Improved lower bounds > 0 for Riemann Zeta function off the critical line using derivatives of Re(Riemann Siegel Z) function.

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Executive Summary

Approximate lower bounds > 0, for $abs(\zeta(s))$ off the critical line are generated using the real part of the Riemann Siegel function Re(Z(t)) and its first derivative, further supporting the Riemann Hypothesis. An approximation to $abs(\zeta(s))$ off the critical line, is then obtained by the rescaled series sum of the Re(Z(t)) and several higher order derivatives.

Introduction

The Riemann Zeta function is defined (1), in the complex plane by the integral

$$\zeta(s) = \frac{\prod(-s)}{2\pi i}\int_{C_{\epsilon,\delta}} \frac{(-x)^s}{(e^x-1)x}dx \tag{1}$$

where $s \,\epsilon\, \mathbb{C}$ and $C_{\epsilon,\delta}$ is the contour about the imaginary poles.

The Riemann Zeta function has been shown to obey the functional equation (2)

$$\zeta(s) = 2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s) \tag{2}$$

Following directly from the form of the functional equation and the properties of the coefficients on the RHS of eqn (2) it has been shown that any zeroes off the critical line would be paired, ie. if $\zeta(s) = 0$ was true then $\zeta(1-s) = 0$.

The Riemann Siegel function is an approximating function (3) for the magnitude of the Riemann Zeta function along the critical line (0.5+it) of the form

$$Z(t) = \zeta(0.5+it)e^{i\theta(t)} \tag{3}$$

where

$$\theta(t) = Im(ln(\Gamma(\frac{1}{4}+\frac{1}{2}it))) - \frac{t}{2}ln(\pi) \tag{4}$$

The transformation $e^{i\theta(t)}$, rotates $\zeta(0.5+it)$ such that Re(Z(t)) contains the entire Riemann Zeta critical line waveform energy and the zeroes of Z(t) correspond with the zeroes of $abs(\zeta(0.5+it))$.

In Martin (4), the properties of the Riemann Zeta conjugate pair ratio function was examined. It is obtained from eqn (2) by dividing by sides by $\zeta(1-s)$.

$$\frac{\zeta(s)}{\zeta(1-s)} = 2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s) \tag{5}$$

As well as showing that the Riemann Zeta conjugate pair ratio function has a simple AM-FM lineshape. It was also illustrated that the absolute magnitude of the Riemann Zeta conjugate pair ratio function is an accurate estimate of the average growth of the Riemann Zeta function, for Re(s) <= 0.5.

In Martin (5), it was shown that

(i) the absolute value of the Riemann Zeta function clearly appears to be a smoothed version of the absolute value of the rescaled Riemann Siegel function,

(ii) the Im(s) behaviour of $\zeta(s)$ is equivalent for the same abs(Re(s)-0.5) except

(iii) the amplitude of Im(s) is asymmetric above and below the critical line, given by the relationship

$$Z_{rescaled}(t) = \begin{cases} Z(t) & for\, Re(s) \geq 0.5 \\ abs(2^s \pi^{s-1} sin(\frac{\pi s}{2})\Gamma(1-s))Z(t) & for\, Re(s) < 0.5 \end{cases} \tag{6}$$

Exploring the idea of smoothing the Riemann Siegel function (or critical line Riemann Zeta function). In this paper, a series expansion formula for the Riemann Zeta function is investigated using the Riemann Siegel Z function and its derivatives.

A smoothing series expansion of the Riemann Siegel Z function about the critical line Riemann Zeta zeroes

Figure 1, below illustrates abs(Re(Z(t))), abs(Im(Z(t))), abs($\zeta(0.5+it)$) and additionally the absolute value of the first derivative of Re(Z(t)) ($abs(Z'(t))$). It is important to note that

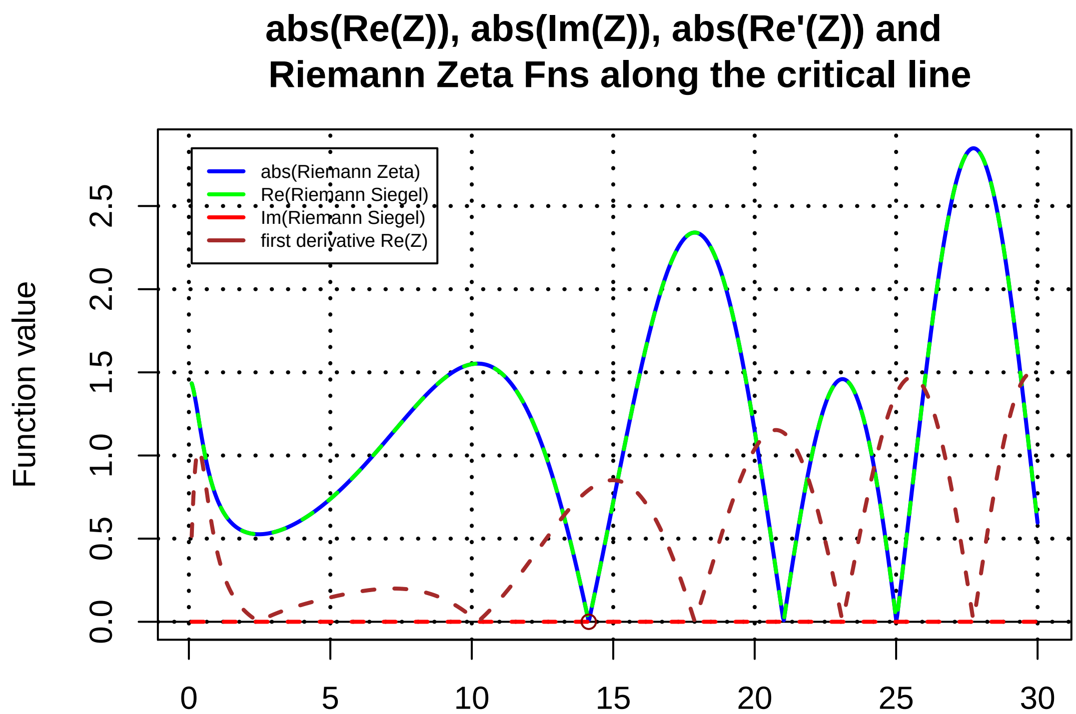
(ii) Im(Z(t)) = 0 except for rounding errors,

(iii) the only zeroes of Re(Z(t)) are the Riemann Zeta critical line zeroes and

(iv) $Re(Z'(t))$ and Re(Z(t)) are $\frac{\pi}{2}$ out of phase so

$$\Sigma(Re(Z(t))^2 + \alpha^2 Re(Z'(t))^2) > 0, \qquad for\, \alpha \,\epsilon\, \mathbb{R} \tag{7}$$

Most of the calculations in this paper involving zeta and its numerical derivatives used the "pracma" r package (6) except when using pari/gp at large imaginary values (7).


