The finite Dirichlet Series expressed as the sum of products of finite geometric series of primes and powers of primes.

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Executive summary

A linear combination of products of finite geometric series of primes and power of primes exhibits excellent performance in reproducing the finite Riemann Zeta Dirichlet Series for known examples. From prime 5 upwards, the number of terms to be calculated for each prime follows Pascal's triangle pattern and for powers of a prime the same number of calculation terms are required as did the prime itself.

Introduction

For $\Re(s) > 1$, the infinite Euler Product of the primes absolutely converges to the infinite Riemann Zeta Dirichlet Series sum [1,2]

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} = \prod_{\rho=2}^{\infty} \frac{1}{(1 - 1/\rho^s)} \qquad \text{for } \Re(s) > 1 \tag{1}$$

Importantly, using the log(1-x) expansion of $\log(\zeta(s))$ [3-5] the Euler product also has the form

$$\prod_{\rho=2}^{\infty} \frac{1}{(1 - 1/\rho^s)} = \exp\left(\sum_{\rho=2}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n \cdot \rho^{ns}} \right) \tag{2}$$

For $\Re(s) \leq 1$, the partial Euler Product diverges, however, using the above equations for finite sums (products) of integers (primes) the following relationship holds

$$\begin{aligned} \sum_{k=1}^{N} \frac{1}{k^s} = 1 &+ \left(\sum_{\rho=2}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n \cdot \rho^{ns}} \cdot \delta(\rho^n \leq N) \right) \\ &+ \frac{1}{2!} \left(\sum_{\rho_1=2}^{\infty} \sum_{n=1}^{\infty} \sum_{\rho_2=2}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n \cdot \rho_1^{ns}} \cdot \frac{1}{m \cdot \rho_2^{ms}} \cdot \delta(\rho_1^n \cdot \rho_2^m \leq N) \right) \\ &+ \frac{1}{3!} \left(\sum_{\rho_1=2}^{\infty} \sum_{n=1}^{\infty} \sum_{\rho_2=2}^{\infty} \sum_{m=1}^{\infty} \sum_{\rho_3=2}^{\infty} \sum_{o=1}^{\infty} \frac{1}{n \cdot \rho_1^{ns}} \cdot \frac{1}{m \cdot \rho_2^{ms}} \cdot \frac{1}{o \cdot \rho_3^{os}} \cdot \delta(\rho_1^n \cdot \rho_2^m \cdot \rho_3^o \leq N) \right) \\ &+ \quad \ldots \end{aligned} \tag{3}$$

where the delta functions play a crucial role in appropriately truncating the Euler Product terms. Hence the above expression can be used with the $N \sim \lfloor \frac{t}{\pi} \rfloor$ and ($N \sim \lfloor \sqrt{\frac{t}{2\pi}} \rfloor$) quiescent regions of the oscillatory divergence of the Riemann Zeta function to obtain useful partial Euler Product based approximations of the Riemann Zeta function in the critical strip (and below) [6].

In [6] empirical calculations showed that the truncated exponential series version of the finite Euler product is a slower running algorithm at the complex plane points presented compared to the simple Dirichlet Series. This is due to the extra multiplication operations and truncation checks that are required at each higher order term of the power series calculation.

In this paper, an alternative series expression for the truncated euler product is given in terms of finite geometric series of primes and powers of primes in the hope of identifying a faster way to calculate a finite Dirichlet Series using only primes. This alternative series looks dense but some calculation simplications will happen in algorithmic practice as many primes (~50%) contributing to a finite dirichlet series will only have the non-zero leading term of 1 so the speed of an algorithm based on the series is yet to be established.

Using finite geometric series of primes to represent the finite Riemann Zeta Dirichlet Series

The pattern whereby a linear combination of products of finite geometric series of primes lower than a given prime can be used to construct a finite dirichlet series was identified by careful examination of the difference in dirichlet series terms between a finite truncated Euler Product $\prod_p^{P\leq N} \frac{(1-(1/p^s)^{\left\lceil \frac{\log(N)}{\log(p)} \right\rceil})}{(1-1/p^s)}$ and the dirichlet series $\sum_{n=1}^{N} \frac{1}{n^s}$

