Homotopy Type Theory

Categories

9.1 Categories and precategories

Definition 9.1.1. A **precategory** A consists of the following.

(i). A type A_0, whose elements are called **objects**. We write $a : A$ for $a : A_0$.

(ii). For each $a, b : A$, a set $\hom_A(a, b)$, whose elements are called **arrows** or **morphisms**.

(iii). For each $a : A$, a morphism $1_a : \hom_A(a, a)$, called the **identity morphism**.

(iv). For each $a, b, c : A$, a function

$$\hom_A(b, c) \to \hom_A(a, b) \to \hom_A(a, c)$$

called **composition**, and denoted infix by $g \mapsto f \mapsto g \circ f$, or sometimes simply by gf.

(v). For each $a, b : A$ and $f : \hom_A(a, b)$, we have $f = 1_b \circ f$ and $f = f \circ 1_a$.

(vi). For each $a, b, c, d : A$ and

$$f : \hom_A(a, b), \qquad g : \hom_A(b, c), \qquad h : \hom_A(c, d),$$

we have $h \circ (g \circ f) = (h \circ g) \circ f$.

Definition 9.1.2. A morphism $f : \hom_A(a, b)$ is an **isomorphism** if there is a morphism $g : \hom_A(b, a)$ such that $g \circ f = 1_a$ and $f \circ g = 1_b$. We write $a \cong b$ for the type of such isomorphisms.

Lemma 9.1.3. *For any $f : \hom_A(a, b)$, the type "f is an isomorphism" is a mere proposition. Therefore, for any $a, b : A$ the type $a \cong b$ is a set.*

Lemma 9.1.4 (idtoiso). *If A is a precategory and $a, b : A$, then*

$$(a = b) \to (a \cong b).$$

Example 9.1.5. There is a precategory $\mathcal{S}et$, whose type of objects is Set, and with $\hom_{\mathcal{S}et}(A, B) :\equiv (A \to B)$. The identity morphisms are identity functions and the composition is function composition. For this precategory, Lemma 9.1.4 is equal to (the restriction to sets of) the map idtoeqv from **??**.
Of course, to be more precise we should call this category $\mathcal{S}et_{\mathcal{U}}$, since its objects are only the *small sets* relative to a universe $\mathcal{U}$.

Definition 9.1.6. A **category** is a precategory such that for all $a, b : A$, the function $\mathsf{idtoiso}_{a,b}$ from Lemma 9.1.4 is an equivalence.

Example 9.1.7. The univalence axiom implies immediately that $\mathcal{S}et$ is a category. One can also show, using univalence, that any precategory of set-level structures such as groups, rings, topological spaces, etc. is a category; see §9.8.

Lemma 9.1.8. *In a category, the type of objects is a 1-type.*

Lemma 9.1.9. *For $p : a = a'$ and $q : b = b'$ and $f : \hom_A(a, b)$, we have*

$$(p, q)_*(f) = \mathsf{idtoiso}(q) \circ f \circ \mathsf{idtoiso}(p)^{-1}. \qquad (9.1.10)$$

Example 9.1.11. A precategory in which each set $\hom_A(a, b)$ is a mere proposition is equivalently a type A_0 equipped with a mere relation "$\leq$" that is reflexive ($a \leq a$) and transitive (if $a \leq b$ and $b \leq c$, then $a \leq c$). We call this a **preorder**.
In a preorder, a witness $f : a \leq b$ is an isomorphism just when there exists some witness $g : b \leq a$. Thus, $a \cong b$ is the mere proposition that $a \leq b$ and $b \leq a$. Therefore, a preorder A is a category just when (1) each type $a = b$ is a mere proposition, and (2) for any $a, b : A_0$ there exists a function $(a \cong b) \to (a = b)$. In other words, A_0 must be a set, and $\leq$ must be antisymmetric (if $a \leq b$ and $b \leq a$, then $a = b$). We call this a **(partial) order** or a **poset**.

Example 9.1.12. If A is a category, then A_0 is a set if and only if for any $a, b : A_0$, the type $a \cong b$ is a mere proposition. This is equivalent to saying that every isomorphism in A is an identity; thus it is rather stronger than the classical notion of "skeletal" category. Categories of this sort are sometimes called **gaunt** [**?**]. There is not really any notion of "skeletality" for our categories, unless one considers Definition 9.1.6 itself to be such.

Example 9.1.13. For any 1-type X, there is a category with X as its type of objects and with $\hom(x, y) :\equiv (x = y)$. If X is a set, we call this the **discrete** category on X. In general, we call it a **groupoid** (see **??**).

