The asymptotic behaviour of the logarithmic derivative of the functional equation of L functions identifies (i) a simple crosscheck on the numbering of the known Riemann Zeta function non-trivial zeroes and (ii) an improved first quiescent region formula for higher degree L functions.

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DRAFT Executive Summary

The first quiescent region of the dirichlet series sum of 1st degree L functions is located at $N_1(d=1, N_c) = \sqrt{(\frac{t}{2\pi} \cdot N_c)}$ where N_c is the L function conductor value. Combining the leading terms of the logarithmic derivative of the functional equation of L functions (on the critical line) proposes an improved first quiescent region formula for higher degree L functions $N_1(d, N_c) = (\frac{t}{2\pi})^{\frac{d}{2}} \cdot \sqrt{N_c}$ where d is the degree of the L function. This first quiescent region is investigated for higher degree L functions. Integrating the logarithmic derivative expression for the Riemann Zeta functional equation on the critical line gives a simple crosscheck expression for the numbering of the known non-trivial zeroes when analysing S function results.

Introduction

In this paper, the asymptotic behaviour of the logarithmic derivative of the functional equation of L function is expressed as a function of the first quiescent region of the dirichlet series of the L function. The first quiescent region of the dirichlet series sum generally contains a useful saddle point for Riemann Siegel formula based approximations of the L function.

Integrating the asymptotic logarithmic derivative on the critical line for the Riemann Zeta function and quantitatively comparing the results for the Rieman Zeta S function the integration constant is derived. This simple $imag(\log(\zeta(1/2 + I \cdot t)) - \log(\zeta(1 - (1/2 + I \cdot t)))$ expression provides a fast crosscheck method of reproducing the S function given the validity of the numbering of the known non-trivial zeroes.

The logarithmic derivative of the functional equation of L functions

Functional equation of the Riemann Zeta function

The Riemann Zeta function has the functional equation

$$\pi^{-\frac{s}{2}} \cdot \Gamma(\frac{s}{2}) \cdot \zeta(s) = \pi^{-\frac{(1-s)}{2}} \cdot \Gamma(\frac{(1-s)}{2}) \cdot \zeta(1-s) \tag{1}$$

where $s = \sigma + I \cdot t$ and $s \in \mathbb{C}$. Rearranging the terms

$$\zeta(s) = \frac{\pi^{-\frac{(1-s)}{2}} \cdot \Gamma(\frac{(1-s)}{2})}{\pi^{-\frac{s}{2}} \cdot \Gamma(\frac{s}{2})} \cdot \zeta(1-s) \tag{2}$$

$$\zeta(s) = \pi^{(s-\frac{1}{2})} \cdot \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \cdot \zeta(1-s) \tag{3}$$

and taking logarithms of both sides

$$\begin{aligned}\log\zeta(s) &= \log(\pi^{(s-\frac{1}{2})} \cdot \frac{\Gamma(\frac{(1-s)}{2})}{\Gamma(\frac{s}{2})} \cdot \zeta(1-s)) && (4)\\ &= (s-\frac{1}{2}) \cdot \log(\pi) + \log(\Gamma(\frac{(1-s)}{2})) - \log(\Gamma(\frac{s}{2})) + \log(\zeta(1-s)) && (5)\end{aligned}$$

after further rearranging terms

$$\log\zeta(s) - \log(\zeta(1-s)) = (s-\frac{1}{2}) \cdot \log(\pi) + \log(\Gamma(\frac{(1-s)}{2})) - \log(\Gamma(\frac{s}{2})) \tag{6}$$

and expanding the RHS using Stirling series for $t \to \infty$

$$\begin{aligned}\log\zeta(s) - \log(\zeta(1-s)) \approx\ & (s-\frac{1}{2}) \cdot \log(\pi)\\ & + \left[\frac{(1-s)}{2} \cdot \log(\frac{(1-s)}{2}) - \frac{(1-s)}{2} - \frac{1}{2} \cdot \log(\frac{(1-s)}{2}) + \frac{1}{2} \cdot \log(2\pi) + \frac{1}{12 \cdot (\frac{(1-s)}{2})} + ...\right]\\ & - \left[\frac{s}{2} \cdot \log(\frac{s}{2}) - \frac{s}{2} - \frac{1}{2} \cdot \log(\frac{s}{2}) + \frac{1}{2} \cdot \log(2\pi) + \frac{1}{12 \cdot (\frac{s}{2})} + ...\right] \quad \text{as } t \to \infty && (7)\\ \approx\ & (s-\frac{1}{2}) \cdot \log(\pi)\\ & + \left[\frac{(1-s)}{2} \cdot \log(\frac{(1-s)}{2}) - \frac{(1-s)}{2} - \frac{1}{2} \cdot \log(\frac{(1-s)}{2}) + \frac{1}{12 \cdot (\frac{(1-s)}{2})} + ...\right]\\ & - \left[\frac{s}{2} \cdot \log(\frac{s}{2}) - \frac{s}{2} - \frac{1}{2} \cdot \log(\frac{s}{2}) + \frac{1}{12 \cdot (\frac{s}{2})} + ...\right] \quad \text{as } t \to \infty && (8)\end{aligned}$$

taking the derivative $\frac{d}{ds}$

$$\begin{aligned}\frac{d}{ds}(\log\zeta(s)) - \frac{d}{ds}(\log(\zeta(1-s))) \approx\ & \frac{d}{ds}\Big((s-\frac{1}{2}) \cdot \log(\pi)\\ & + \left[\frac{(1-s)}{2} \cdot \log(\frac{(1-s)}{2}) - \frac{(1-s)}{2} - \frac{1}{2} \cdot \log(\frac{(1-s)}{2}) + \frac{1}{12 \cdot (\frac{(1-s)}{2})} + ...\right]\\ & - \left[\frac{s}{2} \cdot \log(\frac{s}{2}) - \frac{s}{2} - \frac{1}{2} \cdot \log(\frac{s}{2}) + \frac{1}{12 \cdot (\frac{s}{2})} + ...\right]\Big) \quad \text{as } t \to \infty && (9)\end{aligned}$$

