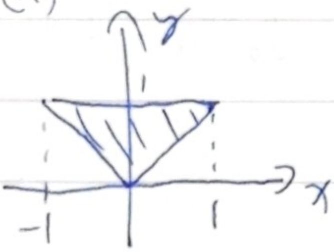
NO.

109550020 Probability HW3 胡昇諭 DATE

1.(a)




$$f_X(x) = \begin{cases} \int_x^1 1\,dy = 1-x, & \text{if } 0<x<y \\ \int_{-x}^1 1\,dy = 1+x, & \text{if } -y<x<0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \int_{-y}^{y} 1\,dx = 2y, & \text{if } 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = \int_{-1}^0 x(1+x)\,dx + \int_0^1 x(1-x)\,dx = \left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-1}^0 + \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1$$

$$= -\frac{1}{2}+\frac{1}{3}+\frac{1}{2}-\frac{1}{3} = 0$$

$$E[Y] = \int_0^1 y\cdot 2y = \frac{2}{3}y^3\Big|_0^1 = \frac{2}{3}$$

$$E[XY] = \int_0^1 y\int_{-y}^{y} x\cdot 1\,dx\,dy = \int_0^1 y\left[\frac{x^2}{2}\right]_{-y}^{y} dy = \int_0^1$$

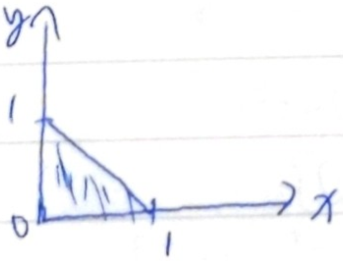
$$= \int_0^1 \frac{y^3}{2}-\frac{y^3}{2}\,dy = 0$$

$$E[XY] = E[X]E[Y] = 0$$

(b) $f_{XY}(\frac{1}{4},\frac{1}{2}) = 1$ $\quad f_X(\frac{1}{4}) = 1-\frac{1}{4} = \frac{3}{4}$ $\quad f_Y(\frac{1}{2}) = 2\times\frac{1}{2} = 1$

$$f_{XY}(\tfrac{1}{4},\tfrac{1}{2}) = 1 \neq \frac{3}{4} = f_X(\tfrac{1}{4})f_Y(\tfrac{1}{2})$$

2. (a)




$$\frac{1\times1\times D}{2} = 1 \qquad f_{XY}(x,y) = \begin{cases} 2, & \text{if } x>0,\ y>0,\ x+y<1 \\ 0, & \text{otherwise} \end{cases}$$

$$D = 2$$

$$f_X(x) = \begin{cases} \int_0^{1-x} 2\,dy = 2(1-x), & \text{if } x>0,\ y>0,\ x+y<1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \int_0^{1-y} 2\,dx = 2(1-y), & \text{if } x>0,\ y>0,\ x+y<1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{X|Y}(x|y) = \frac{2}{2(1-y)} = \frac{1}{1-y}$$

(b) $E[X|Y=y] = \int_0^1 x \frac{1}{1-y}\,dx = \left[\frac{1}{2}\frac{x^2}{1-y}\right]_0^1 = \frac{1}{2(1-y)}$.

$E[X] = E[E[X|Y=y]] = \int_0^1 E[X|Y=y]\cdot f_Y(y)\,dy$

$= \int_0^1 \frac{2(1-y)}{2(1-y)}\,dy = y\Big|_0^1 = 1$

3. (a) $X \sim U(-1, 3)$

$M_X(t) = E[e^{tX}] = \int_{-1}^{3} \frac{1}{4} e^{tX}\,dx = \frac{1}{4t} e^{tX}\Big|_{-1}^{3} = \frac{e^{3t}-e^{-t}}{4t}$

$M_X'(t) = \frac{1}{16t^2}\left[(3e^{3t}+e^{-t})4t - (e^{3t}-e^{-t})4\right]$

$= \frac{1}{4t^2}\left[(3t-1)e^{3t} + (t+1)e^{-t}\right]$

$M_X'(0) = \lim_{t\to 0} M_X'(t) = \lim_{t\to 0} \frac{\overset{\approx -1}{(3t-1)e^{3t}} + \overset{\approx 1}{(t+1)e^{-t}}}{4t^2} \quad \left(\frac{0}{0}\right)$

