Examining Riemann Zeta mesoscale self-similarity along the line $\Re(\zeta) = 1$ near large peaks.

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November 1, 2020

Executive Summary

The self-similarity of the partial Euler Product and Riemann Zeta functions surrounding large peaks on the line S=1+i*T is examined, using pari/gp software, for $(280 < T < 10^{370})$. Empirically, it is observed that there is extended mesoscale structure surrounding the large peaks on the line S=1+i*T reflecting closely a truncated Riemann Zeta function near the real axis and the similarity grows as $T \to \infty$. Exploiting known lower bounds of the growth rate $|\zeta(1 + I * T)| \geq e^{\gamma} \cdot [log_2(T) + log_3(T) + O(1)]$ and the observed mesoscale behaviour, simple expressions are provided to approximate the known lower bound height and local structure about the large peaks (at S=1+i*T) as truncated versions of (i) the Riemann Zeta function near the real axis $|\zeta(1 + I * (T + t))| \sim |(\zeta(1 + I * t)|$ where $|t| > \gamma/(log_2(T) + log_3(T))$ and (ii) its Euler product $|\zeta_{EP}(1 + I * (T + t), N \to \infty)| \sim \left|\prod_{\rho=2}^{\lfloor 2\pi log(\theta(T))\rceil} \left\{1/(1 - 1/p^{(1+i*t)})\right\}\right|$ for $t < T$.

Introduction

On the critical line $\zeta(0.5 + iT)$, the divergence of the partial Euler Product, is weak when only using the lowest primes well away from the real axis. As a result, the partial Euler Product has been actively used as a useful first order approximation to locate the largest peaks when searching for closely (and widely) spaced non-trivial zeroes which represent a interesting test for Riemann Hypothesis behaviour [1-4]. Other authors have added corrections to the partial Euler product to reduce the divergence behaviour and get better agreement with $\zeta(0.5 + iT)$ values [5,6]. These peaks on $\zeta(0.5 + iT)$ are directly related to smaller peaks on the line $\zeta(1 + iT)$

Theoretical work improving the expected lower and upper bounds of the growth of the Riemann Zeta function from Littlewood's bounds [7] (with and without the Riemann Hypothesis respectively) along the line $\zeta(1+iT)$ [8-11] has attained a current lower bound for line 1 growth assuming the Riemann Zeta hypothesis [8] of

$$|\zeta(1 + I * T)| \geq e^{\gamma} \cdot [log_2(T) + log_3(T) - log_4(T) + O(1)] \quad (1)$$

and the lower bound conjecture [8]

$$|\zeta(1 + I * T)| \geq e^{\gamma} \cdot [log_2(T) + log_3(T) + O(1)] \quad (2)$$

In this paper, large Euler Product peaks are presented in the wide range $(280 < T < 10^{370})$ along $S = (1 + i(T + t))$ where $t < T$ and the behaviour is fitted by a truncated Riemann Zeta function and partial Euler Product located at $s = (1 + it)$ about the Riemann Zeta pole. The peak positions were previously identified via pari/gp software [12] employing the Lenstra-Lenstra-Lovász (LLL) basis reduction algorithm [13], to solve the diophantine approximation $\frac{log(2*\mathbb{N}_1)T}{2\pi} \approx n_j$.

The intent of the truncation was to (i) investigate and approximate the effect of the large peaks as arising from a finite set of primes, in constrast to the infinity of primes contributing to the Riemann Zeta pole (and the Riemann Zeta function as $T \to \infty$) while (ii) mimicking that the width of the known large peaks [2-4,14-18] remains $\Delta T \sim 0.4$ for $T > 10^{30}$.

As illustrated in the paper, (i) useful truncation bounds related to equation (2) and/or the Riemann Siegel Theta function result in sensible bounds with respect to observed Riemann Zeta and Euler Product behaviour along S=1+i*T, for $(280 < T < 10^{370})$ and (ii) extensive self-similarity is observed about the large peaks consistent with the Riemann Zeta function behaviour about the pole.

The Riemann Zeta function and partial Euler Product

For $\Re(s) > 1$, the Euler Product of the primes absolutely converges to the Riemann Zeta function sum of the integers [19-20]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\rho=2}^{\infty} \frac{1}{(1 - 1/\rho^s)} \qquad \text{for } \Re(s) > 1 \tag{3}$$

The Riemann Zeta function can be defined for the whole complex plane by the integral [19-20]

$$\zeta(s) = \frac{\prod(-s)}{2\pi i} \int_{C_{\epsilon,\delta}} \frac{(-x)^s}{(e^x - 1)x} dx \qquad \text{for } \mathbb{C} \tag{4}$$

where $s \,\epsilon\, \mathbb{C}$ and $C_{\epsilon,\delta}$ is the contour about the imaginary poles.

