

# Homotopy Type Theory

## Categories

### 9.1 Categories and precategories

**Definition 9.1.1.** A **precategory**  $A$  consists of the following.

- (i). A type  $A_0$ , whose elements are called **objects**. We write  $a : A$  for  $a : A_0$ .
- (ii). For each  $a, b : A$ , a set  $\text{hom}_A(a, b)$ , whose elements are called **arrows** or **morphisms**.
- (iii). For each  $a : A$ , a morphism  $1_a : \text{hom}_A(a, a)$ , called the **identity morphism**.
- (iv). For each  $a, b, c : A$ , a function

$$\text{hom}_A(b, c) \rightarrow \text{hom}_A(a, b) \rightarrow \text{hom}_A(a, c)$$

called **composition**, and denoted infix by  $g \mapsto f \mapsto g \circ f$ , or sometimes simply by  $gf$ .

- (v). For each  $a, b : A$  and  $f : \text{hom}_A(a, b)$ , we have  $f = 1_b \circ f$  and  $f = f \circ 1_a$ .
- (vi). For each  $a, b, c, d : A$  and

$$f : \text{hom}_A(a, b), \quad g : \text{hom}_A(b, c), \quad h : \text{hom}_A(c, d),$$

we have  $h \circ (g \circ f) = (h \circ g) \circ f$ .

**Definition 9.1.2.** A morphism  $f : \text{hom}_A(a, b)$  is an **isomorphism** if there is a morphism  $g : \text{hom}_A(b, a)$  such that  $g \circ f = 1_a$  and  $f \circ g = 1_b$ . We write  $a \cong b$  for the type of such isomorphisms.

**Lemma 9.1.3.** For any  $f : \text{hom}_A(a, b)$ , the type “ $f$  is an isomorphism” is a mere proposition. Therefore, for any  $a, b : A$  the type  $a \cong b$  is a set.

**Lemma 9.1.4 (idtoiso).** If  $A$  is a precategory and  $a, b : A$ , then

$$(a = b) \rightarrow (a \cong b).$$

**Example 9.1.5.** There is a precategory  $\text{Set}$ , whose type of objects is  $\text{Set}$ , and with  $\text{hom}_{\text{Set}}(A, B) := (A \rightarrow B)$ . The identity morphisms are identity functions and the composition is function composition. For this precategory, Lemma 9.1.4 is equal to (the restriction to sets of) the map  $\text{idtoeqv}$  from ??.

Of course, to be more precise we should call this category  $\text{Set}_{\mathcal{U}}$ , since its objects are only the *small sets* relative to a universe  $\mathcal{U}$ .

**Definition 9.1.6.** A **category** is a precategory such that for all  $a, b : A$ , the function  $\text{idtoiso}_{a,b}$  from Lemma 9.1.4 is an equivalence.

**Example 9.1.7.** The univalence axiom implies immediately that  $\text{Set}$  is a category. One can also show, using univalence, that any precategory of set-level structures such as groups, rings, topological spaces, etc. is a category; see §9.8.

**Lemma 9.1.8.** In a category, the type of objects is a 1-type.

**Lemma 9.1.9.** For  $p : a = a'$  and  $q : b = b'$  and  $f : \text{hom}_A(a, b)$ , we have

$$(p, q)_*(f) = \text{idtoiso}(q) \circ f \circ \text{idtoiso}(p)^{-1}. \quad (9.1.10)$$

**Example 9.1.11.** A precategory in which each set  $\text{hom}_A(a, b)$  is a mere proposition is equivalently a type  $A_0$  equipped with a mere relation “ $\leq$ ” that is reflexive ( $a \leq a$ ) and transitive (if  $a \leq b$  and  $b \leq c$ , then  $a \leq c$ ). We call this a **preorder**.

In a preorder, a witness  $f : a \leq b$  is an isomorphism just when there exists some witness  $g : b \leq a$ . Thus,  $a \cong b$  is the mere proposition that  $a \leq b$  and  $b \leq a$ . Therefore, a preorder  $A$  is a category just when (1) each type  $a = b$  is a mere proposition, and (2) for any  $a, b : A_0$  there exists a function  $(a \cong b) \rightarrow (a = b)$ . In other words,  $A_0$  must be a set, and  $\leq$  must be antisymmetric (if  $a \leq b$  and  $b \leq a$ , then  $a = b$ ). We call this a **(partial) order** or a **poset**.

**Example 9.1.12.** If  $A$  is a category, then  $A_0$  is a set if and only if for any  $a, b : A_0$ , the type  $a \cong b$  is a mere proposition. This is equivalent to saying that every isomorphism in  $A$  is an identity; thus it is rather stronger than the classical notion of “skeletal” category. Categories of this sort are sometimes called **gaunt** [?]. There is not really any notion of “skeletality” for our categories, unless one considers Definition 9.1.6 itself to be such.

**Example 9.1.13.** For any 1-type  $X$ , there is a category with  $X$  as its type of objects and with  $\text{hom}(x, y) := (x = y)$ . If  $X$  is a set, we call this the **discrete category** on  $X$ . In general, we call it a **groupoid** (see ??).

