Some high peaks of partial Euler Product of the lowest primes on the Riemann Zeta critical line, in the interval ($10^{20} < T < 10^{400}$) providing a proxy lower bound on Riemann Zeta function growth.

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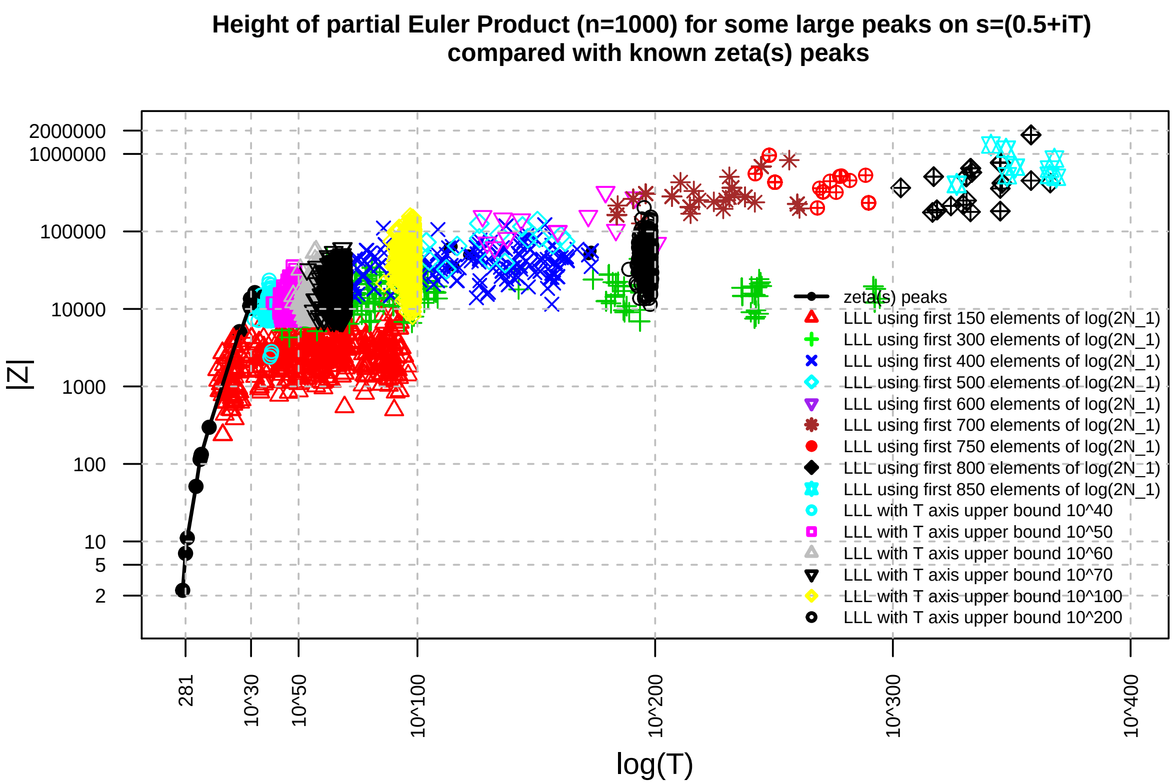
created "Sun, October 7th, 2018" last modified "Fri, October 19th, 2018"

Executive Summary

Using pari/gp software and the Lenstra-Lenstra-Lovász (LLL) basis reduction algorithm, repeated grid searches using larger and larger sets of the irrationals $log(2 * \mathbb{N}_1)/2\pi$ (150-850 elements) have produced many positions on the Riemann Zeta critical line ($10^{20} < T < 10^{400}$), that correspond to large partial Euler product peaks using only the lowest primes (n=1000). In comparison to the known behaviour for large peaks of the Riemann Zeta function ($17 < T < 10^{32}$), the partial Euler product peaks appear to be proxy lower bounds of the $|\zeta(1/2 + iT)|$ growth rate. Several partial Euler product peaks exceed heights of 1,000,000 for $T > 10^{300}$.



