

Homotopy Type Theory

Categories

9.1 Categories and precategories

Definition 9.1.1. A **precategory** \mathbf{A} consists of the following.

- (i). A type \mathbf{A}_0 , whose elements are called **objects**. We write $a : \mathbf{A}$ for $a : \mathbf{A}_0$.
- (ii). For each $a, b : \mathbf{A}$, a set $\text{hom}_{\mathbf{A}}(a, b)$, whose elements are called **arrows** or **morphisms**.
- (iii). For each $a : \mathbf{A}$, a morphism $1_a : \text{hom}_{\mathbf{A}}(a, a)$, called the **identity morphism**.
- (iv). For each $a, b, c : \mathbf{A}$, a function

$$\text{hom}_{\mathbf{A}}(b, c) \rightarrow \text{hom}_{\mathbf{A}}(a, b) \rightarrow \text{hom}_{\mathbf{A}}(a, c)$$

called **composition**, and denoted infix by $g \mapsto f \mapsto g \circ f$, or sometimes simply by gf .

- (v). For each $a, b : \mathbf{A}$ and $f : \text{hom}_{\mathbf{A}}(a, b)$, we have $f = 1_b \circ f$ and $f = f \circ 1_a$.
- (vi). For each $a, b, c, d : \mathbf{A}$ and

$$f : \text{hom}_{\mathbf{A}}(a, b), \quad g : \text{hom}_{\mathbf{A}}(b, c), \quad h : \text{hom}_{\mathbf{A}}(c, d),$$

we have $h \circ (g \circ f) = (h \circ g) \circ f$.

Definition 9.1.2. A morphism $f : \text{hom}_{\mathbf{A}}(a, b)$ is an **isomorphism** if there is a morphism $g : \text{hom}_{\mathbf{A}}(b, a)$ such that $g \circ f = 1_a$ and $f \circ g = 1_b$. We write $a \cong b$ for the type of such isomorphisms.

Lemma 9.1.3. For any $f : \text{hom}_{\mathbf{A}}(a, b)$, the type “ f is an isomorphism” is a mere proposition. Therefore, for any $a, b : \mathbf{A}$ the type $a \cong b$ is a set.

Lemma 9.1.4 (idtoiso). If \mathbf{A} is a precategory and $a, b : \mathbf{A}$, then

$$(a = b) \rightarrow (a \cong b).$$

Example 9.1.5. There is a precategory \mathbf{Set} , whose type of objects is \mathbf{Set} , and with $\text{hom}_{\mathbf{Set}}(A, B) := (A \rightarrow B)$. The identity morphisms are identity functions and the composition is function composition. For this precategory, ?? is equal to (the restriction to sets of) the map idtoeqv from ??.

Of course, to be more precise we should call this category $\mathbf{Set}_{\mathcal{U}}$, since its objects are only the *small sets* relative to a universe \mathcal{U} .

Definition 9.1.6. A **category** is a precategory such that for all $a, b : \mathbf{A}$, the function $\text{idtoiso}_{a,b}$ from ?? is an equivalence.

Example 9.1.7. The univalence axiom implies immediately that \mathbf{Set} is a category. One can also show, using univalence, that any precategory of set-level structures such as groups, rings, topological spaces, etc. is a category; see ??.

Lemma 9.1.8. In a category, the type of objects is a 1-type.

Lemma 9.1.9. For $p : a = a'$ and $q : b = b'$ and $f : \text{hom}_{\mathbf{A}}(a, b)$, we have

$$(p, q)_*(f) = \text{idtoiso}(q) \circ f \circ \text{idtoiso}(p)^{-1}. \quad (9.1.10)$$

Example 9.1.11. A precategory in which each set $\text{hom}_{\mathbf{A}}(a, b)$ is a mere proposition is equivalently a type \mathbf{A}_0 equipped with a mere relation “ \leq ” that is reflexive ($a \leq a$) and transitive (if $a \leq b$ and $b \leq c$, then $a \leq c$). We call this a **preorder**.

In a preorder, a witness $f : a \leq b$ is an isomorphism just when there exists some witness $g : b \leq a$. Thus, $a \cong b$ is the mere proposition that $a \leq b$ and $b \leq a$. Therefore, a preorder \mathbf{A} is a category just when (1) each type $a = b$ is a mere proposition, and (2) for any $a, b : \mathbf{A}_0$ there exists a function $(a \cong b) \rightarrow (a = b)$. In other words, \mathbf{A}_0 must be a set, and \leq must be antisymmetric (if $a \leq b$ and $b \leq a$, then $a = b$). We call this a **(partial) order** or a **poset**.

Example 9.1.12. If \mathbf{A} is a category, then \mathbf{A}_0 is a set if and only if for any $a, b : \mathbf{A}_0$, the type $a \cong b$ is a mere proposition. This is equivalent to saying that every isomorphism in \mathbf{A} is an identity; thus it is rather stronger than the classical notion of “skeletal” category. Categories of this sort are sometimes called **gaunt** [?]. There is not really any notion of “skeletality” for our categories, unless one considers ?? itself to be such.