**Example 9.1.14.** For any type  $X$ , there is a precategory with  $X$  as its type of objects and with  $\text{hom}(x, y) := \|x = y\|_0$ . The composition operation

$$\|y = z\|_0 \rightarrow \|x = y\|_0 \rightarrow \|x = z\|_0$$

is defined by induction on truncation from concatenation

$(y = z) \rightarrow (x = y) \rightarrow (x = z)$ . We call this the **fundamental pregroupoid** of  $X$ . (In fact, we have met it already in ??; see also ??.)

**Example 9.1.15.** There is a precategory whose type of objects is  $\mathcal{U}$  and with  $\text{hom}(X, Y) := \|X \rightarrow Y\|_0$ , and composition defined by induction on truncation from ordinary composition  $(Y \rightarrow Z) \rightarrow (X \rightarrow Y) \rightarrow (X \rightarrow Z)$ . We call this the **homotopy precategory of types**.

**Example 9.1.16.** Let  $\text{Rel}$  be the following precategory:

- Its objects are sets.
- $\text{hom}_{\text{Rel}}(X, Y) = X \rightarrow Y \rightarrow \text{Prop}$ .
- For a set  $X$ , we have  $1_X(x, x') := (x = x')$ .
- For  $R : \text{hom}_{\text{Rel}}(X, Y)$  and  $S : \text{hom}_{\text{Rel}}(Y, Z)$ , their composite is defined by

$$(S \circ R)(x, z) := \left\| \sum_{y:Y} R(x, y) \times S(y, z) \right\|.$$

Suppose  $R : \text{hom}_{\text{Rel}}(X, Y)$  is an isomorphism, with inverse  $S$ . We observe the following.

- (i). If  $R(x, y)$  and  $S(y', x)$ , then  $(R \circ S)(y', y)$ , and hence  $y' = y$ . Similarly, if  $R(x, y)$  and  $S(y, x')$ , then  $x = x'$ .
- (ii). For any  $x$ , we have  $x = x$ , hence  $(S \circ R)(x, x)$ . Thus, there merely exists a  $y : Y$  such that  $R(x, y)$  and  $S(y, x)$ .
- (iii). Suppose  $R(x, y)$ . By (ii), there merely exists a  $y'$  with  $R(x, y')$  and  $S(y', x)$ . But then by (i), merely  $y' = y$ , and hence  $y' = y$  since  $Y$  is a set. Therefore, by transporting  $S(y', x)$  along this equality, we have  $S(y, x)$ . In conclusion,  $R(x, y) \rightarrow S(y, x)$ . Similarly,  $S(y, x) \rightarrow R(x, y)$ .
- (iv). If  $R(x, y)$  and  $R(x, y')$ , then by (iii),  $S(y', x)$ , so that by (ii),  $y = y'$ . Thus, for any  $x$  there is at most one  $y$  such that  $R(x, y)$ . And by (ii), there merely exists such a  $y$ , hence there exists such a  $y$ .

In conclusion, if  $R : \text{hom}_{\text{Rel}}(X, Y)$  is an isomorphism, then for each  $x : X$  there is exactly one  $y : Y$  such that  $R(x, y)$ , and dually. Thus, there is a function  $f : X \rightarrow Y$  sending each  $x$  to this  $y$ , which is an equivalence; hence  $X = Y$ . With a little more work, we conclude that  $\text{Rel}$  is a category.

### 9.2 Functors and transformations

**Definition 9.2.1.** Let  $A$  and  $B$  be precategories. A **functor**  $F : A \rightarrow B$  consists of

- (i). A function  $F_0 : A_0 \rightarrow B_0$ , generally also denoted  $F$ .
- (ii). For each  $a, b : A$ , a function  $F_{a,b} : \text{hom}_A(a, b) \rightarrow \text{hom}_B(Fa, Fb)$ , generally also denoted  $F$ .
- (iii). For each  $a : A$ , we have  $F(1_a) = 1_{Fa}$ .
- (iv). For each  $a, b, c : A$  and  $f : \text{hom}_A(a, b)$  and  $g : \text{hom}_A(b, c)$ , we have

$$F(g \circ f) = Fg \circ Ff.$$

**Definition 9.2.2.** For functors  $F, G : A \rightarrow B$ , a **natural transformation**  $\gamma : F \rightarrow G$  consists of

- (i). For each  $a : A$ , a morphism  $\gamma_a : \text{hom}_B(Fa, Ga)$  (the “components”).
- (ii). For each  $a, b : A$  and  $f : \text{hom}_A(a, b)$ , we have  $Gf \circ \gamma_a = \gamma_b \circ Ff$  (the “naturality axiom”).

**Definition 9.2.3.** For precategories  $A, B$ , there is a precategory  $B^A$ , called the **functor precategory**, defined by

- $(B^A)_0$  is the type of functors from  $A$  to  $B$ .
- $\text{hom}_{B^A}(F, G)$  is the type of natural transformations from  $F$  to  $G$ .

**Lemma 9.2.4.** A natural transformation  $\gamma : F \rightarrow G$  is an isomorphism in  $B^A$  if and only if each  $\gamma_a$  is an isomorphism in  $B$ .