abs(Re(Z)), abs(Im(Z)), abs(Re'(Z)) and
Riemann Zeta Fns along the critical line
abs(Riemann Zeta)
Re(Riemann Siegel)
Im(Riemann Siegel)
first derivative Re(Z)
Function value
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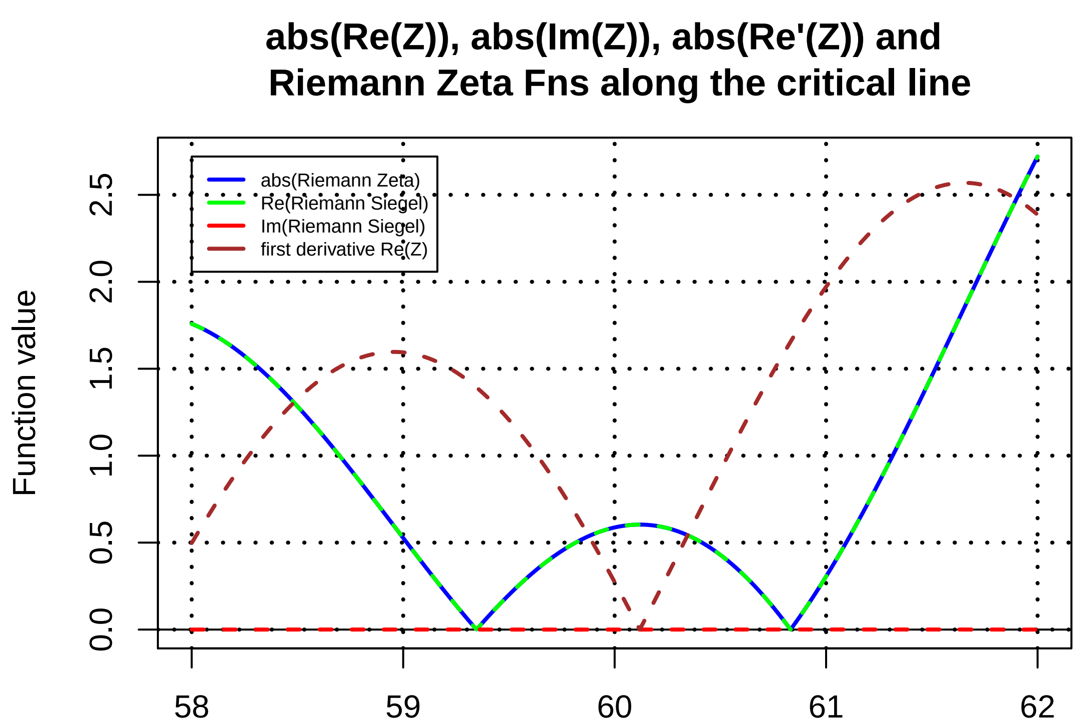
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0.0
0.5
1.0
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Figure 1a: Riemann Siegel and Riemann Zeta function behaviour on the critical line. Figure 1b includes example of tricky adjacent Riemann Zeta zeroes

Given eqn (7), the findings in (5) and the relationships,

$$abs(Z(t)) = abs(\zeta(0.5 + I * t)e^{i\theta(t)}) \tag{8}$$
$$= abs(Re(Z(t))) \tag{9}$$
$$= abs(\zeta(0.5 + I * t)) \tag{10}$$

It has been found that the following series expansion

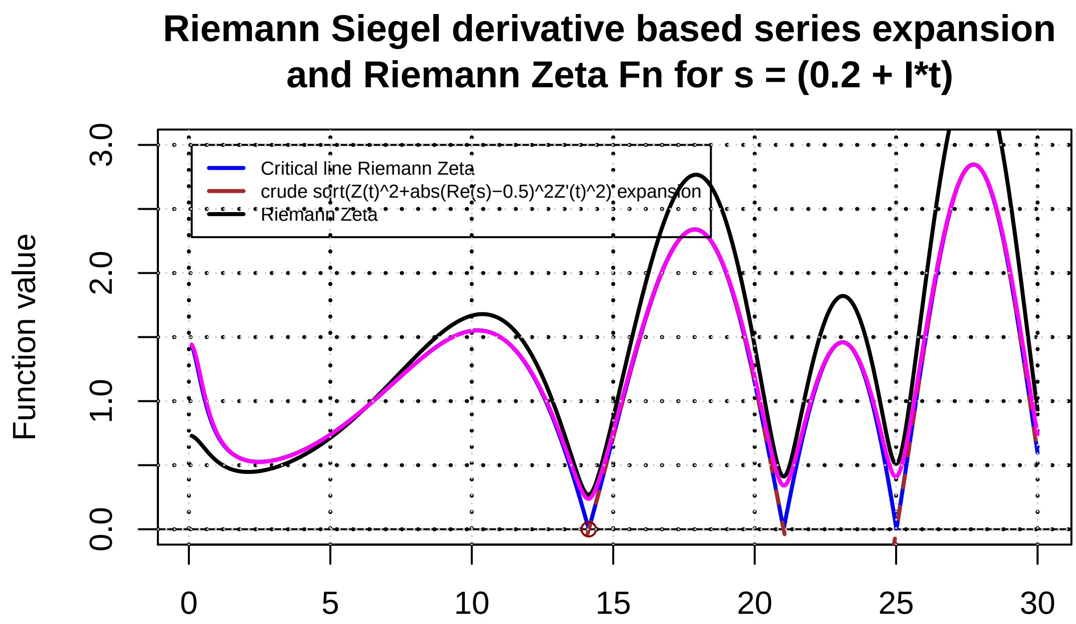
$$abs(\zeta(s)) = \sqrt{Re(Z(t))^2 + \alpha^2 Re(Z)'(t)^2 + \beta^2 Re(Z)''(t) + ...} \tag{11}$$

has good properties to smoothly estimate $abs(\zeta(s))$.

Non-zero lower bound estimate for abs(zeta(s)), for Re(s) < 0.5

After some investigation, Figure 2, below illustrates a crude first order approximation, for Re(s) = 0.2

$$abs(\zeta(s)) \approx (1) \cdot \sqrt{Re(Z(t))^2 + abs(Re(s) - 0.5)^2 Re(Z)'(t)^2 + ...} \qquad for\, Re(s) \, < \, 0.5 \tag{12}$$