On the s=1 line, away from the real axis the Euler Product asymptotically approaches to the Riemann Zeta function value [21]. This behaviour can be seen in equation 4.11.2 of [21]

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + O(\frac{|s|}{N^{\Re(s)}}) + O(\frac{1}{N^{\Re(s)}}) \tag{5}$$

On inspecting the partial Euler Product results for complex values s, in the upper half of the critical strip, the divergence of the $\sum_{n=1}^{N} \frac{1}{n^s}$ term in the above equation and hence the partial Euler Product $\prod_{\rho=2}^{P} \frac{1}{(1-1/\rho^s)}$ in that region, exhibits a dominant $\frac{N^{1-s}}{1-s}$ oscillatory divergence behaviour near the real axis which becomes a trivial difference well away from the real axis.

As exploited by [1,4,16-18] and others, there are large peaks in the Riemann Zeta function on the critical line, co-incident with similar sized peaks in the partial Euler Product

$$\zeta_{EP}(s) = \prod_{\rho=2}^{P} \frac{1}{(1 - 1/\rho^s)} \qquad \text{for } P << \infty \tag{6}$$

when many $\rho^s \approx 1$ at the same value of T. This constraint is described as a diophantine approximation

$$log(p_j)T \approx n_j 2\pi \tag{7}$$

where n_j are integers, for as many primes p_j as possible.

Importantly, these peaks are directly correspond to smaller but dominant peaks on the line $S = 1 + iT$ [16,17,22].

Behaviour of Riemann Zeta function and partial Euler Product, on the line 1+I*(T+t) near large diophatine peaks for the interval $(280 < T < 2.1 \cdot 10^5)$ compared to truncated (and translated) Riemann Zeta and Euler Product 1+I*t functions

In the figures below,

for figures 1-3, the absolute values of the exact Riemann Zeta function (gray) and overlapping partial Euler product (blue) for spans of t=(-2,2) and t=(-45,45) about the peaks at T=280.8, 10025.3 and 200257.65 on S=1+I*(T+t) is compared to

1. (horizontal red) Granville and Soundararajan's [8] conjectured lower bound equation (2),
   $|\zeta(1 + I(T + t))|_{LB} \sim e^{\gamma} \left(log(log(T + t)) + log(log(log(T + t)))\right)$
   under the Riemann Hypothesis,
2. (red) truncated Riemann Zeta function (about the pole) adapting the result of equation (2)
   $\zeta(1 + I(T + t)) \sim \zeta(1 + I(t))$ where $|t| > \frac{\gamma}{(log(log(T+t))+log(log(log(T+t))))}$
3. (green) truncated Euler Product (about the pole) approximated using the Riemann Siegel Theta function
   $|\zeta_{EP}(1 + I * (T + t), \lfloor 2\pi log(\theta(T + t))\rceil| \sim \left|\prod_{\rho=2}^{\lfloor 2\pi log(\theta(T+t))\rceil} \left\{1/(1 - 1/p^{(1+i*t)})\right\}\right|$
4. (horizontal blue) approximate Euler Product height using the average number of Riemann Zeta zeroes (to height $T + t$)
   $|\zeta_{EP}(1 + I(T + t))| \sim e^{\gamma} \left(log(log(\theta(T + t)))\right) \approx e^{\gamma} \left(log(log(\pi \cdot (N(T + t) - 1)))\right)$

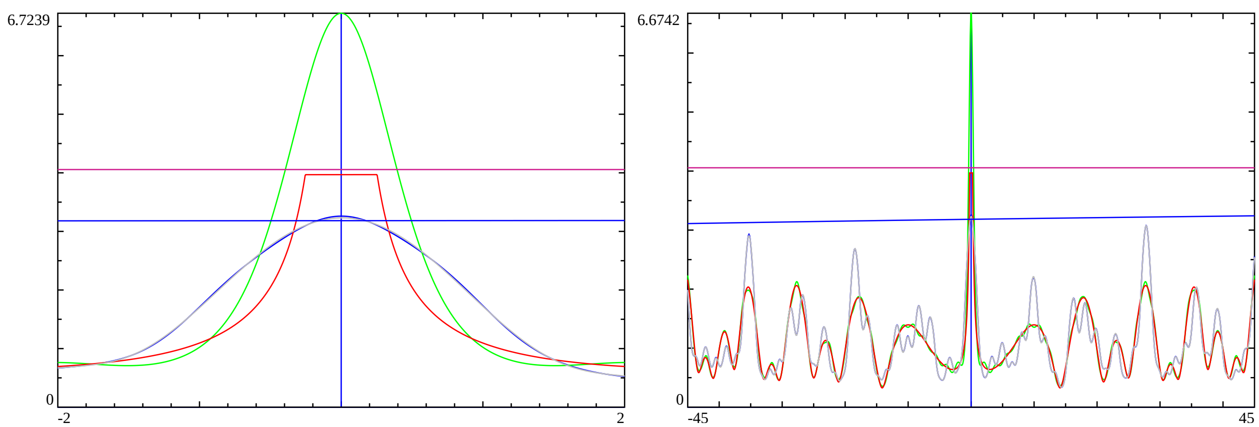

Figure 1: Line 1 Riemann Zeta mesoscale self-similarity about the large peak at location S=1+I*280. Left panel t=(-2,2), Right panel t=(-45,45) about the peak with partial Euler Product P(S,N) (blue), truncated real axis $\zeta(1 + It)$ (red) $|t| > \frac{\gamma}{(log(log(T+t))+log(log(log(T+t))))}$, truncated real axis P(1+I*t,2π $(log(\theta(T + t)))$) (green) versions, $e^{\gamma} \left(log(log(T + t)) + log(log(log(T + t)))\right)$ (horizontal magenta) based growth lower bound, the exact $\zeta(S + t)$ (gray) function and $e^{\gamma} \left(log(log(\theta(T + t)))\right)$ (horizontal blue) based growth. peak=1+I*280

