1179 Probability HW 1

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Problem 1

(a) Let $A = \{x \mid x \in S_n$, for infinitely many $n\}$

(1) $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \subseteq A$: Pick any $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n$, $x \in \bigcup_{n=k}^{\infty} S_n$ for all $k \in \mathbb{N} \Rightarrow x \in A$

(2) $A \subseteq \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty}$: Pick any $y \in A$, then y appears in infinitely many S_n

$\Rightarrow$ for all $k \geq n \in \mathbb{N}$, there must exist $y \in \bigcup_{n=k}^{\infty} S_n$

$\Rightarrow y \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n$

(b) (1) $\bigcap_{n=1}^{\infty} A_n = A_3 \cap A_4 \cap_{n=1}^{\infty} A_n = (B-C) \cap (C-B) \cap_{n=1}^{\infty} A_n = \emptyset \cap_{n=3}^{\infty} A_n = \emptyset$

(2) $\bigcup_{n=1}^{\infty} A_n = A_3 \cup A_4 \cup_{n=1}^{\infty} A_n = (B-C) \cup (C-B) \cup_{n=1}^{\infty} A_n = (B-C) \cup (C-B) = (B \cup C) - (B \cap C)$

(3) from (1) we know $\bigcap_{n=1}^{\infty} A_n = \emptyset$, similarly for all $k \geq 2$,

$\bigcap_{n=k}^{\infty} A_n = (B-C) \cap (C-B) \bigcap_{n=k}^{\infty} A_n = \emptyset$, thus $\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n = \bigcup_{k=1}^{\infty} \emptyset = \emptyset$

(4) from (2) we know $\bigcup_{n=1}^{\infty} A_n = (B \cup C) - (B \cap C)$, similarly for all $k \geq 2$,

$\bigcup_{n=k}^{\infty} A_n = (B-C) \cup (C-B) \bigcap_{n=k}^{\infty} A_n = (B \cup C) - (B \cap C)$, thus $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n = (B \cup C) - (B \cap C)$

(c) Assume that there are countably infinite real numbers in the interval $(0, 1)$, and denoted as $x_1, x_2, x_3 \ldots, x_i$ in decimal expansion like below:

$x_1 = 0.a_{11}a_{12}a_{13}\ldots$

$x_2 = 0.a_{21}a_{22}a_{23}\ldots$

$x_3 = 0.a_{31}a_{32}a_{33}\ldots$

$\vdots$

Then we could find a number y, to choose the diagonal of x_i's, then add one on each digit to form a total different number from the x_i's,

$y = 0.(a_{11}+1)(a_{22}+1)(a_{33}+1)\ldots$ then we add y to the x_i's, and could also form a new y' that is different from all the number before,

so we can tell that real numbers in the interval $(0, 1)$ is uncountablely infinite

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Problem 2

(a) base case : for $N=2$, $P(\bigcup_{n=1}^{2} A_n) = P(A_1)+P(A_2)-P(A_1\cap A_2) \le \sum_{n=1}^{2} P(A_n)$ holds

induction step : Assume for $N=k$ holds, $P(\bigcup_{n=1}^{k+1} A_n) = P(\bigcup_{n=1}^{k} A_n \cup A_{k+1})$

$= P(\bigcup_{n=1}^{k} A_n) + P(A_{k+1}) - P(\bigcup_{n=1}^{k} A_n \cap P(A_{k+1})$

$\le \sum_{n=1}^{k} P(A_n) + P(A_{k+1}) = \sum_{n=1}^{k+1} P(A_n)$ also holds

By mathematical induction $P(\bigcup_{n=1}^{N} A_n) \le \sum_{n=1}^{N} P(A_n)$ holds for $N \in \mathbb{N}$

