Homotopy Type Theory

Homotopy n-types

7.1 Definition of n-types

D 7.1.1. Define the predicate $\mathsf{is}\text{-}n\text{-}\mathsf{type} : \mathcal{U} \to \mathcal{U}$ for $n \geq -2$ by recursion as follows:

$$\mathsf{is}\text{-}n\text{-}\mathsf{type}(X) :\equiv \begin{cases} \mathsf{isContr}(X) & \text{if } n = -2, \\ \prod_{(x,y:X)} \mathsf{is}\text{-}n'\text{-}\mathsf{type}(x =_X y) & \text{if } n = n' + 1. \end{cases}$$

We say that X is an **n-type**, or sometimes that it is *n-truncated*, if $\mathsf{is}\text{-}n\text{-}\mathsf{type}(X)$ is inhabited.

E 7.1.2. We saw that X is a (-1)-type if and only if it is a mere proposition. Therefore, X is a 0-type if and only if it is a set.

T 7.1.3. *Let $p : X \to Y$ be a retraction and suppose that X is an n-type, for any $n \geq -2$. Then Y is also an n-type.*

C 7.1.4. *If $X \simeq Y$ and X is an n-type, then so is Y.*

T 7.1.5. *If $f : X \to Y$ is an embedding and Y is an n-type for some $n \geq -1$, then so is X.*

T 7.1.6. *The hierarchy of n-types is cumulative in the following sense: given a number $n \geq -2$, if X is an n-type, then it is also an $(n+1)$-type.*

7.2 Preservation under constructors

T 7.2.1. *Let $n \geq -2$, and let $A : \mathcal{U}$ and $B : A \to \mathcal{U}$. If A is an n-type and for all $a : A$, $B(a)$ is an n-type, then so is $\sum_{(x:A)} B(x)$.*

T 7.2.2. *Let $n \geq -2$, and let $A : \mathcal{U}$ and $B : A \to \mathcal{U}$. If for all $a : A$, $B(a)$ is an n-type, then so is $\prod_{(x:A)} B(x)$.*

T 7.2.3. *For any $n \geq -2$ and any type X, the type $\mathsf{is}\text{-}n\text{-}\mathsf{type}(X)$ is a mere proposition.*

T 7.2.4. *For any $n \geq -2$, the type $n\text{-}\mathsf{Type}$ is an $(n+1)$-type.*

7.3 Uniqueness of identity proofs and Hedberg's theorem

T 7.3.1. *A type X is a set if and only if it satisfies **Axiom K**: for all $x : X$ and $p : (x =_A x)$ we have $p = \mathsf{refl}_x$.*

T 7.3.2. *Suppose R is a reflexive mere relation on a type X implying identity. Then X is a set, and $R(x,y)$ is equivalent to $x =_X y$ for all $x, y : X$.*

C 7.3.3. *If a type X has the property that $\neg\neg(x = y) \to (x = y)$ for any $x, y : X$, then X is a set.*

L 7.3.4. *For any type A we have $(A + \neg A) \to (\neg\neg A \to A)$.*

T 7.3.5 (Hedberg). *If X has decidable equality, then X is a set.*

T 7.3.6. *The type $\mathbb{N}$ of natural numbers has decidable equality, and hence is a set.*

T 7.3.7. *For any $n \geq -1$, a type X is an $(n+1)$-type if and only if for all $x : X$, the type $\Omega(X, x)$ is an n-type.*

L 7.3.8. *Given $n \geq -1$ and $X : \mathcal{U}$. If, given any inhabitant of X it follows that X is an n-type, then X is an n-type.*

T 7.3.9. *For every $n \geq -1$, a type A is an n-type if and only if $\Omega^{n+1}(A, a)$ is contractible for all $a : A$.*

7.4 Truncations

L 7.4.1. *$\|A\|_n$ is an n-type.*

T 7.4.2. *For any type family $P : \|A\|_n \to \mathcal{U}$ such that each $P(x)$ is an n-type, and any function $g : \prod_{(a:A)} P(|a|_n)$, there exists a section $f : \prod_{(x:\|A\|_n)} P(x)$ such that $f(|a|_n) :\equiv g(a)$ for all $a : A$.*

L 7.4.3 (Universal property of truncations). *Let $n \geq -2$, $A : \mathcal{U}$ and $B : n\text{-}\mathsf{Type}$. The following map is an equivalence:*

$$\begin{cases} (\|A\|_n \to B) & \longrightarrow & (A \to B) \\ g & \longmapsto & g \circ |-|_n \end{cases}$$

L 7.4.4. *Given $f, g : A \to B$ and a homotopy $h : f \sim g$, there is an induced homotopy $\|h\|_n : \|f\|_n \sim \|g\|_n$ such that the composite is equal to $\mathsf{ap}_{|-|_n}(h(a))$.*

C 7.4.5. *A type A is an n-type if and only if $|-|_n : A \to \|A\|_n$ is an equivalence.*

T 7.4.6. *For any types A and B, the induced map $\|A \times B\|_n \to \|A\|_n \times \|B\|_n$ is an equivalence.*