Example 9.1.14. For *any* type X, there is a precategory with X as its type of objects and with $\hom(x, y) :\equiv \|x = y\|_0$. The composition operation

$$\|y = z\|_0 \to \|x = y\|_0 \to \|x = z\|_0$$

is defined by induction on truncation from concatenation $(y = z) \to (x = y) \to (x = z)$. We call this the **fundamental pregroupoid** of X. (In fact, we have met it already in **??**; see also **??**.)

Example 9.1.15. There is a precategory whose type of objects is $\mathcal{U}$ and with $\hom(X, Y) :\equiv \|X \to Y\|_0$, and composition defined by induction on truncation from ordinary composition $(Y \to Z) \to (X \to Y) \to (X \to Z)$. We call this the **homotopy precategory of types**.

Example 9.1.16. Let $\mathcal{R}el$ be the following precategory:

- Its objects are sets.
- $\hom_{\mathcal{R}el}(X, Y) = X \to Y \to \mathsf{Prop}$.
- For a set X, we have $1_X(x, x') :\equiv (x = x')$.
- For $R : \hom_{\mathcal{R}el}(X, Y)$ and $S : \hom_{\mathcal{R}el}(Y, Z)$, their composite is defined by

$$(S \circ R)(x, z) :\equiv \left\| \sum_{y:Y} R(x, y) \times S(y, z) \right\|.$$

Suppose $R : \hom_{\mathcal{R}el}(X, Y)$ is an isomorphism, with inverse S. We observe the following.

(i). If $R(x, y)$ and $S(y', x)$, then $(R \circ S)(y', y)$, and hence $y' = y$. Similarly, if $R(x, y)$ and $S(y, x')$, then $x = x'$.

(ii). For any x, we have $x = x$, hence $(S \circ R)(x, x)$. Thus, there merely exists a $y : Y$ such that $R(x, y)$ and $S(y, x)$.

(iii). Suppose $R(x, y)$. By (ii), there merely exists a y' with $R(x, y')$ and $S(y', x)$. But then by (i), merely $y' = y$, and hence $y' = y$ since Y is a set. Therefore, by transporting $S(y', x)$ along this equality, we have $S(y, x)$. In conclusion, $R(x, y) \to S(y, x)$. Similarly, $S(y, x) \to R(x, y)$.

(iv). If $R(x, y)$ and $R(x, y')$, then by (iii), $S(y', x)$, so that by (i), $y = y'$. Thus, for any x there is at most one y such that $R(x, y)$. And by (ii), there merely exists such a y, hence there exists such a y.

In conclusion, if $R : \hom_{\mathcal{R}el}(X, Y)$ is an isomorphism, then for each $x : X$ there is exactly one $y : Y$ such that $R(x, y)$, and dually. Thus, there is a function $f : X \to Y$ sending each x to this y, which is an equivalence; hence $X = Y$. With a little more work, we conclude that $\mathcal{R}el$ is a category.

9.2 Functors and transformations

Definition 9.2.1. Let A and B be precategories. A **functor** $F : A \to B$ consists of

(i). A function $F_0 : A_0 \to B_0$, generally also denoted F.

(ii). For each $a, b : A$, a function $F_{a,b} : \hom_A(a, b) \to \hom_B(Fa, Fb)$, generally also denoted F.

(iii). For each $a : A$, we have $F(1_a) = 1_{Fa}$.

(iv). For each $a, b, c : A$ and $f : \hom_A(a, b)$ and $g : \hom_A(b, c)$, we have

$$F(g \circ f) = Fg \circ Ff.$$

Definition 9.2.2. For functors $F, G : A \to B$, a **natural transformation** $\gamma : F \to G$ consists of

(i). For each $a : A$, a morphism $\gamma_a : \hom_B(Fa, Ga)$ (the "components").

(ii). For each $a, b : A$ and $f : \hom_A(a, b)$, we have $Gf \circ \gamma_a = \gamma_b \circ Ff$ (the "naturality axiom").

Definition 9.2.3. For precategories A, B, there is a precategory B^A, called the **functor precategory**, defined by

- $(B^A)_0$ is the type of functors from A to B.
- $\hom_{B^A}(F, G)$ is the type of natural transformations from F to G.

Lemma 9.2.4. *A natural transformation $\gamma : F \to G$ is an isomorphism in B^A if and only if each γ_a is an isomorphism in B.*

Theorem 9.2.5. *If A is a precategory and B is a category, then B^A is a category.*

Definition 9.2.6. For functors $F : A \to B$ and $G : B \to C$, their composite $G \circ F : A \to C$ is given by

- The composite $(G_0 \circ F_0) : A_0 \to C_0$
- For each $a, b : A$, the composite

$$(G_{Fa,Fb} \circ F_{a,b}) : \hom_A(a, b) \to \hom_C(GFa, GFb).$$

It is easy to check the axioms.

