The behaviour of non-trivial zeroes in tapered zeroth order Riemann Siegel formula finite Dirichlet Series about the first quiescent region with lower symmetry dirichlet coefficients near high Riemann Zeta function peaks.

John Martin

June 24, 2024

Executive Summary

An investigation of the non-trivial zero behaviour about five known large Riemann Zeta function peaks at t={363991205.178, 673297382.184, 1387123309.985, 2381374874120.454, 4257232978148261.802} for a simple perturbation of the dirichlet coefficients of tapered finite Dirichlet Series of the zeroth order Riemann Siegel formula is reported. The distinctive non-trivial zero behaviour under simple perturbation nearby these particular large peaks appears related to non-trivial zeroes of the Riemann Zeta function with $|S| \gtrsim 1.7$. The same behaviour is observed for the (more accurate approximation of the Riemann Zeta function given by the) tapered finite Dirichlet Series truncated at the second quiescent region for smaller Riemann Zeta function peaks occurring nearby the first three Rosser rule violations.

Introduction

The tapered finite Dirichlet Series truncated about the second quiescent region in the final plateau of the oscillatory divergence of such dirichlet series provides a useful approximation of the mean value of the infinite series sum (i.e., averaging out the oscillatory divergence) [1-4]. Perturbation of such an series approximation function about the second quiescent region provides access to complex plane functions that encompasses lower symmetry than the L functions.

In this paper, the non-trivial zero behaviour of a simple perturbation of the dirichlet coefficients of the Riemann Zeta function [5-8] that does not change the location of the second (and first) quiescent region and (in the oscillatory divergence of Riemann Zeta Dirichlet Series) is investigated for the tapered zeroth order Riemann Siegel function. Perturbation of the zeroth order Riemann Siegel function is attempted to allow (much faster calculation of the) approximate estimates of the lower symmetry behaviour of non-trivial zeroes nearby large Riemann Zeta peaks with heights 100-850.

A simple perturbation of the Riemann Zeta function to produce lower symmetry behaviour

A simple perturbation of the Rieman Zeta function can be achieved by the modified function

$$\zeta(s,\alpha)_{\text{pert}} = (1-\alpha) + \alpha \cdot \zeta(s) \tag{1}$$

The impact of the perturbation is that the symmetry of the function is lowered (compared to the Riemann Zeta function) and non-trivial zeroes can be observed away from the critical line (where in the unmodified function none have been observed to occur).

This perturbation was one of two perturbations examined in [9] where the Riemann Zeta function was approximated by the finite Riemann Zeta Dirichlet Series truncated at the second quiescent region $N = \frac{t}{\pi}$ which is a useful approximation (of the Riemann Zeta function) away from the real axis. The use of the Dirichlet Series approximation allowed first principles calculation of the first and second derivatives of the complex function with respect to the real and imaginary components ($s = \sigma + I \cdot t$) necessary for quadrature based searches of non-trivial zero locations and hence trajectories under perturbation. In terms of the Dirichlet Series the above perturbation can be expressed as

$$\zeta(s,\alpha)_{\text{pert}} \approx (1-\alpha) + \alpha \cdot \sum_{k=1}^{(\lfloor \frac{t}{\pi} \rfloor)} \left(\frac{1}{k^s}\right) = 1 + \alpha \cdot \sum_{k=2}^{(\lfloor \frac{t}{\pi} \rfloor)} \left(\frac{1}{k^s}\right) \tag{2}$$

using truncation at the second quiescent region $N = \frac{t}{\pi}$ to allow approximation of the Riemann Zeta function when $\sigma \leq 1$.

To achieve a more accurate approximation of perturbation of the Riemann Zeta function (away from the real axis) when $\sigma \leq 1$, a 128 tapered finite Riemann Zeta Dirichlet Series truncated at the second quiescent region $N = \frac{t}{\pi}$ is available

$$\zeta(s,\alpha)_{\text{pert}} \approx 1 + \alpha \cdot \left[\sum_{k=2}^{(\lfloor \frac{t}{\pi} \rfloor - p)} \left(\frac{1}{k^s}\right) + \sum_{i=(-p+1)}^{p} \frac{\frac{1}{2^{2p}} \left(2^{2p} - \sum_{k=0}^{i+p-1} \binom{2p}{2p-k}\right)}{(\lfloor \frac{t}{\pi} \rfloor + i)^s} \right] \qquad \text{as } t \to \infty \tag{3}$$

where 2p=128 (for 128 point tapering) which is used in this paper. Results using the above expression are given in the appendix for t $<$ 14,254,000 nearby the first three Rosser rule violations [10,11] and this approach was used in [5].

However, to allow feasible calculations of the approximate behaviour of the Riemann Zeta function at higher values along the imaginary co-ordinate in the complex plane, the perturbed 128 tapered finite zeroth order Riemann Siegel formula is the main expression used in this paper. Firstly given the Riemann Siegel formula [12,13] and its zeroth order component using truncation at the first quiescent region $N = \sqrt{\frac{t}{2\pi}}$

$$\zeta(s) = R(s) + \chi(s) \cdot \bar{R}(s) \tag{4}$$

$$\approx \sum_{k=1}^{(\lfloor \sqrt{\frac{t}{2\pi}} \rfloor)} \left(\frac{1}{k^s}\right) + \chi(s) \cdot \sum_{k=1}^{(\lfloor \sqrt{\frac{t}{2\pi}} \rfloor)} \left(\frac{1}{k^{(1-s)}}\right) \qquad \text{to zeroth order} \tag{5}$$

where (i) $R(s) = \sum_{k=1}^{(\lfloor \sqrt{\frac{t}{2\pi}} \rfloor)} \left(\frac{1}{k^s}\right)$+remainder terms, (ii) the remainder terms can be expressed as an expansion series [12,13] which may be divergent and (iii) $\chi(s)$ is obtained from the functional equation [5-8,12,13] for the Riemann Zeta function

$$\zeta(s) = \chi(s) \cdot \zeta(1-s) \tag{6}$$

Given the above formulae, it is straightforward to derive a tapered finite zeroth order Riemann Siegel function approximation of the form

$$\zeta(s,\alpha)_{\text{pert}} \approx \left\{ 1+\alpha\cdot\left[\sum_{k=2}^{\left(\lfloor\sqrt{\frac{t}{2\pi}}\rfloor-p\right)} \left(\frac{1}{k^s}\right) + \sum_{i=(-p+1)}^{p} \frac{\frac{1}{2^{2p}}\left(2^{2p}-\sum_{k=0}^{i+p-1}\binom{2p}{2p-k}\right)}{(\lfloor\sqrt{\frac{t}{2\pi}}\rfloor+i)^s} \right] \right\}$$
$$+\chi(s)\cdot\left\{ 1+\alpha\cdot\left[\sum_{k=2}^{\left(\lfloor\sqrt{\frac{t}{2\pi}}\rfloor-p\right)} \left(\frac{1}{k^{(1-s)}}\right) + \sum_{i=(-p+1)}^{p} \frac{\frac{1}{2^{2p}}\left(2^{2p}-\sum_{k=0}^{i+p-1}\binom{2p}{2p-k}\right)}{(\lfloor\sqrt{\frac{t}{2\pi}}\rfloor+i)^{(1-s)}} \right] \right\} \quad \text{as } t\to\infty \tag{7}$$

where 2p=128 (for 128 point tapering) is used in this paper. The advantage of using the zeroth order expression for the Riemann Siegel function approximation is to simplify the calculation of first and second order derivatives for execution of the non-trivial zero quadrature search under the perturbation parameter (α).

Results - Examples of non-trivial zero behaviour under equation (7) perturbation, approximating equation (1) away from the real axis (e.g., imag(s) > 26000), about large Riemann Zeta peaks of the 128 point tapered finite Riemann Siegel formula

All the calculations of non-trivial zero locations for the tapered finite zeroth Riemann Siegel function at the first quiescent region (and the finite Riemann Siegel dirichlet series at the second quiescent region) were performed using the pari-gp language [14] as a solution to second order taylor series in real(s) and imag(s) that produces iterative fourth order polynomials for imag(t) and then real(s) respectively. To test and improve code performance and conduct longer searches at higher t intervals the CoCalc platform [15] using pari-gp language was employed. References used for information include, [16-18] on the location of large Riemann peaks and nearby zeroes and [10,11,19] on Rosser rule violations for the Riemann Zeta function to identify starting values for pari-gp calculations. The R language [20] and R-studio IDE [21] were used to piece the pari-gp based results together and produce graphs.

Figures 1,3,5,7,9 display examples of the trajectory of the perturbed location of non-trivial zeroes associated near known large Riemann Zeta function peaks at t={363991205.178, 673297382.184, 1387123309.985, 2381374874120.454, 4257232978148261.802} respectively, which have peak heights Z={114, 123, 148, 368, 855}.

Figures 1,3,5,7,9 have two panels,

- The upper panel displays the real(s) versus imag(s) co-ordinate trajectory of nearby non-trivial zeroes as the perturbation varies from $0.005 < \alpha < 1$ where $\alpha = 1$ represent zero perturbation. As a guide on the upper panel are vertical lines indicating the expected imaginary co-ordinate of the zeroes if Gram's law were perfectly obeyed. Under high perturbation $\alpha = 0.001$ when the Riemann Zeta function contribution is a heavily reduced the imaginary component of the non-trivial zeroes generally head towards these vertical line co-ordinates.
- The lower panel displays the real(s) versus α co-ordinate trajectory of nearby non-trivial zeroes as the perturbation varies from $0.005 < \alpha < 1$.

Figures 2,4,6,8,10 display the S(1/2+it) values of the 128 point tapered finite Riemann Siegel formula near these known large Riemann Zeta function peaks at t={363991205.178, 673297382.184, 1387123309.985, 2381374874120.454, 4257232978148261.802}. Overlayed on these figures is the signed trajectory of the perturbed location of non-trivial zeroes (for $0.005 < \alpha < 1$) in order to see if graphically there is any consistent behaviour between the S(1/2+it) values and the non-trivial zero perturbation trajectories. A signed trajectory just means that if a S(1/2+it) discontinuity has positive (negative) value then the associated non-trivial zero trajectory has a positive (negative) sign assigned (using a ± 1 multiplicative factor). This overlay helps visualise that the graphical evidence of a particular non-trivial zero perturbation trajectory behaviour when |S(1/2+it)| $>$ 1.7.

Located just above figures 2,4,6,8,10 are simple tables comparing known (LMFDB [18]) values of Gram points, Riemann Zeta zeroes co-ordinates and 128 point tapered finite Riemann Siegel formula (using first quiescent region) zeroes co-ordinates for the displayed non-trivial zeroes in the nearby figures. In particular, the 128 point tapered finite Riemann Siegel formula (using first quiescent region) zeroes co-ordinates exhibit less accuracy for closely spaced zeroes but the accuracy is generally to the 3rd decimal place (given 9+ significant digits). (In the appendix, the 128 taper finite Dirichlet Series (using second quiescent region) zeroes co-ordinates are also provided for the first three Rosser rule violation points and the observed accuracy is higher as expected.)

