Problem 1.

(a) ○ (Union bound)

(b) ✗ (Y is still exponential only if $a > 0$)

(c) ○ (HW2, Problem 1(b))

(d). ○

(e) ✗ ($P(\{1,2\}) = 0.3$ and $P(\{2,3\}) = 0.5$ imply that $P(\{3\}) \geq 0.2$. Therefore, the maximum possible value of $P(\{4\})$ is 0.5)

(f) ○ (We can divide the $2n$ objects into 2 groups. Then, choosing Y from $2n$ objects is equivalent to choosing k from group 1 and $(Y-k)$ from group 2, for all $k = 0, 1, \cdots, Y$)

(g) ✗ (By definition, A, B, C are independent if
$$\begin{cases} P(A \cap B) = P(A) P(B) \\ P(A \cap C) = P(A) P(C) \\ P(B \cap C) = P(B) P(C) \\ P(A \cap B \cap C) = P(A) P(B) P(C) \end{cases}$$)

Problem 2:

$$f(x) = \sqrt{2k} \cdot \exp\left(-k^2 x^2 + 2kx - 1\right)$$

$$= \sqrt{2k} \cdot \exp\left(-(kx-1)^2\right) = \sqrt{2k} \cdot \exp\left(-\frac{(x-\frac{1}{k})^2}{(\frac{1}{k})^2}\right)$$

$$\Rightarrow \begin{cases} \sqrt{2k} = \dfrac{1}{\sigma\sqrt{2\pi}} \Rightarrow \dfrac{1}{\sigma^2} = 4k\pi \\[3mm] \dfrac{1}{k^2} = 2\sigma^2 \Rightarrow \dfrac{1}{k^2} = 2 \cdot \dfrac{1}{4k\pi} = \dfrac{1}{2k\pi} \Rightarrow k = 2\pi \\[3mm] \dfrac{1}{k} = \mu \end{cases}$$

Therefore, we conclude that $\begin{cases} \mu = \dfrac{1}{2\pi} \\[4mm] \sigma^2 = \dfrac{1}{8\pi^2} \end{cases}$

Problem 3:

For any integer $k \geq 0$:

$$P(Y=k) = \sum_{m=0}^{k} P(X_1=m) \cdot P(X_2=k-m)$$

$$= \sum_{m=0}^{k} \frac{e^{-\lambda_1 T} \cdot (\lambda_1 T)^m}{m!} \cdot \frac{e^{-\lambda_2 T} \cdot (\lambda_2 T)^{k-m}}{(k-m)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)T} \cdot T^k}{k!} \cdot \sum_{m=0}^{k} \underbrace{\frac{\lambda_1^m \cdot \lambda_2^{k-m}}{m!(k-m)!} \cdot k!}_{C_m^k}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)T} \cdot T^k}{k!} \cdot \underbrace{\sum_{m=0}^{k} C_m^k \cdot \lambda_1^m \cdot \lambda_2^{k-m}}_{= (\lambda_1+\lambda_2)^k}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)T} \cdot \big((\lambda_1+\lambda_2)T\big)^k}{k!}$$

Therefore, the PMF of Y: $P(Y=k) = \begin{cases} \dfrac{e^{-(\lambda_1+\lambda_2)T} \cdot \big((\lambda_1+\lambda_2)T\big)^k}{k!} & , \; k \geq 0 \\ \\ 0 & , \; \text{else} \end{cases}$

Remark: Y is a Poisson random variable with parameters $(\lambda_1+\lambda_2, T)$.

Problem 4: $\quad X \sim \text{Unif}(0,5), \quad Y \sim \text{Exp}(2),$ and X, Y are independent

(a).

