Examining the convergence behaviour of the finite Dirichlet Series and zeroth order Riemann Siegel formula calculated via the sum of products of finite geometric series of primes and powers of primes near the first Rosser point violation

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DRAFT Executive summary

A linear combination of products of finite geometric series of primes and power of primes exhibits excellent performance in reproducing the finite Riemann Zeta Dirichlet Series and zeroth order Riemann Siegel formula results for N=1041 corresponding to the interval containing the first Rosser point violation. It can be observed that the first sqrt(N) of the primes (and their cross products and powers) contribute the majority of the Riemann Zeta large peak at s=0.5+I*6820051. However, all primes up to the first quiescent region $N=\sqrt{\frac{t}{2\pi}}$ are required to obtain accurate zeroth order approximations of the zero locations for closely spaced zeroes.

Introduction

For $\Re(s)>1$, the infinite Euler Product of the primes absolutely converges to the infinite Riemann Zeta Dirichlet Series sum [1,2]

$$\zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k^s}=\prod_{\rho=2}^{\infty}\frac{1}{(1-1/\rho^s)} \qquad \text{for } \Re(s)>1 \tag{1}$$

Importantly, using the log(1-x) expansion of $\log(\zeta(s))$ [3-5] the Euler product also has the form

$$\prod_{\rho=2}^{\infty}\frac{1}{(1-1/\rho^s)}=\exp\left(\sum_{\rho=2}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n\cdot\rho^{ns}}\right) \tag{2}$$

For $\Re(s)\leq 1$, the partial Euler Product diverges, however, using the above equations for finite sums (products) of integers (primes) the following relationship holds

$$
\begin{aligned}
\sum_{k=1}^{N}\frac{1}{k^s} = 1 &+ \left(\sum_{\rho=2}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n\cdot\rho^{ns}}\cdot\delta(\rho^n\leq N)\right)\\
&+\frac{1}{2!}\left(\sum_{\rho_1=2}^{\infty}\sum_{n=1}^{\infty}\sum_{\rho_2=2}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n\cdot\rho_1^{ns}}\cdot\frac{1}{m\cdot\rho_2^{ms}}\cdot\delta(\rho_1^n\cdot\rho_2^m\leq N)\right)\\
&+\frac{1}{3!}\left(\sum_{\rho_1=2}^{\infty}\sum_{n=1}^{\infty}\sum_{\rho_2=2}^{\infty}\sum_{m=1}^{\infty}\sum_{\rho_3=2}^{\infty}\sum_{o=1}^{\infty}\frac{1}{n\cdot\rho_1^{ns}}\cdot\frac{1}{m\cdot\rho_2^{ms}}\cdot\frac{1}{o\cdot\rho_3^{os}}\cdot\delta(\rho_1^n\cdot\rho_2^m\cdot\rho_3^o\leq N)\right)\\
&+\quad\ldots
\end{aligned}
\tag{3}
$$

where the delta functions play a crucial role in appropriately truncating the Euler Product terms. Hence the above expression can be used with the $N\sim\lfloor\frac{t}{\pi}\rfloor$ and $(N\sim\lfloor\sqrt{\frac{t}{2\pi}}\rfloor)$ quiescent regions of the oscillatory divergence of the Riemann Zeta function to obtain useful partial Euler Product based approximations of the Riemann Zeta function in the critical strip (and below) [6].

In [6] empirical calculations showed that the truncated exponential series version of the finite Euler product is a slower running algorithm at the complex plane points presented compared to the simple Dirichlet Series. This is due to the extra multiplication operations and truncation checks that are required at each higher order term of the power series calculation.

In [7], an alternative series expansion for the truncated euler product was given in terms of finite geometric series of primes and powers of primes in the hope of identifying a faster way to calculate a finite Dirichlet Series using only primes. The early terms of the alternative series expansion are shown in the appendix.

In this paper the alternate prime based expansion [7], is used to calculate the finite Dirichlet Series and zeroth order Riemann Siegel formula for the first quiescent region $N=\sqrt{\frac{t}{2\pi}}$ using only a linear combination of finite geometric series of primes and powers of primes near the first Rosser rule violation (gram point 13999525, near by the Riemann Zeta large peak t=6820051).

Convergence of the cumulative sum of the alternate prime based expansion [7] compared to the first principle Dirichlet Series (at the first quiescent region)

The cumulative series sum behaviour of the real part of (i) the finite Riemann Zeta Dirichlet Series sum (green) and (ii) the alternate finite geometric series based linear combination sum (red). For t=6820051, the first quiescent region is at $N=\lfloor\sqrt{\frac{6820051}{2\pi}}\rfloor=1041$. Shown in grey is the Dirichlet Series sum $\sum_{n=1}^{N=1041}\frac{1}{n^s}$ used in the zeroth order term of the Riemann Siegel formula.