**Theorem 9.2.5.** If  $A$  is a precategory and  $B$  is a category, then  $B^A$  is a category.

**Definition 9.2.6.** For functors  $F : A \rightarrow B$  and  $G : B \rightarrow C$ , their composite  $G \circ F : A \rightarrow C$  is given by

- The composite  $(G_0 \circ F_0) : A_0 \rightarrow C_0$
- For each  $a, b : A$ , the composite

$$(G_{Fa, Fb} \circ F_{a,b}) : \text{hom}_A(a, b) \rightarrow \text{hom}_C(GFa, GFb).$$

It is easy to check the axioms.

**Definition 9.2.7.** For functors  $F : A \rightarrow B$  and  $G, H : B \rightarrow C$  and a natural transformation  $\gamma : G \rightarrow H$ , the composite  $(\gamma F) : GF \rightarrow HF$  is given by

- For each  $a : A$ , the component  $\gamma_{Fa}$ .

Naturality is easy to check. Similarly, for  $\gamma$  as above and  $K : C \rightarrow D$ , the composite  $(K\gamma) : KG \rightarrow KH$  is given by

- For each  $b : B$ , the component  $K(\gamma_b)$ .

**Lemma 9.2.8.** For functors  $F, G : A \rightarrow B$  and  $H, K : B \rightarrow C$  and natural transformations  $\gamma : F \rightarrow G$  and  $\delta : H \rightarrow K$ , we have

$$(\delta G)(H\gamma) = (K\gamma)(\delta F).$$

**Lemma 9.2.9.** Composition of functors is associative:  $H(GF) = (HG)F$ .

**Lemma 9.2.10.** Lemma 9.2.9 is coherent, i.e. the following pentagon of equalities commutes:

$$\begin{array}{ccccc} & & K(H(GF)) & & \\ & \swarrow & & \searrow & \\ (KH)(GF) & & & & K((HG)F) \\ \parallel & & & & \parallel \\ ((KH)G)F & \xlongequal{\quad} & & & (K(HG))F \end{array}$$

**Lemma 9.2.11.** For a functor  $F : A \rightarrow B$ , we have equalities  $(1_B \circ F) = F$  and  $(F \circ 1_A) = F$ , such that given also  $G : B \rightarrow C$ , the following triangle of equalities commutes.

$$\begin{array}{ccc} G \circ (1_B \circ F) & \xlongequal{\quad} & (G \circ 1_B) \circ F \\ \parallel & & \parallel \\ G \circ F & \swarrow & \searrow \end{array}$$

## 9.3 Adjunctions

**Definition 9.3.1.** A functor  $F : A \rightarrow B$  is a **left adjoint** if there exists

- A functor  $G : B \rightarrow A$ .
- A natural transformation  $\eta : 1_A \rightarrow GF$  (the **unit**).
- A natural transformation  $\epsilon : FG \rightarrow 1_B$  (the **counit**).
- $(\epsilon F)(F\eta) = 1_F$ .
- $(G\epsilon)(\eta G) = 1_G$ .

**Lemma 9.3.2.** If  $A$  is a category (but  $B$  may be only a precategory), then the type “ $F$  is a left adjoint” is a mere proposition.

## 9.4 Equivalences

**Definition 9.4.1.** A functor  $F : A \rightarrow B$  is an **equivalence of (pre)categories** if it is a left adjoint for which  $\eta$  and  $\epsilon$  are isomorphisms. We write  $A \simeq B$  for the type of equivalences of categories from  $A$  to  $B$ .

**Lemma 9.4.2.** If for  $F : A \rightarrow B$  there exists  $G : B \rightarrow A$  and isomorphisms  $GF \cong 1_A$  and  $FG \cong 1_B$ , then  $F$  is an equivalence of precategories.

**Definition 9.4.3.** We say a functor  $F : A \rightarrow B$  is **faithful** if for all  $a, b : A$ , the function

$$F_{a,b} : \text{hom}_A(a, b) \rightarrow \text{hom}_B(Fa, Fb)$$

is injective, and **full** if for all  $a, b : A$  this function is surjective. If it is both (hence each  $F_{a,b}$  is an equivalence) we say  $F$  is **fully faithful**.

**Definition 9.4.4.** We say a functor  $F : A \rightarrow B$  is **split essentially surjective** if for all  $b : B$  there exists an  $a : A$  such that  $Fa \cong b$ .

**Lemma 9.4.5.** For any precategories  $A$  and  $B$  and functor  $F : A \rightarrow B$ , the following types are equivalent.

- (i).  $F$  is an equivalence of precategories.
- (ii).  $F$  is fully faithful and split essentially surjective.

**Definition 9.4.6.** A functor  $F : A \rightarrow B$  is **essentially surjective** if for all  $b : B$ , there merely exists an  $a : A$  such that  $Fa \cong b$ . We say  $F$  is a **weak equivalence** if it is fully faithful and essentially surjective.

**Lemma 9.4.7.** If  $F : A \rightarrow B$  is fully faithful and  $A$  is a category, then for any  $b : B$  the type  $\sum_{(a:A)}(Fa \cong b)$  is a mere proposition. Hence a functor between categories is an equivalence if and only if it is a weak equivalence.

