Extended Riemann Siegel Theta function further simplified using functional equation factor, for the Riemann Zeta function.

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The extended Riemann Siegel functions on the Riemann Zeta function with further algebraic manipulation

The Riemann Zeta function is defined (1), in the complex plane by the integral

$$\zeta(s) = \frac{\prod(-s)}{2\pi i}\int_{C_{\epsilon,\delta}} \frac{(-x)^s}{(e^x-1)x}dx \tag{1}$$

where $s \,\epsilon\, \mathbb{C}$ and $C_{\epsilon,\delta}$ is the contour about the imaginary poles.

The Riemann Zeta function has been shown to obey the functional equation (2)

$$\zeta(s) = 2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s) \tag{2}$$

Following directly from the form of the functional equation and the properties of the coefficients on the RHS of eqn (2) it has been shown that any zeroes off the critical line would be paired, ie. if $\zeta(s) = 0$ was true then $\zeta(1-s) = 0$.

Along the critical line (0.5+it), the Riemann Siegel function is an exact function (3) for the magnitude of the Riemann Zeta function with two components Z(t) & $\theta(t)$

$$Z(t) = \zeta(0.5+it)e^{i\theta(t)} \tag{3}$$

and

$$\theta(t) = \Im(log(\Gamma(\frac{1}{4}+\frac{1}{2}it))) - \frac{t}{2}log(\pi) \tag{4}$$

In Martin (4) and earlier work, the properties of the Riemann Zeta generating function were investigated and used to develop/map the extended Riemann Siegel function $Z_{ext}(s)$ and $\theta_{ext}(s)$ definitions also applicable away from the critical line.

In the following derivation, the functional equation factor in eqn (2) is reapplied to the ratio of Riemann Zeta funtions in $\theta_{ext}(s)$ to simplfy the expression along with the standard decompositon of the complex value, $s = \sigma + i*t$

$$
\begin{aligned}
\theta_{ext}(s) &= \Im(log(\sqrt{\frac{\zeta(1-s)abs(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))}{\zeta(s)}})) && (5)\\
&= \Im(log(\sqrt{\frac{abs(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))}{2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))}})) && (6)\\
&= \Im(log(\sqrt{\frac{\sqrt{(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))\cdot(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))^*}}{\sqrt{(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))\cdot(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))}}})) && (7)\\
&= \Im(log(\sqrt{\frac{\sqrt{(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))^*}}{\sqrt{(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))}}})) && (8)\\
&= \Im(log(\Big(\frac{(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))^*}{(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))}\Big)^{\frac{1}{4}})) && (9)\\
&= \frac{1}{4}\Im(log((2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))^*) - log((2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s)))) && (10)\\
&= \frac{1}{4}\Im(2\cdot -i\cdot\Im(log((2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))))) && (11)\\
&= -\frac{1}{2}\Im(log((2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s)))) \qquad \text{agrees by computation with eqn (5)} && (12)\\
&= -\frac{1}{2}\Im(s\cdot log(2) + (s-1)log(\pi) + log(sin(\frac{\pi s}{2})) + log(\Gamma(1-s))) && (13)\\
&= -\frac{t}{2}log(2) - \frac{t}{2}log(\pi) - \frac{1}{2}\Im\big(log(\frac{e^{i*\frac{\pi s}{2}} - e^{-i*\frac{\pi s}{2}}}{2*i}\big) - \frac{1}{2}\Im(log(\Gamma(1-s))) && (14)\\
&= -\frac{t}{2}log(2) - \frac{t}{2}log(\pi) - \frac{1}{2}\Im\big(log(e^{i*\frac{\pi s}{2}} - e^{-i*\frac{\pi s}{2}})\big) + \frac{1}{2}\Im\big(log(2*i)\big) - \frac{1}{2}\Im(log(\Gamma(1-s))) && (15)\\
&= -\frac{t}{2}log(2) - \frac{t}{2}log(\pi) - \frac{1}{2}\Im\big(log(e^{i*\frac{\pi s}{2}} - e^{-i*\frac{\pi s}{2}})\big) + \frac{\pi}{4} - \frac{1}{2}\Im(log(\Gamma(1-s))) && (16)
\end{aligned}
$$

This last expression should be compared to eqn (4). A distinctive difference between the $\theta(t)$ & $\theta_{ext}(s)$ functions being that the $\theta(t)$ branch points provide the Gram points which have the tendency of approximately bisecting Riemann Zeta zeroes (1) while the $\theta_{ext}(s)$ branch points are approximately at the position of Riemann Zeta zeroes.

References

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