$$
\begin{aligned}
\frac{\zeta'(s)}{\zeta(s)} + \frac{\zeta'(1-s)}{\zeta(1-s)} &\approx \log(\pi) \\
&+ \left[\frac{-1}{2} \cdot \log(\frac{(1-s)}{2}) + \frac{(1-s)}{2} \cdot \frac{-\frac{1}{2}}{\frac{(1-s)}{2}} + \frac{1}{2} - \frac{1}{2} \cdot \frac{-\frac{1}{2}}{\frac{(1-s)}{2}} + \frac{-1 \cdot -1}{12 \cdot (\frac{(1-s)^2}{2})} + ... \right] \\
&- \left[\frac{1}{2} \cdot \log(\frac{s}{2}) + \frac{s}{2} \cdot \frac{\frac{1}{2}}{\frac{s}{2}} - \frac{1}{2} - \frac{1}{2} \cdot \frac{\frac{1}{2}}{\frac{s}{2}} + \frac{-1}{12 \cdot (\frac{s^2}{2})} + ... \right] \qquad \text{as } t \to \infty \qquad (10) \\
&\approx \log(\pi) \\
&+ \left[\frac{-1}{2} \cdot \log(\frac{(1-s)}{2}) + \frac{1}{2(1-s)} + \frac{1}{6(1-s)^2} + ... \right] \\
&- \left[\frac{1}{2} \cdot \log(\frac{s}{2}) - \frac{1}{2s} - \frac{1}{6s^2} + ... \right] \qquad \text{as } t \to \infty \qquad (11)
\end{aligned}
$$

using the logarithmic derivative identities

$$
\frac{d}{ds} (\log \zeta(s)) = \frac{\zeta'(s)}{\zeta(s)} \qquad (12)
$$

$$
\frac{d}{ds} (\log(\zeta(1-s))) = -\frac{\zeta'(1-s)}{\zeta(1-s)} \qquad (13)
$$

collecting the leading terms together as a single logarithm

$$
\frac{\zeta'(s)}{\zeta(s)} + \frac{\zeta'(1-s)}{\zeta(1-s)} \approx \log\left(\frac{1}{\sqrt{\frac{(1-s)\cdot s}{2\pi}}} \right) + \left[\frac{1}{2(1-s)} + \frac{1}{6(1-s)^2} + \frac{1}{2s} + \frac{1}{6s^2} + ... \right] \qquad \text{as } t \to \infty \qquad (14)
$$

and finally rewriting the leading term using the identity $\log(z) = 2\log(\sqrt{z})$ and combining the minor terms gives the asymptotic behaviour of the logarithmic derivative of the functional equation that explicitly acknowledges the s:(1-s) symmetry of the Riemann Zeta function.

$$
\frac{\zeta'(s)}{\zeta(s)} + \frac{\zeta'(1-s)}{\zeta(1-s)} \approx 2 \cdot \log\left(\frac{1}{\sqrt{\sqrt{\frac{(1-s)\cdot s}{2\pi}}}} \right) + \left[\frac{1}{2 \cdot (1-s) \cdot s} + \frac{(1 - 2 \cdot s + 2 \cdot s^2)}{6 \cdot (1-s)^2 \cdot s^2} + ... \right] \qquad \text{as } t \to \infty \quad (15)
$$

The following three results for the logarithmic derivative of the Riemann Zeta functional equation when s=$\{(0.5 + i \cdot t), (1 + i \cdot t), (0 + i \cdot t)\}$ are obtained from equation (15). The details of the series derivation are given in Appendix A.

$$
\frac{\zeta'(0.5 + i \cdot t)}{\zeta(0.5 + i \cdot t)} + \frac{\zeta'(0.5 - i \cdot t)}{\zeta(0.5 - i \cdot t)} \sim 2 \cdot \log\left(\frac{1}{\sqrt{\frac{t}{2\pi}}} \right) + \frac{1}{24 \cdot t^2} + ... \qquad \text{as } t \to \infty \qquad (16)
$$

$$
\sim 2 \cdot \log\left(\frac{1}{\left(\sqrt{\frac{t}{(2\pi)}}\right)^1 \cdot \sqrt{1}} \right) + \frac{1}{24 \cdot t^2} + ... \qquad \text{as } t \to \infty \qquad (17)
$$

noting in equation (17) that $\zeta(s)$ has degree=1 and $N_C = 1$. In practice the above formula works excellently for t > 2.

Away from the critical line, the logarithmic derivative of the functional equation also has imaginary component.

$$\frac{\zeta'(1+i\cdot t)}{\zeta(1+i\cdot t)}+\frac{\zeta'(0-i\cdot t)}{\zeta(0-i\cdot t)}\sim 2\cdot\log\left(\frac{1}{\sqrt{\frac{t}{2\pi}}}\right)+\frac{i}{2\cdot t}-\frac{1}{12\cdot t^2}+...\qquad \text{as } t\to\infty \tag{18}$$

$$\frac{\zeta'(0+i\cdot t)}{\zeta(0+i\cdot t)}+\frac{\zeta'(1-i\cdot t)}{\zeta(1-i\cdot t)}\sim 2\cdot\log\left(\frac{1}{\sqrt{\frac{t}{2\pi}}}\right)-\frac{i}{2\cdot t}-\frac{1}{12\cdot t^2}+...\qquad \text{as } t\to\infty \tag{19}$$

A simple crosscheck on the numbering of the known non-trivial zeroes in the S function of the Riemann Zeta function