**Definition 9.4.8.** A functor  $F : A \rightarrow B$  is an **isomorphism of (pre)categories** if  $F$  is fully faithful and  $F_0 : A_0 \rightarrow B_0$  is an equivalence of types.

**Lemma 9.4.9.** For precategories  $A$  and  $B$  and  $F : A \rightarrow B$ , the following are equivalent.

- (i).  $F$  is an isomorphism of precategories.
- (ii). There exist  $G : B \rightarrow A$  and  $\eta : 1_A = GF$  and  $\epsilon : FG = 1_B$  such that

$$\text{ap}_{(\Lambda H.FH)}(\eta) = \text{ap}_{(\Lambda K.KF)}(\epsilon^{-1}). \quad (9.4.10)$$

- (iii). There merely exist  $G : B \rightarrow A$  and  $\eta : 1_A = GF$  and  $\epsilon : FG = 1_B$ .

*Proof.* First note that since hom-sets are sets, equalities between equalities of functors are uniquely determined by their object-parts. Thus, by function extensionality, (9.4.10) is equivalent to

$$(F_0)(\eta_0)_a = (\epsilon_0)^{-1}_{Fa}. \quad (9.4.11)$$

for all  $a : A_0$ . Note that this is precisely the triangle identity for  $G_0, \eta_0$ , and  $\epsilon_0$  to be a proof that  $F_0$  is a half adjoint equivalence of types. Now suppose (i). Let  $G_0 : B_0 \rightarrow A_0$  be the inverse of  $F_0$ , with  $\eta_0 : \text{id}_{A_0} = G_0F_0$  and  $\epsilon_0 : F_0G_0 = \text{id}_{B_0}$  satisfying the triangle identity,

which is precisely (9.4.11). Now define  $G_{b,b'} : \text{hom}_B(b, b') \rightarrow \text{hom}_A(G_0b, G_0b')$  by

$$G_{b,b'}(g) := (F_{G_0b, G_0b'})^{-1} \left( \text{idtoiso}((\epsilon_0)^{-1}_{Fa'}) \circ g \circ \text{idtoiso}((\epsilon_0)_b) \right)$$

(using the assumption that  $F$  is fully faithful). Since  $\text{idtoiso}$  takes inverses to inverses and concatenation to composition, and  $F$  is a functor, it follows that  $G$  is a functor.

By definition, we have  $(GF)_0 \equiv G_0F_0$ , which is equal to  $\text{id}_{A_0}$  by  $\eta_0$ . To obtain  $1_A = GF$ , we need to show that when transported along  $\eta_0$ , the identity function of  $\text{hom}_A(a, a')$  becomes equal to the composite  $G_{Fa,Fa'} \circ F_{a,a'}$ . In other words, for any  $f : \text{hom}_A(a, a')$  we must have

$$\begin{aligned} & \text{idtoiso}((\eta_0)_{a'}) \circ f \circ \text{idtoiso}((\eta_0)^{-1}_a) \\ &= (F_{GFa, GFa'})^{-1} \left( \text{idtoiso}((\epsilon_0)^{-1}_{Fa'}) \circ F_{a,a'}(f) \circ \text{idtoiso}((\epsilon_0)_{Fa}) \right). \end{aligned}$$

But this is equivalent to

$$\begin{aligned} & (F_{GFa, GFa'}) \left( \text{idtoiso}((\eta_0)_{a'}) \circ f \circ \text{idtoiso}((\eta_0)^{-1}_a) \right) \\ &= \text{idtoiso}((\epsilon_0)^{-1}_{Fa'}) \circ F_{a,a'}(f) \circ \text{idtoiso}((\epsilon_0)_{Fa}). \end{aligned}$$

which follows from functoriality of  $F$ , the fact that  $F$  preserves  $\text{idtoiso}$ , and (9.4.11). Thus we have  $\eta : 1_A = GF$ .

On the other side, we have  $(FG)_0 \equiv F_0G_0$ , which is equal to  $\text{id}_{B_0}$  by  $\epsilon_0$ . To obtain  $FG = 1_B$ , we need to show that when transported along  $\epsilon_0$ , the identity function of  $\text{hom}_B(b, b')$  becomes equal to the composite  $F_{Gb,Gb'} \circ G_{b,b'}$ . That is, for any  $g : \text{hom}_B(b, b')$  we must have

$$\begin{aligned} & F_{Gb,Gb'} \left( (F_{Gb,Gb'})^{-1} \left( \text{idtoiso}((\epsilon_0)^{-1}_{b'}) \circ g \circ \text{idtoiso}((\epsilon_0)_b) \right) \right) \\ &= \text{idtoiso}((\epsilon_0)^{-1}_{b'}) \circ g \circ \text{idtoiso}((\epsilon_0)_b). \end{aligned}$$

But this is just the fact that  $(F_{Gb,Gb'})^{-1}$  is the inverse of  $F_{Gb,Gb'}$ . And we have remarked that (9.4.10) is equivalent to (9.4.11), so (ii) holds.