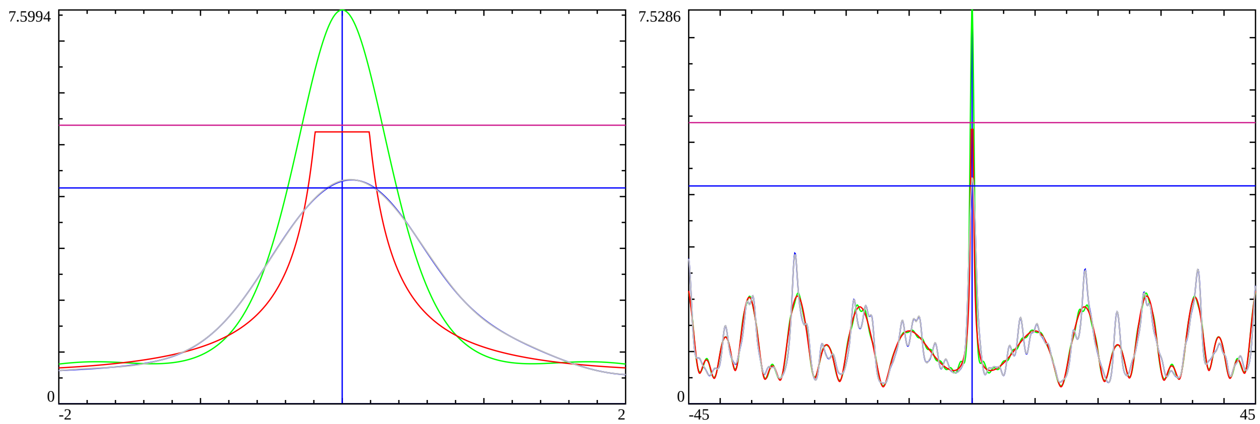

Figure 2: Line 1 Riemann Zeta mesoscale self-similarity about the large peak at location S=1+I*10025.53.

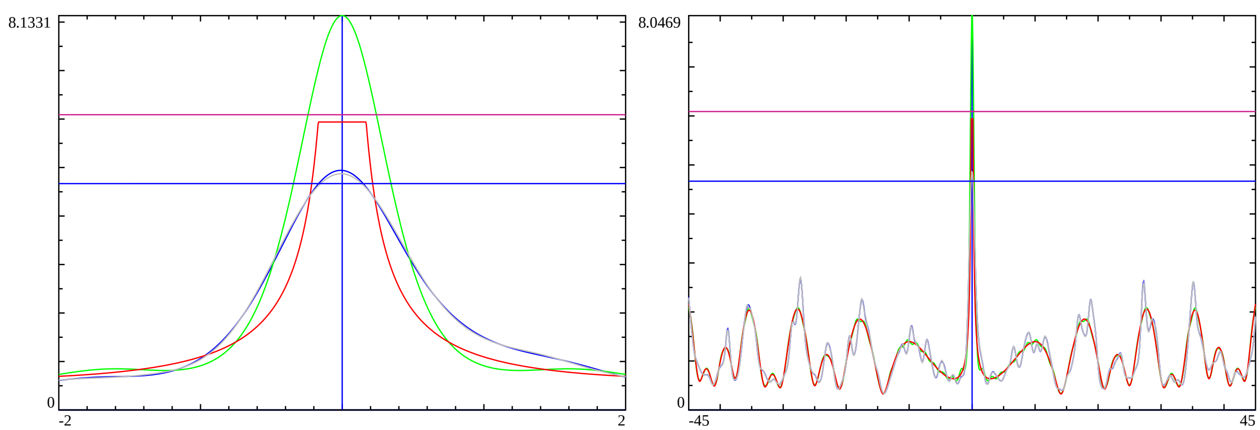

Figure 3: Line 1 Riemann Zeta mesoscale self-similarity about the large peak at location S=1+I*200257.65.

Firstly, it can be seen in figures 1-3, that the Riemann Zeta function (gray) and partial Euler Product (blue) on S=1+I*(T+t) overlap closely. The partial Euler Product was calculated using $10 * \lfloor 2\pi log(\theta(T)) \rceil$ primes which was found to give good convergence (to calculations using much higher numbers of primes) and backed up by the agreement with the exact Riemann Zeta function.

Secondly, there is consistent agreement between the bound calculated by equation (2) (horizontal red) and the plateau (red) of the truncated (and translated) Riemann Zeta function.

Thirdly, the proposed approximate Euler Product height estimate (horizontal blue) $e^{\gamma}\left(log(log(\theta(T+t)))\right)$ for large peaks on $S = 1 + I*(T+t)$ appears below but close to the exact Riemann Zeta function value.

