Problem 1

(a) X

$P(H_1)=\frac{1}{2}$, $P(H_2)=\frac{1}{2}$, $P(D)=\frac{1}{2}$

However, $P(H_1\cap H_2\cap D) = P(\emptyset) = 0 \neq P(H_1)\cdot P(H_2)\cdot P(D)$.

(b) O

Note that $\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} \frac{1}{n^2 \log(n+1)} < \infty$.

By Borel-Cantelli Lemma, we have $P\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n\right) = 0$.

(c) X

Consider the following counterexample:

Let $\Omega = \{1,2\}$, $N=2$, $A_1=\{1\}$, $A_2=\{1,2\}$.

Suppose $P(\{1\})=0$, $P(\{2\})=1$.

Then, $P\left(\bigcup_{n=1}^{2} A_n\right) = P(\{1,2\}) = 1$.

$\sum_{n=1}^{2} P(A_n) = P(A_1)+P(A_2) = 0+1 = 1$.

Therefore, we do not have $P\left(\bigcup_{n=1}^{N} A_n\right) < \sum_{n=1}^{N} P(A_n)$

(d) X

$X = \min\{X_1, \cdots, X_n\}$, $X_i \sim Exp(\lambda_i)$

$X \sim Exp\left(\sum_{i=1}^{n} \lambda_i\right)$ if X_i's are independent.

(e) $\underline{\bigcirc}$

Note that $\{E_n\}$ is a decreasing sequence of events.

By the continuity of probability, we have

$$\underbrace{\lim_{n\to\infty} P(E_n)}_{\substack{\| \\ \lim_{n\to\infty} \frac{3n+2}{5n} \\ \| \\ \frac{3}{5}}} = P\left(\lim_{n\to\infty} E_n\right) = P(\omega \in (0,1]) = P(E)$$

(f) $\underline{\bigcirc}$

$$E\left[|X|^\alpha\right] = \overbrace{\sum_{|x|<1} |x|^\alpha \cdot p_X(x)}^{\text{upper bounded by } 1} + \sum_{|x|\geq 1} |x|^\alpha \cdot p_X(x)$$

Then, $E[|X|^\alpha] = \infty$ implies that $\sum_{|x|\geq 1} |x|^\alpha \cdot p_X(x) = \infty$.

Moreover, for any $\gamma \geq 0$, we know

$$\sum_{|x|\geq 1} |x|^{\alpha+\gamma} \geq \sum_{|x|\geq 1} |x|^\alpha = \infty.$$

Hence, $E\left[|X|^{\alpha+\gamma}\right] = \infty$.

(g) $\underline{X}$

$$F_X(t) = \frac{e^t}{e^t + e^{-t}} = \frac{e^{2t}}{e^{2t}+1}$$

$$F_X^{-1}(t) = \frac{1}{2}\ln\left(\frac{t}{1-t}\right).$$

Therefore, by ITS, we shall construct $X = \frac{1}{2}\ln\left(\frac{U}{1-U}\right)$.

Problem 2 $\quad X \sim \text{Poisson}(\lambda, T)$

The PMF of X is: $\quad P_X(k) = \begin{cases} \dfrac{e^{-\lambda T} \cdot (\lambda T)^k}{k!}, & \text{if } k = 0, 1, 2, \cdots \\ 0, & \text{otherwise} \end{cases}$

Let $k^* = \lfloor \lambda T \rfloor$.

(For any $k \in \mathbb{N} \cup \{0\}$)

We have $$\frac{P_X(k+1)}{P_X(k)} = \frac{\dfrac{e^{-\lambda T} \cdot (\lambda T)^{k+1}}{(k+1)!}}{\dfrac{e^{-\lambda T} \cdot (\lambda T)^k}{k!}} = \frac{\lambda T}{k+1}$$

It is easy to verify that $\begin{cases} \dfrac{P_X(k+1)}{P_X(k)} > 1 & \text{if } k \leq \lfloor \lambda T \rfloor - 1 \\ \dfrac{P_X(k+1)}{P_X(k)} \leq 1 & \text{if } k \geq \lfloor \lambda T \rfloor \end{cases}$

Therefore, we conclude that k^* is a maximizer of $P_X(k)$.