Conversely, suppose given (ii); then the object-parts of  $G$ ,  $\eta$ , and  $\epsilon$  together with (9.4.11) show that  $F_0$  is an equivalence of types. And for  $a, a' : A_0$ , we define  $\bar{G}_{a,a'} : \text{hom}_B(Fa, Fa') \rightarrow \text{hom}_A(a, a')$  by

$$\bar{G}_{a,a'}(g) := \text{idtoiso}(\eta^{-1})_{a'} \circ G(g) \circ \text{idtoiso}(\eta)_a. \quad (9.4.12)$$

By naturality of  $\text{idtoiso}(\eta)$ , for any  $f : \text{hom}_A(a, a')$  we have

$$\begin{aligned} \bar{G}_{a,a'}(F_{a,a'}(f)) &= \text{idtoiso}(\eta^{-1})_{a'} \circ G(F(f)) \circ \text{idtoiso}(\eta)_a \\ &= \text{idtoiso}(\eta^{-1})_{a'} \circ \text{idtoiso}(\eta)_{a'} \circ f \\ &= f. \end{aligned}$$

On the other hand, for  $g : \text{hom}_B(Fa, Fa')$  we have

$$\begin{aligned} F_{a,a'}(\bar{G}_{a,a'}(g)) &= F(\text{idtoiso}(\eta^{-1})_{a'} \circ G(g) \circ \text{idtoiso}(\eta)_a) \\ &= \text{idtoiso}(\epsilon)_{Fa'} \circ F(G(g)) \circ \text{idtoiso}(\epsilon^{-1})_{Fa} \\ &= \text{idtoiso}(\epsilon)_{Fa'} \circ \text{idtoiso}(\epsilon^{-1})_{Fa'} \circ g \\ &= g. \end{aligned}$$

(There are lemmas needed here regarding the compatibility of  $\text{idtoiso}$  and whiskering, which we leave it to the reader to state and prove.) Thus,  $F_{a,a'}$  is an equivalence, so  $F$  is fully faithful; i.e. (i) holds. Now the composite (i)  $\rightarrow$  (ii)  $\rightarrow$  (i) is equal to the identity since (i) is a mere proposition. On the other side, tracing through the above constructions we see that the composite (ii)  $\rightarrow$  (i)  $\rightarrow$  (ii) essentially preserves the object-parts  $G_0, \eta_0, \epsilon_0$ , and the object-part of (9.4.10). And in the latter three cases, the object-part is all there is, since hom-sets are sets. Thus, it suffices to show that we recover the action of  $G$  on hom-sets. In other words, we must show that if  $g : \text{hom}_B(b, b')$ , then

$$G_{b,b'}(g) = \overline{G}_{G_0 b, G_0 b'}(\text{idtoiso}((\epsilon_0)^{-1} b') \circ g \circ \text{idtoiso}((\epsilon_0)_b))$$

where  $\overline{G}$  is defined by (9.4.12). However, this follows from functoriality of  $G$  and the other triangle identity, which we have seen in ?? is equivalent to (9.4.11).

Now since (i) is a mere proposition, so is (ii), so it suffices to show they are logically equivalent to (iii). Of course, (ii)  $\rightarrow$  (iii), so let us assume (iii). Since (i) is a mere proposition, we may assume given  $G, \eta$ , and  $\epsilon$ . Then  $G_0$  along with  $\eta$  and  $\epsilon$  imply that  $F_0$  is an equivalence. Moreover, we also have natural isomorphisms  $\text{idtoiso}(\eta) : 1_A \cong GF$  and  $\text{idtoiso}(\epsilon) : FG \cong 1_B$ , so by Lemma 9.4.2,  $F$  is an equivalence of precategories, and in particular fully faithful.  $\square$

**Example 9.4.13.** Let  $X$  be a type and  $x_0 : X$  an element, and let  $X_{\text{ch}}$  denote the *chaotic* or *indiscrete* precategory on  $X$ . By definition, we have  $(X_{\text{ch}})_0 := X$ , and  $\text{hom}_{X_{\text{ch}}}(x, x') := \mathbf{1}$  for all  $x, x'$ . Then the unique functor  $X_{\text{ch}} \rightarrow \mathbf{1}$  is an equivalence of precategories, but not an isomorphism unless  $X$  is contractible.

This example also shows that a precategory can be equivalent to a category without itself being a category. Of course, if a precategory is *isomorphic* to a category, then it must itself be a category.

**Lemma 9.4.14.** For categories  $A$  and  $B$ , a functor  $F : A \rightarrow B$  is an equivalence of categories if and only if it is an isomorphism of categories.

**Lemma 9.4.15.** If  $A$  and  $B$  are precategories, then the function

$$(A = B) \rightarrow (A \cong B)$$

(defined by induction from the identity functor) is an equivalence of types.

**Theorem 9.4.16.** If  $A$  and  $B$  are categories, then the function

$$(A = B) \rightarrow (A \simeq B)$$

(defined by induction from the identity functor) is an equivalence of types.

## 9.5 The Yoneda lemma

**Definition 9.5.1.** For a precategory  $A$ , its **opposite**  $A^{\text{op}}$  is a precategory with the same type of objects, with  $\text{hom}_{A^{\text{op}}}(a, b) := \text{hom}_A(b, a)$ , and with identities and composition inherited from  $A$ .