Next, the proposed truncated (and translated) Euler Product (green) is higher than the Granville and Soundararajan's [8] conjecture value for low T.

Finally, on the right panel it can be seen that away from the central peak there is a strong self similarity in the envelope of the riemann zeta function about T+t compared to the Riemann Zeta function about t (red) (ie. about the pole). With some evidence that the difference between the gray and red line is diminishing as T increases.

Behaviour of partial Euler Product, on the line 1+I*(T+t) near large diophatine peaks for the interval $(6 \cdot 10^{6} < T < 10^{360})$ compared to truncated (and translated) Riemann Zeta and Euler Product 1+I*t functions

For figures 4-11, only the results of the partial Euler Product (blue) for spans of t=(-2,2) and t=(-45,45) about the peaks at T=6820051 - 8.936e367 on S=1+I*(T+t) are compared to

1. (horizontal red) Granville and Soundararajan's [8] conjectured lower bound equation (2),
   $|\zeta(1+I(T+t))|_{LB} \sim e^{\gamma}\left(log(log(T+t)) + log(log(log(T+t)))\right)$
   under the Riemann Hypothesis,
2. (red) truncated Riemann Zeta function (about the pole) adapting the result of equation (2)
   $\zeta(1+I(T+t)) \sim \zeta(1+I(t))$ where $|t| > \frac{\gamma}{(log(log(T+t))+log(log(log(T+t))))}$
3. (green) truncated Euler Product (about the pole) approximated using the Riemann Siegel Theta function
   $|\zeta_{EP}(1+I*(T+t), \lfloor 2\pi log(\theta(T+t)) \rceil| \sim \left|\prod_{p=2}^{\lfloor 2\pi log(\theta(T+t)) \rceil} \left\{1/(1-1/p^{(1+i*t)})\right\}\right|$
4. (horizontal gray for figs 4-10, green for fig 11) approximate Euler Product height using the average number of Riemann Zeta zeroes (to height $T+t$)
   $|\zeta_{EP}(1+I(T+t))| \sim e^{\gamma}\left(log(log(\theta(T+t)))\right) \approx e^{\gamma}\left(log(log(\pi \cdot (N(T+t)-1)))\right)$ ()

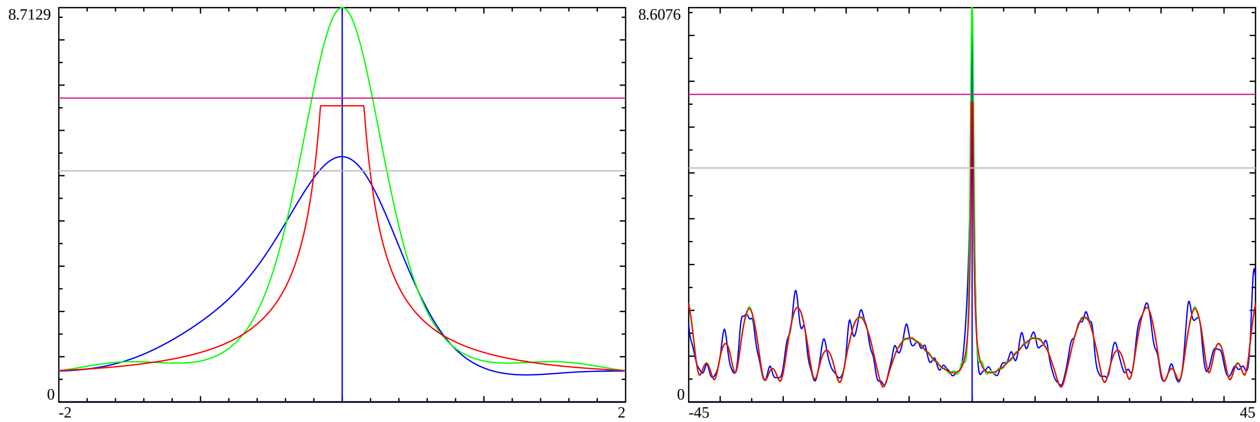

Figure 4: Line 1 Riemann Zeta mesoscale self-similarity about the large peak at location S=1+I*6.82E6. Left panel t=(-2,2), Right panel t=(-45,45) about the peak with partial Euler Product P(S,N) (blue), truncated real axis $\zeta(1+It)$ (red) $|t| > \frac{\gamma}{(log(log(T+t))+log(log(log(T+t))))}$, truncated real axis P(1+I*t,$2\pi\,(log(\theta(T+t)))$) (green) versions, $e^{\gamma}\,(log(log(T+t))+log(log(log(T+t))))$ (horizontal magenta) based growth lower bound and $e^{\gamma}\,(log(log(\theta(T+t)))))$ (horizontal gray) based growth. peak=1+I*6820051

