloor) + \chi(s) \cdot \sum_{p=2}^{P \leq \lfloor \sqrt{\frac{t}{2\pi}} \rfloor} a_p((1-s), \lfloor \sqrt{\frac{t}{2\pi}} \rfloor) \quad (24)$$

$$Z_{\text{primeRS zeroth order}}(s) = \exp(I \cdot \theta(t)) \cdot \left(\sum_{p=2}^{P \leq \lfloor \sqrt{\frac{t}{2\pi}} \rfloor} a_p(s, \lfloor \sqrt{\frac{t}{2\pi}} \rfloor) + \chi(s) \cdot \sum_{p=1}^{P \leq \lfloor \sqrt{\frac{t}{2\pi}} \rfloor} a_p((1-s), \lfloor \sqrt{\frac{t}{2\pi}} \rfloor) \right) \quad (25)$$

where $p \in \text{primes and powers of primes} \leq \lfloor \sqrt{\frac{N}{2\pi}} \rfloor$ and $p \neq 4, 8, 16, \dots$ since the prime powers of 2 are already included the finite geometric series term for prime 2.

Using equations (23) and (25), the impact of number of summed primes (up to N=1041) for the zeroth order Riemann Siegel Z function in the interval $t=(6820048, 6820054)$ can be viewed in figures 2-6.

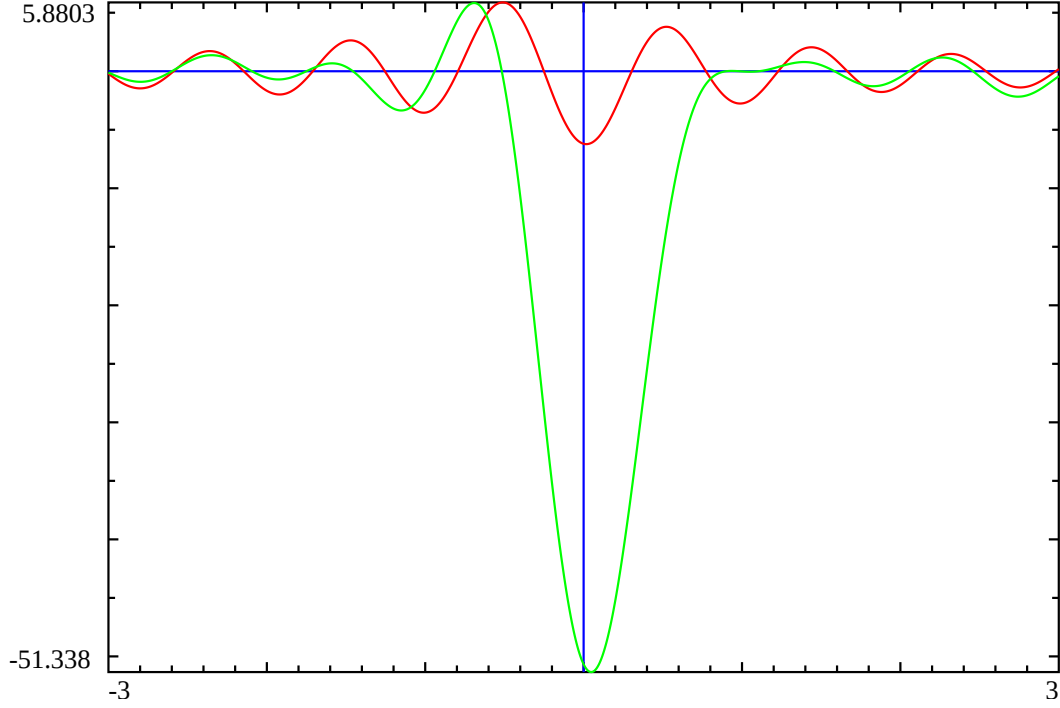


Figure 2. A comparison of the zeroth order Riemann Siegel Z function on the critical line in the interval $t=(6820048, 6820054)$ to a prime based zeroth order Riemann Siegel Z function using only prime 2 and prime 2 powers.