Noting Cauchy-Riemann behaviour for complex functions and that t is the $\Im(s)$, the partial integral of equation (17) with respect to t can be expressed as

$$I\cdot\int\left[2\cdot\log\left(\frac{1}{\left(\sqrt{\frac{t}{(2\pi)}}\right)^1\cdot\sqrt{1}}\right)+\frac{1}{24\cdot t^2}\right]dt=I\cdot\int\left[-\log(t)-\log(2\pi)+\frac{1}{24\cdot t^2}\right]dt \tag{20}$$

$$=I\cdot\left[-(t\cdot\log(t)-t)-t\cdot\log(2\pi)-\frac{1}{24\cdot t}+C\right] \tag{21}$$

$$=I\cdot\left[-t\cdot\log(\frac{t}{2\pi})+t-\frac{1}{24\cdot t}+C\right] \tag{22}$$

where C is the integration constant.

On inspection in the interval 1 < t < 30002, the continuous curve $-3\log(2\pi)+0.0158844-t\cdot\log(\frac{t}{2\pi})+t-\frac{1}{24\cdot t}$ is an excellent match for the slope of the discontinuous function $\log(\zeta(1/2+i\cdot t))-\log(\zeta(1-(1/2+i\cdot t)))=2\cdot\text{imag}\left(\log(\zeta(1/2+i\cdot t))\right)$.

Figure 1, illustrates that the addition of $2*\pi$ for every known Riemann Zeta non-trivial zero ($\gamma\le t$) to the continuous curve $-3\log(2\pi)+0.0158844-t\cdot\log(\frac{t}{2\pi})+t-\frac{1}{24\cdot t}$ brings excellent agreement with the discontinuous curve $2\cdot\text{imag}\left(\log(\zeta(1/2+i\cdot t))\right)$

$$\therefore C=\text{imag}\left[\log(\zeta(1/2+i\cdot t))-\log(\zeta(1-(1/2+i\cdot t)))\right]$$

$$+\left[t\cdot\log(\frac{t}{2\pi})-t+\frac{1}{24\cdot t}+...+\sum_{\gamma_1=14.1347...}^{\gamma_n\le t}2\pi\cdot\delta(\gamma_i\le t)\right] \tag{23}$$

$$\approx -5.497746799 \tag{24}$$

$$\approx -3\log(2\pi)+0.0158844 \tag{25}$$

Figure 2, illustrates that equations (22-25) with n=13999521 for $\gamma_n=6820048.3979145...$ (green line) can be used to very quickly crosscheck the S function calculation (red line) in the interval t=(6820048.91,6820053) thus validating n=13999521 for $\gamma_n=6820048.3979145...$

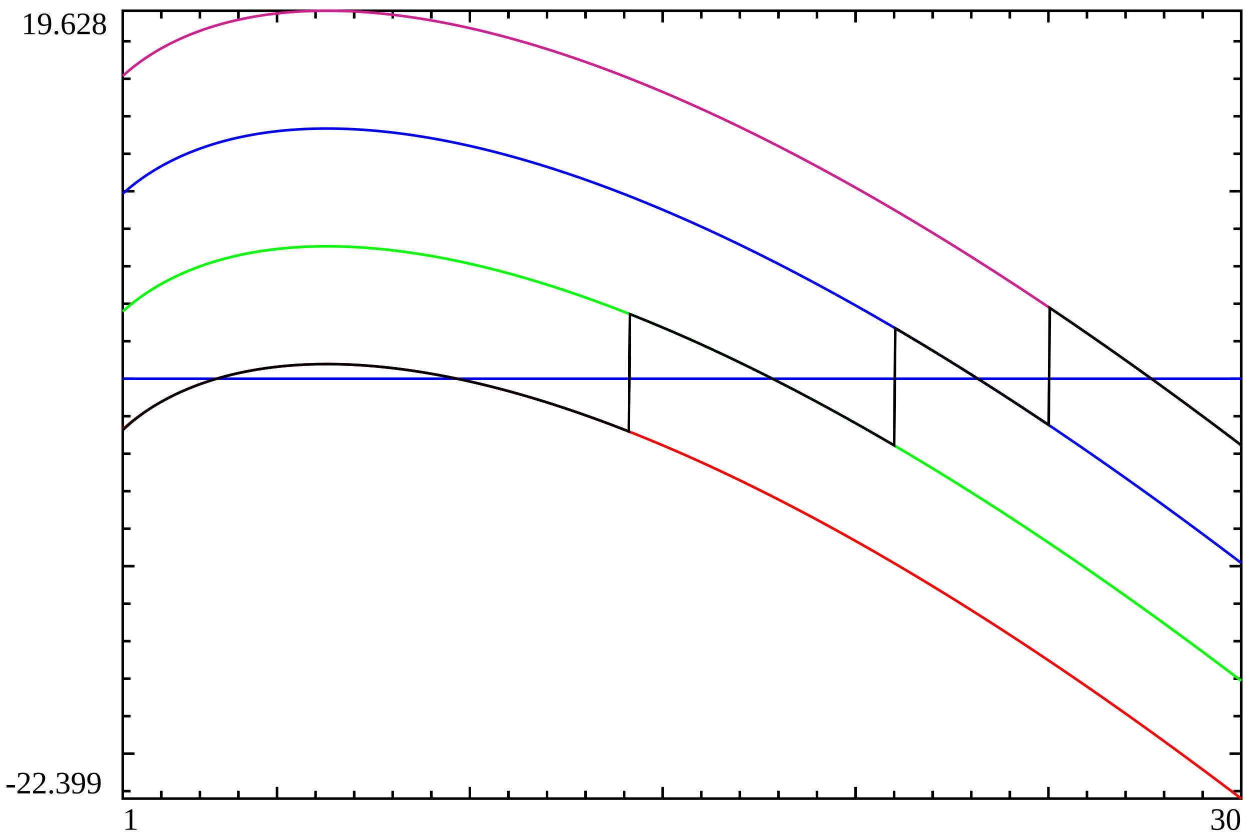

Figure 1: Imaginary component (black) of $(\log(\zeta(1/2+i\cdot t))-\log(\zeta(1-(1/2+i\cdot t)))) = 2$ *Pi*S$(\zeta(1/2+i\cdot t))$ function 1<Im(s)<30 plus shifted versions (+0 red,+2π green,+4π blue,+6π violet-red) of the partial integral (wrt t) of the asymptotic logarithmic derivative of the functional equation

