1. (28 points) Please mark whether each of the following arguments is true (O) or false (X) (note: you are NOT required to provide justification for your answers):
   (a) (4 points) For any sequence of events $A_1, A_2, \cdots, A_N$, we have $P(\bigcup_{n=1}^{N} A_n) \leq \sum_{n=1}^{N} P(A_n)$.
   (b) (4 points) If $X \sim \text{Exp}(\lambda)$, then for any $a \neq 0$, $Y = aX$ is also an exponential random variable.
   (c) (4 points) Let X_1 and X_2 be two independent Geometric random variables with parameters p_1 and p_2, respectively. Define $X = \min(X_1, X_2)$. Then, X is also a Geometric random variable.
   (d) (4 points) Suppose X is an exponential random variable. Then, the memoryless property suggests that $P(s + t_2 > X > s + t_1 | X > s) = P(t_2 > X > t_1)$, for any $s > 0$ and $t_2 > t_1 > 0$.
   (e) (4 points) Consider an experiment with a sample space $\Omega = \{1, 2, 3, 4\}$. Suppose we know $P(\{1, 2\}) = 0.3$, $P(\{2, 3\}) = 0.5$, and $P(\{3, 4\}) = 0.7$. Among all the possible valid probability assignments, the maximum possible value of $P(\{4\})$ is 0.7.
   (f) (4 points) For $r, n \in \mathbb{N}$ and $r \leq n$, we have $C_r^{2n} = C_0^n C_r^n + C_1^n C_{r-1}^n + \cdots + C_r^n C_0^n$.
   (g) (4 points) Let A, B, and C be three events defined on the same sample space. We say that A, B, C are independent if $P(A \cap B \cap C) = P(A)P(B)P(C)$.

2. (10 points) Let X be a normal random variable with the following PDF:

$$f(x) = \sqrt{2k} \exp\Big(- k^2 x^2 + 2kx - 1 \Big), \forall x \in \mathbb{R}.$$

   Find the expected value and variance of X (please express your answers in the form of numerical values without k).

3. (10 points) Let $X_1 \sim \text{Poisson}(\lambda_1, T)$ and $X_2 \sim \text{Poisson}(\lambda_2, T)$, and X_1, X_2 are independent. Define $Y = X_1 + X_2$. Find the PMF of Y. Please clearly justify your answer. (Hint: For any integer $k \geq 0$, $P(Y = k) = \sum_{m=0}^{k} P(X_1 = m)P(X_2 = k - m)$)

4. (26 points) Consider the following two continuous random variables:
   - X is a continuous uniform random variable on $[0, 5]$.
   - Y is an exponential random variable with parameter $\lambda = 2$, independent from X.

   (a) (10 points) Find the expected value and variance of e^X. (Hint: use LOTUS, i.e. $E[g(X)] = \int g(x) f_X(x) dx$)
   (b) (10 points) Find the joint CDF of X and Y. (Hint: use the fact that X, Y are independent. You may want to consider 6 different cases separately)
   (c) (6 points) Define $W = X^2 + 2X + 3$. Find the CDF of W.

5. (24 points) At each time, a binary message ('+' or '$-$') is transmitted as a wireless signal X, which is $+1$ ('+') or -1 ('$-$'). The wireless channel corrupts the transmission with additive noise $Y \sim \mathcal{N}(0, \sigma^2)$, independent from X. The received signal Z can therefore be written as $Z = X + Y$. The receiver concludes that the signal '+' (or '$-$') was transmitted if $Z \geq 0$ (or $Z < 0$, respectively).

   (a) (10 points) Given that $X = +1$, what is the probability of error (i.e. $P(Z < 0|X = +1)$)? Similarly, given that $X = -1$, what is the probability of error (i.e. $P(Z \geq 0|X = -1)$)? (You may use the notation $\Phi(\cdot)$ to denote the CDF of the standard normal distribution)

   (b) (8 points) Suppose for each transmission, the transmitter randomly chooses to send '+' or '$-$' with probability p and $1 - p$, respectively. Given that the receiver gets a '+', what is the probability that '+' is sent by the transmitter? (Hint: Use the error probability derived in 2(a) and apply the Bayes' rule)

   (c) (6 points) Similar to (b), for each transmission, the transmitter randomly chooses to send '+' or '$-$' with probability p and $1 - p$, respectively. Given that the receiver gets "$+-$", what is the probability that "$+-$" is sent by the transmitter?

6. (12 points) David works for a food delivery company, and he is about to set off to deliver a cup of oolong milk tea from Shinemood to the HSR station:

   - When all the traffic lights on his route are green, the delivery time is 15 minutes.
   - There are 6 traffic lights on his route, and each is red with probability 1/3, independent of every other light.
   - Each red traffic light that he encounters adds 1 minute to the delivery time.
   - Define T to be the actual delivery time (in minute), and T is a discrete random variable.

   Find the PMF, expected value, and variance of T. (4 points for each)