# Exact functional dependence for the growth in the magnitude of the Riemann Zeta function within the critical strip, based on curve fitting at low Im(s) so valid only for low Im(s).

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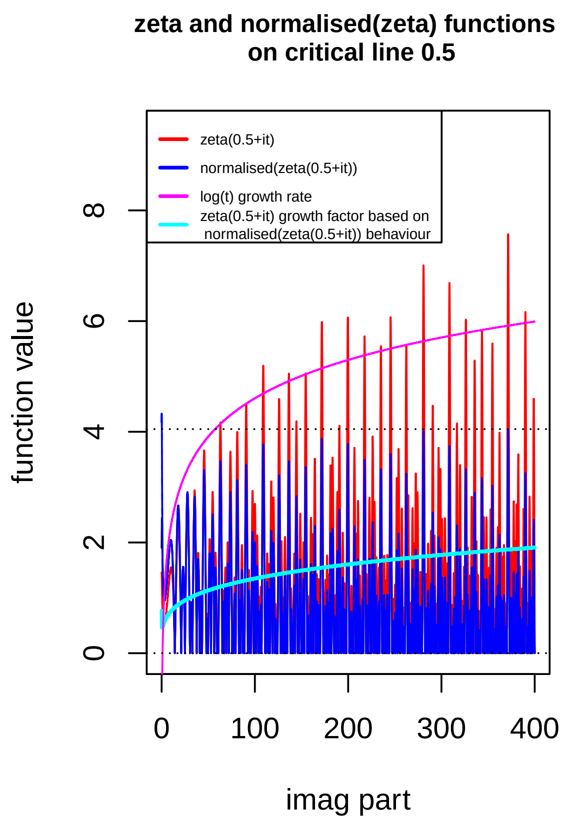
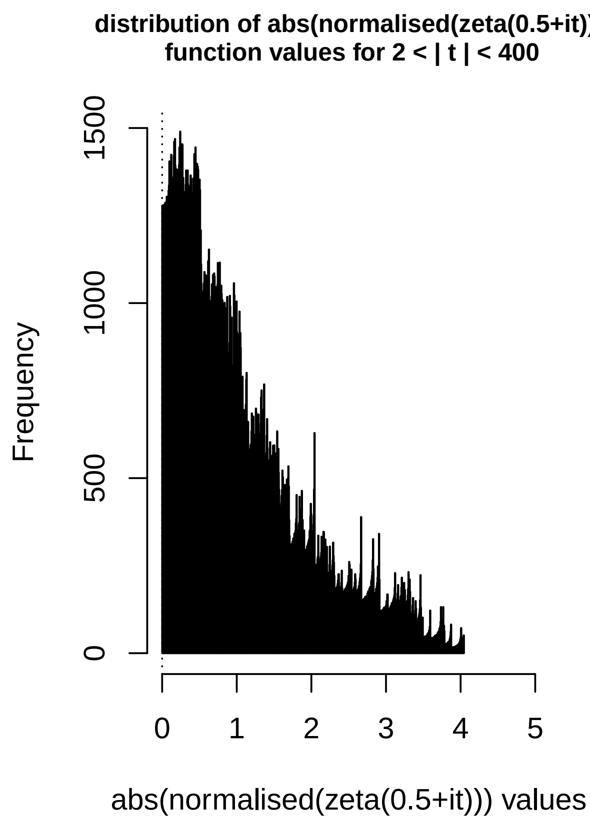
*created:07/01/2018, last modified: 10/01/2018*

## Executive Summary

The normalised Riemann Zeta function $\zeta_{norm}(s) = e^{(-abs(\Re(s)-\frac{1}{2})+abs(\Im(s))\frac{\pi}{4}+\frac{\gamma}{4\pi})}\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = e^{(-abs(\Re(s)-\frac{1}{2})+abs(\Im(s))\frac{\pi}{4}+\frac{\gamma}{4\pi})}\frac{\xi(s)}{\frac{1}{2}s(s-1)}$ for $0 \leq \Re(s) \leq 1$ has finite magnitude ($\lesssim (1+\frac{\gamma}{2})\pi$ for $\Re(s) = \frac{1}{2}$) as $\Im(s) \to \infty$. Using this function the growth of the magnitude in the Riemann Zeta function has the functional dependence $\zeta_{growthfactor}(s) = abs\left(e^{(-abs(\Re(s)-\frac{1}{2})+abs(\Im(s))\frac{\pi}{4}+\frac{\gamma}{4\pi})}\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\right)$ for $0 \leq \Re(s) \leq 1$, ie. within the critical strip. The result is valid for low $\Im(s) < 1000$ based on deviation in performance from $\mu(\frac{1}{2})$ behaviour for the current known Lindelof Hypothesis bound $t^{-0.158}|\zeta(0.5+i\Im(s))|$ which has $max(t^{-0.158}|\zeta(0.5+i\Im(s))|) \sim \pi$ for the largest peaks. In contrast, the curve fitted functional form for $\zeta_{norm}(s)$ given in this paper has $max(|\zeta_{norm}(s)|) \ll \pi, \Im(s) \to \infty$.



