Homotopy Type Theory

Homotopy n-types

7.1 Definition of n-types

D 7.1.1. Define the predicate $\mathsf{is\text{-}n\text{-}type} : \mathcal{U} \to \mathcal{U}$ for $n \geq -2$ by recursion as follows:

$$\mathsf{is\text{-}n\text{-}type}(X) :\equiv \begin{cases} \mathsf{isContr}(X) & \text{if } n = -2, \\ \prod_{(x,y:X)} \mathsf{is\text{-}n'\text{-}type}(x =_X y) & \text{if } n = n' + 1. \end{cases}$$

We say that X is an **n-type**, or sometimes that it is n-truncated, if $\mathsf{is\text{-}n\text{-}type}(X)$ is inhabited.

E 7.1.2. We saw that X is a (-1)-type if and only if it is a mere proposition. Therefore, X is a 0-type if and only if it is a set.

T 7.1.3. Let $p : X \to Y$ be a retraction and suppose that X is an n-type, for any $n \geq -2$. Then Y is also an n-type.

C 7.1.4. If $X \simeq Y$ and X is an n-type, then so is Y.

T 7.1.5. If $f : X \to Y$ is an embedding and Y is an n-type for some $n \geq -1$, then so is X.

T 7.1.6. The hierarchy of n-types is cumulative in the following sense: given a number $n \geq -2$, if X is an n-type, then it is also an $(n+1)$-type.

7.2 Preservation under constructors

T 7.2.1. Let $n \geq -2$, and let $A : \mathcal{U}$ and $B : A \to \mathcal{U}$. If A is an n-type and for all $a : A$, $B(a)$ is an n-type, then so is $\sum_{(x:A)} B(x)$.

T 7.2.2. Let $n \geq -2$, and let $A : \mathcal{U}$ and $B : A \to \mathcal{U}$. If for all $a : A$, $B(a)$ is an n-type, then so is $\prod_{(x:A)} B(x)$.

T 7.2.3. For any $n \geq -2$ and any type X, the type $\mathsf{is\text{-}n\text{-}type}(X)$ is a mere proposition.

T 7.2.4. For any $n \geq -2$, the type $n\text{-}\mathsf{Type}$ is an $(n+1)$-type.

7.3 Uniqueness of identity proofs and Hedberg's theorem

T 7.3.1. A type X is a set if and only if it satisfies **Axiom K**: for all $x : X$ and $p : (x =_A x)$ we have $p = \mathsf{refl}_x$.

T 7.3.2. Suppose R is a reflexive mere relation on a type X implying identity. Then X is a set, and $R(x,y)$ is equivalent to $x =_X y$ for all $x, y : X$.

C 7.3.3. If a type X has the property that $\neg\neg(x = y) \to (x = y)$ for any $x, y : X$, then X is a set.

L 7.3.4. For any type A we have $(A + \neg A) \to (\neg\neg A \to A)$.

T 7.3.5 (Hedberg). If X has decidable equality, then X is a set.

T 7.3.6. The type $\mathbb{N}$ of natural numbers has decidable equality, and hence is a set.

T 7.3.7. For any $n \geq -1$, a type X is an $(n+1)$-type if and only if for all $x : X$, the type $\Omega(X, x)$ is an n-type.

L 7.3.8. Given $n \geq -1$ and $X : \mathcal{U}$. If, given any inhabitant of X it follows that X is an n-type, then X is an n-type.

T 7.3.9. For every $n \geq -1$, a type A is an n-type if and only if $\Omega^{n+1}(A, a)$ is contractible for all $a : A$.

7.4 Truncations

L 7.4.1. $\|A\|_n$ is an n-type.

T 7.4.2. For any type family $P : \|A\|_n \to \mathcal{U}$ such that each $P(x)$ is an n-type, and any function $g : \prod_{(a:A)} P(|a|_n)$, there exists a section $f : \prod_{(x:\|A\|_n)} P(x)$ such that $f(|a|_n) :\equiv g(a)$ for all $a : A$.

