Homotopy Type Theory

Set theory

The category of sets

Limits and colimits

Images

L 10.0.1. *For a morphism $f : \hom_A(a,b)$ in a category A, the following are equivalent.*

(i). *f is a **monomorphism**: for all $x : A$ and $g, h : \hom_A(x,a)$, if $f \circ g = f \circ h$ then $g = h$.*
(ii). *(If A has pullbacks) the diagonal map $a \to a \times_b a$ is an isomorphism.*
(iii). *For all $x : A$ and $k : \hom_A(x,b)$, the type $\sum_{(h:\hom_A(x,a))}(k = f \circ h)$ is a mere proposition.*
(iv). *For all $x : A$ and $g : \hom_A(x,a)$, the type $\sum_{(h:\hom_A(x,a))}(f \circ g = f \circ h)$ is contractible.*

L 10.0.2. *A function $f : A \to B$ between sets is injective if and only if it is a monomorphism in $\mathcal{S}et$.*

L 10.0.3. *Let $f, g : A \to B$ be functions between sets A and B. The set-coequalizer $c_{f,g} : B \to Q$ has the property that, for any set C and any $h : B \to C$ with $h \circ f = h \circ g$, the type*

$$\sum_{k:Q\to C}(k \circ c_{f,g} = h)$$

is contractible.

L 10.0.4. *For any function $f : A \to B$ between sets, the following are equivalent:*

(i). *f is an epimorphism.*
(ii). *Consider the pushout diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "\iota"] \\ \mathbf{1} \arrow[r, "t"'] & C_f \end{tikzcd}$$

in $\mathcal{S}et$ defining the mapping cone. Then the type C_f is contractible.
(iii). *f is surjective.*

T 10.0.5. *The category $\mathcal{S}et$ is regular. Moreover, surjective functions between sets are regular epimorphisms.*

L 10.0.6. *Pullbacks of regular epis in $\mathcal{S}et$ are regular epis.*

Quotients

D 10.0.7. A relation $R : A \to A \to \mathsf{Prop}$ is said to be **effective** if the square

$$\begin{tikzcd} \sum_{(x,y:A)} R(x,y) \arrow[r, "\mathsf{pr}_1"] \arrow[d, "\mathsf{pr}_2"'] & A \arrow[d, "c_R"] \\ A \arrow[r, "c_R"'] & A/R \end{tikzcd}$$

is a pullback.

L 10.0.8. *Suppose (A,R) is an equivalence relation. Then there is an equivalence*

$$(c_R(x) = c_R(y)) \simeq R(x,y)$$

for any $x, y : A$. In other words, equivalence relations are effective.

T 10.0.9. *For any function $f : A \to B$ between any two sets, the relation $\ker(f) : A \to A \to \mathsf{Prop}$ given by $\ker(f,x,y) :\equiv (f(x) = f(y))$ is effective.*

T 10.0.10. *Equivalence relations are effective and there is an equivalence $A/R \simeq A /\!\!/ R$.*

$\mathcal{S}et$ is a $\Pi\mathsf{W}$-pretopos

T 10.0.11. *The category $\mathcal{S}et$ is a $\Pi\mathsf{W}$-pretopos.*

T 10.0.12. *If there is a type $\Omega : \mathcal{U}$ of all mere propositions, then the category $\mathcal{S}et_{\mathcal{U}}$ is an elementary topos.*

The axiom of choice implies excluded middle

L 10.0.13. *If A is a mere proposition then its suspension $\Sigma(A)$ is a set, and A is equivalent to $\mathsf{N} =_{\Sigma(A)} \mathsf{S}$.*

T 10.0.14 (Diaconescu). *The axiom of choice implies the law of excluded middle.*

T 10.0.15. *If the axiom of choice holds then the category $\mathcal{S}et$ is a well-pointed boolean elementary topos with choice.*

NB 10.0.16. The conditions on a category mentioned in the theorem are known as Lawvere's axioms for the *Elementary Theory of the Category of Sets* [?].