Definition 9.2.7. For functors $F : A \to B$ and $G, H : B \to C$ and a natural transformation $\gamma : G \to H$, the composite $(\gamma F) : GF \to HF$ is given by

- For each $a : A$, the component γ_{Fa}.

Naturality is easy to check. Similarly, for γ as above and $K : C \to D$, the composite $(K\gamma) : KG \to KH$ is given by

- For each $b : B$, the component $K(\gamma_b)$.

Lemma 9.2.8. *For functors $F, G : A \to B$ and $H, K : B \to C$ and natural transformations $\gamma : F \to G$ and $\delta : H \to K$, we have*

$$(\delta G)(H\gamma) = (K\gamma)(\delta F).$$

Lemma 9.2.9. *Composition of functors is associative: $H(GF) = (HG)F$.*

Lemma 9.2.10. *Lemma 9.2.9 is coherent, i.e. the following pentagon of equalities commutes:*

$$\begin{tikzcd}
 & K(H(GF)) \arrow[dl, equal] \arrow[dr, equal] & \\
(KH)(GF) \arrow[d, equal] & & K((HG)F) \arrow[d, equal] \\
((KH)G)F \arrow[rr, equal] & & (K(HG))F
\end{tikzcd}$$

Lemma 9.2.11. *For a functor $F : A \to B$, we have equalities $(1_B \circ F) = F$ and $(F \circ 1_A) = F$, such that given also $G : B \to C$, the following triangle of equalities commutes.*

$$\begin{tikzcd}
G \circ (1_B \circ F) \arrow[rr, equal] \arrow[dr, equal] & & (G \circ 1_B) \circ F \arrow[dl, equal] \\
 & G \circ F. &
\end{tikzcd}$$

9.3 Adjunctions

Definition 9.3.1. A functor $F : A \to B$ is a **left adjoint** if there exists

- A functor $G : B \to A$.
- A natural transformation $\eta : 1_A \to GF$ (the **unit**).
- A natural transformation $\epsilon : FG \to 1_B$ (the **counit**).
- $(\epsilon F)(F\eta) = 1_F$.
- $(G\epsilon)(\eta G) = 1_G$.

Lemma 9.3.2. *If A is a category (but B may be only a precategory), then the type "F is a left adjoint" is a mere proposition.*

9.4 Equivalences

Definition 9.4.1. A functor $F : A \to B$ is an **equivalence of (pre)categories** if it is a left adjoint for which η and ϵ are isomorphisms. We write $A \simeq B$ for the type of equivalences of categories from A to B.

Lemma 9.4.2. *If for $F : A \to B$ there exists $G : B \to A$ and isomorphisms $GF \cong 1_A$ and $FG \cong 1_B$, then F is an equivalence of precategories.*

Definition 9.4.3. We say a functor $F : A \to B$ is **faithful** if for all $a, b : A$, the function

$$F_{a,b} : \hom_A(a,b) \to \hom_B(Fa, Fb)$$

is injective, and **full** if for all $a, b : A$ this function is surjective. If it is both (hence each $F_{a,b}$ is an equivalence) we say F is **fully faithful**.

Definition 9.4.4. We say a functor $F : A \to B$ is **split essentially surjective** if for all $b : B$ there exists an $a : A$ such that $Fa \cong b$.

Lemma 9.4.5. *For any precategories A and B and functor $F : A \to B$, the following types are equivalent.*

(i). F is an equivalence of precategories.
(ii). F is fully faithful and split essentially surjective.

Definition 9.4.6. A functor $F : A \to B$ is **essentially surjective** if for all $b : B$, there *merely* exists an $a : A$ such that $Fa \cong b$. We say F is a **weak equivalence** if it is fully faithful and essentially surjective.

Lemma 9.4.7. *If $F : A \to B$ is fully faithful and A is a category, then for any $b : B$ the type $\sum_{(a:A)}(Fa \cong b)$ is a mere proposition. Hence a functor between categories is an equivalence if and only if it is a weak equivalence.*

Definition 9.4.8. A functor $F : A \to B$ is an **isomorphism of (pre)categories** if F is fully faithful and $F_0 : A_0 \to B_0$ is an equivalence of types.

