Homotopy Type Theory

Homotopy theory

D 8.0.1 (Homotopy Groups). Given $n \geq 1$ and (A, a) a pointed type, we define the **homotopy groups** of A at a by

$$\pi_n(A, a) :\equiv \left\| \Omega^n(A, a) \right\|_0$$

$\pi_1(S^1)$

Getting started

The classical proof

The universal cover in type theory

D 8.0.2 (Universal Cover of S^1). Define $\mathsf{code} : \mathsf{S}^1 \to \mathcal{U}$ by circle-recursion, with

$$\mathsf{code}(\mathsf{base}) :\equiv \mathbb{Z}$$
$$\mathsf{ap}_{\mathsf{code}}(\mathsf{loop}) := \mathsf{ua}(\mathsf{succ}).$$

L 8.0.3. *$\mathsf{transport}^{\mathsf{code}}(\mathsf{loop}, x) = x + 1$ and $\mathsf{transport}^{\mathsf{code}}(\mathsf{loop}^{-1}, x) = x - 1$.*

The encode-decode proof

D 8.0.4. Define $\mathsf{encode} : \prod_{(x:\mathsf{S}^1)} (\mathsf{base} = x) \to \mathsf{code}(x)$ by

$$\mathsf{encode}\ p :\equiv \mathsf{transport}^{\mathsf{code}}(p, 0)$$

(we leave the argument x implicit).

D 8.0.5. Define $\mathsf{decode} : \prod_{(x:\mathsf{S}^1)} \mathsf{code}(x) \to (\mathsf{base} = x)$ by circle induction on x. It suffices to give a function $\mathsf{code}(\mathsf{base}) \to (\mathsf{base} = \mathsf{base})$, for which we use loop^{-}, and to show that loop^{-} respects the loop.

L 8.0.6. *For all $x : \mathsf{S}^1$ and $p : \mathsf{base} = x$, $\mathsf{decode}_x(\mathsf{encode}_x(p)) = p$.*

L 8.0.7. *For all $x : \mathsf{S}^1$ and $c : \mathsf{code}(x)$, we have $\mathsf{encode}_x(\mathsf{decode}_x(c)) = c$.*

T 8.0.8. *There is a family of equivalences $\prod_{(x:\mathsf{S}^1)}((\mathsf{base} = x) \simeq \mathsf{code}(x))$.*

C 8.0.9. $\Omega(\mathsf{S}^1, \mathsf{base}) \simeq \mathbb{Z}$.

C 8.0.10. *$\pi_1(\mathsf{S}^1) = \mathbb{Z}$, while $\pi_n(\mathsf{S}^1) = 0$ for $n > 1$.*

The homotopy-theoretic proof

L 8.0.11. *The type $\sum_{(x:\mathsf{S}^1)} \mathsf{code}(x)$ is contractible.*

C 8.0.12. *The map induced by encode:*

$$\textstyle\sum_{(x:\mathsf{S}^1)} (\mathsf{base} = x) \to \sum_{(x:\mathsf{S}^1)} \mathsf{code}(x)$$

is an equivalence.

T 8.0.13. $\Omega(\mathsf{S}^1, \mathsf{base}) \simeq \mathbb{Z}$.

The universal cover as an identity system

T 8.0.14. *The pair $(\mathsf{code}, 0)$ is an identity system at $\mathsf{base} : \mathsf{S}^1$ in the sense of* **??**.

C 8.0.15. *For any $x : \mathsf{S}^1$, we have $(\mathsf{base} = x) \simeq \mathsf{code}(x)$.*

NB 8.0.16. Note that all of the above proofs that $\pi_1(\mathsf{S}^1) \simeq \mathbb{Z}$ use the univalence axiom in an essential way. This is unavoidable: univalence or something like it is *necessary* in order to prove $\pi_1(\mathsf{S}^1) \simeq \mathbb{Z}$. In the absence of univalence, it is consistent to assume the statement “all types are sets” (a.k.a. “uniqueness of identity proofs” or “Axiom K”, as discussed in **??**), and this statement implies instead that $\pi_1(\mathsf{S}^1) \simeq \mathbf{1}$. In fact, the (non)triviality of $\pi_1(\mathsf{S}^1)$ detects exactly whether all types are sets: the proof of **??** showed conversely that if $\mathsf{loop} = \mathsf{refl}_{\mathsf{base}}$ then all types are sets.