It can be seen that the contribution from the early terms of the alternate finite geometric series based linear combination sum is dominant and the series sum much more rapidly approaches the expected finite sum value. This is because the geometric series approach accelerates the series sum by incorporating later terms (in the first principles series sum) into the early series terms.

Explicitly, the real part of the early terms in the alternate finite geometric series linear combination

$$
\Re\left(\sum_{n=1}^{1041}\frac{1}{n^s}\right)=\Re\left(1+\frac{1}{2^s}+\frac{1}{3^s}+...+\frac{1}{1041^s}\right)\tag{4}
$$

$$
=\Re(a_2(s,1041)+a_3(s,1041)+a_5(s,1041)+a_7(s,1041)+a_9(s,1041)+..+a_{p\leq1041}(s,1041))\tag{5}
$$

are for $s=0.5+I\cdot6820051$

$$\Re(a_2(s,1041)) = 3.017649 \tag{6}$$
$$\Re(a_3(s,1041)) = 1.620435 \tag{7}$$
$$\Re(a_5(s,1041)) = 2.045711 \tag{8}$$
$$\Re(a_7(s,1041)) = 1.352946 \tag{9}$$
$$\Re(a_9(s,1041)) = 0.834580 \tag{10}$$
$$\Re(a_{11}(s,1041)) = 2.344479 \tag{11}$$
$$\Re(a_{13}(s,1041)) = 2.484002 \tag{12}$$
$$\Re(a_{17}(s,1041)) = 1.474045 \tag{13}$$
$$\Re(a_{19}(s,1041)) = 1.671007 \tag{14}$$
$$\Re(a_{23}(s,1041)) = 2.034249 \tag{15}$$
$$\Re(a_{25}(s,1041)) = 0.535378 \tag{16}$$
$$\Re(a_{27}(s,1041)) = 0.414098 \tag{17}$$
$$\Re(a_{29}(s,1041)) = 1.388086 \tag{18}$$
$$\Re(a_{31}(s,1041)) = -0.461051 \tag{19}$$

