Homotopy Type Theory

Equivalences

Recall that we wanted $\mathsf{isequiv}(f)$ to have the following properties, which we restate here:

(i) $\mathsf{qinv}(f) \to \mathsf{isequiv}(f)$.
(ii) $\mathsf{isequiv}(f) \to \mathsf{qinv}(f)$.
(iii) $\mathsf{isequiv}(f)$ is a mere proposition.

Here $\mathsf{qinv}(f)$ denotes the type of quasi-inverses to f:

$$\sum_{g:B\to A} \big((f \circ g \sim \mathsf{id}_B) \times (g \circ f \sim \mathsf{id}_A) \big).$$

By function extensionality, it follows that $\mathsf{qinv}(f)$ is equivalent to the type

$$\sum_{g:B\to A} \big((f \circ g = \mathsf{id}_B) \times (g \circ f = \mathsf{id}_A) \big).$$

We will define three different types having properties (i)–(iii), which we call

- half adjoint equivalences,
- bi-invertible maps, and
- contractible functions.

4.1 Quasi-inverses

Lemma 4.1.1. *If $f : A \to B$ is such that $\mathsf{qinv}(f)$ is inhabited, then*

$$\mathsf{qinv}(f) \simeq \Big(\prod_{x:A} (x = x) \Big).$$

Lemma 4.1.2. *Suppose we have a type A with $a : A$ and $q : a = a$ such that*

(i) The type $a = a$ is a set.
(ii) For all $x : A$ we have $\|a = x\|$.
(iii) For all $p : a = a$ we have $p \cdot q = q \cdot p$.

Then there exists $f : \prod_{(x:A)} (x = x)$ with $f(a) = q$.

Theorem 4.1.3. *There exist types A and B and a function $f : A \to B$ such that $\mathsf{qinv}(f)$ is not a mere proposition.*

4.2 Half adjoint equivalences

Definition 4.2.1. A function $f : A \to B$ is a **half adjoint equivalence** if there are $g : B \to A$ and homotopies $\eta : g \circ f \sim \mathsf{id}_A$ and $\epsilon : f \circ g \sim \mathsf{id}_B$ such that there exists a homotopy

$$\tau : \prod_{x:A} f(\eta x) = \epsilon(fx).$$

Lemma 4.2.2. *For functions $f : A \to B$ and $g : B \to A$ and homotopies $\eta : g \circ f \sim \mathsf{id}_A$ and $\epsilon : f \circ g \sim \mathsf{id}_B$, the following conditions are logically equivalent:*

- $\prod_{(x:A)} f(\eta x) = \epsilon(fx)$
- $\prod_{(y:B)} g(\epsilon y) = \eta(gy)$

Theorem 4.2.3. *For any $f : A \to B$ we have $\mathsf{qinv}(f) \to \mathsf{ishae}(f)$.*

Definition 4.2.4. The **fiber** of a map $f : A \to B$ over a point $y : B$ is

$$\mathsf{fib}_f(y) :\equiv \sum_{x:A} (f(x) = y).$$

In homotopy theory, this is what would be called the *homotopy fiber* of f. The path lemmas in **??** yield the following characterization of paths in fibers:

Lemma 4.2.5. *For any $f : A \to B$, $y : B$, and $(x, p), (x', p') : \mathsf{fib}_f(y)$, we have*

$$\big((x, p) = (x', p') \big) \simeq \Big(\sum_{\gamma:x=x'} f(\gamma) \cdot p' = p \Big)$$

Theorem 4.2.6. *If $f : A \to B$ is a half adjoint equivalence, then for any $y : B$ the fiber $\mathsf{fib}_f(y)$ is contractible.*

Definition 4.2.7. Given a function $f : A \to B$, we define the types

$$\mathsf{linv}(f) :\equiv \sum_{g:B\to A} (g \circ f \sim \mathsf{id}_A)$$

$$\mathsf{rinv}(f) :\equiv \sum_{g:B\to A} (f \circ g \sim \mathsf{id}_B)$$

of **left inverses** and **right inverses** to f, respectively. We call f **left invertible** if $\mathsf{linv}(f)$ is inhabited, and similarly **right invertible** if $\mathsf{rinv}(f)$ is inhabited.