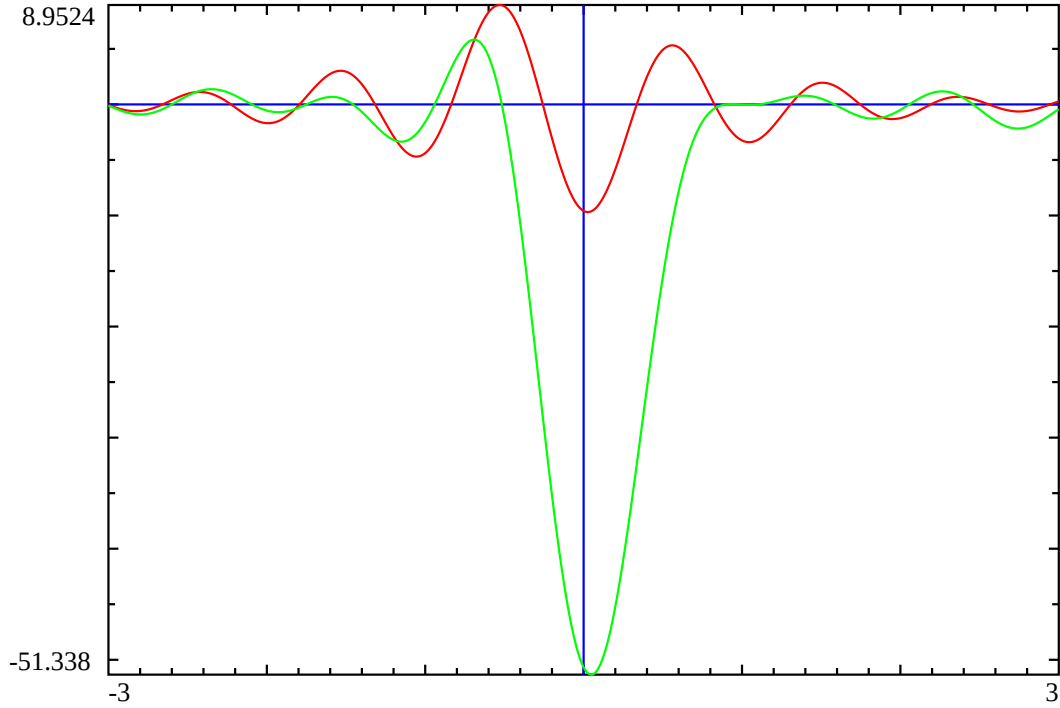


Figure 3. A comparison of the zeroth order Riemann Siegel Z function on the critical line in the interval $t=(6820048, 6820054)$ to a prime based zeroth order Riemann Siegel Z function using only prime 2, prime 3 and prime 2 powers.

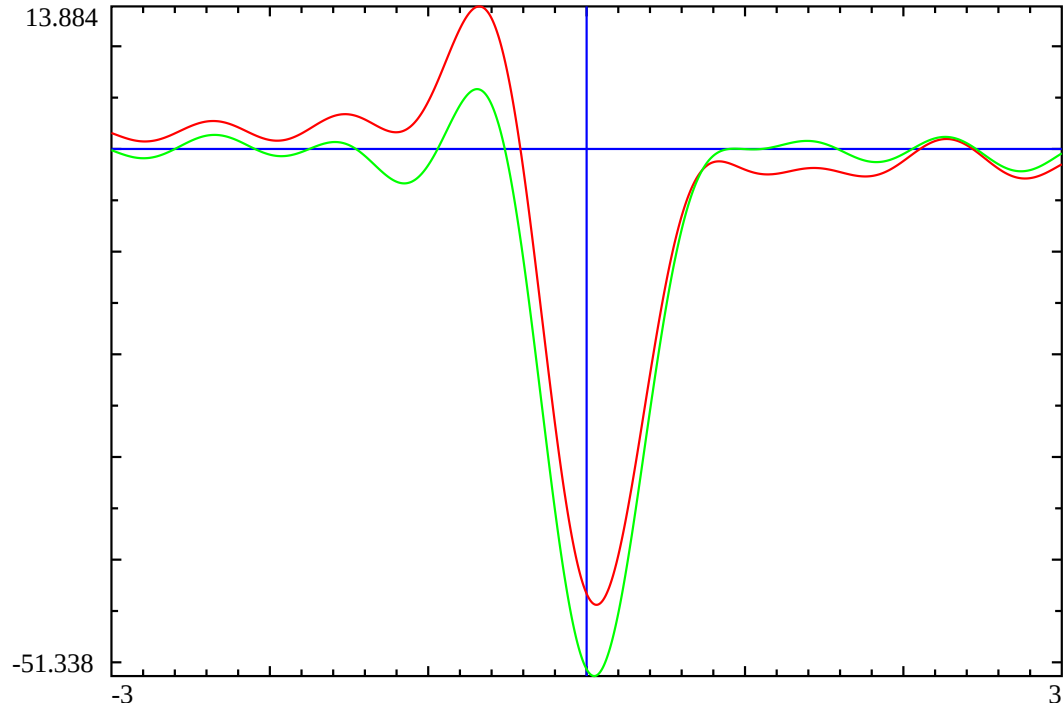


Figure 4. A comparison of the zeroth order Riemann Siegel Z function on the critical line in the interval $t=(6820048, 6820054)$ to a prime based zeroth order Riemann Siegel Z function using terms upto prime=31.