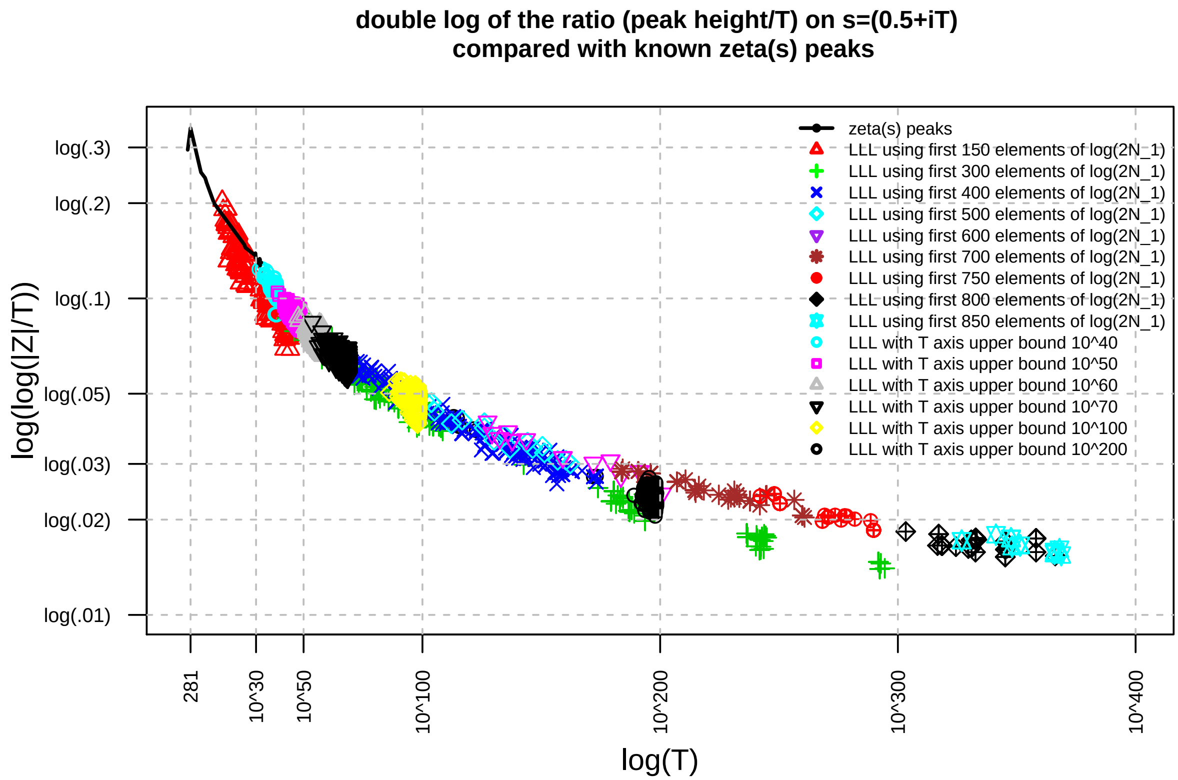


Figure 1. Height of peaks of Partial Euler Product of the lowest 1000 primes along s=(0.5+iT) compared to $|\zeta(s)|$. The positions of the peaks were calculated via two directions of grid searches using the LLL algorithm based (i) on varying the LLL algorithm upper bound for different size sets of the irrationals $log(2*\mathbb{N}_1)/2\pi$ ranging from 150-850 elements (horizontal grid) and (ii) increasing the set of irrationals used for a fixed LLL algorithm upper bound (vertical grid).

Introduction

On the critical line, the divergence of the partial Euler Product, is weak when only using the lowest primes well away from the real axis. As a result, the partial Euler Product has been actively used as a useful first order approximation to locate the largest peaks when searching for closely (and widely) spaced non-trivial zeroes which represent a interesting test for Riemann Hypothesis behaviour [1-4]. Other authors have added corrections to the partial Euler product to reduce the divergence behaviour and get better agreement with $\zeta(0.5+iT)$ values [5,6].