$\overset{L'H}{=} \frac{\overset{\approx 3}{3e^{3t}} + \overset{\approx -3}{3(3t-1)e^{3t}} + \overset{\approx 1}{e^{-t}} + \overset{\approx -1}{-(t+1)e^{-t}}}{8t} \quad \left(\frac{0}{0}\right)$

$\overset{L'H}{=} \frac{9e^{3t} + 9e^{3t} + 9(3t-1)e^{3t} - e^{-t} + (t+1)e^{-t}}{8} = \frac{9+9-9-1-1+1}{8} = 1$

$$M_X''(t) = \frac{1}{4t^4}\left[\left(3e^{3t} + 3(3t-1)e^{3t} + e^{-t} - (t+1)e^{-t}\right)t^2 - 2t\left[(3t-1)e^{3t} + (t+1)e^{-t}\right)\right]$$

$$= \frac{1}{4t^3}\left(3te^{3t} + 9t^2e^{3t} - 3te^{3t} + te^{-t} - t^2e^{-t} - te^{-t} - 6te^{3t} + 2e^{3t} - 2te^{-t} - 2e^{-t}\right)$$

$$= \frac{1}{4t^3}\left[(9t^2 - 6t + 2)e^{3t} - (t^2 + 2t + 2)e^{-t}\right]$$

$$M_X''(0) = \lim_{t\to 0} M_X'(t) = \lim_{t\to 0} \frac{1}{4t^3}\left[(9t^2-6t+2)e^{3t} - (t^2+2t+2)e^{-t}\right] \quad \left(\frac{0}{0}\right)$$

$$\overset{\text{l'H}}{=} \lim_{t\to 0} \frac{1}{12t^2}\left[(18t-6)e^{3t} + 3(9t^2-6t+2)e^{3t} - (2t+2)e^{-t} + (t^2+2t+2)e^{-t}\right) \quad \left(\frac{0}{0}\right)$$

$$\overset{\text{l'H}}{=} \lim_{t\to 0} \frac{1}{24t}\left\{18e^{3t} + 3(18t-6)e^{3t} + 3(18t-6)e^{3t} + 9(9t^2-6t+2)e^{3t} - 2e^{-t} + (2t+2)e^{-t} + (2t+2)e^{-t} - (t^2+2t+2)e^{-t}\right] \quad \left(\frac{0}{0}\right)$$

$$\overset{\text{l'H}}{=} \lim_{t\to 0} \frac{1}{24}\left[54e^{3t} + 2(54e^{3t} + 9(18t-6)e^{3t}) + 9(18t-6)e^{3t} + 27(9t^2-6t+2)e^{3t} + 2e^{-t} + 2(2e^{-t} - (2t+2)e^{-t}) - (2t+2)e^{-t} + (t^2+2t+2)e^{-t}\right]$$

$$= \frac{1}{24}\left(54 + (54-54)\times 2 + 54 - 54 + 2 + 2(2-2) - 2 + 2\right) = \frac{56}{24} = \frac{7}{3}$$

$E[X] = 1$, $E[X^2] = \frac{7}{3}$, $Var(X) = \frac{7}{3} - 1^2 = \frac{4}{3}$

(b) $M_Y(t) = E[e^{tY}] = \sum_{k=1}^{\infty} \frac{6}{\pi^2 k^2} e^{tk}$

for $t \neq 0$, $\lim_{k\to\infty} \left| \frac{p_Y(k+1)e^{t(k+1)}}{p_Y(k)e^{tk}} \right| = \frac{(k+1)^2}{k^2} e^t > 1$, $M_Y(t)$ does not exist

(series converge by ratio test)

4. (a) $M_X(t) = (\frac{1}{3}e^t + \frac{2}{3})^5$

$$= C_0^5 (\frac{2}{3})^5 (\frac{1}{3}e^t)^0 + C_1^5 (\frac{2}{3})^4 (\frac{1}{3}e^t)^1 + \dots + C_5^5 (\frac{2}{3})^0 (\frac{1}{3}e^t)^5$$

$$p_X(k) = \begin{cases} C_k^5 \frac{2^{(5-k)}}{243}, & \text{if } k = 0, 1, 2, 3, 4, 5 \\ 0, & \text{otherwise} \end{cases}$$

