Householder Reflection

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(1) Given any vector $v \neq 0$, we express the Householder Reflection P as $P := 1 - 2|v|^{-2} v v^{+}$.
P is obviously hermitian. P is also unitary, because
$$P_i^{+} P_j = (e_i^{+} - 2|v|^{-2} v^{+} v_i)(e_j - 2|v|^{-2} v v_j) = e_i^{+} e_j - 4|v|^{-2} v_j v_i + 4|v|^{-4} v_i v_j v^{+} v = \delta_{ij}$$
It follows that P is its own inverse.

Let $x \neq 0$ be a vector. We want to construct P such that (2) $y = Px$ with (3) $y_i = 0 \; \forall \, i > 1$.

(4) via (1),(2),(3) $x_i - v_i 2|v|^{-2} v^{+} x = 0 \; \forall \, i > 1$.

(5) via (4) $v_i = \lambda x_i \; \forall \, i > 1$ with some common scalar λ.

(6) We try the simplest possible ansatz $\lambda = 1$ to find out if it yields a solution for v_1 :

(7) via (4), (5), (6) $2|v|^{-2} v^{+} x = 1$ => $2 v^{+} x = |v|^2$ => $2(x^{+} x - |x_1|^2 + v_1 x_1) = (x^{+} x - |x_1|^2 + |v_1|^2)$ =>
$|v_1|^2 - 2 x_1 v_1 - x^{+} x + |x_1|^2 = 0$

(8) This yields the solutions $v_1 = x_1 + \alpha |x|$, $\alpha \in \{1, -1\}$, which verifies our ansatz (6).

Let's try for a compact expression of y_1 :

(9) via (1), (2) $y_1 = x_1 - 2 v_1 |v|^{-2} (v^{+} x)$.

(10) via (8) $|v|^2 = \sum_{i=1}^{n} v_i^2 = v_1^2 + \sum_{i=2}^{n} x_i^2 = 2(x_1^2 + x_1 \alpha |x|)$

(11) via (8) $v^{+} x = \sum_{i=1}^{n} v_i x_i = v_k x_1 + \sum_{i=2}^{n} x_i^2 = (x_1^2 + x_1 \alpha |x|)$

(12) via (10), (11) $|v|^{-2} (v^{+} x) = 1/2$

(13) via (9), (12), (8) $y_1 = x_1 - v_1 = -\alpha |x|$

Hence, the reflection $y = Px$ has the form $(-\alpha |x|, 0, 0, \ldots.)^{+}$.

Supplemental observations:

- The magnitude of (13) could also have been predicted from the fact that transformation P is unitary (which does not change the magnitude of a vector) and that y_1 is the only nonzero element of y .

- It is easy to comprehend that the reflection can also be constructed on any sub-vector of x, e.g a partition. The transformation changes only the components on that sub vector. It is also easily verifiable that multiple reflections can be chained up across subsequent partitions of the vector. Hence, a transformation zeroing the one non-zero component of the previous transformation. This is useful for blocking multiple transformations in a cache-efficient way.

- In 2D, a Householder Reflection is related to a Givens Rotation. Both are convertible into each other via the following Reflection $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.