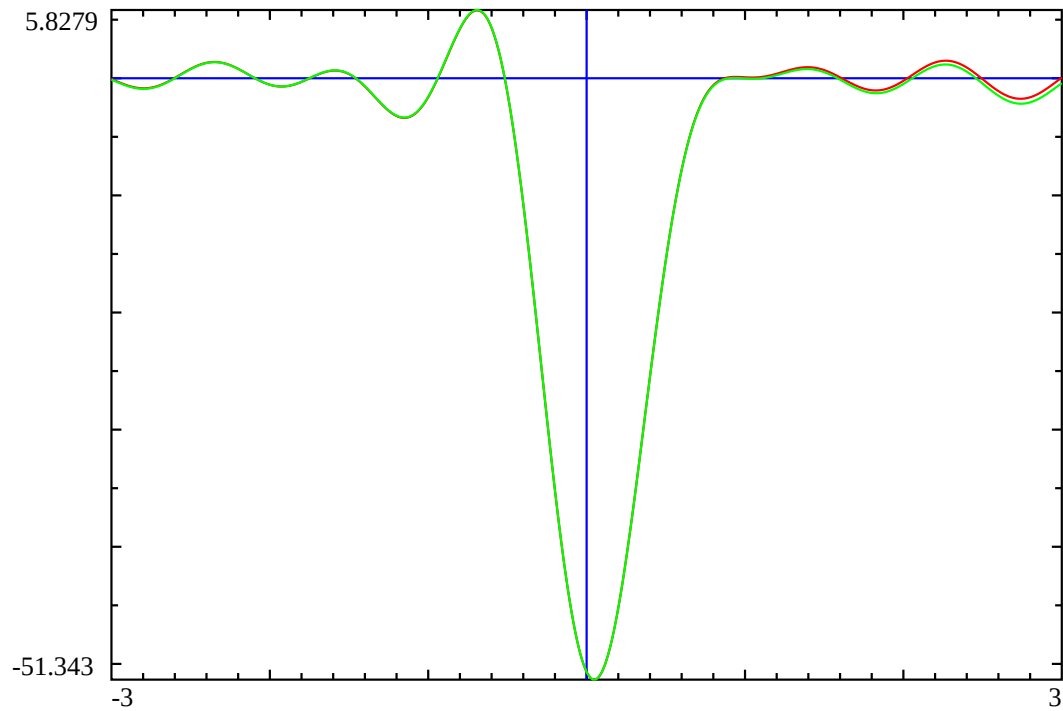


Figure 5. A comparison of the zeroth order Riemann Siegel Z function on the critical line in the interval $t=(6820048, 6820054)$ to a prime based zeroth order Riemann Siegel Z function using terms upto prime 509.

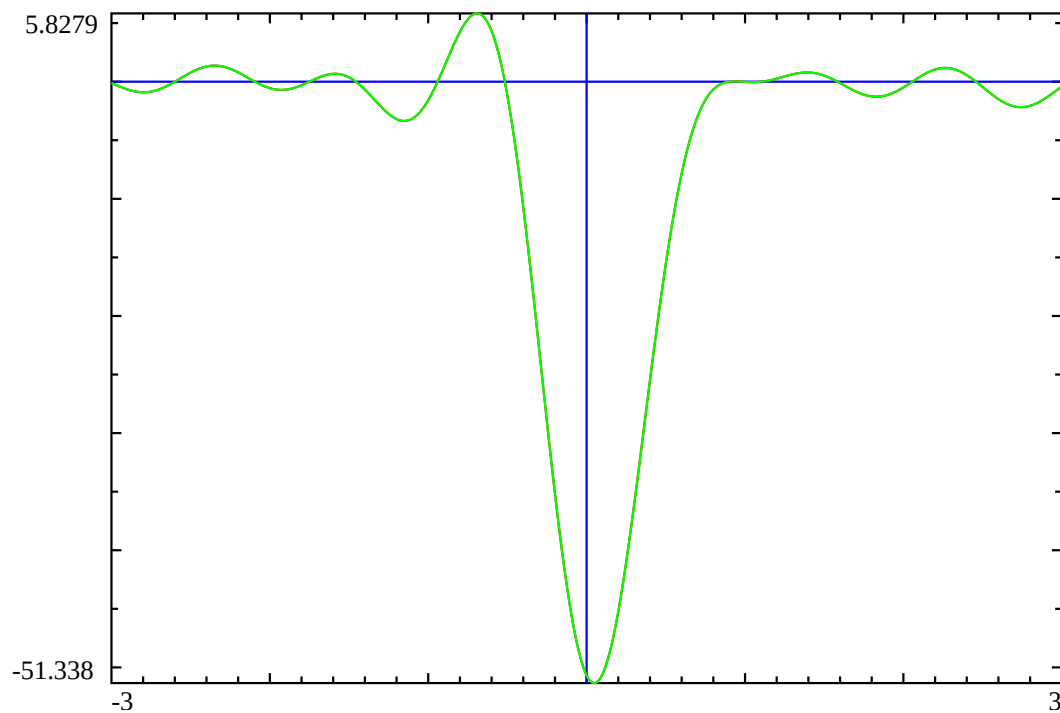


Figure 6. A comparison of the zeroth order Riemann Siegel Z function on the critical line in the interval $t=(6820048, 6820054)$ to a prime based zeroth order Riemann Siegel Z function using all primes up to $p=1039$. Technically 192 terms (primes and powers of primes 3 upwards containing products and finite geometric series) compared to 1041 terms in dirichlet series.

Proposed improvements to the algorithm for the alternate prime based expansion [7]

Examining the first principles algorithm of the alternate prime based expansion for $N=1041$ produces 192 series terms (down from 1041 terms for the finite Dirichlet series) which was laborious and subject to human error. Further reduction in the complexity, length and or speed of the algorithm can be achieved by noting (from the results in appendix B)

1. prime powers series expansion terms do not have a confirmed simplification but there are many fewer terms of this type in the series expansion.

For $N=1041$, there are 15 terms out of 192 that are prime powers where $p \geq 3$. (The prime powers of 2 do not need to be explicitly examined as they are naturally included in $a_2(s, N)$).