Lemma 9.4.9. *For precategories A and B and $F : A \to B$, the following are equivalent.*

(i). F is an isomorphism of precategories.
(ii). There exist $G : B \to A$ and $\eta : 1_A = GF$ and $\epsilon : FG = 1_B$ such that

$$\mathsf{ap}_{(\lambda H.\, FH)}(\eta) = \mathsf{ap}_{(\lambda K.\, KF)}(\epsilon^{-1}). \tag{9.4.10}$$

(iii). There merely exist $G : B \to A$ and $\eta : 1_A = GF$ and $\epsilon : FG = 1_B$.

Proof. First note that since hom-sets are sets, equalities between equalities of functors are uniquely determined by their object-parts. Thus, by function extensionality, (9.4.10) is equivalent to

$$(F_0)(\eta_0)_a = (\epsilon_0)^{-1}{}_{F_0 a}. \tag{9.4.11}$$

for all $a : A_0$. Note that this is precisely the triangle identity for G_0, η_0, and ϵ_0 to be a proof that F_0 is a half adjoint equivalence of types.
Now suppose (i). Let $G_0 : B_0 \to A_0$ be the inverse of F_0, with $\eta_0 : \mathsf{id}_{A_0} = G_0 F_0$ and $\epsilon_0 : F_0 G_0 = \mathsf{id}_{B_0}$ satisfying the triangle identity, which is precisely (9.4.11). Now define $G_{b,b'} : \hom_B(b,b') \to \hom_A(G_0 b, G_0 b')$ by

$$G_{b,b'}(g) :\equiv (F_{G_0 b, G_0 b'})^{-1}\Big(\mathsf{idtoiso}((\epsilon_0)^{-1}{}_{b'}) \circ g \circ \mathsf{idtoiso}((\epsilon_0)_b)\Big)$$

(using the assumption that F is fully faithful). Since $\mathsf{idtoiso}$ takes inverses to inverses and concatenation to composition, and F is a functor, it follows that G is a functor.
By definition, we have $(GF)_0 \equiv G_0 F_0$, which is equal to id_{A_0} by η_0. To obtain $1_A = GF$, we need to show that when transported along η_0, the identity function of $\hom_A(a,a')$ becomes equal to the composite $G_{Fa,Fa'} \circ F_{a,a'}$. In other words, for any $f : \hom_A(a,a')$ we must have

$$\begin{aligned}
&\mathsf{idtoiso}((\eta_0)_{a'}) \circ f \circ \mathsf{idtoiso}((\eta_0)^{-1}{}_a) \\
&\quad = (F_{GFa,GFa'})^{-1}\Big(\mathsf{idtoiso}((\epsilon_0)^{-1}{}_{Fa'}) \circ F_{a,a'}(f) \circ \mathsf{idtoiso}((\epsilon_0)_{Fa})\Big).
\end{aligned}$$

But this is equivalent to

$$\begin{aligned}
&(F_{GFa,GFa'})\Big(\mathsf{idtoiso}((\eta_0)_{a'}) \circ f \circ \mathsf{idtoiso}((\eta_0)^{-1}{}_a)\Big) \\
&\qquad = \mathsf{idtoiso}((\epsilon_0)^{-1}{}_{Fa'}) \circ F_{a,a'}(f) \circ \mathsf{idtoiso}((\epsilon_0)_{Fa}).
\end{aligned}$$

which follows from functoriality of F, the fact that F preserves $\mathsf{idtoiso}$, and (9.4.11). Thus we have $\eta : 1_A = GF$.
On the other side, we have $(FG)_0 \equiv F_0 G_0$, which is equal to id_{B_0} by ϵ_0. To obtain $FG = 1_B$, we need to show that when transported along ϵ_0, the identity function of $\hom_B(b,b')$ becomes equal to the composite $F_{Gb,Gb'} \circ G_{b,b'}$. That is, for any $g : \hom_B(b,b')$ we must have

$$\begin{aligned}
&F_{Gb,Gb'}\Big((F_{Gb,Gb'})^{-1}\Big(\mathsf{idtoiso}((\epsilon_0)^{-1}{}_{b'}) \circ g \circ \mathsf{idtoiso}((\epsilon_0)_b)\Big)\Big) \\
&\qquad = \mathsf{idtoiso}((\epsilon_0{}^{-1})_{b'}) \circ g \circ \mathsf{idtoiso}((\epsilon_0)_b).
\end{aligned}$$

But this is just the fact that $(F_{Gb,Gb'})^{-1}$ is the inverse of $F_{Gb,Gb'}$. And we have remarked that (9.4.10) is equivalent to (9.4.11), so (ii) holds.
Conversely, suppose given (ii); then the object-parts of G, η, and ϵ together with (9.4.11) show that F_0 is an equivalence of types. And for $a, a' : A_0$, we define $\overline{G}_{a,a'} : \hom_B(Fa, Fa') \to \hom_A(a,a')$ by

$$\overline{G}_{a,a'}(g) :\equiv \mathsf{idtoiso}(\eta^{-1})_{a'} \circ G(g) \circ \mathsf{idtoiso}(\eta)_a. \tag{9.4.12}$$