Connectedness of suspensions

T 8.0.17. *If A is n-connected then the suspension of A is $(n+1)$-connected.*

C 8.0.18. *For all $n : \mathbb{N}$, the sphere S^n is $(n-1)$-connected.*

$\pi_{k \leq n}$ of an n-connected space and $\pi_{k<n}(\mathsf{S}^n)$

L 8.0.19. *If A is n-truncated and $a : A$, then $\pi_k(A, a) = \mathbf{1}$ for all $k > n$.*

L 8.0.20. *If A is n-connected and $a : A$, then $\pi_k(A, a) = \mathbf{1}$ for all $k \leq n$.*

C 8.0.21. *$\pi_k(\mathsf{S}^n) = \mathbf{1}$ for $k < n$.*

Fiber sequences and the long exact sequence

D 8.0.22. A **pointed map** between pointed types (X, x_0) and (Y, y_0) is a map $f : X \to Y$ together with a path $f_0 : f(x_0) = y_0$.

D 8.0.23. Given a pointed map between pointed types $f : X \to Y$, we define a pointed map $\Omega f : \Omega X \to \Omega Y$ by

$$(\Omega f)(p) :\equiv {f_0}^{-1} \cdot f(p) \cdot f_0.$$

The path $(\Omega f)_0 : (\Omega f)(\mathsf{refl}_{x_0}) = \mathsf{refl}_{y_0}$, which exhibits Ωf as a pointed map, is the obvious path of type

$${f_0}^{-1} \cdot f(\mathsf{refl}_{x_0}) \cdot f_0 = \mathsf{refl}_{y_0}.$$

D 8.0.24. The **fiber sequence** of a pointed map $f : X \to Y$ is the infinite sequence of pointed types and pointed maps

$$\dots \xrightarrow{f^{(n+1)}} X^{(n+1)} \xrightarrow{f^{(n)}} X^{(n)} \xrightarrow{f^{(n-1)}} \dots \longrightarrow X^{(2)} \xrightarrow{f^{(1)}} X^{(1)} \xrightarrow{f^{(0)}} X^{(0)}$$

defined recursively by

$$X^{(0)} :\equiv Y \qquad X^{(1)} :\equiv X \qquad f^{(0)} :\equiv f$$

and

$$X^{(n+1)} :\equiv \mathsf{fib}_{f^{(n-1)}}(x_0^{(n-1)})$$
$$f^{(n)} :\equiv \mathsf{pr}_1 \qquad : X^{(n+1)} \to X^{(n)}.$$

where $x_0^{(n)}$ denotes the basepoint of $X^{(n)}$, chosen recursively as above.

L 8.0.25. *Let $f : X \to Y$ be a pointed map of pointed spaces. Then:*

(i). The fiber of $f^{(1)} :\equiv \mathsf{pr}_1 : \mathsf{fib}_f(y_0) \to X$ is equivalent to ΩY.

(ii). Similarly, the fiber of $f^{(2)} : \Omega Y \to \mathsf{fib}_f(y_0)$ is equivalent to ΩX.

(iii). Under these equivalences, the pointed map $f^{(3)} : \Omega X \to \Omega Y$ is identified with the pointed map $\Omega f \circ (-)^{-1}$.

D 8.0.26. An **exact sequence of pointed sets** is a (possibly bounded) sequence of pointed sets and pointed maps:

$$\dots \longrightarrow A^{(n+1)} \xrightarrow{f^{(n)}} A^{(n)} \xrightarrow{f^{(n-1)}} A^{(n-1)} \longrightarrow \dots$$

such that for every n, the image of $f^{(n)}$ is equal, as a subset of $A^{(n)}$, to the kernel of $f^{(n-1)}$. In other words, for all $a : A^{(n)}$ we have

$$(f^{(n-1)}(a) = a_0^{(n-1)}) \iff \exists (b : A^{(n+1)}). (f^{(n)}(b) = a).$$

where $a_0^{(n)}$ denotes the basepoint of $A^{(n)}$.

