A normalisation of the ordinary Dirichlet Series in the lower half complex plane that has the equivalent normalised Riemann Zeta function as an detrended envelope function.

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Executive Summary

In the lower half complex plane, $\Re(s) < 1$, a convergent normalisation of the ordinary dirichlet series $\sum_{n=1}^{\infty}(\frac{1}{n^s})$ on the real line is given by $\lim_{N\to\infty} \frac{1}{N^{(1-\Re(s))}} \sum_{n=1}^{N}(\frac{1}{n^{\Re(s)}}) = \frac{1}{1-\Re(s)}$. Elsewhere, across the lower half complex plane, the absolute value of the normalised series $|\frac{1}{N^{(1-\Re(s))}} \sum_{n=1}^{N}(\frac{1}{n^s})|$ has an upper (lower) detrended envelope function of the form $\pm|\frac{\zeta(s)}{N^{(1-\Re(s))}}|$, for $10000 < N < \infty$.

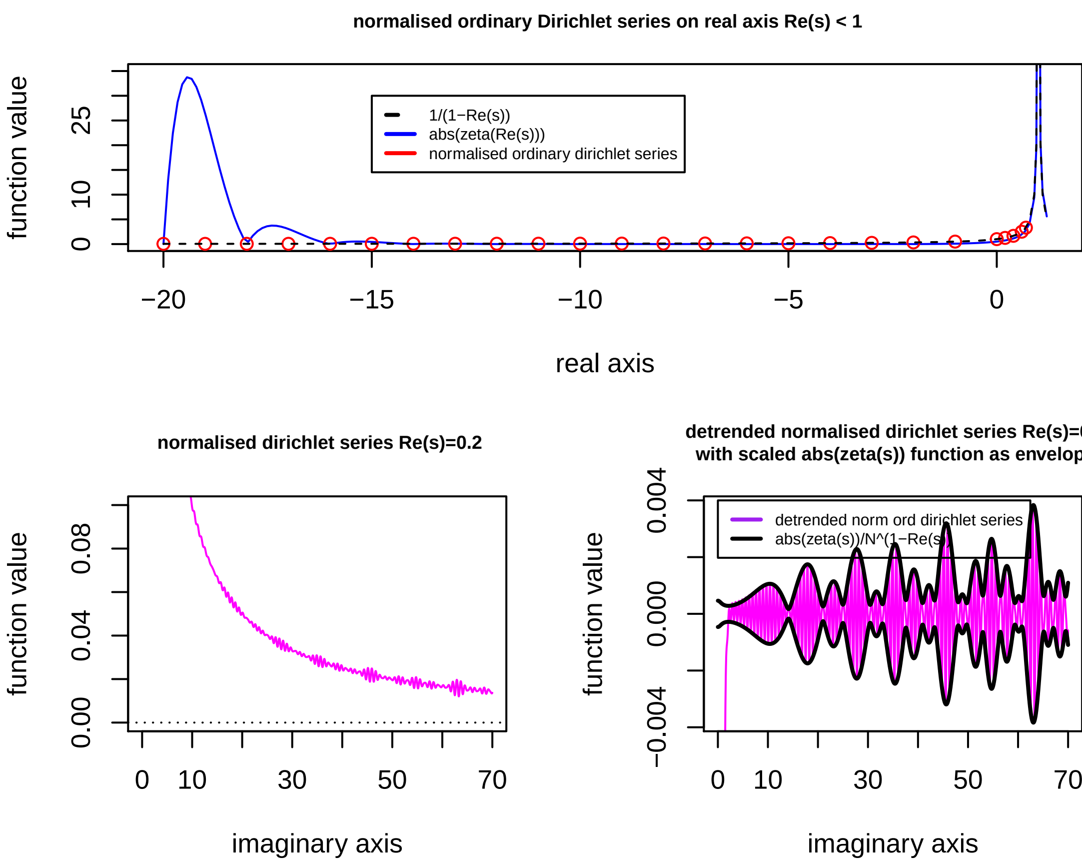


***Figure 1.* Normalised ordinary dirichlet series ($\Re(s) < 1$) along (i) real axis, (ii) imaginary line s=0.2+it and (iii) overlay of equivalent normalised Riemann Zeta function as an envelope function of the detrended series**

Introduction

The ordinary dirichlet series is given by

$$\mathfrak{D}_{id}^{\mathbb{N}} = \sum_{n=1}^{\infty}(\frac{1}{n^s}) \tag{1}$$

On the real positive axis $\Re(s) > 1$, the ordinary dirichlet series is equivalent to the Riemann Zeta function

$$\mathfrak{D}_{id}^{\mathbb{N}} \equiv \zeta(\Re(s)) \qquad \Re(s) > 1 \tag{2}$$

where the Riemann Zeta function is defined (1), in the complex plane by the integral

$$\zeta(s) = \frac{\prod(-s)}{2\pi i}\int_{C_{\epsilon,\delta}} \frac{(-x)^s}{(e^x - 1)x}dx \tag{3}$$

where $s \,\epsilon\, \mathbb{C}$ and $C_{\epsilon,\delta}$ is the contour about the imaginary poles.

The Riemann Zeta function has been shown to obey the functional equation (2)

$$\zeta(s) = \zeta(1-s) * (2^s \pi^{s-1} sin(\frac{\pi s}{2})\Gamma(1-s)) \tag{4}$$

and is convergent on the whole complex plane except for the pole at s=1. In contrast, on the lower half of the real axis $\Re(s) \leq 1$ the ordinary dirichlet series diverges.

$$\mathfrak{D}_{id}^{\mathbb{N}} = \sum_{n=1}^{\infty}(\frac{1}{n^s}) \rightarrow \infty \qquad \Re(s) \leq 1 \tag{5}$$

In this paper, (i) the strong role of the Riemann Zeta function as an detrended envelope function of the normalised $\mathfrak{D}_{id}^{\mathbb{N}}$ series is reported, confirming the applicability of use of the Riemann Zeta function for the analytic continuation of $\mathfrak{D}_{id}^{\mathbb{N}}$ in the lower half of the complex plane (except perhaps for the negative real axis itself) and (ii) a simple derivation of the leading terms of the sum of the logarithms of the positive integers is given using the simple limiting function for the normalised $\mathfrak{D}_{id}^{\mathbb{N}}$ series on the lower half real axis.