**Definition 9.5.2.** For precategories  $A$  and  $B$ , their **product**  $A \times B$  is a precategory with  $(A \times B)_0 := A_0 \times B_0$  and

$$\text{hom}_{A \times B}((a, b), (a', b')) := \text{hom}_A(a, a') \times \text{hom}_B(b, b').$$

Identities are defined by  $1_{(a,b)} := (1_a, 1_b)$  and composition by  $(g, g')(f, f') := ((gf), (g'f'))$ .

**Lemma 9.5.3.** For precategories  $A, B, C$ , the following types are equivalent.

- (i). Functors  $A \times B \rightarrow C$ .
- (ii). Functors  $A \rightarrow C^B$ .

Now for any precategory  $A$ , we have a hom-functor

$$\text{hom}_A : A^{\text{op}} \times A \rightarrow \text{Set}.$$

It takes a pair  $(a, b) : (A^{\text{op}})_0 \times A_0 \equiv A_0 \times A_0$  to the set  $\text{hom}_A(a, b)$ . For a morphism  $(f, f') : \text{hom}_{A^{\text{op}} \times A}((a, b), (a', b'))$ , by definition we have  $f : \text{hom}_A(a', a)$  and  $f' : \text{hom}_A(b, b')$ , so we can define

$$\begin{aligned} (\text{hom}_A)_{(a,b),(a',b')}(f, f') &:= (g \mapsto (f'gf)) \\ &: \text{hom}_A(a, b) \rightarrow \text{hom}_A(a', b'). \end{aligned}$$

Functoriality is easy to check.

**Theorem 9.5.4** (The Yoneda lemma). For any precategory  $A$ , any  $a : A$ , and any functor  $F : \text{Set}^{A^{\text{op}}}$ , we have an isomorphism

$$\text{hom}_{\text{Set}^{A^{\text{op}}}}(ya, F) \cong Fa. \quad (9.5.5)$$

Moreover, this is natural in both  $a$  and  $F$ .

**Corollary 9.5.6.** The Yoneda embedding  $y : A \rightarrow \text{Set}^{A^{\text{op}}}$  is fully faithful.

**Corollary 9.5.7.** If  $A$  is a category, then  $y_0 : A_0 \rightarrow (\text{Set}^{A^{\text{op}}})_0$  is an embedding. In particular, if  $ya = yb$ , then  $a = b$ .

**Definition 9.5.8.** A functor  $F : \text{Set}^{A^{\text{op}}}$  is said to be **representable** if there exists  $a : A$  and an isomorphism  $ya \cong F$ .

**Theorem 9.5.9.** If  $A$  is a category, then the type “ $F$  is representable” is a mere proposition.

**Lemma 9.5.10.** For any precategories  $A$  and  $B$  and a functor  $F : A \rightarrow B$ , the following types are equivalent.

- (i).  $F$  is a left adjoint.
- (ii). For each  $b : B$ , the functor  $(a \mapsto \text{hom}_B(Fa, b))$  from  $A^{\text{op}}$  to  $\text{Set}$  is representable.

**Corollary 9.5.11.** [Lemma 9.3.2] If  $A$  is a category and  $F : A \rightarrow B$ , then the type “ $F$  is a left adjoint” is a mere proposition.

## 9.6 Strict categories

**Definition 9.6.1.** A **strict category** is a precategory whose type of objects is a set.

**Example 9.6.2.** Let  $A$  be a precategory and  $x : A$  an object. Then there is a precategory  $\text{mono}(A, x)$  as follows:

- Its objects consist of an object  $y : A$  and a monomorphism  $m : \text{hom}_A(y, x)$ . (As usual,  $m : \text{hom}_A(y, x)$  is a **monomorphism** (or is **monic**) if  $(m \circ f = m \circ g) \Rightarrow (f = g)$ .)
- Its morphisms from  $(y, m)$  to  $(z, n)$  are arbitrary morphisms from  $y$  to  $z$  in  $A$  (not necessarily respecting  $m$  and  $n$ ).

An equality  $(y, m) = (z, n)$  of objects in  $\text{mono}(A, x)$  consists of an equality  $p : y = z$  and an equality  $p_*(m) = n$ , which by Lemma 9.1.9 is equivalently an equality  $m = n \circ \text{idtoiso}(p)$ . Since hom-sets are sets, the type of such equalities is a mere proposition. But since  $m$  and  $n$  are monomorphisms, the type of morphisms  $f$  such that  $m = n \circ f$  is also a mere proposition. Thus, if  $A$  is a category, then  $(y, m) = (z, n)$  is a mere proposition, and hence  $\text{mono}(A, x)$  is a strict category.

**Example 9.6.3.** Let  $E/F$  be a finite Galois extension of fields, and  $G$  its Galois group. Then there is a strict category whose objects are intermediate fields  $F \subseteq K \subseteq E$ , and whose morphisms are field homomorphisms which fix  $F$  pointwise (but need not commute with the inclusions into  $E$ ). There is another strict category whose objects are subgroups  $H \subseteq G$ , and whose morphisms are morphisms of  $G$ -sets  $G/H \rightarrow G/K$ . The fundamental theorem of Galois theory says that these two precategories are isomorphic (not merely equivalent).