L 7.4.3 (Universal property of truncations). Let $n \geq -2$, $A : \mathcal{U}$ and $B : n\text{-}\mathsf{Type}$. The following map is an equivalence:

$$\begin{cases} (\|A\|_n \to B) & \longrightarrow & (A \to B) \\ g & \longmapsto & g \circ |-|_n \end{cases}$$

L 7.4.4. Given $f, g : A \to B$ and a homotopy $h : f \sim g$, there is an induced homotopy $\|h\|_n : \|f\|_n \sim \|g\|_n$ such that the composite is equal to $\mathsf{ap}_{|-|_n}(h(a))$.

C 7.4.5. A type A is an n-type if and only if $|-|_n : A \to \|A\|_n$ is an equivalence.

T 7.4.6. For any types A and B, the induced map $\|A \times B\|_n \to \|A\|_n \times \|B\|_n$ is an equivalence.

T 7.4.7. Let $P : A \to \mathcal{U}$ be a family of types. Then there is an equivalence

$$\left\| \sum_{x:A} \|P(x)\|_n \right\|_n \simeq \left\| \sum_{x:A} P(x) \right\|_n.$$

C 7.4.8. If A is an n-type and $P : A \to \mathcal{U}$ is any type family, then

$$\sum_{a:A} \|P(a)\|_n \simeq \left\| \sum_{a:A} P(a) \right\|_n$$

T 7.4.9. For any A and $x, y : A$ and $n \geq -2$, the map (7) is an equivalence; thus we have

$$\|x =_A y\|_n \simeq \left(|x|_{n+1} =_{\|A\|_{n+1}} |y|_{n+1} \right).$$

C 7.4.10. Let $n \geq -2$ and (A, a) be a pointed type. Then

$$\|\Omega(A, a)\|_n = \Omega(\|(A, a)\|_{n+1})$$

C 7.4.11. Let $n \geq -2$ and $k \geq 0$ and (A, a) a pointed type. Then

$$\|\Omega^k(A, a)\|_n = \Omega^k(\|(A, a)\|_{n+k}).$$

L 7.4.12. Let $k, n \geq -2$ with $k \leq n$ and $A : \mathcal{U}$. Then $\|\|A\|_n\|_k = \|A\|_k$.

7.5 Colimits of n-types

D 7.5.1. A **span** is a 5-tuple $\mathscr{D} = (A, B, C, f, g)$ with $f : C \to A$ and $g : C \to B$.

D 7.5.2. Given a span $\mathscr{D} = (A, B, C, f, g)$ and a type D, a **cocone under** $\mathscr{D}$ **with base** D is a triple (i, j, h) with $i : A \to D$, $j : B \to D$ and $h : \prod_{(c:C)} i(f(c)) = j(g(c))$: We denote by $\mathsf{cocone}_{\mathscr{D}}(D)$ the type of all such cocones.

D 7.5.3. Given a span $\mathscr{D}$ of n-types, an n-type D, and a cocone $c : \mathsf{cocone}_{\mathscr{D}}(D)$, the pair (D, c) is said to be a **pushout of** $\mathscr{D}$ **in** n**-types** if for every n-type E, the map

$$\begin{cases} (D \to E) & \longrightarrow & \mathsf{cocone}_{\mathscr{D}}(E) \\ t & \longmapsto & t \circ c \end{cases}$$

is an equivalence.

L 7.5.4. If (D, c) and (D', c') are two pushouts of $\mathscr{D}$ in $\mathcal{U}_P$, then $(D, c) = (D', c')$.

C 7.5.5. The type of pushouts of $\mathscr{D}$ in $\mathcal{U}_P$ is a mere proposition. In particular if pushouts merely exist then they actually exist.

D 7.5.6. Let be a span. We denote by $\bigcirc(\mathscr{D})$ the following span of n-types:

D 7.5.7. Let $D : \mathcal{U}$ and $c = (i, j, h) : \mathsf{cocone}_{\mathscr{D}}(D)$. We define

$$\bigcirc(c) = (\bigcirc(i), \bigcirc(j), k) : \mathsf{cocone}_{\bigcirc(\mathscr{D})}(\bigcirc(D))$$

where k is the composite homotopy

$$\bigcirc(i) \circ \bigcirc(f) \sim \bigcirc(i \circ f) \sim \bigcirc(j \circ g) \sim \bigcirc(j) \circ \bigcirc(g)$$

using L 7.4.4 and the functoriality of $\bigcirc(-)$.