Example 9.1.13. For any 1-type X , there is a category with X as its type of objects and with $\text{hom}(x, y) := (x = y)$. If X is a set, we call this the **discrete category** on X . In general, we call it a **groupoid** (see ??).

Example 9.1.14. For any type X , there is a precategory with X as its type of objects and with $\text{hom}(x, y) := \|x = y\|_0$. The composition operation

$$\|y = z\|_0 \rightarrow \|x = y\|_0 \rightarrow \|x = z\|_0$$

is defined by induction on truncation from concatenation

$(y = z) \rightarrow (x = y) \rightarrow (x = z)$. We call this the **fundamental pregroupoid** of X . (In fact, we have met it already in ??; see also ??.)

Example 9.1.15. There is a precategory whose type of objects is \mathbf{U} and with $\text{hom}(X, Y) := \|X \rightarrow Y\|_0$, and composition defined by induction on truncation from ordinary composition

$(Y \rightarrow Z) \rightarrow (X \rightarrow Y) \rightarrow (X \rightarrow Z)$. We call this the **homotopy precategory of types**.

Example 9.1.16. Let \mathbf{Rel} be the following precategory:

- Its objects are sets.
- $\text{hom}_{\mathbf{Rel}}(X, Y) = X \rightarrow Y \rightarrow \mathbf{Prop}$.
- For a set X , we have $1_X(x, x') := (x = x')$.
- For $R : \text{hom}_{\mathbf{Rel}}(X, Y)$ and $S : \text{hom}_{\mathbf{Rel}}(Y, Z)$, their composite is defined by

$$(S \circ R)(x, z) := \left\| \sum_{y:Y} R(x, y) \times S(y, z) \right\|.$$

Suppose $R : \text{hom}_{\mathbf{Rel}}(X, Y)$ is an isomorphism, with inverse S . We observe the following.

- (i). If $R(x, y)$ and $S(y', x)$, then $(R \circ S)(y', y)$, and hence $y' = y$. Similarly, if $R(x, y)$ and $S(y, x')$, then $x = x'$.
- (ii). For any x , we have $x = x$, hence $(S \circ R)(x, x)$. Thus, there merely exists a $y : Y$ such that $R(x, y)$ and $S(y, x)$.
- (iii). Suppose $R(x, y)$. By (i), there merely exists a y' with $R(x, y')$ and $S(y', x)$. But then by (i), merely $y' = y$, and hence $y' = y$ since Y is a set. Therefore, by transporting $S(y', x)$ along this equality, we have $S(y, x)$. In conclusion, $R(x, y) \rightarrow S(y, x)$. Similarly, $S(y, x) \rightarrow R(x, y)$.
- (iv). If $R(x, y)$ and $R(x, y')$, then by (i), $S(y', x)$, so that by (i), $y = y'$. Thus, for any x there is at most one y such that $R(x, y)$. And by (i), there merely exists such a y , hence there exists such a y .

In conclusion, if $R : \text{hom}_{\mathbf{Rel}}(X, Y)$ is an isomorphism, then for each $x : X$ there is exactly one $y : Y$ such that $R(x, y)$, and dually. Thus, there is a function $f : X \rightarrow Y$ sending each x to this y , which is an equivalence; hence $X = Y$. With a little more work, we conclude that \mathbf{Rel} is a category.

9.2 Functors and transformations

Definition 9.2.1. Let \mathbf{A} and \mathbf{B} be precategories. A **functor** $F : \mathbf{A} \rightarrow \mathbf{B}$ consists of

- (i). A function $F_0 : \mathbf{A}_0 \rightarrow \mathbf{B}_0$, generally also denoted F .
- (ii). For each $a, b : \mathbf{A}$, a function $F_{a,b} : \text{hom}_{\mathbf{A}}(a, b) \rightarrow \text{hom}_{\mathbf{B}}(Fa, Fb)$, generally also denoted F .
- (iii). For each $a : \mathbf{A}$, we have $F(1_a) = 1_{Fa}$.
- (iv). For each $a, b, c : \mathbf{A}$ and $f : \text{hom}_{\mathbf{A}}(a, b)$ and $g : \text{hom}_{\mathbf{A}}(b, c)$, we have

$$F(g \circ f) = Fg \circ Ff.$$

Definition 9.2.2. For functors $F, G : \mathbf{A} \rightarrow \mathbf{B}$, a **natural transformation** $\gamma : F \rightarrow G$ consists of

- (i). For each $a : \mathbf{A}$, a morphism $\gamma_a : \text{hom}_{\mathbf{B}}(Fa, Ga)$ (the “components”).
- (ii). For each $a, b : \mathbf{A}$ and $f : \text{hom}_{\mathbf{A}}(a, b)$, we have $Gf \circ \gamma_a = \gamma_b \circ Ff$ (the “naturality axiom”).