By naturality of $\mathsf{idtoiso}(\eta)$, for any $f : \hom_A(a,a')$ we have

$$\begin{aligned}
\overline{G}_{a,a'}(F_{a,a'}(f)) &= \mathsf{idtoiso}(\eta^{-1})_{a'} \circ G(F(f)) \circ \mathsf{idtoiso}(\eta)_a \\
&= \mathsf{idtoiso}(\eta^{-1})_{a'} \circ \mathsf{idtoiso}(\eta)_{a'} \circ f \\
&= f.
\end{aligned}$$

On the other hand, for $g : \hom_B(Fa, Fa')$ we have

$$\begin{aligned}
F_{a,a'}(\overline{G}_{a,a'}(g)) &= F(\mathsf{idtoiso}(\eta^{-1})_{a'}) \circ F(G(g)) \circ F(\mathsf{idtoiso}(\eta)_a) \\
&= \mathsf{idtoiso}(\epsilon)_{Fa'} \circ F(G(g)) \circ \mathsf{idtoiso}(\epsilon^{-1})_{Fa} \\
&= \mathsf{idtoiso}(\epsilon)_{Fa'} \circ \mathsf{idtoiso}(\epsilon^{-1})_{Fa'} \circ g \\
&= g.
\end{aligned}$$

(There are lemmas needed here regarding the compatibility of idtoiso and whiskering, which we leave it to the reader to state and prove.)
Thus, $F_{a,a'}$ is an equivalence, so F is fully faithful; i.e. (i) holds.
Now the composite (i)→(ii)→(i) is equal to the identity since (i) is a mere proposition. On the other side, tracing through the above constructions we see that the composite (ii)→(i)→(ii) essentially preserves the object-parts G_0, η_0, ϵ_0, and the object-part of (9.4.10). And in the latter three cases, the object-part is all there is, since hom-sets are sets.
Thus, it suffices to show that we recover the action of G on hom-sets. In other words, we must show that if $g : \hom_B(b, b')$, then

$$G_{b,b'}(g) = \overline{G}_{G_0 b, G_0 b'}\left(\mathsf{idtoiso}((\epsilon_0)^{-1}{}_{b'}) \circ g \circ \mathsf{idtoiso}((\epsilon_0)_b)\right)$$

where $\overline{G}$ is defined by (9.4.12). However, this follows from functoriality of G and the *other* triangle identity, which we have seen in **??** is equivalent to (9.4.11).
Now since (i) is a mere proposition, so is (ii), so it suffices to show they are logically equivalent to (iii). Of course, (ii)→(iii), so let us assume (iii). Since (i) is a mere proposition, we may assume given G, η, and ϵ. Then G_0 along with η and ϵ imply that F_0 is an equivalence. Moreover, we also have natural isomorphisms $\mathsf{idtoiso}(\eta) : 1_A \cong GF$ and $\mathsf{idtoiso}(\epsilon) : FG \cong 1_B$, so by Lemma 9.4.2, F is an equivalence of precategories, and in particular fully faithful. □

Example 9.4.13. Let X be a type and $x_0 : X$ an element, and let X_{ch} denote the *chaotic* or *indiscrete* precategory on X. By definition, we have $(X_{\mathrm{ch}})_0 :\equiv X$, and $\hom_{X_{\mathrm{ch}}}(x, x') :\equiv \mathbf{1}$ for all x, x'. Then the unique functor $X_{\mathrm{ch}} \to \mathbf{1}$ is an equivalence of precategories, but not an isomorphism unless X is contractible.
This example also shows that a precategory can be equivalent to a category without itself being a category. Of course, if a precategory is *isomorphic* to a category, then it must itself be a category.

Lemma 9.4.14. *For categories A and B, a functor $F : A \to B$ is an equivalence of categories if and only if it is an isomorphism of categories.*

Lemma 9.4.15. *If A and B are precategories, then the function*

$$(A = B) \to (A \cong B)$$

(defined by induction from the identity functor) is an equivalence of types.

Theorem 9.4.16. *If A and B are categories, then the function*

$$(A = B) \to (A \simeq B)$$

(defined by induction from the identity functor) is an equivalence of types.

9.5 The Yoneda lemma

Definition 9.5.1. For a precategory A, its **opposite** A^{op} is a precategory with the same type of objects, with $\hom_{A^{\mathrm{op}}}(a, b) :\equiv \hom_A(b, a)$, and with identities and composition inherited from A.

