Review of[1]
Erdős on Graphs: His Legacy of Unsolved Problems
by Fan Chung and Ron Graham
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Review by
Andy Parrish (atparrish@ucsd.edu)
Dept. of Mathematics, University of California - San Diego

Paul Erdős has posed an incredible number of tantalizing problems in fields ranging from number theory to geometry to combinatorics. Erdős' problems have helped to shape mathematical research, but there seems to be no comprehensive collection of his problems.

Erdős on Graphs is an effort by Fan Chung and Ron Graham to collect some of his problems — those about graphs — which remain unsolved. (Full disclosure: I am a student working with both Chung and Graham, but I believe this review to be fair.) The problems are divided up into chapters by subject: Ramsey theory, extremal graph theory, coloring/packing/covering, random graphs, hypergraphs, and infinite graphs. For each chapter, the authors give relevant definitions, and then dive into the problems.

Since the book is focused on open problems, proofs are rare. However, each problem is followed by whatever progress has been made[2], by whom, and occasionally hints at the techniques used. Of course references are abundant. Since "Uncle Paul" liked to offer cash for solutions to his favorite problems, those awards are listed as well.

The book ends with three personal stories about Erdős from his longtime friend Andy Vázsonyi.

By my count, there are over 170 problems of Erdős discussed in the book, underlining his prowess in formulating them.

In this review, I will discuss the chapters, and some individual problems, with the hope of conveying the breadth and depth of the problems asked.

Ramsey Theory

Graph Ramsey theory explores edge-colorings of large graphs, looking for monochromatic subgraphs. For instance, Ramsey's theorem states that there is a least number $r(k, \ell)$ so that, whenever the edges of the complete graph K_N are colored red and blue (with $N \geq r(k, \ell)$), there is either a red k-clique or a blue ℓ-clique. It is well-known that $2^{n/2} \leq r(n, n) \leq 2^{2n}$. Some of the first questions posed in this book are about the true behavior of $r(n, n)$. Namely, does $\lim r(n, n)^{1/n}$ exist, and what is its value? These infamous questions are valued at \$100 and \$250 respectively. But Erdős also asks some perhaps more approachable questions. For example, Burr and Erdős asked about the local growth of Ramsey numbers: is $r(n + 1, n) > c \cdot r(n, n)$ for some fixed $c > 1$.

Similarly, Erdős and Sós wondered about the growth of $r(3, n)$. Kim and Ajtai-Komlós-Szemerédi showed that $r(3, n) = \Theta\left(\frac{n^2}{\log n}\right)$, but it remains unknown whether the difference between successive values $r(3, n + 1) - r(3, n)$ tends to infinity. (It is easily shown that the difference is less than n.)

There is a more general Ramsey function. Given two graphs G and H, let $r(G, H)$ denote the smallest integer so that, for $N \geq r(G, H)$, every red/blue-coloring of the edges of K_N contains either a red G or a

[2]Because the book came out in 1998, many problems have had further progress, some no doubt *because of* the book.

blue H. This is the graph-Ramsey function. Erdős asked about some very specific bounds to get a handle on this. For example, is $r(C_m, H) \leq 2n + \lceil (m-1)/2 \rceil$, where $m \geq 3$ and H is a connected graph with n edges and no isolated vertices? There are many questions of this form — a particular special case which may help the general problem of understanding $r(G, H)$.

There is has been some scattered progress on several of these problems since the book appeared in 1998. Markedly, Alon and Rödl proved that $r(3, 3, n) = \Theta(n^3 \text{polylog } n)$, strongly affirming the conjecture of Erdős and Sós that $r(3, 3, n)/r(3, n) \to \infty$. (Here $r(a, b, c)$ refers to the 3-color Ramsey number, guaranteeing either a red K_a, a blue K_b, or a yellow K_c.)

Extremal graph theory

The canonical example of a result in extremal graph theory is Turán's theorem. Let $T_{n,k}$ denote the complete k-partite graph on n vertices, whose parts each have size either $\lfloor \frac{n}{k} \rfloor$ or $\lceil \frac{n}{k} \rceil$. Turán's theorem states that any K_{k+1}-free graph G on n vertices has at most as many edges as $T_{n,k}$, with equality only when $G \cong T_{n,k}$. More generally, for a fixed graph H, $t(n, H)$ is used to denote the maximum number of edges in an H-free graph on n vertices[3].

