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On Maintenance of the Opportunity List for Class-Teacher Timetable Problems

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One of the principal components of procedures for the solution of class-teacher timetable problems is that for maintenance of the opportunity list. Opportunity list maintenance methods are based on necessary conditions for the existence of a solution. A general framework for necessary conditions, together with four specific sets of necessary conditions, is given.

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Introduction

The construction of timetables for schools is normally done by hand. It is a frustrating and time-consuming task, and the final timetable often contains many compromises.

In spite of the many attempts to use computers for this work (an extensive bibliography is given by Sefton [10]), there does not seem to be widespread use of computers for the construction of real timetables.

One of the more successful approaches, due to Gotlieb [3], was used for some time for schools in Ontario, Canada [7]. Gotlieb considered a well-structured problem involving the arrangement of the required number of meetings between each class and teacher such that no teacher meets more than one class at any time, and no class has more than one teacher at any one time.

Problem 1. This can be formulated as a zero-one programming problem, where one has to find a feasible solution to:

$$\sum_{k=1}^{n} x_{ijk} = A_{ij} \quad (i = 1, 2, \ldots, l; j = 1, 2, \ldots, m), \tag{1.1}$$

$$\sum_{j=1}^{m} x_{ijk} \leq 1 \quad (i = 1, 2, \ldots, l; k = 1, 2, \ldots, n), \tag{1.2}$$

$$\sum_{i=1}^{l} x_{ijk} \leq 1 \quad (j = 1, 2, \ldots, m; k = 1, 2, \ldots, n), \tag{1.3}$$

$$x_{ijk} = 0, 1 \quad (i = 1, 2, \ldots, l; j = 1, 2, \ldots, m; k = 1, 2, \ldots, n)$$

where:

x_{ijk} = 1 if class i and teacher j meet at time k,
= 0 otherwise;
A_{ij} = the number of meetings required between class i and teacher j;
l = the number of classes;
m = the number of teachers; and
n = the number of time periods.

As stated above, this formulation ignores many difficulties posed by real timetable problems, such as: (i) classes may merge or split for certain activities; (ii) double periods may be required for some lessons and they must not extend across recess breaks; (iii) certain resources (especially dedicated classrooms) may be critically limited; and (iv) certain lessons may have to be given at fixed times.

Such complexities are often avoided by performing some of the timetable function by hand before submitting the problem to the computer. The partially completed timetable may be specified as a set of meetings between specific classes and specific teachers at specific time periods. Members of this set are termed *preassignments*, and must be embedded in the final timetable.

Problem 2. The method of Gotlieb, which was tested by Csima and Gotlieb [1], treats assignments one at a time. An *assignment* is a meeting between a specific class and a specific teacher at a specific time period. Preassignments are successively entered into the timetable before any arbitrary assignments are made. De Werra [2], Sefton [10] and Smith [11] used a slightly different formulation, which allows (but does not require) all preassignments to be treated simultaneously. If all preassignments are treated simultaneously this formulation is: find

x_{ijk} $(i = 1, 2, \ldots, l; j = 1, 2, \ldots, m; k = 1, 2, \ldots, n)$ such that

$$\sum_{k=1}^{n} x_{ijk} = A_{ij} \; (i = 1, 2, \ldots, l; j = 1, 2, \ldots, m), \quad (2.1)$$

$$\sum_{j=1}^{m} x_{ijk} \leq B_{ik} \; (i = 1, 2, \ldots, l; k = 1, 2, \ldots, n), \quad (2.2)$$

$$\sum_{i=1}^{l} x_{ijk} \leq C_{jk} \; (j = 1, 2, \ldots, m; \; k = 1, 2, \ldots, n), \quad (2.3)$$

x_{ijk} $= 0,1$ $(i = 1, 2, \ldots, l; j = 1, 2, \ldots, m; k = 1, 2, \ldots, n)$ where

x_{ijk} = 1 if class i and teacher j meet at time k for a meeting other than a preassignment,
= 0 otherwise;

A_{ij} = the number of meetings (other than preassignments) required between class i and teacher j;

B_{ik} = 0 if class i is preassigned or is unavailable for some other reason at time k,
= 1 if class i is available for an assignment at time k;

C_{jk} = 0 if teacher j is preassigned or is unavailable for some other reason at time k,
= 1 if teacher j is available for an assignment at time k;

l = the number of classes;

m = the number of teachers; and

n = the number of time periods.

