Final exam solution

Problem 1:

(a). O

(b). X (Recall that $Var(X_1+X_2) = Var(X_1)+Var(X_2)+2\cdot Cov(X_1,X_2)$, and $\therefore$ we may not have $Cov(X_1,X_2)=0$)

(c) O

(d) X (Counterexample: Let Y, Z be two independent standard normal r.v.s.

Define $X_1 = |Y|\cdot sign(Z)$

$X_2 = Y$)

(e) O

(f) X (Counterexample: $X_n = \begin{cases} n, & w.p\ \frac{1}{n} \\ 0, & w.p.\ 1-\frac{1}{n} \end{cases}$

Then, $X_n \xrightarrow{P} 0$

However, $E[X_n^4] = n^3$. This suggests that $\lim_{n\to\infty} E[X_n^4] = \infty$)

(g) O

Problem 2:

If $X_i \sim Exp(\lambda)$, then $E[X_i] = \frac{1}{\lambda}$ and $Var[X_i] = \frac{1}{\lambda^2}$

By the problem description, we know $\lambda = 1$.

$$P\left(0.95 < \bar{X} < 1.05\right) = P\left(\frac{28\cdot(0.95-1)}{\sqrt{28\cdot 1}} < \frac{28\cdot\bar{X} - 28}{\sqrt{28\cdot 1}} < \frac{28\,(1.05-1)}{\sqrt{28\cdot 1}}\right)$$

$$= P\left(-\frac{\sqrt{7}}{10} < \frac{28\bar{X}-28}{\sqrt{28}} < \frac{\sqrt{7}}{10}\right)$$

By normal approximation

$$\approx \Phi\left(\frac{\sqrt{7}}{10}\right) - \Phi\left(-\frac{\sqrt{7}}{10}\right)$$

$$\left(= 1 - 2\cdot\Phi\left(-\frac{\sqrt{7}}{10}\right) = 2\Phi\left(\frac{\sqrt{7}}{10}\right) - 1 \right)$$

Problem 3

$$f(x) = \frac{1}{2}\exp(-|x|), \quad -\infty < x < \infty.$$

(a). $$M_X(t) = E[e^{tX}] = \int_{-\infty}^{+\infty} e^{tx} \cdot \frac{1}{2} e^{-|x|}\, dx$$

$$= \int_{-\infty}^{0} \frac{1}{2} e^{(t+1)x}\, dx + \int_{0}^{\infty} \frac{1}{2} e^{(t-1)x}\, dx$$

we need to have $-1 < t < 1$ to ensure both integrals exist

$$= \frac{1}{2(t+1)} e^{(t+1)x}\Big|_{-\infty}^{0} + \frac{1}{2(t-1)} e^{(t-1)x}\Big|_{0}^{\infty}$$

$$= \frac{1}{2(t+1)} - \frac{1}{2(t-1)} \left(= \frac{1}{1-t^2} \right), \text{ for } -1<t<1$$

(b). $$M_X(t) = \frac{1}{1-t^2} = \sum_{r=0}^{\infty} t^{2r}$$

$$\frac{dM_X(t)}{dt} = \sum_{r=1}^{\infty} (2r) \cdot t^{2r-1}$$

Similarly, we have

$$\frac{d^{2n} M_X(t)}{dt^{2n}} = \sum_{r=n}^{\infty} (2r)\cdot(2r-1)\cdots(2r-2n+1)\cdot t^{2r-2n} \Rightarrow E[X^{2n}] = \frac{d^{2n} M_X(t)}{dt^{2n}}\Big|_{t=0}$$

$$= (2n)\cdot(2n-1)\cdots 1$$

$$= (2n)!$$

$$\frac{d^{2n+1} M_X(t)}{dt^{2n+1}} = \sum_{r=n+1}^{\infty} (2r)\cdot(2r-1)\cdots(2r-2n)\cdot t^{2r-(2n+1)} \Rightarrow E[X^{2n+1}] = \frac{d^{2n+1} M_X(t)}{dt^{2n+1}}\Big|_{t=0}$$

$$= 0$$

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Problem 4 We already know the PDFs of X_i's are uniformly bounded by $C>0$.

(a).

$$E[e^{-tX_i}] = \int_0^{+\infty} \overset{\geq 0}{e^{-tx}} \cdot f(x)\,dx$$

$$\leq C \cdot \int_0^{+\infty} e^{-tx}\,dx$$

$$= C \cdot -\frac{1}{t} e^{-tx} \Big|_0^{\infty}$$

$$= \frac{C}{t}, \quad \text{for all } t>0.$$

(b). For any $t>0$, we have

$$P\left(\sum_{i=1}^{N} X_i \leq \varepsilon N\right) = P\left(e^{t\sum_{i=1}^{N} X_i} \leq e^{t\varepsilon N}\right)$$