T 8.0.27. *Let $f : X \to Y$ be a pointed map between pointed spaces with fiber $F :\equiv \mathsf{fib}_f(y_0)$. Then we have the following long exact sequence, which consists of groups except for the last three terms, and abelian groups except for the last six.*

$$\begin{tikzcd}
\vdots & \vdots & \vdots \arrow[dll] \\
\pi_k(F) \arrow[r] & \pi_k(X) \arrow[r] & \pi_k(Y) \arrow[dll] \\
\vdots & \vdots & \vdots \arrow[dll] \\
\pi_2(F) \arrow[r] & \pi_2(X) \arrow[r] & \pi_2(Y) \arrow[dll] \\
\pi_1(F) \arrow[r] & \pi_1(X) \arrow[r] & \pi_1(Y) \arrow[dll] \\
\pi_0(F) \arrow[r] & \pi_0(X) \arrow[r] & \pi_0(Y)
\end{tikzcd}$$

L 8.0.28. *Suppose given an exact sequence of abelian groups:*

$$K \longrightarrow G \xrightarrow{f} H \longrightarrow Q.$$

(i). If $K = 0$, then f is injective.
(ii). If $Q = 0$, then f is surjective.
(iii). If $K = Q = 0$, then f is an isomorphism.

C 8.0.29. *Let $f : A \to B$ be n-connected and $a : A$, and define $b :\equiv f(a)$. Then:*

(i). If $k \le n$, then $\pi_k(f) : \pi_k(A, a) \to \pi_k(B, b)$ is an isomorphism.
(ii). If $k = n + 1$, then $\pi_k(f) : \pi_k(A, a) \to \pi_k(B, b)$ is surjective.

The Hopf fibration

T 8.0.30 (Hopf Fibration)**.** *There is a fibration H over $\mathbb{S}^2$ whose fiber over the basepoint is $\mathbb{S}^1$ and whose total space is $\mathbb{S}^3$.*

C 8.0.31. *We have $\pi_2(\mathbb{S}^2) \simeq \mathbb{Z}$ and $\pi_k(\mathbb{S}^3) \simeq \pi_k(\mathbb{S}^2)$ for every $k \ge 3$ (where the map is induced by the Hopf fibration, seen as a map from the total space $\mathbb{S}^3$ to the base space $\mathbb{S}^2$).*

Fibrations over pushouts

L 8.0.32. *Let $\mathscr{D} = (Y \xleftarrow{j} X \xrightarrow{k} Z)$ be a span and assume that we have*

- *Two fibrations $E_Y : Y \to \mathcal{U}$ and $E_Z : Z \to \mathcal{U}$.*
- *An equivalence e_X between $E_Y \circ j : X \to \mathcal{U}$ and $E_Z \circ k : X \to \mathcal{U}$, i.e.*

$$e_X : \prod_{x:X} E_Y(j(x)) \simeq E_Z(k(x)).$$

Then we can construct a fibration $E : Y \sqcup^X Z \to \mathcal{U}$ such that

- *For all $y : Y$, $E(\mathsf{inl}(y)) \equiv E_Y(y)$.*
- *For all $z : Z$, $E(\mathsf{inr}(z)) \equiv E_Z(z)$.*
- *For all $x : X$, $E(\mathsf{glue}(x)) = \mathsf{ua}(e_X(x))$ (note that both sides of the equation are paths in $\mathcal{U}$ from $E_Y(j(x))$ to $E_Z(k(x))$).*

Moreover, the total space of this fibration fits in the following pushout square:

$$\begin{tikzcd}
\sum_{(x:X)} E_Y(j(x)) \arrow[r, "\mathsf{id} \times e_X", "\sim"'] \arrow[d, "j \times \mathsf{id}"'] & \sum_{(x:X)} E_Z(k(x)) \arrow[r, "k \times \mathsf{id}"] & \sum_{(z:Z)} E_Z(z) \arrow[d, "\mathsf{inr}"] \\
\sum_{(y:Y)} E_Y(y) \arrow[rr, "\mathsf{inl}"'] & & \sum_{(t:Y \sqcup^X Z)} E(t)
\end{tikzcd}$$

The Hopf construction

D 8.0.33. An **H-space** consists of

- a type A,
- a base point $e : A$,
- a binary operation $\mu : A \times A \to A$, and
- for every $a : A$, equalities $\mu(e, a) = a$ and $\mu(a, e) = a$.