Definition 9.2.3. For precategories \mathbf{A}, \mathbf{B} , there is a precategory $\mathbf{B}^{\mathbf{A}}$, called the **functor precategory**, defined by

- $(B^A)_0$ is the type of functors from \mathbf{A} to \mathbf{B} .
- $\text{hom}_{\mathbf{B}^A}(F, G)$ is the type of natural transformations from F to G .

Lemma 9.2.4. A natural transformation $\gamma : F \rightarrow G$ is an isomorphism in B^A if and only if each γ_a is an isomorphism in B .

Theorem 9.2.5. If \mathbf{A} is a precategory and \mathbf{B} is a category, then $\mathbf{B}^{\mathbf{A}}$ is a category.

Definition 9.2.6. For functors $F : \mathbf{A} \rightarrow \mathbf{B}$ and $G : \mathbf{B} \rightarrow \mathbf{C}$, their composite $G \circ F : \mathbf{A} \rightarrow \mathbf{C}$ is given by

- The composite $(G_0 \circ F_0) : \mathbf{A}_0 \rightarrow \mathbf{C}_0$
- For each $a, b : \mathbf{A}$, the composite

$$(G_{Fa, Fb} \circ F_{a,b}) : \text{hom}_{\mathbf{A}}(a, b) \rightarrow \text{hom}_{\mathbf{C}}(GFa, GFb).$$

It is easy to check the axioms.

Definition 9.2.7. For functors $F : A \rightarrow B$ and $G, H : B \rightarrow C$ and a natural transformation $\gamma : G \rightarrow H$, the composite $(\gamma F) : GF \rightarrow HF$ is given by

- For each $a : A$, the component γ_{Fa} .
- Naturality is easy to check. Similarly, for γ as above and $K : C \rightarrow D$, the composite $(K\gamma) : KG \rightarrow KH$ is given by
- For each $b : B$, the component $K(\gamma_b)$.

Lemma 9.2.8. For functors $F, G : A \rightarrow B$ and $H, K : B \rightarrow C$ and natural transformations $\gamma : F \rightarrow G$ and $\delta : H \rightarrow K$, we have

$$(\delta G)(H\gamma) = (K\gamma)(\delta F).$$

Lemma 9.2.9. Composition of functors is associative: $H(GF) = (HG)F$.

Lemma 9.2.10. ?? is coherent, i.e. the following pentagon of equalities commutes:

$$\begin{array}{ccccc} & & K(H(GF)) & & \\ & \swarrow & & \searrow & \\ (KH)(GF) & & & & K((HG)F) \\ \parallel & & & & \parallel \\ ((KH)G)F & \xlongequal{\quad} & & & (K(HG))F \end{array}$$

Lemma 9.2.11. For a functor $F : A \rightarrow B$, we have equalities $(1_B \circ F) = F$ and $(F \circ 1_A) = F$, such that given also $G : B \rightarrow C$, the following triangle of equalities commutes.

$$\begin{array}{ccc} G \circ (1_B \circ F) & \xlongequal{\quad} & (G \circ 1_B) \circ F \\ \swarrow & & \searrow \\ G \circ F & & \end{array}$$

9.3 Adjunctions

Definition 9.3.1. A functor $F : A \rightarrow B$ is a **left adjoint** if there exists

- A functor $G : B \rightarrow A$.
- A natural transformation $\eta : 1_A \rightarrow GF$ (the **unit**).
- A natural transformation $\epsilon : FG \rightarrow 1_B$ (the **counit**).
- $(c_F)(F\eta) = 1_F$.
- $(c_G)(\eta G) = 1_G$.

Lemma 9.3.2. If A is a category (but B may be only a precategory), then the type “ F is a left adjoint” is a mere proposition.

9.4 Equivalences

Definition 9.4.1. A functor $F : A \rightarrow B$ is an **equivalence of (pre)categories** if it is a left adjoint for which η and ϵ are isomorphisms. We write $A \simeq B$ for the type of equivalences of categories from A to B .

Lemma 9.4.2. If for $F : A \rightarrow B$ there exists $G : B \rightarrow A$ and isomorphisms $GF \cong 1_A$ and $FG \cong 1_B$, then F is an equivalence of precategories.

Definition 9.4.3. We say a functor $F : A \rightarrow B$ is **faithful** if for all $a, b : A$, the function

$$F_{a,b} : \text{hom}_A(a, b) \rightarrow \text{hom}_B(Fa, Fb)$$

is injective, and **full** if for all $a, b : A$ this function is surjective. If it is both (hence each $F_{a,b}$ is an equivalence) we say F is **fully faithful**.

Definition 9.4.4. We say a functor $F : A \rightarrow B$ is **split essentially surjective** if for all $b : B$ there exists an $a : A$ such that $Fa \cong b$.