Cardinal numbers

D 10.0.17. The **type of cardinal numbers** is the 0-truncation of the type Set of sets:

$$\mathsf{Card} :\equiv \|\mathsf{Set}\|_0$$

Thus, a **cardinal number**, or **cardinal**, is an inhabitant of $\mathsf{Card} \equiv \|\mathsf{Set}\|_0$. Technically, of course, there is a separate type $\mathsf{Card}_{\mathcal{U}}$ associated to each universe $\mathcal{U}$.

D 10.0.18. The operation of **cardinal addition**

$$(- + -) : \mathsf{Card} \to \mathsf{Card} \to \mathsf{Card}$$

is defined by induction on truncation:

$$|A|_0 + |B|_0 :\equiv |A + B|_0.$$

D 10.0.19. Similarly, the operation of **cardinal multiplication**

$$(- \cdot -) : \mathsf{Card} \to \mathsf{Card} \to \mathsf{Card}$$

is defined by induction on truncation:

$$|A|_0 \cdot |B|_0 :\equiv |A \times B|_0$$

L 10.0.20. *Card is a commutative semiring, i.e. for $\alpha, \beta, \gamma : \mathsf{Card}$ we have the following.*

$$\begin{aligned} (\alpha + \beta) + \gamma &= \alpha + (\beta + \gamma) \\ \alpha + 0 &= \alpha \\ \alpha + \beta &= \beta + \alpha \\ (\alpha \cdot \beta) \cdot \gamma &= \alpha \cdot (\beta \cdot \gamma) \\ \alpha \cdot 1 &= \alpha \\ \alpha \cdot \beta &= \beta \cdot \alpha \\ \alpha \cdot (\beta + \gamma) &= \alpha \cdot \beta + \alpha \cdot \gamma \end{aligned}$$

where $0 :\equiv |\mathbf{0}|_0$ and $1 :\equiv |\mathbf{1}|_0$.

D 10.0.21. The operation of **cardinal exponentiation** is also defined by induction on truncation:

$$|A|_0^{|B|_0} :\equiv |B \to A|_0.$$

L 10.0.22. *For $\alpha, \beta, \gamma : \mathsf{Card}$ we have*

$$\begin{aligned} \alpha^0 &= 1 \\ 1^\alpha &= 1 \\ \alpha^1 &= \alpha \\ \alpha^{\beta+\gamma} &= \alpha^\beta \cdot \alpha^\gamma \\ \alpha^{\beta\cdot\gamma} &= (\alpha^\beta)^\gamma \\ (\alpha \cdot \beta)^\gamma &= \alpha^\gamma \cdot \beta^\gamma \end{aligned}$$

D 10.0.23. The relation of **cardinal inequality**

$$(- \leq -) : \mathsf{Card} \to \mathsf{Card} \to \mathsf{Prop}$$

is defined by induction on truncation:

$$|A|_0 \leq |B|_0 :\equiv \|\mathsf{inj}(A,B)\|$$

where $\mathsf{inj}(A,B)$ is the type of injections from A to B. In other words, $|A|_0 \leq |B|_0$ means that there merely exists an injection from A to B.

L 10.0.24. *Cardinal inequality is a preorder, i.e. for $\alpha, \beta : \mathsf{Card}$ we have*

$$\alpha \leq \alpha$$
$$(\alpha \leq \beta) \to (\beta \leq \gamma) \to (\alpha \leq \gamma)$$

L 10.0.25. *Consider the following statements:*

(i). *There is an injection $A \to B$.*
(ii). *There is a surjection $B \to A$.*

Then, assuming excluded middle:

- *Given $a_0 : A$, we have* ($\to$(.
- *Therefore, if A is merely inhabited, we have* ($\to$ *merely* (.
- *Assuming the axiom of choice, we have* ($\to$ *merely* (.