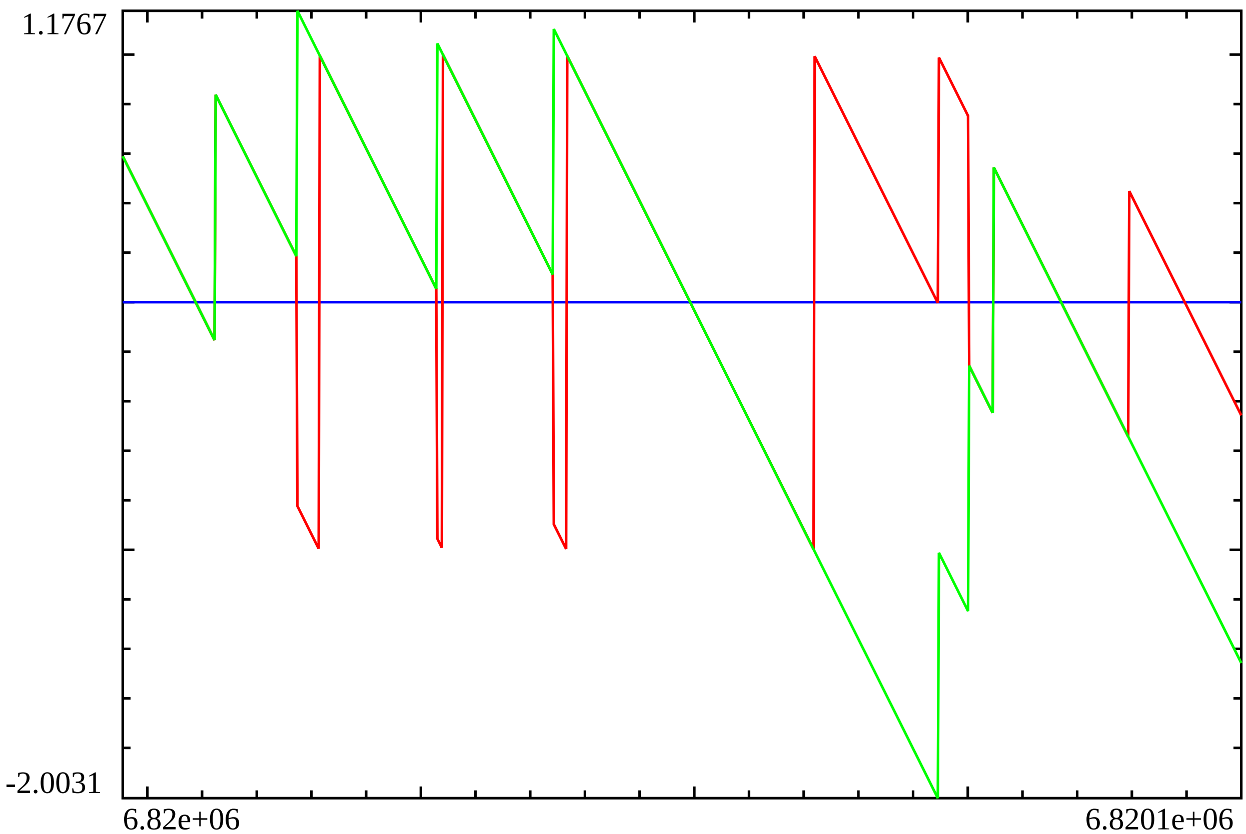

Figure 2: Imaginary component (red line) of imag$(\log(\zeta(1/2+i\cdot t))-\log(\zeta(1-(1/2+i\cdot t))))/2/\pi = \mathrm{S}(\zeta(1/2+i\cdot t))$ function calculated using tapered Dirichlet series zeroth order Riemann Siegel formula in the interval 6820048.91<Im(s)<6820053 crosschecked by (equations (22-25) , n=13999521 for $\gamma_n = 6820048.3979145...$ and $2\cdot\pi$ for each of the following non-trivial zeroes) all scaled by $1/2/\pi$.

Functional equation of the Elliptic curve with LMFDB label 27.a4

Minimal Weierstrass equation

$$y^2 + y = x^3 \tag{26}$$

L-function 2-3e3-1.1-c1-0-0

Degree 2

Conductor $27 = 3^3$

Selberg data (2, 27, (:1/2),1)