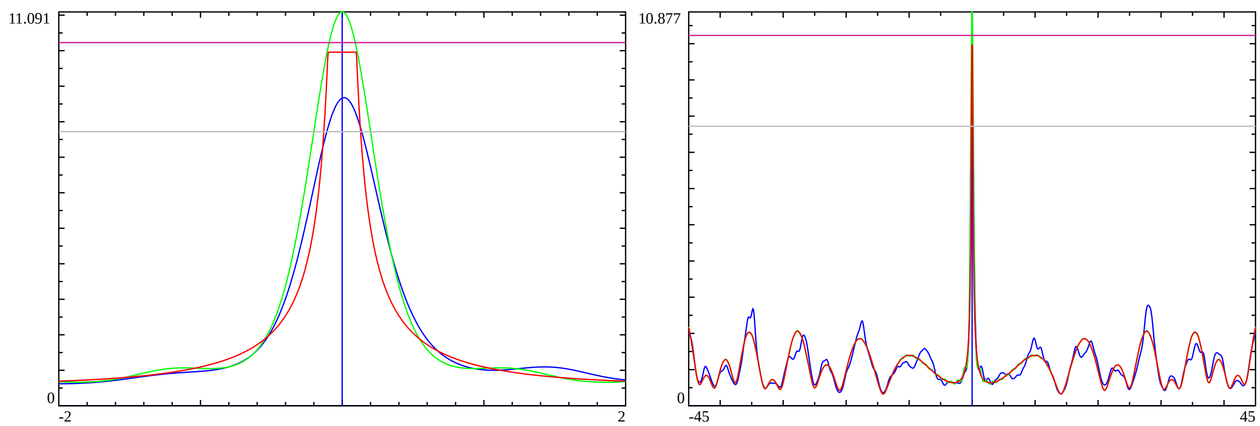

Figure 5: Line 1 Riemann Zeta mesoscale self-similarity about the large peak at location S=1+I*3.924676E31.peak=1+I*39246764589894309155251169284104

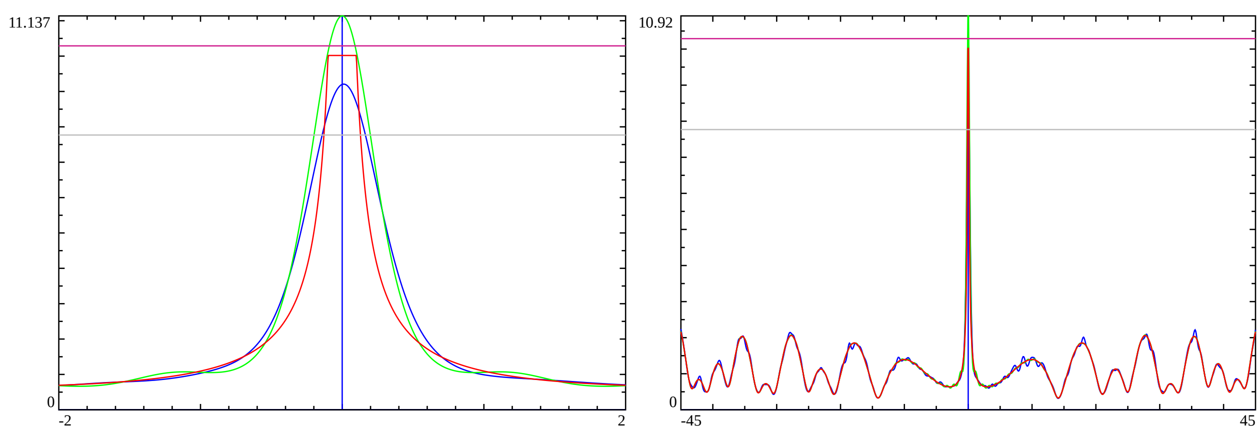

Figure 6: Line 1 Riemann Zeta mesoscale self-similarity about the large peak at location S=1+I*3.924676E31.peak=1+I*310678833629083965667540576593682.0582

In these figures, the partial Euler Product was again calculated using $10 * \lfloor 2\pi log(\theta(T)) \rceil$ primes and found to give reasonable convergence (to calculations using a much higher numbers of primes). Likewise, there is consistent agreement between the bound calculated by equation (2) (horizontal red) and the plateau (red) of the truncated (and translated) Riemann Zeta function. Also the proposed approximate Euler Product height estimate (horizontal blue) $e^{\gamma}\left(log(log(\theta(T+t)))\right)$ for large peaks on $S = 1 + I * (T + t)$ continues to appear below the partial Euler Product function value.

Importantly, on the right panel it can be seen that away from the central peak there is stronger self similarity in the envelope of the riemann zeta function about T+t compared to the Riemann Zeta function about t (red) (ie. about the pole) is only getting stronger as T increases. Within $|t| < 2$, the truncated (and translated) partial Euler product contains some oscillations due to its known divergence from the Riemann Zeta function close to t=0.