T 10.0.26 (Schroeder–Bernstein). *Assuming excluded middle, for sets A and B we have*

$$\mathsf{inj}(A, B) \to \mathsf{inj}(B, A) \to (A \cong B)$$

C 10.0.27. *Assuming excluded middle, cardinal inequality is a partial order, i.e. for $\alpha, \beta : \mathsf{Card}$ we have*

$$(\alpha \leq \beta) \to (\beta \leq \alpha) \to (\alpha = \beta).$$

T 10.0.28 (Cantor). *For $A : \mathsf{Set}$, there is no surjection $A \to (A \to \mathbf{2})$.*

C 10.0.29. *Assuming excluded middle, for any $\alpha : \mathsf{Card}$, there is a cardinal β such that $\alpha \leq \beta$ and $\alpha \neq \beta$.*

Ordinal numbers

D 10.0.30. Let A be a set and

$$(- < -) : A \to A \to \mathsf{Prop}$$

a binary relation on A. We define by induction what it means for an element $a : A$ to be **accessible** by $<$:

- If b is accessible for every $b < a$, then a is accessible.

We write $\mathsf{acc}(a)$ to mean that a is accessible.

L 10.0.31. *Accessibility is a mere property.*

D 10.0.32. A binary relation $<$ on a set A is **well-founded** if every element of A is accessible.

L 10.0.33. *Well-foundedness is a mere property.*

E 10.0.34. Perhaps the most familiar well-founded relation is the usual strict ordering on $\mathbb{N}$. To show that this is well-founded, we must show that n is accessible for each $n : \mathbb{N}$. This is just the usual proof of "strong induction" from ordinary induction on $\mathbb{N}$.
Specifically, we prove by induction on $n : \mathbb{N}$ that k is accessible for all $k \leq n$. The base case is just that 0 is accessible, which is vacuously true since nothing is strictly less than 0. For the inductive step, we assume that k is accessible for all $k \leq n$, which is to say for all $k < n + 1$; hence by definition $n + 1$ is also accessible.
A different relation on $\mathbb{N}$ which is also well-founded is obtained by setting only $n < \mathsf{succ}(n)$ for all $n : \mathbb{N}$. Well-foundedness of this relation is almost exactly the ordinary induction principle of $\mathbb{N}$.

E 10.0.35. Let $A : \mathsf{Set}$ and $B : A \to \mathsf{Set}$ be a family of sets. Recall from **??** that the W-type $\mathsf{W}_{(a:A)} B(a)$ is inductively generated by the single constructor

- $\mathsf{sup} : \prod_{(a:A)} (B(a) \to \mathsf{W}_{(x:A)} B(x)) \to \mathsf{W}_{(x:A)} B(x)$

We define the relation $<$ on $\mathsf{W}_{(x:A)} B(x)$ by recursion on its second argument:

- For any $a : A$ and $f : B(a) \to \mathsf{W}_{(x:A)} B(x)$, we define $w < \mathsf{sup}(a, f)$ to mean that there merely exists a $b : B(a)$ such that $w = f(b)$.

Now we prove that every $w : \mathsf{W}_{(x:A)} B(x)$ is accessible for this relation, using the usual induction principle for $\mathsf{W}_{(x:A)} B(x)$. This means we assume given $a : A$ and $f : B(a) \to \mathsf{W}_{(x:A)} B(x)$, and also a lifting $f' : \prod_{(b:B(a))} \mathsf{acc}(f(b))$. But then by definition of $<$, we have $\mathsf{acc}(w)$ for all $w < \mathsf{sup}(a, f)$; hence $\mathsf{sup}(a, f)$ is accessible.