L 8.0.34. *Let A be a connected H-space. Then for every $a : A$, the maps $\mu(a, -) : A \to A$ and $\mu(-, a) : A \to A$ are equivalences.*

D 8.0.35. Let A be a connected H-space. We define a fibration over ΣA using **??**.
Given that ΣA is the pushout $\mathbf{1} \sqcup^A \mathbf{1}$, we can define a fibration over ΣA by specifying

- two fibrations over $\mathbf{1}$ (i.e. two types F_1 and F_2), and
- a family $e : A \to (F_1 \simeq F_2)$ of equivalences between F_1 and F_2, one for every element of A.

We take A for F_1 and F_2, and for $a : A$ we take the equivalence $\mu(a, -)$ for $e(a)$.

L 8.0.36. *Given a connected H-space A, there is a fibration, called the **Hopf construction**, over ΣA with fiber A and total space $A * A$.*

The Hopf fibration

L 8.0.37. *There is an H-space structure on the circle $\mathbb{S}^1$.*

L 8.0.38. *The operation of join is associative: if A, B and C are three types then we have an equivalence $(A * B) * C \simeq A * (B * C)$.*

L 8.0.39. *For any type A, there is an equivalence $\Sigma A \simeq \mathbf{2} * A$.*

T 8.0.40. *There is a fibration over $\mathbb{S}^2$ of fiber $\mathbb{S}^1$ and total space $\mathbb{S}^3$.*

The Freudenthal suspension theorem

L 8.0.41. *If $f : A \to B$ is n-connected and $P : B \to k\text{-}\mathsf{Type}$ is a family of k-types for $k \ge n$, then the induced function*

$$(- \circ f) : \Big(\prod_{b:B} P(b)\Big) \to \Big(\prod_{a:A} P(f(a))\Big)$$

is $(k - n - 2)$-truncated.

L 8.0.42 (Wedge connectivity lemma)**.** *Suppose that (A, a_0) and (B, b_0) are n- and m-connected pointed types, respectively, with $n, m \ge 0$, and let*

$$P : A \to B \to (n + m)\text{-}\mathsf{Type}.$$

Then for any $f : \prod_{(a:A)} P(a, b_0)$ and $g : \prod_{(b:B)} P(a_0, b)$ with $p : f(a_0) = g(b_0)$, there exists $h : \prod_{(a:A)} \prod_{(b:B)} P(a, b)$ with homotopies

$$q : \prod_{a:A} h(a, b_0) = f(a) \qquad \text{and} \qquad r : \prod_{b:B} h(a_0, b) = g(b)$$

such that $p = q(a_0)^{-1} \cdot r(b_0)$.

NB 8.0.43. In classical algebraic topology, one considers the *reduced suspension*, in which the path $\mathsf{merid}(x_0)$ is collapsed down to a point, identifying N and S. The reduced and unreduced suspensions are homotopy equivalent, so the distinction is invisible to our purely homotopy-theoretic eyes — and higher inductive types only allow us to "identify" points up to a higher path anyway, there is no purpose to considering reduced suspensions in homotopy type theory. However, the "unreducedness" of our suspension is the reason for the (possibly unexpected) appearance of $\mathsf{merid}(x_0)^{-1}$ in the definition of σ.