Definition 9.5.2. For precategories A and B, their **product** $A \times B$ is a precategory with $(A \times B)_0 :\equiv A_0 \times B_0$ and

$$\hom_{A\times B}((a, b), (a', b')) :\equiv \hom_A(a, a') \times \hom_B(b, b').$$

Identities are defined by $1_{(a,b)} :\equiv (1_a, 1_b)$ and composition by $(g, g')(f, f') :\equiv ((gf), (g'f'))$.

Lemma 9.5.3. *For precategories A, B, C, the following types are equivalent.*

(i). Functors $A \times B \to C$.
(ii). Functors $A \to C^B$.

Now for any precategory A, we have a hom-functor

$$\hom_A : A^{\mathrm{op}} \times A \to \mathcal{S}et.$$

It takes a pair $(a, b) : (A^{\mathrm{op}})_0 \times A_0 \equiv A_0 \times A_0$ to the set $\hom_A(a, b)$. For a morphism $(f, f') : \hom_{A^{\mathrm{op}}\times A}((a, b), (a', b'))$, by definition we have $f : \hom_A(a', a)$ and $f' : \hom_A(b, b')$, so we can define

$$(\hom_A)_{(a,b),(a',b')}(f, f') :\equiv (g \mapsto (f' g f)) \\ : \hom_A(a, b) \to \hom_A(a', b').$$

Functoriality is easy to check.

Theorem 9.5.4 (The Yoneda lemma)**.** *For any precategory A, any $a : A$, and any functor $F : \mathcal{S}et^{A^{\mathrm{op}}}$, we have an isomorphism*

$$\hom_{\mathcal{S}et^{A^{\mathrm{op}}}}(\mathbf{y}a, F) \cong Fa. \tag{9.5.5}$$

Moreover, this is natural in both a and F.

Corollary 9.5.6. *The Yoneda embedding $\mathbf{y} : A \to \mathcal{S}et^{A^{\mathrm{op}}}$ is fully faithful.*

Corollary 9.5.7. *If A is a category, then $\mathbf{y}_0 : A_0 \to (\mathcal{S}et^{A^{\mathrm{op}}})_0$ is an embedding. In particular, if $\mathbf{y}a = \mathbf{y}b$, then $a = b$.*

Definition 9.5.8. A functor $F : \mathcal{S}et^{A^{\mathrm{op}}}$ is said to be **representable** if there exists $a : A$ and an isomorphism $\mathbf{y}a \cong F$.

Theorem 9.5.9. *If A is a category, then the type "F is representable" is a mere proposition.*

Lemma 9.5.10. *For any precategories A and B and a functor $F : A \to B$, the following types are equivalent.*

(i). F is a left adjoint.
(ii). For each $b : B$, the functor $(a \mapsto \hom_B(Fa, b))$ from A^{op} to $\mathcal{S}et$ is representable.

Corollary 9.5.11. *[Lemma 9.3.2] If A is a category and $F : A \to B$, then the type "F is a left adjoint" is a mere proposition.*

9.6 Strict categories

Definition 9.6.1. A **strict category** is a precategory whose type of objects is a set.

Example 9.6.2. Let A be a precategory and $x : A$ an object. Then there is a precategory $\mathsf{mono}(A, x)$ as follows:

- Its objects consist of an object $y : A$ and a monomorphism $m : \hom_A(y, x)$. (As usual, $m : \hom_A(y, x)$ is a **monomorphism** (or is **monic**) if $(m \circ f = m \circ g) \Rightarrow (f = g)$.)
- Its morphisms from (y, m) to (z, n) are arbitrary morphisms from y to z in A (not necessarily respecting m and n).

An equality $(y, m) = (z, n)$ of objects in $\mathsf{mono}(A, x)$ consists of an equality $p : y = z$ and an equality $p_*(m) = n$, which by Lemma 9.1.9 is equivalently an equality $m = n \circ \mathsf{idtoiso}(p)$. Since hom-sets are sets, the type of such equalities is a mere proposition. But since m and n are monomorphisms, the type of morphisms f such that $m = n \circ f$ is also a mere proposition. Thus, if A is a category, then $(y, m) = (z, n)$ is a mere proposition, and hence $\mathsf{mono}(A, x)$ is a strict category.

Example 9.6.3. Let E/F be a finite Galois extension of fields, and G its Galois group. Then there is a strict category whose objects are intermediate fields $F \subseteq K \subseteq E$, and whose morphisms are field homomorphisms which fix F pointwise (but need not commute with the inclusions into E). There is another strict category whose objects are subgroups $H \subseteq G$, and whose morphisms are morphisms of G-sets $G/H \to G/K$. The fundamental theorem of Galois theory says that these two precategories are isomorphic (not merely equivalent).