L 10.0.36. *Suppose B is a set and we have a function*

$$g : \mathcal{P}(B) \to B$$

Then if $<$ is a well-founded relation on A, there is a function $f : A \to B$ such that for all $a : A$ we have

$$f(a) = g\Big(\{ f(a') \mid a' < a \} \Big).$$

L 10.0.37. *Assuming excluded middle, $<$ is well-founded if and only if every nonempty subset $B : \mathcal{P}(A)$ merely has a minimal element.*

D 10.0.38. A well-founded relation $<$ on a set A is **extensional** if for any $a, b : A$, we have

$$\Big(\forall (c : A). (c < a) \Leftrightarrow (c < b)\Big) \to (a = b).$$

T 10.0.39. *The type of extensional well-founded relations is a set.*

D 10.0.40. If $(A, <)$ and $(B, <)$ are extensional and well-founded, a **simulation** is a function $f : A \to B$ such that

(i). if $a < a'$, then $f(a) < f(a')$, and
(ii). for all $a : A$ and $b : B$, if $b < f(a)$, then there merely exists an $a' < a$ with $f(a') = b$.

L 10.0.41. *Any simulation is injective.*

C 10.0.42. *If $f : A \to B$ is a simulation, then for all $a : A$ and $b : B$, if $b < f(a)$, there* purely *exists an $a' < a$ with $f(a') = b$.*

T 10.0.43. *For a set A, let $P(A)$ be the type of extensional well-founded relations on A. If $<_A : P(A)$ and $<_B : P(B)$ and $f : A \to B$, let $H_{<_A <_B}(f)$ be the mere proposition that f is a simulation. Then (P, H) is a standard notion of structure over $\mathcal{S}et$ in the sense of* **??**.

C 10.0.44. *There is a category whose objects are sets equipped with extensional well-founded relations, and whose morphisms are simulations.*

L 10.0.45. *For extensional and well-founded $(A, <)$ and $(B, <)$, there is at most one simulation $f : A \to B$.*

D 10.0.46. An **ordinal** is a set A with an extensional well-founded relation which is *transitive*, i.e. satisfies $\forall (a, b, c : A). (a < b) \to (b < c) \to (a < c)$.

E 10.0.47. Of course, the usual strict order on $\mathbb{N}$ is transitive. It is easily seen to be extensional as well; thus it is an ordinal. As usual, we denote this ordinal by ω.

D 10.0.48. For ordinals A and B, a simulation $f : A \to B$ is said to be **bounded** if there exists $b : B$ such that $A = B_{/b}$.

T 10.0.49. *$(\mathsf{Ord}, <)$ is an ordinal.*

L 10.0.50. *Let $\mathcal{U}$ be a universe. For any $A : \mathsf{Ord}_{\mathcal{U}}$, there is a $B : \mathsf{Ord}_{\mathcal{U}}$ such that $A < B$.*

L 10.0.51. *Let $\mathcal{U}$ be a universe. For any $X : \mathcal{U}$ and $F : X \to \mathsf{Ord}_{\mathcal{U}}$, there exists $B : \mathsf{Ord}_{\mathcal{U}}$ such that $Fx \leq B$ for all $x : X$.*

Classical well-orderings

L 10.0.52. *Assuming excluded middle, every ordinal is trichotomous:*

$$\forall (a, b : A). (a < b) \vee (a = b) \vee (b < a).$$

L 10.0.53. *A well-founded relation contains no cycles, i.e.*

$$\forall (n : \mathbb{N}). \forall (a : \mathbb{N}_n \to A). \neg \Big((a_0 < a_1) \wedge \cdots \wedge (a_{n-1} < a_n) \wedge (a_n < a_0) \Big).$$

T 10.0.54. *Assuming excluded middle, $(A, <)$ is an ordinal if and only if every nonempty subset $B \subseteq A$ has a least element.*

T 10.0.55. *Assuming excluded middle, the following are equivalent.*

(i). For every set X, there merely exists a function $f : \mathcal{P}_+(X) \to X$ such that $f(Y) \in Y$ for all $Y : \mathcal{P}_+(X)$.
(ii). Every set merely admits the structure of an ordinal.