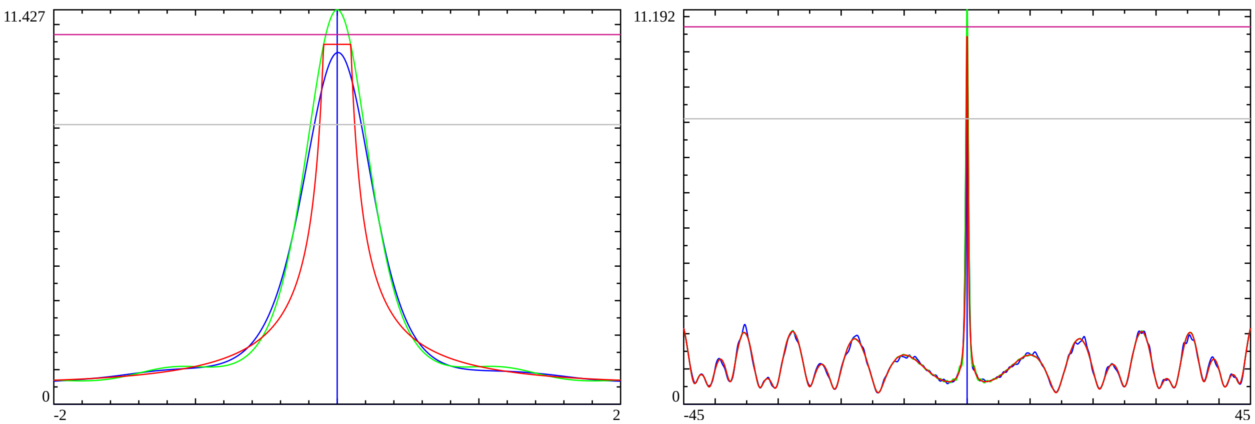

Figure 7: Line 1 Riemann Zeta mesoscale self-similarity about the large peak at location S=1+I*2.3022029198E39. peak=1+I*2302202919833091938191454510490853528294

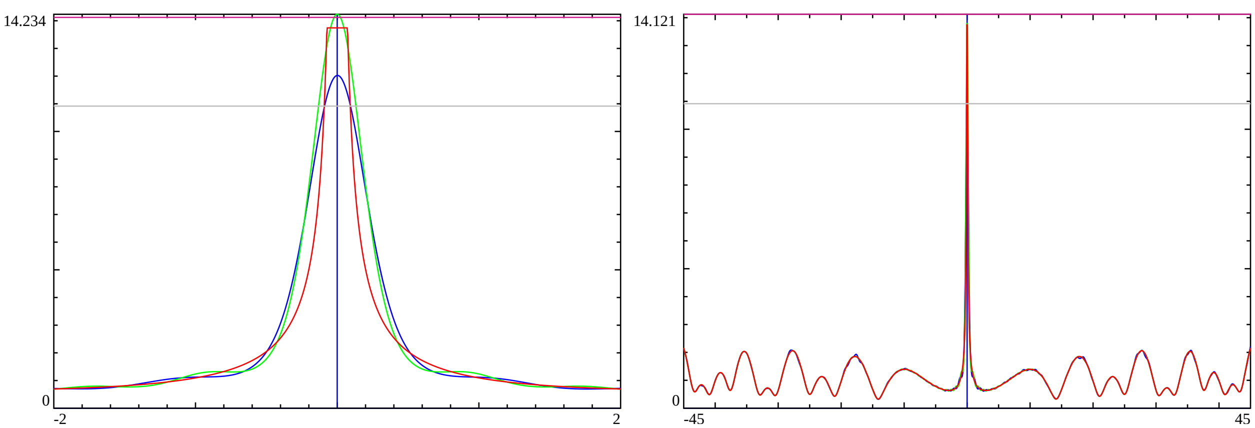

Figure 8: Line 1 Riemann Zeta mesoscale self-similarity about the large peak at location S=1+I*1.0111676E197. peak=1+I*1011167639015609843855339199226907281807356950724220026634786454197018261785784295862873546988680978372570433
18579891969574348341457647060139453285798560070823624340849132140635153744986427167920455

In figures 5-7, there is a larger difference between the approximate Euler Product height estimate (horizontal gray) $e^{\gamma}\left(log(log(\theta(T+t)))\right)$ and the partial Euler Product function value but but for higher peaks 8-11 so far discovered, the difference becomes small again.

In figures 5-8, the truncated (and translated) partial Euler Product comes closer to the Granville and Soundararajan's [8] conjecture lower bound $\left(e^{\gamma}\left(log(log(T+t)) + log(log(log(T+t)))\right)\right)$. In figures 9-11, for $T > 10^{274}$ the $\left(e^{\gamma}\left(log(log(T+t)) + log(log(log(T+t)))\right)\right)$ bound is higher hence the horizontal red line is right at the top of these last 3 figures.

Based on calculations, the (Riemann Hypothesis based) known lower bound equation (1) exceeds the proposed truncated (and translated) partial Euler Product

$$e^{\gamma} \cdot [log_2(T) + log_3(T) - log_4(T) + O(1)] > \left| \prod_{\rho=2}^{\lfloor 2\pi log(\theta(T+t)) \rceil} \left\{ 1/(1 - 1/p^{(1+i*t)}) \right\} \right| \tag{8}$$

only after $T > 10^{500000}$.