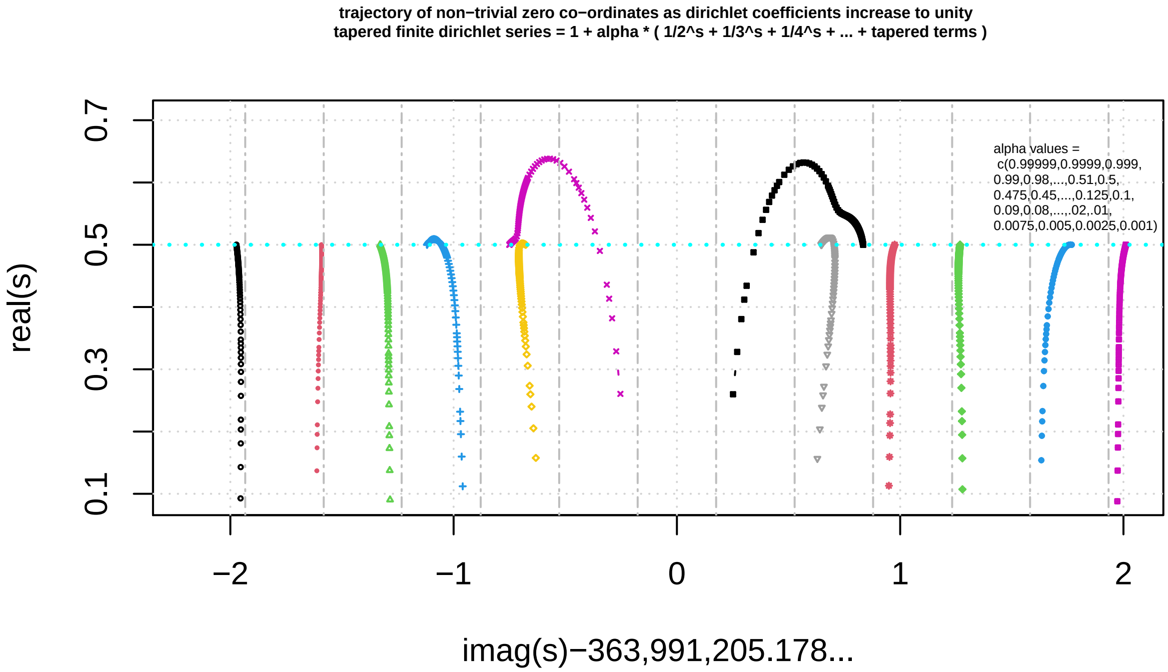
Figure 1 displays the behaviour of twelve low-lying non-trivial zeroes for $0.005 < \alpha < 1$ for equation (7) perturbation near the large peak [16] at t=363991205.1788358 of the Riemann Zeta function. It can be observed that under perturbation equation (7), the Riemann Zeta non-trivial zeroes located at t=363991204.427948... (gram point 977571899) and t=363991206.013777... (gram point 977571902) overshoot the critical line to reach real(s)~0.68 when α~0.25 finally settling back to real(s)=0.5 when $\alpha = 1$ changing the sequence of the non-trivial zeroes. Under high perturbation $\alpha = 0.001$ when the Riemann Zeta function contribution is a heavily reduced the imaginary component of the non-trivial zeroes generally head towards these vertical line co-ordinates but the inner zeroes closest to the Riemann Zeta function peak clearly take longer to approach the Gram's law expected zeroes co-ordinates. The upper panel shows the complex plane {imag(s),real(s)} trajectory of the non-trivial zero under the perturbation while the lower panel shows the {α, real(s)} trajectory.

Figure 2 displays the behaviour of (the 128 point tapered finite Riemann Siegel formula) S(1/2+it) function values for the Riemann Zeta zeroes near the large peak at t=363991205.1788358 of the Riemann Zeta function. Comparing the signed trajectory (red lines) of the non-trivial zeroes under perturbation equation (7) to the S(1/2+it) values provides preliminary evidence that the non-trivial zero trajectories that overshoot their Gram point order are associated with S(1/2+it) function values $>$ 1.7

1874184807 673297382.9696185572189753149955533350492 673297382.970139...

Figure 3 displays the behaviour of twelve low-lying non-trivial zeroes for $0.005 < \alpha < 1$ for equation (7) perturbation near the large peak [16] at t=673297382.184 of the Riemann Zeta function. It can be observed that under perturbation equation (7), the Riemann Zeta non-trivial zeroes located at t=673297382.9696185... (gram point 1874184807) overshoots the critical line to reach real(s)~0.65 when α~0.25 finally settling back to real(s)=0.5 when $\alpha = 1$ changing the sequence of the non-trivial zeroes. Under high perturbation $\alpha = 0.001$ when the Riemann Zeta function contribution is a heavily reduced the imaginary component of the non-trivial zeroes generally head towards these vertical line co-ordinates but the inner zeroes closest to the Riemann Zeta function peak clearly take longer to approach the Gram's law expected zeroes co-ordinates. The upper panel shows the complex plane {imag(s),real(s)} trajectory of the non-trivial zero under the perturbation while the lower panel shows the {α, real(s)} trajectory. The empirical observation that only a zero on the upper side of the peak exhibit overshoot on perturbation equation (7) provides a useful counterexample to figure 1.

Figure 4 displays the behaviour of (the 128 point tapered finite Riemann Siegel formula) S(1/2+it) function values for the Riemann Zeta zeroes near the large peak at t=673297382.184 of the Riemann Zeta function. Comparing the signed trajectory (red lines) of the non-trivial zeroes under perturbation equation (7) to the S(1/2+it) values provides preliminary evidence that the non-trivial zero trajectories that overshoot their Gram point order are associated with S(1/2+it) function values $>$ 1.7. The observation that the two zeroes on the lower side of the peak have |S|<1.7 and don't exhibit overshoot in the perturbation equation (7) behaviour helps establish an clear association about the viability of the proposed |S| threshold and observed perturbation equation (7) behaviour.



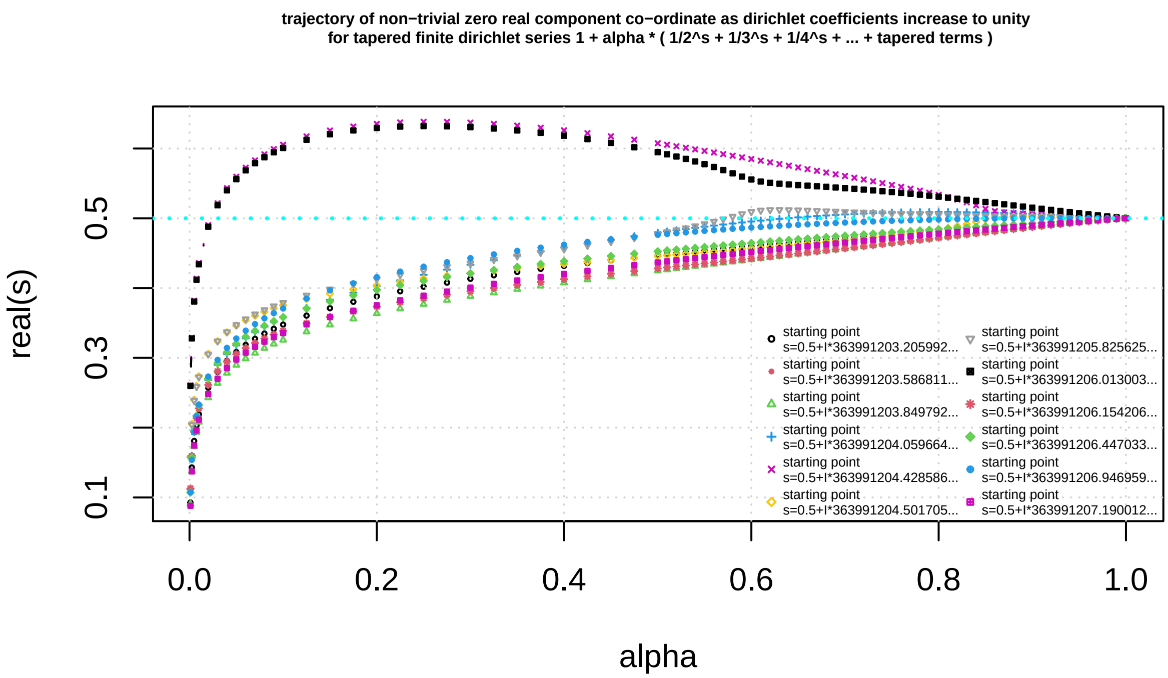


***Figure 1.** The trajectory of (twelve) non-trivial zero co-ordinates around the high Riemann Zeta function critical line peak (t=363991205.178835806079) as the magnitude (α) of the 2nd, 3rd, 4th, ... etc dirichlet coefficients of the tapered finite dirichlet series = 1 + α * ($1/2^s$ + $1/3^s$ + $1/4^s$ + ... + tapered terms) increases to unity.*

central value of high peak as determined by Gram point behaviour $\Theta(t)/\pi = n \in \mathbb{Z}$

```
vtheta(363991205.178835806079)/Pi = 977571899.00000000000058717719830939971

Comparing LMFDB known zero positions and 128 tapered zeroth order Riemann Siegel function zero position

Gram number LMFDB Riemann Zeta function zero position  128 taper zeroth RS function zero position
977571895   363991203.2055154467633210651142654369833 363991203.205992...
977571896   363991203.5874647959267013230604889887420 363991203.586811...
977571897   363991203.8488736843312366201235116884576 363991203.849792...
977571898   363991204.0603526518254301862245829360622 363991204.059664...
977571899   363991204.4279482944857544444239351841440 363991204.428586...
977571900   363991204.5021544954473942593777291793608 363991204.501705...
977571901   363991205.8254203516208204866431702777920 363991205.825625...
977571902   363991206.0137777335580653648668989590724 363991206.013003...
977571903   363991206.1533735936313339776340519741543 363991206.154206...
977571904   363991206.4474212120057061586462351591596 363991206.447033...
977571905   363991206.9464869474181906360545628091899 363991206.946959...
977571906   363991207.1906684348656300487139505824776 363991207.190012...
```

