An alternative first order series expansion correction to the Riemann Zeta Dirichlet Series about the second quiescent region $\frac{t}{\pi}$.

John Martin

September 22, 2023

Executive Summary

On the critical line, the following first order series expansion of the Dirichlet series for the Riemann Zeta function about the (second) quiescent region $N \approx \lfloor \frac{t}{\pi} \rfloor$ appears to asymptotically approach the Riemann Zeta function value

$$\sum_{n=1}^{\lfloor \frac{t}{\pi} \rfloor} \frac{1}{n^{(1/2+I*t)}} + \cos(t-\pi(\frac{t}{\pi}-\lfloor \frac{t}{\pi} \rfloor)) \exp(I(-t\log(t)+(1+\log(\pi))t+\pi+O(\frac{1}{t})) \cdot \frac{1}{\left(2\cdot(\frac{t}{\pi})^{\frac{1}{2}}\right)} + O\left(\frac{1}{(\frac{t}{\pi})^{\left(1+(\frac{1}{2})\right)}}\right) \approx$$

$\zeta(\frac{1}{2}+I*t)$ as $t \to \infty$

Across the complex plane, the first order series expansion about the (second) quiescent region $N \approx \lfloor \frac{t}{\pi} \rfloor$ is of the more general form

$$\sum_{n=1}^{\lfloor \frac{t}{\pi} \rfloor} \frac{1}{n^s} + \frac{\cos(t-\pi(\frac{t}{\pi}-\lfloor \frac{t}{\pi} \rfloor)) \exp(I(-\theta_{ext}(s)-\frac{1}{2}t\log(t)+\frac{1}{2}(1+\log(\pi)-log(2))t+\frac{7\pi}{8}+O(\frac{1}{t}))}{2^{\left(\frac{(\sigma-\frac{1}{2})}{2}\right)}} \cdot \frac{1}{\left(2\cdot(\frac{t}{\pi})^{(\frac{1}{4}+\frac{\sigma}{2})}\right)} + O(\frac{1}{(\frac{t}{\pi})^{\left(1+(\frac{1}{4}+\frac{\sigma}{2})\right)}}) \approx$$

$\zeta(\sigma+I*t)$ as $t \to \infty$

and also provides a good approximation of the Riemann Zeta function away from the real axis, where $\theta_{ext}(s)$ is the extended Riemann-Siegel Theta function which contains both real and imaginary parts for $\sigma \neq \frac{1}{2}$.