The required timetable is given jointly by the preassignments and the

x_{ijk} $(i = 1, 2, \ldots, l; j = 1, 2, \ldots, m; k = 1, 2, \ldots, n)$.

This paper deals with some aspects of Problem 2, although the results apply equally to Problem 1 since it is a special case of Problem 2.

Some Aspects of Solution Methods

Each stage in the determination of a solution involves: (i) the selection of an assignment from the list of possible assignments (called the *opportunity list*); and (ii) the consequent maintenance of the opportunity list.

Lions [8] pointed out that several published solution methods rely almost entirely on one or other of these two components, and he conjectured that the best procedure would lie between these extremes. For example, a simple assignment strategy might be combined with an efficient method for opportunity list maintenance.

Methods for maintenance of the opportunity list are based on necessary conditions for the existence of a solution, and one possible consequence of updating the opportunity list is the discovery that no solution exists to the unsolved portion of the timetable. This is called an *infeasibility*.

This paper will examine several sets of necessary conditions which could be used as the basis for opportunity list maintenance.

A General Framework for the Necessary Conditions for the Existence of a Solution

Adding slack variables to the constraints (2) yields the constraint set

$$\sum_{k=1}^{n} x_{ijk} = A_{ij} \; (i = 1, 2, \ldots, l; \; j = 1, 2, \ldots, m), \quad (3.1)$$

$$\sum_{j=1}^{m} x_{ijk} + \beta_{ik} = B_{ik} \; (i = 1, 2, \ldots, l; \; k = 1, 2, \ldots, n), \quad (3.2)$$

$$\sum_{i=1}^{l} x_{ijk} + \gamma_{jk} = C_{jk} \; (j = 1, 2, \ldots, m; \; k = 1, 2, \ldots, n), \quad (3.3)$$

from which it is evident that

$$\sum_{k=1}^{n} \beta_{ik} = \sum_{k=1}^{n} B_{ik} - \sum_{j=1}^{m} A_{ij} \; (i = 1, 2, \ldots, l), \quad (4.1)$$

$$\sum_{k=1}^{n} \gamma_{jk} = \sum_{k=1}^{n} C_{jk} - \sum_{i=1}^{l} A_{ij} \; (j = 1, 2, \ldots, m). \quad (4.2)$$

Table I gives nomenclature for the upper and lower bounds for each variable.

Table I. Upper and Lower Bounds

Variable	Lower bound	Upper bound
x_{ijk} $(i = 1, 2, \cdots, l; j = 1, 2, \cdots, m; k = 1, 2, \cdots, n)$	M_{ijk}	m_{ijk}
β_{ik} $(i = 1, 2, \cdots, l; k = 1, 2, \cdots, n)$	M_{ik}^{β}	m_{ik}^{β}
γ_{jk} $(j = 1, 2, \cdots, m; k = 1, 2, \cdots, n)$	M_{jk}^{γ}	m_{jk}^{γ}

Each of the variables x_{ijk}, β_{ik}, γ_{jk} must be either 0 or 1, which initially leads to lower bounds of zero and upper bounds of 1. However, tighter upper bounds can be simply obtained.

Since no x_{ijk} can be greater than the corresponding A_{ij}, B_{ik}, or C_{jk}, an upper bound on x_{ijk} is:

$m_{ijk} = \min(A_{ij}, B_{ik}, C_{jk})$.

Similarly, upper bounds on β_{ik} and γ_{jk} are:

$$m_{ik}^{\beta} = \min\left[B_{ik}, \left(\sum_{k'=1}^{n} B_{ik'} - \sum_{j=1}^{m} A_{ij}\right)\right],$$

$$m_{jk}^{\gamma} = \min\left[C_{jk}, \left(\sum_{k'=1}^{n} C_{jk'} - \sum_{i=1}^{l} A_{ij}\right)\right].$$

Let

$$(\kappa)^+ = \kappa \quad \text{if } \kappa \geq 0,$$
$$= 0 \quad \text{if } \kappa < 0,$$

and

$$(\kappa)^- = 0 \quad \text{if } \kappa \geq 0,$$
$$= -\kappa \text{ if } \kappa < 0.$$