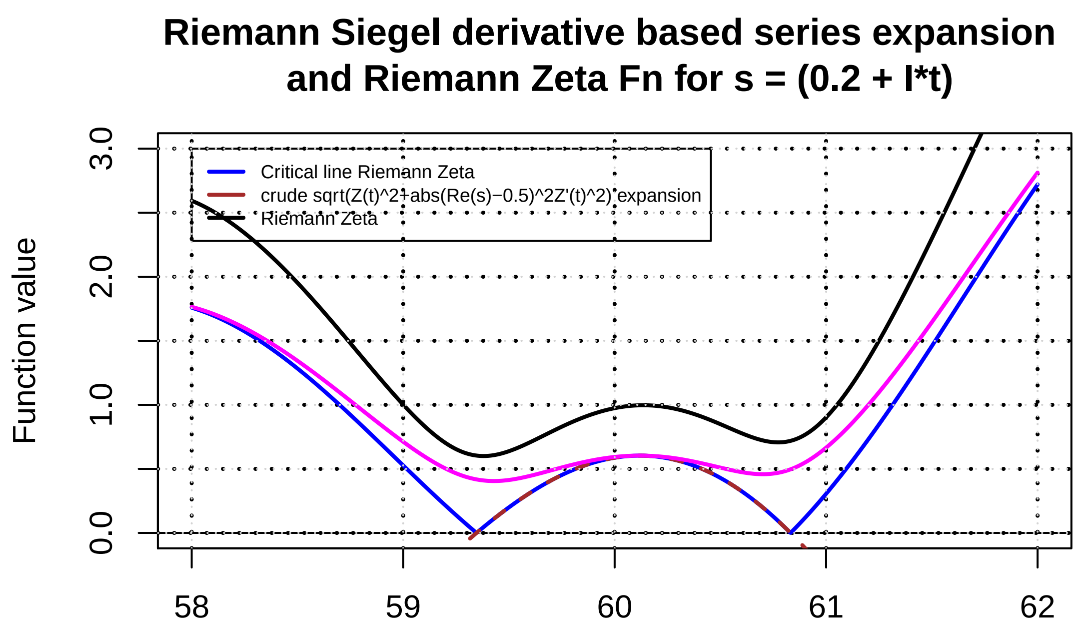


Figure 2a,b: First order Riemann Siegel series expansion and Riemann Zeta function behaviour, below the critical line for s = (0.2 + It)

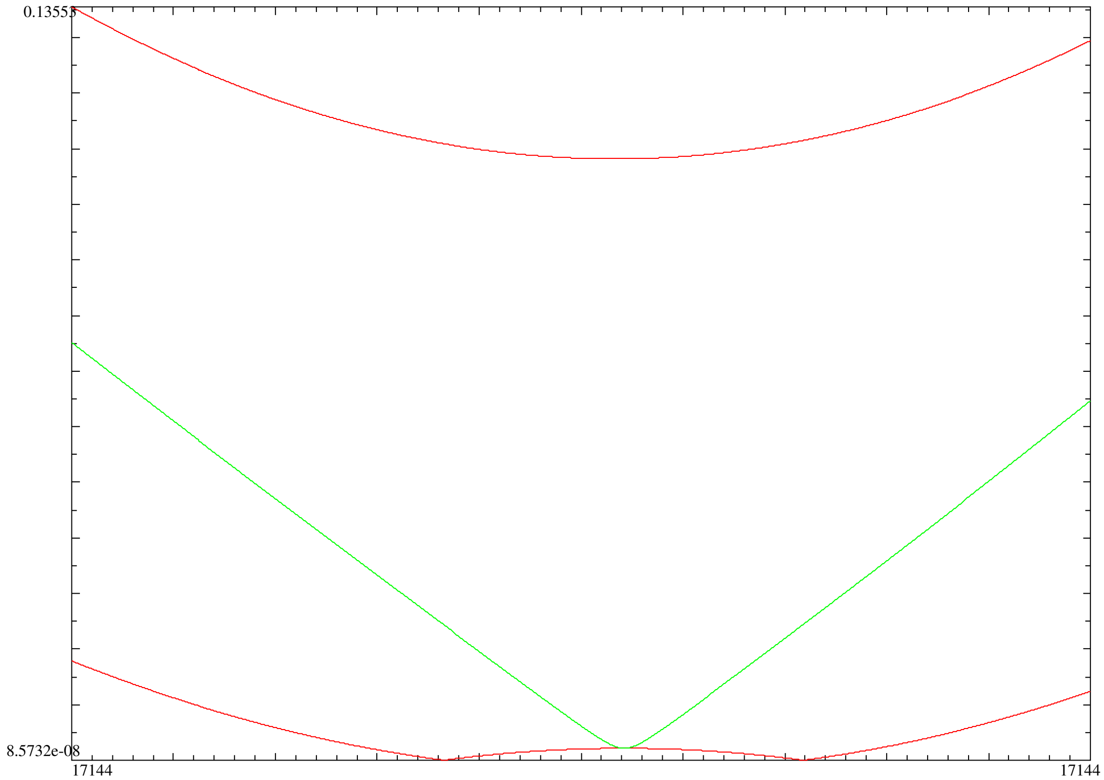


Figure 2c: For Re(s) = 0.4, the non-zero lower bound minimum (green) is below minima in Riemann Zeta function below the critical line (red) at tricky large Im(s) value 17,143.8

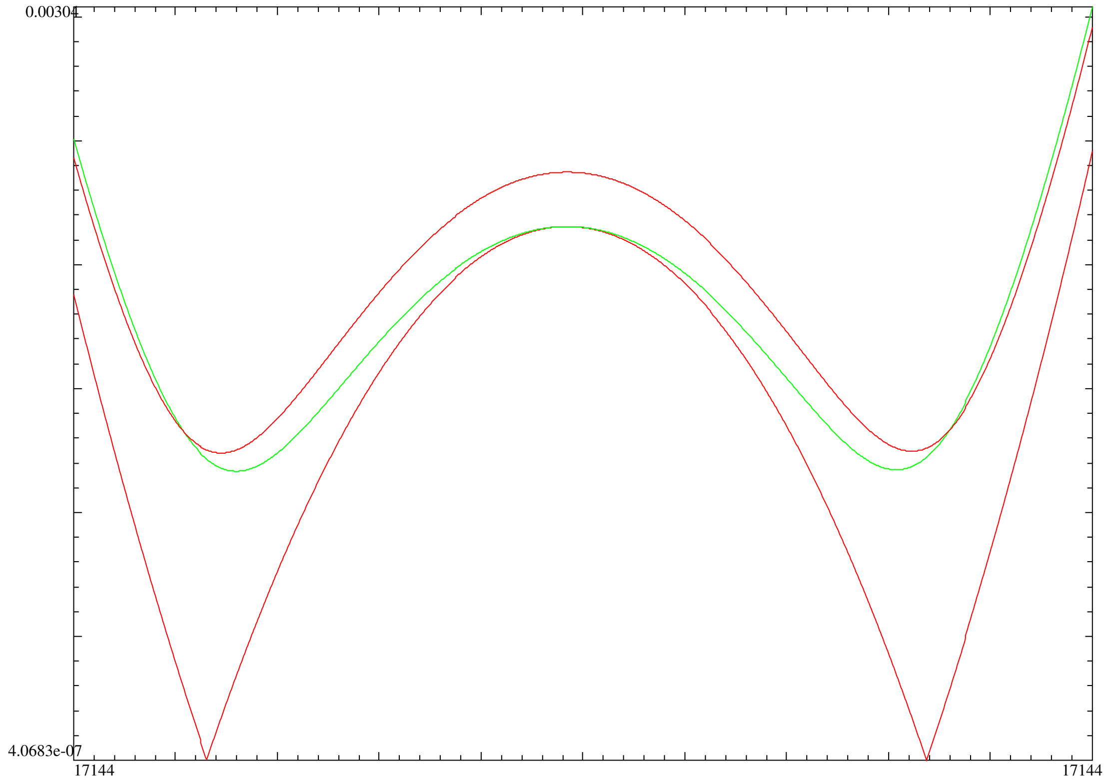


Figure 2d: For Re(s) = 0.49, the non-zero lower bound twin minima (green) are lower than the twin minima in Riemann Zeta function below the critical line (red) at tricky large Im(s) value 17,143.8