L(27.a4,s) has the functional equation

$$27^{\frac{s}{2}} \cdot 2 \cdot (2\pi)^{-s} \cdot \Gamma(s) \cdot L(27.a4, s) = 27^{\frac{(2-s)}{2}} \cdot 2 \cdot (2\pi)^{-(2-s)} \cdot \Gamma(2-s) \cdot L(27.a4, 2-s) \tag{27}$$

where $s = \sigma + I \cdot t$ and $s \in \mathbb{C}$. Rearranging the terms

$$L(27.a4, s) = \frac{27^{\frac{(2-s)}{2}} \cdot 2 \cdot (2\pi)^{-(2-s)} \cdot \Gamma(2-s)}{27^{\frac{s}{2}} \cdot 2 \cdot (2\pi)^{-s} \cdot \Gamma(s)} \cdot L(27.a4, 2-s) \tag{28}$$

$$L(27.a4, s) = 27^{(1-s)} \cdot (2\pi)^{2\cdot(s-1)} \cdot \frac{\Gamma(2-s)}{\Gamma(s)} \cdot L(27.a4, 2-s) \tag{29}$$

and taking logarithms of both sides

$$\log L(27.a4, s) = \log\left(27^{(1-s)} \cdot (2\pi)^{2\cdot(s-1)} \cdot \frac{\Gamma(2-s)}{\Gamma(s)} \cdot L(27.a4, 2-s)\right) \tag{30}$$

$$= (1-s) \cdot \log(27) + 2 \cdot (s-1) \cdot \log(2\pi) + \log(\Gamma(2-s)) - \log(\Gamma(s)) + \log(L(27.a4, 2-s)) \tag{31}$$

after further rearranging terms

$$\log L(27.a4, s) - \log(L(27.a4, 2-s)) = (1-s) \cdot \log(27) + 2 \cdot (s-1) \cdot \log(2\pi) + \log(\Gamma(2-s)) - \log(\Gamma(s)) \tag{32}$$

and expanding the RHS using Stirling series for $t \to \infty$

$$\begin{aligned} \log L(27.a4, s) - \log(L(27.a4, 2-s)) \approx\ & (1-s) \cdot \log(27) + 2 \cdot (s-1) \cdot \log(2\pi) \\ & + \left[(2-s) \cdot \log(2-s) - (2-s) - \frac{1}{2} \cdot \log(2-s) + \frac{1}{2} \cdot \log(2\pi) + \frac{1}{12 \cdot (2-s)} + ...\right] \\ & - \left[s \cdot \log(s) - s - \frac{1}{2} \cdot \log(s) + \frac{1}{2} \cdot \log(2\pi) + \frac{1}{12 \cdot (s)} + ...\right] \qquad \text{as } t \to \infty \end{aligned} \tag{33}$$

$$\begin{aligned} \approx\ & (1-s) \cdot \log(27) + 2 \cdot (s-1) \cdot \log(2\pi) \\ & + \left[(2-s) \cdot \log(2-s) - (2-s) - \frac{1}{2} \cdot \log(2-s) + \frac{1}{12 \cdot (2-s)} + ...\right] \\ & - \left[s \cdot \log(s) - s - \frac{1}{2} \cdot \log(s) + \frac{1}{12 \cdot (s)} + ...\right] \qquad \text{as } t \to \infty \end{aligned} \tag{34}$$

taking the derivative $\frac{d}{ds}$

$$\begin{aligned}\frac{d}{ds}\left(\log L(27.a4,s)\right)-\frac{d}{ds}\left(\log(L(27.a4,2-s))\right)\approx \frac{d}{ds}\Big(&(1-s)\cdot\log(27)+2\cdot(s-1)\cdot\log(2\pi)\\ &+\left[(2-s)\cdot\log(2-s)-(2-s)-\frac{1}{2}\cdot\log(2-s)+\frac{1}{12\cdot(2-s)}+...\right]\\ &-\left[s\cdot\log(s)-s-\frac{1}{2}\cdot\log(s)+\frac{1}{12\cdot(s)}+...\right]\Big)\qquad \text{as } t\to\infty\end{aligned} \tag{35}$$

$$\begin{aligned}\frac{L'(27.a4,s)}{L(27.a4,s)}+\frac{L(27.a4,2-s)}{L(27.a4,2-s)}\approx&-\log(27)+2\cdot\log(2\pi)\\ &+\left[-\log(2-s)+(2-s)\cdot\frac{-1}{(2-s)}+1-\frac{1}{2}\cdot\frac{-1}{(2-s)}+\frac{-1\cdot-1}{12\cdot(2-s)^2}+...\right]\\ &-\left[\log(s)+s\cdot\frac{1}{s}-1-\frac{1}{2}\cdot\frac{1}{s}+\frac{-1}{12\cdot s^2}+...\right]\qquad \text{as } t\to\infty\end{aligned} \tag{36}$$

$$\begin{aligned}\approx&-\log(27)+2\cdot\log(2\pi)\\ &+\left[-\log(2-s)+\frac{1}{2\cdot(2-s)}+\frac{1}{12\cdot(2-s)^2}+...\right]\\ &-\left[\log(s)-\frac{1}{2\cdot s}-\frac{1}{12\cdot s^2}+...\right]\qquad \text{as } t\to\infty\end{aligned} \tag{37}$$

using the logarithmic derivative identities

$$\frac{d}{ds}\left(\log L(27.a4,s)\right)=\frac{L'(27.a4,s)}{L(27.a4,s)} \tag{38}$$

$$\frac{d}{ds}\left(\log(L(27.a4,2-s))\right)=-\frac{L'(27.a4,2-s)}{L(27.a4,2-s)} \tag{39}$$

collecting the leading terms together as a single logarithm

$$\frac{L'(27.a4,s)}{L(27.a4,s)}+\frac{L(27.a4,2-s)}{L(27.a4,2-s)}\approx\log\left(\frac{1}{\frac{27\cdot(2-s)\cdot s}{(2\pi)^2}}\right)+\left[\frac{1}{2(2-s)}+\frac{1}{2s}+\frac{1}{12(2-s)^2}+\frac{1}{12s^2}+...\right]\qquad \text{as } t\to\infty \tag{40}$$

and finally rewriting the leading term using the identity $\log(z)=2\log(\sqrt{z})$ and combining the minor terms gives the asymptotic behaviour of the logarithmic derivative of the functional equation that explicitly acknowledges the s:(1-s) symmetry of the Riemann Zeta function.