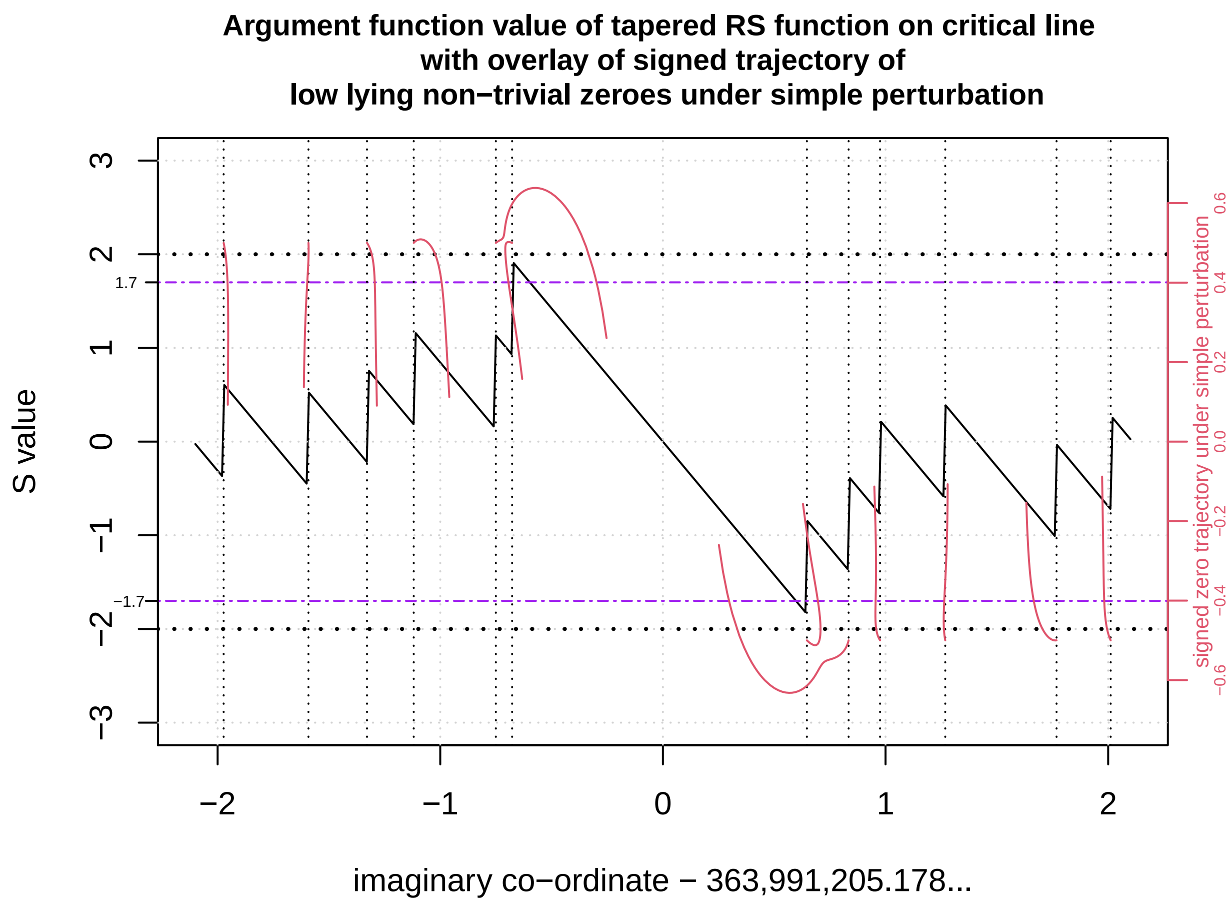


***Figure 2.** Comparing non-trivial zero positions of (LMFDB known) Riemann Zeta function (vertical lines), signed trajectory of 128 taper zeroth order Riemann Siegel function Dirichlet Series under simple perturbation (red lines) and 128 taper zeroth order Riemann Siegel function argument function (S values).*

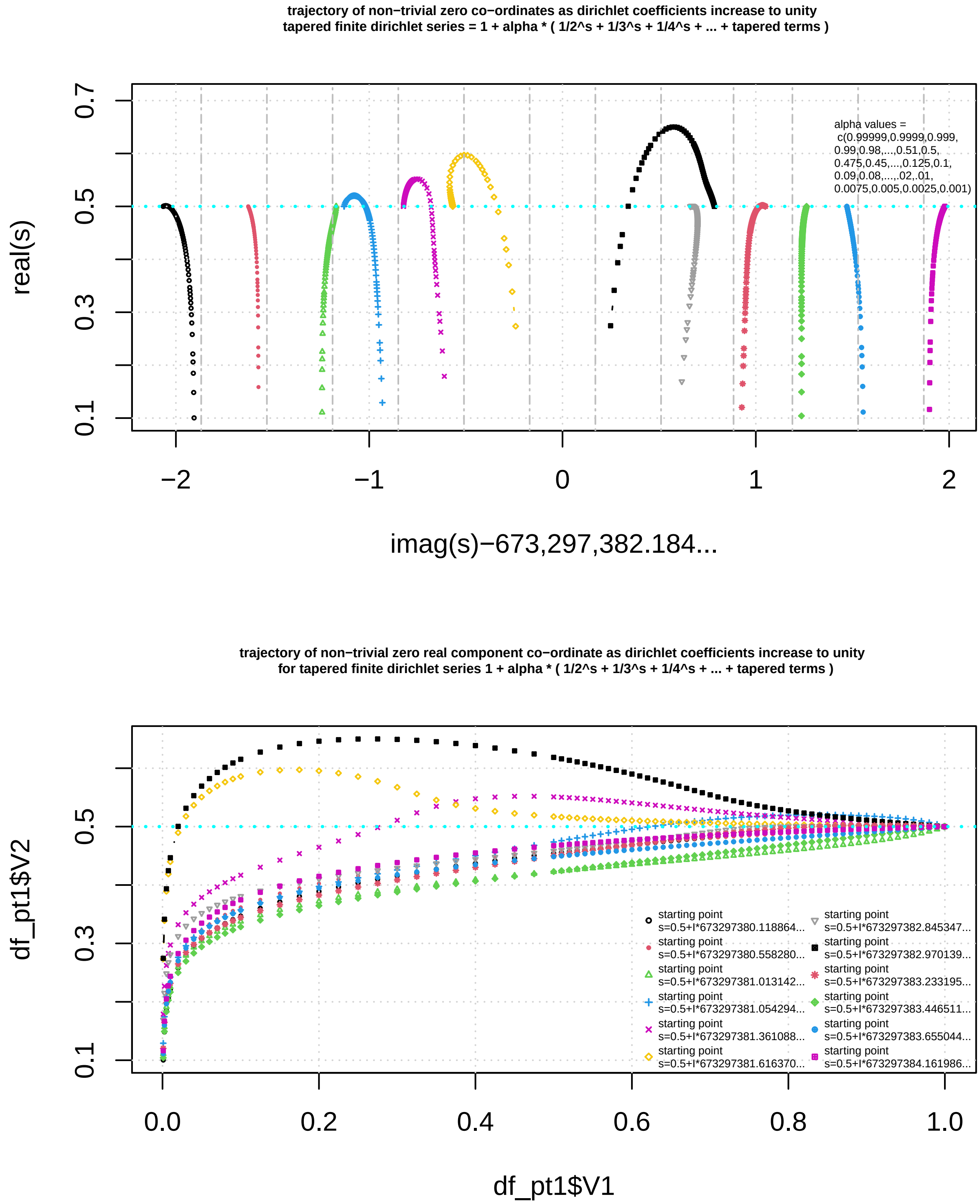


***Figure 3.** The trajectory of (twelve) non-trivial zero co-ordinates around the high Riemann Zeta function critical line peak (t=673297382.18411690675) as the magnitude (α) of the 2nd, 3rd, 4th, ... etc dirichlet coefficients of the tapered finite dirichlet series = 1 + α * ($1/2^s$ + $1/3^s$ + $1/4^s$ + ... + tapered terms) increases to unity.*

central value of high peak as determined by Gram point behaviour $\Theta(t)/\pi = n \in \mathbb{Z}$

```
vtheta(673297382.18411690675)/Pi = 1874184803.9999999999099687090291338101

Comparing LMFDB known zero positions and 128 tapered zeroth order Riemann Siegel function zero position

Gram number LMFDB Riemann Zeta function zero position  128 taper zeroth RS function zero position
1874184800  673297380.1204220106459432001639480717505 673297380.118864...
1874184801  673297380.5579411133578600756665722665752 673297380.558280...
1874184802  673297381.0156516577310389172169518439777 673297381.013142...
1874184803  673297381.0517448151100000683765537098245 673297381.054294...
1874184804  673297381.3613566360415483828914493488505 673297381.361088...
1874184805  673297381.6162983265734247695998007886312 673297381.616370...
1874184806  673297382.8455950979397768447096468519234 673297382.845347...
1874184807  673297382.96961855572189753149955533350492 673297382.970139...
1874184808  673297383.2339462554021997531146141081394 673297383.233195...
1874184809  673297383.4456351591856742140892733070548 673297383.446511...
1874184810  673297383.6555569853220632771090395331945 673297383.655044...
1874184811  673297384.1616895542181501225748466307027 673297384.161986...
```