The original approach [1] for finding candidate $\zeta(0.5+iT)$ large peaks involved (i) using the Lenstra-Lenstra-Lovász (LLL) basis reduction algorithm [7] to identify when $log(p_j)T \approx n_j 2\pi$ where n_j are integers, for multiple primes p_j based on the properties of the partial Euler Product formula, (ii) then calculating the partial Euler product as a guide whether the $\zeta(0.5+iT)$ peak height is expected to be large and (iii) performing the $\zeta(0.5+iT)$ calculation for these large partial Euler Product peaks. More recent work [4,8,9] investigating $\zeta(0.5+iT)$ behaviour in the region $T \sim 10^{30+}$ developed the RS-PEAK algorithm to more efficiently identify the increased number of large peaks as the Riemann Zeta function (and partial Euler Product) grows in magnitude higher along the critical line.

In this paper extending the approach used in [10,11], large Euler Product peaks are presented in the wide range ($10^{20} < T < 10^{400}$). The peak positions are firstly identified via pari/gp software [12] employing the Lenstra-Lenstra-Lovász (LLL) basis reduction algorithm [7], involving repeated grid searches using larger and larger sets of the irrationals (150-850 elements) to solve the diophantine approximation $\frac{log(2*\mathbb{N}_1)T}{2\pi} \approx n_j$. Figure 1, illustrates the broad critical line dependence of the partial Euler Product $|\zeta_{EP}(0.5+iT)|$ peak height compared to known $\zeta(0.5+iT)$ peaks and the decreasing ratio $\phi = (\frac{|\zeta_{EP}(0.5+iT)|}{T})$ of peak height to position T as $T \to \infty$. Then the partial Euler product and the extended Riemann Siegel Z function forms of the partial Euler product [10] are examined for some peaks and compared on the lines $s = (1+iT)$ and $s = (0.5+iT)$.

The Riemann Zeta function and partial Euler Product

For $\Re(s) > 1$, the Euler Product of the primes absolutely converges to the Riemann Zeta function sum of the integers [13,14]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\rho=2}^{\infty} \frac{1}{(1-1/\rho^s)} \qquad \text{for } \Re(s) > 1 \tag{1}$$

The Riemann Zeta function can be defined for the whole complex plane by the integral [13,14]

$$\zeta(s) = \frac{\prod(-s)}{2\pi i} \int_{C_{\epsilon,\delta}} \frac{(-x)^s}{(e^x-1)x} dx \qquad \text{for } \mathbb{C} \tag{2}$$

where $s \,\epsilon\, \mathbb{C}$ and $C_{\epsilon,\delta}$ is the contour about the imaginary poles.

On the s=1 line, away from the real axis the Euler Product asymptotically approaches to the Riemann Zeta function value [15]. This behaviour can be seen in equation 4.11.2 of [15]

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + O(\frac{|s|}{N^{\Re(s)}}) + O(\frac{1}{N^{\Re(s)}}) \tag{3}$$

On inspecting the partial Euler Product results for complex values s, in the upper half of the critical strip, the divergence of the $\sum_{n=1}^{N} \frac{1}{n^s}$ term in the above equation and hence the partial Euler Product $\prod_{\rho=2}^{P} \frac{1}{(1-1/\rho^s)}$ in that region, exhibits a dominant $\frac{N^{1-s}}{1-s}$ oscillatory divergence behaviour near the real axis which becomes weak divergence well away from the real axis for small N or P.

As exploited by [1,4] and others, there are large peaks in the Riemann Zeta function on the critical line, co-incident with similar sized peaks in the partial Euler Product

$$\zeta_{EP}(s) = \prod_{\rho=2}^{P} \frac{1}{(1-1/\rho^s)} \qquad \text{for } P << \infty \tag{4}$$

when many $\rho^s \approx 1$ at the same value of t. This constraint is described as a diophantine approximation

$$log(p_j)T \approx n_j 2\pi \tag{5}$$

where n_j are integers, for as many primes p_j as possible.