$$= P\left(e^{-t\sum_{i=1}^{N} X_i} \geq e^{-t\varepsilon N}\right)$$

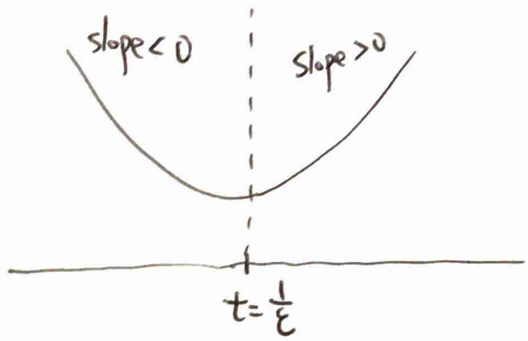
$$\leq \frac{E\left[e^{-t\sum_{i=1}^{N} X_i}\right]}{e^{-t\varepsilon N}}$$

$$= \frac{\prod E[e^{-tX_i}]}{e^{-t\varepsilon N}}$$

$$\leq \left(\frac{C}{t} e^{t\varepsilon}\right)^N$$

Next, we minimize (*) over $t>0$:

Define $h(t) = \left(\frac{C}{t} e^{t\varepsilon}\right)^N$



Then, $\ln h(t) = N \cdot (\ln C - \ln t + t\varepsilon)$

$$\frac{d \ln h(t)}{dt} = N\left(-\frac{1}{t} + \varepsilon\right)$$

Therefore, we know the minimizer of $\ln h(t)$ and $h(t)$ is $t = \frac{1}{\varepsilon}$.

Hence, by the Chernoff technique, we conclude that

$$P\left(\sum_{i=1}^{N} X_i \leq \varepsilon N\right) \leq \left(\frac{C}{\frac{1}{\varepsilon}} e^{\frac{1}{\varepsilon}\cdot\varepsilon}\right)^N = (Ce\varepsilon)^N.$$

Problem 5

(a).

Define $A = \{\omega: X_n(\omega) \text{ does not converge to } a\}$

$B = \{\omega: Y_n(\omega) \text{ does not converge to } b\}$

Since $X_n \xrightarrow{a.s} a$, then $P(A) = 0$

Similarly, as $Y_n \xrightarrow{a.s} b$, then $P(B) = 0$.

Define $C = \left\{\omega: \frac{X_n(\omega)}{Y_n(\omega)} \text{ does not converge to } \frac{a}{b}\right\}$.

By the definition of the events A, B, C, we know $C \subseteq (A \cup B)$

Therefore, $P(C) \leq P(A \cup B) \underset{\substack{\uparrow \\ \text{union} \\ \text{bound}}}{\leq} P(A) + P(B) = 0$

Hence, $\frac{X_n}{Y_n} \xrightarrow{a.s.} \frac{a}{b}$

$\square$

(b).

$$Z_n := \frac{U_1 + \cdots + U_n}{V_1 + \cdots + V_n} = \frac{\frac{U_1 + \cdots + U_n}{n}}{\frac{V_1 + \cdots + V_n}{n}}$$

For every $n \in \mathbb{N}$, define $\tilde{U}_n = \frac{1}{n}(U_1 + \cdots + U_n)$

$$\tilde{V}_n = \frac{1}{n}(V_1 + \cdots + V_n)$$

Since U_i's are i.i.d., then by SLLN we know $\tilde{U}_n \xrightarrow{a.s.} E[U]$

Similarly, by SLLN we have $\tilde{V}_n \xrightarrow{a.s.} E[V]$

Now, by the result of (a), if we view $\tilde{U}_n$ as X_n and $\tilde{V}_n$ as Y_n, then

we can conclude that $\frac{\tilde{U}_n}{\tilde{V}_n} \xrightarrow{a.s.} \frac{E[U]}{E[V]}$

Hence, $Z_n \xrightarrow{a.s.} \frac{E[U]}{E[V]}$

$\square$

Problem 6

(a). Likelihood function $= \prod_{i=1}^{5} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)$,

where $x_1 = 200$, $x_2 = 150$, $x_3 = 250$, $x_4 = 350$, $x_5 = 400$.

By taking the derivative of the log-likelihood function,

it is easy to verify that $\mu_{MLE} = \frac{x_1+x_2+x_3+x_4+x_5}{5} = \frac{1350}{5} = \underline{270}$ #

(b). Posterior $= \overset{\text{a normalizing constant}}{C_0} \cdot \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma_0}\exp\left(-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right)}_{\text{prior}} \cdot \underbrace{\prod_{i=1}^{5}\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)}_{\text{likelihood.}}$

$$\text{log-posterior} = \underbrace{\ln C_0 + \ln\left(\frac{1}{\sqrt{2\pi}\,\sigma_0}\right) + 5\cdot\ln\frac{1}{(\sqrt{2\pi})\sigma} - \frac{(\mu-\mu_0)^2}{2\sigma_0^2} - \sum_{i=1}^{5}\frac{(x_i-\mu)^2}{2\sigma^2}}_{\text{a quadratic function of } \mu}$$

By taking the derivative of the log-posterior,

it is easy to verify that $\mu_{MAP} = \dfrac{\frac{\mu_0}{\sigma_0^2} + \frac{x_1+x_2+x_3+x_4+x_5}{\sigma^2}}{\frac{1}{\sigma_0^2} + \frac{5}{\sigma^2}} = \frac{1650}{6} = \underline{275}$ #