The identified expansion in terms of finite geometric series of primes is as follows

$$\sum_{n=1}^{N} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + ... + \frac{1}{N^s} \tag{4}$$

$$= a_2(s,N) + a_3(s,N) + a_5(s,N) + a_7(s,N) + a_9(s,N) + .. + a_{p\leq N}(s,N) \tag{5}$$

where

$$a_2(s,N) = 1 + \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\left(\left\lfloor \frac{\log(\max\{1,N\})}{\log(2)} \right\rfloor\right)})}{(1-\frac{1}{2^s})} \tag{6}$$

$$a_3(s,N) = \frac{1}{3^s} \cdot \left[\delta(N \geq 3) + \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\left\lfloor \frac{\log(\max\{1,\frac{N}{3}\})}{\log(2)} \right\rfloor})}{(1-\frac{1}{2^s})}\right] \tag{7}$$

$$a_5(s,N) = \frac{1}{5^s} \cdot \left[\delta(N \geq 5) + \frac{1}{2^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\left\lfloor \frac{\log(\max\{1,\frac{N}{5}\})}{\log(2)} \right\rfloor})}{(1-\frac{1}{2^s})} + \frac{1}{3^s} \cdot \frac{(1 - (\frac{1}{3^s})^{\left\lfloor \frac{\log(\max\{1,\frac{N}{5}\})}{\log(3)} \right\rfloor})}{(1-\frac{1}{3^s})} + \frac{1}{2^s} \cdot \frac{1}{3^s} \cdot \frac{(1 - (\frac{1}{2^s})^{\left\lfloor \frac{\log(\max\{1,\frac{N}{3\cdot 5}\})}{\log(2)} \right\rfloor})}{(1-\frac{1}{2^s})} \cdot \frac{(1 - (\frac{1}{3^s})^{\left\lfloor \frac{\log(\max\{1,\frac{N}{2\cdot 5}\})}{\log(3)} \right\rfloor})}{(1-\frac{1}{3^s})}\right] \tag{8}$$

$$a_7(s,N) = \frac{1}{7^s} \cdot \Bigg[\delta(N \geq 7) + \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{7}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})}$$

$$+\frac{1}{3^s} \cdot \frac{(1-(\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{7}\})}{\log(3)} \rfloor})}{(1-\frac{1}{3^s})}$$

$$+\frac{1}{5^s} \cdot \frac{(1-(\frac{1}{5^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{7}\})}{\log(5)} \rfloor})}{(1-\frac{1}{5^s})}$$

$$+\frac{1}{2^s} \cdot \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{3\cdot 7}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} \cdot \frac{(1-(\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{2\cdot 7}\})}{\log(3)} \rfloor})}{(1-\frac{1}{3^s})}$$

$$+\frac{1}{2^s} \cdot \frac{1}{5^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{5\cdot 7}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} \cdot \frac{(1-(\frac{1}{5^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{2\cdot 7}\})}{\log(5)} \rfloor})}{(1-\frac{1}{5^s})}$$

$$+\frac{1}{3^s} \cdot \frac{1}{5^s} \cdot \frac{(1-(\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{5\cdot 7}\})}{\log(3)} \rfloor})}{(1-\frac{1}{3^s})} \cdot \frac{(1-(\frac{1}{5^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{3\cdot 7}\})}{\log(5)} \rfloor})}{(1-\frac{1}{5^s})}$$

$$+\frac{1}{2^s} \cdot \frac{1}{3^s} \cdot \frac{1}{5^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{3\cdot 5\cdot 7}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} \cdot \frac{(1-(\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{2\cdot 5\cdot 7}\})}{\log(3)} \rfloor})}{(1-\frac{1}{3^s})} \cdot \frac{(1-(\frac{1}{5^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{2\cdot 3\cdot 7}\})}{\log(5)} \rfloor})}{(1-\frac{1}{5^s})}\Bigg] \tag{9}$$

$$a_9(s,N) = \left(\frac{1}{3^s}\right)^2 \cdot \left[\delta(N \geq 3^2) + \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{3^2}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})}\right] \tag{10}$$