***Behaviour of the normalised Riemann Zeta function magnitude and the growth factor of the Riemann Zeta function on the critical line.***

## Introduction

On the critical line, in the limit $\Im(0.5+it) \to \infty$, the magnitude of the Riemann Zeta $\zeta(0.5+it) \to \infty$ and the Riemann Zeta $\xi(0.5+it) \to 0$ (rapidly) (1-3). These functions also share the non-trivial zeroes within the critical strip and obey similar functional equations. Starting from inspection of the extended Riemann Siegel function behaviour (4-6) of the $\xi$ function, simple multiplicative factor adjustments were successfully made to the $\xi$ functional equation to decrease the rapid damping along the imaginary axis such that $\zeta(1+it)$ was recreated (for $\Im(s) > 2\pi$) and the values for the other upper critical strip Re(s) values produced very similar Riemann Zeta function lineshapes but with deflated magnitudes.

In this paper, the normalised Riemann Zeta function $\zeta_{norm}(s)$ within the critical strip is defined, the behaviour is presented and an exact growth factor for the Riemann Zeta function obtained which is valid only for low $\Im(s)$.

### The extended Riemann Siegel functions and usage for Argument Principle calculations on the Riemann Zeta function

The Riemann Zeta function is defined (1), in the complex plane by the integral

$$\zeta(s) = \frac{\prod(-s)}{2\pi i} \int_{C_{\epsilon,\delta}} \frac{(-x)^s}{(e^x-1)x} dx \tag{1}$$

where $s \,\epsilon\, \mathbb{C}$ and $C_{\epsilon,\delta}$ is the contour about the imaginary poles.

The Riemann Zeta function has been shown to obey the functional equation (2)

$$\zeta(s) = 2^s \pi^{s-1} sin(\frac{\pi s}{2}) \Gamma(1-s) \zeta(1-s) \tag{2}$$

Following directly from the form of the functional equation and the properties of the coefficients on the RHS of eqn (2) it has been shown that any zeroes off the critical line would be paired, ie. if $\zeta(s) = 0$ was true then $\zeta(1-s) = 0$.

Along the critical line (0.5+it), the Riemann Siegel function is an exact function (3) for the magnitude of the Riemann Zeta function with two components Z(t) & $\theta(t)$

$$Z(t) = \zeta(0.5+it) e^{i\theta(t)} \tag{3}$$

and

$$\theta(t) = \Im(log(\Gamma(\frac{1}{4} + \frac{1}{2}it))) - \frac{t}{2} log(\pi) \tag{4}$$

In Martin (4-6) and earlier work, the properties of the Riemann Zeta generating function were investigated and used to develop/map the extended Riemann Siegel function $Z_{ext}(s)$ and $\theta_{ext}(s)$ definitions also applicable away from the critical line,

$$\theta_{ext}(s) = \Im(log(\sqrt{\frac{\zeta(1-s)abs(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))}{\zeta(s)}})) \tag{5}$$

$$= -\frac{1}{2}\Im(log((2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s)))) \tag{6}$$

$$Z_{ext}(s) = \sqrt{\zeta(s) * \zeta(1-s) * abs(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))} \tag{7}$$

The Riemann Zeta $\xi(s)$ function was specifically defined as

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \tag{8}$$

to (i) remove the pole at $s = 1$ and (ii) obey the simple functional equation

$$\xi(s) = \xi(1-s) \tag{9}$$

After inspecting the extended Riemann Siegel function behaviour of $\xi(s)$ using (4-6), further multiplicative factors were explicitly trialled to counteract the rapid damping of the $\xi(s)$ function magnitude.