Continuation in the lower half complex plane using the normalised $\mathfrak{D}_{id}^{\mathbb{N}}$ series

A continuation of the $\mathfrak{D}_{id}^{\mathbb{N}}$ series can be constructed in the whole complex plane, using the function

$$\mathfrak{D}_{id}^{\mathbb{N}} = \begin{cases} \sum_{n=1}^{\infty}(\frac{1}{n^s}) & \Re(s) > 1 \\ \lim\limits_{N\rightarrow\infty} \frac{1}{N^{(1-\Re(s))}} \sum_{n=1}^{N}(\frac{1}{n^s}) & \Re(s) \leq 1 \end{cases} \tag{6}$$

In the strict limit, $N \rightarrow \infty$ on the real axis, it is proposed using the Laurent series of the ζ function, in the lower half complex plane

$$\lim_{N\rightarrow\infty} \frac{1}{N^{(1-\Re(s))}}[\sum_{n=1}^{N}(\frac{1}{n^s}) - \gamma - \sum_{n=1}^{N}(\gamma_n(1-\Re(s))^n)] = \frac{1}{1-\Re(s)} \qquad \Re(s) \leq 1, \Im(s) = 0 \tag{7}$$

For $10^4 < N < \infty$ and $\Re(s) < 1$, an analytic continuation in the lower half complex plane can be summarised, on the real axis as

$$\frac{1}{N^{(1-\Re(s))}}\sum_{n=1}^{N}(\frac{1}{n^s}) \to \frac{1}{1-\Re(s)} \qquad \Re(s)<1, \Im(s)=0 \tag{8}$$

Off the real axis, the absolute magnitude of the normalised series can be fit with an envelope function including an equivalent $\frac{1}{N^{(1-\Re(s))}}$ normalised version of the Riemann Zeta function

$$Envelope[|\frac{1}{N^{(1-\Re(s))}}\sum_{n=1}^{N}(\frac{1}{n^s})|] \approx trend \pm |\frac{\zeta(s)}{N^{(1-\Re(s))}}| \qquad \Re(s)\leq 1, \Im(s)\neq 0 \tag{9}$$

where for small values of $|\Re(s)|$, at least covering the Riemann Zeta function critical strip and near below, the trend component behaves as a nonlinear decaying version of the real axis normalised ordinary dirichlet series value given by eqn (8)

$$trend \sim [\frac{1}{N^{(1-\Re(s))}}\sum_{n=1}^{N}(\frac{1}{n^{\Re(s)}})] * [\frac{(1-\Re(s))}{\Im(s)} - \frac{(1-\Re(s))^3}{2\Im(s)^3} + \frac{3(1-\Re(s))^5}{8\Im(s)^5}] \qquad -1<\Re(s)<1 \tag{10}$$

Convergence of normalised $\frac{\mathfrak{D}_{id}^{\mathbb{N}}}{N^{(1-\Re(s))}}$ series on the real axis

Figure 2 illustrates a comparison of finite convergence results of the normalised $\frac{\mathfrak{D}_{id}^{\mathbb{N}}}{N^{(1-\Re(s))}}$ series for various values along the real axis above and below s=1 pole.

The s=1 case highlighted in red, with constant slope is not convergent.

The s=0 case, is trivally convergent

$$\lim_{N\to\infty}\frac{1}{N^{(1-\Re(0))}}\sum_{n=1}^{N}(\frac{1}{n^0}) = \frac{N}{N} = 1 \tag{11}$$

Figure 3, illustrates the striking $\frac{1}{1-\Re(s)}$ result given in eqn (7) for the normalised ordinary Dirichlet series, along the lower real axis $\mathbb{N}<1$, which appears to be a new series expansion.