NB 10.0.56. If we had given the wrong proof of **??** or **??**, then the resulting proof of **??** would be invalid: there would be no way to consistently assign universe levels. As it is, we require propositional resizing (which follows from LEM) to ensure that X' lives in the same universe as X (up to equivalence).

C 10.0.57. *Assuming the axiom of choice, the function $\mathsf{Ord} \to \mathsf{Set}$ (which forgets the order structure) is a surjection.*

C 10.0.58. *Assuming AC, $\mathcal{S}et$ admits a weak equivalence functor from a strict category.*

T 10.0.59. *Assuming AC, the surjection $\mathsf{Ord} \to \mathsf{Card}$ has a section.*

The cumulative hierarchy

D 10.0.60. The **cumulative hierarchy** V relative to a type universe $\mathcal{U}$ is the higher inductive type generated by the following constructors.

(i). For every $A : \mathcal{U}$ and $f : A \to V$, there is an element $\mathsf{set}(A, f) : V$.
(ii). For all $A, B : \mathcal{U}$, $f : A \to V$ and $g : B \to V$ such that

$$(\forall (a : A).\, \exists (b : B).\, f(a) =_V g(b)) \wedge \\ (\forall (b : B).\, \exists (a : A).\, f(a) =_V g(b)) \qquad (10.0.61)$$

there is a path $\mathsf{set}(A, f) =_V \mathsf{set}(B, g)$.
(iii). The 0-truncation constructor: for all $x, y : V$ and $p, q : x = y$, we have $p = q$.

D 10.0.62. Define the **bisimulation** relation

$$\sim\, : V \times V \longrightarrow \mathsf{Prop}_{\mathcal{U}}$$

by double induction over V, where for $\mathsf{set}(A, f)$ and $\mathsf{set}(B, g)$ we let:

$$\mathsf{set}(A, f) \sim \mathsf{set}(B, g) :\equiv$$
$$(\forall (a : A).\, \exists (b : B).\, f(a) \sim g(b)) \wedge (\forall (b : B).\, \exists (a : A).\, f(a) \sim g(b)).$$

L 10.0.63. *For any $u, v : V$ we have $(u =_V v) = (u \sim v)$.*

L 10.0.64. *For every $u : V$ there is a given $A_u : \mathcal{U}$ and monic $m_u : A_u \rightarrowtail V$ such that $u = \mathsf{set}(A_u, m_u)$.*

D 10.0.65. For $u : V$, the just constructed monic presentation $m_u : A_u \rightarrowtail V$ such that $u = \mathsf{set}(A_u, m_u)$ may be called the **type of members** of u and denoted $m_u : [u] \rightarrowtail V$, or even $[u] \rightarrowtail V$. We can think of $[u]$ as the "subclass of V consisting of members of u".

T 10.0.66. *The following hold for $(V, \in)$:*

(i). extensionality:

$$\forall (x, y : V).\, x \subseteq y \wedge y \subseteq x \Leftrightarrow x = y.$$

(ii). empty set: *for all $x : V$, we have $\neg(x \in \emptyset)$.*
(iii). pairing: *for all $u, v : V$, the class $\{u, v\} :\equiv \{\, x \mid x = u \vee x = v \,\}$ is a V-set.*
(iv). infinity: *there is a $v : V$ with $\emptyset \in v$ and $x \in v$ implies $x \cup \{x\} \in v$.*
(v). union: *for all $v : V$, the class $\cup v :\equiv \{\, x \mid \exists (u : V).\, x \in u \in v \,\}$ is a V-set.*
(vi). function set: *for all $u, v : V$, the class $v^u :\equiv \{\, x \mid x : u \to v \,\}$ is a V-set.*
(vii). $\in$-induction: *if $\mathsf{C} : V \to \mathsf{Prop}$ is a class such that $\mathsf{C}(a)$ holds whenever*