$\square$

Problem 3: $F_1(t), F_2(t)$ are CDFs, $p \in (0,1)$, $F(t) = pF_1(t) + (1-p)F_2(t)$.

(a). To verify whether $F(t)$ is a valid CDF, we need to check the following:

① $F(t)$ is non-decreasing:

As $F_1(t), F_2(t)$ are CDFs, they are non-decreasing in t.

Given that $F(t) = p \cdot F_1(t) + (1-p)F_2(t)$ with $p \in (0,1)$, $F(t)$ is also a non-decreasing function.

② $\lim_{t\to\infty} F(t) = 1$:

As $F_1(t), F_2(t)$ are CDFs, we have $\lim_{t\to\infty} F_1(t) = 1$, $\lim_{t\to\infty} F_2(t) = 1$

Therefore, $\lim_{t\to\infty} F(t) = \lim_{t\to\infty} p \cdot F_1(t) + (1-p)F_2(t)$

$$= p \cdot \lim_{t\to\infty} F_1(t) + (1-p) \cdot \lim_{t\to\infty} F_2(t) = 1$$

③ $\lim_{t\to-\infty} F(t) = 0$:

As $F_1(t), F_2(t)$ are CDFs, we must have $\lim_{t\to-\infty} F_1(t) = 0$, $\lim_{t\to-\infty} F_2(t) = 0$.

This implies that $\lim_{t\to-\infty} F(t) = \lim_{t\to-\infty} p \cdot F_1(t) + (1-p)F_2(t) = 0$.

④ $F(t)$ is right-continuous:

– $F(t)$ is a linear combination of $F_1(t), F_2(t)$, which are right-continuous.

As right-continuity is preserved under linear combination, $F(t)$ is therefore also right-continuous.

(Cont.).

(b) Find of CDF of X:

total probability theorem

For any $t \in \mathbb{R}$, $P(X \le t) = P(X \le t \mid \text{getting a head}) \cdot p$

$$+ P(X \le t \mid \text{getting a tail}) \cdot (1-p).$$

$$= p \cdot F_1(t) + (1-p) \cdot F_2(t)$$

$$= F(t).$$

$\square$

Problem 4: $X \sim N(0,1)$, $f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$, $\forall x \in \mathbb{R}$

(a). $Y = X^2$.

$$- E[Y] = E[X^2] = Var[X] + \left(E[X]\right)^2 = 1 + 0^2 = 1$$

$$- Var[Y] = E[Y^2] - \left(E[Y]\right)^2 = E[Y^2] - 1$$

$$E[Y^2] = E[X^4] = \int_{-\infty}^{+\infty} x^4 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

$$= \underbrace{\left(-x^3 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\right)\Bigg|_{-\infty}^{\infty}}_{= 0} - \int_{-\infty}^{+\infty} -3x^2 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

$$= 3 \cdot \int_{-\infty}^{+\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

$$= 3 \cdot E[X^2]$$

$$= 3$$

Therefore, $Var[Y] = 3 - 1 = 2$

$\square$

(Cont.).

(b) $Z = \begin{cases} 2X+3, & \text{if } Y<1 \\ X-5, & \text{otherwise} \end{cases}$, $Y = X^2$.

