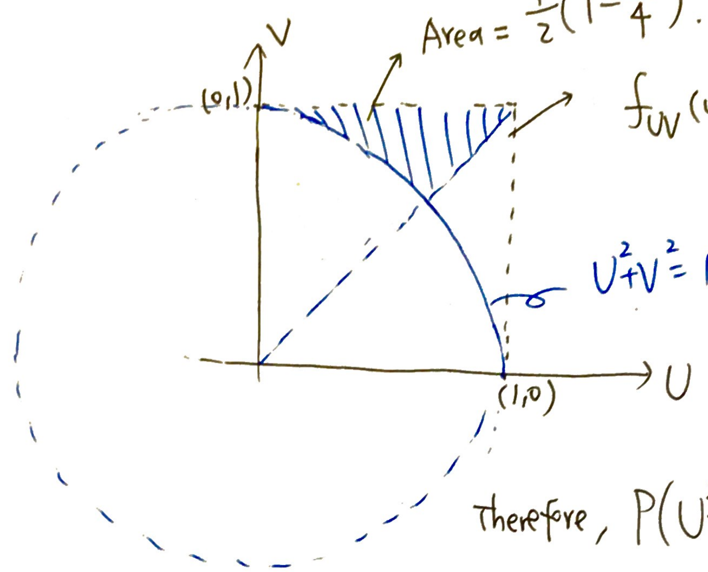
Problem 1

(a) O

(b). X



$$f_{UV}(u,v) = \begin{cases} 1, & 0<u<1,\ 0<v<1 \\ 0, & \text{else} \end{cases}$$

Therefore, $P(U^2+V^2 \geq 1,\ U \leq V)$

$$= \frac{1}{2}\left(1-\frac{\pi}{4}\right)$$

(c) X

Markov's inequality is applicable only to non-negative random variables.

(d) O

(e) X

Since X_1, X_2 are Bernoulli and $E[X_1^2]=\frac{1}{2}$, $E[X_2^2]=\frac{1}{2}$,

then $X_1 = \begin{cases} 1, & \text{w.p. } \frac{1}{2} \\ 0, & \text{w.p. } \frac{1}{2} \end{cases}$, $X_2 = \begin{cases} 1, & \text{w.p. } \frac{1}{2} \\ 0, & \text{w.p. } \frac{1}{2} \end{cases}$

One possible case is that $X_2 = 1 - X_1$.

Then, $E[X_1X_2]=0$, $Cov(X_1,X_2) = E[X_1X_2] - E[X_1]E[X_2]$

$$= 0 - \frac{1}{2}\times\frac{1}{2}$$

$$= -\frac{1}{4}$$

Therefore, it is possible that X_1, X_2 are negatively correlated.

(f) $\underline{\times}$

$$P\left(\left\{\omega : \lim_{n\to\infty} Y_n(\omega) \geq \varepsilon\right\}\right) = 0 \text{ for all } \varepsilon > 0.$$

Hence, $P\left(\left\{\omega : \lim_{n\to\infty} Y_n(\omega) = 0\right\}\right) = 1$, i.e., $Y_n \xrightarrow{a.s.} 0$

(g) $\underline{\bigcirc}$

(h) $\underline{\times}$

$$P(\theta=0.3 \mid THT) = \frac{P(\theta=0.3)\cdot P(THT \mid \theta=0.3)}{P(THT)} = \frac{0.4 \times (0.7)^2 \times (0.3)}{P(THT)}$$

$$P(\theta=0.7 \mid THT) = \frac{P(\theta=0.7)\cdot P(THT \mid \theta=0.7)}{P(THT)} = \frac{0.6 \times (0.3)^2 \times (0.7)}{P(THT)}$$

Therefore, $P(\theta=0.3 \mid THT) > P(\theta=0.7 \mid THT)$.

Problem 2

(a) For $t \neq 0$:

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{+\infty} e^{tx} f_X(x)\,dx$$

$$= \int_{-1}^{3} e^{tx} \frac{1}{4}\,dx = \frac{1}{4t} e^{tx}\Big|_{-1}^{3} = \frac{1}{4t}(e^{3t} - e^{-t})$$

For $t=0$:

$$M_X(t) = \int_{-1}^{3} e^{0} \cdot \frac{1}{4}\,dx = 1$$

Therefore,

$$M_X(t) = \begin{cases} \frac{1}{4t}(e^{3t} - e^{-t}), & t \neq 0 \\ 1, & t = 0. \end{cases}$$

(b).

$$E[X] = M_X'(0) = \lim_{h \to 0} \frac{M_X(h) - M_X(0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{4h}(e^{3h} - e^{-h}) - 1}{h}$$

$$= \lim_{h \to 0} \frac{e^{3h} - e^{-h} - 4h}{4h^2}$$

L'Hôpital's rule

$$= \lim_{h \to 0} \frac{3e^{3h} + e^{-h} - 4}{8h}$$

$$= \lim_{h \to 0} \frac{9e^{3h} - e^{-h}}{8}$$

$$= 1$$

(Cont.)

