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1. (a.) $E[e^{-tX_i}] = \int_0^\infty e^{-tx_i} \underbrace{f_{X_i}(x_i)}_{\leq 1} \, dx_i$

$$\leq \int_0^\infty e^{-tx_i} \, dx_i$$

$$= \left[-\frac{1}{t} e^{-tx_i}\right]_0^\infty$$

$$= \frac{1}{t}$$

(b.) $P\left(\sum_{i=1}^N X_i \leq \varepsilon N\right) = P\left(e^{t\sum_{i=1}^N X_i} \leq e^{t\varepsilon N}\right) = P\left(\underbrace{e^{-t\sum_{i=1}^N X_i}}_{=Y} \geq e^{-t\varepsilon N}\right)$

$$\leq \frac{E(Y)}{e^{-t\varepsilon N}} = \frac{E[e^{-t\sum_{i=1}^N X_i}]}{e^{-t\varepsilon N}} = \frac{E[\prod_{i=1}^N e^{-tX_i}]}{e^{-t\varepsilon N}} = e^{t\varepsilon N} \underbrace{\prod_{i=1}^N E[e^{-tX_i}]}_{\leq \frac{1}{t}}$$

$$\leq \frac{e^{t\varepsilon N}}{t^N} = \left(\frac{e^{t\varepsilon}}{t}\right)^N = (e\varepsilon)^N \quad (\text{choose } t = \tfrac{1}{\varepsilon})$$

2. let $A = \{\omega : \lim_{n\to\infty} X_n(\omega) = a\}$

$B = \{\omega : \lim_{n\to\infty} Y_n(\omega) = b\}$

$P(A) = 1, \quad P(B) = 1$

$P(A \cap B) = P((A' \cup B')') \geq 1 - P(A') - P(B') = 1$

let $Z_n = X_n \cdot Y_n$, for $n \in \mathbb{N}$

$C = \{\omega : \lim_{n\to\infty} Z_n(\omega) = a \cdot b\}$

$$\lim_{n\to\infty} Z_n = \lim_{n\to\infty} (X_n \cdot Y_n) = \lim_{n\to\infty} X_n \cdot \lim_{n\to\infty} Y_n$$

$$= a \cdot b$$

thus $A \cap B$ implies C, $1 = P(A \cap B) \leq P(C)$

$P(C) = 1$ ⋕

(Markov's)

3. (a) for any $\varepsilon > 0$, $P(|X_n - c| \geq \varepsilon) = P(|X_n - c|^2 \geq \varepsilon^2) \leq \dfrac{E[|X_n - c|^2]}{\varepsilon^2}$

$\lim\limits_{n\to\infty} P(|X - c| \geq \varepsilon) \leq \lim\limits_{n\to\infty} \dfrac{E[|X - c|^2]}{\varepsilon^2} = 0$, if convergence in mean square #

(b) Consider a sequence of random variable $Y_1, Y_2, \ldots$

define as $Y_n = \begin{cases} n^2 & \text{with probability } \frac{1}{n} \\ 0 & \text{with probability } 1 - \frac{1}{n} \end{cases}$

for any $\varepsilon > 0$, $\lim\limits_{n\to\infty} P(|X_n| \geq \varepsilon) = \lim\limits_{n\to\infty} P(X_n = n^2) = \lim\limits_{n\to\infty} \frac{1}{n} = 0$

$\lim\limits_{n\to\infty} E[(X_n - 0)^2] = \lim\limits_{n\to\infty} \left(n^4 \cdot \frac{1}{n} + 0 \cdot (1 - \frac{1}{n})\right) = \lim\limits_{n\to\infty} n^3$.

which converge in probability but doesn't converge in the square mean