PDF of X: $\quad f_X(x) = \begin{cases} \frac{1}{5}, & 0 < x < 5 \\ 0, & \text{else} \end{cases}$

$$E[e^X] = \int_{-\infty}^{+\infty} e^x \cdot f_X(x)\,dx = \int_0^5 e^x \cdot \frac{1}{5}\,dx = \frac{1}{5}e^x\Big|_0^5 = \frac{1}{5}(e^5 - 1) \quad \text{\#}$$

$$E[(e^X)^2] = E[e^{2X}] = \int_{-\infty}^{+\infty} e^{2X} \cdot f_X(x)\,dx$$

$$= \int_0^5 e^{2X} \cdot \frac{1}{5}\,dx = \frac{1}{10}e^{2X}\Big|_0^5 = \frac{1}{10}(e^{10} - 1).$$

$$Var[e^X] = E[(e^X)^2] - (E[e^X])^2$$

$$= \frac{1}{10}(e^{10} - 1) - \left(\frac{1}{5}(e^5 - 1)\right)^2$$

$$= \frac{3}{50}e^{10} + \frac{2}{25}e^5 - \frac{7}{50} \quad \text{\#}$$

(b). Let $F_{XY}(t,u)$ denote the joint CDF of X and Y.

The CDF of X:

$$F_X(t) = \begin{cases} 0 & , \text{ if } t<0 \\ \dfrac{t}{5} & , \text{ if } 0 \le t \le 5 \\ 1 & , \text{ if } t>5 \end{cases}$$

The CDF of Y:

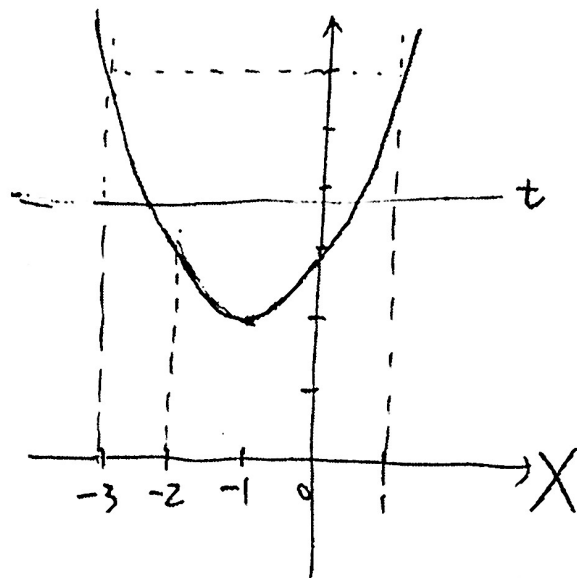$$F_Y(u) = \begin{cases} 1-e^{-2u} & , \text{ if } u \ge 0 \\ 0 & , \text{ if } u<0 \end{cases}$$

Since X and Y are independent, then we know $F_{XY}(t,u) = F_X(t) \cdot F_Y(u)$.

$F_{XY}(t,u)$	$t<0$	$0 \le t \le 5$	$t>5$
$u \ge 0$	0	$\frac{t}{5}(1-e^{-2u})$	$1-e^{-2u}$
$u<0$	0	0	0

(c).

$$W = X^2 + 2X + 3, \qquad X \sim \text{Unif}(0,5)$$

<u>The CDF of W:</u>

$$F_W(t) = P(W \le t)$$

$$= P(X^2 + 2X + 3 \le t)$$

$$= P((X+1)^2 + 2 \le t)$$

We discuss the following cases:

① $\underline{t < 2}$:   The event $\left\{ (X+1)^2 + 2 \le t \right\}$ is an empty set.

② $\underline{t \ge 2}$: $P\left((X+1)^2 + 2 \le t\right) = P\left((X+1)^2 \le t-2\right)$

$$= P\left(|X+1| \le \sqrt{t-2}\right)$$

$$= P\left(X \le -1+\sqrt{t-2} \text{ and } X \ge -1-\sqrt{t-2}\right)$$
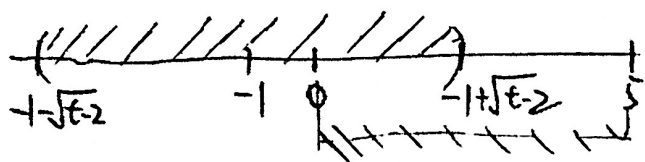
$$= \begin{cases} 0 & , \ 2 \le t < 3 \\ \frac{1}{5}\left(-1+\sqrt{t-2}\right) & , \ 3 \le t \le 38 \\ 1 & , \ t > 38 \end{cases}$$