## 9.7 $\dagger$ -categories

**Definition 9.7.1.** A  **$\dagger$ -precategory** is a precategory  $A$  together with the following.

- (i). For each  $x, y : A$ , a function  $(-)^\dagger : \text{hom}_A(x, y) \rightarrow \text{hom}_A(y, x)$ .
- (ii). For all  $x : A$ , we have  $(1_x)^\dagger = 1_x$ .
- (iii). For all  $f, g$  we have  $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$ .
- (iv). For all  $f$  we have  $(f^\dagger)^\dagger = f$ .

**Definition 9.7.2.** A morphism  $f : \text{hom}_A(x, y)$  in a  $\dagger$ -precategory is **unitary** if  $f^\dagger \circ f = 1_x$  and  $f \circ f^\dagger = 1_y$ .

**Lemma 9.7.3.** If  $p : (x = y)$ , then  $\text{idtoiso}(p)$  is unitary.

**Definition 9.7.4.** A  **$\dagger$ -category** is a  $\dagger$ -precategory such that for all  $x, y : A$ , the function

$$(x = y) \rightarrow (x \cong^\dagger y)$$

from Lemma 9.7.3 is an equivalence.

**Example 9.7.5.** The category  $\mathcal{R}\ell$  from Example 9.1.16 becomes a  $\dagger$ -precategory if we define  $(R^\dagger)(y, x) := R(x, y)$ . The proof that  $\mathcal{R}\ell$  is a category actually shows that every isomorphism is unitary; hence  $\mathcal{R}\ell$  is also a  $\dagger$ -category.

**Example 9.7.6.** Any groupoid becomes a  $\mathbb{t}$ -category if we define  $f^\dagger := f^{-1}$ .

**Example 9.7.7.** Let  $\mathcal{Hilb}$  be the following precategory.

- Its objects are finite-dimensional vector spaces equipped with an inner product  $\langle -, - \rangle$ .
- Its morphisms are arbitrary linear maps.

By standard linear algebra, any linear map  $f : V \rightarrow W$  between finite dimensional inner product spaces has a uniquely defined adjoint  $f^\dagger : W \rightarrow V$ , characterized by  $\langle fv, w \rangle = \langle v, f^\dagger w \rangle$ . In this way,  $\mathcal{Hilb}$  becomes a  $\mathbb{t}$ -precategory. Moreover, a linear isomorphism is unitary precisely when it is an **isometry**, i.e.  $\langle fv, fw \rangle = \langle v, w \rangle$ . It follows from this that  $\mathcal{Hilb}$  is a  $\mathbb{t}$ -category, though it is not a category (not every linear isomorphism is unitary).

## 9.8 The structure identity principle

**Definition 9.8.1.** A notion of structure  $(P, H)$  over  $X$  consists of the following.

- A type family  $P : X_0 \rightarrow \mathcal{U}$ . For each  $x : X_0$  the elements of  $Px$  are called  **$(P, H)$ -structures** on  $x$ .
- For  $x, y : X_0$  and  $\alpha : Px$ ,  $\beta : Py$ , to each  $f : \text{hom}_X(x, y)$  a mere proposition
 
$$H_{\alpha\beta}(f).$$
 If  $H_{\alpha\beta}(f)$  is true, we say that  $f$  is a  **$(P, H)$ -homomorphism** from  $\alpha$  to  $\beta$ .
- For  $x : X_0$  and  $\alpha : Px$ , we have  $H_{\alpha\alpha}(1_x)$ .
- For  $x, y, z : X_0$  and  $\alpha : Px$ ,  $\beta : Py$ ,  $\gamma : Pz$ , if  $f : \text{hom}_X(x, y)$  and  $g : \text{hom}_X(y, z)$ , we have
 
$$H_{\alpha\beta}(f) \rightarrow H_{\beta\gamma}(g) \rightarrow H_{\alpha\gamma}(g \circ f).$$

When  $(P, H)$  is a notion of structure, for  $\alpha, \beta : Px$  we define

$$(\alpha \leq_x \beta) := H_{\alpha\beta}(1_x).$$

By (iii) and (iv), this is a preorder (Example 9.1.11) with  $Px$  its type of objects. We say that  $(P, H)$  is a **standard notion of structure** if this preorder is in fact a partial order, for all  $x : X$ .

**Theorem 9.8.2** (Structure identity principle). *If  $X$  is a category and  $(P, H)$  is a standard notion of structure over  $X$ , then the precategory  $\text{Str}_{(P,H)}(X)$  is a category.*

**Example 9.8.3.** Let  $A$  be a precategory and  $B$  a category. There is a precategory  $B^{A_0}$  whose objects are functions  $A_0 \rightarrow B_0$ , and whose set of morphisms from  $F_0 : A_0 \rightarrow B_0$  to  $G_0 : A_0 \rightarrow B_0$  is  $\prod_{(a:A_0)} \text{hom}_B(F_0a, G_0a)$ . Composition and identities are inherited directly from those in  $B$ . It is easy to show that  $\gamma : \text{hom}_{B^{A_0}}(F_0, G_0)$  is an isomorphism exactly when each component  $\gamma_a$  is an isomorphism, so that we have  $(F_0 \cong G_0) \simeq \prod_{(a:A_0)} (F_0a \cong G_0a)$ . Moreover, the map  $\text{idtoiso} : (F_0 = G_0) \rightarrow (F_0 \cong G_0)$  of  $B^{A_0}$  is equal to the composite

$$(F_0 = G_0) \longrightarrow \prod_{a:A_0} (F_0a = G_0a) \longrightarrow \prod_{a:A_0} (F_0a \cong G_0a) \longrightarrow (F_0 \cong G_0)$$

in which the first map is an equivalence by function extensionality, the second because it is a dependent product of equivalences (since  $B$  is a category), and the third as remarked above. Thus,  $B^{A_0}$  is a category.