$$
\begin{aligned}
a_{11}(s,N) = \frac{1}{11^s} \cdot \Bigg[\delta(N \geq 11) &+ \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{11}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} \\
&+ \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{11}\})}{\log(3)} \rfloor})}{(1-\frac{1}{3^s})} \\
&+ \frac{1}{5^s} \cdot \frac{(1-(\frac{1}{5^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{11}\})}{\log(5)} \rfloor})}{(1-\frac{1}{5^s})} \\
&+ \frac{1}{7^s} \cdot \frac{(1-(\frac{1}{7^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{11}\})}{\log(7)} \rfloor})}{(1-\frac{1}{7^s})} \\
&+ \frac{1}{2^s} \cdot \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{3\cdot 11}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} \cdot \frac{(1-(\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{2\cdot 11}\})}{\log(3)} \rfloor})}{(1-\frac{1}{3^s})} \\
&+ \frac{1}{2^s} \cdot \frac{1}{5^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{5\cdot 11}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} \cdot \frac{(1-(\frac{1}{5^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{2\cdot 11}\})}{\log(5)} \rfloor})}{(1-\frac{1}{5^s})} \\
&+ \frac{1}{2^s} \cdot \frac{1}{7^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{7\cdot 11}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} \cdot \frac{(1-(\frac{1}{7^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{2\cdot 11}\})}{\log(7)} \rfloor})}{(1-\frac{1}{7^s})} \\
&+ \frac{1}{3^s} \cdot \frac{1}{5^s} \cdot \frac{(1-(\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{5\cdot 11}\})}{\log(3)} \rfloor})}{(1-\frac{1}{3^s})} \cdot \frac{(1-(\frac{1}{5^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{3\cdot 11}\})}{\log(5)} \rfloor})}{(1-\frac{1}{5^s})} \\
&+ \frac{1}{3^s} \cdot \frac{1}{7^s} \cdot \frac{(1-(\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{7\cdot 11}\})}{\log(3)} \rfloor})}{(1-\frac{1}{3^s})} \cdot \frac{(1-(\frac{1}{7^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{3\cdot 11}\})}{\log(7)} \rfloor})}{(1-\frac{1}{7^s})} \\
&+ \frac{1}{5^s} \cdot \frac{1}{7^s} \cdot \frac{(1-(\frac{1}{5^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{7\cdot 11}\})}{\log(5)} \rfloor})}{(1-\frac{1}{5^s})} \cdot \frac{(1-(\frac{1}{7^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{5\cdot 11}\})}{\log(7)} \rfloor})}{(1-\frac{1}{7^s})} \\
&+ \frac{1}{2^s} \cdot \frac{1}{3^s} \cdot \frac{1}{5^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{3\cdot 5\cdot 11}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} \cdot \frac{(1-(\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{2\cdot 5\cdot 11}\})}{\log(3)} \rfloor})}{(1-\frac{1}{3^s})} \cdot \frac{(1-(\frac{1}{5^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{2\cdot 3\cdot 11}\})}{\log(5)} \rfloor})}{(1-\frac{1}{5^s})} \\
&+ \frac{1}{2^s} \cdot \frac{1}{3^s} \cdot \frac{1}{7^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{3\cdot 7\cdot 11}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} \cdot \frac{(1-(\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{2\cdot 7\cdot 11}\})}{\log(3)} \rfloor})}{(1-\frac{1}{3^s})} \cdot \frac{(1-(\frac{1}{7^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{2\cdot 3\cdot 11}\})}{\log(7)} \rfloor})}{(1-\frac{1}{7^s})} \\
&+ \frac{1}{2^s} \cdot \frac{1}{5^s} \cdot \frac{1}{7^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{5\cdot 7\cdot 11}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} \cdot \frac{(1-(\frac{1}{5^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{2\cdot 7\cdot 11}\})}{\log(5)} \rfloor})}{(1-\frac{1}{5^s})} \cdot \frac{(1-(\frac{1}{7^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{2\cdot 5\cdot 11}\})}{\log(7)} \rfloor})}{(1-\frac{1}{7^s})} \\
&+ \frac{1}{3^s} \cdot \frac{1}{5^s} \cdot \frac{1}{7^s} \cdot \frac{(1-(\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{5\cdot 7\cdot 11}\})}{\log(3)} \rfloor})}{(1-\frac{1}{3^s})} \cdot \frac{(1-(\frac{1}{5^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{3\cdot 7\cdot 11}\})}{\log(5)} \rfloor})}{(1-\frac{1}{5^s})} \cdot \frac{(1-(\frac{1}{7^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{3\cdot 5\cdot 11}\})}{\log(7)} \rfloor})}{(1-\frac{1}{7^s})} \\
&+ \frac{1}{2^s} \cdot \frac{1}{3^s} \cdot \frac{1}{5^s} \cdot \frac{1}{7^s} \cdot \frac{(1-(\frac{1}{2^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{3\cdot 5\cdot 7\cdot 11}\})}{\log(2)} \rfloor})}{(1-\frac{1}{2^s})} \cdot \frac{(1-(\frac{1}{3^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{2\cdot 5\cdot 7\cdot 11}\})}{\log(3)} \rfloor})}{(1-\frac{1}{3^s})} \cdot \frac{(1-(\frac{1}{5^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{2\cdot 3\cdot 7\cdot 11}\})}{\log(5)} \rfloor})}{(1-\frac{1}{5^s})} \\
&\cdot \frac{(1-(\frac{1}{7^s})^{\lfloor \frac{\log(\max\{1,\frac{N}{2\cdot 3\cdot 5\cdot 11}\})}{\log(7)} \rfloor})}{(1-\frac{1}{7^s})} \Bigg]
\end{aligned} \tag{11}
$$