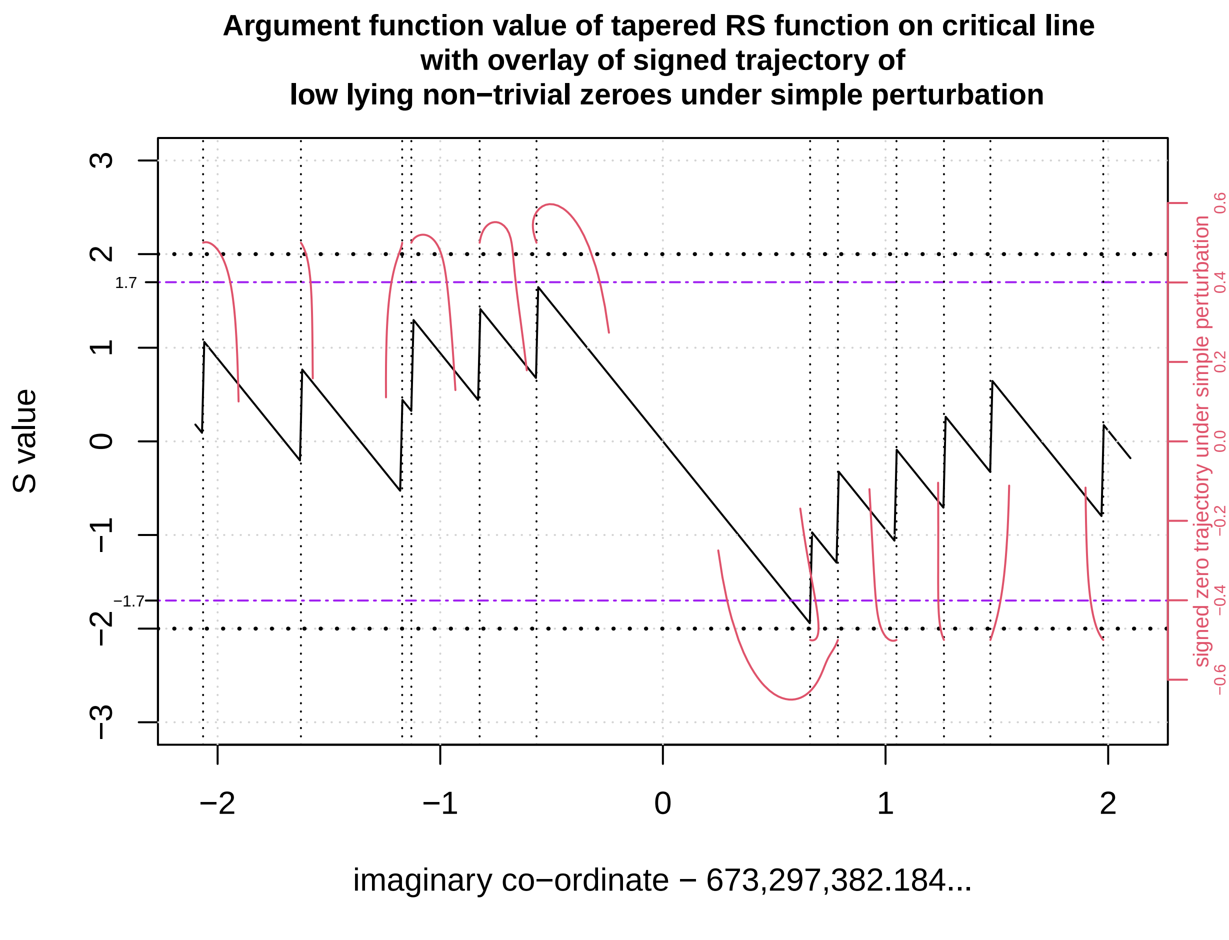


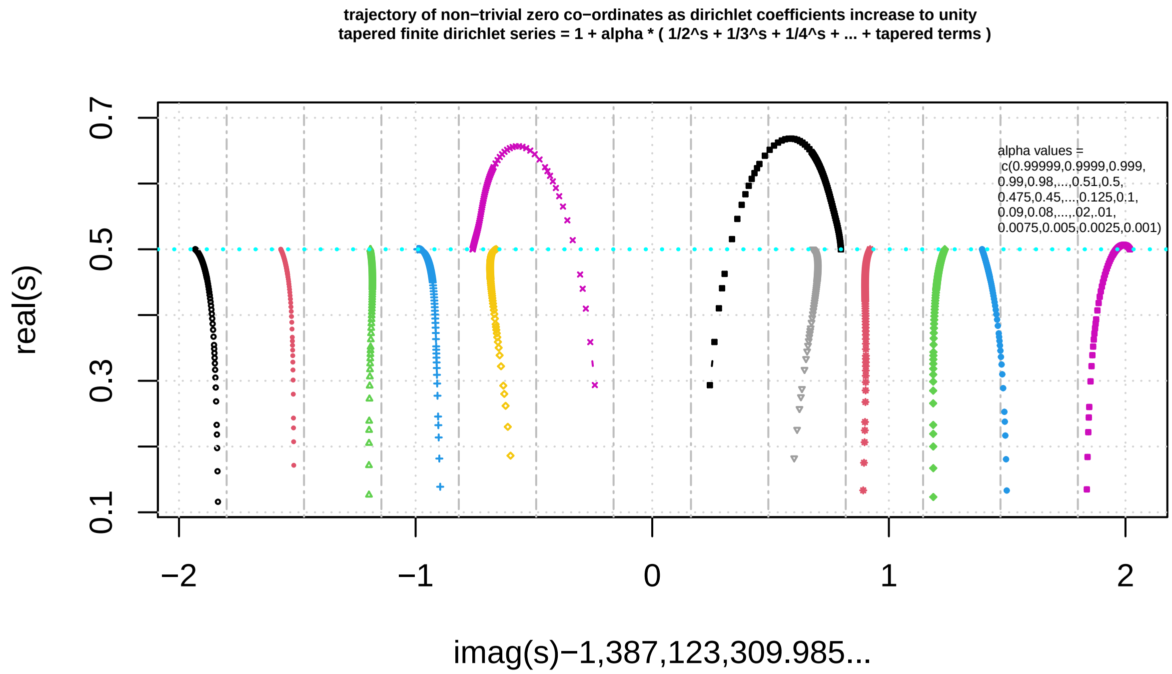
Figure 4. Comparing non-trivial zero positions of (LMFDB known) Riemann Zeta function (vertical lines), signed trajectory of 128 taper zeroth order Riemann Siegel function Dirichlet Series under simple perturbation (red lines) and 128 taper zeroth order Riemann Siegel function argument function (S values).

Figure 5 displays the behaviour of twelve low-lying non-trivial zeroes for $0.005 < \alpha < 1$ for equation (7) perturbation near the large peak [16] at t=1387123309.985 of the Riemann Zeta function. It can be observed that under perturbation equation (7), the Riemann Zeta non-trivial zeroes located at t=1387123309.2256571... (gram point 4020755338) and t=1387123310.7835436... (gram point 4020755341) overshoot the critical line to reach real(s)~0.65, 0.67 respectively when α~0.25 finally settling back to real(s)=0.5 when $\alpha = 1$ changing the sequence of the non-trivial zeroes. Under high perturbation $\alpha = 0.001$ when the Riemann Zeta function contribution is a heavily reduced the imaginary component of the non-trivial zeroes generally head towards these vertical line co-ordinates but the inner zeroes closest to the Riemann Zeta function peak clearly take longer to approach the Gram's law expected zeroes co-ordinates. The upper panel shows the complex plane {imag(s),real(s)} trajectory of the non-trivial zero under the perturbation while the lower panel shows the {α, real(s)} trajectory.

Figure 6 displays the behaviour of (the 128 point tapered finite Riemann Siegel formula) S(1/2+it) function values for the Riemann Zeta zeroes near the large peak at t=1387123309.985 of the Riemann Zeta function. Comparing the signed trajectory (red lines) of the non-trivial zeroes under perturbation equation (7) to the S(1/2+it) values provides ongoing evidence that the non-trivial zero trajectories that overshoot their Gram point order are associated with S(1/2+it) function values > 1.7

Figure 7 displays the behaviour of twelve low-lying non-trivial zeroes for $0.005 < \alpha < 1$ for equation (7) perturbation near the large peak [16] at t=2381374874120.454 of the Riemann Zeta function. It can be observed that under perturbation equation (7), the Riemann Zeta non-trivial zeroes located at t=2381374874119.853140... and t=2381374874121.057263... overshoot the critical line to reach real(s)~0.67, 0.68 respectively when α~0.25 finally settling back to real(s)=0.5 when $\alpha = 1$ changing the sequence of the non-trivial zeroes. Under high perturbation $\alpha = 0.001$ when the Riemann Zeta function contribution is a heavily reduced the imaginary component of the non-trivial zeroes generally head towards these vertical line co-ordinates but the inner zeroes closest to the Riemann Zeta function peak clearly take longer to approach the Gram's law expected zeroes co-ordinates. The upper panel shows the complex plane {imag(s),real(s)} trajectory of the non-trivial zero under the perturbation while the lower panel shows the {α, real(s)} trajectory.

Figure 8 displays the behaviour of (the 128 point tapered finite Riemann Siegel formula) S(1/2+it) function values for the Riemann Zeta zeroes near the large peak at t=2381374874120.454 of the Riemann Zeta function. Comparing the signed trajectory (red lines) of the non-trivial zeroes under perturbation equation (7) to the S(1/2+it) values provides ongoing evidence that the non-trivial zero trajectories that overshoot their Gram point order are associated with S(1/2+it) function values > 1.7



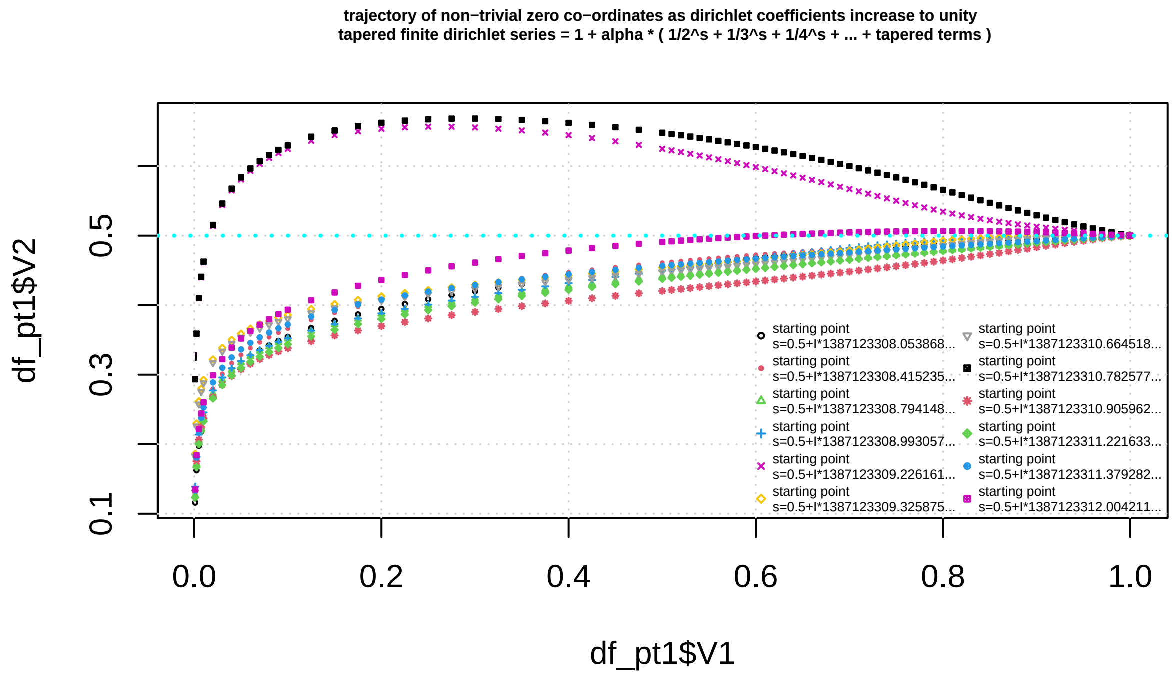


Figure 5. The trajectory of (twelve) non-trivial zero co-ordinates around the high Riemann Zeta function critical line peak (t=1387123309.985231567137) as the magnitude (α) of the 2nd, 3rd, 4th, . . . etc dirichlet coefficients of the tapered finite dirichlet series = 1 + α * ($1/2^s$ + $1/3^s$ + $1/4^s$ + . . . + tapered terms) increases to unity.

central value of high peak as determined by Gram point behaviour $\Theta(t)/\pi = n \in \mathbb{Z}$

```
vtheta(1387123309.985231567137)/Pi = 4020755338.000000000015693381555683816

Comparing LMFDB known zero positions and 128 tapered zeroth order Riemann Siegel function zero position
```

Gram number	LMFDB Riemann Zeta function zero position	128 taper zeroth RS function zero position
4020755334	1387123308.053386578581022573109666420934 1	1387123308.053868...
4020755335	1387123308.4155330874982095975359105556222	1387123308.415235...
4020755336	1387123308.7936984130815743272450539535073	1387123308.794148...
4020755337	1387123308.9936133902126949185682078466466	1387123308.993057...
4020755338	1387123309.2256571405199546473206714032527	1387123309.226161...
4020755339	1387123309.3261365486237198942298603710861	1387123309.325875...
4020755340	1387123310.6641909722304114462764657048521	1387123310.664518...
4020755341	1387123310.7835436614709630258565144810545	1387123310.782577...
4020755342	1387123310.9050690789555236712733011203872	1387123310.905962...
4020755343	1387123311.2222578449095928600763092848845	1387123311.221633...
4020755344	1387123311.3788456414209246982726571898255	1387123311.379282...
4020755345	1387123312.0047494775963503459761374606242	1387123312.004211...