Equivalently, we have $Z = \begin{cases} 2X+3, & -1<X<1 \\ X-5, & \text{otherwise} \end{cases}$

Case 1: $t \le -6$

empty set under $t<-6$

$$P(Z \le t) = P\left(\{Z \le t \text{ and } Y<1\} \cup \{Z \le t \text{ and } Y \ge 1\}\right)$$

$$= P(Z \le t \text{ and } X \le -1)$$

$$= P(X-5 \le t \text{ and } X \le -1)$$

$$= P(X \le 5+t)$$

$$= \Phi(5+t).$$

Case 2: $-6<t\le -4$

empty set under $-6<t\le -4$

$$P(Z \le t) = P\left(\{Z \le t \text{ and } Y<1\} \cup \{Z \le t \text{ and } Y \ge 1\}\right)$$

$$= P(Z \le t \text{ and } X \le -1)$$

$$= P(X-5 \le t \text{ and } X \le -1)$$

$$= P(X \le -1)$$

$$= \Phi(-1).$$

<u>Case3:</u> $-4 < t \le 1$

Still empty under $-4 < t \le 1$

$$P(Z \le t) = P\left(\{Z \le t \text{ and } Y < 1\} \cup \{Z \le t \text{ and } Y \ge 1\}\right)$$

$$= P\left(\underbrace{\{Z \le t \text{ and } X \le -1\}} \cup \underbrace{\{Z \le t \text{ and } X \ge 1\}}\right)$$

mutually exclusive

$$= P\left(\{Z \le t \text{ and } X \le -1\}\right) + P\left(\{Z \le t \text{ and } X \ge 1\}\right)$$

$$= P(X \le -1) + P(1 \le X \le 5+t)$$

$$= \Phi(-1) + \left(\Phi(5+t) - \Phi(1)\right)$$

<u>Case4:</u> $1 < t < 5$

$$P(Z \le t) = P\left(\{Z \le t \text{ and } Y < 1\} \cup \{Z \le t \text{ and } Y \ge 1\}\right)$$

$$= P\left(\{Z \le t \text{ and } Y < 1\}\right) + P\left(\{Z \le t \text{ and } Y \ge 1\}\right)$$

$$= P\left(\{2X+3 \le t \text{ and } -1 < X < 1\}\right) + \left[P\left(\{X-5 \le t \text{ and } X \le -1\}\right) + P\left(\{X-5 \le t \text{ and } X \ge 1\}\right)\right]$$

$$= P\left(\left\{-1 < X \le \tfrac{t-3}{2}\right\}\right) + \left[P(X \le -1) + P(1 \le X \le 5+t)\right]$$

$$= \left(\Phi\left(\tfrac{t-3}{2}\right) - \cancel{\Phi(-1)}\right) + \left(\cancel{\Phi(-1)} + \Phi(5+t) - \Phi(1)\right)$$

$$= \Phi\left(\tfrac{t-3}{2}\right) - \Phi(1) + \Phi(5+t)$$

Case 5: $t \geq 5$

$$P(Z \leq t) = P(\{Z \leq t \text{ and } Y < 1\} \cup \{Z \leq t \text{ and } Y \geq 1\})$$

$$= P(\{2X+3 \leq t \text{ and } -1 < X < 1\})$$

$$+ \left[P(\{X-5 \leq t \text{ and } X \leq -1\}) + P(\{X-5 \leq t \text{ and } X \geq 1\}) \right]$$

$$= P(\{-1 < X < 1\}) + \left[P(\{X \leq -1\}) + P(\{1 \leq X \leq 5+t\}) \right]$$

$$= \left(\Phi(1) - \Phi(-1) \right) + \left[\Phi(-1) + \left(\Phi(5+t) - \Phi(1) \right) \right]$$

$$= \Phi(5+t).$$

In summary, the CDF of Z is

$$F_Z(t) = \begin{cases} \Phi(5+t) & , \quad t \leq -6 \\ \Phi(-1) & , \quad -6 < t \leq -4 \\ \Phi(-1) + \Phi(5+t) - \Phi(1) & , \quad -4 < t \leq 1 \\ \Phi\left(\frac{t-3}{2}\right) - \Phi(1) + \Phi(5+t) & , \quad 1 < t < 5 \\ \Phi(5+t) & , \quad t \geq 5 \end{cases}$$

(cont.)