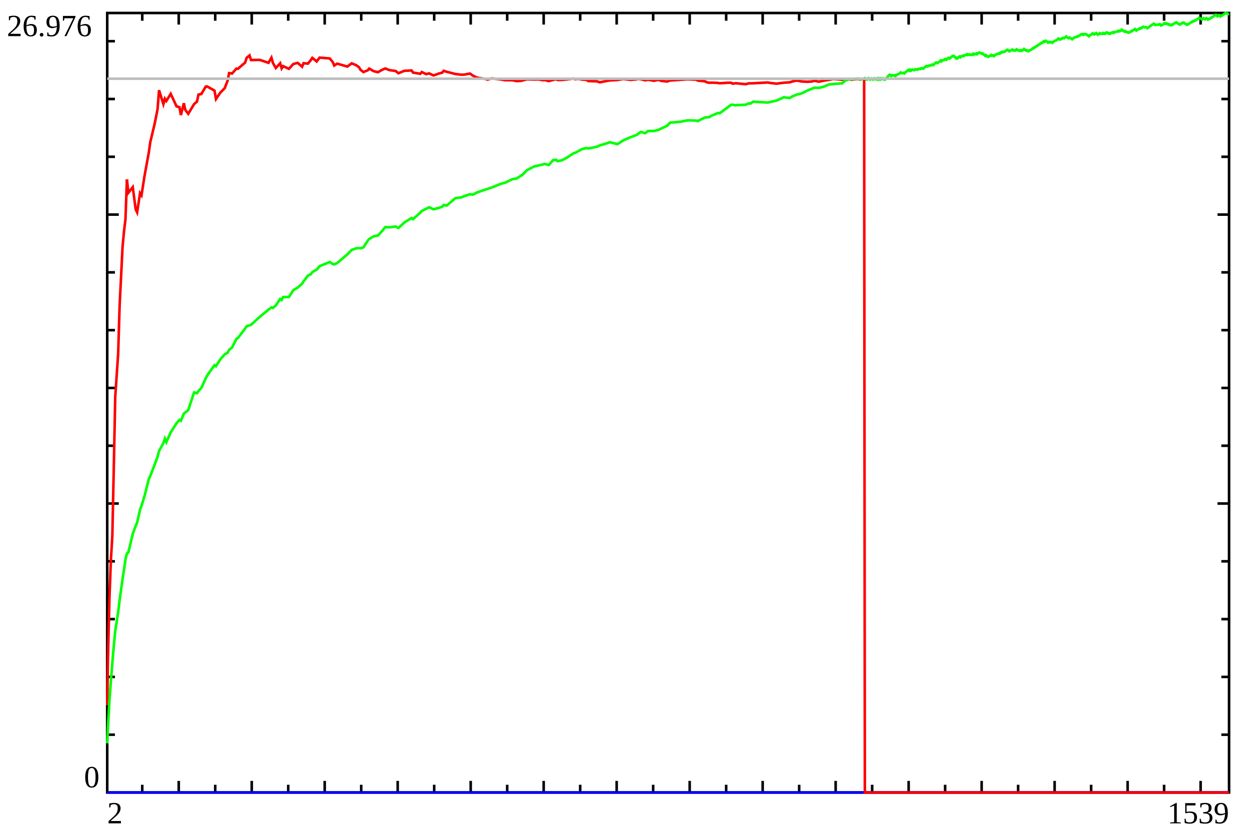

Figure 1. A comparison of the convergence of the cumulative sums (up to and just beyond the first quiescent region of the Riemann Zeta Dirchlet Series sum) for $s = 0.5 + I \cdot 6820051$. At this point on the critical line the Riemann Zeta function has a large peak $|\zeta(s)| \sim 50$ (and the zeroth order finite dirichlet series sum is 24.7 (approximately one half of 50 as expected). The finite geometric series linear combination based sum is only calculated up to the i=1039 term, belonging to the nearest prime below N=1041.

Using the alternate prime based expansion [7] for generating Riemann Siegel type calculations

Based on the first quiescent region, the interval for N=1041 calculations spans the interval

$$1041^2 \cdot 2 \cdot \pi \leq t < 1042^2 \cdot 2 \cdot \pi) \tag{20}$$

$$[6808968.5368 \leq t < 6822056.4118) \tag{21}$$

so the same alternate prime based expansion for N=1041 can be reused over a sizeable interval which offsets the cost of setting up the N=1041 expansion.

An approximation of Riemann Zeta function behaviour can be obtained by the finite dirichlet series based zeroth order Riemann Siegel formula [1,2] at the first quiescent region $N_1 = \sqrt{\frac{t}{2\pi}}$ of the oscillatory divergence of the dirichlet series

$$\zeta_{RS \text{ zeroth order}}(s) = \sum_{n=1}^{\lfloor\sqrt{\frac{N}{2\pi}}\rfloor} \frac{1}{n^s} + \chi(s) \cdot \sum_{n=1}^{\lfloor\sqrt{\frac{N}{2\pi}}\rfloor} \frac{1}{n^{(1-s)}} \tag{22}$$

$$Z_{RS \text{ zeroth order}}(s) = \exp\left(I \cdot \theta(t)\right) \cdot \left(\sum_{n=1}^{\lfloor\sqrt{\frac{N}{2\pi}}\rfloor} \frac{1}{n^s} + \chi(s) \cdot \sum_{n=1}^{\lfloor\sqrt{\frac{N}{2\pi}}\rfloor} \frac{1}{n^{(1-s)}} \right) \tag{23}$$

where (i) $\chi(s)$ is the multiplicative factor of the Riemann Zeta functional equation $\zeta(s) = \chi(s)\zeta(1-s)$, (ii) $\theta(t)$ the Riemann Siegel Theta function is a continuous function and (iii) equation (23) is the zeroth order Riemann Siegel Z function [1-3].

Since the alternate finite geometric series based linear combination is equivalent to the finite dirichlet series the following Riemann Siegel prime based formulae can be written

$$\zeta_{\text{primeRS zeroth order}}(s) = \sum_{p=2}^{P \leq \lfloor\sqrt{\frac{N}{2\pi}}\rfloor} a_p(s,N) + \chi(s) \cdot \sum_{p=2}^{P \leq \lfloor\sqrt{\frac{N}{2\pi}}\rfloor} a_p((1-s),N) \tag{24}$$

$$Z_{\text{primeRS zeroth order}}(s) = \exp\left(I \cdot \theta(t)\right) \cdot \left(\sum_{p=2}^{P \leq \lfloor\sqrt{\frac{N}{2\pi}}\rfloor} a_p(s,N) + \chi(s) \cdot \sum_{p=1}^{P \leq \lfloor\sqrt{\frac{N}{2\pi}}\rfloor} a_p((1-s),N) \right) \tag{25}$$

where $p \in$ primes and powers of primes $\leq \lfloor\sqrt{\frac{N}{2\pi}}\rfloor$ and $p \neq 4, 8, 16, ..$ since the prime powers of 2 are already included the finite geometric series term for prime 2.