THEOREM 1. *For any real values of:*

u_{ij} $(i = 1, 2, \ldots, l; j = 1, 2, \ldots, m)$,

v_{ik} $(i = 1, 2, \ldots, l; k = 1, 2, \ldots, n)$,

w_{jk} $(j = 1, 2, \ldots, m; k = 1, 2, \ldots, n)$,

$$\text{if } \delta = \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} M_{ijk}(-u_{ij} + v_{ik} + w_{jk})^+ - \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} m_{ijk}(-u_{ij} + v_{ik} + w_{jk})^- + \sum_{i=1}^{l} \sum_{k=1}^{n} M_{ik}^{\beta}(v_{ik})^+ - \sum_{i=1}^{l} \sum_{k=1}^{n} m_{ik}^{\beta}(v_{ik})^- + \sum_{j=1}^{m} \sum_{k=1}^{n} M_{jk}^{\gamma}(w_{jk})^+ - \sum_{j=1}^{m} \sum_{k=1}^{n} m_{jk}^{\gamma}(w_{jk})^-,$$

$$\eta = -\sum_{i=1}^{l} \sum_{j=1}^{m} u_{ij} \cdot A_{ij} + \sum_{i=1}^{l} \sum_{k=1}^{n} v_{ik} \cdot B_{ik} + \sum_{j=1}^{m} \sum_{k=1}^{n} w_{jk} \cdot C_{jk}, \text{ and}$$

$$\theta = \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} m_{ijk}(-u_{ij} + v_{ik} + w_{jk})^+ - \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} M_{ijk}(-u_{ij} + v_{ik} + w_{jk})^- + \sum_{i=1}^{l} \sum_{k=1}^{n} m_{ik}^{\beta}(v_{ik})^+ - \sum_{i=1}^{l} \sum_{k=1}^{m} M_{ik}^{\beta}(v_{ik})^- + \sum_{j=1}^{m} \sum_{k=1}^{n} m_{jk}^{\gamma}(w_{jk})^+ - \sum_{j=1}^{m} \sum_{k=1}^{n} M_{jk}^{\gamma}(w_{jk})^-,$$

then the relation

$$\delta \leq \eta \leq \theta \tag{5}$$

is necessary for the existence of a solution.

PROOF. Substituting eqs (3.1), (3.2), and (3.3) into the definition for η, one obtains:

$$\begin{aligned}\eta &= -\sum_{i=1}^{l} \sum_{j=1}^{m} u_{ij} \left(\sum_{k=1}^{n} x_{ijk} \right) + \sum_{i=1}^{l} \sum_{k=1}^{n} v_{ik} \left(\sum_{j=1}^{m} x_{ijk} + \beta_{ik} \right) + \sum_{j=1}^{m} \sum_{k=1}^{n} w_{jk} \left(\sum_{i=1}^{l} x_{ijk} + \gamma_{jk} \right), \\ &= \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} x_{ijk}(-u_{ij} + v_{ik} + w_{jk}) + \sum_{i=1}^{l} \sum_{k=1}^{n} \beta_{ik} \cdot v_{ik} + \sum_{j=1}^{m} \sum_{k=1}^{n} \gamma_{jk} \cdot w_{jk}, \\ &= \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} x_{ijk}(-u_{ij} + v_{ik} + w_{jk})^+ - \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} x_{ijk}(-u_{ij} + v_{ik} + w_{jk})^- \\ &\quad + \sum_{i=1}^{l} \sum_{k=1}^{n} \beta_{ik}(v_{ik})^+ - \sum_{i=1}^{l} \sum_{k=1}^{n} \beta_{ik}(v_{ik})^- + \sum_{j=1}^{m} \sum_{k=1}^{n} \gamma_{jk}(w_{jk})^+ - \sum_{j=1}^{m} \sum_{k=1}^{n} \gamma_{jk}(w_{jk})^-.\end{aligned} \tag{6}$$

There is a one to one correspondence between the terms in the right hand side of (6) and the terms in the definitions of both δ and θ. In every case, for nonnegative values of x_{ijk}, β_{ik} and γ_{jk}, the value of each term in δ is less than or equal to the value of the corresponding term on the right hand side of (6), which is in turn less than or equal to the value of the corresponding term in θ.

Therefore $\delta \leq \eta \leq \theta$.

It will be shown subsequently that several sets of necessary conditions can be generated by constraining u_{ij}, v_{ik}, and w_{jk} in the relation (5). Each such set of necessary conditions is a subset of the conditions (5).