4(c)

At $t=2$, we have $F_Z(t) = \Phi\left(\frac{t-3}{2}\right) + \Phi(5+t) - \Phi(1)$

$$f_Z(2) = \left.\frac{dF_Z(t)}{dt}\right|_{t=2} = \left.\frac{d\left(\Phi\left(\frac{t-3}{2}\right) + \Phi(t+3) - \Phi(1)\right)}{dt}\right|_{t=2}$$

$$= \left.\Phi'\left(\frac{t-3}{2}\right) + \Phi'(t+5)\right|_{t=2}$$

$$= \frac{1}{2}\Phi'\left(\frac{-1}{2}\right) + \Phi'(7)$$

$$= \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{8}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{7^2}{2}}$$

Problem 5

(a). For ease of notation, define the following events:

$E_{ZA} = \{ \text{T'Challa got vaccinated with ZA} \}$

$E_{TNB} = \{ \text{T'Challa got vaccinated with TNB} \}$

$E_r = \{ \text{T'Challa suffers from vaccine reactions} \}$

$E_{inf} = \{ \text{T'Challa gets infected in the next 6 months} \}$

We know $P(E_{ZA}) = p$, $P(E_{TNB}) = 1-p$.

$$P(E_r \mid E_{ZA}) = \alpha_Z, \quad P(E_{inf} \mid E_{ZA}) = \beta_Z$$

$$P(E_r \mid E_{TNB}) = \alpha_T, \quad P(E_{inf} \mid E_{TNB}) = \beta_T$$

$$P(E_{ZA} \mid E_r \cap E_{inf}) = \frac{P(E_{ZA} \cap E_r \cap E_{inf})}{P(E_r \cap E_{inf})}$$

$$= \frac{P(E_{ZA} \cap E_r \cap E_{inf})}{P(E_{ZA} \cap E_r \cap E_{inf}) + P(E_{TNB} \cap E_r \cap E_{inf})}$$

$$= \frac{P(E_{ZA}) \cdot P(E_r \cap E_{inf} \mid E_{ZA})}{P(E_{ZA}) P(E_r \cap E_{inf} \mid E_{ZA}) + P(E_{TNB}) \cdot P(E_r \cap E_{inf} \mid E_{TNB})}$$

Since E_r and E_{inf} are not necessarily independent, we have

$$\begin{cases} P(E_r \cap E_{inf} \mid E_{ZA}) \leq \min\{\alpha_Z, \beta_Z\} = \beta_Z \\ P(E_r \cap E_{inf} \mid E_{ZA}) \geq \max\{\alpha_Z + \beta_Z - 1, 0\} \end{cases}$$

Similarly, we have

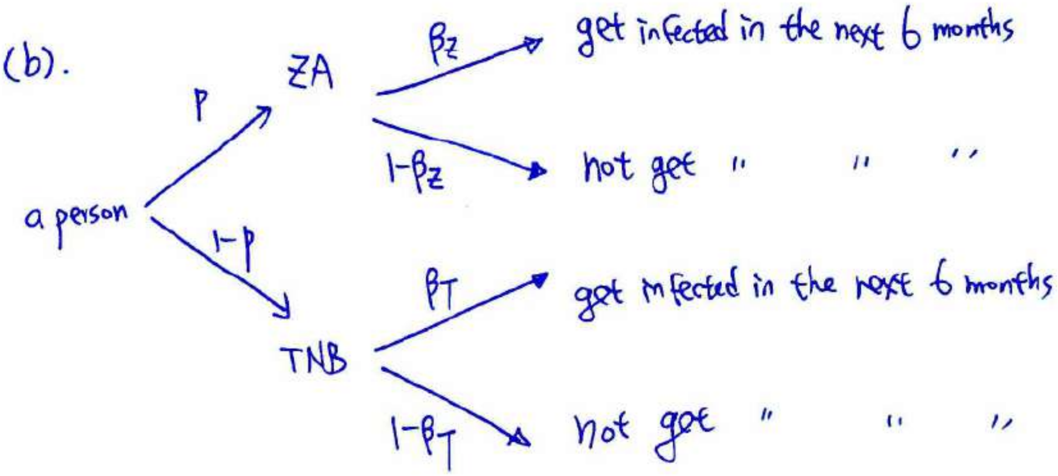
$$\begin{cases} P(E_r \cap E_{inf} \mid E_{TNB}) \leq \min\{\alpha_T, \beta_T\} = \beta_T \\ P(E_r \cap E_{inf} \mid E_{TNB}) \geq \max\{\alpha_T + \beta_T - 1, 0\}. \end{cases}$$