(b) From total Probability Theorem, $\sum_{n=1}^{5} P(A_n) = 1$

Then we know that $P(\{5\}) = 1 - P(\{1,2,4\}) - P(\{3\}) = 0.3$

$P(\{1\}) = P(\{1,5\}) - P(\{5\}) = 0.2$

$P(\{2,4\}) = P(\{1,2,4\}) - P(\{1\}) = 0.2$

$\min(P(\{2,3,5\}) = \min(P(\{2\})) + P(\{3\}) + P(\{5\}) = 0 + 0.3 + 0.3 = 0.6$

Problem 3

(a) Since $P(\bigcup_{n=k}^{\infty} A_k)$ is decreasing in k, $P(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n) = P(\lim_{k\to\infty} \bigcup_{n=k}^{\infty} A_n)$

$= \lim_{k\to\infty} P(\bigcup_{n=k}^{\infty} A_n)$ (P is continuous) $\le \lim_{k\to\infty} \sum_{n=k}^{\infty} P(A_n)$

We know that $\lim_{k\to\infty} \sum_{n=k}^{\infty} A_n = 0$ because $\sum_{n=1}^{\infty} P(A_n) < \infty$ (converge)

(b) Note that the k-th toss is head as h_k,

$P(I) = P(\bigcap_{k=1}^{\infty} h_k) < \sum_{k=1}^{\infty} P(h_k) = \sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} 100 k^{-N}$

$\lim_{k\to\infty} \left| \frac{100(k+1)^{-N}}{100 k^{-N}} \right| = \lim_{k\to\infty} \frac{k^N}{(k+1)^N} < 1$ when $N > 1$, then the series converge, $\sum_{k=1}^{\infty} p_k < \infty$

By the Borel-Cantelli Lemma, $p(I) = 0$

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Problem 4

(a) $$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{\underbrace{P(A_1)P(B|A_1)}_{=x_1} + \underbrace{P(A_2)P(B|A_2)}_{=x_2} + \underbrace{P(A_3)P(B|A_3)}_{=x_3}}$$

$$= \frac{\frac{1}{3}\times 0.1}{\frac{1}{3}\times 0.1 + \frac{1}{3}\times 0.3 + \frac{1}{3}\times 0.6} = 0.1$$

$$P(A_2|B) = \frac{x_2}{x_1+x_2+x_3} = \frac{\frac{1}{3}\times 0.6}{\frac{1}{3}\times 1} = 0.6$$

$$P(A_3|B) = \frac{x_3}{x_1+x_2+x_3} = \frac{\frac{1}{3}\times 0.3}{\frac{1}{3}\times 1} = 0.3$$

(each toss independent)

(b) $$P(A_1|C) = \frac{\frac{1}{3}(0.1)^4(0.3)^7(0.6)}{\underbrace{\frac{1}{3}(0.1)^4(0.3)^7(0.6)}_{=y_1} + \underbrace{\frac{1}{3}(0.3)^4(0.6)^7(0.1)}_{=y_2} + \underbrace{\frac{1}{3}(0.6)^4(0.3)^7(0.1)}_{=y_3}}$$

$$= \frac{(0.1)^3(0.3)^3}{(0.1)^3(0.3)^3 + (0.6)^6 + (0.6)^3(0.3)^3} = 0.000514138817$$

$$P(A_2|C) = \frac{y_2}{y_1+y_2+y_3} = 0.8884318766$$

$$P(A_3|C) = \frac{y_3}{y_1+y_2+y_3} = 0.1110539846$$

(c) $$P(A_1|C) = \frac{\frac{3}{5}(0.1)^4(0.3)^7(0.6)}{\underbrace{\frac{3}{5}(0.1)^4(0.3)^7(0.6)}_{=z_1} + \underbrace{\frac{1}{5}(0.3)^4(0.6)^7(0.1)}_{=z_2} + \underbrace{\frac{1}{5}(0.6)^4(0.3)^7(0.1)}_{=z_3}}$$

$$= \frac{3(0.1)^3(0.3)^3}{3(0.1)^3(0.3)^3 + (0.6)^6 + (0.6)^3(0.3)^3} \doteq 0.001540832049306663$$

$$P(A_2|C) = \frac{z_2}{z_1+z_2+z_3} = 0.8875192604$$

$$P(A_3|C) = \frac{z_3}{z_1+z_2+z_3} = 0.1109399076$$