Lemma 9.4.5. For any precategories A and B and functor $F : A \rightarrow B$, the following types are equivalent.

- (i). F is an equivalence of precategories.
- (ii). F is fully faithful and split essentially surjective.

Definition 9.4.6. A functor $F : A \rightarrow B$ is **essentially surjective** if for all $b : B$, there merely exists an $a : A$ such that $Fa \cong b$. We say F is a **weak equivalence** if it is fully faithful and essentially surjective.

Lemma 9.4.7. If $F : A \rightarrow B$ is fully faithful and A is a category, then for any $b : B$ the type $\sum_{(a:A)}(Fa \cong b)$ is a mere proposition. Hence a functor between categories is an equivalence if and only if it is a weak equivalence.

Definition 9.4.8. A functor $F : A \rightarrow B$ is an **isomorphism of (pre)categories** if F is fully faithful and $F_0 : A_0 \rightarrow B_0$ is an equivalence of types.

Lemma 9.4.9. For precategories A and B and $F : A \rightarrow B$, the following are equivalent.

- (i). F is an isomorphism of precategories.
- (ii). There exist $G : B \rightarrow A$ and $\eta : 1_A = GF$ and $\epsilon : FG = 1_B$ such that

$$\text{ap}_{(\lambda H.FH)}(\eta) = \text{ap}_{(\lambda K.KF)}(\epsilon^{-1}). \quad (9.4.10)$$

- (iii). There merely exist $G : B \rightarrow A$ and $\eta : 1_A = GF$ and $\epsilon : FG = 1_B$.

Proof. First note that since hom-sets are sets, equalities between equalities of functors are uniquely determined by their object-parts. Thus, by function extensionality, (9) is equivalent to

$$(F_0)(\eta_0)_a = (\epsilon_0)^{-1}_{Fa}. \quad (9.4.11)$$

for all $a : A_0$. Note that this is precisely the triangle identity for G_0, η_0 , and ϵ_0 to be a proof that F_0 is a half adjoint equivalence of types. Now suppose (. Let $G_0 : B_0 \rightarrow A_0$ be the inverse of F_0 , with $\eta_0 : \text{id}_{A_0} = G_0 F_0$ and $\epsilon_0 : F_0 G_0 = \text{id}_{B_0}$ satisfying the triangle identity,

which is precisely (9). Now define $G_{b,b'} : \text{hom}_B(b, b') \rightarrow \text{hom}_A(G_0 b, G_0 b')$ by

$$G_{b,b'}(g) := (F_{G_0 b, G_0 b'})^{-1}(\text{idtoiso}((\epsilon_0)^{-1}_{Fa}) \circ g \circ \text{idtoiso}((\epsilon_0)_b))$$

(using the assumption that F is fully faithful). Since idtoiso takes inverses to inverses and concatenation to composition, and F is a functor, it follows that G is a functor.

By definition, we have $(GF)_0 \equiv G_0 F_0$, which is equal to id_{A_0} by η_0 . To obtain $1_A = GF$, we need to show that when transported along η_0 , the identity function of $\text{hom}_A(a, a')$ becomes equal to the composite $G_{Fa,Fa'} \circ F_{a,a'}$. In other words, for any $f : \text{hom}_A(a, a')$ we must have

$$\begin{aligned} \text{idtoiso}((\eta_0)_{a'}) \circ f \circ \text{idtoiso}((\eta_0)^{-1}_a) \\ = (F_{GFa,GFa'})^{-1}(\text{idtoiso}((\epsilon_0)^{-1}_{Fa'}) \circ F_{a,a'}(f) \circ \text{idtoiso}((\epsilon_0)_{Fa})) \end{aligned}$$

But this is equivalent to

$$\begin{aligned} (F_{GFa,GFa'}) \left(\text{idtoiso}((\eta_0)_{a'}) \circ f \circ \text{idtoiso}((\eta_0)^{-1}_a) \right) \\ = \text{idtoiso}((\epsilon_0)^{-1}_{Fa'}) \circ F_{a,a'}(f) \circ \text{idtoiso}((\epsilon_0)_{Fa}) \end{aligned}$$

which follows from functoriality of F , the fact that F preserves idtoiso , and (9). Thus we have $\eta : 1_A = GF$.

On the other side, we have $(FG)_0 \equiv F_0 G_0$, which is equal to id_{B_0} by ϵ_0 . To obtain $FG = 1_B$, we need to show that when transported along ϵ_0 , the identity function of $\text{hom}_B(b, b')$ becomes equal to the composite $F_{Gb,Gb'} \circ G_{b,b'}$. That is, for any $g : \text{hom}_B(b, b')$ we must have

$$\begin{aligned} F_{Gb,Gb'} \left((F_{Gb,Gb'})^{-1}(\text{idtoiso}((\epsilon_0)^{-1}_{b'})) \circ g \circ \text{idtoiso}((\epsilon_0)_b) \right) \\ = \text{idtoiso}((\epsilon_0)^{-1}_{b'}) \circ g \circ \text{idtoiso}((\epsilon_0)_b) \end{aligned}$$

But this is just the fact that $(F_{Gb,Gb'})^{-1}$ is the inverse of $F_{Gb,Gb'}$. And we have remarked that (9) is equivalent to (9), so (holds).