To conclude:

$$F_W(t) = \begin{cases} 0 & , \quad t < 3 \\ \frac{1}{5}\left(-1+\sqrt{t-2}\right), & 3 \le t \le 38 \\ 1 & , \quad t > 38 \end{cases}$$

Problem 5

(a).

$$P(Z<0 \mid X=+1) = P(Y<-1 \mid X=+1)$$

$$= P(Y<-1)$$

the noise is independent from X

$$= \Phi\left(\frac{-1}{\sigma}\right). \quad \left(\text{or } 1-\Phi\left(\frac{1}{\sigma}\right)\right)$$

Similarly, $P(Z \geq 0 \mid X=-1) = P(Y \geq 1 \mid X=-1)$

$$= P(Y \geq 1)$$

$$= \Phi\left(\frac{-1}{\sigma}\right). \quad \left(\text{or } 1-\Phi\left(\frac{1}{\sigma}\right)\right)$$

(b). $P\left(\text{"+" is sent} \mid \text{"+" is received}\right) = \dfrac{P\left(\text{"+" is sent and "+" is received}\right)}{P\left(\text{"+" is received}\right)}$

Bayes' rule

$$\underline{\downarrow} \quad \frac{P\left(\text{"+" is sent}\right) \cdot P\left(\text{"+" is received} \mid \text{"+" is sent}\right)}{P\left(\text{"+" is sent}\right) P\left(\text{"+" is received} \mid \text{"+" is sent}\right) + P\left(\text{"-" is sent}\right) \cdot P\left(\text{"+" is recv} \mid \text{"-" is sent}\right)}$$

$$= \frac{P \cdot \left(1-\Phi\left(\frac{-1}{\sigma}\right)\right)}{P \cdot \left(1-\Phi\left(\frac{-1}{\sigma}\right)\right) + (1-p) \cdot \left(\Phi\left(\frac{1}{\sigma}\right)\right)}.$$

(c).

For ease of notation, use S and R to denote the sent bits and the received bits, respectively.

$$P(\ S="+-"\ |\ R="+-"\ )$$

Bayes' rule
$\downarrow$

$$= \frac{P(S="+-")\cdot P(R="+-"\ |\ S="+-")}{\begin{array}{l} P(S="+-")\cdot P(R="+-"|S="+-") + P(S="++")\cdot P(R="+-"|S="++") + \\ \qquad P(S="-+")\cdot P(R="+-"|S="-+") + P(S="--")\cdot P(R="+-"|S="--") \end{array}}$$

$$= \frac{p\cdot(1-p)\cdot\left(1-\Phi\left(\frac{-1}{\sigma}\right)\right)^2}{p(1-p)\left(1-\Phi\left(\frac{-1}{\sigma}\right)\right)^2 + p^2\left(1-\Phi\left(\frac{-1}{\sigma}\right)\right)\cdot\Phi\left(\frac{-1}{\sigma}\right) + (1-p)\cdot p\left(\Phi\left(\frac{-1}{\sigma}\right)\right)^2 + (1-p)^2\Phi\left(\frac{-1}{\sigma}\right)\cdot\left(1-\Phi\left(\frac{-1}{\sigma}\right)\right)}$$

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Problem 6 :   Let X be the number of red lights, and $X \sim \text{Binomial}(6, \frac{1}{3})$.

① $T = 15 + X$ ⇒ The PMF of T:

$$P(T=k) = \begin{cases} C_{k-15}^{6} \left(\frac{1}{3}\right)^{k-15} \cdot \left(\frac{2}{3}\right)^{21-k}, & 15 \le k \le 21 \\\\ 0, & \text{else.} \end{cases}$$

————————————————— ✳

② $E[T] = 15 + E[X]$ (by the linearity of expected value).

$$= 15 + 6 \times \frac{1}{3}$$

$$= \underline{17}.$$
   ✳

③ $\text{Var}[X] = \text{Var}[X]$ (since translation does not change the variance)

$$= 6 \times \frac{1}{3} \times \left(1 - \frac{1}{3}\right)$$

$$= \frac{4}{3}$$
—————————— ✳