T 7.5.8. Let $\mathscr{D}$ be a span and (D, c) its pushout. Then $(\|D\|_n, \|c\|_n)$ is a pushout of $\|\mathscr{D}\|_n$ in n-types.

7.6 Connectedness

D 7.6.1. A function $f : A \to B$ is said to be n**-connected** if for all $b : B$, the type $\left\| \mathsf{fib}_f(b) \right\|_n$ is contractible:

$$\mathsf{conn}_n(f) :\equiv \prod_{b:B} \mathsf{isContr}(\left\| \mathsf{fib}_f(b) \right\|_n).$$

A type A is said to be n**-connected** if the unique function $A \to \mathbf{1}$ is n-connected, i.e. if $\|A\|_n$ is contractible.

L 7.6.2. *A function f is (-1)-connected if and only if it is surjective in the sense of mono-surj.*

NB 7.6.3. While our notion of n-connectedness for types agrees with the standard notion in homotopy theory, our notion of n-connectedness for *functions* is off by one from a common indexing in classical homotopy theory. Whereas we say a function f is n-connected if all its fibers are n-connected, some classical homotopy theorists would call such a function $(n+1)$-connected. (This is due to a historical focus on *cofibers* rather than fibers.)

L 7.6.4. *Suppose that g is a retract of a n-connected function f. Then g is n-connected.*

C 7.6.5. *If g is homotopic to a n-connected function f, then g is n-connected.*

C 7.6.6. *For any A, the canonical function $|-|_n : A \to \|A\|_n$ is n-connected.*

Exer. 7.1.

(i). Use T 7.3.2 to show that if $\|A\| \to A$ for every type A, then every type is a set.

(ii). Show that if every surjective function (purely) splits, i.e. if

$$\prod_{b:B} \left\| \mathsf{fib}_f(b) \right\| \to \prod_{b:B} \mathsf{fib}_f(b)$$

for every $f : A \to B$, then every type is a set.

Exer. 7.2. For this exercise, we consider the following general notion of colimit. Define a **graph** Γ to consist of a type Γ_0 and a family $\Gamma_1 : \Gamma_0 \to \Gamma_0 \to \mathcal{U}$. A **diagram** (of types) over a graph Γ consists of a family $F : \Gamma_0 \to \mathcal{U}$ together with for each $x, y : \Gamma_0$, a function $F_{x,y} : \Gamma_1(x,y) \to F(x) \to F(y)$. The **colimit** of such a diagram is the higher inductive type $\mathsf{colim}(F)$ generated by

- for each $x : \Gamma_0$, a function $\mathsf{inc}_x : F(x) \to \mathsf{colim}(F)$, and
- for each $x, y : \Gamma_0$ and $\gamma : \Gamma_1(x,y)$ and $a : F(x)$, a path $\mathsf{inc}_y(F_{x,y}(\gamma, a)) = \mathsf{inc}_x(a)$.

There are more general kinds of colimits as well (see e.g. **??**), but this is good enough for many purposes.

(i). Exhibit a graph Γ such that colimits of Γ-diagrams can be identified with pushouts as defined in **??**. In other words, each span should induce a diagram over Γ whose colimit is the pushout of the span.

(ii). Exhibit a graph Γ and a diagram F over Γ such that $F(x) = \mathbf{1}$ for all x, but such that $\mathsf{colim}(F) = \mathsf{S}^2$. Note that $\mathbf{1}$ is a (-2)-type, while S^2 is not expected to be an n-type for any finite n. See also **??**.

Exer. 7.3. Show that if A is an n-type and $B : A \to n\text{-Type}$ is a family of n-types, where $n \geq -1$, then the W-type $W_{(a:A)}B(a)$ (see **??**) is also an n-type.

Exer. 7.4. Use **??** to extend **??** to any section-retraction pair.

Exer. 7.5. Show that **??** also works as a characterization in the other direction: B is an n-type if and only if every map into B from an n-connected type is constant. Ideally, your proof should work for any modality as in **??**.

