Homotopy Type Theory

Sets and logic

Sets and n-types

D 3.0.1. A type A is a **set** if for all $x, y : A$ and all $p, q : x = y$, we have $p = q$.

More precisely, the proposition $\mathsf{isSet}(A)$ is defined to be the type

$$\mathsf{isSet}(A) :\equiv \prod_{(x,y:A)} \prod_{(p,q:x=y)} (p = q).$$

E 3.0.2. The type $\mathbf{1}$ is a set. For any $x, y : \mathbf{1}$ the type $(x = y)$ is equivalent to $\mathbf{1}$. Since any two elements of $\mathbf{1}$ are equal, this implies that any two elements of $x = y$ are equal.

E 3.0.3. The type $\mathbf{0}$ is a set, for given any $x, y : \mathbf{0}$ we may deduce anything we like, by the induction principle of $\mathbf{0}$.

E 3.0.4. The type $\mathbb{N}$ of natural numbers is also a set. Since all equality types $x =_{\mathbb{N}} y$ are equivalent to either $\mathbf{1}$ or $\mathbf{0}$, and any two inhabitants of $\mathbf{1}$ or $\mathbf{0}$ are equal.

Most of the type forming operations we have considered so far also preserve sets.

E 3.0.5. If A and B are sets, then so is $A \times B$. For given $x, y : A \times B$ and $p, q : x = y$, then we have $p = \mathsf{pair}^{=}(\mathsf{ap}_{\mathsf{pr}_1}(p), \mathsf{ap}_{\mathsf{pr}_2}(p))$ and $q = \mathsf{pair}^{=}(\mathsf{ap}_{\mathsf{pr}_1}(q), \mathsf{ap}_{\mathsf{pr}_2}(q))$. But $\mathsf{ap}_{\mathsf{pr}_1}(p) = \mathsf{ap}_{\mathsf{pr}_1}(q)$ since A is a set, and $\mathsf{ap}_{\mathsf{pr}_2}(p) = \mathsf{ap}_{\mathsf{pr}_2}(q)$ since B is a set; hence $p = q$.
Similarly, if A is a set and $B : A \to \mathcal{U}$ is such that each $B(x)$ is a set, then $\sum_{(x:A)} B(x)$ is a set.

E 3.0.6. If A is *any* type and $B : A \to \mathcal{U}$ is such that each $B(x)$ is a set, then the type $\prod_{(x:A)} B(x)$ is a set. For suppose $f, g : \prod_{(x:A)} B(x)$ and $p, q : f = g$. By function extensionality, we have

$$p = \mathsf{funext}(x \mapsto \mathsf{happly}(p, x)) \quad \text{and} \quad q = \mathsf{funext}(x \mapsto \mathsf{happly}(q, x)).$$

But for any $x : A$, we have

$$\mathsf{happly}(p, x) : f(x) = g(x) \qquad \text{and} \qquad \mathsf{happly}(q, x) : f(x) = g(x),$$

so since $B(x)$ is a set we have $\mathsf{happly}(p, x) = \mathsf{happly}(q, x)$. Now using function extensionality again, the dependent functions $(x \mapsto \mathsf{happly}(p, x))$ and $(x \mapsto \mathsf{happly}(q, x))$ are equal, and hence (applying $\mathsf{ap}_{\mathsf{funext}}$) so are p and q.

D 3.0.7. A type A is a **1-type** if for all $x, y : A$ and $p, q : x = y$ and $r, s : p = q$, we have $r = s$.

L 3.0.8. *If A is a set (that is, $\mathsf{isSet}(A)$ is inhabited), then A is a 1-type.*

Propositions as types?

NB 3.0.9. (Statement) If for all $x : X$ there exists an $a : A(x)$ such that $P(x, a)$, then there exists a function $g : \prod_{(x:A)} A(x)$ such that for all $x : X$ we have $P(x, g(x))$".
This looks like the classical *axiom of choice*, is always true under this reading.

NB 3.0.10. The classical *law of double negation* and *law of excluded middle* are incompatible with the univalence axiom.

T 3.0.11. *It is not the case that for all $A : \mathcal{U}$ we have $\neg(\neg A) \to A$.*

NB 3.0.12. For any A, $\neg\neg\neg A \to \neg A$ for any A.