Using equations (23) and (25), the impact of number of summed primes (up to N=1041) for the zeroth order Riemann Siegel Z function in the interval t=(6820047,6820053) can be viwed in figures 2-6.

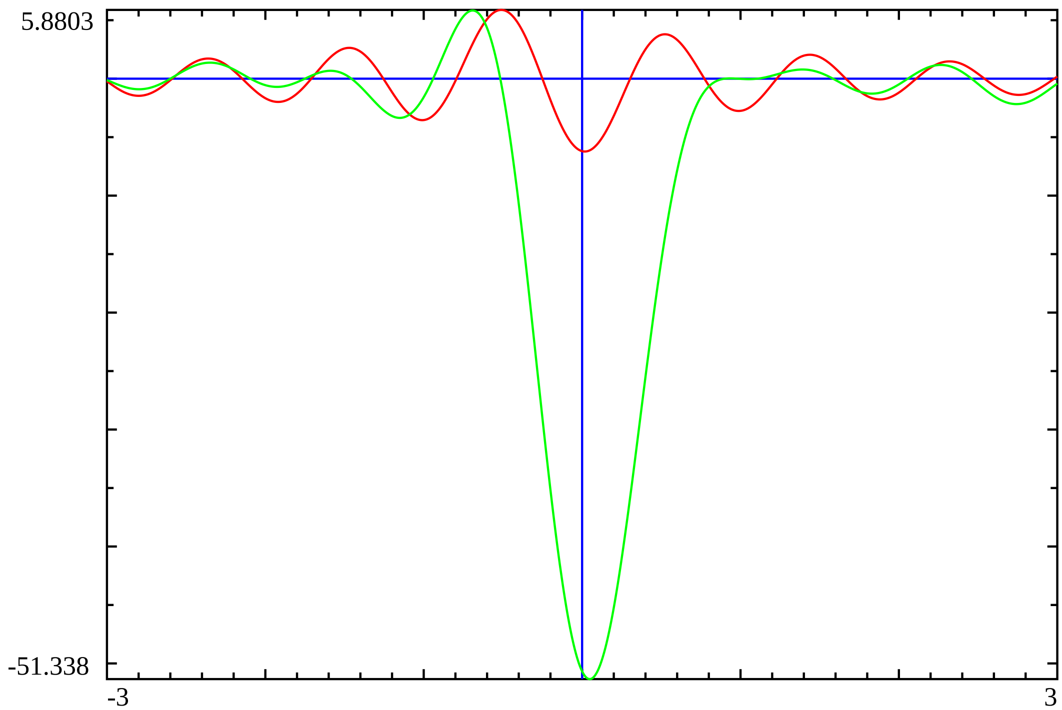

Figure 2. A comparison of the zeroth order Riemann Siegel Z function on the critical line in the interval t=(6820047, 6820053) to a prime based zeroth order Riemann Siegel Z function using only prime 2 and prime 2 powers.