$$a_{25}(s,N)=\left(\frac{1}{5^s}\right)^2\cdot\Bigg[\delta(N\geq 5^2)+\frac{1}{2^s}\cdot\frac{(1-(\frac{1}{2^s})^{\lfloor\frac{\log(\max\{1,\frac{N}{5^2}\})}{\log(2)}\rfloor})}{(1-\frac{1}{2^s})}$$

$$+\frac{1}{3^s}\cdot\frac{(1-(\frac{1}{3^s})^{\lfloor\frac{\log(\max\{1,\frac{N}{5^2}\})}{\log(3)}\rfloor})}{(1-\frac{1}{3^s})}$$

$$+\frac{1}{2^s}\cdot\frac{1}{3^s}\cdot\frac{(1-(\frac{1}{2^s})^{\lfloor\frac{\log(\max\{1,\frac{N}{3\cdot 5^2}\})}{\log(2)}\rfloor})}{(1-\frac{1}{2^s})}\cdot\frac{(1-(\frac{1}{3^s})^{\lfloor\frac{\log(\max\{1,\frac{N}{2\cdot 5^2}\})}{\log(3)}\rfloor})}{(1-\frac{1}{3^s})}\Bigg] \tag{12}$$

$$a_{27}(s,N)=\left(\frac{1}{3^s}\right)^3\cdot\left[\delta(N\geq 3^3)+\frac{1}{2^s}\cdot\frac{(1-(\frac{1}{2^s})^{\lfloor\frac{\log(\max\{1,\frac{N}{3^3}\})}{\log(2)}\rfloor})}{(1-\frac{1}{2^s})}\right] \tag{13}$$

$$a_{81}(s,N)=\left(\frac{1}{3^s}\right)^4\cdot\left[\delta(N\geq 3^4)+\frac{1}{2^s}\cdot\frac{(1-(\frac{1}{2^s})^{\lfloor\frac{\log(\max\{1,\frac{N}{3^4}\})}{\log(2)}\rfloor})}{(1-\frac{1}{2^s})}\right] \tag{14}$$

$$a_{125}(s,N)=\left(\frac{1}{5^s}\right)^3\cdot\Bigg[\delta(N\geq 5^3)+\frac{1}{2^s}\cdot\frac{(1-(\frac{1}{2^s})^{\lfloor\frac{\log(\max\{1,\frac{N}{5^3}\})}{\log(2)}\rfloor})}{(1-\frac{1}{2^s})}$$

$$+\frac{1}{3^s}\cdot\frac{(1-(\frac{1}{3^s})^{\lfloor\frac{\log(\max\{1,\frac{N}{5^3}\})}{\log(3)}\rfloor})}{(1-\frac{1}{3^s})}$$

$$+\frac{1}{2^s}\cdot\frac{1}{3^s}\cdot\frac{(1-(\frac{1}{2^s})^{\lfloor\frac{\log(\max\{1,\frac{N}{3\cdot 5^3}\})}{\log(2)}\rfloor})}{(1-\frac{1}{2^s})}\cdot\frac{(1-(\frac{1}{3^s})^{\lfloor\frac{\log(\max\{1,\frac{N}{2\cdot 5^3}\})}{\log(3)}\rfloor})}{(1-\frac{1}{3^s})}\Bigg] \tag{15}$$

Simple examples

N=1

$$a_2(s,1)=1+\frac{1}{2^s}\cdot\frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})}=1 \tag{16}$$

$$a_3(s,1)=\frac{1}{3^s}\cdot\left[0+\frac{1}{2^s}\cdot\frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})}\right]=\frac{1}{3^s}\cdot\left[0+0\right]=0 \tag{17}$$

$$a_5(s,1)=\frac{1}{5^s}\cdot\left[0+\frac{1}{2^s}\cdot\frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})}+\frac{1}{3^s}\cdot\frac{(1-(\frac{1}{3^s})^0)}{(1-\frac{1}{3^s})}+\frac{1}{2^s}\cdot\frac{1}{3^s}\cdot\frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})}\cdot\frac{(1-(\frac{1}{3^s})^0)}{(1-\frac{1}{3^s})}\right]=0 \tag{18}$$

$$a_{(p>2)}(s,1)=0 \tag{19}$$

$$\therefore\qquad a_2(s,1)+..+a_{p\leq 1}(s,1)=1=\sum_{n=1}^{1}\frac{1}{n^s} \tag{20}$$