As described in [10,11], extended Riemann Siegel function $Z_{ext}(s, hybrid)$ and $\theta_{ext}(s, hybrid)$ can be constructed and used to understand the number of primes producing constructive interference at the large peaks in addition to simply using $\zeta_{EP}(s)$,

$$\theta_{extEP}(s, hybrid) = \Im\Big(log\Big(\sqrt{\frac{\zeta_{EP}(1-s)abs(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))}{\zeta_{EP}(s)}}\Big)\Big) \tag{6}$$

$$= -\frac{1}{2}\Im\big(log(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))\big) \tag{7}$$

$$= \theta_{ext}(s) \tag{8}$$

$$Z_{extEP}(s, hybrid) = \sqrt{\zeta_{EP}(s) * \zeta_{EP}(1-s) * abs(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))} \tag{9}$$

$$= \sqrt{\zeta_{EP}(s) * \frac{\zeta_{EP}(s)}{(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))} * abs(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))} \tag{10}$$

$$= \sqrt{\frac{\zeta_{EP}(s)^2}{(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))} * abs(2^s\pi^{s-1}sin(\frac{\pi s}{2})\Gamma(1-s))} \tag{11}$$

It is noted that the branch points of $\Im(\log(Z_{extEP}(s, hybrid)))$ only correctly identify first order shifts in Gram points if the calculation inside the square root function of equation (11) is completed first.

Modifying the Diophantine approximation for identifying candidate large partial Euler Product peaks

As mentioned, the original approach [1] for finding candidate $\zeta(0.5 + iT)$ large peaks involved using the Lenstra-Lenstra-Lovász (LLL) basis reduction algorithm [7] to solve equation (5). The numerical performance of this method however peters out beyond 10-50 primes as T increases and the number of large peaks increases, so other approaches [4,8,9] have been adopted.

In looking for a novel solution, this author started examining alternate simpler Euler Products, their peak positions and seeing if the LLL algorithm numerical performance was better for such functions. In particular, two obvious functions that have similar periodicity to $\zeta_{EP}(s)$ are

$$\zeta_{EP_{int}}(s) = \prod_{i=2}^{N}\frac{1}{(1-1/i^s)} \qquad \text{for } N << \infty \tag{12}$$

with the diophantine approximation for large peaks occurring when

$$\frac{log(i)T}{2\pi} \approx n_i \tag{13}$$

where n_i are integers, for as many integers $\mathbb{N}_2$ as possible.

$$\zeta_{EP_{N_1}}(s) = \prod_{i=1}^{N}\frac{1}{(1-1/(2i)^s)} \qquad \text{for } N << \infty \tag{14}$$

$$\frac{log(2i)T}{2\pi} \approx m_i \tag{15}$$

where m_i are integers, for as many integers $\mathbb{N}_1$ as possible.

since for $s = 0.5 + iT$ & $s = 1 + iT$ the dominant terms of all three functions for high peaks are the leading values $\frac{1}{(1-1/(2)^s)}$ and $\frac{1}{(1-1/(3)^s)}$. In practice, it was easy to observe by plotting that the large peaks of $\zeta_{EP_{int}}(s)$ & $\zeta_{EP_{N_1}}(s)$ were very close to important known $\zeta_{EP}(s)$ (and $\zeta(s)$) peaks. The peak heights for (13) & (15) were found to be dominant for important known $\zeta(s)$ peaks (T=6820051, first Rosser rule violation) & (T=310678833629083965667540576593682.065, highest published peak height) and orders of magnitude greater than the $\zeta(s)$ value. This suggested that the constructive interference for $\zeta_{EP_{int}}(s)$ & $\zeta_{EP_{N_1}}(s)$ for the highest peaks was more concentrated compared to $\zeta_{EP}(s)$. Again the close agreement on peak position can be easily attributed because of importance of the $\frac{1}{(1-1/(2)^s)}$ term.