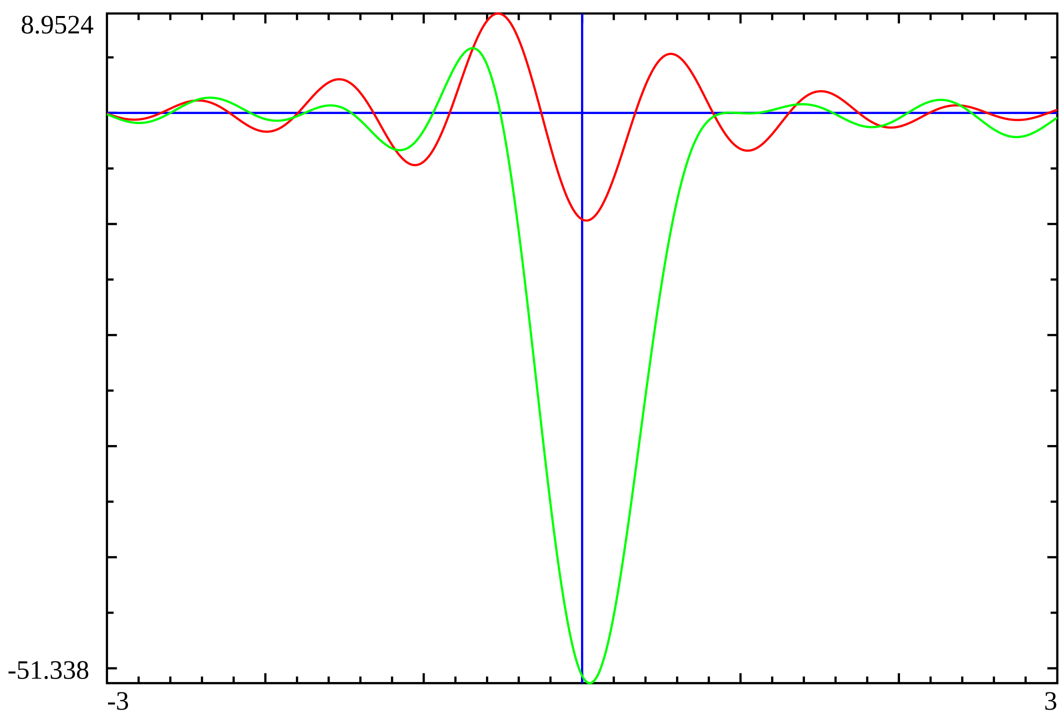

Figure 3. A comparison of the zeroth order Riemann Siegel Z function on the critical line in the interval t=(6820047, 6820053) to a prime based zeroth order Riemann Siegel Z function using only prime 2, prime 3 and prime 2 powers.

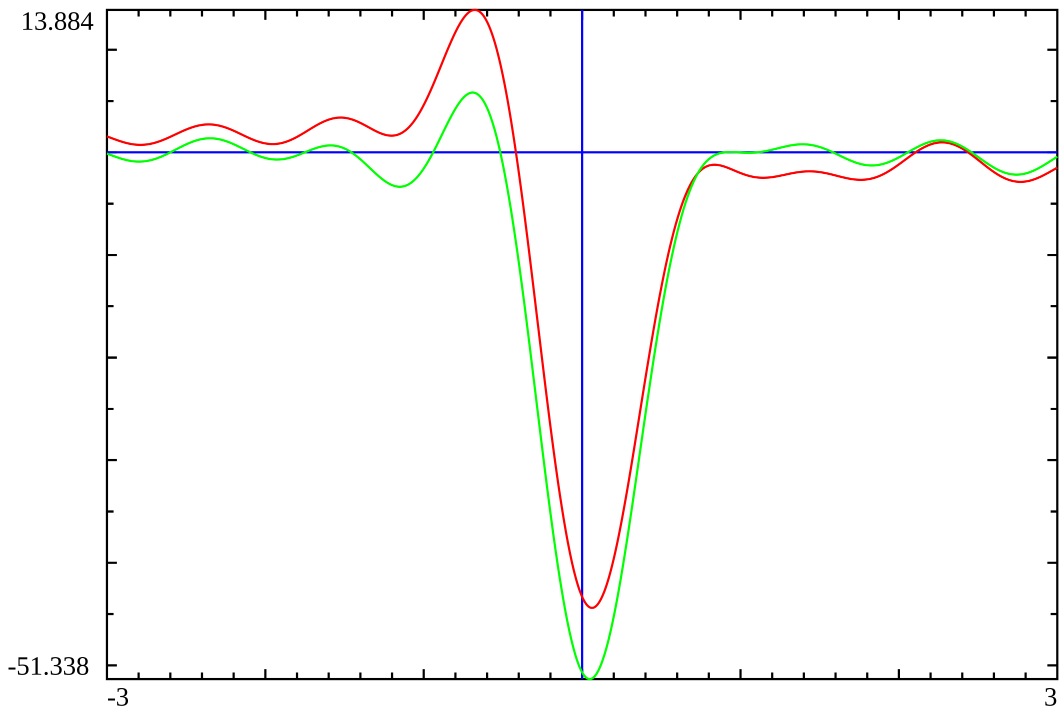

Figure 4. A comparison of the zeroth order Riemann Siegel Z function on the critical line in the interval t=(6820047, 6820053) to a prime based zeroth order Riemann Siegel Z function using terms upto prime=31.