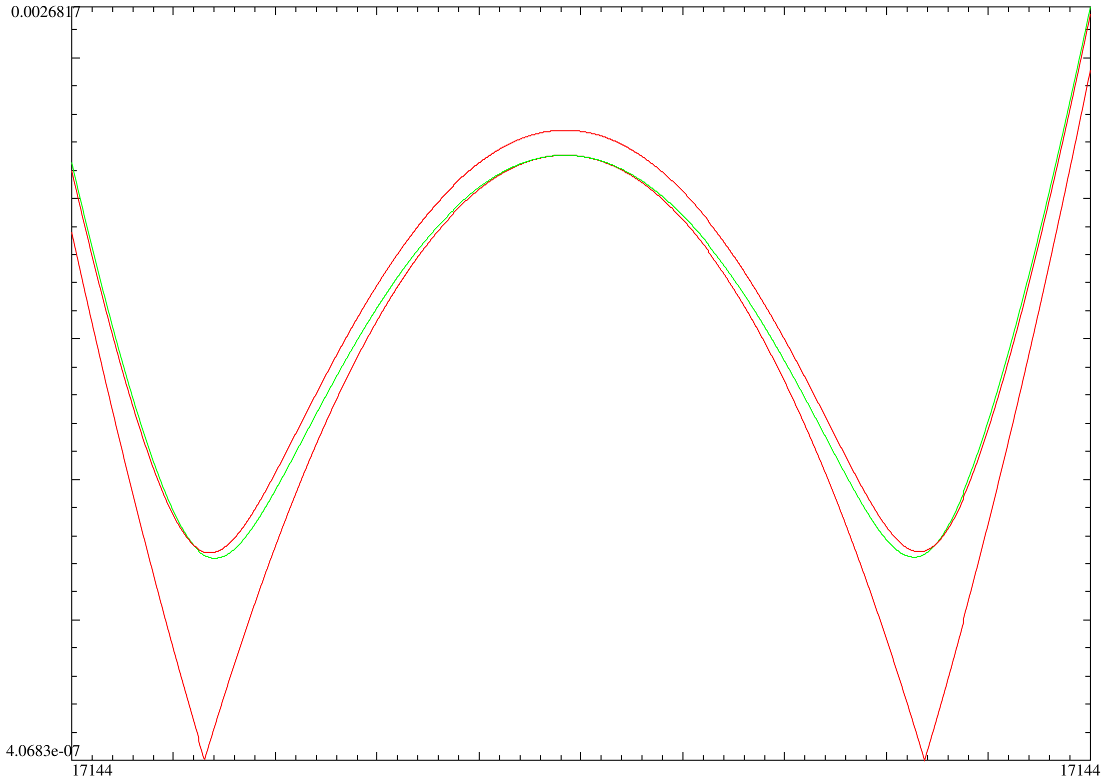
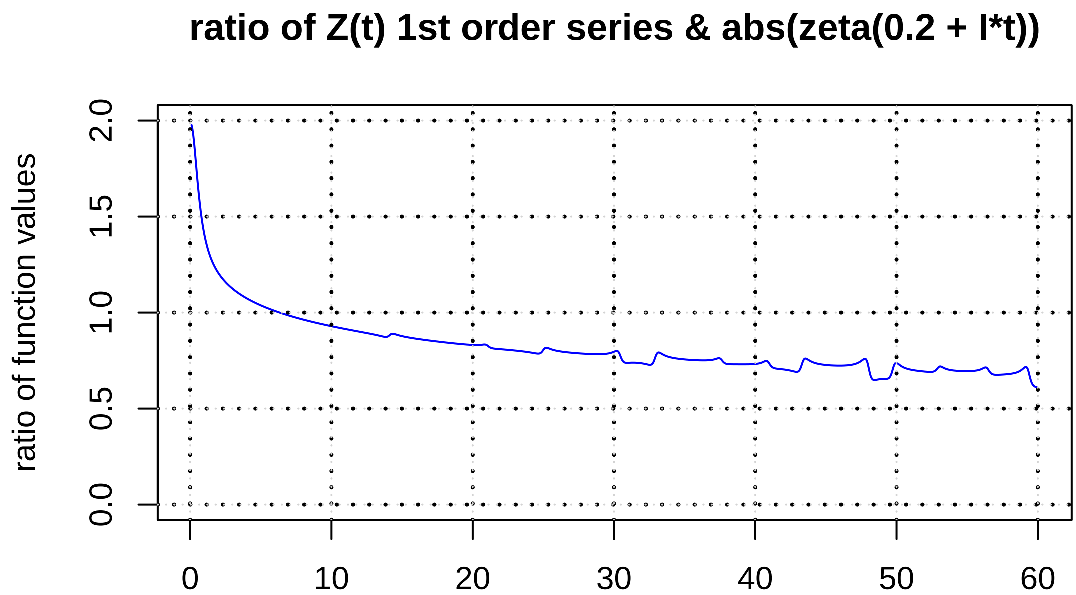


Figure 2e: For Re(s) = 0.497, the non-zero lower bound twin minima (green) are lower and converging below the twin minima in Riemann Zeta function below the critical line (red) at tricky large Im(s) value 17,143.8

For t >= (the first Riemann Zeta zero position) & Re(s) < 0.5, the crude series expansion eqn (12) will have local minima located over each critical line Riemann Zeta zero due to the properties of abs(Z(t)) and $abs(Z'(t))$. These local minima as indicated in figure 3 form a strictly positive lower bound for the Riemann Zeta function below the critical line.



Given that

(i) the minimum of the Riemann Zeta function $abs(\zeta(s))$ appear above the zeroes of the rescaled Riemann Siegel function $abs(Z(t))$ from (5),

(ii) the minimum of the Riemann Zeta function $abs(\zeta(s))$ for Re(s) < 0.5, appear above the strictly positive minima of the 1st order Riemann Siegel derivative series expansion eqn (12) and

(iii) zeroes off the critical line are expected to occur in pairs (1-3),

indicates that the Riemann Hypothesis is valid.

While the 1st order expansion gives a conservative lower estimate and hence can be used a lower bound of the actual Riemann Zeta magnitude, it underestimates the the magnitudes of the Riemann Zeta peaks In the following section, higher order derivatives are added to the expansion.