Lemma 4.2.8. *If $f : A \to B$ has a quasi-inverse, then so do*

$$(f \circ -) : (C \to A) \to (C \to B)$$
$$(- \circ f) : (B \to C) \to (A \to C).$$

Lemma 4.2.9. *If $f : A \to B$ has a quasi-inverse, then the types $\mathsf{rinv}(f)$ and $\mathsf{linv}(f)$ are contractible.*

Definition 4.2.10. For $f : A \to B$, a left inverse $(g, \eta) : \mathsf{linv}(f)$, and a right inverse $(g, \epsilon) : \mathsf{rinv}(f)$, we denote

$$\mathsf{lcoh}_f(g, \eta) :\equiv \sum_{(\epsilon:f\circ g\sim\mathsf{id}_B)} \prod_{(y:B)} g(\epsilon y) = \eta(gy),$$

$$\mathsf{rcoh}_f(g, \epsilon) :\equiv \sum_{(\eta:g\circ f\sim\mathsf{id}_A)} \prod_{(x:A)} f(\eta x) = \epsilon(fx).$$

Lemma 4.2.11. *For any f, g, ϵ, η, we have*

$$\mathsf{lcoh}_f(g, \eta) \simeq \prod_{y:B} (fgy, \eta(gy)) =_{\mathsf{fib}_g(gy)} (y, \mathsf{refl}_{gy}),$$

$$\mathsf{rcoh}_f(g, \epsilon) \simeq \prod_{x:A} (gfx, \epsilon(fx)) =_{\mathsf{fib}_f(fx)} (x, \mathsf{refl}_{fx}).$$

Lemma 4.2.12. *If f is a half adjoint equivalence, then for any $(g, \epsilon) : \mathsf{rinv}(f)$, the type $\mathsf{rcoh}_f(g, \epsilon)$ is contractible.*

Theorem 4.2.13. *For any $f : A \to B$, the type $\mathsf{ishae}(f)$ is a mere proposition.*

4.3 Bi-invertible maps

Definition 4.3.1. We say $f : A \to B$ is **bi-invertible** if it has both a left inverse and a right inverse:

$$\mathsf{biinv}(f) :\equiv \mathsf{linv}(f) \times \mathsf{rinv}(f).$$

Theorem 4.3.2. *For any $f : A \to B$, the type $\mathsf{biinv}(f)$ is a mere proposition.*

Corollary 4.3.3. *For any $f : A \to B$ we have $\mathsf{biinv}(f) \simeq \mathsf{ishae}(f)$.*

4.4 Contractible fibers

Definition 4.4.1 (Contractible maps). A map $f : A \to B$ is **contractible** if for all $y : B$, the fiber $\mathsf{fib}_f(y)$ is contractible.

Theorem 4.4.2. *For any $f : A \to B$ we have $\mathsf{isContr}(f) \to \mathsf{ishae}(f)$.*

Lemma 4.4.3. *For any f, the type $\mathsf{isContr}(f)$ is a mere proposition.*

Theorem 4.4.4. *For any $f : A \to B$ we have $\mathsf{isContr}(f) \simeq \mathsf{ishae}(f)$.*

Corollary 4.4.5. *If $f : A \to B$ is such that $B \to \mathsf{isequiv}(f)$, then f is an equivalence.*

4.5 On the definition of equivalences

We have shown that all three definitions of equivalence satisfy the three desirable properties and are pairwise equivalent:

$$\mathsf{isContr}(f) \simeq \mathsf{ishae}(f) \simeq \mathsf{biinv}(f).$$

4.6 Surjections and embeddings

When A and B are sets and $f : A \to B$ is an equivalence, we also call it as **isomorphism** or a **bijection**. (We avoid these words for types that are not sets, since in homotopy theory and higher category theory they often denote a stricter notion of "sameness" than homotopy equivalence.) In set theory, a function is a bijection just when it is both injective and surjective. The same is true in type theory, if we formulate these conditions appropriately. For clarity, when dealing with types that are not sets, we will speak of *embeddings* instead of injections.