Therefore, the maximum possible $P(E_{ZA} \mid E_r \cap E_{inf})$ is

$$\frac{p \cdot \beta_Z}{p \cdot \beta_Z + (1-p) \cdot \max\{\alpha_T + \beta_T - 1, 0\}}.$$

$\square$

(b).



Define X = number of people (in the selected group of 100 people) get infected in the next 6 months

Then, X is a Binomial random variable with $n=100$,

success probability $= p \cdot \beta_Z + (1-p) \cdot \beta_T$

P(none of these 100 people get infected in the next 6 months)

$= P(X=0)$

$= \left[1-\left(p \cdot \beta_Z + (1-p) \cdot \beta_T\right)\right]^{100}$

$\square$

Problem 6

(a) For simplicity, define the following events:

$E_n = \{ \text{the n-th trial is a success} \}$

$A_n = \{ \text{Version A is assigned to user } n \}$

$B_n = \{ \text{Version B is assigned to user } n \}$

$$P(E_n) = P(E_n|A_n)\cdot P(A_n) + P(E_n|B_n)\cdot P(B_n) \quad \text{.... total probability theorem}$$

$$= a\cdot\alpha_n + b\cdot(1-\alpha_n)$$

$$= (a-b)\cdot\alpha_n + b$$

by total probability theorem

$$P(A_{n+1}) = P(A_{n+1}|A_n\cap E_n)\cdot P(A_n\cap E_n) + P(A_{n+1}|B_n\cap E_n^c)\cdot P(B_n\cap E_n^c)$$

$$= 1\cdot\alpha_n\cdot a + 1\cdot(1-\alpha_n)\cdot(1-b)$$

$$= (a+b-1)\alpha_n + 1-b.$$

$\square$

(Cont.).

(b). $P_n = (a-b)\alpha_n + b \Rightarrow \alpha_n = \dfrac{P_n - b}{a-b}.$

Given that $\alpha_{n+1} = (a+b-1)\alpha_n + 1 - b$,

we have $\dfrac{P_{n+1}-b}{a-b} = (a+b-1)\cdot\dfrac{P_n - b}{a-b} + 1 - b$

$\Leftrightarrow \quad P_{n+1} - b = (a+b-1)(P_n - b) + (1-b)(a-b)$

$\Leftrightarrow \quad P_{n+1} = (a+b-1)P_n + a + b - 2ab.$ ——— (*)

To find $\lim\limits_{n\to\infty} P_n$, we can rewrite (*) as

$$\left(P_{n+1} - q\right) = (a+b-1)\cdot\left(P_n - q\right), \quad \text{where } q = \frac{a+b-2ab}{2-a-b}$$

Then, we have

$$\begin{aligned}
(P_n - q) &= (a+b-1)\cdot\cancel{(P_{n-1} - q)} \\
\cancel{(P_{n-1} - q)} &= (a+b-1)\cdot\cancel{(P_{n-2} - q)} \\
&\ \ \vdots \qquad\qquad \vdots \\
\times)\ \cancel{(P_1 - q)} &= (a+b-1)\cdot(P_0 - q) \\
\hline
P_n - q &= (a+b-1)^{n-1}\cdot(P_0 - q)
\end{aligned}$$

Hence, $\lim\limits_{n\to\infty} P_n = q = \dfrac{a+b-2ab}{2-a-b}$

□.