109550020 Prob HW2 胡景誠

Problem 1

(a) $p_X(k) = \frac{e^{-\lambda T}(\lambda T)^k}{k!}$ $\quad p_X(\lambda T) = \frac{e^{-\lambda T}(\lambda T)^{\lambda T}}{(\lambda T)!}$

let $\Delta k \geq 0, \in \mathbb{N}$,

$$p_X(\lambda T + \Delta k) = \frac{e^{-\lambda T}(\lambda T)^{(\lambda T + \Delta k)}}{(\lambda T + \Delta k)!} = \frac{e^{-\lambda T}(\lambda T)^{\lambda T}(\lambda T)^{\Delta k}}{(\lambda T)! \cdot \prod_{n=1}^{\Delta k}(\lambda T - n)}$$

$$= p_X(\lambda T)\frac{(\lambda T)^{\Delta k}}{\prod_{n=1}^{\Delta k}(\lambda T - n)}$$, monotonically decrease as Δk increase

Similarly

$$p_X(\lambda T - \Delta k) = p_X(\lambda T)\frac{(\lambda T)^{\Delta k}}{\prod_{n=1}^{\Delta k}(\lambda T + n)}$$, monotonically non-decrease as Δk decrease

$$\Rightarrow k^* = \lfloor \lambda T \rfloor = \arg\max(k \in \mathbb{N} \cup \{0\} \; p_X(k))$$

(b) $X = \min(X_1, X_2, \ldots, X_n)$

the CDF of X is $F_X(k) = P(X \leq k) = P(\text{at least one of } X_i \leq k)$

$$= 1 - P(\text{all } X_i > k)$$

$$= 1 - P(X_1 > k)P(X_2 > k) \ldots P(X_n > k) \text{ (independent)}$$

$$= 1 - P(X_1 > k)^n \text{ (share the same } p\text{)}$$

$$= 1 - [1 - F_{X_1}(k)]^n$$ CDF of X_1

$$F_{X_1}(k) = 1 - (1-p)^k$$

$$p_X(k) = F_X(k) - F_X(k-1)$$

$$= 1 - [1 - F_{X_1}(k)]^n - \{1 - [1 - F_{X_1}(k-1)]^n\}$$

$$= [1 - F_{X_1}(k-1)]^n - [1 - F_{X_1}(k)]^n$$

$$= \{1 - [1 - (1-p)^{k-1}]\}^n - \{1 - [1 - (1-p)^k]\}^n$$

$$= (1-p)^{n(k-1)} - (1-p)^{nk}$$

$$= (1-p)^{n(k-1)}(1 - (1-p)) = p(1-p)^{n(k-1)}$$

X is a Geometric Random Variable

Problem 2

let X_i = # of balls in box i, $i \in \mathbb{N}$

$X_1 + X_2 + \dots + X_n = r$, there are $H^n_r = C^{n+r-1}_r$ cases

(a) if $X = X_1 = k \in \mathbb{N} \cup \{0\}$

$X_2 + X_3 + \dots + X_n = r-k$, there are $H^{n-1}_{r-k} = C^{(n-1)+r-k-1}_{r-k}$ cases

thus $q_k = \dfrac{C^{n+r-k-2}_{r-k}}{C^{n+r-1}_r}$ (since all cases have equal probabilities)

(b) $q_k = \dfrac{(n+r-k-2)!}{(r-k)!} \Big/ \dfrac{(n+r-1)!}{r!} = \dfrac{(n+r-1-k-1)!}{(n+r-1)!} \times \dfrac{r!}{(r-k)!}$

$= \dfrac{r(r-1)\cdots(r-k+1)}{(n+r-1)(n+r-2)\cdots(n+r-1-k)} = \prod_{i=0}^{k} \left(\dfrac{r-i}{n+r-1-i}\right) \times \left(\dfrac{1}{r-k}\right)$

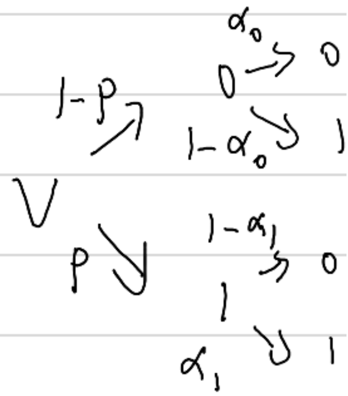
$\lim\limits_{\substack{r\to\infty \\ n\to\infty}} q_k = \lim\limits_{\substack{r\to\infty \\ n\to\infty}} \prod_{i=0}^{k} \left(\dfrac{\frac{r}{n} - \frac{i}{n}}{\frac{n}{n} + \frac{r}{n} - \frac{1}{n} - \frac{i}{n}}\right) \times \left(\dfrac{\frac{1}{n}}{\frac{r}{n} - \frac{k}{n}}\right)$, $\dfrac{r}{n} = \lambda$ as $r \to \infty$, $n \to \infty$