N=2

$$a_2(s,2) = 1 + \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^1)}{(1-\frac{1}{2^s})} = 1 + \frac{1}{2^s} \tag{21}$$

$$a_3(s,2) = \frac{1}{3^s} \cdot \left[0 + \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})}\right] = \frac{1}{3^s} \cdot [0+0] = 0 \tag{22}$$

$$a_5(s,2) = \frac{1}{5^s} \cdot \left[0 + \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})} + \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{3^s})^0)}{(1-\frac{1}{3^s})} + \frac{1}{2^s} \cdot \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})} \cdot \frac{(1-(\frac{1}{3^s})^0)}{(1-\frac{1}{3^s})}\right] = 0 \tag{23}$$

$$a_{(p>2)}(s,2) = 0 \tag{24}$$

$$\therefore \qquad a_2(s,2) + .. + a_{p\le 2}(s,2) = 1 + \frac{1}{2^s} = \sum_{n=1}^{2} \frac{1}{n^s} \tag{25}$$

N=3

$$a_2(s,3) = 1 + \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^1)}{(1-\frac{1}{2^s})} = 1 + \frac{1}{2^s} \tag{26}$$

$$a_3(s,3) = \frac{1}{3^s} \cdot \left[1 + \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})}\right] = \frac{1}{3^s} \cdot [1+0] = \frac{1}{3^s} \tag{27}$$

$$a_5(s,3) = \frac{1}{5^s} \cdot \left[0 + \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})} + \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{3^s})^0)}{(1-\frac{1}{3^s})} + \frac{1}{2^s} \cdot \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})} \cdot \frac{(1-(\frac{1}{3^s})^0)}{(1-\frac{1}{3^s})}\right] = 0 \tag{28}$$

$$a_{(p>3)}(s,3) = 0 \tag{29}$$

$$\therefore \qquad a_2(s,3) + .. + a_{p\le 3}(s,3) = 1 + \frac{1}{2^s} + \frac{1}{3^s} = \sum_{n=1}^{3} \frac{1}{n^s} \tag{30}$$

N=4

$$a_2(s,4) = 1 + \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^2)}{(1-\frac{1}{2^s})} = 1 + \frac{1}{2^s} + \frac{1}{4^s} \tag{31}$$

$$a_3(s,4) = \frac{1}{3^s} \cdot \left[1 + \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})}\right] = \frac{1}{3^s} \cdot [1+0] = \frac{1}{3^s} \tag{32}$$

$$a_5(s,4) = \frac{1}{5^s} \cdot \left[0 + \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})} + \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{3^s})^0)}{(1-\frac{1}{3^s})} + \frac{1}{2^s} \cdot \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})} \cdot \frac{(1-(\frac{1}{3^s})^0)}{(1-\frac{1}{3^s})}\right] = 0 \tag{33}$$

$$a_{(p>4)}(s,4) = 0 \tag{34}$$

$$\therefore \qquad a_2(s,4) + .. + a_{p\le 4}(s,4) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} = \sum_{n=1}^{4} \frac{1}{n^s} \tag{35}$$

N=10

$$a_2(s,10) = 1 + \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^3)}{(1-\frac{1}{2^s})} = 1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} \tag{36}$$

$$a_3(s,10) = \frac{1}{3^s} \cdot \left[1 + \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^1)}{(1-\frac{1}{2^s})}\right] = \frac{1}{3^s} \cdot [1 + \frac{1}{2^s}] = \frac{1}{3^s} + \frac{1}{6^s} \tag{37}$$

$$\begin{aligned} a_5(s,10) &= \frac{1}{5^s} \cdot \left[1 + \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^1)}{(1-\frac{1}{2^s})} + \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{3^s})^0)}{(1-\frac{1}{3^s})} + \frac{1}{2^s} \cdot \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})} \cdot \frac{(1-(\frac{1}{3^s})^0)}{(1-\frac{1}{3^s})}\right] \\ &= \frac{1}{5^s}\left[1 + \frac{1}{2^s} + 0 + 0\right] = \frac{1}{5^s} + \frac{1}{10^s} \end{aligned} \tag{38}$$

$$\begin{aligned} a_7(s,10) &= \frac{1}{7^s} \cdot [1 + \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})} + \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{3^s})^0)}{(1-\frac{1}{3^s})} + \frac{1}{5^s} \cdot \frac{(1-(\frac{1}{5^s})^0)}{(1-\frac{1}{5^s})} \\ &+ \frac{1}{2^s} \cdot \frac{1}{3^s} \cdot \frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})} \cdot \frac{(1-(\frac{1}{3^s})^0)}{(1-\frac{1}{3^s})} + \frac{1}{2^s} \cdot \frac{1}{5^s} \cdot \frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})} \cdot \frac{(1-(\frac{1}{5^s})^0)}{(1-\frac{1}{5^s})} \\ &+ \frac{1}{3^s} \cdot \frac{1}{5^s} \cdot \frac{(1-(\frac{1}{3^s})^0)}{(1-\frac{1}{3^s})} \cdot \frac{(1-(\frac{1}{5^s})^0)}{(1-\frac{1}{5^s})} \\ &+ \frac{1}{2^s} \cdot \frac{1}{3^s} \cdot \frac{1}{5^s} \cdot \frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})} \cdot \frac{(1-(\frac{1}{3^s})^0)}{(1-\frac{1}{3^s})} \cdot \frac{(1-(\frac{1}{5^s})^0)}{(1-\frac{1}{5^s})}] \\ &= \frac{1}{7^s}[1 + 0 + 0 + 0 + 0 + 0 + 0 + 0] = \frac{1}{7^s} \end{aligned} \tag{39}$$