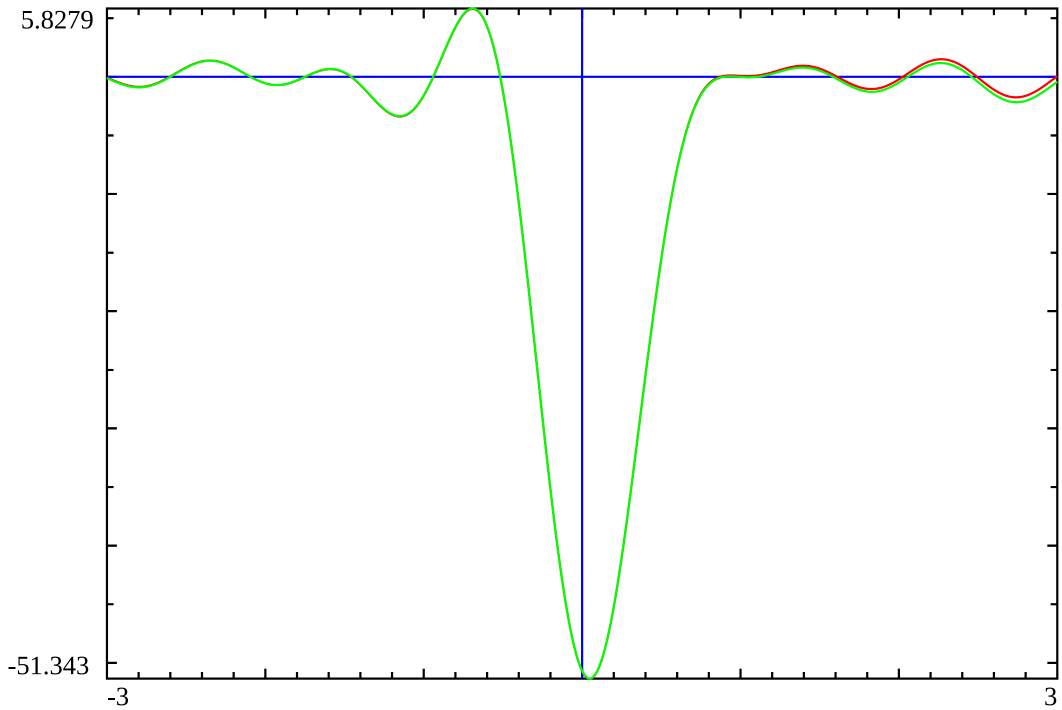

Figure 3. A comparison of the zeroth order Riemann Siegel Z function on the critical line in the interval t=(6820047, 6820053) to a prime based zeroth order Riemann Siegel Z function using terms upto prime 509.

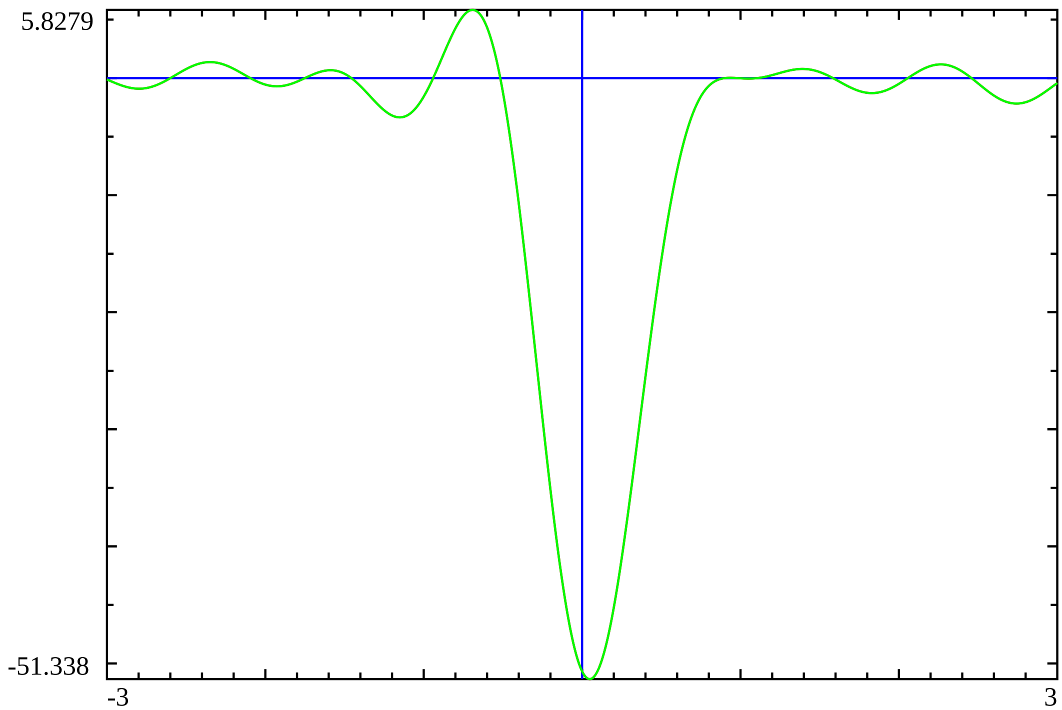

Figure 3. A comparison of the zeroth order Riemann Siegel Z function on the critical line in the interval t=(6820047, 6820053) to a prime based zeroth order Riemann Siegel Z function using all primes up to p=1039. Technically 185 terms (primes and powers of primes 3 upwards containing products and finite geometric series) compared to 1041 terms in dirichlet series.