This approach was fruitful and the following normalised Riemann Zeta function $\zeta_{norm}$ was thus defined

$$\zeta_{norm}(s) = e^{(-abs(\Re(s)-\frac{1}{2})+abs(\Im(s))\frac{\pi}{4}+\frac{\gamma}{4\pi})}\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \tag{10}$$

which obeys the following approximate functional equation due to asymmetry caused by the pole at s=1+0t.

$$abs(\zeta_{norm}(s)) \approx abs(\zeta_{norm}(1-s)) \quad \text{for } 0 \leq \Re(s) \leq 1,\, \Im(s) > 2\pi \tag{11}$$

As shown in the next section, given the agreement with $\zeta(1) = \zeta_{norm}(1)$ for $\Im(s) > 2\pi$ and sharing the same non-trivial zeroes, the exact growth factor of the Reimann Zeta function, within the critical strip, can be obtained as the ratio of the two functions

$$\zeta_{growthfactor}(s) = abs\Big(e^{(-abs(\Re(s)-\frac{1}{2})+abs(\Im(s))\frac{\pi}{4}+\frac{\gamma}{4\pi})}\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\Big) \quad \text{for } 0 \leq \Re(s) \leq 1 \tag{12}$$

where using the Stirling Series behaviour (7) for $\Gamma(\frac{s}{2})$ , as $\Im(s) \to \infty$

$$\begin{aligned}\zeta_{growthfactor}(s) = abs\Big(exp[-abs(\Re(s)-\frac{1}{2}) + abs(\Im(s))\frac{\pi}{4} + \frac{\gamma}{4\pi} - \frac{s}{2}log(\pi) \\ +\frac{1}{2}log(2\pi) + (\frac{s}{2}-\frac{1}{2})log(\frac{s}{2}) - \frac{s}{2} + \frac{1}{12(\frac{s}{2})} - ...]\Big) \quad \text{for } 0 \leq \Re(s) \leq 1\end{aligned} \tag{13}$$

The fascinating property of the growth factor eqn (13) is that for $\Im(s) \to \infty$

$$\Re\big(log(\Gamma(\frac{s}{2})\big) \sim -\Re\big(log(e^{(-abs(\Re(s)-\frac{1}{2})+abs(\Im(s))\frac{\pi}{4}+\frac{\gamma}{4\pi})}\pi^{-\frac{s}{2}})\big) \qquad \text{for } \Im(s) \to \infty \tag{14}$$

in series form

$$\Re\big(\frac{1}{2}log(2\pi) + (\frac{s}{2} - \frac{1}{2})log(\frac{s}{2}) - \frac{s}{2} + \frac{1}{12(\frac{s}{2})} - ...)\big) \sim -\Re\big( - abs(\Re(s) - \frac{1}{2}) + abs(\Im(s))\frac{\pi}{4} + \frac{\gamma}{4\pi} - \frac{s}{2}log(\pi)\big) \tag{15}$$

$$\text{for } \Im(s) \to \infty$$

and determines the slow growth of the Riemann Zeta function.

The Lindelof hypothesis (1) originally conjectured that the Riemann Zeta growth factor needs to be $t^{-0}|\zeta(0.5+i\Im(s))|$ for the Riemann Hypothesis to be confirmed so this exact growth factor needs to be checked against that hypothesis.

The latest known bounds for the Lindelof hypothesis (10) has $t^{-0.158}|\zeta(0.5 + i\Im(s))|$. In calculations, $\zeta_{growthfactor}(s)$ exceeds $1/t^{-0.158}$ well above $\Im(s) > 1000$ with the discerning characteristic being that $t^{-0.158}|\zeta(0.5 + i\Im(s))|$ maintains $max(t^{-0.158}|\zeta(0.5 + i\Im(s))|) \sim \pi$ for the largest peaks while eqn (10) has $max(|\zeta_{norm}(s)|) \ll \pi, \Im(s) \to \infty$ for the largest peaks.

### The behaviour of the normalised Riemann Zeta function across the critical strip

The following graphs show the close correspondence of the normalised Riemann Zeta function eqn (10) for $\Re(s) = 1, 0.75, 0.5, 0.25, 0$ respectively. In the lefthand figures, the functions $\zeta(s)$, $\zeta_{norm}(s)$, $log(\Im(s))$ and $\zeta_{growthfactor}(s)$ are displayed. In the righthand figure, the histogram of the $abs(\zeta_{norm}(s))$ is shown for $2 < \Im(s) < 200$. The calculations were perform using Julia language (8)