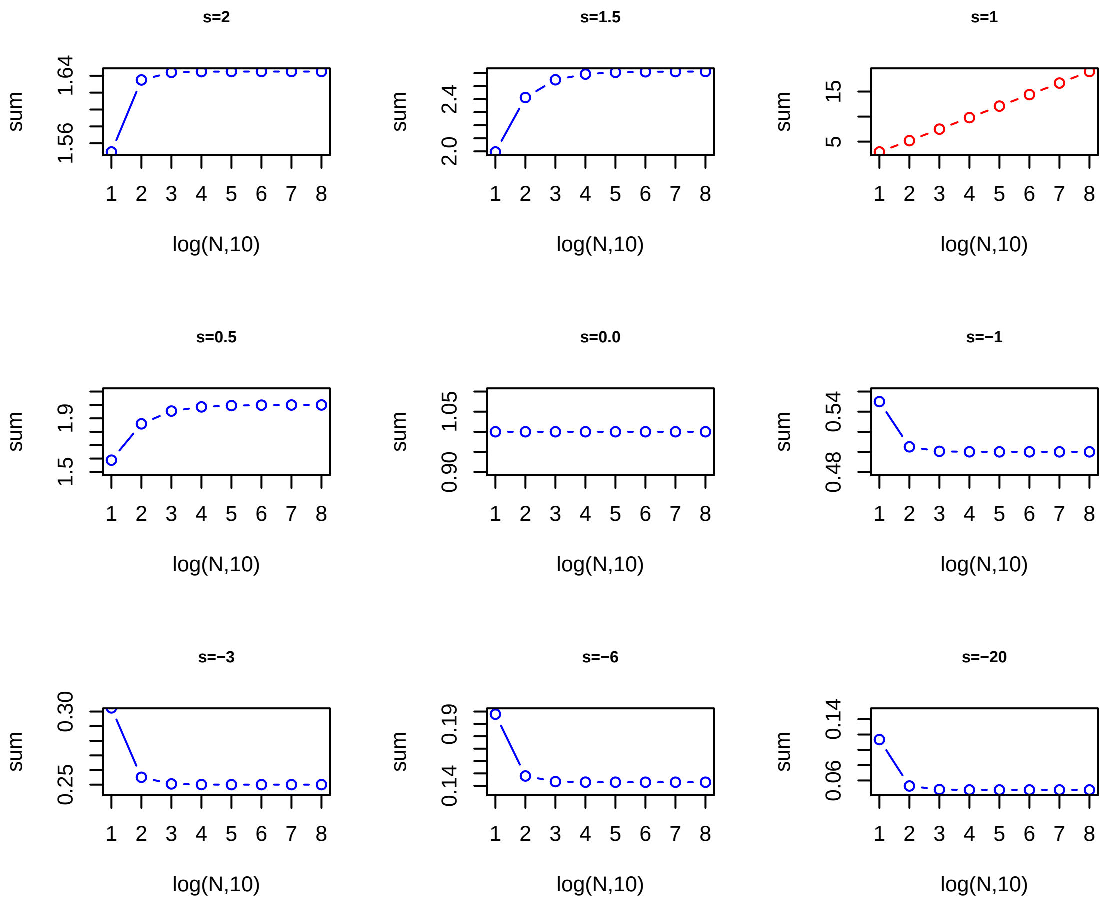


Figure 2. Convergence of normalised ordinary dirichlet series

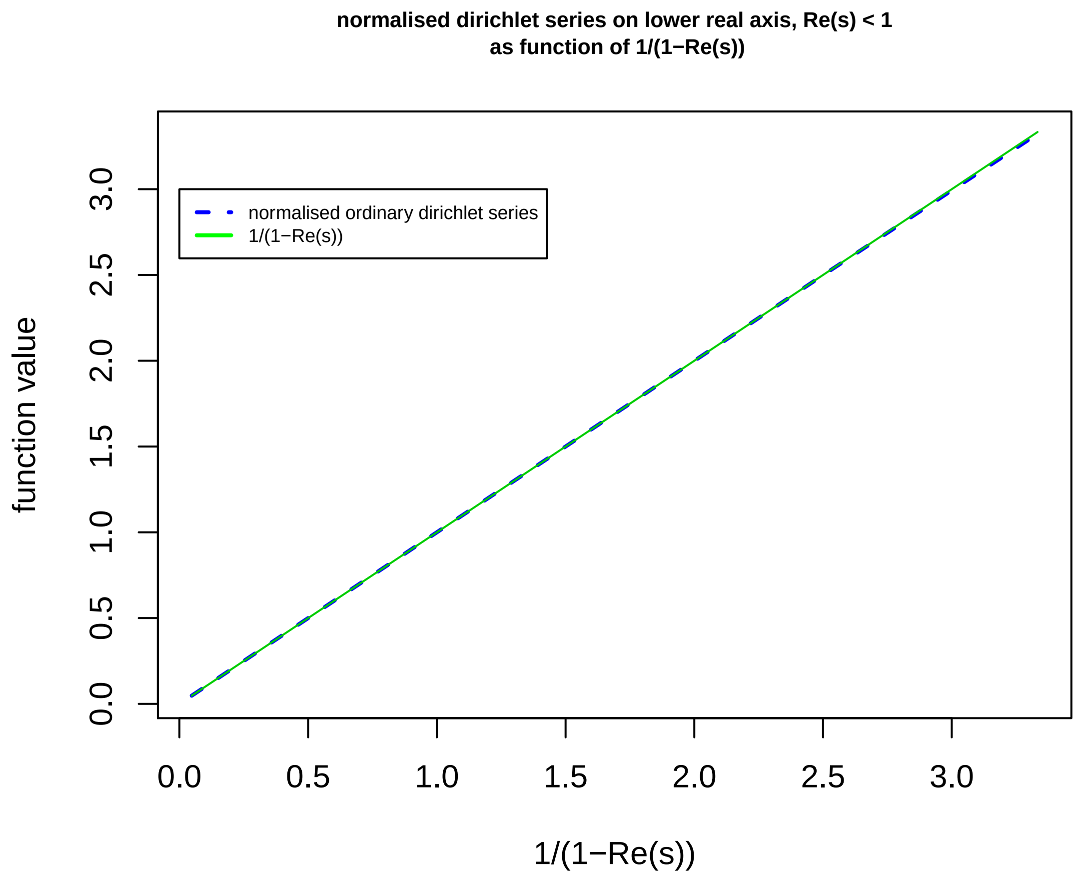
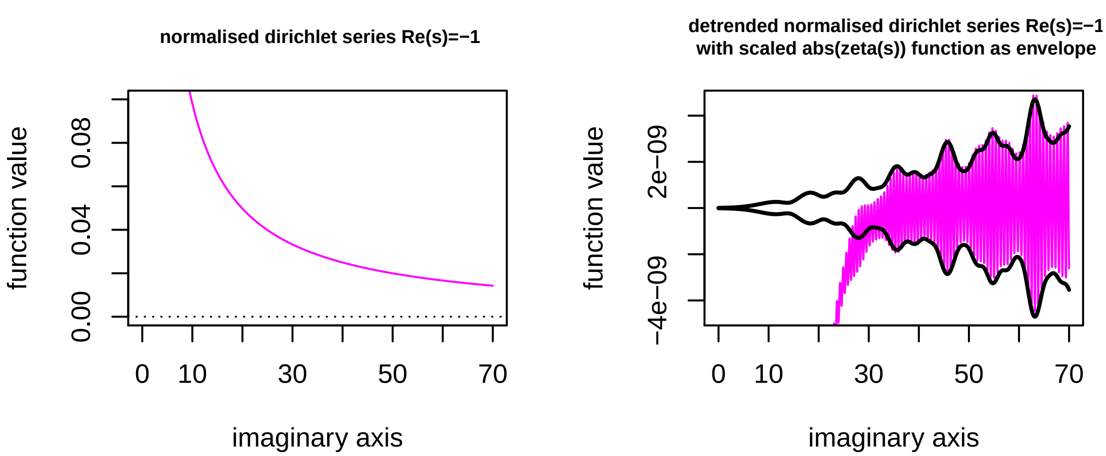
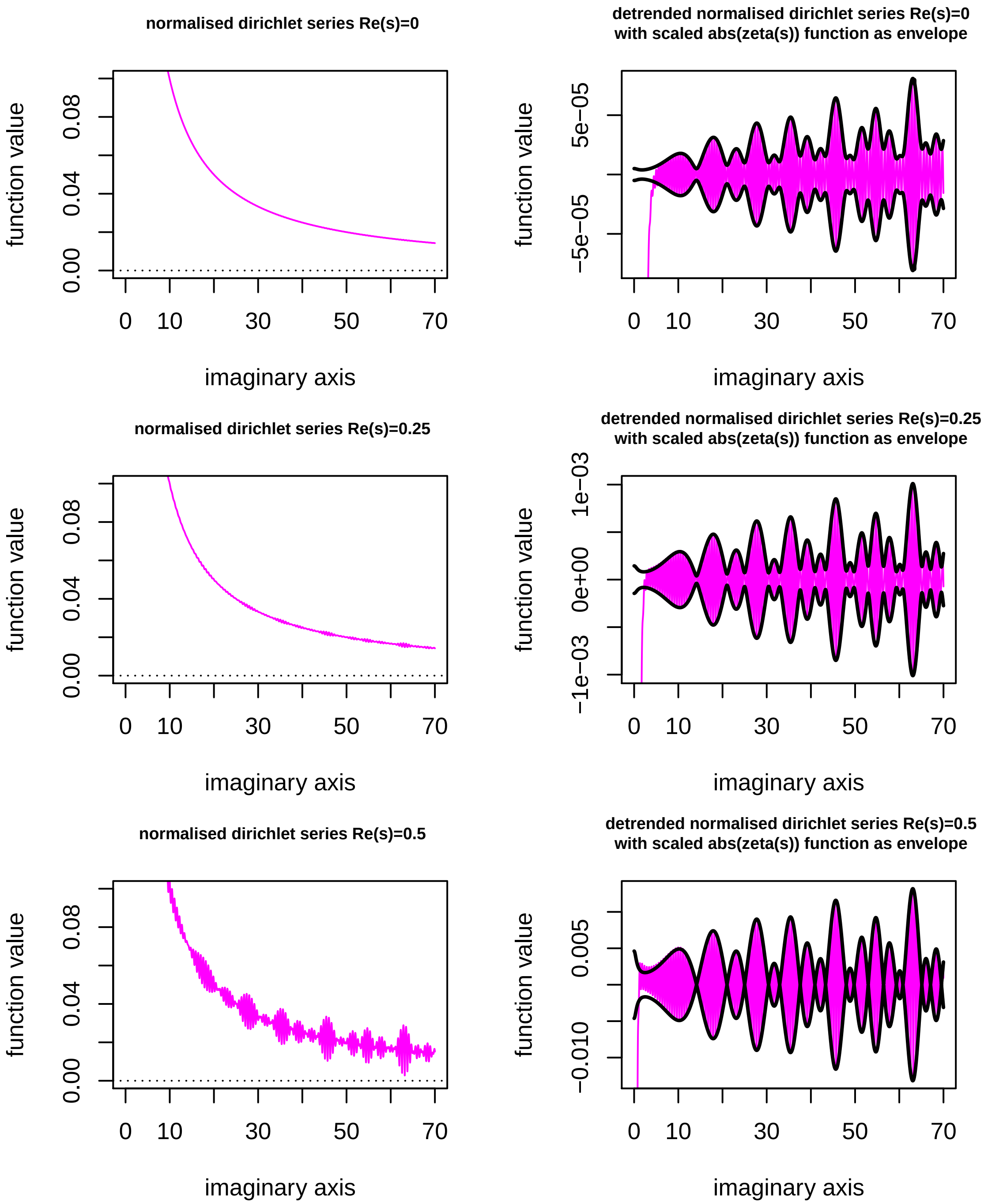