Conversely, suppose given (. Then the object-parts of G, η , and ϵ together with (9) show that F_0 is an equivalence of types. And for $a, a' : A_0$, we define $\overline{G}_{a,a'} : \text{hom}_B(Fa, Fa') \rightarrow \text{hom}_A(a, a')$ by

$$\overline{G}_{a,a'}(g) := \text{idtoiso}(\eta^{-1})_{a'} \circ G(g) \circ \text{idtoiso}(\eta)_a. \quad (9.4.12)$$

By naturality of $\text{idtoiso}(\eta)$, for any $f : \text{hom}_A(a, a')$ we have

$$\begin{aligned} \overline{G}_{a,a'}(F_{a,a'}(f)) &= \text{idtoiso}(\eta^{-1})_{a'} \circ G(F(f)) \circ \text{idtoiso}(\eta)_a \\ &= \text{idtoiso}(\eta^{-1})_{a'} \circ \text{idtoiso}(\eta)_{a'} \circ f \\ &= f. \end{aligned}$$

On the other hand, for $g : \text{hom}_B(Fa, Fa')$ we have

$$\begin{aligned} F_{a,a'}(\overline{G}_{a,a'}(g)) &= F(\text{idtoiso}(\eta^{-1})_{a'} \circ G(g)) \circ F(\text{idtoiso}(\eta)_a) \\ &= \text{idtoiso}(\epsilon)_{Fa'} \circ F(G(g)) \circ \text{idtoiso}(\epsilon^{-1})_{Fa} \\ &= \text{idtoiso}(\epsilon)_{Fa'} \circ \text{idtoiso}(\epsilon^{-1})_{Fa'} \circ g \\ &= g. \end{aligned}$$

(There are lemmas needed here regarding the compatibility of idtoiso and whiskering, which we leave it to the reader to state and prove.) Thus, $F_{a,a'}$ is an equivalence, so F is fully faithful; i.e. (holds.

Now the composite $(\rightarrow)(\rightarrow)$ is equal to the identity since (is a mere proposition. On the other side, tracing through the above constructions we see that the composite $(\rightarrow)(\rightarrow)$ essentially preserves the object-parts G_0, η_0, ϵ_0 , and the object-part of (9). And in the latter three cases, the object-part is all there is, since hom-sets are sets.

Thus, it suffices to show that we recover the action of G on hom-sets. In other words, we must show that if $g : \hom_B(b, b')$, then

$$G_{b,b'}(g) = \overline{G}_{G_0 b, G_0 b'}(\text{idtoiso}((\epsilon_0)^{-1} b') \circ g \circ \text{idtoiso}((\epsilon_0)_b))$$

where \overline{G} is defined by (9). However, this follows from functoriality of G and the other triangle identity, which we have seen in ?? is equivalent to (9).

Now since (is a mere proposition, so is (, so it suffices to show they are logically equivalent to (. Of course, (\rightarrow) , so let us assume (. Since (is a mere proposition, we may assume given G_0, η , and ϵ . Then G_0 along with η and ϵ imply that F_0 is an equivalence. Moreover, we also have natural isomorphisms $\text{idtoiso}(\eta) : 1_A \cong GF$ and $\text{idtoiso}(\epsilon) : FG \cong 1_B$, so by ??, F is an equivalence of precategories, and in particular fully faithful. \square

Example 9.4.13. Let X be a type and $x_0 : X$ an element, and let X_{ch} denote the chaotic or indiscrete precategory on X . By definition, we have $(X_{\text{ch}})_0 := X$, and $\hom_{X_{\text{ch}}}(x, x') := 1$ for all x, x' . Then the unique functor $X_{\text{ch}} \rightarrow 1$ is an equivalence of precategories, but not an isomorphism unless X is contractible.

This example also shows that a precategory can be equivalent to a category without itself being a category. Of course, if a precategory is isomorphic to a category, then it must itself be a category.

Lemma 9.4.14. For categories A and B , a functor $F : A \rightarrow B$ is an equivalence of categories if and only if it is an isomorphism of categories.

Lemma 9.4.15. If A and B are precategories, then the function

$$(A = B) \rightarrow (A \cong B)$$

(defined by induction from the identity functor) is an equivalence of types.

Theorem 9.4.16. If A and B are categories, then the function

$$(A = B) \rightarrow (A \simeq B)$$

(defined by induction from the identity functor) is an equivalence of types.