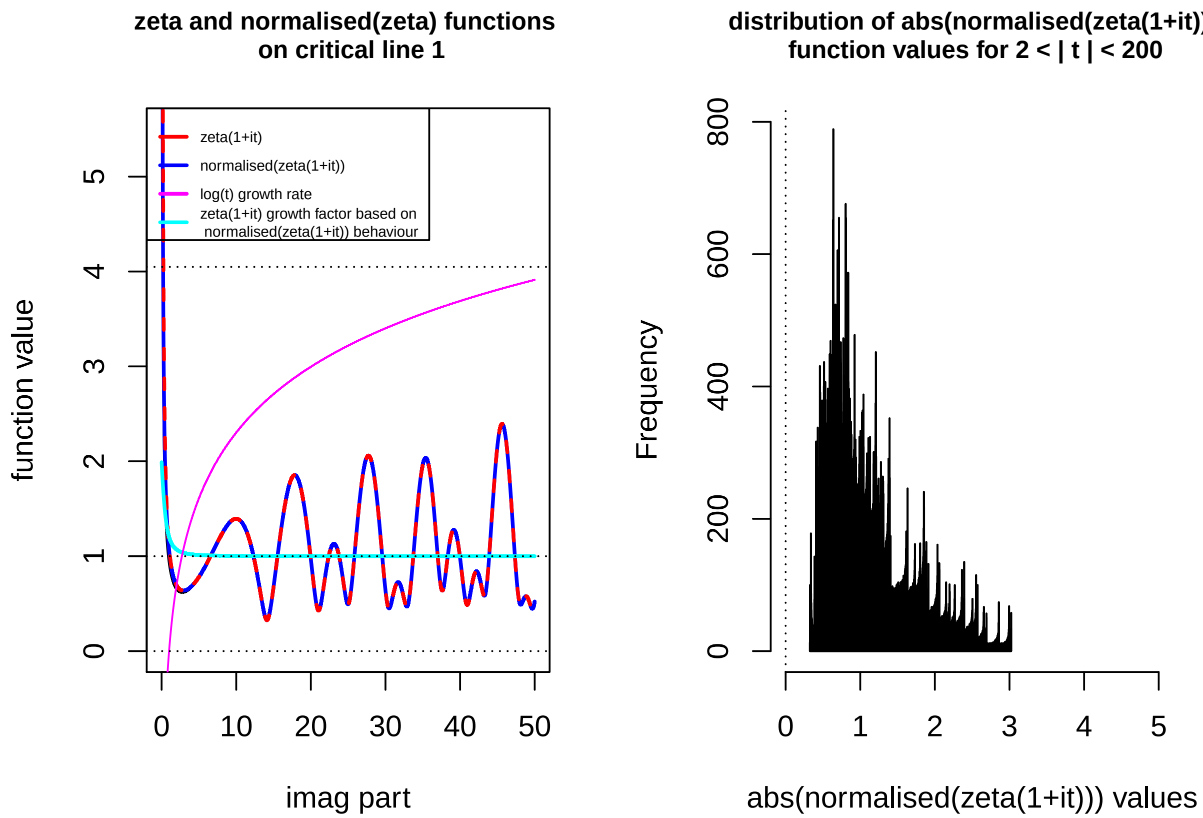
In figure 1, the normalised Riemann Zeta function agrees closely to the Riemann Zeta function except near the pole at s=1+0t. The growth factor for this line in the critical strip is 1 in agreement with previous results.

Comparing figures 1-5,

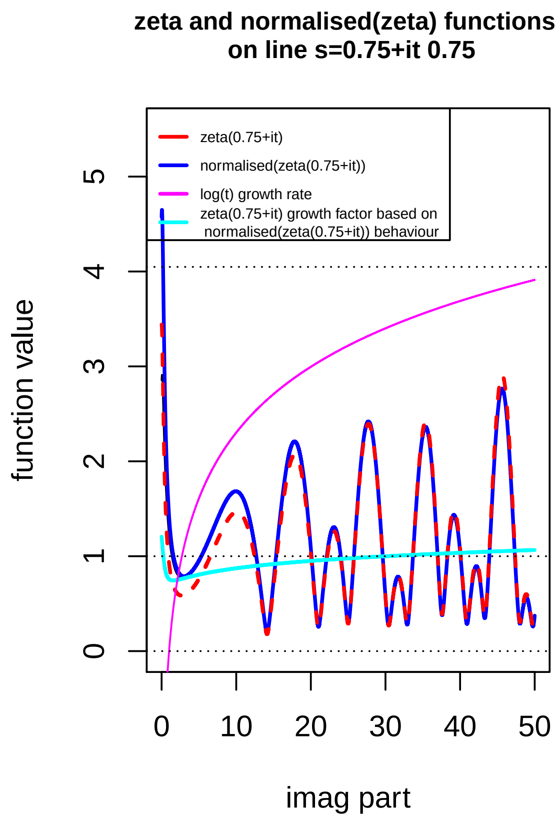
(i) the normalised Riemann Zeta function is symmetric in magnitude about the critical line,