A nominal rescaled 4th order Riemann Siegel derivative series expansion

Using higher order derivatives, noting the symmetry of lineshape for Abs(Re(s)) mentioned in (5) and identifying an appropriate rescaling factor from the Riemann Zeta conjugate pair ratio function. A rescaled 4th order Riemann Siegel derivative series expansion that gives approximate Riemann Zeta estimates in the critical strip is

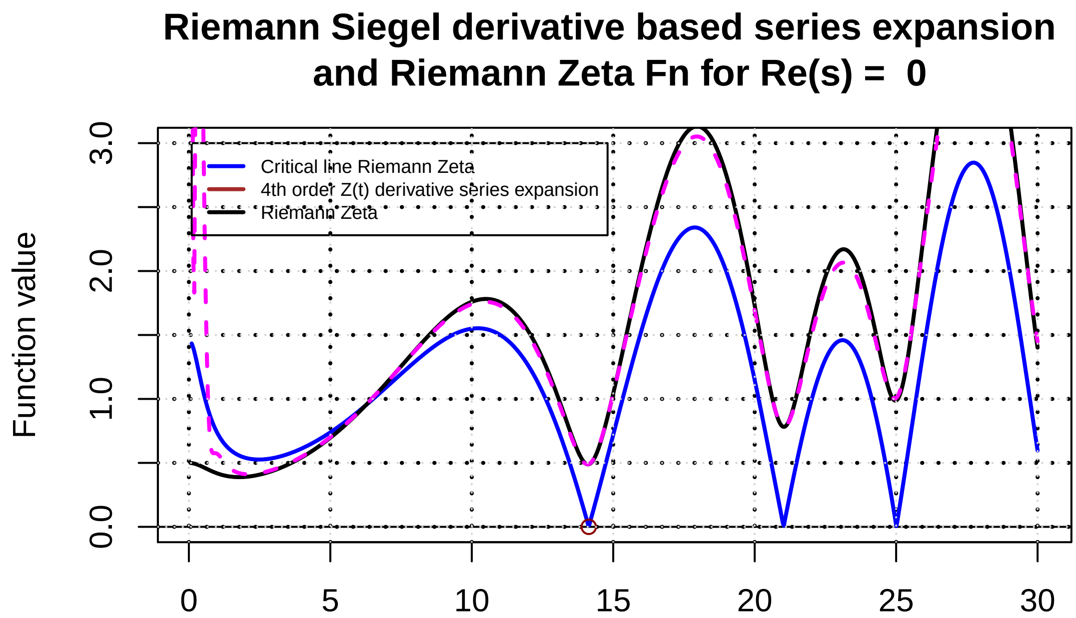
$$abs(\zeta(s)) \approx A \cdot \sqrt{Re(Z(t))^2 + B^2 Re(Z)'(t)^2 + B^3 Re(Z)''(t)^2 + B^4 Re(Z)'''(t)^2 + ...} \tag{13}$$

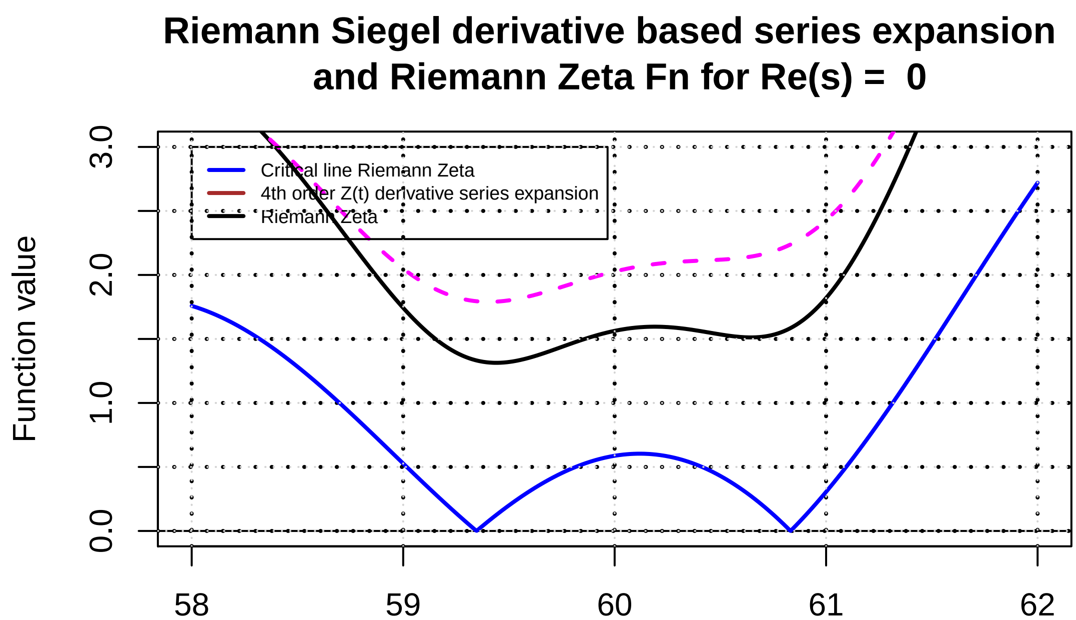
where

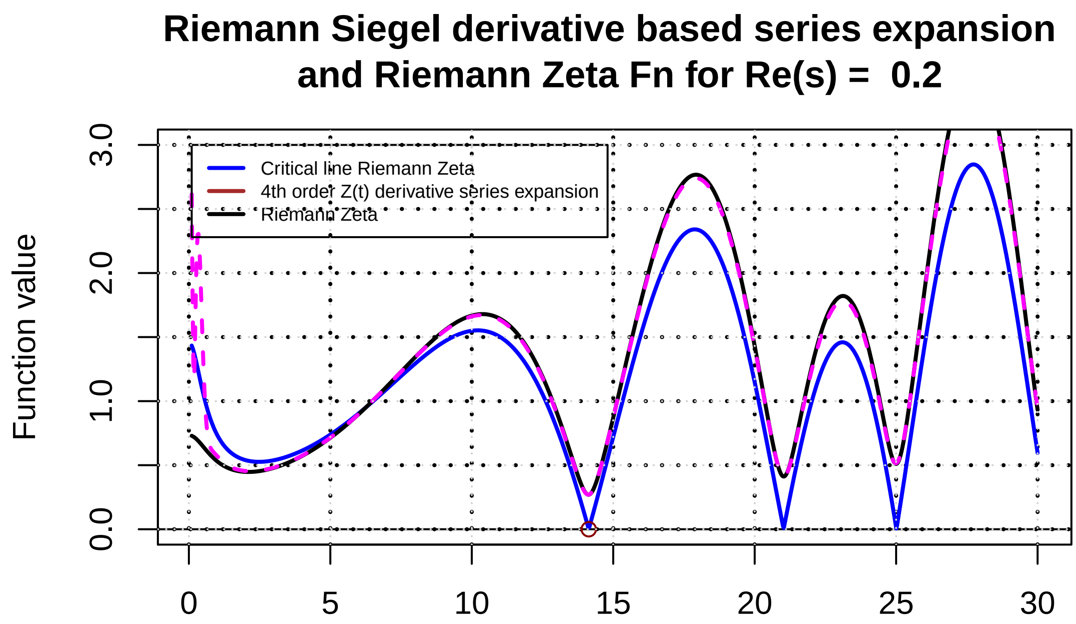
$$A = \sqrt{abs(2^s \pi^{(s-1)} sin(\frac{s}{2}\pi)\Gamma(1-s))} \tag{14}$$