$$E[X^2] = M_X''(0) = \lim_{h\to 0} \frac{M_X'(h) - M_X'(0)}{h}$$

$$= \lim_{h\to 0} \frac{-\frac{1}{4h^2}(e^{3h}-e^{-h}) + \frac{1}{4h}(3e^{3h}+e^{-h}) - 1}{h}$$

$$= \lim_{h\to 0} \frac{-(e^{3h}-e^{-h}) + h\cdot(3e^{3h}+e^{-h}) - 4h^2}{4h^3}$$

L'Hôpital's rule

$$= \lim_{h\to 0} \frac{\cancel{-(3e^{3h}+e^{-h})} + \cancel{(3e^{3h}+e^{-h})} + h(9e^{3h}-e^{-h}) - 8h}{12h^2}$$

$$= \lim_{h\to 0} \frac{(9e^{3h}-e^{-h}) + h(27e^{3h}+e^{-h}) - 8}{24h}$$

$$= \lim_{h\to 0} \frac{(27e^{3h}+e^{-h}) + (27e^{3h}+e^{-h}) + h(81e^{3h}-e^{-h})}{24}$$

$$= \frac{(27+1)+(27+1)}{24}$$

$$= \frac{7}{3}.$$

Therefore, $\mathrm{Var}[X] = E[X^2] - (E[X])^2$

$$= \frac{7}{3} - 1^2$$

$$= \frac{4}{3}$$

Problem 3

$X_1, X_2, \cdots, X_n$ are i.i.d. standard normal.

Then, $X_1^2, X_2^2, \cdots, X_n^2$ are also i.i.d. with:

$$E[X_i^2] = Var[X_i] + (E[X_i])^2 = 1$$

$$E[X_i^4] = \left.\frac{d^4 M_{X_i}(t)}{dt^4}\right|_{t=0} = 3.$$

$$Var[X_i^2] = E[X_i^4] - (E[X_i^2])^2 = 3 - 1 = 2$$

$$M_{X_i}(t) = e^{\frac{t^2}{2}}$$

$$M'_{X_i}(t) = t \cdot e^{\frac{t^2}{2}}$$

$$M''_{X_i}(t) = e^{\frac{t^2}{2}} + t^2 \cdot e^{\frac{t^2}{2}}$$

$$M'''_{X_i}(t) = te^{\frac{t^2}{2}} + 2t \cdot e^{\frac{t^2}{2}} + t^3 \cdot e^{\frac{t^2}{2}} = 3te^{\frac{t^2}{2}} + t^3 e^{\frac{t^2}{2}}$$

$$M''''_{X_i}(t) = 3e^{\frac{t^2}{2}} + 3t^2 e^{\frac{t^2}{2}} + 3t^2 e^{\frac{t^2}{2}} + t^4 e^{\frac{t^2}{2}}$$

Hence, $M''''_{X_i}(0) = 3$.

Accordingly, by CLT, we can approximate $S_n = X_1^2 + \cdots + X_n^2$ as follows: For any $a \in \mathbb{R}$,

$$P\left(\frac{S_n - n \cdot E[X_i^2]}{\sqrt{n \cdot Var[X_i^2]}} \leq a\right) = P\left(\frac{S_n - n}{\sqrt{2n}} \leq a\right) \approx \Phi(a)$$

Therefore, we have $P(S_n \leq n + \sqrt{2n}) = P\left(\frac{S_n - n}{\sqrt{2n}} \leq 1\right) \approx \Phi(1)$.

$\square$

Problem 4

$X \sim N(0,1)$, $U \sim \text{Bernoulli}(0.5)$, $Z = (2U-1)X$.

(a). $P(Z \le t) = P(U=1 \text{ and } X \le t) + P(U=0 \text{ and } X \ge -t)$

by the independence of U and X $\searrow$
$$= P(U=1)\cdot P(X \le t) + P(U=0)\cdot P(X \ge -t)$$
$$= \frac{1}{2}\cdot \Phi(t) + \frac{1}{2}\underbrace{(1-\Phi(-t))}_{=\Phi(t) \text{ by the symmetry of } \Phi(\cdot)}$$
$$= \frac{1}{2}\Phi(t) + \frac{1}{2}\Phi(t)$$
$$= \Phi(t).$$

(b) $$\mathrm{Cov}(X,Z) = E[XZ] - \overset{0}{\overset{\parallel}{E[X]}}E[Z]$$
$$= E[XZ].$$
$$= P(U=0)\cdot E[XZ \mid U=0] + P(U=1)\cdot E[XZ \mid U=1]$$
$$= P(U=0)\cdot E[-X^2 \mid U=0] + P(U=1)\cdot E[X^2 \mid U=1]$$
$$= 0.$$

(c). Recall that two random variables X_1, X_2 are independent if for arbitrary sets of real numbers A, B, we have $P(X\in A, Y \in B) = P(X \in A)\cdot P(X \in B)$.