$$a_9(s,10) = (\frac{1}{3^s})^2 \cdot \left[1 + \frac{1}{2^s} \cdot \frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})}\right] = \frac{1}{9^s} \cdot [1 + 0] = \frac{1}{9^s} \tag{40}$$

$$a_{(p>10)}(s,10) = 0 \tag{41}$$

$$\begin{aligned} \therefore \quad a_2(s,10) + .. + a_{p\leq 10}(s,10) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} \\ + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + \frac{1}{9^s} + \frac{1}{10^s} &= \sum_{n=1}^{10} \frac{1}{n^s} \end{aligned} \tag{42}$$

Serious examples

The Riemann Zeta function has a peak when

$$\zeta(0.5 + 280.8 \cdot I) = 7.002850509 + 0.0323221757 \cdot I \tag{43}$$

This peak can be approximated by a finite dirichlet series sum at the second quiescent region $N_2 = \frac{t}{\pi}$ of the oscillatory divergence of the series [6].

$$\sum_{n=1}^{\lfloor \frac{280.8}{\pi} \rfloor} \frac{1}{n^{(0.5+280.8\cdot I)}} = \sum_{n=1}^{89} \frac{1}{n^{(0.5+280.8\cdot I)}} = 6.9603700512 + 0.0637616537 \cdot I \tag{44}$$

This peak can also be approximated by the finite dirichlet series Riemann Siegel formula [1,2] at the first quiescent region $N_1 = \sqrt{\frac{t}{2\pi}}$ of the oscillatory divergence of the dirichlet series

$$
\begin{aligned}
&\sum_{n=1}^{\lfloor\sqrt{\frac{280.8}{2\pi}}\rfloor} \frac{1}{n^{(0.5+280.8\cdot I)}} + \chi(0.5+280.8\cdot I)\cdot \sum_{n=1}^{\lfloor\sqrt{\frac{280.8}{2\pi}}\rfloor} \frac{1}{n^{1-(0.5+280.8\cdot I)}} \\
&= \sum_{n=1}^{6} \frac{1}{n^{(0.5+280.8\cdot I)}} + \chi(0.5+280.8\cdot I)\cdot \sum_{n=1}^{6} \frac{1}{n^{1-(0.5+280.8\cdot I)}} \\
&= 6.8311940570 + 0.0315298826\cdot I \qquad (45)
\end{aligned}
$$

where $\chi(s)$ is the multiplicative factor of the Riemann Zeta functional equation $\zeta(s) = \chi(s)\zeta(1-s)$

So using equation (5), the above dirichlet series can be generated by a linear combination of products of geometric series of primes and powers of primes

Firstly for the Riemann Siegel formula approach

N=6

$$a_2(s,6) = 1 + \frac{1}{2^s}\cdot\frac{(1-(\frac{1}{2^s})^2)}{(1-\frac{1}{2^s})} = 1 + \frac{1}{2^s} + \frac{1}{4^s} \qquad (46)$$

$$a_3(s,6) = \frac{1}{3^s}\cdot\left[1 + \frac{1}{2^s}\cdot\frac{(1-(\frac{1}{2^s})^1)}{(1-\frac{1}{2^s})}\right] = \frac{1}{3^s}\cdot\left[1+\frac{1}{2^s}\right] = \frac{1}{3^s} + \frac{1}{6^s} \qquad (47)$$

$$a_5(s,6) = \frac{1}{5^s}\cdot\left[1 + \frac{1}{2^s}\cdot\frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})} + \frac{1}{3^s}\cdot\frac{(1-(\frac{1}{3^s})^0)}{(1-\frac{1}{3^s})} + \frac{1}{2^s}\cdot\frac{1}{3^s}\cdot\frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})}\cdot\frac{(1-(\frac{1}{3^s})^0)}{(1-\frac{1}{3^s})}\right] = \frac{1}{5^s} \qquad (48)$$

$$a_{(p>6)}(s,6) = 0 \qquad (49)$$

$$\therefore \qquad a_2(s,6) + .. + a_{p\le 6}(s,6) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} = \sum_{n=1}^{6}\frac{1}{n^s} \qquad (50)$$