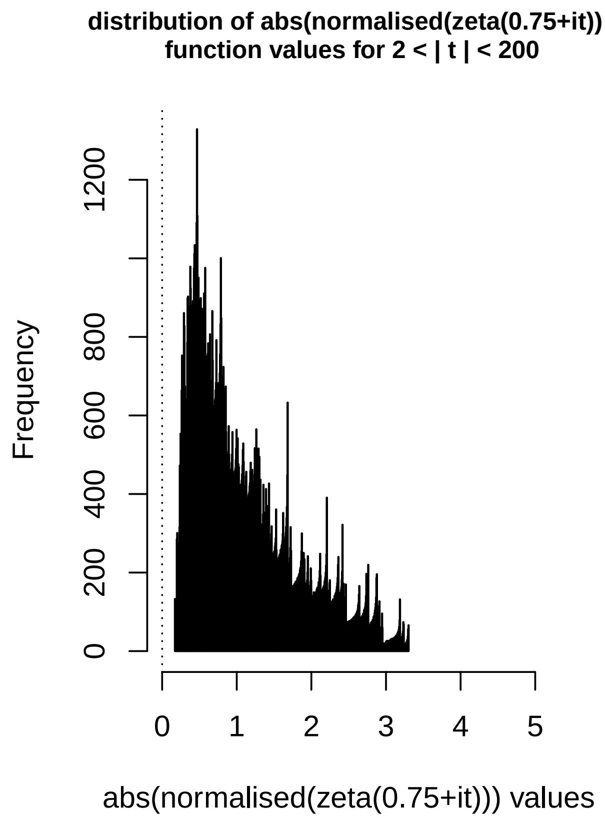
(ii) the largest spread of values in the normalised Riemann Zeta function (see righthand figures) occurs on the critical line and

(iii) the $\zeta_{growthfactor}(s)$ rapidly grows below the critical line.

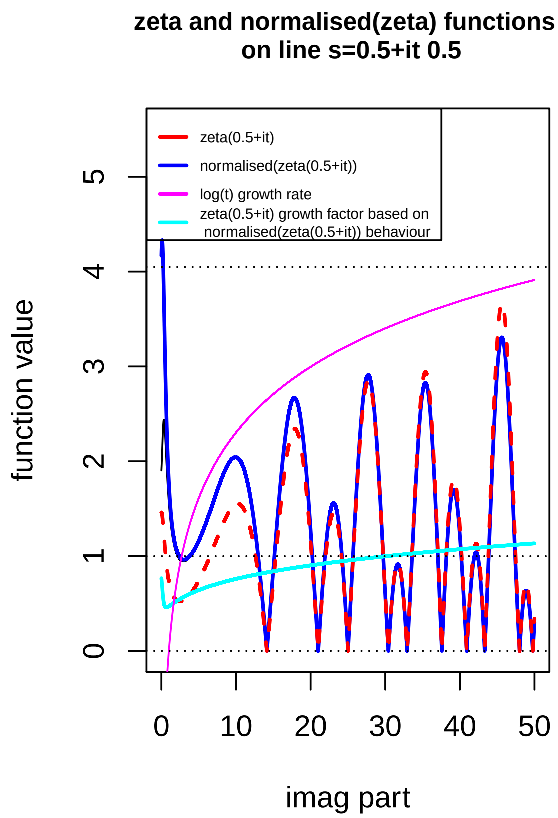


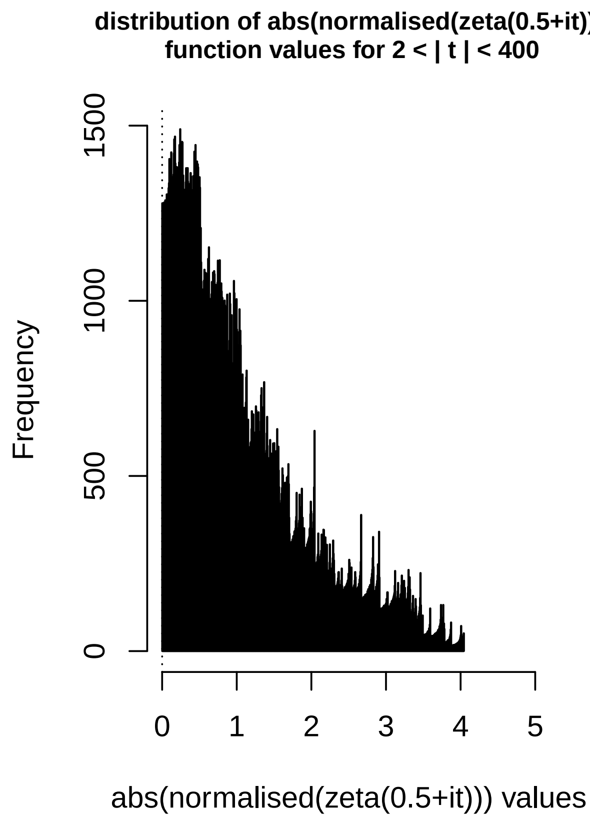

***Behaviour of the normalised Riemann Zeta function magnitude and the growth factor of the Riemann Zeta function on the line s=1+it***