$$\frac{L'(27.a4,s)}{L(27.a4,s)}+\frac{L(27.a4,2-s)}{L(27.a4,2-s)}\approx 2\cdot\log\left(\frac{1}{\sqrt{\frac{27\cdot(2-s)\cdot s}{(2\pi)^2}}}\right)+\left[\frac{1}{(2-s)\cdot s}+\frac{(1-2\cdot s+s^2)}{6\cdot(2-s)^2\cdot s^2}+...\right]\qquad \text{as } t\to\infty \tag{41}$$

on the critical line for L(27.a4,s) $s = 1 + i \cdot t$

$$\frac{L'(27.a4, 1+i\cdot t)}{L(27.a4, 1+i\cdot t)} + \frac{L(27.a4, 1-i\cdot t)}{L(27.a4, 1-i\cdot t)} \approx 2\cdot\log\left(\frac{1}{\sqrt{\frac{27\cdot(1+t^2)}{(2\pi)^2}}}\right) + \left[\frac{1}{(1+t^2)} + \frac{-t^2}{6\cdot(1+t^2)^2} + ...\right] \quad \text{as } t\to\infty \tag{42}$$

$$\approx 2\cdot\log\left(\frac{1}{\sqrt{\frac{27\cdot t^2}{(2\pi)^2}}\cdot(1+\frac{1}{t^2})^{\frac{1}{2}}}\right) + \left[\frac{1}{(1+t^2)} + \frac{-t^2}{6\cdot(1+t^2)^2} + ...\right] \quad \text{as } t\to\infty \tag{43}$$

$$\approx 2\cdot\log\left(\frac{1}{\frac{t\cdot\sqrt{27}}{(2\pi)}}\right) - \log(1+\frac{1}{t^2}) + \left[\frac{1}{(1+t^2)} + \frac{-t^2}{6\cdot(1+t^2)^2} + ...\right] \quad \text{as } t\to\infty \tag{44}$$

$$\sim 2\cdot\log\left(\frac{1}{\frac{t\cdot\sqrt{27}}{(2\pi)}}\right) - \frac{1}{t^2} + \left[\frac{1}{t^2} - \frac{1}{6\cdot t^2} + ...\right] \quad \text{as } t\to\infty \tag{45}$$

$$\sim 2\cdot\log\left(\frac{1}{\frac{t\cdot\sqrt{27}}{(2\pi)}}\right) - \frac{1}{6\cdot t^2} + ... \quad \text{as } t\to\infty \tag{46}$$

$$\sim 2\cdot\log\left(\frac{1}{\left(\sqrt{\frac{t}{(2\pi)}}\right)^2\cdot\sqrt{27}}\right) - \frac{1}{6\cdot t^2} + ... \quad \text{as } t\to\infty \tag{47}$$

noting in equation (47) that L(27.a4,s) has degree=2 and $N_C = 27$. In practice the above formula works excellently for t > 2.

Functional equation of the number field 14.0.9095120158391.1

Normalized defining polynomial

$$x^{14} - 7x^{13} + 25x^{12} - 59x^{11} + 103x^{10} - 141x^9 + 159x^8 - 153x^7 + 129x^6 - 95x^5 + 58x^4 - 27x^3 + 10x^2 - 3x + 1 \tag{48}$$

L-function number field 14.0.9095120158391.1

Degree 14

Discriminant $-71^7 = -9095120158391$

Following the previous results it is expected on the critical line for L(14.0.9095120158391.1,s) $s = 0.5 + i \cdot t$

$$\frac{L'(14.0.9095120158391.1, 0.5+i\cdot t)}{L(14.0.9095120158391.1, 0.5+i\cdot t)} + \frac{L'(14.0.9095120158391.1, 0.5-i\cdot t)}{L(14.0.9095120158391.1, 0.5-i\cdot t)} \sim 2\cdot\log\left(\frac{1}{\left(\sqrt{\frac{t}{(2\pi)}}\right)^{14}\cdot\sqrt{9095120158391}}\right) + ... \quad \text{as } t\to\infty \tag{49}$$

noting in equation (49) that L(14.0.9095120158391.1,s) has degree=14 and |discriminant| = 9095120158391.

In practice the above formula works excellently for t > 2 as shown in figure 3

Code snippets

```
from https://www.lmfdb.org/NumberField/14.0.9095120158391.1
\\ Pari/GP code for working with number field 14.0.9095120158391.1

\\ Some of these functions may take a long time to execute (this depends on the field).

\\ Define the number field:
K = bnfinit(y^14 - 7*y^13 + 25*y^12 - 59*y^11 + 103*y^10 - 141*y^9 + 159*y^8
- 153*y^7 + 129*y^6 - 95*y^5 + 58*y^4 - 27*y^3 + 10*y^2 - 3*y + 1, 1)
```

Straightforward pari gp code for producing a graph of the logarithmic derivative of the functional equation and comparison to series approximation

```
ploth(t=1,30,[real(lfun(K,0.5+I*t,1)/lfun(K,0.5+I*t)+lfun(K,1-(0.5+I*t),1)/lfun(K,1-(0.5+I*t))),
2*log(1/(sqrt(t/2/Pi))^14/sqrt(9095120158391))])
```