Exer. 7.6. Prove that for $n \geq -1$, a type A is n-connected if and only if it is merely inhabited and for all $a, b : A$ the type $a =_A b$ is $(n-1)$-connected. Thus, since every type is (-2)-connected, n-connectedness of types can be defined inductively using only propositional truncations. (In particular, A is 0-connected if and only if $\|A\|$ and $\prod_{(a,b:A)} \|a = b\|$.)

Exer. 7.7. For $-1 \leq n, m \leq \infty$, let $\mathsf{LEM}_{n,m}$ denote the statement

$$\prod_{A:n\text{-Type}} \|A + \neg A\|_m,$$

where $\infty\text{-Type} :\equiv \mathcal{U}$ and $\|X\|_\infty :\equiv X$. Show that:

(i). If $n = -1$ or $m = -1$, then $\mathsf{LEM}_{n,m}$ is equivalent to LEM from **??**.

(ii). If $n \geq 0$ and $m \geq 0$, then $\mathsf{LEM}_{n,m}$ is inconsistent with univalence.

Exer. 7.8. For $-1 \leq n, m \leq \infty$, let $\mathsf{AC}_{n,m}$ denote the statement

$$\prod_{(X:\mathsf{Set})} \prod_{(Y:X \to n\text{-Type})} \left(\prod_{x:X} \|Y(x)\|_m \right) \to \left\| \prod_{x:X} Y(x) \right\|_m,$$

with conventions as in **??**. Thus $\mathsf{AC}_{0,-1}$ is the axiom of choice from **??**, while $\mathsf{AC}_{\infty,\infty}$ is the identity function. (If we had formulated $\mathsf{AC}_{n,m}$ analogously to (3) rather than (3), $\mathsf{AC}_{\infty,\infty}$ would be like **??**.) It is known that $\mathsf{AC}_{\infty,-1}$ is consistent with univalence, since it holds in Voevodsky's simplicial model.

(i). Without using univalence, show that $\mathsf{LEM}_{n,\infty}$ implies $\mathsf{AC}_{n,m}$ for all m. (On the other hand, in **??** we will show that $\mathsf{AC} = \mathsf{AC}_{0,-1}$ implies $\mathsf{LEM} = \mathsf{LEM}_{-1,-1}$.)

(ii). Of course, $\mathsf{AC}_{n,m} \Rightarrow \mathsf{AC}_{k,m}$ if $k \leq n$. Are there any other implications between the principles $\mathsf{AC}_{n,m}$? Is $\mathsf{AC}_{n,m}$ consistent with univalence for any $m \geq 0$ and any n? (These are open questions.)

Exer. 7.9. Show that $\mathsf{AC}_{n,-1}$ implies that for any n-type A, there merely exists a set B and a surjection $B \to A$.

Exer. 7.10. Define the n**-connected axiom of choice** to be the statement

If X is a set and $Y : X \to \mathcal{U}$ is a family of types such that each $Y(x)$ is n-connected, then $\prod_{(x:X)} Y(x)$ is n-connected.

Note that the (-1)-connected axiom of choice is $\mathsf{AC}_{\infty,-1}$ from **??**.

(i). Prove that the (-1)-connected axiom of choice implies the n-connected axiom of choice for all $n \geq -1$.

(ii). Are there any other implications between the n-connected axioms of choice and the principles $\mathsf{AC}_{n,m}$? (This is an open question.)

Exer. 7.11. Show that the n-truncation modality is not left exact for any $n \geq -1$. That is, exhibit a pullback which it fails to preserve.

Exer. 7.12. Show that $X \mapsto (\neg\neg X)$ is a modality.

Exer. 7.13. Let P be a mere proposition.

(i). Show that $X \mapsto (P \to X)$ is a left exact modality. This is called the **open modality** associated to P.

(ii). Show that $X \mapsto P * X$ is a left exact modality, where $*$ denotes the join (see **??**). This is called the **closed modality** associated to P.

Exer. 7.14. Let $f : A \to B$ be a map; a type Z is f**-local** if $(- \circ f) : (B \to Z) \to (A \to Z)$ is an equivalence.

(i). Prove that the f-local types form a reflective subuniverse. You will want to use a higher inductive type to define the reflector (localization).

(ii). Prove that if $B = \mathbf{1}$, then this subuniverse is a modality.