and

$$B = abs(Re(s) - 0.5) \tag{15}$$



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and Riemann Zeta Fn for Re(s) = 0.2
Critical line Riemann Zeta
4th order Z(t) derivative series expansion
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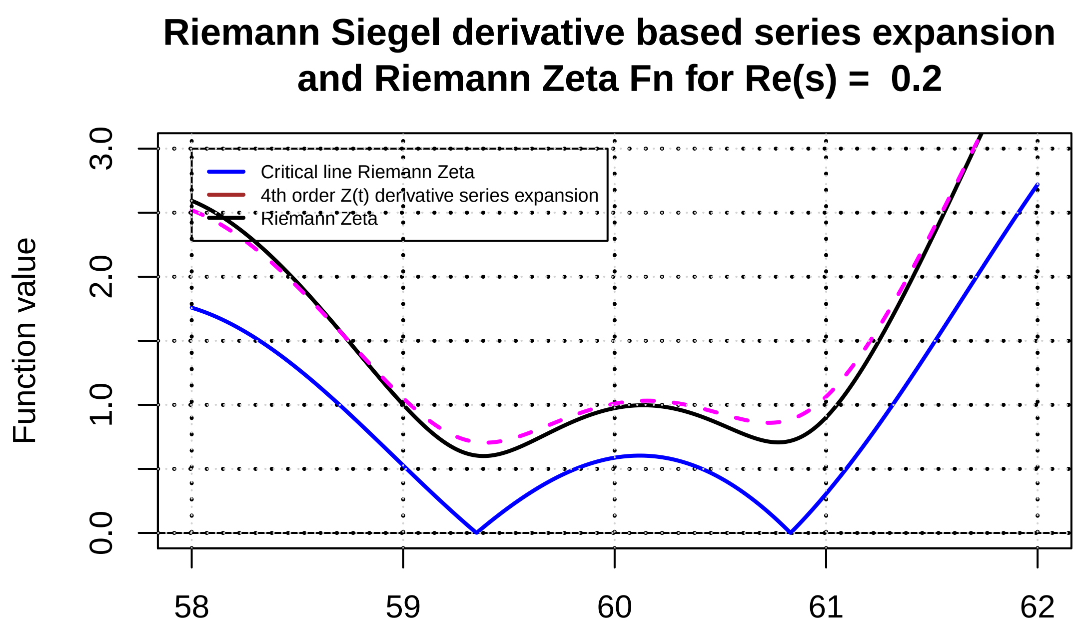
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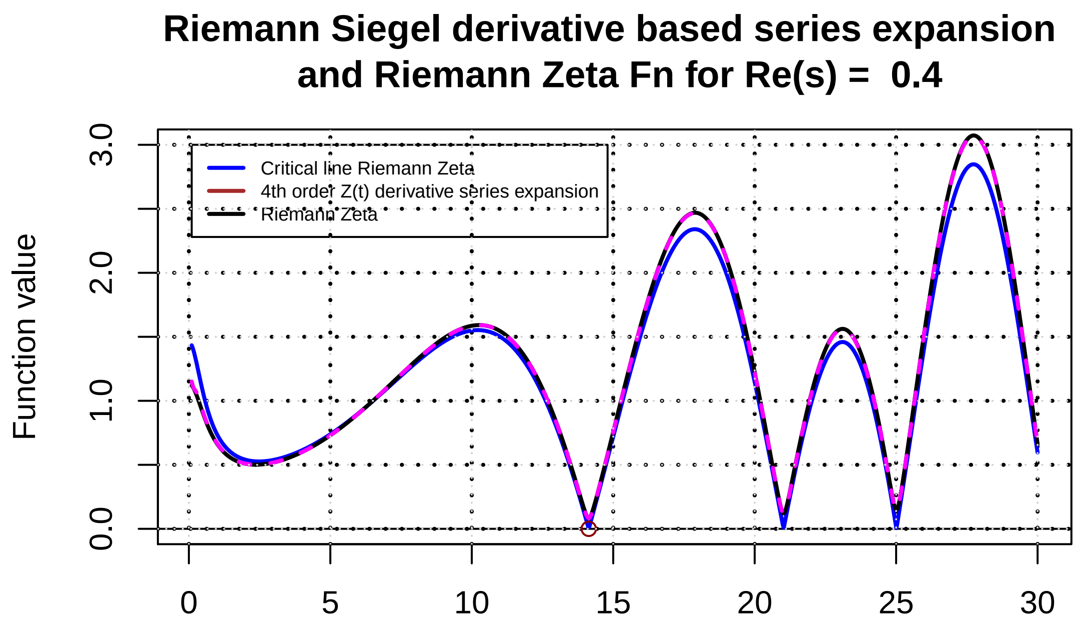
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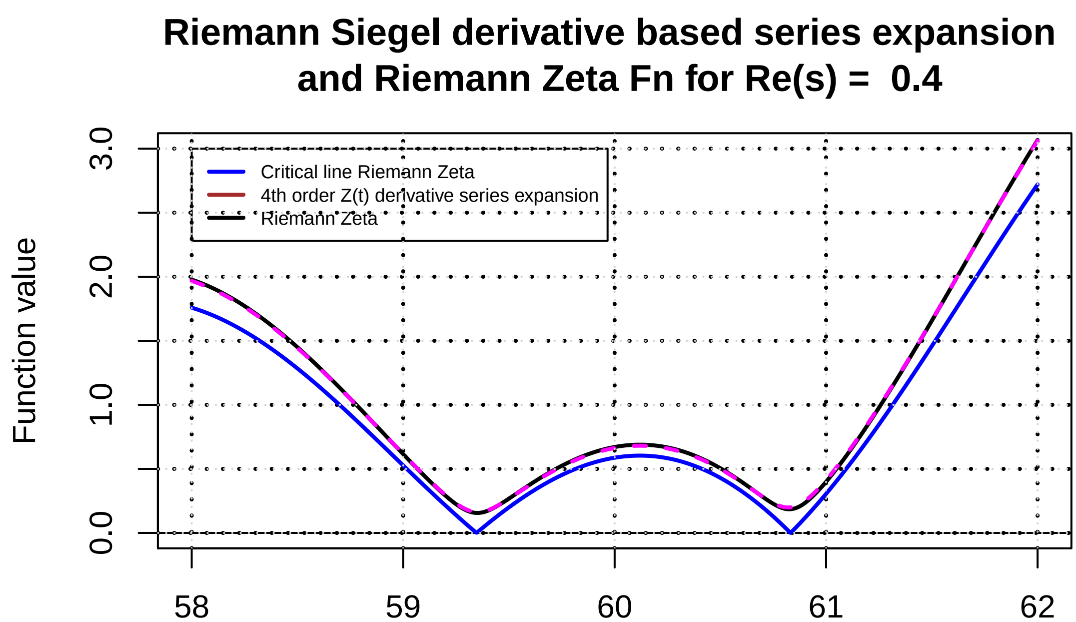
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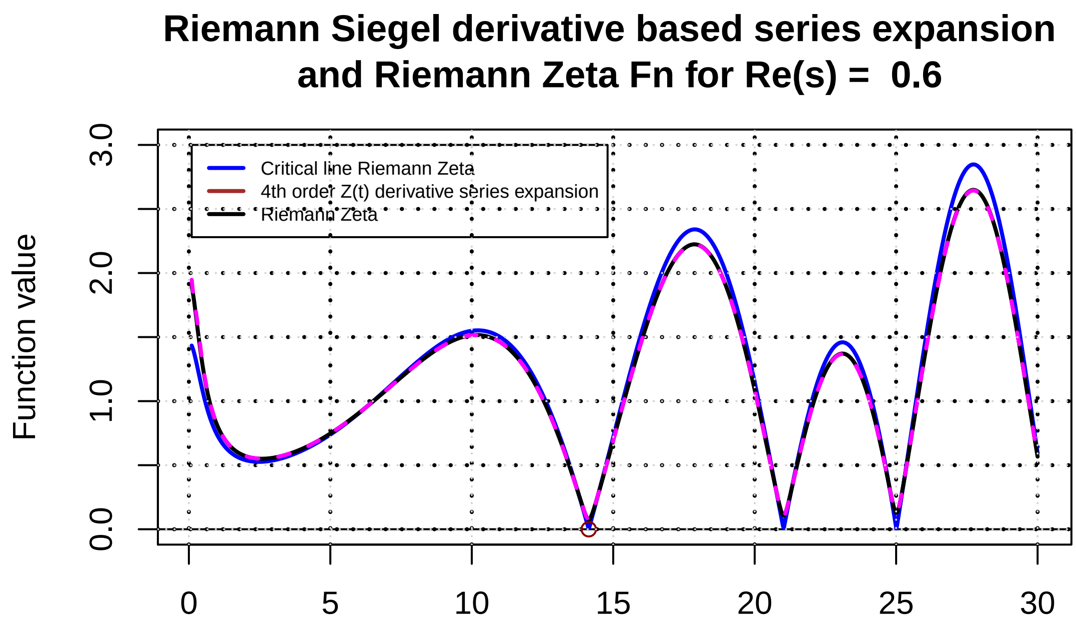
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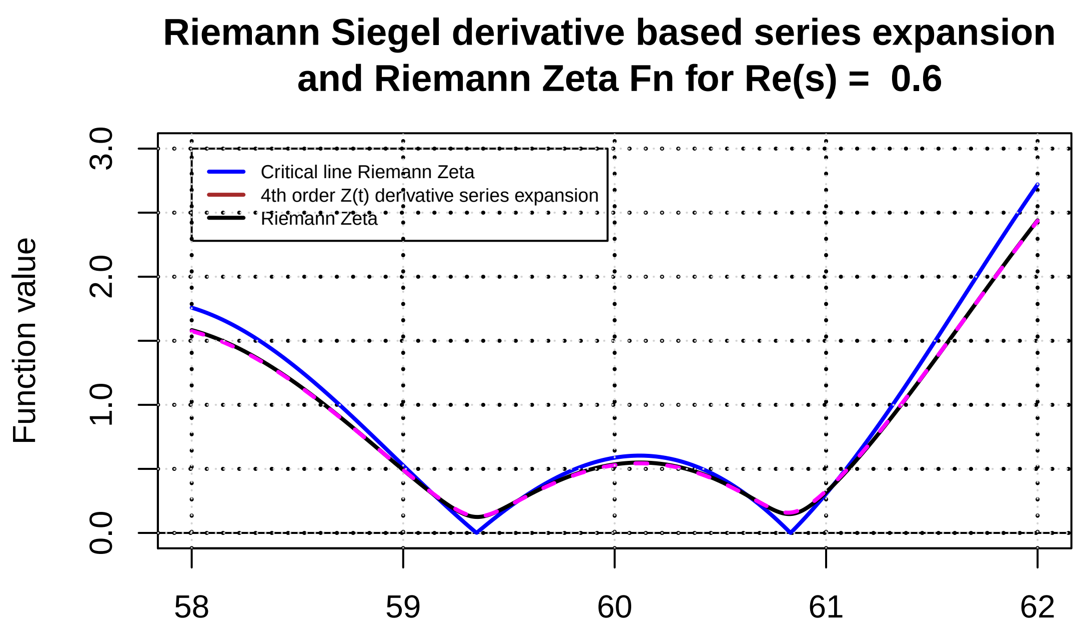
Riemann Siegel derivative based series expansion
and Riemann Zeta Fn for Re(s) = 0.6
Function value
Critical line Riemann Zeta
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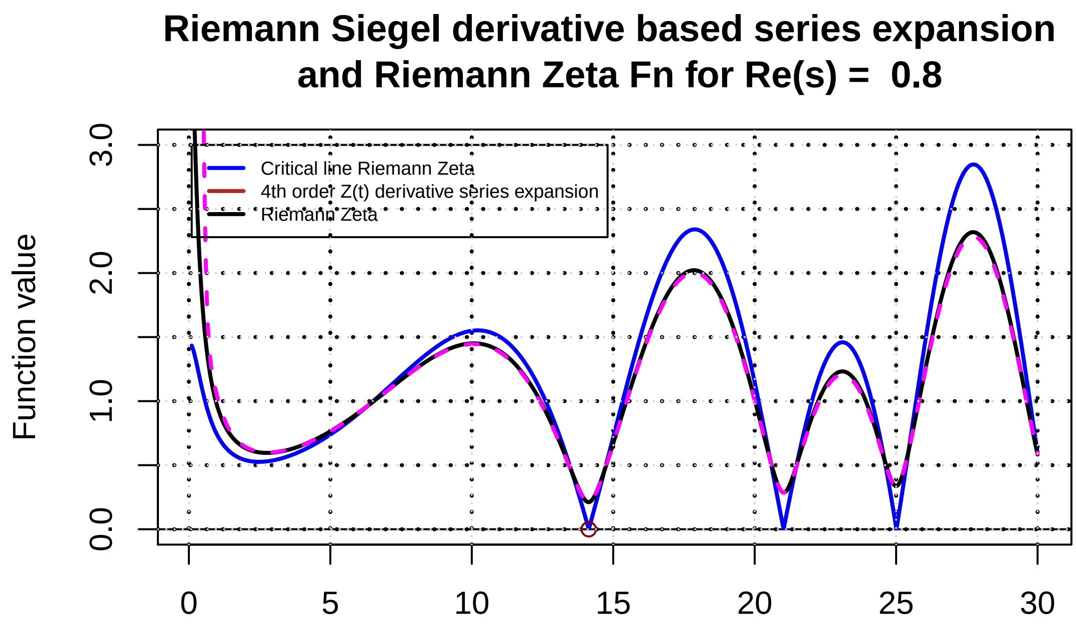