Figure 3. $\frac{1}{1-\Re(s)}$ ***dependence of normalised ordinary dirichlet series on real axis*** $\Re(s) < 1$

Lower half complex plane real axis behaviour of the normalised ordinary dirichlet series

Figure 4, illustrates the behaviour of magnitude of $|\frac{\mathfrak{D}^{\mathbb{N}}_{id}}{N^{(1-\Re(s))}}|$ series for various $\Re(s)$ values in the complex plane. The left hand figure, shows the $\frac{\mathfrak{D}^{\mathbb{N}}_{id}}{N^{(1-\Re(s))}}$ for N =100,000 which is comfortably accurate in estimating the real axis value (see figure 2). The right hand figure is the approximately detrended version of $|\frac{\mathfrak{D}^{\mathbb{N}}_{id}}{N^{(1-\Re(s))}}|$ overlayed by $\pm|\frac{\zeta(s)}{N^{(1-\Re(s))}}|$ as an envelope function. As $\Im(s) -> \infty$, the normalisation limit, N needs to rise rapdily to get accurate results for the symmetry of the detrended function.





normalised dirichlet series Re(s)=0
function value
0.00
0.04
0.08
0
10
30
50
70
imaginary axis
detrended normalised dirichlet series Re(s)=0
with scaled abs(zeta(s)) function as envelope
function value
−5e−05
5e−05
0
10
30
50
70
imaginary axis
normalised dirichlet series Re(s)=0.25
function value
0.00
0.04
0.08
0
10
30
50
70
imaginary axis
detrended normalised dirichlet series Re(s)=0.25
with scaled abs(zeta(s)) function as envelope
function value
−1e−03
0e+00
1e−03
0
10
30
50
70
imaginary axis
normalised dirichlet series Re(s)=0.5
function value
0.00
0.04
0.08
0
10
30
50
70
imaginary axis
detrended normalised dirichlet series Re(s)=0.5
with scaled abs(zeta(s)) function as envelope
function value
−0.010
0.005
0
10
30
50
70
imaginary axis