9.5 The Yoneda lemma

Definition 9.5.1. For a precategory A , its **opposite** A^{op} is a precategory with the same type of objects, with $\hom_{A^{\text{op}}}(a, b) := \hom_A(b, a)$, and with identities and composition inherited from A .

Definition 9.5.2. For precategories A and B , their **product** $A \times B$ is a precategory with $(A \times B)_0 := A_0 \times B_0$ and

$$\hom_{A \times B}((a, b), (a', b')) := \hom_A(a, a') \times \hom_B(b, b')$$

Identities are defined by $1_{(a,b)} := (1_a, 1_b)$ and composition by $(g, g')(f, f') := ((gf), (g'f'))$.

Lemma 9.5.3. For precategories A, B, C , the following types are equivalent.

- (i). Functors $A \times B \rightarrow C$.
- (ii). Functors $A \rightarrow C^B$.

Now for any precategory A , we have a hom-functor

$$\hom_A : A^{\text{op}} \times A \rightarrow \text{Set}.$$

It takes a pair $(a, b) : (A^{\text{op}})_0 \times A_0 \equiv A_0 \times A_0$ to the set $\hom_A(a, b)$. For a morphism $(f, f') : \hom_{A^{\text{op}} \times A}((a, b), (a', b'))$, by definition we have $f : \hom_A(a', a)$ and $f' : \hom_A(b, b')$, so we can define

$$\begin{aligned} (\hom_A)_{(a,b),(a',b')}(f, f') &\equiv (g \mapsto (f'gf)) \\ &: \hom_A(a, b) \rightarrow \hom_A(a', b'). \end{aligned}$$

Functoriality is easy to check.

Theorem 9.5.4 (The Yoneda lemma). For any precategory A , any $a : A$, and any functor $F : \text{Set}^{A^{\text{op}}}$, we have an isomorphism

$$\hom_{\text{Set}^{A^{\text{op}}}}(ya, F) \cong Fa. \quad (9.5.5)$$

Moreover, this is natural in both a and F .

Corollary 9.5.6. The Yoneda embedding $y : A \rightarrow \text{Set}^{A^{\text{op}}}$ is fully faithful.

Corollary 9.5.7. If A is a category, then $y_0 : A_0 \rightarrow (\text{Set}^{A^{\text{op}}})_0$ is an embedding. In particular, if $ya = yb$, then $a = b$.

Definition 9.5.8. A functor $F : \text{Set}^{A^{\text{op}}}$ is said to be **representable** if there exists $a : A$ and an isomorphism $ya \cong F$.

Theorem 9.5.9. If A is a category, then the type “ F is representable” is a mere proposition.

Lemma 9.5.10. For any precategories A and B and a functor $F : A \rightarrow B$, the following types are equivalent.

- (i). F is a left adjoint.
- (ii). For each $b : B$, the functor $(a \mapsto \hom_B(Fa, b))$ from A^{op} to Set is representable.

Corollary 9.5.11. [??] If A is a category and $F : A \rightarrow B$, then the type “ F is a left adjoint” is a mere proposition.

9.6 Strict categories

Definition 9.6.1. A **strict category** is a precategory whose type of objects is a set.

Example 9.6.2. Let A be a precategory and $x : A$ an object. Then there is a precategory $\text{mono}(A, x)$ as follows:

- Its objects consist of an object $y : A$ and a monomorphism $m : \hom_A(y, x)$. (As usual, $m : \hom_A(y, x)$ is a **monomorphism** (or is **monic**) if $(m \circ f = m \circ g) \Rightarrow (f = g)$.)
- Its morphisms from (y, m) to (z, n) are arbitrary morphisms from y to z in A (not necessarily respecting m and n).

An equality $(y, m) = (z, n)$ of objects in $\text{mono}(A, x)$ consists of an equality $p : y = z$ and an equality $p_*(m) = n$, which by ?? is equivalently an equality $m = n \circ \text{idtoiso}(p)$. Since hom-sets are sets, the type of such equalities is a mere proposition. But since m and n are monomorphisms, the type of morphisms f such that $m = n \circ f$ is also a mere proposition. Thus, if A is a category, then $(y, m) = (z, n)$ is a mere proposition, and hence $\text{mono}(A, x)$ is a strict category.

Example 9.6.3. Let E/F be a finite Galois extension of fields, and G its Galois group. Then there is a strict category whose objects are intermediate fields $F \subseteq K \subseteq E$, and whose morphisms are field homomorphisms which fix F pointwise (but need not commute with the inclusions into E). There is another strict category whose objects are subgroups $H \subseteq G$, and whose morphisms are morphisms of G -sets $G/H \rightarrow G/K$. The fundamental theorem of Galois theory says that these two precategories are isomorphic (not merely equivalent).