C 3.0.13. *It is not the case that for all $A : \mathcal{U}$ we have $A + (\neg A)$.*

Mere propositions

D 3.0.14. A type P is a **mere proposition** if for all $x, y : P$ we have $x = y$.

Specifically, for any $P : \mathcal{U}$, the type $\mathsf{isProp}(P)$ is defined to be

$$\mathsf{isProp}(P) :\equiv \prod_{x,y:P} (x = y).$$

L 3.0.15. *If P is a mere proposition and $x_0 : P$, then $P \simeq \mathbf{1}$.*

L 3.0.16. *If P and Q are mere propositions such that $P \to Q$ and $Q \to P$, then $P \simeq Q$.*

L 3.0.17. *Every mere proposition is a set.*

L 3.0.18. *For any type A, the types $\mathsf{isProp}(A)$ and $\mathsf{isSet}(A)$ are mere propositions.*

Classical vs. intuitionistic logic

With the notion of mere proposition in hand, we can now give the proper formulation of the **law of excluded middle** in homotopy type theory:

$$\mathsf{LEM} :\equiv \prod_{A:\mathcal{U}} \Big(\mathsf{isProp}(A) \to (A + \neg A)\Big). \tag{3.0.19}$$

Similarly, the **law of double negation** is

$$\prod_{A:\mathcal{U}} \Big(\mathsf{isProp}(A) \to (\neg\neg A \to A)\Big). \tag{3.0.20}$$

D 3.0.21.

(i). A type A is called **decidable** if $A + \neg A$.

(ii). Similarly, a type family $B : A \to \mathcal{U}$ is **decidable** if

$$\prod_{a:A} (B(a) + \neg B(a)).$$

(iii). In particular, A has **decidable equality** if

$$\prod_{a,b:A} ((a = b) + \neg(a = b)).$$

Subsets and propositional resizing

L 3.0.22. *Suppose $P : A \to \mathcal{U}$ is a type family such that $P(x)$ is a mere proposition for all $x : A$. If $u, v : \sum_{(x:A)} P(x)$ are such that $\mathsf{pr}_1(u) = \mathsf{pr}_1(v)$, then $u = v$.*

For instance, recall that

$$(A \simeq B) :\equiv \sum_{f:A\to B} \mathsf{isequiv}(f),$$

where each type $\mathsf{isequiv}(f)$ was supposed to be a mere proposition. It follows that if two equivalences have equal underlying functions, then they are equal as equivalences.
If $P : A \to \mathcal{U}$ is a family of mere propositions (i.e. each $P(x)$ is a mere proposition), we may write

$$\{\, x : A \mid P(x) \,\} \tag{3.0.23}$$

as an alternative notation for $\sum_{(x:A)} P(x)$. We may define the "subuniverses" of sets and of mere propositions in a universe $\mathcal{U}$:

$$\mathsf{Set}_{\mathcal{U}} :\equiv \{\, A : \mathcal{U} \mid \mathsf{isSet}(A) \,\},$$
$$\mathsf{Prop}_{\mathcal{U}} :\equiv \{\, A : \mathcal{U} \mid \mathsf{isProp}(A) \,\}.$$

An element of $\mathsf{Set}_{\mathcal{U}}$ is a type $A : \mathcal{U}$ together with evidence $s : \mathsf{isSet}(A)$, and similarly for $\mathsf{Prop}_{\mathcal{U}}$.

A 3.0.24 (Propositional resizing)**.** *The map $\mathsf{Prop}_{\mathcal{U}_i} \to \mathsf{Prop}_{\mathcal{U}_{i+1}}$ is an equivalence.*

With propositional resizing, we can define the power set to be

$$\mathcal{P}(A) :\equiv (A \to \Omega),$$

which is then independent of $\mathcal{U}$.

The logic of mere propositions

E 3.0.25. If A and B are mere propositions, so is $A \times B$. This is easy to show using the characterization of paths in products, just like **??** but simpler. Thus, the connective "and" preserves mere propositions.