Conclusions

The above procedure using products of finite geometric series of primes (and their powers) accurately calculates the finite dirichlet series for N=1041 applicable to Riemann Siegel calculation near the first Rosser rule violation.

The cost of the prime based exercise is generating the formula for a given N but the formula does apply over a interval on the imaginary axis. The speed of the calculation is slightly quicker than the finite Dirichlet series (given N=1041) but the code could be further improved by removing repeated calculations.

References

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4. S. M. Gonek, C. P. Hughes, and J. P. Keating, A hybrid Euler-Hadamard product for the Riemann zeta function, Duke Math. J. 136 (2007), no. 3, 507–549. MR 2309173
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6. Martin, J.P.D. "Truncated Exponential Series based partial Euler Product calculations at quiescent regions of oscillatory divergence to produce approximations of the Riemann Siegel Z function." (2021) http://dx.doi.org/10.6084/m9.figshare.14842803

Appendix: Alternate expansion of finite Dirichlet Series based on finite geometric series of primes and powers of primes

The identified alternate expansion in terms of finite geometric series of primes is as follows

$$\sum_{n=1}^{N} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + ... + \frac{1}{N^s} \tag{26}$$

$$= a_2(s,N) + a_3(s,N) + a_5(s,N) + a_7(s,N) + a_9(s,N) + .. + a_{p \leq N}(s,N) \tag{27}$$

where

$$a_2(s,N) = 1 + \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\left(\left\lfloor \frac{\log(\max\{1,N\})}{\log(2)} \right\rfloor\right)})}{(1 - \frac{1}{2^s})} \tag{28}$$

$$a_3(s,N) = \frac{1}{3^s} \cdot \left[\delta(N \geq 3) + \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\left\lfloor \frac{\log(\max\{1,\frac{N}{3}\})}{\log(2)} \right\rfloor})}{(1 - \frac{1}{2^s})} \right] \tag{29}$$

and more generally for prime powers of 3

$$a_{3^m}(s,N) = \left(\frac{1}{3^s}\right)^m \cdot \left[\delta(N \geq 3^m) + \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\left\lfloor \frac{\log(\max\{1,\frac{N}{3^m}\})}{\log(2)} \right\rfloor})}{(1 - \frac{1}{2^s})} \right] \tag{30}$$

$$a_5(s,N) = \frac{1}{5^s} \cdot \Bigg[\delta(N \geq 5) + \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{5}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} + \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{5}\})}{\log(3)} \rfloor})}{(1-\frac{1}{3^s})} + \sum_{q=1} \left(\frac{1}{3^s}\right)^q \cdot \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{3^q \cdot 5}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})}\Bigg] \tag{31}$$

and more generally for prime powers of 5

$$a_{5^m}(s,N) = \left(\frac{1}{5^s}\right)^m \cdot \Bigg[\delta(N \geq 5^m) + \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{5^m}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} + \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{5^m}\})}{\log(3)} \rfloor})}{(1-\frac{1}{3^s})} + \sum_{q=1} \left(\frac{1}{3^s}\right)^q \cdot \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{3^q \cdot 5^m}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})}\Bigg] \tag{32}$$

$$\begin{aligned} a_7(s,N) = \frac{1}{7^s} \cdot \Bigg[&\delta(N \geq 7) + \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{7}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} + \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{7}\})}{\log(3)} \rfloor})}{(1-\frac{1}{3^s})} + \frac{1}{5^s} \cdot \frac{(1-(\frac{1}{5^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{7}\})}{\log(5)} \rfloor})}{(1-\frac{1}{5^s})} \\ &+ \sum_{q=1} \left(\frac{1}{3^s}\right)^q \cdot \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{3^q \cdot 7}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} + \sum_{p=1} \left(\frac{1}{5^s}\right)^p \cdot \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{5^p \cdot 7}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} \\ &+ \sum_{p=1} \left(\frac{1}{5^s}\right)^p \cdot \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{5^p \cdot 7}\})}{\log(3)} \rfloor})}{(1-\frac{1}{3^s})} \\ &+ \sum_{p=1} \sum_{q=1} \left(\frac{1}{5^s}\right)^p \cdot \left(\frac{1}{3^s}\right)^q \cdot \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{3^q \cdot 5^p \cdot 7}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})}\Bigg] \end{aligned} \tag{33}$$