Definition 4.6.1. Let $f : A \to B$.

(i) We say f is **surjective** (or a **surjection**) if for every $b : B$ we have $\|\mathsf{fib}_f(b)\|$.
(ii) We say f is an **embedding** if for every $x, y : A$ the function $\mathsf{ap}_f : (x =_A y) \to (f(x) =_B f(y))$ is an equivalence.

In other words, f is surjective if every fiber of f is merely inhabited, or equivalently if for all $b : B$ there merely exists an $a : A$ such that $f(a) = b$. In traditional logical notation, f is surjective if $\forall(b : B). \exists(a : A). (f(a) = b)$. This must be distinguished from the stronger assertion that $\prod_{(b:B)} \sum_{(a:A)} (f(a) = b)$; if this holds we say that f is a **split surjection**. (Since this latter type is equivalent to $\sum_{(g:B\to A)} \prod_{(b:B)} (f(g(b)) = b)$, being a split surjection is the same as being a *retraction* as defined in **??**.)

The axiom of choice from **??** says exactly that every surjection *between sets* is split. However, in the presence of the univalence axiom, it is simply false that *all* surjections are split. In **??** we constructed a type family $Y : X \to \mathcal{U}$ such that $\prod_{(x:X)} \|Y(x)\|$ but $\neg \prod_{(x:X)} Y(x)$; for any such family, the first projection $(\sum_{(x:X)} Y(x)) \to X$ is a surjection that is not split.

If A and B are sets, then by **??**, f is an embedding just when

$$\prod_{x,y:A} (f(x) =_B f(y)) \to (x =_A y). \tag{4.6.2}$$

In this case we say that f is **injective**, or an **injection**. We avoid these word for types that are not sets, because they might be interpreted as (4.6.2), which is an ill-behaved notion for non-sets. It is also true that any function between sets is surjective if and only if it is an *epimorphism* in a suitable sense, but this also fails for more general types, and surjectivity is generally the more important notion.

Theorem 4.6.3. *A function $f : A \to B$ is an equivalence if and only if it is both surjective and an embedding.*

Corollary 4.6.4. *For any $f : A \to B$ we have*

$$\mathsf{isequiv}(f) \simeq (\mathsf{isEmbedding}(f) \times \mathsf{isSurjective}(f)).$$

4.7 Closure properties of equivalences

Theorem 4.7.1 (The 2-out-of-3 property)**.** *Suppose $f : A \to B$ and $g : B \to C$. If any two of f, g, and $g \circ f$ are equivalences, so is the third.*

Definition 4.7.2. *A function $g : A \to B$ is said to be a **retract** of a function $f : X \to Y$ if there is a diagram*

$$\begin{array}{ccccc}
A & \xrightarrow{s} & X & \xrightarrow{r} & A \\
\downarrow g & & \downarrow f & & \downarrow g \\
B & \xrightarrow{s'} & Y & \xrightarrow{r'} & B
\end{array}$$

for which there are

(i) a homotopy $R : r \circ s \sim \mathsf{id}_A$.
(ii) a homotopy $R' : r' \circ s' \sim \mathsf{id}_B$.

(iii) a homotopy $L : f \circ s \sim s' \circ g$.
(iv) a homotopy $K : g \circ r \sim r' \circ f$.
(v) for every $a : A$, a path $H(a)$ witnessing the commutativity of the square

$$\begin{array}{ccc}
g(r(s(a))) & \overset{K(s(a))}{=\!=\!=} & r'(f(s(a))) \\
\| {\scriptstyle g(R(a))} & & \| {\scriptstyle r'(L(a))} \\
g(a) & \underset{R'(g(a))^{-1}}{=\!=\!=} & r'(s'(g(a)))
\end{array}$$

Lemma 4.7.3. *If a function $g : A \to B$ is a retract of a function $f : X \to Y$, then $\mathsf{fib}_g(b)$ is a retract of $\mathsf{fib}_f(s'(b))$ for every $b : B$, where $s' : B \to Y$ is as in Definition 4.7.2.*