9.7 †-categories

Definition 9.7.1. A **†-precategory** is a precategory A together with the following.

(i). For each $x, y : A$, a function $(-)^\dagger : \hom_A(x, y) \to \hom_A(y, x)$.
(ii). For all $x : A$, we have $(1_x)^\dagger = 1_x$.
(iii). For all f, g we have $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$.
(iv). For all f we have $(f^\dagger)^\dagger = f$.

Definition 9.7.2. A morphism $f : \hom_A(x, y)$ in a †-precategory is **unitary** if $f^\dagger \circ f = 1_x$ and $f \circ f^\dagger = 1_y$.

Lemma 9.7.3. *If $p : (x = y)$, then $\mathsf{idtoiso}(p)$ is unitary.*

Definition 9.7.4. A **†-category** is a †-precategory such that for all $x, y : A$, the function

$$(x = y) \to (x \cong^\dagger y)$$

from Lemma 9.7.3 is an equivalence.

Example 9.7.5. The category $\mathcal{R}el$ from Example 9.1.16 becomes a †-precategory if we define $(R^\dagger)(y, x) :\equiv R(x, y)$. The proof that $\mathcal{R}el$ is a category actually shows that every isomorphism is unitary; hence $\mathcal{R}el$ is also a †-category.

Example 9.7.6. Any groupoid becomes a †-category if we define $f^\dagger :\equiv f^{-1}$.

Example 9.7.7. Let $\mathcal{H}ilb$ be the following precategory.

- Its objects are finite-dimensional vector spaces equipped with an inner product $\langle -,- \rangle$.
- Its morphisms are arbitrary linear maps.

By standard linear algebra, any linear map $f : V \to W$ between finite dimensional inner product spaces has a uniquely defined adjoint $f^\dagger : W \to V$, characterized by $\langle fv, w\rangle = \langle v, f^\dagger w\rangle$. In this way, $\mathcal{H}ilb$ becomes a †-precategory. Moreover, a linear isomorphism is unitary precisely when it is an **isometry**, i.e. $\langle fv, fw\rangle = \langle v, w\rangle$. It follows from this that $\mathcal{H}ilb$ is a †-category, though it is not a category (not every linear isomorphism is unitary).

9.8 The structure identity principle

Definition 9.8.1. A **notion of structure** (P, H) over X consists of the following.

(i). A type family $P : X_0 \to \mathcal{U}$. For each $x : X_0$ the elements of Px are called (P, H)**-structures** on x.

(ii). For $x, y : X_0$ and $\alpha : Px,\ \beta : Py$, to each $f : \hom_X(x, y)$ a mere proposition
$$H_{\alpha\beta}(f).$$
If $H_{\alpha\beta}(f)$ is true, we say that f is a (P, H)**-homomorphism** from α to β.

(iii). For $x : X_0$ and $\alpha : Px$, we have $H_{\alpha\alpha}(1_x)$.

(iv). For $x, y, z : X_0$ and $\alpha : Px,\ \beta : Py,\ \gamma : Pz$, if $f : \hom_X(x, y)$ and $g : \hom_X(y, z)$, we have
$$H_{\alpha\beta}(f) \to H_{\beta\gamma}(g) \to H_{\alpha\gamma}(g \circ f).$$

When (P, H) is a notion of structure, for $\alpha, \beta : Px$ we define
$$(\alpha \leq_x \beta) :\equiv H_{\alpha\beta}(1_x).$$

By (iii) and (iv), this is a preorder (Example 9.1.11) with Px its type of objects. We say that (P, H) is a **standard notion of structure** if this preorder is in fact a partial order, for all $x : X$.

Theorem 9.8.2 (Structure identity principle). *If X is a category and (P, H) is a standard notion of structure over X, then the precategory $\mathsf{Str}_{(P,H)}(X)$ is a category.*