Secondly for the second quiescent region based approximation (which requires a longer dirichlet series)

N=89

To save computational time in evaluating equation (5) for N=89,

Step 1. the primes and powers of primes are identified for $p \le 89$. (Powers of prime 2 do not need to be explicitly listed.)

$$p \le 89 = \{2,3,5,7,9,11,13,17,19,23,25,27,29,31,37,41,43,47,49,53,59,61,67,71,73,79,81,83,89\} \qquad (51)$$

step 2. the divisors of $\frac{89}{p}$ are obtained and the set of primes and powers of primes are grouped by greatest common divisor

$$p_{(gcd=1,N=89)} = \{47, 53, 59, 61, 67, 71, 73, 79, 81, 83, 89\} \quad (52)$$
$$p_{(gcd=2,N=89)} = \{31, 37, 41, 43\} \quad (53)$$
$$p_{(gcd=3,N=89)} = \{23, 29\} \quad (54)$$
$$p_{(gcd=4,N=89)} = \{19\} \quad (55)$$
$$p_{(gcd=5,N=89)} = \{17\} \quad (56)$$
$$p_{(gcd=6,N=89)} = \{13\} \quad (57)$$
$$p_{(gcd=8,N=89)} = \{11\} \quad (58)$$
$$p_{(gcd=12,N=89)} = \{7\} \quad (59)$$
$$p_{(gcd=17,N=89)} = \{5\} \quad (60)$$
$$p_{(gcd=29,N=89)} = \{3\} \quad (61)$$
$$p_{(gcd=89,N=89)} = \{1\} \quad (62)$$
$$\text{powers}_{(p=3,N=89)} = \{9, 27, 81\} \quad (63)$$
$$\text{powers}_{(p=5,N=89)} = \{25\} \quad (64)$$
$$\text{powers}_{(p=7,N=89)} = \{49\} \quad (65)$$

step 3. the required contributions to the dirichet series can be generated as follows

$$\sum a_{p\in gcd=1}(s, 89) = (1)\cdot(\frac{1}{47^s} + \frac{1}{53^s} + \frac{1}{59^s} + \frac{1}{61^s} + \frac{1}{67^s} + \frac{1}{71^s} + \frac{1}{73^s} + \frac{1}{79^s} + \frac{1}{81} + \frac{1}{83^s} + \frac{1}{89^s}) \quad (66)$$

$$\sum a_{p\in gcd=2}(s, 89) = (1 + \frac{1}{2^s})\cdot(\frac{1}{31^s} + \frac{1}{37^s} + \frac{1}{41^s} + \frac{1}{43^s}) \quad (67)$$

$$\sum a_{p\in gcd=3}(s, 89) = (a_2(s, 3) + a_3(s, 3))\cdot(\frac{1}{23^s} + \frac{1}{29^s}) \quad (68)$$
$$= (\sum_{n=1}^{3}\frac{1}{n^s})\cdot(\frac{1}{23^s} + \frac{1}{29^s}) \quad (69)$$

$$\sum a_{p\in gcd=4}(s, 89) = (a_2(s, 4) + a_3(s, 4))\cdot(\frac{1}{19^s}) \quad (70)$$
$$= (\sum_{n=1}^{4}\frac{1}{n^s})\cdot(\frac{1}{19^s}) \quad (71)$$

$$\sum a_{p\in gcd=5}(s, 89) = (a_2(s, 5) + a_3(s, 5) + a_3(s, 5))\cdot(\frac{1}{17^s}) \quad (72)$$
$$= (\sum_{n=1}^{5}\frac{1}{n^s})\cdot(\frac{1}{17^s}) \quad (73)$$

$$\sum a_{p\in gcd=6}(s, 89) = (a_2(s, 6) + a_3(s, 6) + a_5(s, 6))\cdot(\frac{1}{13^s}) \quad (74)$$
$$= (\sum_{n=1}^{6}\frac{1}{n^s})\cdot(\frac{1}{13^s}) \quad (75)$$

$$\sum a_{p\in gcd=8}(s,89) = a_{11}(s,89) \tag{76}$$

$$\sum a_{p\in gcd=12}(s,89) = a_7(s,89) \tag{77}$$

$$\begin{aligned}\sum a_{p\in gcd=17}(s,89) &= a_5(s,89)\\ &= \frac{1}{5^s}\cdot\left[1+\frac{1}{2^s}\cdot\frac{(1-(\frac{1}{2^s})^4)}{(1-\frac{1}{2^s})}+\frac{1}{3^s}\cdot\frac{(1-(\frac{1}{3^s})^2)}{(1-\frac{1}{3^s})}+\frac{1}{2^s}\cdot\frac{1}{3^s}\cdot\frac{(1-(\frac{1}{2^s})^2)}{(1-\frac{1}{2^s})}\cdot\frac{(1-(\frac{1}{3^s})^1)}{(1-\frac{1}{3^s})}\right]\end{aligned} \tag{78}$$