Theorem 4.7.4. *If g is a retract of an equivalence f, then g is also an equivalence.*

Definition 4.7.5. *Given type families $P, Q : A \to \mathcal{U}$ and a map $f : \prod_{(x:A)} P(x) \to Q(x)$, we define*

$$\mathsf{total}(f) :\equiv \lambda w. (\mathsf{pr}_1 w, f(\mathsf{pr}_1 w, \mathsf{pr}_2 w)) : \sum_{x:A} P(x) \to \sum_{x:A} Q(x).$$

Theorem 4.7.6. *Suppose that f is a fiberwise transformation between families P and Q over a type A and let $x : A$ and $v : Q(x)$. Then we have an equivalence*

$$\mathsf{fib}_{\mathsf{total}(f)}((x,v)) \simeq \mathsf{fib}_{f(x)}(v).$$

Theorem 4.7.7. *Suppose that f is a fiberwise transformation between families P and Q over a type A. Then f is a fiberwise equivalence if and only if $\mathsf{total}(f)$ is an equivalence.*

4.8 The object classifier

Lemma 4.8.1. *For any type family $B : A \to \mathcal{U}$, the fiber of $\mathsf{pr}_1 : \sum_{(x:A)} B(x) \to A$ over $a : A$ is equivalent to $B(a)$:*

$$\mathsf{fib}_{\mathsf{pr}_1}(a) \simeq B(a)$$

Lemma 4.8.2. *For any function $f : A \to B$, we have $A \simeq \sum_{(b:B)} \mathsf{fib}_f(b)$.*

Theorem 4.8.3. *For any type B there is an equivalence*

$$\chi : \left(\sum_{A:\mathcal{U}} (A \to B) \right) \simeq (B \to \mathcal{U}).$$

Theorem 4.8.4. *Let $f : A \to B$ be a function. Then the diagram*

$$\begin{array}{ccc}
A & \xrightarrow{\vartheta_f} & \mathcal{U}_\bullet \\
\downarrow {\scriptstyle f} & & \downarrow {\scriptstyle \mathsf{pr}_1} \\
B & \xrightarrow{\chi_f} & \mathcal{U}
\end{array}$$

*is a pullback square (see **??**). Here the function ϑ_f is defined by*

$$\lambda a. (\mathsf{fib}_f(f(a)), (a, \mathsf{refl}_{f(a)})).$$

4.9 Univalence implies function extensionality

Definition 4.9.1. *The **weak function extensionality principle** asserts that there is a function*

$$\left(\prod_{x:A} \mathsf{isContr}(P(x)) \right) \to \mathsf{isContr}\left(\prod_{x:A} P(x) \right)$$

for any family $P : A \to \mathcal{U}$ of types over any type A.

Lemma 4.9.2. *Assuming $\mathcal{U}$ is univalent, for any $A, B, X : \mathcal{U}$ and any $e : A \simeq B$, there is an equivalence*

$$(X \to A) \simeq (X \to B)$$

of which the underlying map is given by post-composition with the underlying function of e.

Corollary 4.9.3. *Let $P : A \to \mathcal{U}$ be a family of contractible types, i.e.*

$$\prod_{x:A} \mathsf{isContr}(P(x)).$$

Then the projection $\mathsf{pr}_1 : (\sum_{(x:A)} P(x)) \to A$ is an equivalence. Assuming $\mathcal{U}$ is univalent, it follows immediately that post-composition with pr_1 gives an equivalence

$$\alpha : \left(A \to \sum_{x:A} P(x) \right) \simeq (A \to A).$$

Theorem 4.9.4. *In a univalent universe $\mathcal{U}$, suppose that $P : A \to \mathcal{U}$ is a family of contractible types and let α be the function of Corollary 4.9.3. Then $\prod_{(x:A)} P(x)$ is a retract of $\mathsf{fib}_\alpha(\mathsf{id}_A)$. As a consequence, $\prod_{(x:A)} P(x)$ is contractible. In other words, the univalence axiom implies the weak function extensionality principle.*

Theorem 4.9.5. *Weak function extensionality implies the function extensionality **??**.*