2. many prime terms in the expansion however do share the same multiplicative factor. For example, for primes greater than \sqrt{N}

$$a_{p>\sqrt{N}}(s, N) = \frac{1}{p^s} \cdot \left(\sum_{i=1}^{\lfloor \frac{N}{p} \rfloor} \frac{1}{i^s} \right) \text{ where } \sqrt{N} < p \leq N \quad (26)$$

where for N large many primes share the same $\lfloor \frac{N}{p} \rfloor$ value. The multiplicative factor $\sum_{i=1}^{\lfloor \frac{N}{p} \rfloor} \frac{1}{i^s}$ can itself also be expressed solely in terms of primes and prime powers using the alternate prime based expansion for $N' = \lfloor \frac{N}{p} \rfloor$. (Note the critical threshold $p > \sqrt{N}$ since below that threshold value any attempt at deriving a finite summation term like on the RHS of equation (26) would be missing terms like $1/p^s$).

The above improvements should reduce the number of repeated calculations and length of the code. For example, in practice for $N=1041$, out of 192 series terms, 78 terms for primes shared the multiplicative factor 1, 28 terms for primes shared the multiplicative factor $(1 + 1/2^s)$, 14 terms for primes shared the multiplicative factor $(1 + 1/2^s + 1/3^s)$ and these three simplifications were used in calculations in this paper.

As a second example, consider $N=164755715$ which using a Riemann Siegel formula (based on the $\sqrt{\frac{t}{2\pi}}$ first quiescent region) could be used for approximating the Riemann Zeta function in the interval

$$t = [N^2 \cdot 2 \cdot \pi, (N+1)^2 \cdot 2 \cdot \pi] = [170553581923548213.7840, 170553583993929595.5909]$$

Given $N=164755715$, a finite dirichlet series based Riemann Siegel formula would have $N=164755715$ terms for $\sum_{n=1}^N \frac{1}{n^s}$ and $\sum_{n=1}^N \frac{1}{n^{(1-s)}}$.

The finite geometric series prime and prime powers linear combination based series expansion would have for $N=164755715$

9226067 primes and 1700 prime powers for prime 3 upwards

Adopting the proposed improvements to the finite geometric series prime and prime powers linear combination based series expansion algorithm described in this section means

a critical threshold of $\sqrt{164755715} = 12835.72$ for primes with simple multiplicative factors as in equation (26)

therefore 9224537 prime terms $\frac{1}{p^s}$ would be collected into 6266 respective grouped terms with different $\lfloor \frac{N}{p} \rfloor$ values and hence different common multiplicative factors (as in equation (26)).

That leaves the ~ 1710 (lowest) primes and 1700 prime powers (spread throughout 1-N) for which the first principles finite geometric series prime and prime powers linear combination based series expansion terms needs to be derived.

Conclusions

The above procedure using products of finite geometric series of primes (and their powers) accurately calculates the finite dirichlet series for $N=1041$ applicable to Riemann Siegel calculation near the first Rosser rule violation.

The cost of the prime based exercise is generating the formula for a given N but the formula does apply over a interval on the imaginary axis. The speed of the calculation is slightly quicker than the finite Dirichlet series (given $N=1041$) but the code could be further improved by removing repeated calculations.

References

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Appendix A: Alternate expansion of finite Dirichlet Series based on finite geometric series of primes and powers of primes

The identified alternate expansion in terms of finite geometric series of primes is as follows

$$\sum_{n=1}^N \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots + \frac{1}{N^s} \quad (27)$$

$$= a_2(s, N) + a_3(s, N) + a_5(s, N) + a_7(s, N) + a_9(s, N) + \dots + a_{p \leq N}(s, N) \quad (28)$$

where

$$a_2(s, N) = 1 + \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, N\})}{\log(2)} \rfloor})}{(1 - \frac{1}{2^s})} \quad (29)$$

$$a_3(s, N) = \frac{1}{3^s} \cdot \left[\delta(N \geq 3) + \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{3}\})}{\log(2)} \rfloor})}{(1 - \frac{1}{2^s})} \right] \quad (30)$$

and more generally for prime powers of 3

$$a_{3^m}(s, N) = \left(\frac{1}{3^s} \right)^m \cdot \left[\delta(N \geq 3^m) + \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{3^m}\})}{\log(2)} \rfloor})}{(1 - \frac{1}{2^s})} \right] \quad (31)$$

$$\begin{aligned}
a_5(s, N) = \frac{1}{5^s} \cdot \left[\delta(N \geq 5) + \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{5}\})}{\log(2)} \rfloor})}{(1 - \frac{1}{2^s})} \right. \\
+ \frac{1}{3^s} \cdot \frac{(1 - (\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{5}\})}{\log(3)} \rfloor})}{(1 - \frac{1}{3^s})} \\
\left. + \sum_{q=1} \left(\frac{1}{3^s} \right)^q \cdot \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{3^q \cdot 5}\})}{\log(2)} \rfloor})}{(1 - \frac{1}{2^s})} \right] \quad (32)
\end{aligned}$$

and more generally for prime powers of 5

$$\begin{aligned}
a_{5^m}(s, N) = \left(\frac{1}{5^s} \right)^m \cdot \left[\delta(N \geq 5^m) + \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{5^m}\})}{\log(2)} \rfloor})}{(1 - \frac{1}{2^s})} \right. \\
+ \frac{1}{3^s} \cdot \frac{(1 - (\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{5^m}\})}{\log(3)} \rfloor})}{(1 - \frac{1}{3^s})} \\
\left. + \sum_{q=1} \left(\frac{1}{3^s} \right)^q \cdot \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{3^q \cdot 5^m}\})}{\log(2)} \rfloor})}{(1 - \frac{1}{2^s})} \right] \quad (33)
\end{aligned}$$