Now we define a notion of structure on  $B^{A_0}$  for which  $P(F_0)$  is the type of operations  $F : \prod_{(a,a':A_0)} \text{hom}_A(a, a') \rightarrow \text{hom}_B(F_0a, F_0a')$  which extend  $F_0$  to a functor (i.e. preserve composition and identities). This is a set since each  $\text{hom}_B(-, -)$  is so. Given such  $F$  and  $G$ , we define  $\gamma : \text{hom}_{B^{A_0}}(F_0, G_0)$  to be a homomorphism if it forms a natural transformation. In Definition 9.2.3 we essentially verified that this is a notion of structure. Moreover, if  $F$  and  $F'$  are both structures on  $F_0$  and the identity is a natural transformation from  $F$  to  $F'$ , then for any  $f : \text{hom}_A(a, a')$  we have  $F'f = F'f \circ 1_{F_0a} = 1_{F_0a} \circ Ff = Ff$ . Applying function extensionality, we conclude  $F = F'$ . Thus, we have a **standard notion of structure**, and so by Theorem 9.8.2, the precategory  $B^A$  is a category.

**Definition 9.8.4.**

- For each  $\mathcal{U}$ -small set  $x$  define

$$Px := P_0x \times P_1x.$$

Here

$$\begin{aligned} P_0x &:= \prod_{\omega:\Omega_0} x^{|\omega|} \rightarrow x, \text{ and} \\ P_1x &:= \prod_{\omega:\Omega_1} x^{|\omega|} \rightarrow \text{Prop}_{\mathcal{U}}, \end{aligned}$$

- For  $\mathcal{U}$ -small sets  $x, y$  and  $\alpha : P^\omega x$ ,  $\beta : P^\omega y$ ,  $f : x \rightarrow y$ , define

$$H_{\alpha\beta}(f) := H_{0,\alpha\beta}(f) \wedge H_{1,\alpha\beta}(f).$$

Here

$$\begin{aligned} H_{0,\alpha\beta}(f) &:= \forall(\omega : \Omega_0). \forall(u : x^{|\omega|}). f(\alpha u) = \beta(f \circ u), \text{ and} \\ H_{1,\alpha\beta}(f) &:= \forall(\omega : \Omega_1). \forall(u : x^{|\omega|}). \alpha u \rightarrow \beta(f \circ u). \end{aligned}$$

## 9.9 The Rezk completion

**Lemma 9.9.1.** *If  $A, B, C$  are precategories and  $H : A \rightarrow B$  is an essentially surjective functor, then  $(- \circ H) : C^B \rightarrow C^A$  is faithful.*

**Lemma 9.9.2.** *If  $A, B, C$  are precategories and  $H : A \rightarrow B$  is essentially surjective and full, then  $(- \circ H) : C^B \rightarrow C^A$  is fully faithful.*

**Theorem 9.9.3.** *If  $A, B$  are precategories,  $C$  is a category, and  $H : A \rightarrow B$  is a weak equivalence, then  $(- \circ H) : C^B \rightarrow C^A$  is an isomorphism.*

Therefore, if a precategory  $A$  admits a weak equivalence functor  $A \rightarrow \widehat{A}$  into a category, then that is its “reflection” into categories: any functor from  $A$  into a category will factor essentially uniquely through  $\widehat{A}$ . We now give two constructions of such a weak equivalence.

**Theorem 9.9.4.** *For any precategory  $A$ , there is a category  $\widehat{A}$  and a weak equivalence  $A \rightarrow \widehat{A}$ .*

**Example 9.9.5.** Recall from Example 9.1.14 that for any type  $X$  there is a pregroupoid with  $X$  as its type of objects and  $\text{hom}(x, y) := \|x = y\|_0$ . Its Rezk completion is the **fundamental groupoid** of  $X$ . Recalling that groupoids are equivalent to 1-types, it is not hard to identify this groupoid with  $\|X\|_1$ .

**Example 9.9.6.** Recall from Example 9.1.15 that there is a precategory whose type of objects is  $\mathcal{U}$  and with  $\text{hom}(X, Y) := \|X \rightarrow Y\|_0$ . Its Rezk completion may be called the **homotopy category of types**. Its type of objects can be identified with  $\|\mathcal{U}\|_1$  (see ??).

**Theorem 9.9.7.** *A precategory  $C$  is a category if and only if for every weak equivalence of precategories  $H : A \rightarrow B$ , the induced functor  $(- \circ H) : C^B \rightarrow C^A$  is an isomorphism of precategories.*