and more generally for prime powers of 7

$$
\begin{aligned}
a_{7^m}(s,N) = \left(\frac{1}{7^s}\right)^m \cdot \Bigg[\delta(N \geq 7^m) &+ \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{7^m}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} + \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{7^m}\})}{\log(3)} \rfloor})}{(1-\frac{1}{3^s})} \\
&+ \frac{1}{5^s} \cdot \frac{(1-(\frac{1}{5^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{7^m}\})}{\log(5)} \rfloor})}{(1-\frac{1}{5^s})} \\
+ \sum_{q=1} \left(\frac{1}{3^s}\right)^q \cdot \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{3^q \cdot 7^m}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} &+ \sum_{p=1} \left(\frac{1}{5^s}\right)^p \cdot \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{5^p \cdot 7^m}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} \\
&+ \sum_{p=1} \left(\frac{1}{5^s}\right)^p \cdot \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{5^p \cdot 7^m}\})}{\log(3)} \rfloor})}{(1-\frac{1}{3^s})} \\
&+ \sum_{p=1} \sum_{q=1} \left(\frac{1}{5^s}\right)^p \cdot \left(\frac{1}{3^s}\right)^q \cdot \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{3^q \cdot 5^p \cdot 7^m}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} \Bigg] \qquad (34)
\end{aligned}
$$

and generally for prime powers of 11 including m=1

$$
\begin{aligned}
a_{11^m}(s,N) = \left(\frac{1}{11^s}\right)^m \cdot \Bigg[& \delta(N \geq 11^m) + \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{11^m}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} + \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{11^m}\})}{\log(3)} \rfloor})}{(1-\frac{1}{3^s})} \\
& + \frac{1}{5^s} \cdot \frac{(1-(\frac{1}{5^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{11^m}\})}{\log(5)} \rfloor})}{(1-\frac{1}{5^s})} + \frac{1}{7^s} \cdot \frac{(1-(\frac{1}{7^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{11^m}\})}{\log(7)} \rfloor})}{(1-\frac{1}{7^s})} \\
& + \sum_{q=1} \left(\frac{1}{3^s}\right)^q \cdot \frac{1}{2^s} \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{3^q \cdot 11^m}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} + \sum_{q=1} \left(\frac{1}{5^s}\right)^q \cdot \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{5^q \cdot 11^m}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} \\
& + \sum_{q=1} \left(\frac{1}{7^s}\right)^q \cdot \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{7^q \cdot 11^m}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} \\
& + \sum_{q=1} \left(\frac{1}{5^s}\right)^q \cdot \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{5^q \cdot 11^m}\})}{\log(3)} \rfloor})}{(1-\frac{1}{3^s})} + \sum_{q=1} \left(\frac{1}{7^s}\right)^q \cdot \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{7^q \cdot 11^m}\})}{\log(3)} \rfloor})}{(1-\frac{1}{3^s})} \\
& + \sum_{q=1} \left(\frac{1}{7^s}\right)^q \cdot \frac{1}{5^s} \cdot \frac{(1-(\frac{1}{5^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{7^q \cdot 11^m}\})}{\log(5)} \rfloor})}{(1-\frac{1}{5^s})} \\
& + \sum_{p=1} \sum_{q=1} \left(\frac{1}{5^s}\right)^p \cdot \left(\frac{1}{3^s}\right)^q \cdot \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{3^q \cdot 5^p \cdot 11^m}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} \\
& + \sum_{p=1} \sum_{q=1} \left(\frac{1}{7^s}\right)^p \cdot \left(\frac{1}{3^s}\right)^q \cdot \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{3^q \cdot 7^p \cdot 11^m}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} \\
& + \sum_{p=1} \sum_{q=1} \left(\frac{1}{7^s}\right)^p \cdot \left(\frac{1}{5^s}\right)^q \cdot \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{5^q \cdot 7^p \cdot 11^m}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} \\
& + \sum_{p=1} \sum_{q=1} \left(\frac{1}{7^s}\right)^p \cdot \left(\frac{1}{5^s}\right)^q \cdot \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{5^q \cdot 7^p \cdot 11^m}\})}{\log(3)} \rfloor})}{(1-\frac{1}{3^s})} \\
& + \sum_{p=1} \sum_{q=1} \sum_{r=1} \left(\frac{1}{7^s}\right)^p \cdot \left(\frac{1}{5^s}\right)^q \cdot \left(\frac{1}{3^s}\right)^r \cdot \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{3^r \cdot 5^q \cdot 7^p \cdot 11^m}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} \Bigg]
\end{aligned}
\tag{35}
$$