$$\begin{aligned}
a_7(s, N) = \frac{1}{7^s} \cdot \left[\delta(N \geq 7) + \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{7}\})}{\log(2)} \rfloor})}{(1 - \frac{1}{2^s})} + \frac{1}{3^s} \cdot \frac{(1 - (\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{7}\})}{\log(3)} \rfloor})}{(1 - \frac{1}{3^s})} \right. \\
+ \frac{1}{5^s} \cdot \frac{(1 - (\frac{1}{5^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{7}\})}{\log(5)} \rfloor})}{(1 - \frac{1}{5^s})} \\
+ \sum_{q=1} \left(\frac{1}{3^s} \right)^q \cdot \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{3^q \cdot 7}\})}{\log(2)} \rfloor})}{(1 - \frac{1}{2^s})} + \sum_{p=1} \left(\frac{1}{5^s} \right)^p \cdot \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{5^p \cdot 7}\})}{\log(2)} \rfloor})}{(1 - \frac{1}{2^s})} \\
+ \sum_{p=1} \left(\frac{1}{5^s} \right)^p \cdot \frac{1}{3^s} \cdot \frac{(1 - (\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{5^p \cdot 7}\})}{\log(3)} \rfloor})}{(1 - \frac{1}{3^s})} \\
\left. + \sum_{p=1} \sum_{q=1} \left(\frac{1}{5^s} \right)^p \cdot \left(\frac{1}{3^s} \right)^q \cdot \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{3^q \cdot 5^p \cdot 7}\})}{\log(2)} \rfloor})}{(1 - \frac{1}{2^s})} \right] \quad (34)
\end{aligned}$$

and more generally for prime powers of 7

$$\begin{aligned}
a_{7^m}(s, N) = & \left(\frac{1}{7^s}\right)^m \cdot \left[\delta(N \geq 7^m) + \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{7^m}\})}{\log(2)} \rfloor)}{(1 - \frac{1}{2^s})} + \frac{1}{3^s} \cdot \frac{(1 - (\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{7^m}\})}{\log(3)} \rfloor)}{(1 - \frac{1}{3^s})} \right. \\
& + \frac{1}{5^s} \cdot \frac{(1 - (\frac{1}{5^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{7^m}\})}{\log(5)} \rfloor)}{(1 - \frac{1}{5^s})} \\
& + \sum_{q=1} \left(\frac{1}{3^s}\right)^q \cdot \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{3^q \cdot 7^m}\})}{\log(2)} \rfloor)}{(1 - \frac{1}{2^s})} + \sum_{p=1} \left(\frac{1}{5^s}\right)^p \cdot \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{5^p \cdot 7^m}\})}{\log(2)} \rfloor)}{(1 - \frac{1}{2^s})} \\
& + \sum_{p=1} \left(\frac{1}{5^s}\right)^p \cdot \frac{1}{3^s} \cdot \frac{(1 - (\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{5^p \cdot 7^m}\})}{\log(3)} \rfloor)}{(1 - \frac{1}{3^s})} \\
& \left. + \sum_{p=1} \sum_{q=1} \left(\frac{1}{5^s}\right)^p \cdot \left(\frac{1}{3^s}\right)^q \cdot \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{3^q \cdot 5^p \cdot 7^m}\})}{\log(2)} \rfloor)}{(1 - \frac{1}{2^s})} \right] \quad (35)
\end{aligned}$$