Example 9.8.3. Let A be a precategory and B a category. There is a precategory B^{A_0} whose objects are functions $A_0 \to B_0$, and whose set of morphisms from $F_0 : A_0 \to B_0$ to $G_0 : A_0 \to B_0$ is $\prod_{(a:A_0)} \hom_B(F_0 a, G_0 a)$. Composition and identities are inherited directly from those in B. It is easy to show that $\gamma : \hom_{B^{A_0}}(F_0, G_0)$ is an isomorphism exactly when each component γ_a is an isomorphism, so that we have $(F_0 \cong G_0) \simeq \prod_{(a:A_0)}(F_0 a \cong G_0 a)$. Moreover, the map $\mathsf{idtoiso} : (F_0 = G_0) \to (F_0 \cong G_0)$ of B^{A_0} is equal to the composite
$$(F_0 = G_0) \longrightarrow \prod_{a:A_0}(F_0 a = G_0 a) \longrightarrow \prod_{a:A_0}(F_0 a \cong G_0 a) \longrightarrow (F_0 \cong G_0)$$
in which the first map is an equivalence by function extensionality, the second because it is a dependent product of equivalences (since B is a category), and the third as remarked above. Thus, B^{A_0} is a category. Now we define a notion of structure on B^{A_0} for which $P(F_0)$ is the type of operations $F : \prod_{(a,a':A_0)} \hom_A(a, a') \to \hom_B(F_0 a, F_0 a')$ which extend F_0 to a functor (i.e. preserve composition and identities). This is a set since each $\hom_B(-,-)$ is so. Given such F and G, we define $\gamma : \hom_{B^{A_0}}(F_0, G_0)$ to be a homomorphism if it forms a natural transformation. In Definition 9.2.3 we essentially verified that this is a notion of structure. Moreover, if F and F' are both structures on F_0 and the identity is a natural transformation from F to F', then for any $f : \hom_A(a, a')$ we have $F'f = F'f \circ 1_{F_0 a} = 1_{F_0 a} \circ Ff = Ff$. Applying function extensionality, we conclude $F = F'$. Thus, we have a *standard* notion of structure, and so by Theorem 9.8.2, the precategory B^A is a category.

Definition 9.8.4.

(i). For each $\mathcal{U}$-small set x define
$$Px :\equiv P_0 x \times P_1 x.$$
Here
$$P_0 x :\equiv \prod_{\omega:\Omega_0} x^{|\omega|} \to x, \text{ and}$$
$$P_1 x :\equiv \prod_{\omega:\Omega_1} x^{|\omega|} \to \mathsf{Prop}_{\mathcal{U}},$$

(ii). For $\mathcal{U}$-small sets x, y and $\alpha : P^\omega x,\ \beta : P^\omega y,\ f : x \to y$, define
$$H_{\alpha\beta}(f) :\equiv H_{0,\alpha\beta}(f) \wedge H_{1,\alpha\beta}(f).$$
Here
$$H_{0,\alpha\beta}(f) :\equiv \forall(\omega : \Omega_0).\ \forall(u : x^{|\omega|}).\ f(\alpha u) = \beta(f \circ u), \text{ and}$$
$$H_{1,\alpha\beta}(f) :\equiv \forall(\omega : \Omega_1).\ \forall(u : x^{|\omega|}).\ \alpha u \to \beta(f \circ u).$$

9.9 The Rezk completion

Lemma 9.9.1. *If A, B, C are precategories and $H : A \to B$ is an essentially surjective functor, then $(- \circ H) : C^B \to C^A$ is faithful.*

Lemma 9.9.2. *If A, B, C are precategories and $H : A \to B$ is essentially surjective and full, then $(- \circ H) : C^B \to C^A$ is fully faithful.*

Theorem 9.9.3. *If A, B are precategories, C is a category, and $H : A \to B$ is a weak equivalence, then $(- \circ H) : C^B \to C^A$ is an isomorphism.*

Therefore, if a precategory A admits a weak equivalence functor $A \to \widehat{A}$ into a category, then that is its "reflection" into categories: any functor from A into a category will factor essentially uniquely through $\widehat{A}$. We now give two constructions of such a weak equivalence.

Theorem 9.9.4. *For any precategory A, there is a category $\widehat{A}$ and a weak equivalence $A \to \widehat{A}$.*

Example 9.9.5. Recall from Example 9.1.14 that for any type X there is a pregroupoid with X as its type of objects and $\hom(x, y) :\equiv \|x = y\|_0$. Its Rezk completion is the *fundamental groupoid* of X. Recalling that groupoids are equivalent to 1-types, it is not hard to identify this groupoid with $\|X\|_1$.

Example 9.9.6. Recall from Example 9.1.15 that there is a precategory whose type of objects is $\mathcal{U}$ and with $\hom(X, Y) :\equiv \|X \to Y\|_0$. Its Rezk completion may be called the **homotopy category of types**. Its type of objects can be identified with $\|\mathcal{U}\|_1$ (see **??**).

Theorem 9.9.7. *A precategory C is a category if and only if for every weak equivalence of precategories $H : A \to B$, the induced functor $(- \circ H) : C^B \to C^A$ is an isomorphism of precategories.*