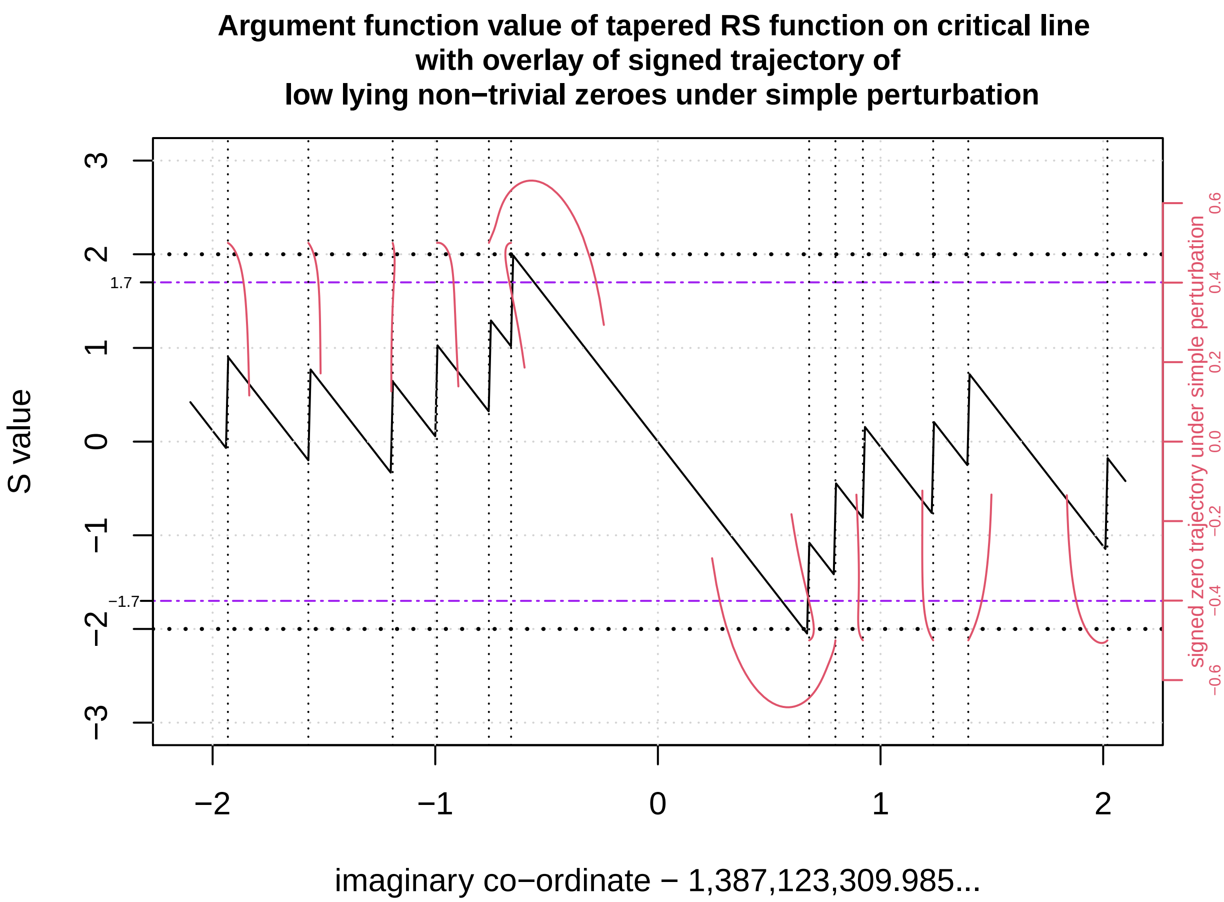
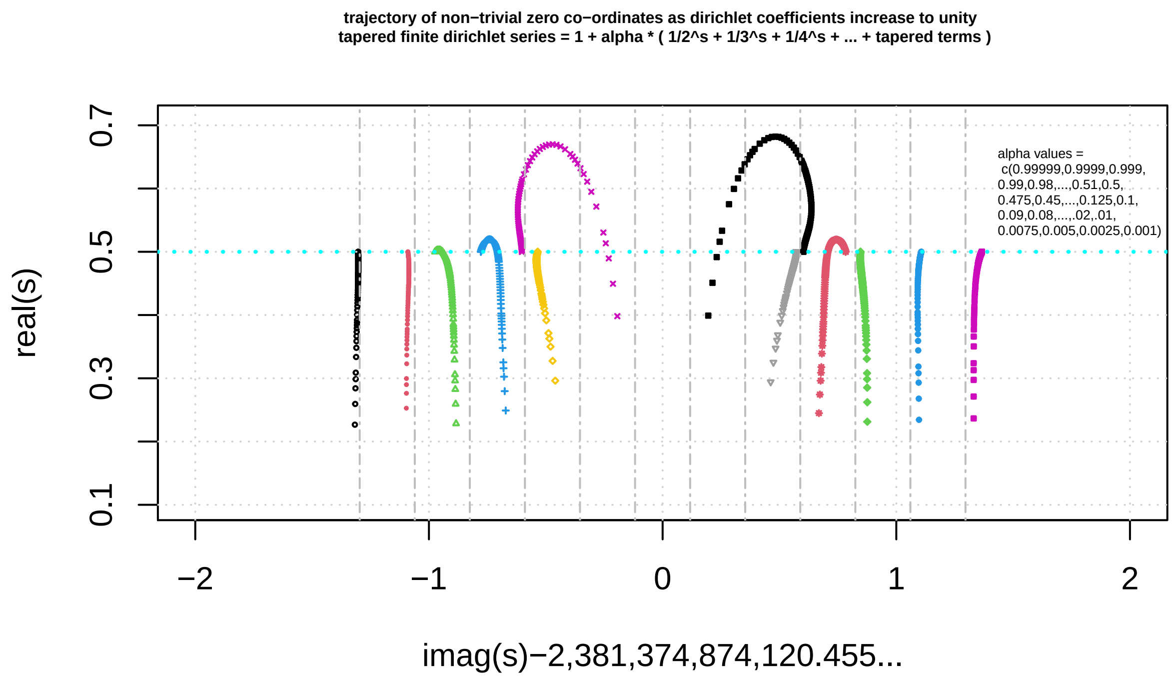


Figure 6. ***Comparing non-trivial zero positions of (LMFDB known) Riemann Zeta function (vertical lines), signed trajectory of 128 taper zeroth order Riemann Siegel function Dirichlet Series under simple perturbation (red lines) and 128 taper zeroth order Riemann Siegel function argument function (S values).***



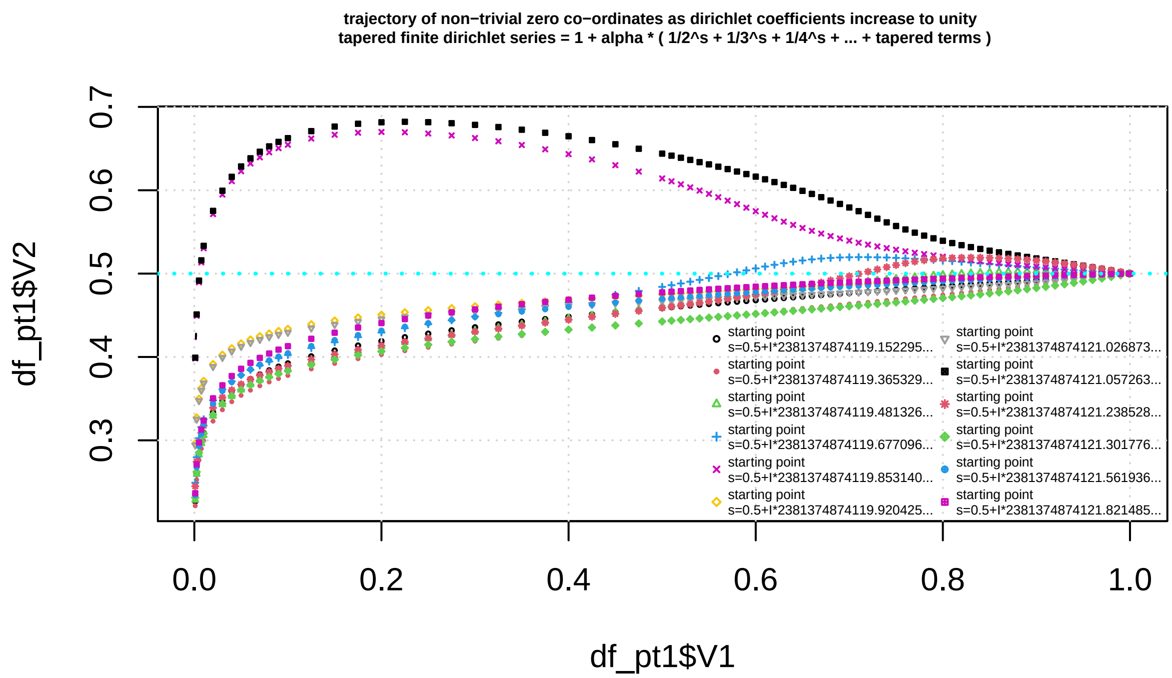


***Figure 7.** The trajectory of (twelve) non-trivial zero co-ordinates around the high Riemann Zeta function critical line peak (t=2381374874120.45494462613) as the magnitude (α) of the 2nd, 3rd, 4th, ... etc dirichlet coefficients of the tapered finite dirichlet series = 1 + α * ($1/2^s$ + $1/3^s$ + $1/4^s$ + ... + tapered terms) increases to unity.*

central value of high peak as determined by Gram point behaviour $\Theta(t)/\pi = n \in \mathbb{Z}$

```
vtheta(2381374874120.45494462613)/Pi = 9725646131432.00000000000017684199347928
```

```
Conjectured Gram number using 128 tapered zeroth order Riemann Siegel function zero positions
under perturbation

Conjectured Gram number 128 taper zeroth RS function zero position
------------            2381374874119.152295...
------------            2381374874119.365329...
------------            2381374874119.481326...
------------            2381374874119.677096...
9725646131432???        2381374874119.853140...
------------            2381374874119.920425...
------------            2381374874121.026873...
------------            2381374874121.057263...
------------            2381374874121.238528...
------------            2381374874121.301776...
------------            2381374874121.561936...
------------            2381374874121.821485...
```