Erdős was very fond of extremal problems, and the book gives a short account of his "near miss" of discovering the field in 1938, two years before Turán. The Erdős-Simonovits-Stone theorem gives the asymptotics for general graphs: $t(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right) \binom{n}{2} + o(n^2)$, essentially settling the question of $t(n, H)$ except in the bipartite case, when the bound becomes $o(n^2)$. A key conjecture then is that $t(n, K_{r,r}) > c_r n^{2-1/r}$ for $r \geq 2$, where $c_r > 0$ is some constant depending only on r. This would match the upper bound given independently by Kővári-Sós-Turán and Erdős. It has been proven for $r = 2, 3$.

Erdős and Simonovits propose the behavior of the Turán number for even cycles: $t(n, C_{2k}) \geq cn^{1+1/k}$, to complement the known upper bound of $ckn^{1+1/k}$. The lower bound is known for $k = 2, 3, 5$.

Erdős and Simonovits also proposed a more general bound on the Turán number for a bipartite graph H: $t(n, H) = O(n^{3/2})$ if and only if every subgraph of H has some vertex of degree ≤ 2. Erdős offered \$250 for a proof, or \$100 for a counterexample.

Some further problems in this chapter, getting away from Turán numbers, ask whether all connected graphs on n vertices can be edge-partitioned into at most $\lfloor (n+1)/2 \rfloor$ paths (Erdős-Gallai), and whether every graph with $1 + \lfloor n^2/4 \rfloor$ edges has $2n^2/9$ of its edges each within some pentagon (Erdős).

Coloring, packing and covering

A graph G is k-colorable of it is possible to paint all vertices of G using up to k colors so that no neighboring vertices are the same color. The chromatic number of a graph G, denoted by $\chi(G)$, is the smallest value k so that G is k-colorable.

Although intuition suggests that graphs with large chromatic number should have many edges within the neighborhood of each vertex, Tutte found a family of graphs with arbitrarily large chromatic number, yet no triangles. Erdős improved this result by proving that, for any k and n, there is a graph on n vertices with chromatic number k and girth[4] at least $\frac{\log n}{4 \log k}$.

Letting $g_k(n)$ be the largest girth of a graph on n vertices with chromatic number at least k, Erdős asked whether g_k grows smoothly — does $\lim \frac{g_k(n)}{\log n}$ exist, for k fixed? The above result, together with an upper

[3] Some texts call this number $\text{ex}(n, H)$.

[4] The girth of a graph is the length of its shortest cycle.

bound by Erdős, show that the limit would have to be between $\frac{1}{4\log k}$ and $\frac{2}{\log(k-2)}$.

Erdős and Lovász asked whether, if $a+b=k+1$, and G has chromatic number k but is K_k-free, does G contain two vertex-disjoint subgraphs with chromatic numbers a and b?

Beyond chromatic number, there are various problems asking about chromatic index, list-coloring, and acyclic chromatic number.

For a taste of a packing problem, I'll briefly mention the final question of this chapter, which seems particularly tantalizing. Does every graph G with $n(n+1)/2$ edges (ignoring the number of vertices) have an "ascending subgraph decomposition?" This is a decomposition into n edge-disjoint subgraphs $G_1,\ldots,G_n$ so that G_i is isomorphic to a proper subgraph of G_{i+1}. The number of edges of G requires that G_i has exactly i edges. For example, it is easy to see that K_{n-1}, which has $n(n+1)/2$ edges, may be decomposed as a sequence of star graphs.

Random graphs and graph enumeration

Erdős led the way in studying random graphs. He gave some of the first results in (deterministic) graph theory. Then, working with Rényi, he explored the evolution of the random graph. The primary model, though not what they studied, is $G_{n,p}$, where there are n labeled vertices, and each edge appears with probability p. Chung and Graham recount the fascinating details of how $G_{n,p}$ changes while p grows as a function of n. I will mention a few problems of Erdős for $p=\frac{1}{2}$. In this case, $G_{n,p}$ is simply called the "random graph," since it is uniformly chosen from all possible graphs on n vertices.

It is known that the chromatic number of the random graph is asymptotically $n/\log n$. How well concentrated is this? Shamir and Spencer showed that there is an interval of length $O(n^{1/2})$ so that, as n grows, the probability of the chromatic number being in this range tends to 1. Erdős asks us to prove that the "true" range is $\omega(1)$. In contrast to what we believe here, the range for the clique number of the random graph contains only 1 or 2 values (depending on n).

A very nice conjecture proposed by Erdős and Bollobás is that the random graph on 2^d vertices contains a d-dimensional hypercube with probability tending to 1. Alon and Füredi showed that the conjecture holds when the edge probability is greater than $\frac{1}{2}$.

This chapter also poses questions about subgraph enumeration — counting subgraphs of a certain type. For example, Erdős and Simonovits asked whether every graph on n vertices and $t(n,C_4)$ edges contains at least two copies of C_4, rather than just the one guaranteed. If we ask the same C_4 by a triangle, Rademacher showed that there are actually $\lfloor n/2\rfloor$ triangles in such a graph.