If, for any values of u_{ij}, v_{ik}, and w_{jk} there is an equality in the conditions (5), then it may be possible to reduce upper bounds and/or increase lower bounds.

For instance, if $\delta = \eta$, then since it is necessary that:

(i) $M_{ijk}(-u_{ij} + v_{ik} + w_{jk})^+ \leq x_{ijk}(-u_{ij} + v_{ik} + w_{jk})^+$ $(i = 1, 2, \ldots, l; j = 1, 2, \ldots, m; k = 1, 2, \ldots, n)$

(ii) $-m_{ijk}(-u_{ij} + v_{ik} + w_{jk})^- \leq -x_{ijk}(-u_{ij} + v_{ik} + w_{jk})^-$ $(i = 1, 2, \ldots, l; j = 1, 2, \ldots, m; k = 1, 2, \ldots, n)$

(iii) $M_{ik}^{\beta}(v_{ik})^+ \leq \beta_{ik}(v_{ik})^+$ $(i = 1, 2, \ldots, l; k = 1, 2, \ldots, n)$

(iv) $-m_{ik}^{\beta}(v_{ik})^- \leq -\beta_{ik}(v_{ik})^-$ $(i = 1, 2, \ldots, l; k = 1, 2, \ldots, n)$

(v) $M_{jk}^{\gamma}(w_{jk})^+ \leq \gamma_{jk}(w_{jk})^+$ $(j = 1, 2, \ldots, m; k = 1, 2, \ldots, n)$

(vi) $-m_{jk}^{\gamma}(w_{jk})^- \leq -\gamma_{jk}(w_{jk})^-$ $(j = 1, 2, \ldots, m; k = 1, 2, \ldots, n)$

and overall equality of the sum of the above relations has been achieved, each relation must be satisfied as an equality.

Each of these equations states that a variable is equal to one of its bounds unless the multiplier of both the variable and the bound is zero. This leads to the following adjustment of bounds:

(i) m_{ijk} is reduced to M_{ijk} if $-u_{ij} + v_{ik} + w_{jk} > 0$;
(ii) M_{ijk} is increased to m_{ijk} if $-u_{ij} + v_{ik} + w_{jk} < 0$;
(iii) m_{ik}^{β} is reduced to M_{ik}^{β} if $v_{ik} > 0$;
(iv) M_{ik}^{β} is increased to m_{ik}^{β} if $v_{ik} < 0$;
(v) m_{jk}^{γ} is reduced to M_{jk}^{γ} if $w_{jk} > 0$;
(vi) M_{jk}^{γ} is increased to m_{jk}^{γ} if $w_{jk} < 0$.

Similarly if in the condition (5) $\eta = \theta$, then the adjustments to the bounds are:

(i) M_{ijk} is increased to m_{ijk} if $-u_{ij} + v_{ik} + w_{jk} > 0$;
(ii) m_{ijk} is reduced to M_{ijk} if $-u_{ij} + v_{ik} + w_{jk} < 0$;
(iii) M_{ik}^{β} is increased to m_{ik}^{β} if $v_{ik} > 0_i$;
(iv) m_{ik}^{β} is reduced to M_{ik}^{β} if $v_{ik} < 0$;
(v) M_{jk}^{γ} is increased to m_{jk}^{γ} if $w_{jk} > 0$;
(vi) m_{jk}^{γ} is reduced to M_{jk}^{γ} if $w_{jk} < 0$.

These adjustments to the bounds may result in equalities in (5) for other values of u_{ij}, v_{ik}, and w_{jk}, which in turn may lead to further adjustments to the bounds.

The process of bound adjustment can be continued until either a condition (5) is violated (in which case no solution exists), or the bounds reach their final values.