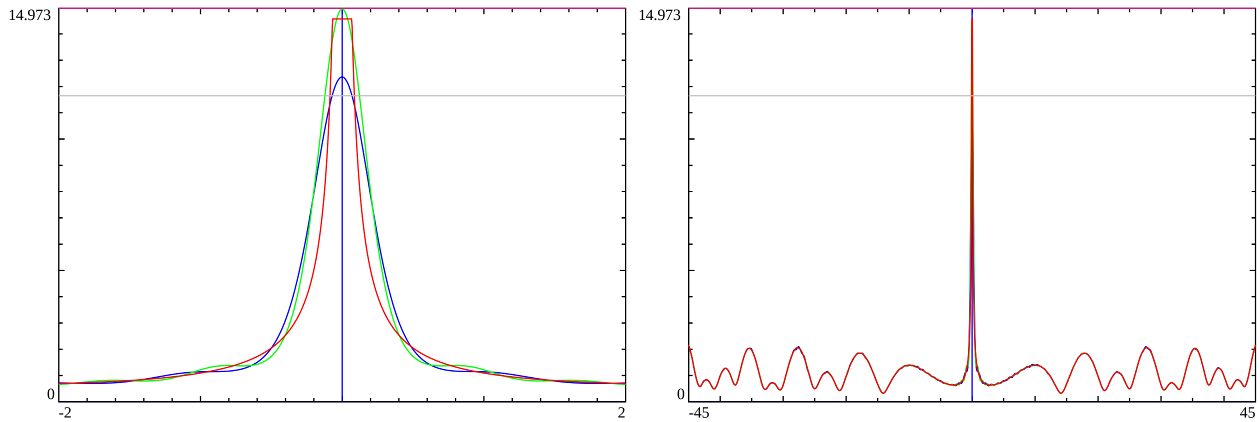

Figure 9: Line 1 Riemann Zeta mesoscale self-similarity about the large peak at location S=1+I*6.00286505291648E297. peak=1+I*600286505291648556627271828666580782546239513318062720906814891751491465674024849210916984486639789217441603802749902965358902350052338458231124297318923513447146746407996791105366995037335707698554237789614274918235051574545430872531715481901827683670069719702093897844693983372865977014061674l673

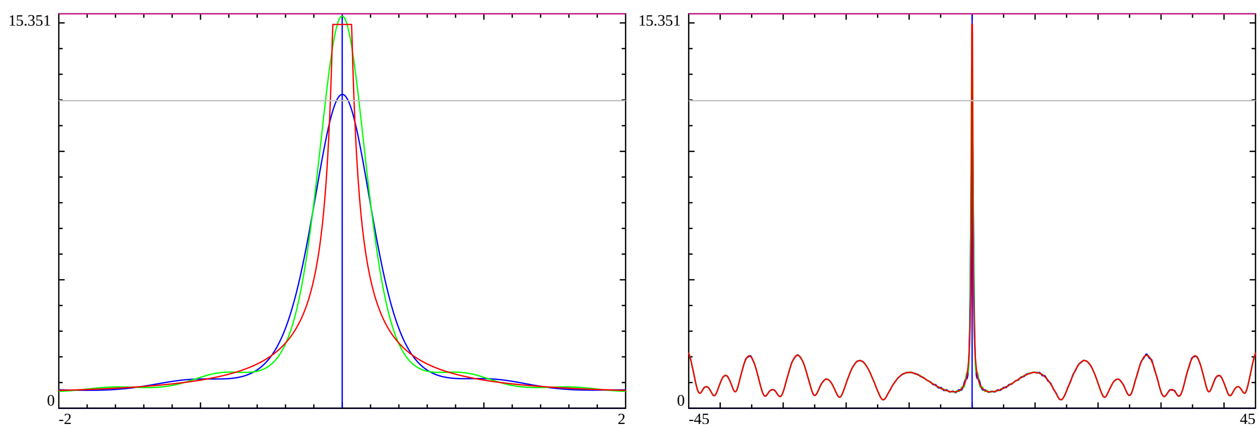

Figure 10: Line 1 Riemann Zeta mesoscale self-similarity about the large peak at location S=1+I*1.30692476E358. peak=1+I*1306924762236183621747256408098679390308733217148558609197453134822792948179260665737952006257485448239655228161598834526314966714359281270036828256552282963288004484244631141247035380584382485693856319843119263004813235910441021382902564735700391541791595967759325456756061116888438106866188650517987836701708773518940421719284118699521814670281231052110912 8

Extensive range of self-similarity for large peaks at high T

For high T, the range of the self-similarity is very extensive. For example, it is simple enough with $T > 10^{10}$ to see a side feature at $T \pm 6820051$ (as in figure 4) and for $T > 10^{35}$ to see a side feature at $T \pm 310678833629083965667540576593682.0582$ (as in figure 6) which are reflections of the large peaks about the pole at 6820051 and 310678833629083965667540576593682.0582. That is, for high T the peaks are repeating as satellite peaks about larger peaks as multiple diophatine integer solutions describing two peaks can simply be added to form a composite solution.