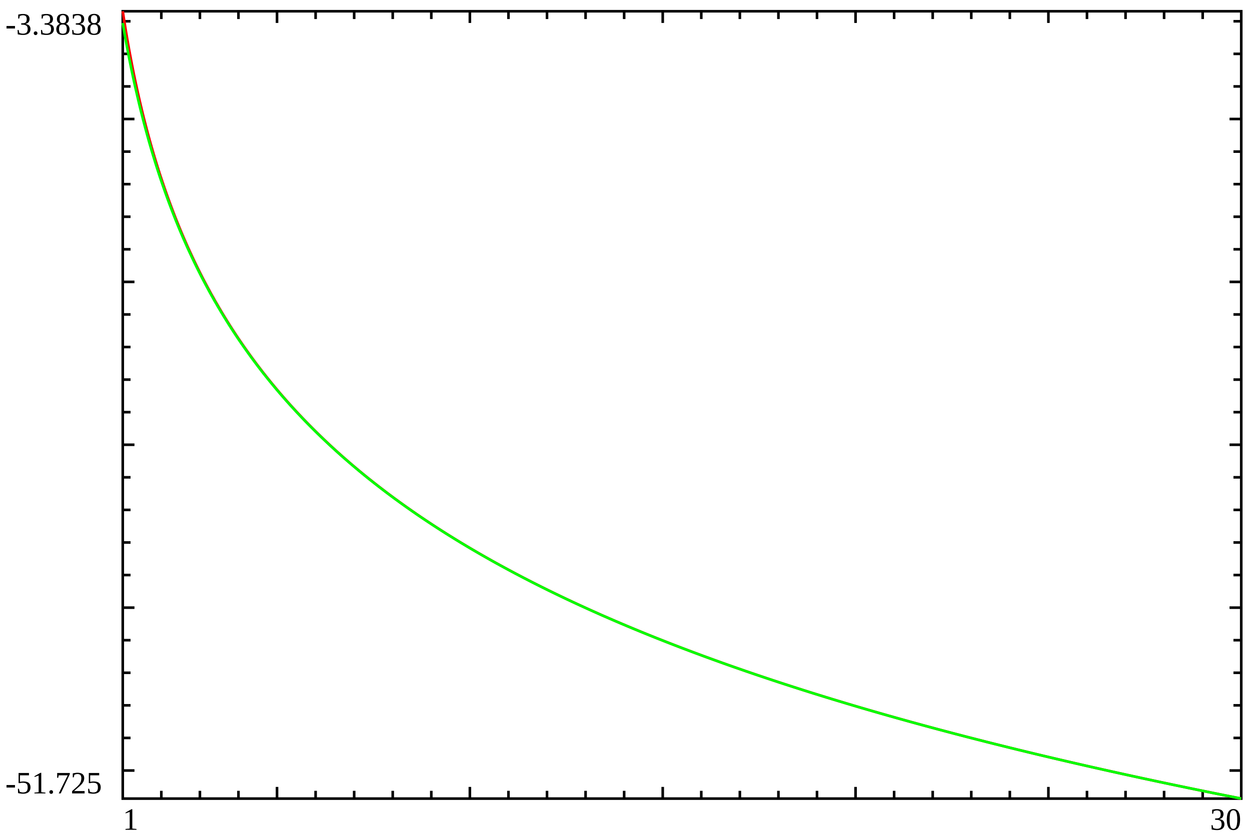

Figure 3: Logarithmic derivative of the functional equation (red line) of the L function 14.0.9095120158391.1 in the interval t=(1,30) using pari gp functions for L function and their first derivative. Green line is RHS equation (49).

Conclusions

The leading term of the logarithmic derivative of the functional equation on the critical line has a dependence on $\left(\sqrt{\frac{t}{2\pi}}\right)^{d/2}\sqrt{N_C}$ which can also be identified as the first quiescent region of the Dirichlet series of L function. Integrating the critical line behaviour of the logarithmic derivative of the functional equation with respect to $\Im(s)$ can produce a simple expression for the Riemman Zeta function that can be used in conjunction with S function calculations to check the numbering of non-trivial zeroes.

References

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Appendix A: Asymptotic behaviour of logarithimic derivative of Riemann Zeta functional equation as $t \to \infty$

Using series expansions for sqrt and log factors and retaining the leading terms as $t \to \infty$.

Inspecting the asymptotic behaviour on the critical line $s = 0.5 + i \cdot t$

$$\frac{\zeta'(0.5+i\cdot t)}{\zeta(0.5+i\cdot t)} + \frac{\zeta'(0.5-i\cdot t)}{\zeta(0.5-i\cdot t)} \approx 2\cdot\log\left(\frac{1}{\sqrt{\frac{\sqrt{(0.5-i\cdot t)\cdot(0.5+i\cdot t)}}{2\pi}}}\right) + \left[\frac{1}{2\cdot(0.5-i\cdot t)\cdot(0.5+i\cdot t)} + \frac{(1-2\cdot(0.5+i\cdot t)+2\cdot(0.5+i\cdot t)^2)}{6\cdot(0.5-i\cdot t)^2\cdot(0.5+i\cdot t)^2} + ...\right] \qquad \text{as } t\to\infty \tag{50}$$

$$\approx 2\cdot\log\left(\frac{1}{\sqrt{\frac{\sqrt{(0.25+t^2)}}{2\pi}}}\right) + \left[\frac{1}{2\cdot(0.25+t^2)} + \frac{(0.5-2\cdot t^2)}{6\cdot(0.25+t^2)^2} + ...\right] \qquad \text{as } t\to\infty \tag{51}$$

$$\sim 2\cdot\log\left(\frac{1}{\sqrt{\frac{t}{2\pi}}\cdot(1+\frac{1}{4\cdot t^2})^{\frac{1}{4}}}\right) + \left[\frac{1}{2\cdot t^2} - \frac{1}{3\cdot t^2} + ...\right] \qquad \text{as } t\to\infty \tag{52}$$

$$\sim 2\cdot\log\left(\frac{1}{\sqrt{\frac{t}{2\pi}}}\right) - \frac{1}{2}\cdot\log(1+\frac{1}{4t^2}) + \frac{1}{6\cdot t^2} + ... \qquad \text{as } t\to\infty \tag{53}$$

$$\sim 2\cdot\log\left(\frac{1}{\sqrt{\frac{t}{2\pi}}}\right) - \frac{1}{8\cdot t^2} + \frac{1}{6\cdot t^2} + ... \qquad \text{as } t\to\infty \tag{54}$$

$$\sim 2\cdot\log\left(\frac{1}{\sqrt{\frac{t}{2\pi}}}\right) + \frac{1}{24\cdot t^2} + ... \qquad \text{as } t\to\infty \tag{55}$$