Exer. 7.15. Show that in contrast to **??**, we could equivalently define $\|A\|_n$ to be generated by a function $|-|_n : A \to \|A\|_n$ together with for each $r : \mathsf{S}^{n+1} \to \|A\|_n$ and each $x : \mathsf{S}^{n+1}$, a path $s_r(x) : r(x) = r(\mathsf{base})$.

Exer. 7.16. In this exercise, we consider a slightly fancier notion of colimit than in **??**. Define a **graph with composition** Γ to be a graph as in **??** together with for each $x, y, z : \Gamma_0$, a function $\Gamma_1(y,z) \to \Gamma_1(x,y) \to \Gamma_1(x,z)$, written as $\delta \mapsto \gamma \mapsto \delta \circ \gamma$. (For instance, any precategory as in **??** is a graph with composition.) A **diagram** F over a graph with composition Γ consists of a diagram over the underlying graph, together with for each $x, y, z : \Gamma_0$ and $\gamma : \Gamma_1(x,y)$ and $\delta : \Gamma_1(y,z)$, a homotopy $\mathsf{cmp}_{x,y,z}(\delta, \gamma) : F_{y,z}(\delta) \circ F_{x,y}(\gamma) \sim F_{x,z}(\delta \circ \gamma)$. The **colimit** of such a diagram is the higher inductive type $\mathsf{colim}(F)$ generated by

- for each $x : \Gamma_0$, a function $\mathsf{inc}_x : F(x) \to \mathsf{colim}(F)$,
- for each $x, y : \Gamma_0$ and $\gamma : \Gamma_1(x,y)$ and $a : F(x)$, a path $\mathsf{glue}_{x,y}(\gamma, a) : \mathsf{inc}_y(F_{x,y}(\gamma, a)) = \mathsf{inc}_x(a)$, and
- for each $x, y, z : \Gamma_0$ and $\gamma : \Gamma_1(x,y)$ and $\delta : \Gamma_1(y,z)$ and $a : F(x)$, a path

$$\mathsf{inc}_z\big(\mathsf{cmp}_{x,y,z}(\delta, \gamma, a)\big) \cdot \mathsf{glue}_{x,z}(\delta \circ \gamma, a) = \mathsf{glue}_{y,z}(\delta, F_{x,y}(\gamma, a)) \cdot \mathsf{glue}_{x,y}(\gamma, a).$$

(This is a "second-order approximation" to a fully homotopy-theoretic notions of diagram and colimit, which ought to involve "coherence paths" of this sort at all higher levels. Defining such things in type theory is an important open problem.)

Exhibit a graph with composition Γ such that Γ_0 is a set and each type $\Gamma_1(x,y)$ is a mere proposition, and a diagram F over Γ such that $F(x) = \mathbf{1}$ for all x, for which $\mathsf{colim}(F) = \mathsf{S}^2$.

Exer. 7.17. Comparing **????**, one might be tempted to conjecture that if $f : A \to B$ is n-connected and $g : \prod_{(a:A)} P(a) \to Q(f(a))$ induces an n-connected map $\left(\sum_{(a:A)} P(a) \right) \to \left(\sum_{(b:B)} Q(b) \right)$, then g is fiberwise n-connected. Give a counterexample to show that this is false. (In fact, when generalized to modalities, this property characterizes the left exact ones; see **??**.)

Exer. 7.18. Show that if $f : A \to B$ is n-connected, then

$\|f\|_k : \|A\|_k \to \|B\|_k$ is also n-connected.

Exer. 7.19. We say a type A is **categorically connected** if for every types B, C the canonical map
$e_{A,B,C} : ((A \to B) + (A \to C)) \to (A \to B + C)$ defined by

$$e_{A,B,C}(\mathsf{inl}(g)) :\equiv \lambda x.\, \mathsf{inl}(g(x)),$$
$$e_{A,B,C}(\mathsf{inr}(g)) :\equiv \lambda x.\, \mathsf{inr}(g(x))$$

is an equivalence.

(i). Show that any connected type is categorically connected.
(ii). Show that all categorically connected types are connected if and only if LEM holds. (Hint: consider $A :\equiv \Sigma P$ such that $\neg\neg P$ holds.)