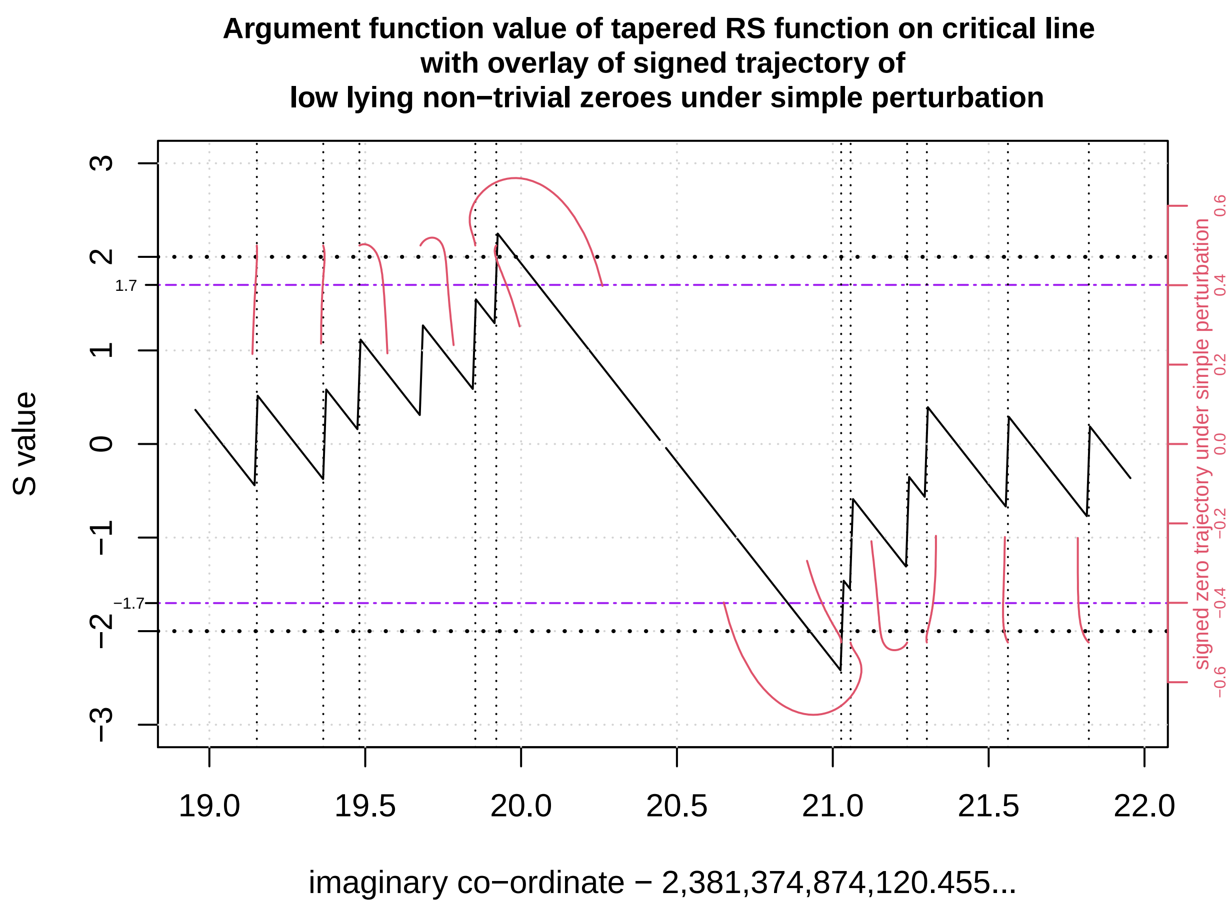
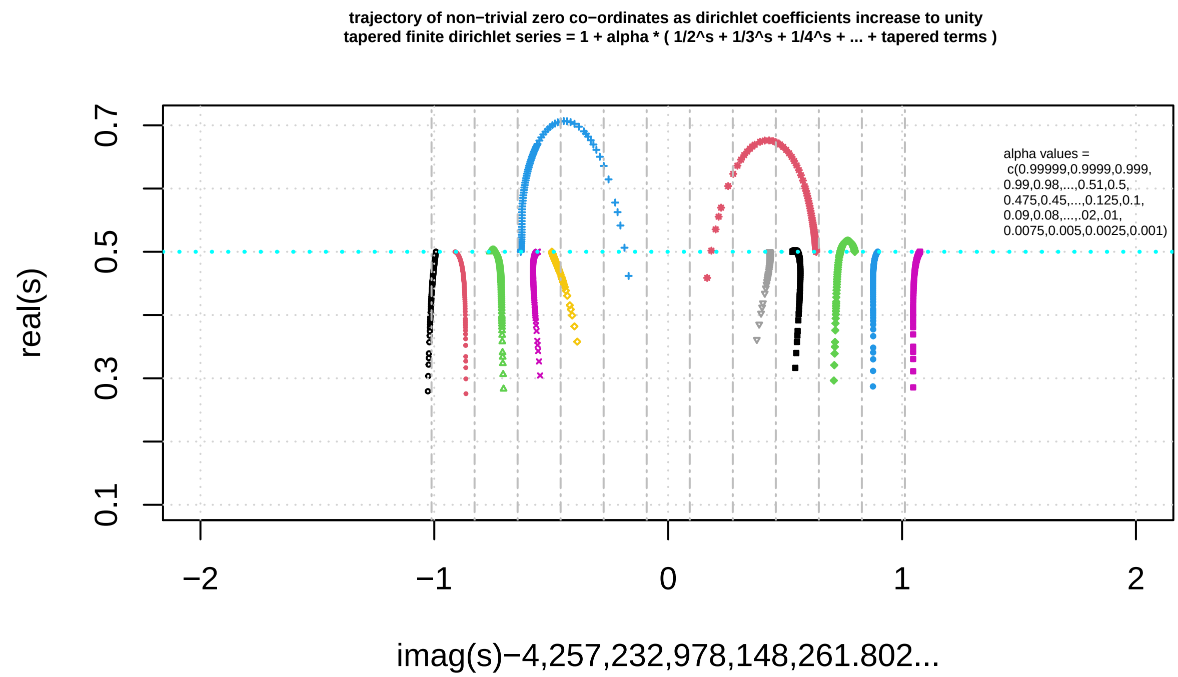


Figure 8. Comparing signed trajectory of 128 taper zeroth order Riemann Siegel function Dirichlet Series under simple perturbation (red lines) and 128 taper zeroth order Riemann Siegel function argument function (S values).

Figure 9 displays the behaviour of twelve low-lying non-trivial zeroes for $0.005 < \alpha < 1$ for equation (7) perturbation near the large peak [17] at t=4257232978148261.802 of the Riemann Zeta function. It can be observed that under perturbation equation (7), the Riemann Zeta non-trivial zeroes located at t=4257232978148261.305478… and t=4257232978148262.236684… both overshoot (by two zero positions) the critical line to reach real(s)~0.70, 0.68 respectively when α~0.2 finally settling back to real(s)=0.5 when $\alpha = 1$ changing the sequence of the non-trivial zeroes. Under high perturbation $\alpha = 0.001$ when the Riemann Zeta function contribution is a heavily reduced the imaginary component of the non-trivial zeroes generally head towards these vertical line co-ordinates but the inner zeroes closest to the Riemann Zeta function peak clearly take longer to approach the Gram's law expected zeroes co-ordinates. The upper panel shows the complex plane {imag(s),real(s)} trajectory of the non-trivial zero under the perturbation while the lower panel shows the {α, real(s)} trajectory.

Figure 10 displays the behaviour of (the 128 point tapered finite Riemann Siegel formula) S(1/2+it) function values for the Riemann Zeta zeroes near the large peak at t=4257232978148261.802 of the Riemann Zeta function. Comparing the signed trajectory (red lines) of the non-trivial zeroes under perturbation equation (7) to the S(1/2+it) values provides useful additional evidence that the non-trivial zero trajectories that overshoot their Gram point order are associated with S(1/2+it) function values > 1.7, since there are four Riemann Zeta zeroes with $|S| \gtrsim 1.7$ two on each side of the peak (due to the large size of the peak, height = 855 [17]).



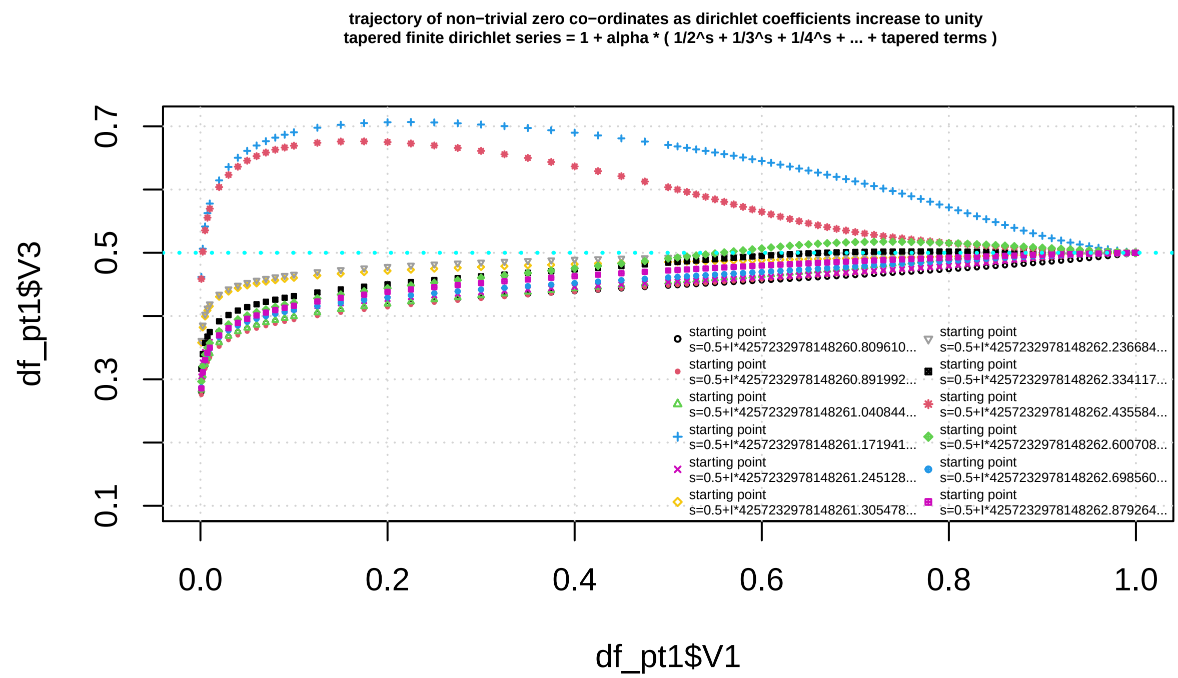


Figure 9. *The trajectory of (twelve) non-trivial zero co-ordinates around the high Riemann Zeta function critical line peak (t=4257232978148261.8026511493) as the magnitude (α) of the 2nd, 3rd, 4th, . . . etc dirichlet coefficients of the tapered finite dirichlet series = 1 + α * (1/2^s + 1/3^s + 1/4^s + . . . + tapered terms) increases to unity.*

central value of high peak as determined by Gram point behaviour $\Theta(t)/\pi = n \in \mathbb{Z}$

```
vtheta(4257232978148261.8026511493)/Pi = 22460777057990021.999999999882523875960

Conjectured Gram number using 128 tapered zeroth order Riemann Siegel function zero positions
under perturbation

Conjectured Gram number 128 taper zeroth RS function zero position
-----------------       4257232978148260.809610...
-----------------       4257232978148260.891992...
-----------------       4257232978148261.040844...
-----------------       4257232978148261.171941...
22460777057990022???    4257232978148261.245128...
-----------------       4257232978148261.305478...
-----------------       4257232978148262.236684...
-----------------       4257232978148262.334117...
-----------------       4257232978148262.435584...
-----------------       4257232978148262.600708...
-----------------       4257232978148262.698560...
-----------------       4257232978148262.879264...
```

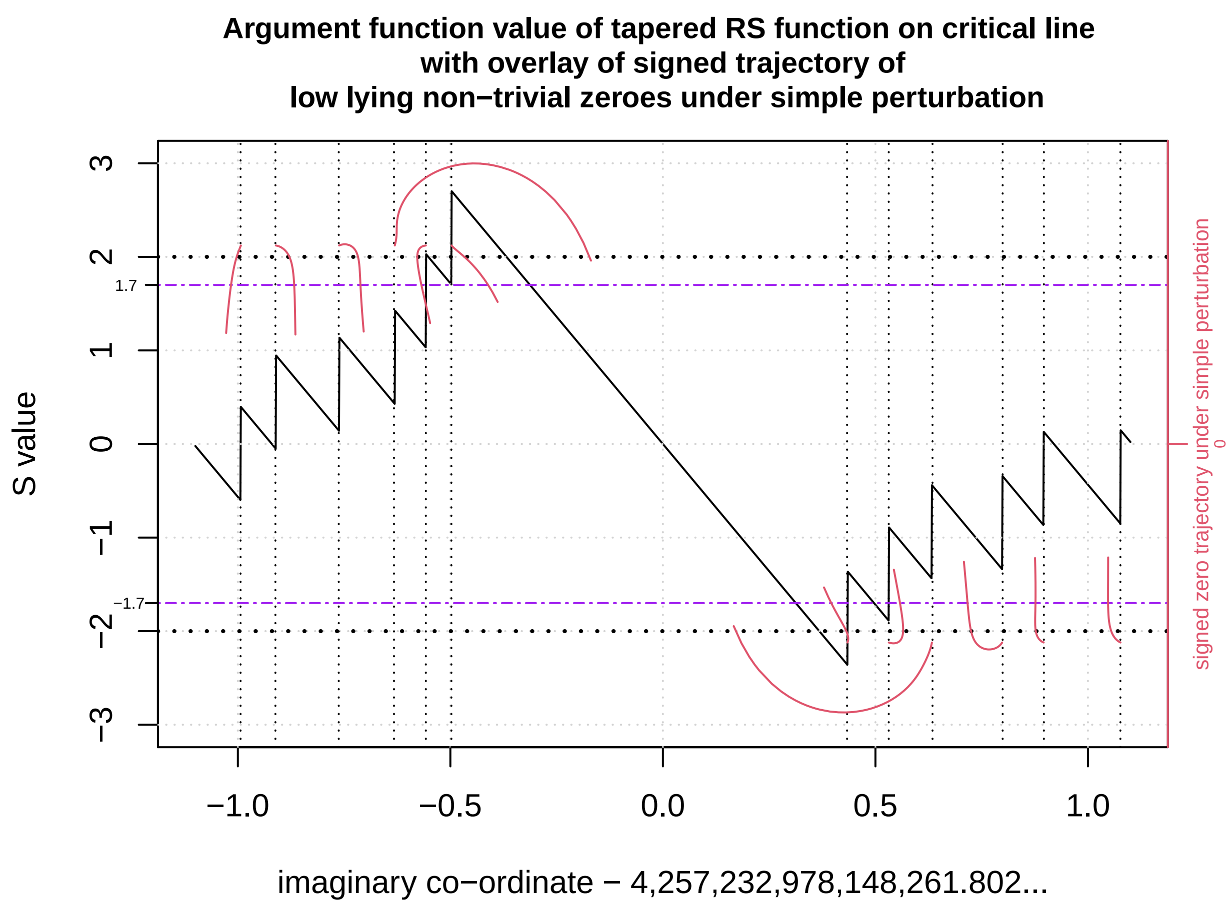


Figure 10. Comparing signed trajectory of 128 taper zeroth order Riemann Siegel function Dirichlet Series under simple perturbation (red lines) and 128 taper zeroth order Riemann Siegel function argument function (S values).