E 3.0.26. If A is any type and $B : A \to \mathcal{U}$ is such that for all $x : A$, the type $B(x)$ is a mere proposition, then $\prod_{(x:A)} B(x)$ is a mere proposition. The proof is just like **??** but simpler: given $f, g : \prod_{(x:A)} B(x)$, for any $x : A$ we have $f(x) = g(x)$ since $B(x)$ is a mere proposition. But then by function extensionality, we have $f = g$.
In particular, if B is a mere proposition, then so is $A \to B$ regardless of what A is. In even more particular, since $\mathbf{0}$ is a mere proposition, so is $\neg A \equiv (A \to \mathbf{0})$. Thus, the connectives "implies" and "not" preserve mere propositions, as does the quantifier "for all".

Propositional truncation

The *propositional truncation*, also called the (-1)*-truncation*, *bracket type*, or *squash type*, is an additional type former which "squashes" or "truncates" a type down to a mere proposition, forgetting all information contained in inhabitants of that type other than their existence.
More precisely, for any type A, there is a type $\|A\|$. It has two constructors:

- For any $a : A$ we have $|a| : \|A\|$.
- For any $x, y : \|A\|$, we have $x = y$.

The recursion principle of $\|A\|$ says that:

- If B is a mere proposition and we have $f : A \to B$, then there is an induced $g : \|A\| \to B$ such that $g(|a|) \equiv f(a)$ for all $a : A$.

D 3.0.27. We define **traditional logical notation** using truncation as follows, where P and Q denote mere propositions (or families thereof):

$$\begin{aligned} \top &:\equiv \mathbf{1} \\ \bot &:\equiv \mathbf{0} \\ P \wedge Q &:\equiv P \times Q \\ P \Rightarrow Q &:\equiv P \to Q \\ P \Leftrightarrow Q &:\equiv P = Q \\ \neg P &:\equiv P \to \mathbf{0} \\ P \vee Q &:\equiv \|P + Q\| \\ \forall (x : A).\, P(x) &:\equiv \prod_{x:A} P(x) \\ \exists (x : A).\, P(x) &:\equiv \Big\| \sum_{x:A} P(x) \Big\| \end{aligned}$$

The notations $\wedge$ and $\vee$ are also used in homotopy theory for the smash product and the wedge of pointed spaces.

$$\begin{aligned} \{\, x : A \mid P(x) \,\} \cap \{\, x : A \mid Q(x) \,\} &:\equiv \{\, x : A \mid P(x) \wedge Q(x) \,\}, \\ \{\, x : A \mid P(x) \,\} \cup \{\, x : A \mid Q(x) \,\} &:\equiv \{\, x : A \mid P(x) \vee Q(x) \,\}, \\ A \setminus \{\, x : A \mid P(x) \,\} &:\equiv \{\, x : A \mid \neg P(x) \,\}. \end{aligned}$$

Of course, in the absence of LEM, the latter are not "complements" in the usual sense: we may not have $B \cup (A \setminus B) = A$ for every subset B of A.

The axiom of choice

$$A : X \to \mathcal{U} \qquad \text{and} \qquad P : \prod_{x:X} A(x) \to \mathcal{U},$$

and moreover that

- X is a set,
- $A(x)$ is a set for all $x : X$, and
- $P(x, a)$ is a mere proposition for all $x : X$ and $a : A(x)$.

The **axiom of choice** AC asserts that under these assumptions,

$$\Big(\prod_{x:X} \Big\| \sum_{a:A(x)} P(x,a) \Big\| \Big) \to \Big\| \sum_{(g:\prod_{(x:X)} A(x))} \prod_{(x:X)} P(x, g(x)) \Big\|. \tag{3.0.28}$$

Of course, this is a direct translation of (3) where we read "there exists $x : A$ such that $B(x)$" as $\|\sum_{(x:A)} B(x)\|$, so we could have written the statement in the familiar logical notation as

$$\Big(\forall (x : X).\, \exists (a : A(x)).\, P(x,a)\Big) \Rightarrow \Big(\exists \big(g : \prod_{x:X} A(x)\big).\, \forall (x : X).\, P(x, g(x))\Big).$$