Riemann Siegel derivative based series expansion
and Riemann Zeta Fn for Re(s) = 0.8
Critical line Riemann Zeta
4th order Z(t) derivative series expansion
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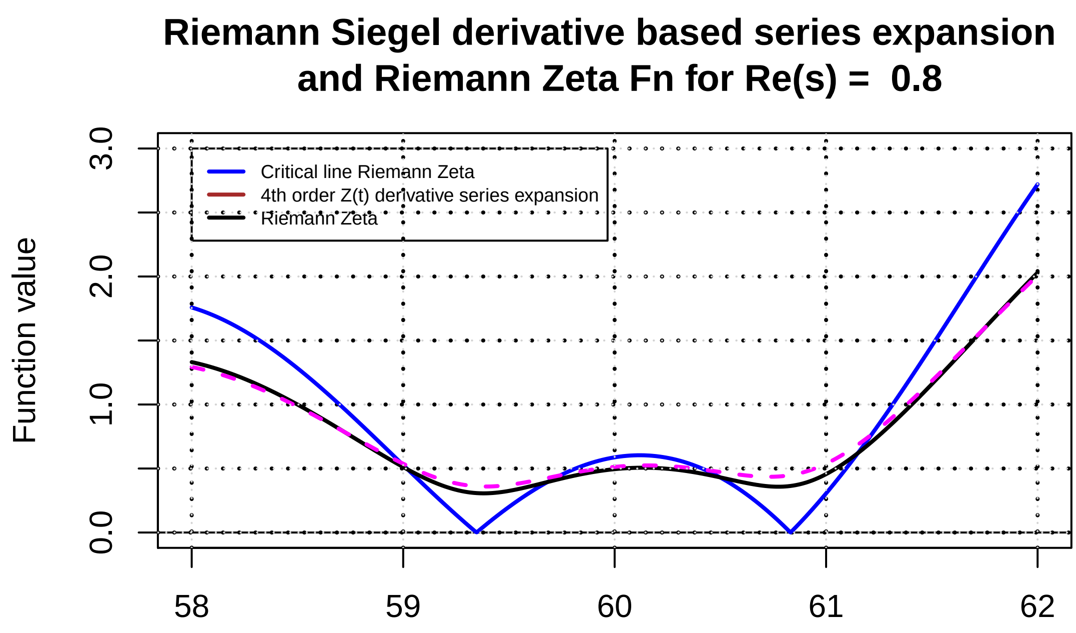