It is easy to check that $P(X \in [-1,1], Z \in [2,3]) = 0$, but $P(X \in [-1,1]) > 0$
$P(Z \in [2,3]) > 0$.

Hence, X and Z are not independent. $\square$

Problem 5 $\quad X \sim \text{Exp}(\lambda = 1)$

$$Y_n = \begin{cases} 1, & \text{if } X > n \\ 0, & \text{else} \end{cases}$$

For any $\varepsilon > 1$, we know $P(\{\omega : |Y_n(\omega) - 0| \geq \varepsilon\}) = 0$, for all $n \in \mathbb{N}$

For any $\varepsilon \in (0, 1]$, we know $P(\{\omega : |Y_n(\omega) - 0| \geq \varepsilon\})$

$$= P(\{\omega : X(\omega) > n\})$$

$$= e^{-n} \quad \text{, for all } n \in \mathbb{N}$$

Therefore, we know that for any $\varepsilon > 0$,

$$\lim_{n \to \infty} P(\{\omega : |Y_n(\omega) - 0| \geq \varepsilon\}) = 0$$

Hence, $Y_n \xrightarrow{P} 0$.

$\square$

Problem 6

(a). By Chebyshev's inequality, we have

$$P\left(\left|\frac{X_{A,N}}{N} - E\left[\frac{X_{A,N}}{N}\right]\right| > \delta\right) \leq \frac{Var\left[\frac{X_{A,N}}{N}\right]}{\delta^2}$$

Moreover, we know $E\left[\frac{X_{A,N}}{N}\right] = \theta_a$.

$$Var\left[\frac{X_{A,N}}{N}\right] = \frac{\theta_a \cdot (1-\theta_a)}{N}$$

Then, we conclude that for any $\delta > 0$,

$$P\left(\frac{X_{A,N}}{N} \leq \theta_a - \delta\right) \leq P\left(\left|\frac{X_{A,N}}{N} - \theta_a\right| > \delta\right) \leq \frac{\theta_a (1-\theta_a)}{N\delta^2}$$

$\square$

(b).

$$P\left(\left\{\frac{X_{A,N}}{N} > \frac{X_{B,N}}{N}\right\}\right) \geq P\left(\left\{\frac{X_{A,N}}{N} > \theta_a - \frac{\Delta}{2}\right\} \cap \left\{\frac{X_{B,N}}{N} \leq \theta_b + \frac{\Delta}{2}\right\}\right)$$

by independence of $X_{A,N}$ and $X_{B,N}$

$$= P\left(\left\{\frac{X_{A,N}}{N} > \theta_a - \frac{\Delta}{2}\right\}\right) \cdot P\left(\left\{\frac{X_{B,N}}{N} \leq \theta_b + \frac{\Delta}{2}\right\}\right)$$

$$= \underbrace{\left[1 - P\left(\left\{\frac{X_{A,N}}{N} \leq \theta_a - \frac{\Delta}{2}\right\}\right)\right]}_{\geq 1 - \frac{\theta_a(1-\theta_a)}{N\cdot\left(\frac{\Delta}{2}\right)^2} \text{ by (a)}} \cdot \underbrace{\left[1 - P\left(\left\{\frac{X_{B,N}}{N} > \theta_b + \frac{\Delta}{2}\right\}\right)\right]}_{\geq 1 - \frac{\theta_b(1-\theta_b)}{N\cdot\left(\frac{\Delta}{2}\right)^2} \text{ by }(*)}$$

Note that by Chebyshev's inequality:

$$P\left(\left\{\frac{X_{B,N}}{N} > \theta_b + \frac{\Delta}{2}\right\}\right) \leq P\left(\left|\frac{X_{B,N}}{N} - \theta_b\right| > \frac{\Delta}{2}\right) \leq \frac{\theta_b(1-\theta_b)}{N\cdot\left(\frac{\Delta}{2}\right)^2} \quad (*)$$

Therefore, we have

$$P\left(\left\{\frac{X_{A,N}}{N} > \frac{X_{B,N}}{N}\right\}\right) \geq \left(1 - \frac{\theta_a(1-\theta_a)}{N\left(\frac{\Delta}{2}\right)^2}\right) \cdot \left(1 - \frac{\theta_b(1-\theta_b)}{N\cdot\left(\frac{\Delta}{2}\right)^2}\right).$$

$\square$