(b) $M_X(t) = e^{5(e^t-1)}$ $\quad X \sim \text{poission } (5)$

$$p_X(k) = \frac{e^{-5} 5^k}{k!}, \text{ if } k = 0, 1, 2 \dots$$

5. $X_1 = \sigma_1 Z + \mu_1 = g_1(z, w)$

$X_2 = \sigma_2(\rho Z + \sqrt{1-\rho^2} W) + \mu_2 = g_2(z, w)$

$z = g_1^{-1}(z, w) = \frac{x_1 - \mu_1}{\sigma_1}$

$w = g_2^{-1}(z, w) = \frac{\frac{x_2 - \mu_2}{\sigma_2} - \rho \frac{x_1 - \mu_1}{\sigma_1}}{\sqrt{1-\rho^2}}$

$$J = \begin{vmatrix} \frac{1}{\sigma_1} & 0 \\ \frac{-\rho}{\sqrt{1-\rho^2}\sigma_1} & \frac{1}{\sqrt{1-\rho^2}\sigma_2} \end{vmatrix} = \frac{1}{\sqrt{1-\rho^2}\sigma_1 \sigma_2}$$

$$f_{ZW}(z,w) = f_Z(z)\cdot f_W(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} = \frac{1}{2\pi} e^{-\frac{1}{2}(z^2+w^2)}$$

$$= \frac{1}{2\pi} \exp\left[ -\frac{1}{2}\left( \frac{(x_1-\mu_1)^2}{\delta_1^2} + \left( \frac{(x_1-\mu_1)^2}{\delta_1^2} + \frac{1}{1-\rho^2}\left( \frac{(x_2-\mu_2)^2}{\delta_2^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\delta_1\delta_2} + \frac{\rho^2(x_1-\mu_1)^2}{\delta_1^2} \right)\right)\right)\right]$$

$$= \frac{1}{2\pi} \exp\left[ -\frac{1}{2(1-\rho)^2}\left( \frac{(x_1-\mu_1)^2}{\delta_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\delta_1\delta_2} + \frac{(x_2-\mu_2)^2}{\delta_2^2} \right)\right]$$

$$f_{X_1X_2}(x_1,x_2) = J\cdot f_{ZW}(z,w) = \frac{1}{2\pi\delta_1\delta_2\sqrt{1-\rho^2}} \exp\left[ -\frac{\left( \frac{(x_1-\mu_1)^2}{\delta_1} - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\delta_1\delta_2} + \frac{(x_2-\mu_2)^2}{\delta_2} \right)}{2(1-\rho^2)} \right]$$

Q.E.D

6. $E[|XY|] \le E\{|X|^p]^{\frac{1}{p}} E[|Y|^q]^{\frac{1}{q}}$ — (1)

$\Rightarrow \sum_{x\in X, y\in Y} |xy| \le (\sum_{x\in X} |x|^p)^{\frac{1}{p}} (\sum_{y\in Y} |y|^q)^{\frac{1}{q}}$ — (2) holds homogeneously i.e.

$\Rightarrow \sum_{x\in X, y\in Y} |(\alpha x)(\beta y)| = |\alpha\beta| \sum_{x\in X, y\in Y} |xy|$

$\le |\alpha\beta| (\sum_{x\in X} |x|^p)^{\frac{1}{p}} (\sum_{y\in Y} |y|^q)^{\frac{1}{q}} = (\sum_{x\in X} |\alpha x|^p)^{\frac{1}{p}} (\sum_{y\in Y} |\beta y|^q)^{\frac{1}{q}}$ — (3)

(3) holds when X and Y satisfying $\sum_{x\in X} |x|^p = 1 = \sum_{y\in Y} |y|^q$ — (4)

then we can reduce (2) to $\sum_{x\in X, y\in Y} |xy| \le 1$

by Young's inequality, $\sum_{x\in X, y\in Y} |xy| \le \sum_{x\in X, y\in Y} (\frac{|x|^p}{p} + \frac{|y|^q}{q})$

$= \frac{1}{p} \sum_{x\in X} |x|^p + \frac{1}{q} \sum_{y\in Y} |y|^q$

$\overset{\text{by (4)}}{=} \frac{1}{p} + \frac{1}{q} = 1$

Q.E.D