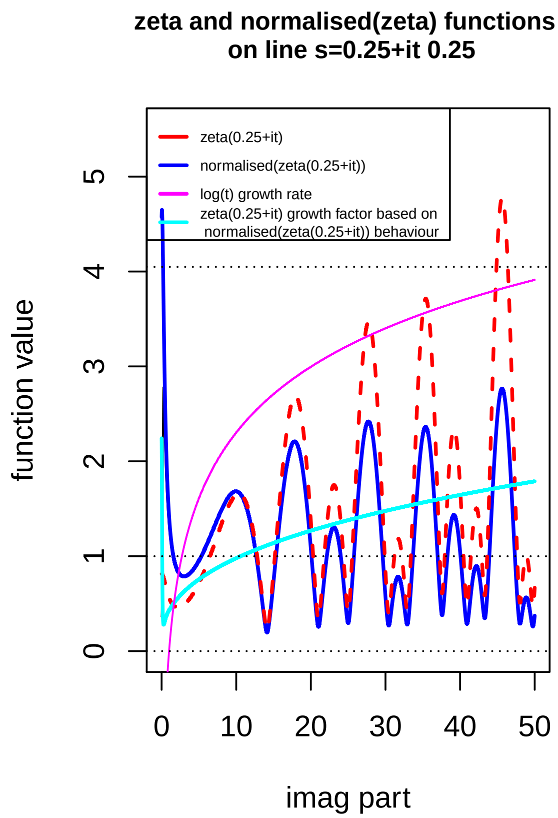



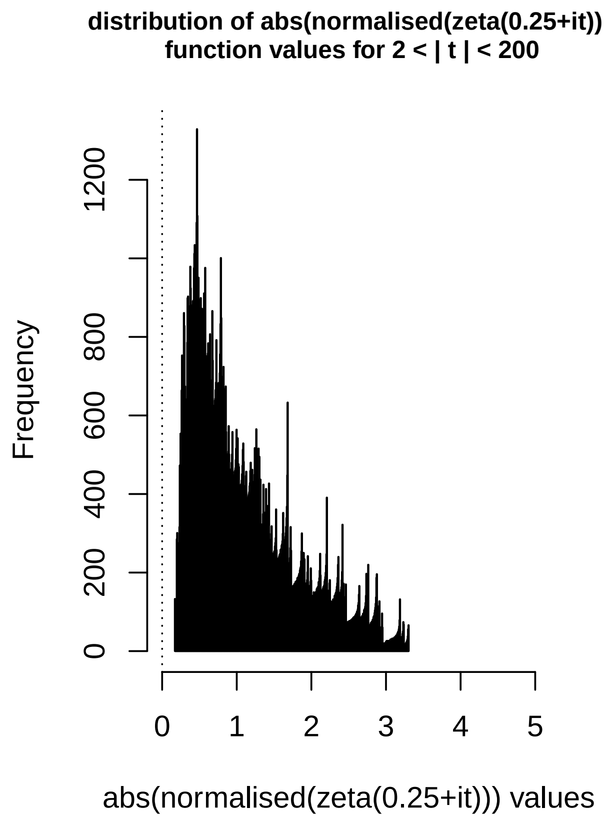

***Behaviour of the normalised Riemann Zeta function magnitude and the growth factor of the Riemann Zeta function on the line s=0.75+it***