In then attempting LLL basis reduction algorithm with equations (13) & (15) using pari/gp software the numerical performance for large and larger sets of $\mathbb{N}_1$ proved fruitful as shown [10,11] and the current results, consistently producing critical line positions with large $\zeta_{EP}(s)$ peaks. This improved performance of basis reduction is probably related to the denser distribution of $\mathbb{N}_1$ compared to the set of primes $\mathbb{P}$. The LLL performance of solving for equation (15) seems to produce slightly higher $\zeta_{EP}(s)$ peaks and lower T positions compared to equation (13) but grid searches using either equations and different sized sets of $\mathbb{N}_1$ produces additional candidate peaks.

The idea for trialling diophantine equation (15) arose first because (i) the Dirichlet $\eta(s)$ function [1] is known to be convergent on the critical line, (ii) can be expressed in terms of the $\zeta(s)$ function and (iii) contains the infinite even integer summand $\Sigma\frac{1}{(2i)^2}$ which is the summation counterpart of the product equation (14). The first diophantine equation (13) was then also trialled since the $\frac{1}{(1-1/(2)^s)}$ term remained the leading term.

Behaviour of partial Euler product, on the critical line for the interval $(10^{20} < T < 10^{400})$

In Figure 1a, the partial Euler Product using the first 1000 primes $|\zeta_{EP}(0.5 + iT)|$ peak heights organised by LLL grid search $\mathbb{N}_1$ sets, are compared to known $\zeta(0.5 + iT)$ peaks.

In Figure 1b, the ratios $\phi = \left(\frac{|\zeta_{EP}(0.5+iT)|}{T}\right)$ and $\phi = \left(\frac{|\zeta(0.5+iT)|}{T}\right)$ of peak height to position T as $T \to \infty$ are presented.

It can be seen that,

(i) for a given input $\mathbb{N}_1$ set, equation (15) using the LLL algorithm produces clusters of large $|\zeta_{EP}(0.5+iT)|$ peaks of similar height, fairly independent of T. This behaviour tends to indicate the possible maximum collective intereference possible from the first 150, 300, etc primes,

(ii) the peak height increases as the diophantine approximation involves more integers and hence allows more primes in $|\zeta_{EP}(0.5 + iT)|$ to constructively interfer at higher T peak positions,

(iii) while the authors in [9] observe a decreasing peak height ratio $\phi = \left(\frac{|\zeta(0.5+iT)|}{T}\right)$ between $(10^{30} < T < 10^{40})$, Figure 1b using data from [4,16] clearly shows the maximum ϕ for $\zeta(0.5 + iT)$ occurs around T=281 the first Gram point violation and the relative peak size has been decreasing from that early point,

(iv) the peak heights for $|\zeta_{EP}(0.5 + iT)|$ beyond $T = 10^{30}$ seem to be somewhat lower than where the $\zeta(0.5 + iT)$ would extrapolate. One reason for this could be that the diophantine approximation produces peak positions with similar deviation from pure integers m_j whereas for known $\zeta(0.5 + iT)$ peaks such as T=310678833629083965667540576593682.065, there are particular low integers which do not satisfy the diophantine approximation, for example $\frac{log(53)310678833629083965667540576593682.065}{2\pi}$ is much further away from being an integer than the other integers in the interval (2,70) apart from 53. This leaves open the possibility of higher peaks occurring outside the strictness of equation (15) constraints and exploiting higher order interactions in equation (4),

(v) overall though these $|\zeta_{EP}(0.5 + iT)|$ peaks provide a consistent set of peaks to assess the likely T dependence of the growth in $|\zeta(0.5 + iT)|$. Using Figure 1b, a coarse estimate of the decrease in peak height ratio is $log(log(\phi)) \sim log(12) - log(log(T))$.