$$\sum a_{p\in gcd=29}(s,89) = a_3(s,89) = \frac{1}{3^s}\cdot\left[1+\frac{1}{2^s}\cdot\frac{(1-(\frac{1}{2^s})^4)}{(1-\frac{1}{2^s})}\right] \tag{79}$$

$$\sum a_{p\in gcd=89}(s,89) = a_2(s,89) = 1+\frac{1}{2^s}\cdot\frac{(1-(\frac{1}{2^s})^6)}{(1-\frac{1}{2^s})} \tag{80}$$

$$\begin{aligned}\sum a_{\text{powers}(p=3,N=89)}(s,89) &= a_9(s,89)+a_{27}(s,89)+a_{81}(s,89)\\ &= (\frac{1}{3^s})^2\cdot\left[1+\frac{1}{2^s}\cdot\frac{(1-(\frac{1}{2^s})^3)}{(1-\frac{1}{2^s})}\right]\\ &+ (\frac{1}{3^s})^3\cdot\left[1+\frac{1}{2^s}\cdot\frac{(1-(\frac{1}{2^s})^1)}{(1-\frac{1}{2^s})}\right]\\ &+ (\frac{1}{3^s})^4\cdot\left[1+\frac{1}{2^s}\cdot\frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})}\right]\\ &= \frac{1}{9^s}+\frac{1}{18^s}+\frac{1}{27^s}+\frac{1}{36^s}+\frac{1}{54^s}+\frac{1}{81^s}\end{aligned} \tag{81}$$

$$\begin{aligned}\sum a_{\text{powers}(p=5,N=89)} &= a_{25}(s,89)\\ &= (\frac{1}{5^s})^2\cdot\left[1+\frac{1}{2^s}\cdot\frac{(1-(\frac{1}{2^s})^1)}{(1-\frac{1}{2^s})}+\frac{1}{3^s}\cdot\frac{(1-(\frac{1}{3^s})^1)}{(1-\frac{1}{3^s})}+\frac{1}{2^s}\cdot\frac{1}{3^s}\cdot\frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})}\cdot\frac{(1-(\frac{1}{3^s})^0)}{(1-\frac{1}{3^s})}\right]\\ &= \frac{1}{25^s}+\frac{1}{50^s}+\frac{1}{75^s}\end{aligned} \tag{82}$$

$$\begin{aligned}\sum a_{\text{powers}(p=7,N=89)} &= a_{49}(s,89)\\ &= \frac{1}{7^s}\cdot[1+\frac{1}{2^s}\cdot\frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})}+\frac{1}{3^s}\cdot\frac{(1-(\frac{1}{3^s})^0)}{(1-\frac{1}{3^s})}+\frac{1}{5^s}\cdot\frac{(1-(\frac{1}{5^s})^0)}{(1-\frac{1}{5^s})}\\ &+\frac{1}{2^s}\cdot\frac{1}{3^s}\cdot\frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})}\cdot\frac{(1-(\frac{1}{3^s})^0)}{(1-\frac{1}{3^s})}+\frac{1}{2^s}\cdot\frac{1}{5^s}\cdot\frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})}\cdot\frac{(1-(\frac{1}{5^s})^0)}{(1-\frac{1}{5^s})}\\ &+\frac{1}{3^s}\cdot\frac{1}{5^s}\cdot\frac{(1-(\frac{1}{3^s})^0)}{(1-\frac{1}{3^s})}\cdot\frac{(1-(\frac{1}{5^s})^0)}{(1-\frac{1}{5^s})}\\ &+\frac{1}{2^s}\cdot\frac{1}{3^s}\cdot\frac{1}{5^s}\cdot\frac{(1-(\frac{1}{2^s})^0)}{(1-\frac{1}{2^s})}\cdot\frac{(1-(\frac{1}{3^s})^0)}{(1-\frac{1}{3^s})}\cdot\frac{(1-(\frac{1}{5^s})^0)}{(1-\frac{1}{5^s})}] = \frac{1}{49^s}\end{aligned} \tag{83}$$

such that step 3 contains the equivalent of all the finite dirichlet series terms in $\sum_{n=1}^{89} \frac{1}{n^s}$.

Conclusions

The above procedure using products of finite geometric series of primes (and their powers) seems dense but in principle, only the prime and powers of primes are being used for the calculations and each term only contains primes smaller in magnitude whereas equation (3) in comparison (which is known to be slow to calculate) requires examination of all triple, quadruple, etc combinations of primes.

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