9.7 \ddagger -categories

Definition 9.7.1. A **\ddagger -precategory** is a precategory A together with the following:

- (i). For each $x, y : A$, a function $(-)^\dagger : \hom_A(x, y) \rightarrow \hom_A(y, x)$.
- (ii). For all $x : A$, we have $(1_x)^\dagger = 1_x$.
- (iii). For all f, g we have $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$.
- (iv). For all f we have $(f^\dagger)^\dagger = f$.

Definition 9.7.2. A morphism $f : \hom_A(x, y)$ in a \ddagger -precategory is **unitary** if $f^\dagger \circ f = 1_x$ and $f \circ f^\dagger = 1_y$.

Lemma 9.7.3. If $p : (x = y)$, then $\text{idtoiso}(p)$ is unitary.

Definition 9.7.4. A **\ddagger -category** is a \ddagger -precategory such that for all $x, y : A$, the function

$$(x = y) \rightarrow (x \cong^\dagger y)$$

from ?? is an equivalence.

Example 9.7.5. The category Rel from ?? becomes a \ddagger -precategory if we define $(R^\dagger)(y, x) := R(x, y)$. The proof that Rel is a category actually shows that every isomorphism is unitary; hence Rel is also a \ddagger -category.

Example 9.7.6. Any groupoid becomes a \ddagger -category if we define $f^\dagger := f^{-1}$.

Example 9.7.7. Let Hilb be the following precategory.

- Its objects are finite-dimensional vector spaces equipped with an inner product $\langle -, - \rangle$.
- Its morphisms are arbitrary linear maps.

By standard linear algebra, any linear map $f : V \rightarrow W$ between finite dimensional inner product spaces has a uniquely defined adjoint $f^\dagger : W \rightarrow V$, characterized by $\langle fv, w \rangle = \langle v, f^\dagger w \rangle$. In this way, Hilb becomes a \ddagger -precategory. Moreover, a linear isomorphism is unitary precisely when it is an **isometry**, i.e. $\langle fv, fw \rangle = \langle v, w \rangle$. It follows from this that Hilb is a \ddagger -category, though it is not a category (not every linear isomorphism is unitary).

9.8 The structure identity principle

Definition 9.8.1. A notion of structure (P, H) over \mathbf{X} consists of the following.

- (i). A type family $P : X_0 \rightarrow \mathcal{U}$. For each $x : X_0$ the elements of Px are called (P, H) -structures on x .
- (ii). For $x, y : X_0$ and $\alpha : Px$, $\beta : Py$, to each $f : \text{hom}_X(x, y)$ a mere proposition

$$H_{\alpha\beta}(f).$$

If $H_{\alpha\beta}(f)$ is true, we say that f is a (P, H) -homomorphism from α to β .

- (iii). For $x : X_0$ and $\alpha : Px$, we have $H_{\alpha\alpha}(1_x)$.
- (iv). For $x, y, z : X_0$ and $\alpha : Px$, $\beta : Py$, $\gamma : Pz$, if $f : \text{hom}_X(x, y)$ and $g : \text{hom}_X(y, z)$, we have

$$H_{\alpha\beta}(f) \rightarrow H_{\beta\gamma}(g) \rightarrow H_{\alpha\gamma}(g \circ f).$$

When (P, H) is a notion of structure, for $\alpha, \beta : Px$ we define

$$(\alpha \leq_x \beta) := H_{\alpha\beta}(1_x).$$

By (and (, this is a preorder (??) with Px its type of objects. We say that (P, H) is a standard notion of structure if this preorder is in fact a partial order, for all $x : X$.

Theorem 9.8.2 (Structure identity principle). If \mathbf{X} is a category and (P, H) is a standard notion of structure over \mathbf{X} , then the precategory $\text{Str}_{(P,H)}(\mathbf{X})$ is a category.

Example 9.8.3. Let \mathbf{A} be a precategory and \mathbf{B} a category. There is a precategory \mathbf{B}^{A_0} whose objects are functions $A_0 \rightarrow B_0$, and whose set of morphisms from $F_0 : A_0 \rightarrow B_0$ to $G_0 : A_0 \rightarrow B_0$ is

$\prod_{(a:A_0)} \text{hom}_{\mathbf{B}}(F_0a, G_0a)$. Composition and identities are inherited directly from those in \mathbf{B} . It is easy to show that $\gamma : \text{hom}_{\mathbf{B}^{A_0}}(F_0, G_0)$ is an isomorphism exactly when each component γ_a is an isomorphism, so

that we have $(F_0 \cong G_0) \simeq \prod_{(a:A_0)} (F_0a \cong G_0a)$. Moreover, the map $\text{idtoiso} : (F_0 = G_0) \rightarrow (F_0 \cong G_0)$ of \mathbf{B}^{A_0} is equal to the composite

$$(F_0 = G_0) \longrightarrow \prod_{a:A_0} (F_0a = G_0a) \longrightarrow \prod_{a:A_0} (F_0a \cong G_0a) \longrightarrow (F_0 \cong G_0)$$