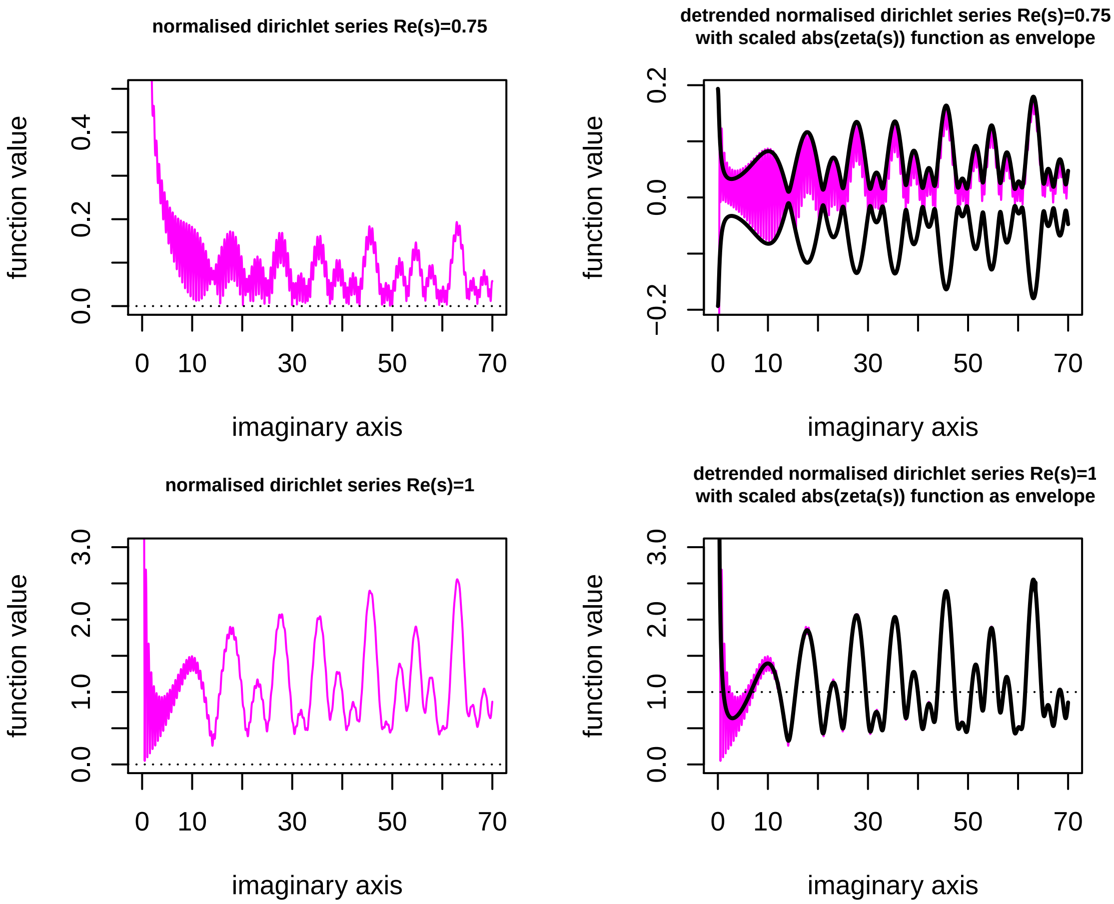


Figure 4. Lower half complex plane behaviour of the normalised ordinary dirichlet series and its detrended version with an overlay of the equivalent normalised Riemann Zeta function as an envelope function

It can be observed there is excellent correspondence with the normalised Riemann Zeta function as an envelope function gives strong direct confirmation of the Riemann Zeta analytical continuation. The discrepancy for values close to the real axis may be due to (i) inaccuracies in the detrending approximation eqn (10) and/or (ii) possible differences between applicability of Riemann Zeta function to the prime counting function for $\Im(s) < 2$ which could be related to the differences between $\zeta(\Re(s))$ and $|\frac{\mathfrak{D}^{\mathbb{N}}_{id}}{N^{(1-\Re(s))}}|$ in the critical strip.

An alternative derivation of the leading terms of the sum of logarithms of the positive integers

Using eqn (8), it is straightforward to derive the leading terms of the sum of the logarithm of the positive integers.

Firstly, the derivative of the LHS and RHS of eqn (8) are obtained on the lower half real axis

$$\frac{d}{d\Re(s)}\left(\frac{1}{N^{(1-\Re(s))}}\sum_{n=1}^{N}\left(\frac{1}{n^{\Re(s)}}\right)\right) = \frac{ln(N)}{N^{(1-\Re(s))}}\sum_{n=1}^{N}\left(\frac{1}{n^{\Re(s)}}\right) - \frac{1}{N^{(1-\Re(s))}}\sum_{n=1}^{N}\left(\frac{ln(n)}{n^{\Re(s)}}\right) \tag{12}$$

$$\frac{d}{d\Re(s)}\left(\frac{1}{1-\Re(s)}\right) = \frac{1}{(1-\Re(s))^2} \tag{13}$$

in the limit of $\mathbb{N} \to \infty$ for $\Im(s) = 0$ and $\Re(s) < 1$ therefore, equating the two derivatives

$$\frac{ln(N)}{N^{(1-\Re(s))}}\sum_{n=1}^{N}(\frac{1}{n^{\Re(s)}})-\frac{1}{N^{(1-\Re(s))}}\sum_{n=1}^{N}(\frac{ln(n)}{n^{\Re(s)}})\rightarrow\frac{1}{(1-\Re(s))^2}\qquad,\mathbb{N}\rightarrow\infty \tag{14}$$

For the particular value s=0, the equation simplifies to

$$ln(N)-\frac{1}{N}\sum_{n=1}^{N}ln(n)\rightarrow 1\qquad,\mathbb{N}\rightarrow\infty \tag{15}$$

which can be rearranged to give the leading terms of the sum of the logarithm of the positive integers

$$\sum_{n=1}^{N}ln(n)\rightarrow Nln(N)-N\qquad,\mathbb{N}\rightarrow\infty \tag{16}$$

which agrees with the leading terms, of the known Stirling formula and Euler-Maclaurin Sum Formula expansions.