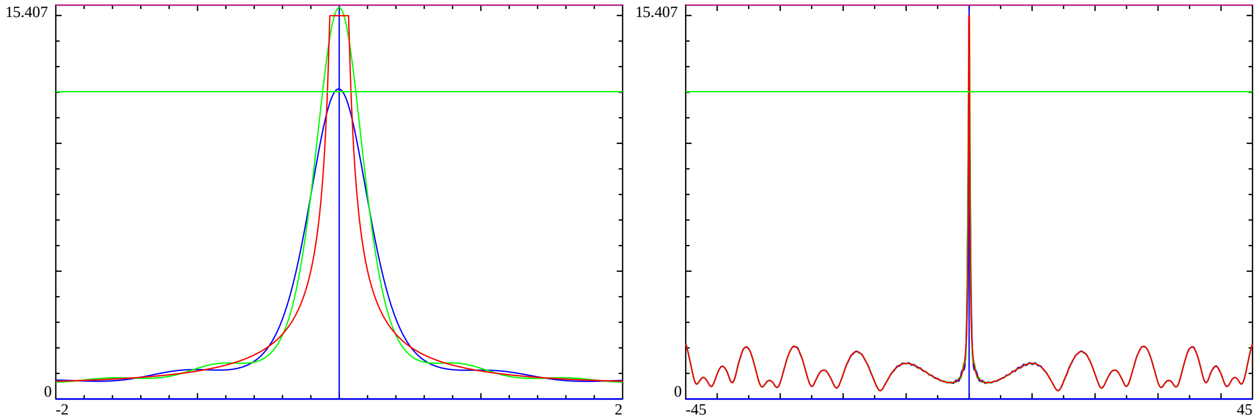

Figure 11: Line 1 Riemann Zeta mesoscale self-similarity about the large peak at location S=1+I*8.9362664425717E367. For this figure, the approximate height $e^{\gamma}\left(log(log(\theta(T+t)))\right)$ based growth is (horizontal green) peak=1+I*89362664425717139931337319204795671911467588339374631383495864144917601033831697216069013588193128049831564795716183589856967569098711364121699707297466237143643620370445717940725178020802166983309882614353917997632030763303096449720006300803931391668954761077509037742762061816566952782742617040349300888139692496123035856135839737951819455965721791490627713793766874

Figure 12, shows satellite peaks at 6820051 and 310678833629083965667540576593682.0582 associated with the large peak peak=1+I*23022029198330919381914545104908535282 94, where the left panel should be compared to figure 4 and right panel to figure 6 respectively. These satellite peaks are then surrounded by a second reflection of the riemann zeta function envelope pattern about the pole.

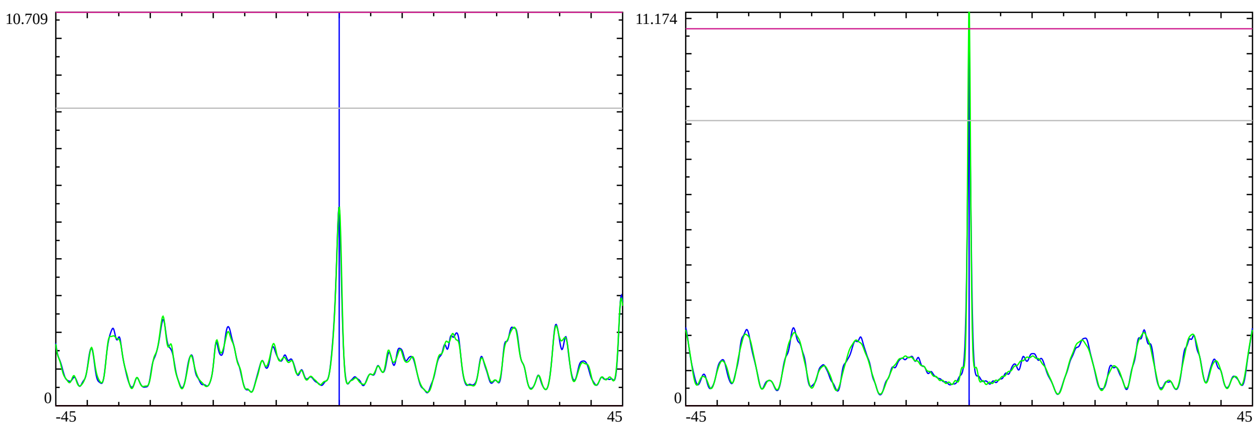

Figure 12: Line 1 Riemann Zeta mesoscale self-similarity about the large peak at location S=1+I*2.3022029198E39. peak=1+I*2302202919833091938191454510490853528294 Left panel t=(-45,45)+6820051, Right panel t=(-45,45)+310678833629083965667540576593682.0582.

Conclusions

By examining the known lower bounds for growth on the line $S=1+I*T$ the Riemann Zeta function and Euler Product about the pole can be approximately truncated (and translated) to explain the relatively constant width (~0.4) of large peaks at high T and the striking self-similarity in the envelope of the local structure (with Riemann Zeta behavior near the real axis).

An approximate estimate of the lower bound of the growth along the line $S=1+I*T$, $\left|\prod_{\rho=2}^{\lfloor 2\pi log(\theta(T+t))\rceil}\left\{1/(1-1/p^{(1+i*t)})\right\}\right|, t\to 0$ gives an interpretation of the growth of $|\zeta(1+I*t)|$ in terms of the average number of Riemann Zeta zeroes contained in the critical strip up to height T ($\bar{N}(T)\sim\theta(T)/\pi+1$).

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