and generally for prime powers of 11 including $m=1$

$$\begin{aligned}
a_{11^m}(s, N) = & \left(\frac{1}{11^s} \right)^m \cdot \left[\delta(N \geq 11^m) + \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{11^m}\})}{\log(2)} \rfloor)}{(1 - \frac{1}{2^s})} + \frac{1}{3^s} \cdot \frac{(1 - (\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{11^m}\})}{\log(3)} \rfloor)}{(1 - \frac{1}{3^s})} \right. \\
& + \frac{1}{5^s} \cdot \frac{(1 - (\frac{1}{5^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{11^m}\})}{\log(5)} \rfloor)}{(1 - \frac{1}{5^s})} + \frac{1}{7^s} \cdot \frac{(1 - (\frac{1}{7^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{11^m}\})}{\log(7)} \rfloor)}{(1 - \frac{1}{7^s})} \\
& + \sum_{q=1} \left(\frac{1}{3^s} \right)^q \cdot \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{3^q \cdot 11^m}\})}{\log(2)} \rfloor)}{(1 - \frac{1}{2^s})} + \sum_{q=1} \left(\frac{1}{5^s} \right)^q \cdot \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{5^q \cdot 11^m}\})}{\log(2)} \rfloor)}{(1 - \frac{1}{2^s})} \\
& + \sum_{q=1} \left(\frac{1}{7^s} \right)^q \cdot \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{7^q \cdot 11^m}\})}{\log(2)} \rfloor)}{(1 - \frac{1}{2^s})} \\
& + \sum_{q=1} \left(\frac{1}{5^s} \right)^q \cdot \frac{1}{3^s} \cdot \frac{(1 - (\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{5^q \cdot 11^m}\})}{\log(3)} \rfloor)}{(1 - \frac{1}{3^s})} + \sum_{q=1} \left(\frac{1}{7^s} \right)^q \cdot \frac{1}{3^s} \cdot \frac{(1 - (\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{7^q \cdot 11^m}\})}{\log(3)} \rfloor)}{(1 - \frac{1}{3^s})} \\
& + \sum_{q=1} \left(\frac{1}{7^s} \right)^q \cdot \frac{1}{5^s} \cdot \frac{(1 - (\frac{1}{5^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{7^q \cdot 11^m}\})}{\log(5)} \rfloor)}{(1 - \frac{1}{5^s})} \\
& + \sum_{p=1} \sum_{q=1} \left(\frac{1}{5^s} \right)^p \cdot \left(\frac{1}{3^s} \right)^q \cdot \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{3^q \cdot 5^p \cdot 11^m}\})}{\log(2)} \rfloor)}{(1 - \frac{1}{2^s})} \\
& + \sum_{p=1} \sum_{q=1} \left(\frac{1}{7^s} \right)^p \cdot \left(\frac{1}{3^s} \right)^q \cdot \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{3^q \cdot 7^p \cdot 11^m}\})}{\log(2)} \rfloor)}{(1 - \frac{1}{2^s})} \\
& + \sum_{p=1} \sum_{q=1} \left(\frac{1}{7^s} \right)^p \cdot \left(\frac{1}{5^s} \right)^q \cdot \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{5^q \cdot 7^p \cdot 11^m}\})}{\log(2)} \rfloor)}{(1 - \frac{1}{2^s})} \\
& + \sum_{p=1} \sum_{q=1} \left(\frac{1}{7^s} \right)^p \cdot \left(\frac{1}{5^s} \right)^q \cdot \frac{1}{3^s} \cdot \frac{(1 - (\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{5^q \cdot 7^p \cdot 11^m}\})}{\log(3)} \rfloor)}{(1 - \frac{1}{3^s})} \\
& + \sum_{p=1} \sum_{q=1} \sum_{r=1} \left(\frac{1}{7^s} \right)^p \cdot \left(\frac{1}{5^s} \right)^q \cdot \left(\frac{1}{3^s} \right)^r \cdot \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1, \frac{N}{3^r \cdot 5^q \cdot 7^p \cdot 11^m}\})}{\log(2)} \rfloor)}{(1 - \frac{1}{2^s})} \left. \right] \quad (36)
\end{aligned}$$

Appendix B: N=1041 example of alternate expansion of finite Dirichlet Series based on finite geometric series of primes and powers of primes

An verbose version of pari-gp code snippet with the functions defining the finite geometric based series expansion terms for N=1041. For larger primes there were opportunities for code efficiency by grouping together terms by the same multiplicative factor. Powers of primes had their own functions as occasionally some higher cross products are not present in the power of primes function having already being generated in a lower series expansion term. Some code bugs occurred and were fixed during the code writing but the final quality assurance did produce exact agreement between the first principles finite dirichlet series calculation and the finite geometric series linear combination based calculation.

As the same prime calculations repeatedly occurs in many series terms a further speedup in code performance is expected from pre-calculating (for a given $s = \sigma + I \cdot t$) these repeated components.