L 3.0.29. *The axiom of choice (3) is equivalent to the statement that for any set X and any $Y : X \to \mathcal{U}$ such that each $Y(x)$ is a set, we have*

$$\Big(\prod_{x:X} \big\| Y(x) \big\| \Big) \to \Big\| \prod_{x:X} Y(x) \Big\|. \tag{3.0.30}$$

NB 3.0.31. The right side of (3) always implies the left. Since both are mere propositions, by **??** the axiom of choice is also equivalent to asking for an equivalence

$$\Big(\prod_{x:X} \big\| Y(x) \big\| \Big) \simeq \Big\| \prod_{x:X} Y(x) \Big\|$$

L 3.0.32. *There exists a type X and a family $Y : X \to \mathcal{U}$ such that each $Y(x)$ is a set, but such that (3) is false.*

The principle of unique choice

L 3.0.33. *If P is a mere proposition, then $P \simeq \|P\|$.*

C 3.0.34 (The principle of unique choice). *Suppose a type family $P : A \to \mathcal{U}$ such that*

(i). For each x, the type $P(x)$ is a mere proposition, and
(ii). For each x we have $\|P(x)\|$.

Then we have $\prod_{(x:A)} P(x)$.

Contractibility

D 3.1.1. A type A is **contractible**, or a **singleton**, if there is $a : A$, called the **center of contraction**, such that $a = x$ for all $x : A$. We denote the specified path $a = x$ by contr_x.

In other words, the type $\mathsf{isContr}(A)$ is defined to be

$$\mathsf{isContr}(A) :\equiv \sum_{(a:A)} \prod_{(x:A)} (a = x).$$

L 3.1.2. *For a type A, the following are logically equivalent.*

*(i). A is contractible in the sense of **??**.*
(ii). A is a mere proposition, and there is a point $a : A$.
(iii). A is equivalent to $\mathbf{1}$.

L 3.1.3. *For any type A, the type $\mathsf{isContr}(A)$ is a mere proposition.*

C 3.1.4. *If A is contractible, then so is $\mathsf{isContr}(A)$.*

L 3.1.5. *If $P : A \to \mathcal{U}$ is a type family such that each $P(a)$ is contractible, then $\prod_{(x:A)} P(x)$ is contractible.*

Of course, if A is equivalent to B and A is contractible, then so is B. More generally, it suffices for B to be a *retract* of A. By definition, a **retraction** is a function $r : A \to B$ such that there exists a function $s : B \to A$, called its **section**, and a homotopy $\epsilon : \prod_{(y:B)} (r(s(y)) = y)$; then we say that B is a **retract**of A.

L 3.1.6. *If B is a retract of A, and A is contractible, then so is B.*

L 3.1.7. *For any A and any $a : A$, the type $\sum_{(x:A)} (a = x)$ is contractible.*

L 3.1.8. *Let $P : A \to \mathcal{U}$ be a type family.*

(i). If each $P(x)$ is contractible, then $\sum_{(x:A)} P(x)$ is equivalent to A.
(ii). If A is contractible with center a, then $\sum_{(x:A)} P(x)$ is equivalent to $P(a)$.

L 3.1.9. *A type A is a mere proposition if and only if for all $x, y : A$, the type $x =_A y$ is contractible.*

Exer. 3.1. Prove that if $A \simeq B$ and A is a set, then so is B.

Exer. 3.2. Prove that if A and B are sets, then so is $A + B$.

Exer. 3.3. Prove that if A is a set and $B : A \to \mathcal{U}$ is a type family such that $B(x)$ is a set for all $x : A$, then $\sum_{(x:A)} B(x)$ is a set.

Exer. 3.4. Show that A is a mere proposition if and only if $A \to A$ is contractible.

Exer. 3.5. Show that $\mathsf{isProp}(A) \simeq (A \to \mathsf{isContr}(A))$.

Exer. 3.6. Show that if A is a mere proposition, then so is $A + (\neg A)$. Thus, there is no need to insert a propositional truncation in (3).

Exer. 3.7. More generally, show that if A and B are mere propositions and $\neg(A \times B)$, then $A + B$ is also a mere proposition.

Exer. 3.8. Show that if A and B are mere propositions such that $A \to B$ and $B \to A$, then $A \simeq B$.