Conclusions

Perturbing the dirichlet coefficients of the 128 tapered zeroth order Riemann Siegel function about the first quiescent region of its dirichlet series using equation (7) can provide useful insights into the origin and behaviour of non-trivial zeroes of the Riemann Zeta function. A similar investigation will be attempted for large peaks of 5 periodic Davenport Heilbron functions.

References

1. Martin, J.P.D. "A quiescent region about $\frac{t}{\pi}$ in the oscillating divergence of the Riemann Zeta Dirichlet Series inside the critical strip." (2021) http://dx.doi.org/10.6084/m9.figshare.14213516
2. Martin, J.P.D. "Tapered end point weighting of finite Riemann Zeta Dirichlet Series using partial sums of binomial coefficients to produce higher order approximations of the Riemann Siegel Z function." (2021) http://dx.doi.org/10.6084/m9.figshare.14702760
3. Martin, J.P.D. "Truncated Exponential Series based partial Euler Product calculations at quiescent regions of oscillatory divergence to produce approximations of the Riemann Siegel Z function." (2021) http://dx.doi.org/10.6084/m9.figshare.14842803
4. Martin, J.P.D. "Examples of quiescent regions in the oscillatory divergence of several 1st degree L functions and their Davenport Heilbronn counterparts." (2021) https://dx.doi.org/10.6084/m9.figshare.14956053
5. Riemann, Bernhard (1859). "*Ü*ber die Anzahl der Primzahlen unter einer gegebenen Gr *ö*sse". Monatsberichte der Berliner Akademie.. In Gesammelte Werke, Teubner, Leipzig (1892), Reprinted by Dover, New York (1953).
6. Edwards, H.M. (1974). Riemann's zeta function. Pure and Applied Mathematics 58. New York-London: Academic Press. ISBN 0-12-242750-0. Zbl 0315.10035.
7. Titchmarsh, E.C. (1986) The Theory of the Riemann Zeta Function. 2nd Revised (Heath-Brown, D.R.) Edition, Oxford University Press, Oxford.
8. Katz, N. M., & Sarnak, P. (1999). Zeroes of Zeta functions and symmetry. Bulletin of the American Mathematical Society, 36(1), 1-26. https://doi.org/10.1090/s0273-0979-99-00766-1
9. Martin, J.P.D. "The behaviour of non-trivial zeroes in tapered finite Dirichlet Series about the second quiescent region with lower symmetry dirichlet coefficients near bad Gram points" (2024) https://dx.doi.org/10.6084/m9.figshare.25734099
10. Brent R.P., van de Lune J., te Riele H.J.J. and Winter D.T. (1982) "On the Zeros of the Riemann Zeta function in the Critical Strip. II" Mathematics of Computation, Volume 39, Number 160, October 1982, Pages 681-688
11. Violations of Rosser's rule: numbers n such that the Gram block [g(n), g(n+k)) contains fewer than k points t such that Z(t) = 0, where Z(t) is the Riemann-Siegel Z-function. 2022 https://oeis.org/A216700/internal
12. M.V. Berry, "Riemann's Saddle-point Method and the Riemann-Siegel Formula" Vol 35.1, pp. 69–78, The Legacy of Bernhard Riemann After One Hundred and Fifty Years, Advanced Lectures in Mathematics 2016
13. J. Arias De Reyna, "High precision computation of Riemann's Zeta function by the Riemann-Siegel formula", Mathematics of Computation Vol 80, no. 274, 2011, Pages 995–1009
14. The PARI~Group, PARI/GP version 2.12.0, Univ. Bordeaux, 2018, http://pari.math.u-bordeaux.fr/.
15. CoCalc – Collaborative Calculation and Data Science, Sagemath, Inc., (2024) https://cocalc.com}

16. Gourdon X. and Sebah P. (2010) "Maximum values of abs(Z(t)) encountered" upper limit ~ 2.45e12 http://numbers.computation.free.fr/Constants/constants.html

17. Kotnik T. (2003) "Computational estimation of the order of $\zeta(1/2 + it)$" Mathematics of Computation, Volume 73, Number 246, Pages 949-956

18. The LMFDB Collaboration, The L-functions and Modular Forms Database, http://www.lmfdb.org, 2019, [Online; accessed January 2020].

19. The On-Line Encyclopedia of Integer Sequences, published electronically at https://oeis.org, 2010, Sequence

20. R Core Team (2017). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. https://www.R-project.org/.

21. RStudio Team (2015). RStudio: Integrated Development for R. RStudio, Inc., Boston, MA http://www.rstudio.com/.

Appendix - Second quiescent region based calculations of non-trivial zero trajectory under simple perturbation

In figure A1-A3, the non-trivial zero behaviour of the 128 tapered Riemann Zeta Dirichlet series (truncated at the second quiescent region $N = \frac{t}{\pi}$) under simple perturbation, nearby the first three Rosser rule violations (gram points n=13999525, 30783329, 30930929) is also shown to move the ordering of the non-trivial zeroes when the argument function magnitude $S \gtrsim 1.7$. Where the tapered Riemann Zeta Dirichlet series truncated at the second quiescent region $N = \frac{t}{\pi}$ is a closer approximation of the Riemann Zeta function than the tapered zeroth order Rieman Siegel function truncated at the first quiescent region $N = \sqrt{\frac{t}{2\pi}}$.

For visual comparison only, signed perturbation trajectories of the 128 tapered Riemann Zeta Dirichlet series (truncated at the second quiescent region $N = \frac{t}{\pi}$) shown as red lines are used in figures A1-A3, whereby the pertubation trajectories of the non-trivial zeroes under perturbation have inverted magnitudes (i.e., a -1 multiplicative factor is applied) so that the red trajectory lines appear with the same sign as the S value lineshapes (produced using 128 taper zeroth order Riemann Siegel function).

Figure A1 displays the behaviour of five low-lying non-trivial zeroes for $0.005 < \alpha < 1$ for equation (7) perturbation near the second Rosser rule violation (gram point number 13999525, t=6820050.0586698...) of the Riemann Zeta function. It can be observed that under perturbation equation (7), the Riemann Zeta non-trivial zero located at t=6820052.0041220... (gram point 13999528) overshoots the critical line to reach real(s)~0.689 when α~0.5 finally settling back to real(s)=0.5 when $\alpha = 1$ changing the sequence of the non-trivial zeroes. The overshoot trajectory is associated with |S|=-2.004138 for the Riemann Zeta function zero [10].

Figure A2 displays the behaviour of five low-lying non-trivial zeroes for $0.005 < \alpha < 1$ for equation (7) perturbation near the second Rosser rule violation (gram point number 30783329, t=14190356.9683576...) of the Riemann Zeta function. It can be observed that under perturbation equation (7), the Riemann Zeta non-trivial zero located at t=14190358.8694475... (gram point 30783332) overshoots the critical line to reach real(s)~0.689 when α~0.5 finally settling back to real(s)=0.5 when $\alpha = 1$ changing the sequence of the non-trivial zeroes. The overshoot trajectory is associated with |S|=-2.00594 for the Riemann Zeta function zero [10].

Figure A3 displays the behaviour of seven low-lying non-trivial zeroes for $0.005 < \alpha < 1$ for equation (7) perturbation near the third Rosser rule violation (gram point number 30930927, t=14253736.0289697...) of the Riemann Zeta function. It can be observed that under perturbation equation (7), the Riemann Zeta non-trivial zero located at t=14253736.6001908... (gram point 30930930) overshoots the critical line to reach real(s)~0.68 when α~0.45 finally settling back to real(s)=0.5 when $\alpha = 1$ changing the sequence of the non-trivial zeroes. The overshoot trajectory is associated with |S|=+2.050625 for the Riemann Zeta function zero [10].

So for the first three Rosser rule violation locations it is observed that a nearby non-trivial zero that initially heavily overshoots the critical line before settling back to the critical line as $\alpha \to 1$. For these Rosser rule violation locations and the evidence gathered for higher peaks in figures 1-10, the sequence of the non-trivial zeroes appears to change going from a low symmetry dirichlet series to a higher symmetry dirichlet series when $|S| \gtrsim 1.7$ of a Riemann Zeta function zero.

central value of high peak as determined by Gram point behaviour $\Theta(t)/\pi = n \in \mathbb{Z}$

```
vtheta(6820050.98489667)/Pi = 13999524.99998452...

Comparing non-trivial zero positions of (LMFDB known) Riemann Zeta function,
128 taper second quiescent region Dirichlet Series and
128 taper zeroth order Riemann Siegel function.
```

Gram number	LMFDB value euler-maclaurin calc	128 taper Dirichlet Series second quiescent region	128 taper zeroth RS function first quiescent region
13999523	6820049.2465292299...	6820049.2465292299...	6820049.2465293813...
13999524	6820049.5452492498...	6820049.5452492498...	6820049.5452491577...
13999525	6820050.0586698640...	6820050.0586698640...	6820049.0586698962...
13999526	6820050.4836581572...	6820050.4836581572...	6820049.4836581437...
13999527	6820051.8909855008...	6820051.8909855008...	6820051.8909858945...
13999528	6820052.0041220270...	6820052.0041220270...	6820052.0041208789...
13999529	6820052.0917739836...	6820052.0917739836...	6820052.0917748155...
13999530	6820052.5865356504...	6820052.5865356504...	6820052.5865355231...

