Identification of the odd positive integers and prime numbers using a square wave version of Riemann Zeta generating function

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Executive Summary

Using a modified version of the Riemann Zeta generating function (based on the extended Riemann Siegel functional form) produces a square wave version of the Riemann Zeta function along the positive real axis. This function can be used to highlight and identify the positive odd integers and prime numbers.

Introduction

The Riemann Zeta function is defined (1), in the complex plane by the integral

$$\zeta(s) = \frac{\prod(-s)}{2\pi i}\int_{C_{\epsilon,\delta}} \frac{(-x)^s}{(e^x-1)x}dx \tag{1}$$

where $s \,\epsilon\, \mathbb{C}$ and $C_{\epsilon,\delta}$ is the contour about the imaginary poles.

The Riemann Zeta function has been shown to obey the functional equation (2)

$$\zeta(s) = \zeta(1-s) * (2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s)) \tag{2}$$

Following directly from the form of the functional equation and the properties of the coefficients on the RHS of eqn (2) it has been shown that any zeroes off the critical line would be paired, ie. if $\zeta(s) = 0$ was true then $\zeta(1-s) = 0$.

The Riemann Siegel function is an exact function (3) for the magnitude of the Riemann Zeta function along the critical line (0.5+it) of the form

$$Z(t) = \zeta(0.5+it)e^{i\theta(t)} \tag{3}$$

where

$$\theta(t) = \Im(ln(\Gamma(\frac{1}{4}+\frac{1}{2}it))) - \frac{t}{2}ln(\pi) \tag{4}$$

In Martin (4) and earlier work, the properties of the Riemann Zeta generating function were investigated and used to define and investigate the extended Riemann Siegel $Z_{ext}(s)$ and $\theta_{ext}(s)$ functions away from the critical line.

$$e^{i\theta_{ext}(s)} = \sqrt{\frac{\zeta(1-s)abs(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))}{\zeta(s)}} \tag{5}$$

$$\theta_{ext}(s) = \Im(log(e^{i*\theta_{ext}(s)})) \tag{6}$$

$$\tag{7}$$

$$= \Im(log(\sqrt{\frac{\zeta(1-s)abs(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))}{\zeta(s)}})) \tag{8}$$

$$Z_{ext}(s) = \sqrt{\zeta(s)*\zeta(1-s)*abs(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))} \tag{9}$$

where consistent with eqn (3)

$$\zeta(s) = Z_{ext}(s)*e^{-i\theta_{ext}(s)} \tag{10}$$

In particular, only on the critical line s=(0.5+i*t) is the cancellation exact between the coupled, extended Riemann Siegel functions eqns (5) and (9), resulting in Riemann Zeta zeroes.

In this paper from eqns (5)-(9),

$$\frac{Z_{ext}(s)^2}{\zeta(s)} = Z_{ext}(s)*e^{+i\theta_{ext}(s)} \tag{11}$$

is used to create a square wave Riemann Zeta generating function, which highlights and identifies the odd positive integers and prime numbers.

Square wave version of Riemann Zeta generating function

Using eqn eqns (5)-(9) it can be shown that

$$\frac{Z_{ext}(s)^2}{\zeta(s)} = Z_{ext}(s)*e^{+i\theta_{ext}(s)} = \zeta(1-s)*abs(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s)) \tag{12}$$

which is almost the Riemann Zeta generating function eqn (2) except for the use of the absolute value for the second term in eqn (12).

As shown in figure 1, the waveform generated is asymptotically a square wave, along the positive x axis since $\zeta(s)$ rapidly tends to 1 on that line.

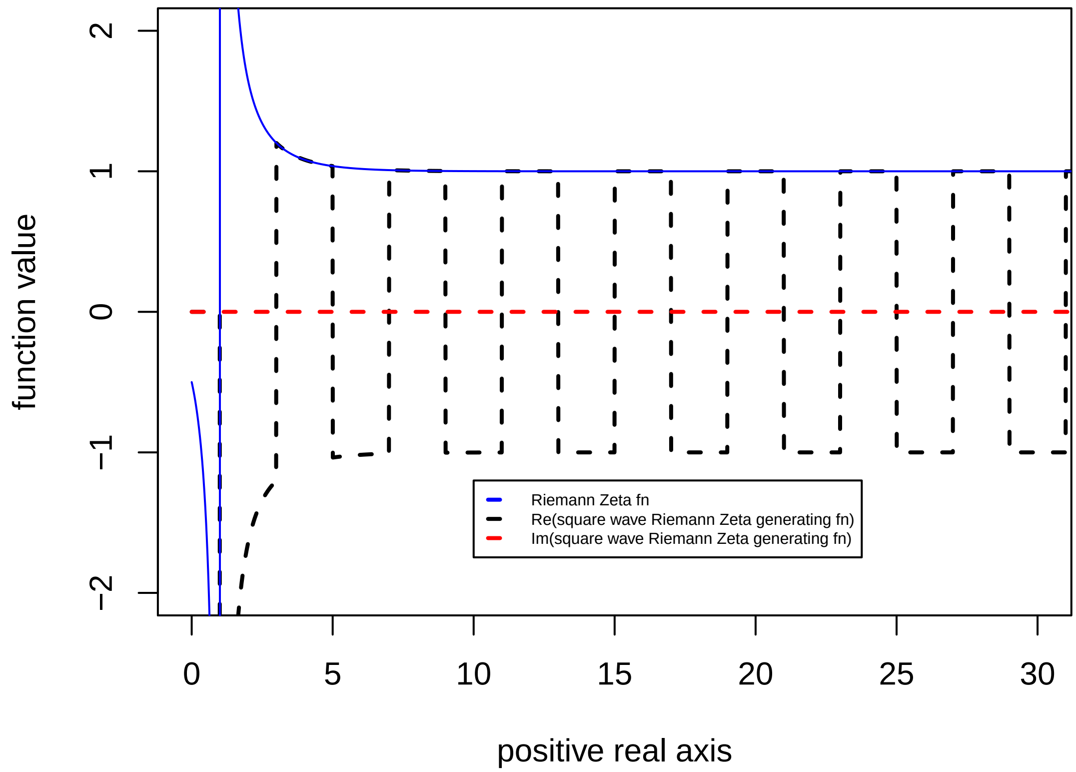

Fig.1:Riemann Zeta fn and a square wave Riemann Zeta generating fn on positive real axis.