Exer. 3.9. Show that for any type A, the types $\mathsf{isProp}(A)$ and $\mathsf{isSet}(A)$ are mere propositions.

Exer. 3.10. Show that if A is already a mere proposition, then $A \simeq \|A\|$.

Exer. 3.11. Assuming that some type $\mathsf{isequiv}(f)$ satisfies conditions **??**–**??** of **??**, show that the type $\|\mathsf{qinv}(f)\|$ satisfies the same conditions and is equivalent to $\mathsf{isequiv}(f)$.

Exer. 3.12. Show that if LEM holds, then the type $\mathsf{Prop} :\equiv \sum_{(A:\mathcal{U})} \mathsf{isProp}(A)$ is equivalent to $\mathbf{2}$.

Exer. 3.13. Show that if $\mathcal{U}_{i+1}$ satisfies LEM, then the canonical inclusion $\mathsf{Prop}_{\mathcal{U}_i} \to \mathsf{Prop}_{\mathcal{U}_{i+1}}$ is an equivalence.

Exer. 3.14. Show that it is not the case that for all $A : \mathcal{U}$ we have $\|A\| \to A$. (However, there can be particular types for which $\|A\| \to A$. **??** implies that $\mathsf{qinv}(f)$ is such.)

Exer. 3.15. Show that if LEM holds, then for all $A : \mathcal{U}$ we have $\|(\|A\| \to A)\|$. (This property is a very simple form of the axiom of choice, which can fail in the absence of LEM; see [?].)

Exer. 3.16. We showed in **??** that the following naive form of LEM is inconsistent with univalence:

$$\prod_{A:\mathcal{U}} (A + (\neg A))$$

In the absence of univalence, this axiom is consistent. However, show that it implies the axiom of choice (3).

Exer. 3.17. Show that assuming LEM, the double negation $\neg\neg A$ has the same universal property as the propositional truncation $\|A\|$, and is therefore equivalent to it. Thus, under LEM, the propositional truncation can be defined rather than taken as a separate type former.

Exer. 3.18. Show that if we assume propositional resizing as in **??**, then the type

$$\prod_{P:\mathsf{Prop}} \Big((A \to P) \to P\Big)$$

has the same universal property as $\|A\|$. Thus, we can also define the propositional truncation in this case.

Exer. 3.19. Assuming LEM, show that double negation commutes with universal quantification of mere propositions over sets. That is, show that if X is a set and each $Y(x)$ is a mere proposition, then LEM implies

$$\Big(\prod_{x:X} \neg\neg Y(x)\Big) \simeq \Big(\neg\neg \prod_{x:X} Y(x)\Big). \tag{3.1.10}$$

Observe that if we assume instead that each $Y(x)$ is a set, then (3) becomes equivalent to the axiom of choice (3).

Exer. 3.20. Show that the rules for the propositional truncation given in **??** are sufficient to imply the following induction principle: for any type family $B : \|A\| \to \mathcal{U}$ such that each $B(x)$ is a mere proposition, if for every $a : A$ we have $B(|a|)$, then for every $x : \|A\|$ we have $B(x)$.

Exer. 3.21. Show that the law of excluded middle (3) and the law of double negation (3) are logically equivalent.

Exer. 3.22. Suppose $P : \mathbb{N} \to \mathcal{U}$ is a decidable family (see **????**) of mere propositions. Prove that

$$\Big\|\sum_{n:\mathbb{N}} P(n)\Big\| \to \sum_{n:\mathbb{N}} P(n).$$

Exer. 3.23. Prove **??**(: if A is contractible with center a, then $\sum_{(x:A)} P(x)$ is equivalent to $P(a)$.

Exer. 3.24. Prove that $\mathsf{isProp}(P) \simeq (P \simeq \|P\|)$.

Exer. 3.25. As in classical set theory, the finite version of the axiom of choice is a theorem. Prove that the axiom of choice (3) holds when X is a finite type $\mathsf{Fin}(n)$ (as defined in **??**).

Exer. 3.26. Show that the conclusion of **??** is true if $P : \mathbb{N} \to \mathcal{U}$ is any decidable family.