Particular partial Euler Product peaks

Large peaks on the critical line

Some particularly large partial Euler product peaks on the critical line, using the first 1000 primes and positions identified by diophantine approximation equation (15) to the nearest integer, are

T=13069247622361836217472564080986793903087332171485586091974531348227929481792606657379520062574
8544823965522816159883452631496671435928127003682825655228296328800448424463114124703538058438248
5693856319843119263004813235910441021382902564735700391541791595967759325456756061116888438106866
1886505179878367017087735189404217192841186995218146702812310521109128

T=1.3069247622361836217 E358

$|\zeta_{EP}(0.5+iT)|$=1760647.31

T=1770955249619635557629753091583681186551045291452819186153408225838800562255237424907491616622
1977145825517973962702405987909642116495127100467209314425718781195455461007487705792014552242068
9319308648064009302484615583904988999819009222632837011391471710900941461416536536243895029218914
63672677138331113520370573697674938007451282538036225

T=1.7709552496196355576 E341

$|\zeta_{EP}(0.5+iT)|$=1303696.36

T=25461213667558826585077573310184571709268718487072937682163830390280740449012248541205182985864
5512911999800061121068805831146886200950615300137489446783067281752310150354925705554531737170631
01234372766685983191415214430508795518110768167130897674497810394

T=2.5461213667558826584 E256

$|\zeta_{EP}(0.5+iT)|$=832699.18

T=53765569482805537253004333443111555328116058497868143243715677017417410542510699878014047776816
9067872903009314420373993472870095901668376504754391413546308088933984162479158665942751396109916
2586279492504427838724505855912431184749792569901931187845361948175378195487012023561124835453587
59461867283393750628433256202272428916832331

T=5.376556948280553725 E332

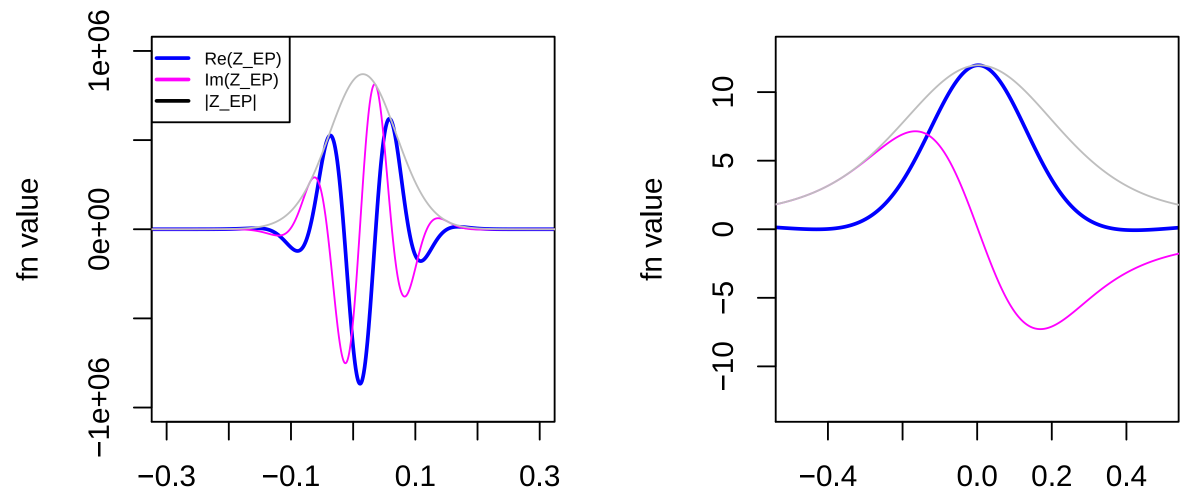
$|\zeta_{EP}(0.5+iT)|$=652677.79

Contrasting peak height, width and riemann siegel component behaviour between the s=1+iT and S=0.5+iT lines

Figures 2 & 3 below, show the behaviour of the extended Riemann Siegel Z function and the partial Euler Product based versions (using 1000 primes) on both the critical line s=0.5+iT and the line s=1+iT, for T=2.5461213667558826584 E256.