The explicit maintenance of lower bounds can be avoided if desired. If for example some of the bounds M_{ijk} $(i = 1, 2, \ldots, l; j = 1, 2, \ldots, m; k = 1, 2, \ldots, n)$ are nonzero, a new problem in x'_{ijk}, β'_{ik} and γ'_{jk} with all lower bounds equal to zero can be created using the standard linear programming transformation for lower bounds as follows:

$$x'_{ijk} = x_{ijk} = M_{ijk} \quad (i = 2, \ldots, l; j = 1, 2, \ldots, m; k = 1, 2, \ldots, n);$$
$$\beta'_{ik} = \beta_{ik} - M^{\beta}_{ik} \; (i = 1, 2, \ldots, l; k = 1, 2, \ldots, n);$$
$$\gamma'_{jk} = \gamma_{jk} - M^{\gamma}_{jk} \; (j = 1, 2, \ldots, m; k = 1, 2, \ldots, n);$$
$$A'_{ij} = A_{ij} - \sum_{k=1}^{n} M_{ijk} \quad (i = 1, 2, \ldots, l; j = 1, 2, \ldots, m);$$
$$B'_{ik} = B_{ik} - \sum_{j=1}^{m} M_{ijk} - M^{\beta}_{ik} \quad (i = 1, 2, \ldots, l; k = 1, 2, \ldots, n);$$
$$C'_{jk} = C_{jk} - \sum_{i=1}^{l} M_{ijk} - M^{\gamma}_{jk} \quad (j = 1, 2, \ldots, m; k = 1, 2, \ldots, n);$$
$$m'_{ijk} = m_{ijk} - M_{ijk} \; (i = 1, 2, \ldots, l; j = 1, 2, \ldots, m; k = 1, 2, \ldots, n);$$
$$m^{\beta\prime}_{ik} = m^{\beta}_{ik} - M^{\beta}_{ik} \; (i = 1, 2, \ldots, l; k = 1, 2, \ldots, n);$$
$$m^{\gamma\prime}_{jk} = m^{\gamma}_{jk} - M^{\gamma}_{jk} \; (j = 1, 2, \ldots, m; k = 1, 2, \ldots, n).$$

It is evident that the constraints

$$\sum_{k=1}^{n} x'_{ijk} = A'_{ij} \quad (i = 1, 2, \ldots, l; j = 1, 2, \ldots, m),$$
$$\sum_{j=1}^{m} x'_{ijk} + \beta'_{ik} = B'_{ik} \quad (i = 1, 2, \ldots, l; k = 1, 2, \ldots, n),$$
$$\sum_{i=1}^{l} x'_{ijk} + \gamma'_{jk} = C'_{jk} \quad (j = 1, 2, \ldots, m; k = 1, 2, \ldots, n),$$
$$\sum_{k=1}^{n} \beta'_{ik} = \sum_{k=1}^{n} B'_{ik} - \sum_{j=1}^{m} A'_{ij} \quad (i = 1, 2, \ldots, l),$$
$$\sum_{k=1}^{n} \gamma'_{jk} = \sum_{k=1}^{n} C'_{jk} - \sum_{i=1}^{l} A'_{ij} \quad (j = 1, 2, \ldots, m),$$

specify the unsolved portion of the original timetable problem, and that all variables have lower bounds of zero. The information contained in the lower bounds must, of course, be retained, since ultimately these lower bounds specify the solution. This can be done by maintaining a teacher by time matrix with entries of class number if the lower bounds become 1, or entries of zero otherwise.

Some special cases of the conditions (5) will now be given.

Table II. Subscripts in Condition (5) That Give Condition (8)

Condition	Subscripts for which $u_{ij} = 1$	Subscripts for which $v_{ik} = 1$	Subscripts for which $w_{jk} = 1$
(8.1)	$i = p; j = q$		
(8.2)		$i = p; k = r$	
(8.3)			$j = q; k = r$
(8.4)	$i = p;$ $j = 1, 2, \cdots, m$	$i = p;$ $k = 1, 2, \cdots, n$	
(8.5)	$i = 1, 2, \cdots, l;$ $j = q$		$j = q;$ $k = 1, 2, \cdots, n$

Table III. Subscripts in Condition (5) That Give Condition (11)

(11.1)	$i = p; j \in J$	$i = p; k \in \bar{K}$	
(11.2)	$i \in I; j = q$		$j = q; k \in \bar{K}$