T 8.0.44 (The Freudenthal suspension theorem)**.** *Suppose that X is n-connected and pointed, with $n \ge 0$. Then the map $\sigma : X \to \Omega\Sigma(X)$ is $2n$-connected.*

D 8.0.45. If X is n-connected and pointed with $n \ge 0$, then there is a family

$$\mathsf{code} : \prod_{y:\Sigma X} (\mathsf{N} = y) \to \mathcal{U} \tag{8.0.46}$$

such that

$$\mathsf{code}(\mathsf{N}, p) :\equiv \|\mathsf{fib}_\sigma(p)\|_{2n} \equiv \Big\|\textstyle\sum_{(x:X)} (\mathsf{merid}(x) \cdot \mathsf{merid}(x_0)^{-1} = p)\Big\|_{2n} \tag{8.0.47}$$

$$\mathsf{code}(\mathsf{S}, q) :\equiv \|\mathsf{fib}_{\mathsf{merid}}(q)\|_{2n} \equiv \big\|\textstyle\sum_{(x:X)} (\mathsf{merid}(x) = q)\big\|_{2n}. \tag{8.0.48}$$

L 8.0.49. *Let $A : \mathcal{U}$, $B : A \to \mathcal{U}$, and $C : \prod_{(a:A)} B(a) \to \mathcal{U}$, and also $a_1, a_2 : A$ with $m : a_1 = a_2$ and $b : B(a_2)$. Then the function*

$$\mathsf{transport}^{\widehat{C}}(\mathsf{pair}^{=}(m, t), -) : C(a_1, \mathsf{transport}^B(m^{-1}, b)) \to C(a_2, b),$$

where $t : \mathsf{transport}^B(m, \mathsf{transport}^B(m^{-1}, b)) = b$ is the obvious coherence path and $\widehat{C} : (\sum_{(a:A)} B(a)) \to \mathcal{U}$ is the uncurried form of C, is equal to the equivalence obtained by univalence from the composite

$$\begin{aligned} C(a_1, \mathsf{transport}^B(m^{-1}, b)) &= \mathsf{transport}^{\lambda a.\, B(a) \to \mathcal{U}}(m, C(a_1))(b) && \text{(by (2))} \\ &= C(a_2, b). && \text{(by } \mathsf{happly}(\mathsf{apd}_C(m), b)\text{)} \end{aligned}$$

C 8.0.50 (Freudenthal Equivalence)**.** *Suppose that X is n-connected and pointed, with $n \ge 0$. Then $\|X\|_{2n} \simeq \|\Omega\Sigma(X)\|_{2n}$.*

C 8.0.51 (Stability for Spheres)**.** *If $k \le 2n - 2$, then $\pi_{k+1}(S^{n+1}) = \pi_k(S^n)$.*

T 8.0.52. *$\pi_n(\mathbb{S}^n) = \mathbb{Z}$ for every $n \ge 1$.*

C 8.0.53. *$\mathbb{S}^{n+1}$ is not an n-type for any $n \ge -1$.*

C 8.0.54. *$\pi_3(\mathbb{S}^2) = \mathbb{Z}$.*

The van Kampen theorem

Naive van Kampen

NB 8.0.55. One might expect to see in the definition of code some additional generating equations for the set-quotient, such as

$$(\ldots, p_{k-1} \cdot fw, x'_k, q_k, \ldots) = (\ldots, p_{k-1}, x_k, gw \cdot q_k, \ldots) \qquad (\text{for } w : \Pi_1 A(x_k, x'_k))$$

$$(\ldots, q_k \cdot gw, y'_k, p_k, \ldots) = (\ldots, q_k, y_k, fw \cdot p_k, \ldots). \qquad (\text{for } w : \Pi_1 A(y_k, y'_k))$$

However, these are not necessary! In fact, they follow automatically by path induction on w. This is the main difference between the "naive" van Kampen theorem and the more refined one we will consider in the next subsection.

T 8.0.56 (Naive van Kampen theorem)**.** *For all $u, v : P$ there is an equivalence*

$$\Pi_1 P(u, v) \simeq \mathsf{code}(u, v).$$

E 8.0.57. Let $A :\equiv \mathbf{2}$, $B :\equiv \mathbf{1}$, and $C :\equiv \mathbf{1}$. Then $P \simeq S^1$. Inspecting the definition of, say, $\mathsf{code}(i(\star), i(\star))$, we see that the paths all may as well be trivial, so the only information is in the sequence of elements $x_1, y_1, \ldots, x_n, y_n : \mathbf{2}$. Moreover, if we have $x_k = y_k$ or $y_k = x_{k+1}$ for any k, then the set-quotient relations allow us to excise both of those elements. Thus, every such sequence is equal to a canonical *reduced* one in which no two adjacent elements are equal. Clearly such a reduced sequence is uniquely determined by its length (a natural number n) together with, if $n > 1$, the information of whether x_1 is $0_\mathbf{2}$ or $1_\mathbf{2}$, since that determines the rest of the sequence uniquely. And these data can, of course, be identified with an integer, where n is the absolute value and x_1 encodes the sign. Thus we recover $\pi_1(S^1) \cong \mathbb{Z}$.