Eqn (14) can be differentiated a second time, to yield

$$\begin{aligned}&ln(N)[\frac{ln(N)}{N^{(1-\Re(s))}}\sum_{n=1}^{N}(\frac{1}{n^{\Re(s)}})-\frac{1}{N^{(1-\Re(s))}}\sum_{n=1}^{N}(\frac{ln(n)}{n^{\Re(s)}})]\\&-[\frac{ln(N)}{N^{(1-\Re(s))}}\sum_{n=1}^{N}(\frac{ln(n)}{n^{\Re(s)}})-\frac{1}{N^{(1-\Re(s))}}\sum_{n=1}^{N}(\frac{ln(n)^2}{n^{\Re(s)}})]\rightarrow\frac{2}{(1-\Re(s))^3}\qquad,\mathbb{N}\rightarrow\infty\end{aligned} \tag{17}$$

which when simplified has the form

$$\frac{ln(N)^2}{N^{(1-\Re(s))}}\sum_{n=1}^{N}(\frac{1}{n^{\Re(s)}})-2\frac{ln(N)}{N^{(1-\Re(s))}}\sum_{n=1}^{N}(\frac{ln(n)}{n^{\Re(s)}})+\frac{1}{N^{(1-\Re(s))}}\sum_{n=1}^{N}(\frac{ln(n)^2}{n^{\Re(s)}})\rightarrow\frac{2}{(1-\Re(s))^3}\qquad,\mathbb{N}\rightarrow\infty \tag{18}$$

For the particular value s=0, eqn (18) simplifies to

$$ln(N)^2-2\frac{ln(N)}{N}\sum_{n=1}^{N}(ln(n))+\frac{1}{N}\sum_{n=1}^{N}(ln(n)^2)\rightarrow 2\qquad,\mathbb{N}\rightarrow\infty \tag{19}$$

which can be rearranged, using eqn (16) for $\sum_{n=1}^{N}(ln(n))$ to give the asymptotic leading terms of the sum of the square of the logarithms of the positive integers

$$\begin{aligned}\sum_{n=1}^{N}ln(n)^2&\rightarrow N(2-ln(N)^2+2\frac{ln(N)}{N}\sum_{n=1}^{N}(ln(n)))\\&\rightarrow N(2-ln(N)^2+2\frac{ln(N)}{N}(Nln(N)-N))\\&\rightarrow Nln(N)^2-2Nln(N)+2N\qquad,\mathbb{N}\rightarrow\infty\end{aligned} \tag{20}$$

This result agrees with the known terms of the indefinite integral of $\int ln(x)^2dx$ excluding the integration constant

General formulae for positive integer sums of $\sum_{n=1}^{N} \frac{ln(n)}{n^{\Re(s)}}$ and $\sum_{n=1}^{N} \frac{ln(n)^2}{n^{\Re(s)}}$, in the lower half real axis of $\Re(s)$

In the limit of $\mathbb{N} \to \infty$ for $\Im(s) = 0$ and $\Re(s) < 1$, where $\sigma \equiv \Re(s)$, using the first and second derivatives of eqn (8), ie. eqns (14) & (18), the following general formulae apply

$$\sum_{n=1}^{N} \frac{ln(n)}{n^{\sigma}} \to N^{(1-\sigma)}\left(\frac{ln(N)}{(1-\sigma)} - \frac{1}{(1-\sigma)^2}\right) \qquad ,\mathbb{N} \to \infty, \qquad \sigma < 1 \tag{21}$$

$$\sum_{n=1}^{N} \frac{ln(n)^2}{n^{\sigma}} \to N^{(1-\sigma)}\left(\frac{2}{(1-\sigma)^3} - \frac{ln(N)^2}{(1-\sigma)} + 2\frac{ln(N)}{N^{(1-\sigma)}}\sum_{n=1}^{N} \frac{ln(n)}{n^{\sigma}}\right) \tag{22}$$

$$\to N^{(1-\sigma)}\left(\frac{ln(N)^2}{(1-\sigma)} - 2\frac{ln(N)}{(1-\sigma)^2} + \frac{2}{(1-\sigma)^3}\right) \qquad ,\mathbb{N} \to \infty, \qquad \sigma < 1 \tag{23}$$

which saves a bit of effort, compared to integrating by parts, when deriving the equivalent looking results (excluding integration constant), for $\int \frac{ln(x)}{x^{\sigma}} dx$ and $\int \frac{ln(x)^2}{x^{\sigma}} dx$

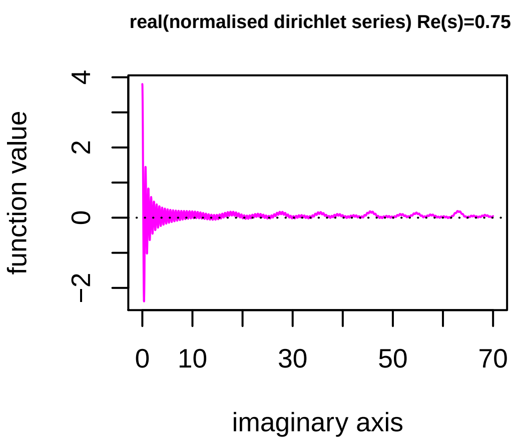
Behaviour of real and imaginary components of $\frac{\mathfrak{D}_{id}^{\mathbb{N}}}{N^{(1-\Re(s))}}$

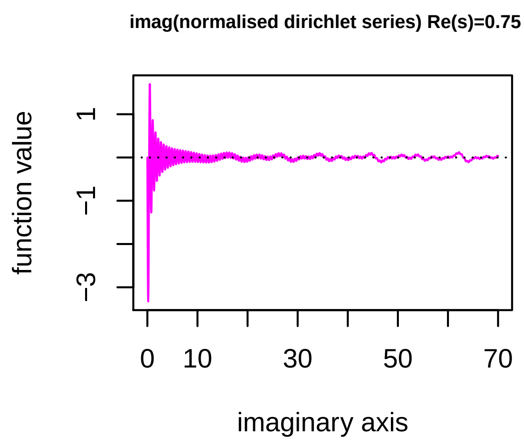
Given the above information, does it add any further insight into the Riemann Hypothesis (1,3). The following graphs, show the real and imaginary components of the absolute value function results discussed above.