The printed code below contains line breaks and tabs to achieve prettier formatting. Such line breaks and tabs would need to be removed before using the code below in further calculations.

```
func2(t,res)=1+1/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(10))
func3(t,res)=1/3^(res+I*t)*(1+1/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(8)))
func5(t,res)=1/5^(res+I*t)*(1+1/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(7))
+1/3^(res+I*t)/(1-1/3^(res+I*t))*(1-1/3^(res+I*t))^(4))
+1/3^(res+I*t)/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(6))
+(1/3^(res+I*t))^2/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(4))
+(1/3^(res+I*t))^3/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(2))
+(1/3^(res+I*t))^4/2^(res+I*t))
func7(t,res)=1/7^(res+I*t)*(1+1/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(7))
+1/3^(res+I*t)/(1-1/3^(res+I*t))*(1-1/3^(res+I*t))^(4))
+1/5^(res+I*t)/(1-1/5^(res+I*t))*(1-1/5^(res+I*t))^(3))
+1/3^(res+I*t)/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(5))
+(1/3^(res+I*t))^2/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(4))
+(1/3^(res+I*t))^3/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(2))
+1/5^(res+I*t)/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(4))
+(1/5^(res+I*t))^2/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(2))
+1/5^(res+I*t)/3^(res+I*t)/(1-1/3^(res+I*t))*(1-1/3^(res+I*t))^(3))
+(1/5^(res+I*t))^2/3^(res+I*t))
+1/5^(res+I*t)/3^(res+I*t)/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(3))
+1/5^(res+I*t)/(3^(res+I*t))^2/2^(res+I*t))
func9(t,res)=+1/3^(res+I*t))^2*(1+1/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(6)))
func11(t,res)=+1/11^(res+I*t)*(1+1/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(6))
+1/3^(res+I*t)/(1-1/3^(res+I*t))*(1-1/3^(res+I*t))^(4))
+1/5^(res+I*t)/(1-1/5^(res+I*t))*(1-1/5^(res+I*t))^(2))
+1/7^(res+I*t)/(1-1/7^(res+I*t))*(1-1/7^(res+I*t))^(2))
+1/3^(res+I*t)/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(4))
+(1/3^(res+I*t))^2/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(3))
+(1/3^(res+I*t))^3/2^(res+I*t))
+1/5^(res+I*t)/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(4))
+(1/5^(res+I*t))^2/2^(res+I*t))
+1/7^(res+I*t)/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(3))
+1/5^(res+I*t)/3^(res+I*t)/(1-1/3^(res+I*t))*(1-1/3^(res+I*t))^(2))
+(1/5^(res+I*t))^2/3^(res+I*t))
+1/7^(res+I*t)/3^(res+I*t)/(1-1/3^(res+I*t))*(1-1/3^(res+I*t))^(2))
+1/7^(res+I*t)/5^(res+I*t))
+1/5^(res+I*t)/3^(res+I*t)/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(2))
+1/5^(res+I*t)/(3^(res+I*t))^2/2^(res+I*t))
+1/7^(res+I*t)/3^(res+I*t)/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(2))
+1/7^(res+I*t)/5^(res+I*t)/2^(res+I*t))
func13(t,res)=+1/13^(res+I*t)*(1+1/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(6))
+1/3^(res+I*t)/(1-1/3^(res+I*t))*(1-1/3^(res+I*t))^(3))
+1/5^(res+I*t)/(1-1/5^(res+I*t))*(1-1/5^(res+I*t))^(2))
+1/7^(res+I*t)/(1-1/7^(res+I*t))*(1-1/7^(res+I*t))^(2))
+1/11^(res+I*t))
+1/3^(res+I*t)/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(4))
+(1/3^(res+I*t))^2/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(3))
+(1/3^(res+I*t))^3/2^(res+I*t))
+1/5^(res+I*t)/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(4))
+(1/5^(res+I*t))^2/2^(res+I*t))
+1/7^(res+I*t)/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(3))
+1/11^(res+I*t)/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(2))
+1/5^(res+I*t)/3^(res+I*t)/(1-1/3^(res+I*t))*(1-1/3^(res+I*t))^(2))
+(1/5^(res+I*t))^2/3^(res+I*t))
+1/7^(res+I*t)/3^(res+I*t)/(1-1/3^(res+I*t))*(1-1/3^(res+I*t))^(2))
+1/11^(res+I*t)/3^(res+I*t))
+1/7^(res+I*t)/5^(res+I*t))
+1/11^(res+I*t)/5^(res+I*t))
```

15

17

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+1/5^(res+I*t)
+1/7^(res+I*t)
+1/11^(res+I*t)
+1/13^(res+I*t)
+1/3^(res+I*t)/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t))^(2))
+1/5^(res+I*t)/2^(res+I*t)
+1/7^(res+I*t)/2^(res+I*t)
func79(t,res)=+1/79^(res+I*t)*(1+1/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t))^(3))
+1/3^(res+I*t)/(1-1/3^(res+I*t))*(1-(1/3^(res+I*t))^(2))
+1/5^(res+I*t)
+1/7^(res+I*t)
+1/11^(res+I*t)
+1/13^(res+I*t)
+1/3^(res+I*t)/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t))^(2))
+1/5^(res+I*t)/2^(res+I*t)
func81(t,res)=(1/3^(res+I*t))^4*(1+1/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t))^(3)))
func83(t,res)=+1/83^(res+I*t)*(1+1/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t))^(3))
+1/3^(res+I*t)/(1-1/3^(res+I*t))*(1-(1/3^(res+I*t))^(2))
+1/5^(res+I*t)
+1/7^(res+I*t)
+1/11^(res+I*t)
+1/3^(res+I*t)/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t))^(2))
+1/5^(res+I*t)/2^(res+I*t)
func89(t,res)=+1/89^(res+I*t)*(1+1/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t))^(3))
+1/3^(res+I*t)/(1-1/3^(res+I*t))*(1-(1/3^(res+I*t))^(2))
+1/5^(res+I*t)
+1/7^(res+I*t)
+1/11^(res+I*t)
+1/3^(res+I*t)/2^(res+I*t)
+1/5^(res+I*t)/2^(res+I*t)
func97(t,res)=+1/97^(res+I*t)*(1+1/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t))^(3))
+1/3^(res+I*t)/(1-1/3^(res+I*t))*(1-(1/3^(res+I*t))^(2))
+1/5^(res+I*t)
+1/7^(res+I*t)
+1/3^(res+I*t)/2^(res+I*t)
+1/5^(res+I*t)/2^(res+I*t)
func101(t,res)=+1/101^(res+I*t)*(1+1/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t))^(3))
+1/3^(res+I*t)/(1-1/3^(res+I*t))*(1-(1/3^(res+I*t))^(2))
+1/5^(res+I*t)
+1/7^(res+I*t)
+1/3^(res+I*t)/2^(res+I*t)
+1/5^(res+I*t)/2^(res+I*t)
func103(t,res)=+1/103^(res+I*t)*(1+1/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t))^(3))
+1/3^(res+I*t)/(1-1/3^(res+I*t))*(1-(1/3^(res+I*t))^(2))
+1/5^(res+I*t)
+1/7^(res+I*t)
+1/3^(res+I*t)/2^(res+I*t)
+1/5^(res+I*t)/2^(res+I*t)
func107to113(t,res)=(+1/113^(res+I*t)+1/109^(res+I*t)+1/107^(res+I*t))*
(1+1/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t))^(3))
+1/3^(res+I*t)/(1-1/3^(res+I*t))*(1-(1/3^(res+I*t))^(2))
+1/5^(res+I*t)
+1/7^(res+I*t)
+1/3^(res+I*t)/2^(res+I*t)
func121(t,res)=(+1/11^(res+I*t))^2*(1+1/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t))^(3))
+1/3^(res+I*t)
+1/5^(res+I*t)
+1/7^(res+I*t)
+1/3^(res+I*t)/2^(res+I*t)
func125(t,res)=(+1/5^(res+I*t))^3*(1+1/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t))^(3))
+1/3^(res+I*t)+1/3^(res+I*t)/2^(res+I*t)
func127(t,res)=+1/127^(res+I*t)*(1+1/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t))^(3))
+1/3^(res+I*t)
+1/5^(res+I*t)
+1/7^(res+I*t)
+1/3^(res+I*t)/2^(res+I*t)
func131to139(t,res)=(+1/139^(res+I*t)+1/137^(res+I*t)+1/131^(res+I*t))
*(1+1/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t))^(2))
+1/3^(res+I*t)
+1/5^(res+I*t)
+1/7^(res+I*t)
+1/3^(res+I*t)/2^(res+I*t)
func149to173(t,res)=(+1/173^(res+I*t)+1/167^(res+I*t)+1/163^(res+I*t)
+1/157^(res+I*t)+1/151^(res+I*t)+1/149^(res+I*t))*
(1+1/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t))^(2))
+1/3^(res+I*t)
+1/5^(res+I*t)
+1/3^(res+I*t)/2^(res+I*t)
func169(t,res)=(+1/13^(res+I*t))^2*(1+1/2^(res+I*t)/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t))^(2))
+1/3^(res+I*t)
+1/5^(res+I*t)