Another question along these lines: Ramsey's theorem guarantees that any graph on n vertices contains either an empty or complete subgraph on $c\log n$ vertices. Erdős, Fajtowicz, and Staton ask whether we can find a larger induced subgraph, if we allow any regular graph.

Hypergraphs

A hypergraph has the same structure as a graph: there is a set of vertices, and edges connecting those vertices. The only difference is that the edges are now allowed to be any size. Most often considered are uniform hypergraphs, where all edges have the same size. For example, graphs are exactly the 2-uniform hypergraphs.

Many questions asked about graphs can be asked just as easily for hypergraphs (though answering them may be a different story). For example, the Turán problem of maximizing edges in a graph while avoiding a particular subgraph translates directly: for an r-regular hypergraph H, let $t(n,H)$ be the maximum number of edges r-regular hypergraph avoiding H. It is easy to show that $\lim t(n,H)/\binom{n}{r}$ exists, but the limit is not known for any $r \geq 3$. Though Turán posed the question, Erdős offered $1000 for its resolution.

Relating to a problem mentioned above, Erdős conjectured that every 3-uniform hypergraph on n vertices with more than $t(n, K_k^{(3)})$ edges must contain $K_{k+1}^{(3)}$ minus an edge. Here $K_k^{(3)}$ refers to the 3-uniform hypergraph on k vertices containing all possible edges. This question is among many other extremal questions about hypergraphs posed in the book.

Much of the rest of the chapter involves hypergraph coloring. A proper coloring of a hypergraph is a vertex-coloring so that no edge is monochromatic. As for graphs, $\chi(H)$ is the chromatic number — the minimum number of colors needed to color H properly. Erdős and Lovász discovered that any 3-chromatic r-uniform hypergraph always has at least three different sizes of edge-intersections (for r large enough). They ask whether the number of sizes increases without bound as r grows, and suggested that the true number is $r-2$.

Infinite graphs

Erdős did a lot of work on infinite graphs, and several of these results are mentioned by Chung and Graham. Many of the open problems in this section are on infinite Ramsey theory, so we just mention the one most valued by Erdős. Let ω be the smallest infinite ordinal. For which ordinals α does it hold that every 2-coloring of the complete graph on ω^α vertices must either contain a red clique on ω^α vertices, or a blue triangle. Equivalently, for which α does every triangle-free graph on ω^α vertices contain an independent set of size ω^α? Erdős offered $1000 for the solution.

Stories of Erdős

The reader who works through all the problems, rather than jumping around the book, with likely finish with the sense that Erdős was utterly devoted to these problems (nevermind his rich work in other fields). It is therefore fitting that the book closes with Andy Vázsonyi's stories of Erdős. As teenagers, the first time they met, Erdős demanded a four-digit number, which he then squared, and followed it up by proving that the reals are uncountable. As the stories go on, we learn that Erdős never held a job, owned almost no possessions, and was in fact homeless, relying on colleagues to give him a place to stay for as long as they could tolerate him. Though much of this is common knowledge, reading the stories makes it seem far more real. It makes clear just how incredible a person he must have been to live and thrive this way.

Opinion

This book has two main sources of value. Of course it is a great repository of open problems. Those listed above are a quick sampling of the many posed and described in the book. In collecting them all in one place, Chung and Graham have done a service to anyone interested in Erdős' style of graph theory who is looking for a difficult problem to work on. Though the book gives the necessary definitions, it is mostly geared toward people already familiar with the topics discussed, so it would not make a good stand-alone reference to graph theory. However, a mathematician with a light background in graph theory should be able to at least make sense of most of the problems, and begin to think about how to approach them.

For those planning to work on these problems, I should mention that three are no longer open. I already mentioned that Alon and Rödl showed that $r(3,3,n)$ grows much faster than $r(3,n)$. Additionally, Gerken showed that every large set of non-collinear points in the plane contains an empty hexagon with none of the other points in its interior. This closed a question related to the Erdős-Szekeres theorem, Erdős' first foray into graph theory. Additionally, Lu found a relatively small graph with no 4-clique, with the property that any 2-coloring of the edges creates a monochromatic triangle, earning $100 by reducing the number of vertices from 3 billion to 9697. Erdős asked whether a million vertices was enough. (Dudek-Rödl independently reduced the number to 130,000).

The other source of value of the book is that it helps give an understanding of Erdős the man. In detailing some of his results, many of his problems, and a bit about his life, I think any reader will learn a great deal about what allowed this hero of mathematics to succeed so spectacularly.