The identification of the positive odd integers occurs trivally because of the $sin(\frac{\pi s}{2})$ term but interestingly the product of $\zeta(1-s)$ and $abs(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))$ plus $\zeta(s)$ rapidly tending to 1 produces the square wave behaviour.

This square wave behaviour allows the positive odd integers to be identified without recourse to arbitrarily defining a staircase function. That is, the function values for $\Re$ between the odd positive integers now have a piecewise continuous function defining their values.

This function when extended with a quotient argument s/n where n the denominator is the integer,

$$\frac{Z_{ext}(\frac{s}{n})^2}{\zeta(\frac{s}{n})} = Z_{ext}(\frac{s}{n}) * e^{+i\theta_{ext}(\frac{s}{n})} = \zeta(1-\frac{s}{n}) * abs(2^{\frac{s}{n}}\pi^{\frac{s}{n}-1}sin(\frac{\pi\frac{s}{n}}{2})\Gamma(1-\frac{s}{n})), \qquad \text{where } n \in \mathbb{N} \tag{13}$$

can be used to test for prime numbers (inefficiently).

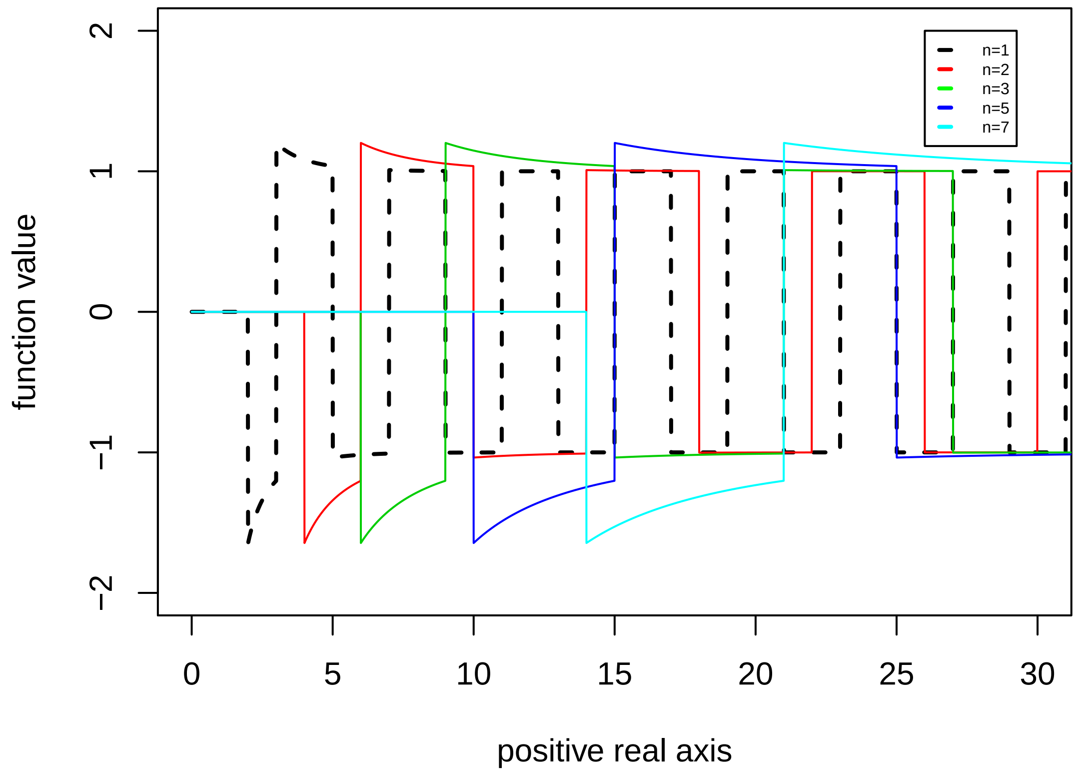

Fig.2:Testing for prime numbers, dashed black lines without overlap of coloured lines are primes

Conclusions

The extended Riemann Siegel functions, can be used to define

(i) the Riemann Zeta function as an interference pattern $\zeta(s) = Z_{ext}(s) * e^{-i\theta_{ext}(s)}$,

(ii) the square wave Riemann Zeta function $Z_{ext}(s) * e^{+i\theta_{ext}(s)} = \zeta(1-s) * abs(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))$ which identifies the positive odd integers and

(iii) through quotient argument form (where the denominator n is integer) $\frac{Z_{ext}(\frac{s}{n})^2}{\zeta(\frac{s}{n})} = Z_{ext}(\frac{s}{n}) * e^{+i\theta_{ext}(\frac{s}{n})}$ which identifies (inefficiently) prime numbers.

This square wave function which allows input values (analytical continuation) off the positive real axis may give useful insights to the prime-counting function.

References

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