# Density Dependent Lotka–Volterra Models

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## 1 Problem motivation

This project will build upon two canonical models that we have discussed in class - the logistic equation for growth of a species with finite carrying capacity and the Lotka-Volterra system for predator-prey dynamics. The logistic differential equation for population models is given by,

$$\dot{N} = rN\left(1 - \frac{N}{K}\right), \tag{1}$$

where $N$ the species population, $r$ the growth rate, and $K$ the carrying capacity. Solutions to equation 1 have density dependent growth rate and will asymptotically approach the carrying capacity $K$. The Lotka-Volterra system for modeling predator prey dynamics is given by,

$$\begin{aligned} \dot{N}_1 &= \alpha N_1 - \beta N_1 N_2 \\ \dot{N}_2 &= \gamma N_1 N_2 - \delta N_2, \end{aligned} \tag{2}$$

where $N_1$ and $N_2$ are the population of predator and prey, respectively, and $\alpha$, $\beta$, $\gamma$, and $\delta$ are positive constants. We observe that for a nonexistent predator population, the Lotka-Volterra model predicts unbounded exponential growth of the prey species. For our project, we will motivate improved models of the Lotka-Volterra system which include more reasonable growth rates for the prey species. It seems intuitive that a more accurate model may be obtained by making the prey-species obey a logistic growth law in the absence of any predators. Doing so yields the following system of equations,

$$\begin{aligned} \dot{N}_1 &= \alpha N_1\left(1 - \frac{N_1}{K}\right) - \beta N_1 N_2 \\ \dot{N}_2 &= \gamma N_1 N_2 - \delta N_2 \end{aligned} \tag{3}$$

Of particular interest will be the stability of fixed points and structural stability of equation 3 and a rigorous quantitative analysis of when the system has sustained oscillations. If time permits we will include structural perturbations to equation 3 to include factors such as harvesting or more sophisticated interaction terms.

## 2 Mathematical methods

Initially, we will focus on a basic analysis of the equations including finding and analyzing the stability of equation 3. Finding the fixed points of the equation is trivial, but their stability will be dependent on 5 parameters in the equation; $\alpha$, $\beta$, $\gamma$, $\delta$, and $K$. We have observed that this may be reduced to 2 parameters by making the change of variables $x = N_1/K$, $y = \delta N_2/\beta$, and $\tau = t/\delta$ which gives,

$$\begin{aligned} \dot{x} &= \tilde{\alpha} x (1 - x) - xy \\ \dot{y} &= \tilde{\gamma} xy - y, \end{aligned} \tag{4}$$

where $\tilde{\gamma} = \gamma K \delta^{-1}$ and $\tilde{\alpha} = \alpha \delta^{-1}$. Trivially, we know that in the limit $K \to \infty$, the equation becomes Lotka-Volterra and has a center around both $x$ and $y > 0$. However, preliminary study of the system given in equation 4 shows that for $\gamma K < \delta$ or equivalently $\tilde{\gamma} < 1$, there is no fixed point with $y > 0$. We interpret this as showing that for some parameters, the predator species is driven to extinction. We are particularly interested in the transition from a model indicating extinction of the predator species to a model with the classic oscillatory behavior observed in Lotka-Volterra. To study this, we will plot the stability of fixed points of equation 4 on the $\tilde{\alpha}, \tilde{\gamma}$ plane, look for bifurcation parameters made from linear combinations of $\tilde{\alpha}$ and $\tilde{\gamma}$ and will construct bifurcation diagrams. This may include further study of bifurcations and 2 dimensional nonlinear systems of equations. We will use the mathematical analysis to discuss the real world interpretations of the model and it's parameters.

## 3 Computational methods

We will support the analysis in this project with some numerical simulation. Computational work will include creating plots of the vector field of the density dependent Lotka-Volterra system given in equation 4, numerical simulations of the dynamics, and plots of the eigenvalues of the Jacobian at any fixed points in the system as a function of $\tilde{\gamma}$ and $\tilde{\alpha}$. We do not anticipate any significant challenges in the computational portion of the project.