$= \prod_{i=0}^{k} \left(\dfrac{\lambda}{1-\lambda}\right) \times \dfrac{1}{\lambda} = \left(\dfrac{\lambda}{1-\lambda}\right)^{k+1} \times \left(\dfrac{1}{\lambda}\right) = \dfrac{\lambda^k}{(1-\lambda)^{k+1}}$

X in the limit is a Geometric Random Variable

Problem 3

(a) let V = total transmitted bits



$$p_V(k) = \frac{e^{-\lambda T}(\lambda T)^k}{k!}$$

X = # of "1" transmitted

$$p_X(k) = P(X=k)$$

$$= \sum_{n=0}^{\infty} P(X=k \mid V=k+n) \cdot P(V=k+n)$$

$$= \sum_{n=0}^{\infty} C_k^{k+n} p^k (1-p)^n \times p_V(k+n)$$

$$= \sum_{n=0}^{\infty} \frac{(k+n)!}{k!\,n!} p^k (1-p)^n \times \frac{e^{-\lambda T}(\lambda T)^{k+n}}{(k+n)!}$$

$$= e^{-\lambda T} \frac{p^k (\lambda T)^k}{k!} \sum_{n=0}^{\infty} \frac{(1-p)^n (\lambda T)^n}{n!}$$

$$= e^{-\lambda T} \frac{p^k (\lambda T)^k}{k!} e^{(1-p)\lambda T} = \frac{e^{-\lambda p T}(\lambda p T)^k}{k!}$$

$\Rightarrow$ average rate is λp

(b) Similarly, let X' = # of 0's transmitted

$$p_X(n) = P(X'=n)$$

$$= \sum_{x=0}^{\infty} P(X=n \mid V=k+n) \cdot P(V=k+n)$$

$$= \frac{e^{-\lambda T(1-p)} (\lambda T)^n (1-p)^n}{n!}$$

$$p_X(Y) = P(\text{receive } 1 \mid \text{sent } 1) + P(\text{receive } 1 \mid \text{sent } 0)$$

$$= (1-\alpha_0) \frac{e^{-\lambda T p}(\lambda T p)^k}{k!} + \alpha_1 \frac{e^{-\lambda T(1-p)}(\lambda T(1-p))^n}{n!}$$

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Problem 4

(a) $E[X] = \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} p$

$= p\left[\sum_{k=1}^{\infty} (1-p)^{k-1} + \sum_{k=2}^{\infty} (1-p)^{k-1} + \sum_{k=3}^{\infty} (1-p)^{k-1} + \dots\right]$

$= p\left[\frac{1}{p} + \frac{(1-p)}{p} + \frac{(1-p)^2}{p} + \dots\right] = 1 + (1-p) + (1-p)^2 + \dots = \lim_{n\to\infty} \frac{1-(1-p)^n}{1-(1-p)} = \frac{1}{p}$ ※

$Var[X] = E[X^2] - E[X]^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$ ※

$E(X^2) = \sum_{n=1}^{\infty} n^2 q^{n-1} p \quad (q = 1-p)$

$= p \sum_{n=1}^{\infty} \frac{d}{dq} (n q)^n = p \frac{d}{dq} \sum_{n=1}^{\infty} n(q)^n = p \frac{d}{dq}\left\{\frac{q}{1-q} E[X]\right\}$

$= p \frac{d}{dq} \{q(1-q)^{-2}\} = p\left\{\frac{1}{(1-q)^2} - \frac{-2q}{(1-q)^3}\right\}$

$= \frac{1}{p} + \frac{2(1-p)}{p^2} = \frac{2-p}{p^2}$

(b) Since $E[X^m] = E[Y^m]$ for all $m \in \mathbb{N}$

X, Y share the same moment generating function

let $S = \{a_1, a_2, \dots, a_n\}$

$\Rightarrow \sum_{x \in S} e^{tx} P_X(x) = \sum_{y \in S} e^{ty} P_Y(y)$

let $A = e^t$

$\sum_{x \in S} A^x P_X(x) - \sum_{y \in S} A^y P_Y(y) = 0$

$\sum_{x \in S} A^x P_X(x) - \sum_{x \in S} A^x P_Y(x) = 0$

$A^x \sum_{x \in S} P_X(X) - P_Y(X) = 0$

$\Leftrightarrow P_X(X) = P_Y(X) \ \forall x \in S$, the PMF of X, Y are the same

(c)

$E[Z^2] = \sum_{n=1}^{\infty} z_n^2 \frac{6}{\pi^2 n^2} = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{n}{n^2} = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow$ diverges

Var[Z] does not exist

$E[Z^3] = \sum_{n=1}^{\infty} z_n^3 = \frac{6}{\pi^2} \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$

since $0 < \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$, $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

the series converge by alternating series test, $E[Z^3]$ exist

$E[Z^{10}] = \sum_{n=1}^{\infty} z_n^{10} = \frac{6}{\pi^2} \sum_{n=1}^{\infty} n^3$, the series diverge $E[Z^{10}]$ does not exist