E 8.0.58. More generally, let $B :\equiv \mathbf{1}$ and $C :\equiv \mathbf{1}$ but A be arbitrary, so that P is the suspension of A. Then once again the paths p_k and q_k are trivial, so that the only information in a path code is a sequence of elements $x_1, y_1, \ldots, x_n, y_n : A$. The first two generating equalities say that adjacent equal elements can be canceled, so it makes sense to think of this sequence as a word of the form

$$x_1 y_1^{-1} x_2 y_2^{-1} \cdots x_n y_n^{-1}$$

in a group. Indeed, it looks similar to the free group on A (or equivalently on $\|A\|_0$; see **??**), but we are considering only words that start with a non-inverted element, alternate between inverted and non-inverted elements, and end with an inverted one. This effectively reduces the size of the generating set by one. For instance, if A has a point $a : A$, then we can identify $\pi_1(\Sigma A)$ with the group presented by $\|A\|_0$ as generators with the relation $|a|_0 = e$; see **????** for details.

E 8.0.59. Let $A :\equiv \mathbf{1}$ and B and C be arbitrary, so that f and g simply equip B and C with basepoints b and c, say. Then P is the *wedge* $B \vee C$ of B and C (the coproduct in the category of based spaces). In this case, it is the elements x_k and y_k which are trivial, so that the only information is a sequence of loops $(p_0, q_1, p_1, \ldots, p_n)$ with $p_k : \pi_1(B, b)$ and $q_k : \pi_1(C, c)$. Such sequences, modulo the equivalence relation we have imposed, are easily identified with the explicit description of the *free product* of the groups $\pi_1(B, b)$ and $\pi_1(C, c)$, as constructed in **??**. Thus, we have $\pi_1(B \vee C) \cong \pi_1(B) * \pi_1(C)$.

The van Kampen theorem with a set of basepoints

L 8.0.60. *$\bar{k}$ is 0-connected.*

NB 8.0.61. T can be regarded as the (homotopy) coequalizer of the "kernel pair" of k. If S and A were sets, then the (-1)-connectivity of k would imply that A is the 0-truncation of this coequalizer (see **??**). For general types, higher topos theory suggests that (-1)-connectivity of k will imply instead that A is the colimit (a.k.a. "geometric realization") of the "simplicial kernel" of k. The type T is the colimit of the "1-skeleton" of this simplicial kernel, so it makes sense that it improves the connectivity of k by 1. More generally, we might expect the colimit of the n-skeleton to improve connectivity by n.

T 8.0.62 (van Kampen with a set of basepoints)**.** *For all $u, v : P$ there is an equivalence*

$$\Pi_1 P(u, v) \simeq \mathsf{code}(u, v).$$

with code defined as in this section.

E 8.0.63. Suppose $S :\equiv \mathbf{1}$, so that A has a basepoint $a :\equiv k(\star)$ and is connected. Then code for loops in the pushout can be identified with alternating sequences of loops in $\pi_1(B, f(a))$ and $\pi_1(C, g(a))$, modulo an equivalence relation which allows us to slide elements of $\pi_1(A, a)$ between them (after applying f and g respectively). Thus, $\pi_1(P)$ can be identified with the *amalgamated free product* $\pi_1(B) *_{\pi_1(A)} \pi_1(C)$ (the pushout in the category of groups), as constructed in **??**. This (in the case when B and C are open subspaces of P and A their intersection) is probably the most classical version of the van Kampen theorem.