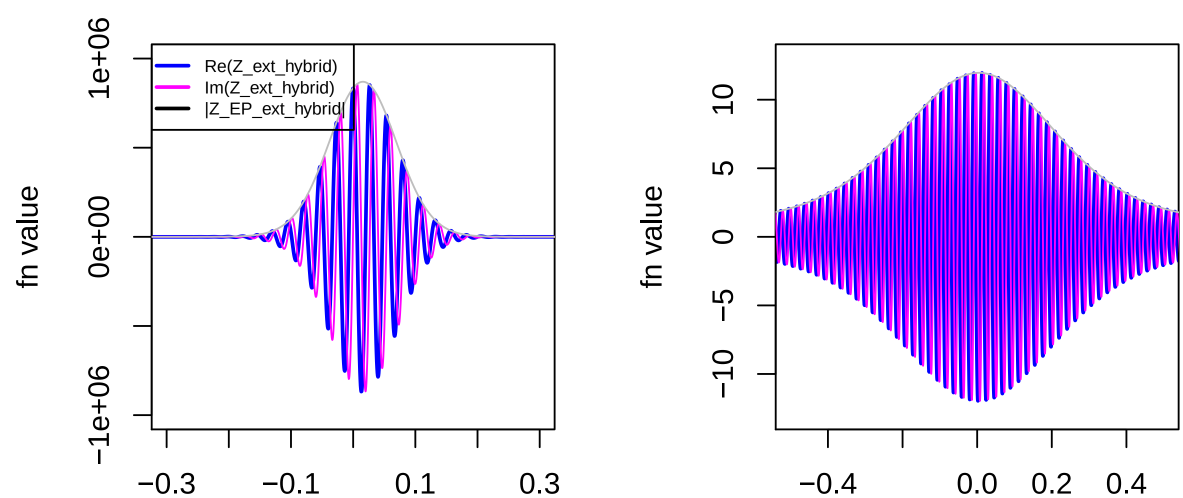
As with previous results [10,11], it can be readily observed that large peaks on the critical line also has dominant peak (but much smaller) on the s=1 line. The smaller peak on the s=1+iT line is consistent with the lower growth expected on that line. However, the peak height of $|Z_EP| \sim 12+ >\approx 4*\pi$ on s=1+iT where the partial Euler product converges, is evidence for potential $|arg(\zeta(s))| > 4$ behaviour on the critical line for T=2.5461213667558826584 E256. There is also supporting evidence for $|arg(\zeta(s))| > 4$ from the fine detail $Im(Z_{extEP}(s, hybrid))$ behaviour (not shown) using the analysis discussed in [11].

Note that the maximum peak position in the figures below is not at zero on the x-axis, indicating the integer peak position provided above should be adjusted by a small amount to detail the true peak position.



(a) imag part s=(0.5+it) line + 2.546e256 (b) imag part s=(1+it) line + 2.546e256

***Figure 2.** The behaviour of partial Euler Product estimates $Z_{EP}(s)$, using first 1000 primes, on the lines s=(0.5+iT) & s=(1+iT), around T=2.5461213667558826584 E256.*



(a) imag part s=(0.5+it) line + 2.546e256 (b) imag part s=(1+it) line + 2.546e256

***Figure 3.** The behaviour of partial Euler Product based Riemann Siegel Z function estimates $Z_{extEP}(s, hybrid)$, using first 1000 primes, on the lines s=(0.5+iT) & s=(1+iT), around T=2.5461213667558826584 E256.*

More work will be considered around whether using more primes in the partial Euler Product at these very high T values may increase the peak height.

It is remarkable that the width of these large resonance peaks is still as broad as ~0.4 at $T = 10^{300+}$ which must mean something about the influence of the $\frac{1}{(1-1/(2)^s)}$ term to the partial Euler product behaviour or perhaps arising from the nature of the interference between the Riemann Siegel components $\theta(s)$ and Z(s).

Conclusions

By using a proxy diophantine approximation equation (15), the LLL algorithm usefulness for Riemann Zeta function investigations has been extended to identify large partial Euler Product peaks at very high T around 10^{340} and beyond, where interesting tests of the Riemann Hypothesis may occur.

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