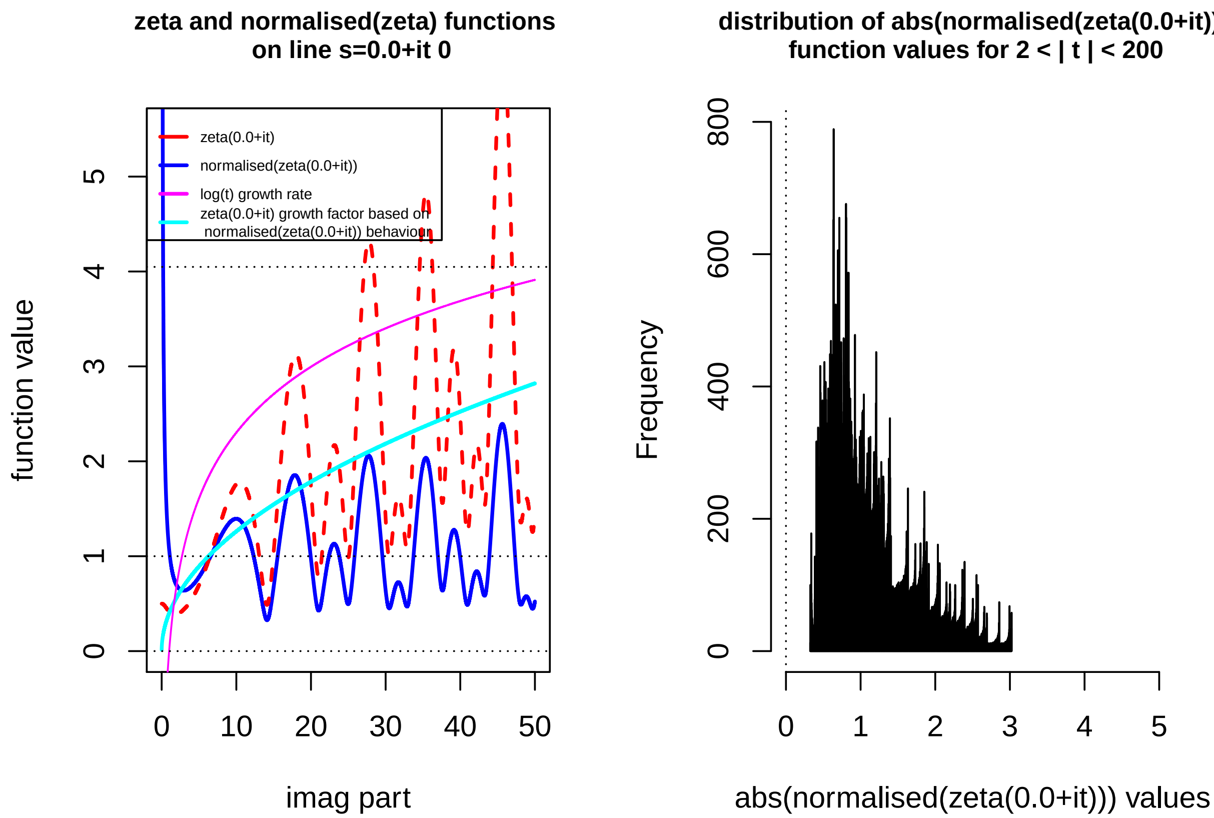




***Behaviour of the normalised Riemann Zeta function magnitude and the growth factor of the Riemann Zeta function on the line s=0.5+it***




***Behaviour of the normalised Riemann Zeta function magnitude and the growth factor of the Riemann Zeta function on the line s=0.25+it***




***Behaviour of the normalised Riemann Zeta function magnitude and the growth factor of the Riemann Zeta function on the line s=0.0+it***

Using a known list of $\zeta(s)$ maxima (9) and the series version of $\zeta_{growthfactor}(s)$ eqn (13), the $\zeta_{norm}(s)$ values remain consistent with the empirical bound $(1+\gamma/2)\pi$ in the figure 3 histogram for the critical line.

s=1/2+1i*363991205.178840, $\zeta(s) = 114.446$, $\zeta_{norm}(s) = 1.942389$

s=1/2+1i*18168214001.68199, $\zeta(s) = 190.043$, $\zeta_{norm}(s) = 1.213477$

s=1/2+1i*1102897584486.13647, $\zeta(s) = 265.440$, $\zeta_{norm}(s) = 0.60725$

s=1/2+1i*2430099556096.07812, $\zeta(s) = 322.062$, $\zeta_{norm}(s) = 0.60570$

s=1/2+1i*2445745756408.35596, $\zeta(s) = 297.435$, $\zeta_{norm}(s) = 0.55727$

## Conclusions

The normalised Riemann Zeta function shares the Riemann Zeta function poles and zeroes and has a finite magnitude behaviour within the critical strip as $\Im(s) \to \infty$ and can be used to derive an exact growth factor estimate for the magnitude of the Riemann Zeta function, valid for low $\Im(s)$.

Comparing to calculations using the latest known Lindelof Hypothesis bound results $t^{-0.158}|\zeta(0.5+i\Im(s))|$, a fully consistent $\zeta_{norm}(s)$ functional form for all large $\Im(s)$ within the critical strip, is expected to have the bounded behaviour $max(|\zeta_{norm}(s)|) \sim (\pi+\gamma/2), \Im(s) \to \infty$ for the largest peaks.