Riemann Siegel derivative based series expansion
and Riemann Zeta Fn for Re(s) = 0.8
Critical line Riemann Zeta
4th order Z(t) derivative series expansion
Riemann Zeta
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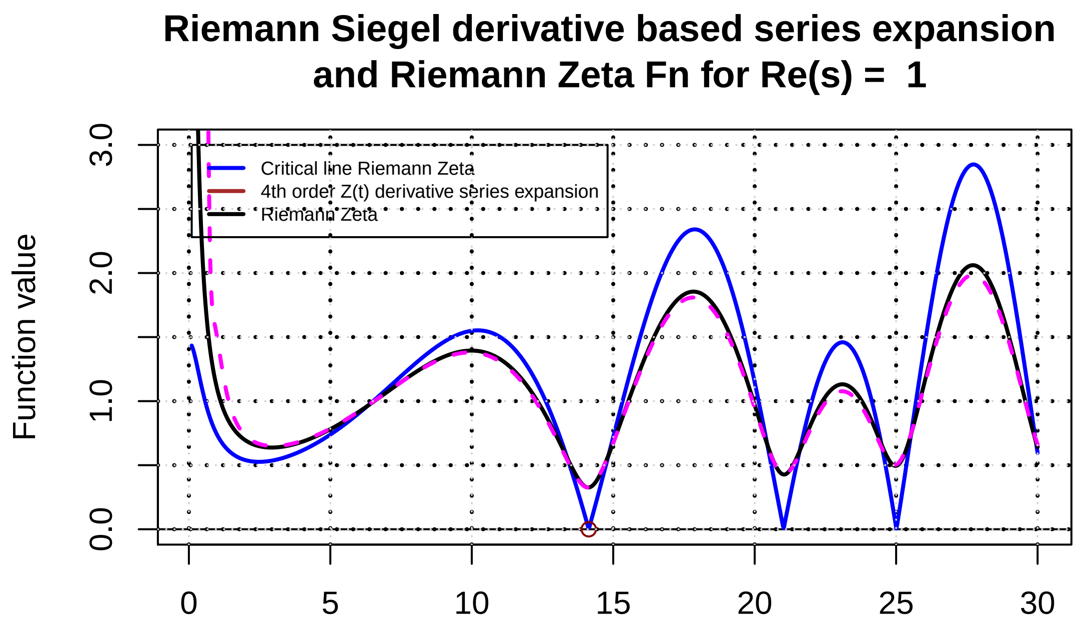

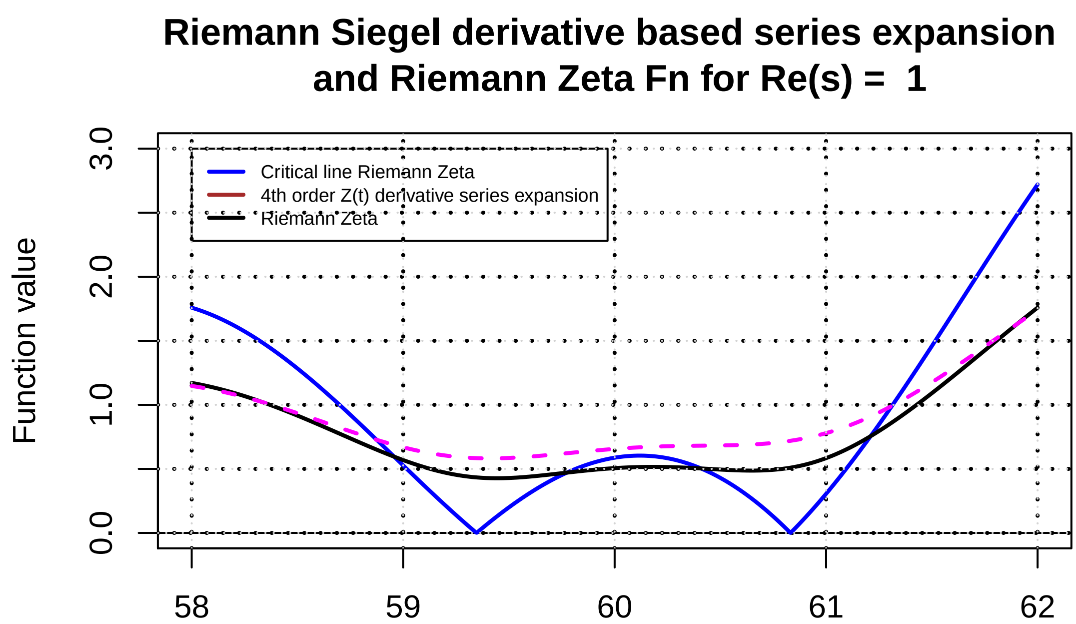

Figure 4: 4th order Riemann Siegel series expansion and Riemann Zeta function behaviour, across the critical strip

Conclusions

An useful series expansion estimator for the Riemann Zeta function has constructed from the real part of the Riemann Siegel Z function and its derivatives and indicates the validity of the Riemann Hypothesis.

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