in which the first map is an equivalence by function extensionality, the second because it is a dependent product of equivalences (since \mathbf{B} is a category), and the third as remarked above. Thus, \mathbf{B}^{A_0} is a category. Now we define a notion of structure on \mathbf{B}^{A_0} for which $P(F_0)$ is the type of operations $F : \prod_{(a,a':A_0)} \text{hom}_A(a, a') \rightarrow \text{hom}_{\mathbf{B}}(F_0a, F_0a')$ which extend F_0 to a functor (i.e. preserve composition and identities). This is a set since each $\text{hom}_{\mathbf{B}}(-, -)$ is so. Given such F and G , we define $\gamma : \text{hom}_{\mathbf{B}^{A_0}}(F_0, G_0)$ to be a homomorphism if it forms a natural transformation. In ?? we essentially verified that this is a notion of structure. Moreover, if F and F' are both structures on F_0 and the identity is a natural transformation from F to F' , then for any $f : \text{hom}_A(a, a')$ we have $F'f = F'f \circ 1_{F_0a} = 1_{F_0a} \circ Ff = Ff$. Applying function extensionality, we conclude $F = F'$. Thus, we have a standard notion of structure, and so by ??, the precategory \mathbf{B}^A is a category.

Definition 9.8.4.

- (i). For each \mathcal{U} -small set x define

$$Px := P_0x \times P_1x.$$

Here

$$P_0x := \prod_{\omega:\Omega_0} x^{|\omega|} \rightarrow x, \text{ and}$$

$$P_1x := \prod_{\omega:\Omega_1} x^{|\omega|} \rightarrow \text{Prop}_{\mathcal{U}},$$

- (ii). For \mathcal{U} -small sets x, y and $\alpha : P^\omega x$, $\beta : P^\omega y$, $f : x \rightarrow y$, define

$$H_{\alpha\beta}(f) := H_{0,\alpha\beta}(f) \wedge H_{1,\alpha\beta}(f).$$

Here

$$H_{0,\alpha\beta}(f) := \forall(\omega : \Omega_0). \forall(u : x^{|\omega|}). f(\alpha u) = \beta(f \circ u), \text{ and}$$

$$H_{1,\alpha\beta}(f) := \forall(\omega : \Omega_1). \forall(u : x^{|\omega|}). \alpha u \rightarrow \beta(f \circ u).$$

9.9 The Rezk completion

Lemma 9.9.1. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are precategories and $H : \mathbf{A} \rightarrow \mathbf{B}$ is an essentially surjective functor, then $(-\circ H) : \mathbf{C}^{\mathbf{B}} \rightarrow \mathbf{C}^{\mathbf{A}}$ is faithful.

Lemma 9.9.2. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are precategories and $H : \mathbf{A} \rightarrow \mathbf{B}$ is essentially surjective and full, then $(-\circ H) : \mathbf{C}^{\mathbf{B}} \rightarrow \mathbf{C}^{\mathbf{A}}$ is fully faithful.

Theorem 9.9.3. If \mathbf{A}, \mathbf{B} are precategories, \mathbf{C} is a category, and $H : \mathbf{A} \rightarrow \mathbf{B}$ is a weak equivalence, then $(-\circ H) : \mathbf{C}^{\mathbf{B}} \rightarrow \mathbf{C}^{\mathbf{A}}$ is an isomorphism.

Therefore, if a precategory \mathbf{A} admits a weak equivalence functor $\mathbf{A} \rightarrow \widehat{\mathbf{A}}$ into a category, then that is its “reflection” into categories: any functor from \mathbf{A} into a category will factor essentially uniquely through $\widehat{\mathbf{A}}$. We now give two constructions of such a weak equivalence.

Theorem 9.9.4. For any precategory \mathbf{A} , there is a category $\widehat{\mathbf{A}}$ and a weak equivalence $\mathbf{A} \rightarrow \widehat{\mathbf{A}}$.

Example 9.9.5. Recall from ?? that for any type \mathbf{X} there is a pregroupoid with \mathbf{X} as its type of objects and $\text{hom}(x, y) := \|x = y\|_0$. Its Rezk completion is the fundamental groupoid of \mathbf{X} . Recalling that groupoids are equivalent to 1-types, it is not hard to identify this groupoid with $\|\mathbf{X}\|_1$.

Example 9.9.6. Recall from ?? that there is a precategory whose type of objects is \mathcal{U} and with $\text{hom}(X, Y) := \|X \rightarrow Y\|_0$. Its Rezk completion may be called the homotopy category of types. Its type of objects can be identified with $\|\mathcal{U}\|_1$ (see ??).

Theorem 9.9.7. A precategory \mathbf{C} is a category if and only if for every weak equivalence of precategories $H : \mathbf{A} \rightarrow \mathbf{B}$, the induced functor $(-\circ H) : \mathbf{C}^{\mathbf{B}} \rightarrow \mathbf{C}^{\mathbf{A}}$ is an isomorphism of precategories.