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## Appendix A: Basic Julia code used to calculate real and imaginary components of $\zeta$, $\xi$ and $\zeta_{norm}$

```
outfile = "outfile_hurwitz_zeta_upper_with_xi_0_5.dat"
f = open(outfile, "w")
res = 0.5
out = [[i/1000,real(zeta(res+i/1000*im)),imag(zeta(res+i/1000*im)),
real((0.5*((res+i/1000*im))*((res+i/1000*im)-1)*(pi^(-(res+i/1000*im)/2)*gamma((res+i/1000*im)/2)))*zeta(res+i/1000*im)),
imag((0.5*((res+i/1000*im))*((res+i/1000*im)-1)*(pi^(-(res+i/1000*im)/2)*gamma((res+i/1000*im)/2)))*zeta(res+i/1000*im)),
real((exp((-abs(res-1/2)+abs(i)/1000)*pi/4+.57721/pi/4)*(pi^(-(res+i/1000*im)/2)*gamma((res+i/1000*im)/2)))*zeta(res+i/1000*im)),
imag((exp((-abs(res-1/2)+abs(i)/1000)*pi/4+.57721/pi/4)*(pi^(-(res+i/1000*im)/2)*gamma((res+i/1000*im)/2)))*zeta(res+i/1000*im)),
real((exp((-abs(res-1/2)+abs(i)/1000)*pi/4+.57721/pi/4+log(pi)*(-(res+i/1000*im)/2)+(1/2*log(2*pi)+
((res+i/1000*im)/2-1/2)*log((res+i/1000*im)/2)-(res+i/1000*im)/2-1/12/(res+i/1000*im)*2-1/360/(res+i/1000*im)^3*8)))*zeta(res+i/1000*im)),
imag((exp((-abs(res-1/2)+abs(i)/1000)*pi/4+.57721/pi/4+log(pi)*(-(res+i/1000*im)/2)+(1/2*log(2*pi)+
((res+i/1000*im)/2-1/2)*log((res+i/1000*im)/2)-(res+i/1000*im)/2-1/12/(res+i/1000*im)*2-1/360/(res+i/1000*im)^3*8)))*zeta(res+i/1000*im)),i] for i in 1:400000]
for i in 1:400000; println(f,out[i,]);end;
close(f)

outfile = "outfile_hurwitz_zeta_lower_with_xi_0_5.dat"
f = open(outfile, "w")
out = [[i/1000,real(zeta(1-(res+i/1000*im))),imag(zeta(1-(res+i/1000*im))),
real((0.5*(1-(res+i/1000*im))*((1-(res+i/1000*im))-1)*(pi^(-(1-(res+i/1000)*im)/2)*gamma((1-(res+i/1000*im))/2)))*zeta(1-(res+i/1000*im))),
imag((0.5*(1-(res+i/1000*im))*((1-(res+i/1000*im))-1)*(pi^(-(1-(res+i/1000)*im)/2)*gamma((1-(res+i/1000*im))/2)))*zeta(1-(res+i/1000*im))),
real((exp((-abs(1-res-1/2)+abs(i)/1000)*pi/4+.57721/pi/4)*(pi^(-(1-(res+i/1000)*im)/2)*gamma((1-(res+i/1000*im))/2)))*zeta(1-(res+i/1000*im))),
imag((exp((-abs(1-res-1/2)+abs(i)/1000)*pi/4+.57721/pi/4)*(pi^(-(1-(res+i/1000)*im)/2)*gamma((1-(res+i/1000*im))/2)))*zeta(1-(res+i/1000*im))),
real((exp((-abs(res-1/2)+abs(i)/1000)*pi/4+.57721/pi/4+log(pi)*(-(1-(res+i/1000*im))/2)+(1/2*log(2*pi)+
((1-(res+i/1000*im))/2-1/2)*log((1-(res+i/1000*im))/2)-(1-(res+i/1000*im))/2-1/12/(1-(res+i/1000*im))*2-1/360/(1-(res+i/1000*im))^3*8)))*zeta(1-(res+i/1000*im))),
imag((exp((-abs(res-1/2)+abs(i)/1000)*pi/4+.57721/pi/4+log(pi)*(-(1-(res+i/1000*im))/2)+(1/2*log(2*pi)+
((1-(res+i/1000*im))/2-1/2)*log((1-(res+i/1000*im))/2)-(1-(res+i/1000*im))/2-1/12/(1-(res+i/1000*im))*2-1/360/(1-(res+i/1000*im))^3*8)))*zeta(1-(res+i/1000*im))),i] for i in 1:400000]
for i in 1:400000; println(f,out[i,]);end;
close(f)
```