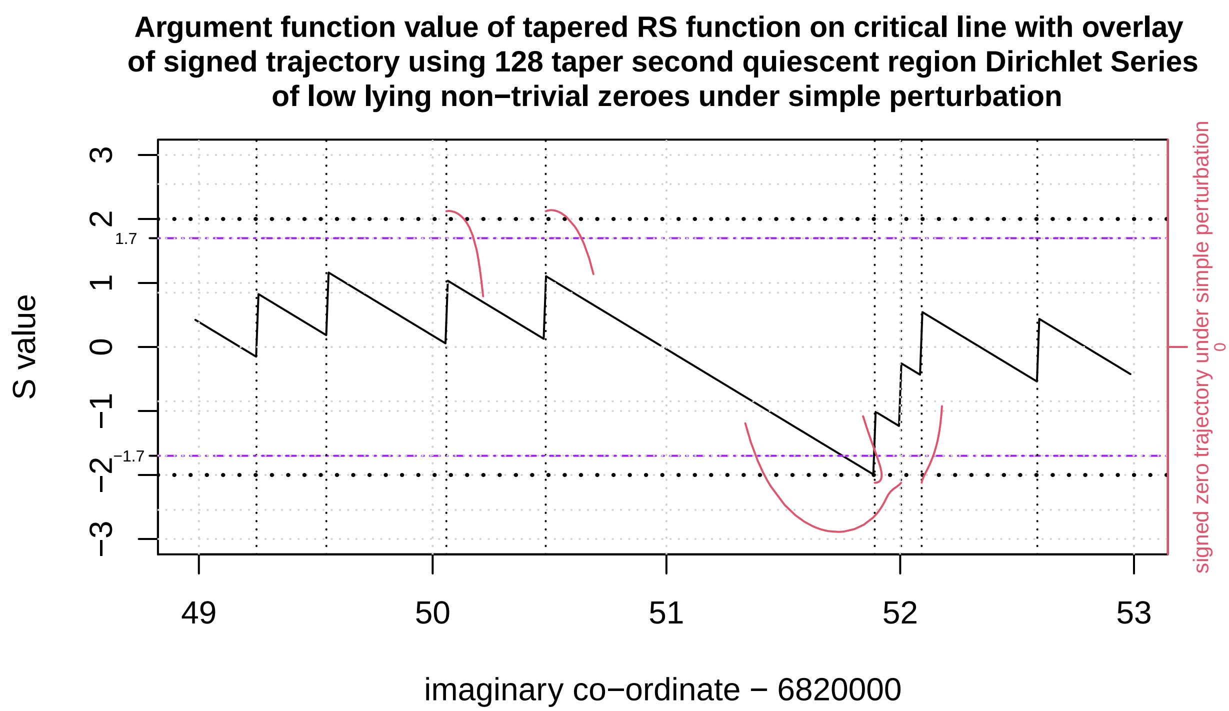


***Figure A1.** Comparing non-trivial zero positions of (LMFDB known) Riemann Zeta function (vertical lines), signed trajectory of 128 taper second quiescent region Dirichlet Series under simple perturbation (red lines) and 128 taper zeroth order Riemann Siegel function argument function (S values).*

central value of peak as determined by Gram point behaviour $\Theta(t)/\pi = n \in \mathbb{Z}$

Note, that the approximate centre of the peak based on central point of S function at imag(s)=14190358.101, near the second Rosser point does not closely follow the above behaviour (see n value below).

```
vtheta(14190358.101)/Pi = 30783329.648272

Comparing non-trivial zero positions of (LMFDB known) Riemann Zeta function,
128 taper second quiescent region Dirichlet Series and
128 taper zeroth order Riemann Siegel function.
```

Gram number	LMFDB value euler-maclaurin calc	128 taper Dirichlet Series second quiescent region	128 taper zeroth RS function first quiescent region
30783327	14190356.0139831934...	14190356.0139831934...	14190356.0139814875...
30783328	14190356.6356740515...	14190356.6356740515...	14190356.6356747129...
30783329	14190356.9683576921...	14190356.9683576921...	14190356.9683572075...
30783330	14190357.5200062615...	14190357.5200062615...	14190357.5200064510...
30783331	14190358.6832662323...	14190358.6832662323...	14190358.6832607679...
30783332	14190358.8694475296...	14190358.8694475296...	14190358.8695183786...
30783333	14190358.8972486583...	14190358.8972486583...	14190358.8971799919...
30783334	14190359.2737659068...	14190359.2737659068...	14190359.2737703909...
30783335	14190359.7451417968...	14190359.7451417968...	14190359.7451394812...
30783336	14190360.1589028494...	14190360.1589028494...	14190360.1589047048...

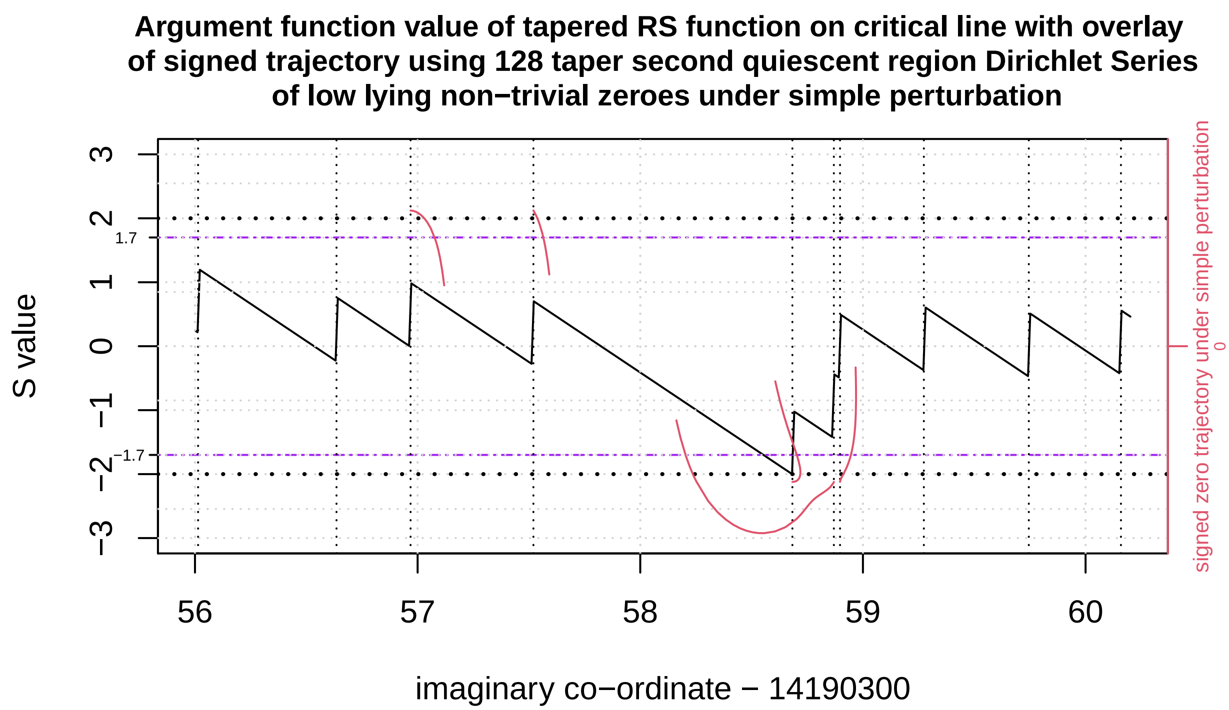


Figure A2. Comparing non-trivial zero positions of (LMFDB known) Riemann Zeta function (vertical lines), signed trajectory of 128 taper second quiescent region Dirichlet Series under simple perturbation (red lines) and 128 taper zeroth order Riemann Siegel function argument function (S values).

central value of peak as determined by Gram point behaviour $\Theta(t)/\pi = n \in \mathbb{Z}$

Note, that the approximate centre of the peak based on central point of S function at imag(s)=14253737.175, near the second Rosser point does not closely follow the above behaviour (see n value below).

```
vtheta(14253737.175)/Pi = 30930928.28820723...

Comparing non-trivial zero positions of (LMFDB known) Riemann Zeta function,
128 taper second quiescent region Dirichlet Series and
128 taper zeroth order Riemann Siegel function.
```

Gram number	LMFDB value euler-maclaurin calc	128 taper Dirichlet Series second quiescent region	128 taper zeroth RS function first quiescent region
30930925	14253735.1775158718...	14253735.1775158718...	14253735.1775306586...
30930926	14253735.5498261436...	14253735.5498261436...	14253735.5498061205...
30930927	14253736.0289697112...	14253736.0289697112...	14253736.0290054839...
30930928	14253736.3735853331...	14253736.3735853331...	14253736.3734679390...
30930929	14253736.5251151771...	14253736.5251151771...	14253736.5253214801...
30930930	14253736.6001908701...	14253736.6001908701...	14253736.6000760390...
30930931	14253737.7532407871...	14253737.7532407871...	14253737.7532421646...
30930932	14253738.4429227673...	14253738.4429227673...	14253738.4429175101...
30930933	14253738.7432317843...	14253738.7432317843...	14253738.7432416932...
30930934	14253739.1229203415...	14253739.1229203415...	14253739.1229096618...

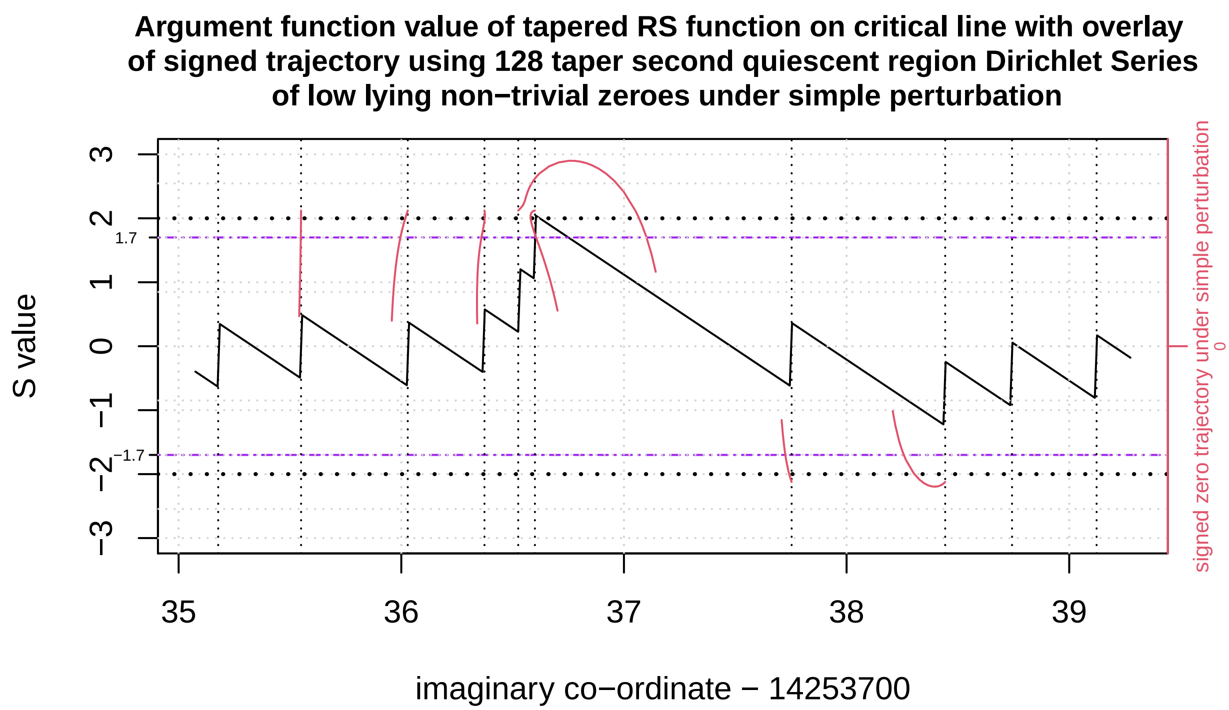


Figure A3. Comparing non-trivial zero positions of (LMFDB known) Riemann Zeta function (vertical lines), signed trajectory of 128 taper second quiescent region Dirichlet Series under simple perturbation (red lines) and 128 taper zeroth order Riemann Siegel function argument function (S values).