T 7.4.7. *Let $P : A \to \mathcal{U}$ be a family of types. Then there is an equivalence*

$$\Big\| \sum_{x:A} \|P(x)\|_n \Big\|_n \simeq \Big\| \sum_{x:A} P(x) \Big\|_n.$$

C 7.4.8. *If A is an n-type and $P : A \to \mathcal{U}$ is any type family, then*

$$\sum_{a:A} \|P(a)\|_n \simeq \Big\| \sum_{a:A} P(a) \Big\|_n$$

T 7.4.9. *For any A and $x, y : A$ and $n \geq -2$, the map (7) is an equivalence; thus we have*

$$\|x =_A y\|_n \simeq \left(|x|_{n+1} =_{\|A\|_{n+1}} |y|_{n+1} \right).$$

C 7.4.10. *Let $n \geq -2$ and (A, a) be a pointed type. Then*

$$\|\Omega(A,a)\|_n = \Omega(\|(A,a)\|_{n+1})$$

C 7.4.11. *Let $n \geq -2$ and $k \geq 0$ and (A, a) a pointed type. Then*

$$\|\Omega^k(A,a)\|_n = \Omega^k(\|(A,a)\|_{n+k}).$$

L 7.4.12. *Let $k, n \geq -2$ with $k \leq n$ and $A : \mathcal{U}$. Then $\|\|A\|_n\|_k = \|A\|_k$.*

7.5 Colimits of n-types

D 7.5.1. A **span** is a 5-tuple $\mathscr{D} = (A, B, C, f, g)$ with $f : C \to A$ and $g : C \to B$.

D 7.5.2. Given a span $\mathscr{D} = (A, B, C, f, g)$ and a type D, a **cocone under $\mathscr{D}$ with base** D is a triple (i, j, h) with $i : A \to D$, $j : B \to D$ and $h : \prod_{(c:C)} i(f(c)) = j(g(c))$: We denote by $\mathsf{cocone}_{\mathscr{D}}(D)$ the type of all such cocones.

D 7.5.3. Given a span $\mathscr{D}$ of n-types, an n-type D, and a cocone $c : \mathsf{cocone}_{\mathscr{D}}(D)$, the pair (D, c) is said to be a **pushout of $\mathscr{D}$ in n-types** if for every n-type E, the map

$$\begin{cases} (D \to E) & \longrightarrow & \mathsf{cocone}_{\mathscr{D}}(E) \\ t & \longmapsto & t \circ c \end{cases}$$

is an equivalence.

L 7.5.4. *If (D, c) and (D', c') are two pushouts of $\mathscr{D}$ in $\mathcal{U}_P$, then $(D, c) = (D', c')$.*

C 7.5.5. *The type of pushouts of $\mathscr{D}$ in $\mathcal{U}_P$ is a mere proposition. In particular if pushouts merely exist then they actually exist.*

D 7.5.6. Let be a span. We denote by $\bigcirc(\mathscr{D})$ the following span of n-types:

D 7.5.7. Let $D : \mathcal{U}$ and $c = (i, j, h) : \mathsf{cocone}_{\mathscr{D}}(D)$. We define

$$\bigcirc(c) = (\bigcirc(i), \bigcirc(j), k) : \mathsf{cocone}_{\bigcirc(\mathscr{D})}(\bigcirc(D))$$

where k is the composite homotopy

$$\bigcirc(i) \circ \bigcirc(f) \sim \bigcirc(i \circ f) \sim \bigcirc(j \circ g) \sim \bigcirc(j) \circ \bigcirc(g)$$

using L 7.4.4 and the functoriality of $\bigcirc(-)$.

T 7.5.8. *Let $\mathscr{D}$ be a span and (D, c) its pushout. Then $(\|D\|_n, \|c\|_n)$ is a pushout of $\|\mathscr{D}\|_n$ in n-types.*

7.6 Connectedness

D 7.6.1. A function $f : A \to B$ is said to be n**-connected** if for all $b : B$, the type $\left\|\mathsf{fib}_f(b)\right\|_n$ is contractible:

$$\mathsf{conn}_n(f) :\equiv \prod_{b:B} \mathsf{isContr}(\left\|\mathsf{fib}_f(b)\right\|_n).$$

A type A is said to be n**-connected** if the unique function $A \to \mathbf{1}$ is n-connected, i.e. if $\|A\|_n$ is contractible.

L 7.6.2. *A function f is (-1)-connected if and only if it is surjective in the sense of mono-surj.*

NB 7.6.3. While our notion of n-connectedness for types agrees with the standard notion in homotopy theory, our notion of n-connectedness for *functions* is off by one from a common indexing in classical homotopy theory. Whereas we say a function f is n-connected if all its fibers are n-connected, some classical homotopy theorists would call such a function $(n+1)$-connected. (This is due to a historical focus on *cofibers* rather than fibers.)

L 7.6.4. *Suppose that g is a retract of a n-connected function f. Then g is n-connected.*

C 7.6.5. *If g is homotopic to a n-connected function f, then g is n-connected.*

C 7.6.6. *For any A, the canonical function $|-|_n : A \to \|A\|_n$ is n-connected.*