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+1/3^(res+I*t)/2^(res+I*t))
func179to199(t,res)=(1/199^(res+I*t)+1/197^(res+I*t)+1/193^(res+I*t)
+1/191^(res+I*t)+1/181^(res+I*t)+1/179^(res+I*t))
*(1+1/2^(res+I*t))/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(2))
+1/3^(res+I*t)
+1/5^(res+I*t))
func211to257(t,res)=(1/211^(res+I*t)+1/223^(res+I*t)+1/227^(res+I*t)
+1/229^(res+I*t)+1/233^(res+I*t)+1/239^(res+I*t)+1/241^(res+I*t)+1/251^(res+I*t)+1/257^(res+I*t))
*(1+1/2^(res+I*t))/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(2))
+1/3^(res+I*t))
func243(t,res)=(1/3^(res+I*t))^5*(1+1/2^(res+I*t))/(1-1/2^(res+I*t))*(1-(1/2^(res+I*t)))^(2)))
func263to347(t,res)=(1/263^(res+I*t)+1/269^(res+I*t)+1/271^(res+I*t)+1/277^(res+I*t)+1/281^(res+I*t)
+1/283^(res+I*t)+1/293^(res+I*t)+1/307^(res+I*t)+1/311^(res+I*t)+1/313^(res+I*t)+1/317^(res+I*t)
+1/331^(res+I*t)+1/337^(res+I*t)+1/347^(res+I*t))
*(1+1/2^(res+I*t)+1/3^(res+I*t))
func289(t,res)=1/289^(res+I*t)*(1+1/2^(res+I*t)
+1/3^(res+I*t))
func343(t,res)=1/343^(res+I*t)*(1+1/2^(res+I*t)
+1/3^(res+I*t))
func349to509(t,res)=(1/349^(res+I*t)+1/353^(res+I*t)+1/359^(res+I*t)+1/367^(res+I*t)+1/373^(res+I*t)
+1/379^(res+I*t)+1/383^(res+I*t)+1/389^(res+I*t)+1/397^(res+I*t)+1/401^(res+I*t)+1/409^(res+I*t)
+1/419^(res+I*t)+1/421^(res+I*t)+1/431^(res+I*t)+1/433^(res+I*t)+1/439^(res+I*t)+1/443^(res+I*t)
+1/449^(res+I*t)+1/457^(res+I*t)+1/461^(res+I*t)+1/463^(res+I*t)+1/467^(res+I*t)+1/479^(res+I*t)
+1/487^(res+I*t)+1/491^(res+I*t)+1/499^(res+I*t)+1/503^(res+I*t)+1/509^(res+I*t))
*(1+1/2^(res+I*t))
func361(t,res)=+1/361^(res+I*t)*(1+1/2^(res+I*t))
func521to1039(t,res)=sum(i=98,175,1/prime(i)^(res+I*t))
func529(t,res)=1/529^(res+I*t)
func625(t,res)=1/625^(res+I*t)
func729(t,res)=1/729^(res+I*t)
func841(t,res)=1/841^(res+I*t)
func961(t,res)=1/961^(res+I*t)

```