E 8.0.64. As a special case of **??**, suppose additionally that $C :\equiv \mathbf{1}$, so that P is the cofiber B/A. Then every loop in C is equal to reflexivity, so the relations on path codes allow us to collapse all sequences to a single loop in B. The additional relations require that multiplying on the left, right, or in the middle by an element in the image of $\pi_1(A)$ is the identity. We can thus identify $\pi_1(B/A)$ with the quotient of the group $\pi_1(B)$ by the normal subgroup generated by the image of $\pi_1(A)$.

E 8.0.65. As a further special case of **??**, let $B :\equiv S^1 \vee S^1$, let $A :\equiv S^1$, and let $f : A \to B$ pick out the composite loop $p \cdot q \cdot p^{-1} \cdot q^{-1}$, where p and q are the generating loops in the two copies of S^1 comprising B. Then P is a presentation of the torus T^2. Indeed, it is not hard to identify P with the presentation of T^2 as described in **??**, using the cone on a particular loop. Thus, $\pi_1(T^2)$ is the quotient of the free group on two generators (i.e., $\pi_1(B)$) by the relation $p \cdot q \cdot p^{-1} \cdot q^{-1} = 1$. This clearly yields the free *abelian* group on two generators, which is $\mathbb{Z} \times \mathbb{Z}$.

E 8.0.66. More generally, any CW complex can be obtained by repeatedly "coning off" spheres, as described in **??**. That is, we start with a set X_0 of points ("0-cells"), which is the "0-skeleton" of the CW complex. We take the pushout

$$\begin{tikzcd} S_1 \times \mathbb{S}^0 \arrow[r, "f_1"] \arrow[d] & X_0 \arrow[d] \\ \mathbf{1} \arrow[r] & X_1 \end{tikzcd}$$

for some set S_1 of 1-cells and some family f_1 of "attaching maps", obtaining the "1-skeleton" X_1. Then we take the pushout

$$\begin{tikzcd} S_2 \times \mathbb{S}^1 \arrow[r, "f_2"] \arrow[d] & X_1 \arrow[d] \\ \mathbf{1} \arrow[r] & X_2 \end{tikzcd}$$

for some set S_2 of 2-cells and some family f_2 of attaching maps, obtaining the 2-skeleton X_2, and so on. The fundamental group of each pushout can be calculated from the van Kampen theorem: we obtain the group presented by generators derived from the 1-skeleton, and relations derived from S_2 and f_2. The pushouts after this stage do not alter the fundamental group, since $\pi_1(\mathbb{S}^n)$ is trivial for $n > 1$ (see **??**).

E 8.0.67. In particular, suppose given any presentation of a (set-)group $G = \langle X \mid R \rangle$, with X a set of generators and R a set of words in these generators. Let $B :\equiv \bigvee_X S^1$ and $A :\equiv \bigvee_R S^1$, with $f : A \to B$ sending each copy of S^1 to the corresponding word in the generating loops of B. It follows that $\pi_1(P) \cong G$; thus we have constructed a connected type whose fundamental group is G. Since any group has a presentation, any group is the fundamental group of some type. If we 1-truncate such a type, we obtain a type whose only nontrivial homotopy group is G; this is called an **Eilenberg–Mac Lane space** $K(G, 1)$.

Whitehead's theorem and Whitehead's principle

T 8.0.68. *Suppose $f : A \to B$ is a function such that*

(i). *$\|f\|_0 : \|A\|_0 \to \|B\|_0$ is surjective, and*
(ii). *for any $x, y : A$, the function $\mathsf{ap}_f : (x =_A y) \to (f(x) =_B f(y))$ is an equivalence.*

Then f is an equivalence.

C 8.0.69. *Suppose $f : A \to B$ is a function such that*

(i). *$\|f\|_0 : \|A\|_0 \to \|B\|_0$ is a bijection, and*
(ii). *for any $x : A$, the function $\Omega f : \Omega(A, x) \to \Omega(B, f(x))$ is an equivalence.*

Then f is an equivalence.

T 8.0.70. *Suppose A and B are n-types and $f : A \to B$ is such that*

(i). *$\|f\|_0 : \|A\|_0 \to \|B\|_0$ is a bijection, and*
(ii). *$\pi_k(f) : \pi_k(A, x) \to \pi_k(B, f(x))$ is a bijection for all $k \geq 1$ and all $x : A$.*

Then f is an equivalence.