From, figure 5, the first observation is the "ringing" behaviour of the lineshapes away from the real axis.

From figure 6, above the critical value $\Re(s) = 0.5$, the ringing is clearly asymmetric for the real component. Secondly, the small modulation on top of the ringing is the normalised Riemann Zeta function.

From figure 7, this modulation will reduce as $\mathbb{N} \to \infty$ the normalisation limit increases (in this case from 10ˆ5 to 10ˆ6 compared to figure 6).





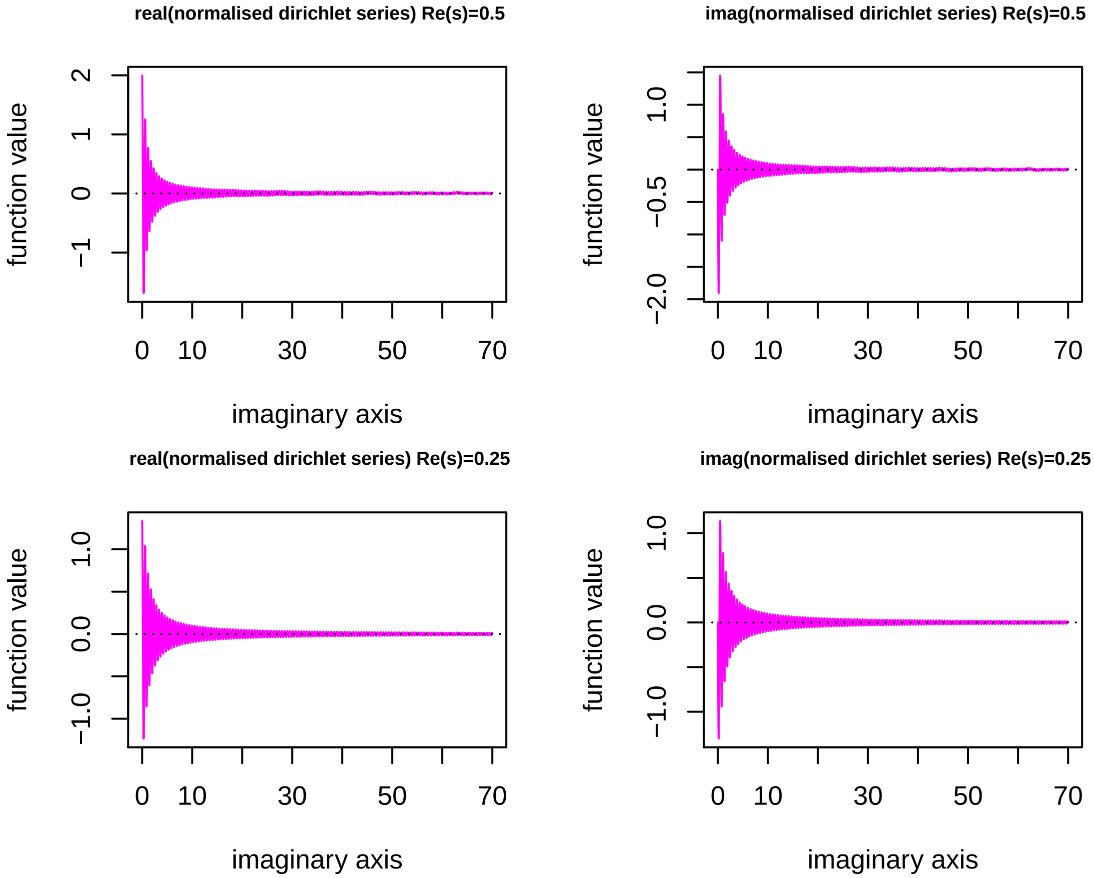
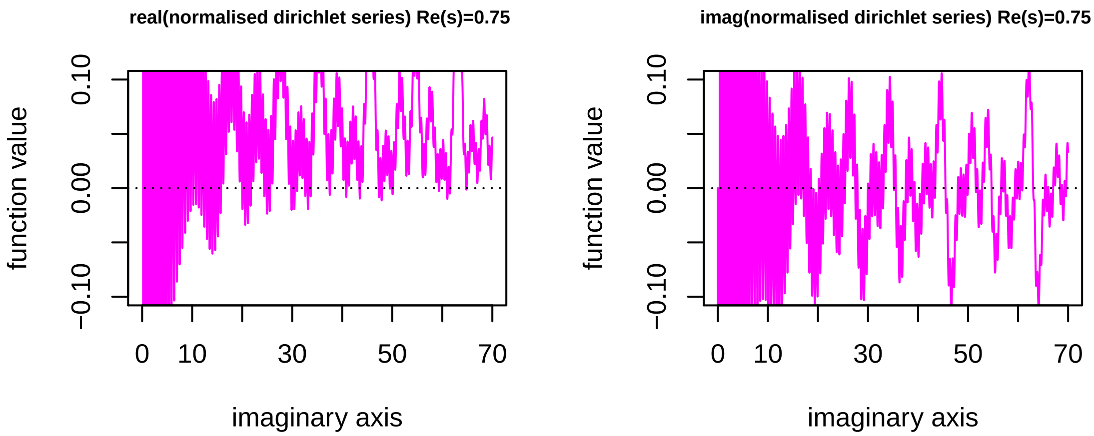