Inspecting the asymptotic behaviour for $s = 1 + i \cdot t$

$$\frac{\zeta'(1+i\cdot t)}{\zeta(1+i\cdot t)} + \frac{\zeta'(0-i\cdot t)}{\zeta(0-i\cdot t)} \approx 2\cdot\log\left(\frac{1}{\sqrt{\frac{\sqrt{(0-i\cdot t)\cdot(1+i\cdot t)}}{2\pi}}}\right) + \left[\frac{1}{2\cdot(0-i\cdot t)\cdot(1+i\cdot t)} + \frac{(1-2\cdot(1+i\cdot t)+2\cdot(1+i\cdot t)^2)}{6\cdot(0-i\cdot t)^2\cdot(1+i\cdot t)^2} + ...\right] \quad \text{as } t\to\infty \tag{56}$$

$$\approx 2\cdot\log\left(\frac{1}{\sqrt{\frac{\sqrt{(-i\cdot t+t^2)}}{2\pi}}}\right) + \left[\frac{1}{2\cdot(-i\cdot t+t^2)} + \frac{(i\cdot 2\cdot t-2\cdot t^2)}{6\cdot(-i\cdot t+t^2)^2} + ...\right] \quad \text{as } t\to\infty \tag{57}$$

$$\sim 2\cdot\log\left(\frac{1}{\sqrt{\frac{t\cdot\sqrt{(1-\frac{i}{t})}}{2\pi}}}\right) + \left[\frac{1}{2\cdot t^2} - \frac{1}{3\cdot t^2} + ...\right] \quad \text{as } t\to\infty \tag{58}$$

$$\sim 2\cdot\log\left(\frac{1}{\sqrt{\frac{t}{2\pi}}\cdot(1-\frac{i}{t})^{\frac{1}{4}}}\right) + \frac{1}{6\cdot t^2} + ... \quad \text{as } t\to\infty \tag{59}$$

$$\sim 2\cdot\log\left(\frac{1}{\sqrt{\frac{t}{2\pi}}}\right) - \frac{1}{2}\cdot\log(1-\frac{i}{t}) + \frac{1}{6\cdot t^2} + ... \quad \text{as } t\to\infty \tag{60}$$

$$\sim 2\cdot\log\left(\frac{1}{\sqrt{\frac{t}{2\pi}}}\right) + \frac{i}{2\cdot t} - \frac{1}{4\cdot t^2} + \frac{1}{6\cdot t^2} + ... \quad \text{as } t\to\infty \tag{61}$$

$$\sim 2\cdot\log\left(\frac{1}{\sqrt{\frac{t}{2\pi}}}\right) + \frac{i}{2\cdot t} - \frac{1}{12\cdot t^2} + ... \quad \text{as } t\to\infty \tag{62}$$

Inspecting the asymptotic behaviour for $s = 0 + i \cdot t$

$$\frac{\zeta'(0+i\cdot t)}{\zeta(0+i\cdot t)} + \frac{\zeta'(1-i\cdot t)}{\zeta(1-i\cdot t)} \approx 2\cdot\log\left(\frac{1}{\sqrt{\frac{\sqrt{(1-i\cdot t)\cdot(0+i\cdot t)}}{2\pi}}}\right) + \left[\frac{1}{2\cdot(1-i\cdot t)\cdot(0+i\cdot t)} + \frac{(1-2\cdot(0+i\cdot t)+2\cdot(0+i\cdot t)^2)}{6\cdot(1-i\cdot t)^2\cdot(0+i\cdot t)^2} + ...\right] \qquad \text{as } t\to\infty \tag{63}$$

$$\approx 2\cdot\log\left(\frac{1}{\sqrt{\frac{\sqrt{(i\cdot t+t^2)}}{2\pi}}}\right) + \left[\frac{1}{2\cdot(i\cdot t+t^2)} + \frac{(-i\cdot 2\cdot t-2\cdot t^2)}{6\cdot(i\cdot t+t^2)^2} + ...\right] \qquad \text{as } t\to\infty \tag{64}$$

$$\sim 2\cdot\log\left(\frac{1}{\sqrt{\frac{t\cdot\sqrt{(1+\frac{i}{t})}}{2\pi}}}\right) + \left[\frac{1}{2\cdot t^2} - \frac{1}{3\cdot t^2} + ...\right] \qquad \text{as } t\to\infty \tag{65}$$

$$\sim 2\cdot\log\left(\frac{1}{\sqrt{\frac{t}{2\pi}}\cdot(1+\frac{i}{t})^{\frac{1}{4}}}\right) + \frac{1}{6\cdot t^2} + ... \qquad \text{as } t\to\infty \tag{66}$$

$$\sim 2\cdot\log\left(\frac{1}{\sqrt{\frac{t}{2\pi}}}\right) - \frac{1}{2}\cdot\log{(1+\frac{i}{t})} + \frac{1}{6\cdot t^2} + ... \qquad \text{as } t\to\infty \tag{67}$$

$$\sim 2\cdot\log\left(\frac{1}{\sqrt{\frac{t}{2\pi}}}\right) - \frac{i}{2\cdot t} - \frac{1}{4\cdot t^2} + \frac{1}{6\cdot t^2} + ... \qquad \text{as } t\to\infty \tag{68}$$

$$\sim 2\cdot\log\left(\frac{1}{\sqrt{\frac{t}{2\pi}}}\right) - \frac{i}{2\cdot t} - \frac{1}{12\cdot t^2} + ... \qquad \text{as } t\to\infty \tag{69}$$