Fig.5:real and imaginary components of $|\frac{\mathfrak{D}_{id}^{\mathbb{N}}}{N^{(1-\Re(s))}}|$ ***at full scale,*** $\mathbb{N} = 10^5$



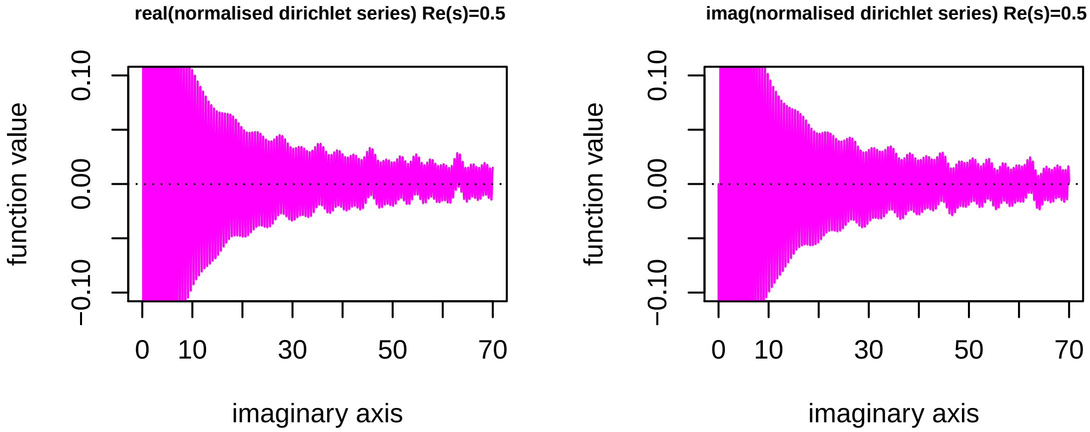
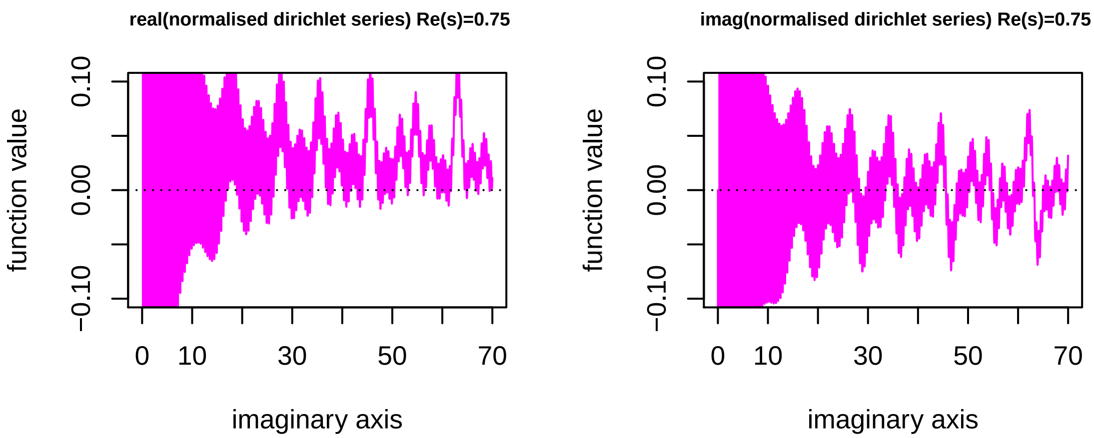


Fig.6:real and imaginary components of $\frac{\mathfrak{D}^{\mathbb{N}}_{id}}{N^{(1-\Re(s))}}$ ***at magnified scale,*** $\mathbb{N} = 10^5$



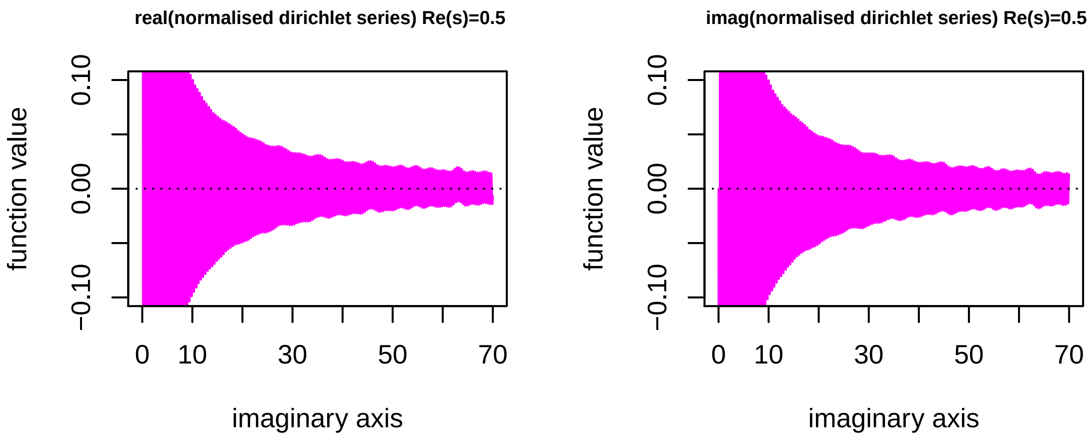


Fig.7:Effect of increasing $\mathbb{N}$ ***from (***$10^5 \to 10^6$***) on decreasing the Riemann Zeta envelope function, with the real and imaginary components of*** $\frac{\mathfrak{D}_{id}^{\mathbb{N}}}{N^{(1-\Re(s))}}$

Conclusions

The normalised Riemann Zeta function presenting as an envelope function for the normalised ordinary Dirichlet series $\frac{\mathfrak{D}_{id}^{\mathbb{N}}}{N^{(1-\Re(s))}}$ gives strong direct confirmation of the Riemann Zeta analytical continuation of $\mathfrak{D}_{id}^{\mathbb{N}}$ across the lower half complex plane.

The lower half plane real axis values of the normalised ordinary Dirichlet series gives (i) an interesting series expansion which may have other potential uses and (ii) an alternative regularisation of divergent sums to zeta function regularization.

References

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