The Simple Conditions

Many published solution methods for the timetable problem (see for example De Werra [2]) have used the conditions:

$$\sum_{k=1}^{n} M_{pqk} \le A_{pq} \le \sum_{k=1}^{n} m_{pqk} \quad (p = 1, 2, \ldots, l; q = 1, 2, \ldots, m), \tag{7.1}$$
$$\sum_{j=1}^{m} M_{pjr} \le B_{pr} \quad (p = 1, 2, \ldots, l; r = 1, 2, \ldots, n), \tag{7.2}$$
$$\sum_{i=1}^{l} M_{iqr} \le C_{qr} \quad (q = 1, 2, \ldots, m; r = 1, 2, \ldots, n), \tag{7.3}$$
$$0 \le \sum_{k=1}^{n} B_{pk} - \sum_{j=1}^{m} A_{pj} \; (p = 1, 2, \ldots, l), \tag{7.4}$$
$$0 \le \sum_{k=1}^{n} C_{qk} - \sum_{i=1}^{l} A_{iq} \; (q = 1, 2, \ldots, m), \tag{7.5}$$

without explicit maintenance of the lower bounds.

The conditions (7) examine each class-teacher, class-time, and teacher-time combination independently. A more thorough way of checking these combinations independently is given by the conditions:

$$\sum_{k=1}^{n} M_{pqk} \le A_{pq} \le \sum_{k=1}^{n} m_{pqk} \quad (p = 1, 2, \ldots, l; q = 1, 2, \ldots, m), \tag{8.1}$$
$$\sum_{j=1}^{m} M_{pjr} + M^{\beta}_{pr} \le B_{pr} \le \sum_{j=1}^{n} m_{pjr} + m^{\beta}_{pr} \quad (p = 1, 2, \ldots, l; r = 1, 2, \ldots, n), \tag{8.2}$$
$$\sum_{i=1}^{l} M_{iqr} + M^{\gamma}_{qr} \le C_{qr} \le \sum_{i=1}^{l} m_{iqr} + m^{\gamma}_{qr} \quad (q = 1, 2, \ldots, m; r = 1, 2, \ldots, n), \tag{8.3}$$
$$\sum_{k=1}^{n} M^{\beta}_{pk} \le \sum_{k=1}^{n} B_{pk} - \sum_{j=1}^{m} A_{pj} \le \sum_{k=1}^{n} m^{\beta}_{pk} \quad (p = 1, 2, \ldots, l), \tag{8.4}$$
$$\sum_{k=1}^{n} M^{\gamma}_{qk} \le \sum_{k=1}^{n} C_{qk} - \sum_{i=1}^{l} A_{iq} \le \sum_{k=1}^{n} m^{\gamma}_{qk} \quad (q = 1, 2, \ldots, m). \tag{8.5}$$

The conditions (8) are the conditions (5) having the u_{ij}, v_{ik}, and w_{jk} specified by Table II equal to 1: all other u_{ij}, v_{ik}, and w_{jk} are zero. The conditions (7) follow directly from the conditions (8).

The Gotlieb Conditions

The Gotlieb conditions [3] are based on Hall's conditions for set distinct representatives [5]. The form of Gotlieb's conditions given below is based on work by Sefton [10].

Define the following sets:

$L = \{1, 2, \ldots, l\}$; $M = \{1, 2, \ldots, m\}$; $N = \{1, 2, \ldots, n\}$; and the following arbitrary (possibly empty) subsets:

$$I \subseteq L; \bar{I} = L - I;$$
$$J \subseteq M; \bar{J} = M - J;$$
$$K \subseteq N; \bar{K} = N - K.$$

then the Gotlieb conditions are:

$$\sum_{j\in J} A_{pj} \le \sum_{k=1}^{n} \{\min(\sum_{j\in J} m_{pjk}, B_{pk})\} \quad (p = 1, 2, \ldots, l; \text{all } J), \tag{9.1}$$

$$\sum_{i\in I} A_{iq} \le \sum_{k=1}^{n} \{\min(\sum_{i\in I} m_{iqk}, C_{qk})\} \quad (q = 1, 2, \ldots, m; \text{all } I), \tag{9.2}$$

or

$$\sum_{j\in J} A_{pj} \le \sum_{j\in J}\sum_{k\in K} m_{pjk} + \sum_{k\in\bar{K}} B_{pk} \quad (p = 1, 2, \ldots, l; \text{all } J, K)$$

$$\sum_{i\in I} A_{iq} \le \sum_{i\in I}\sum_{k\in K} m_{iqk} + \sum_{k\in\bar{K}} C_{qk} \quad (q = 1, 2, \ldots, m; \text{all } I, K)$$

which, on rearrangement, leads to:

$$-\sum_{j\in J}\sum_{k\in K} m_{pjk} \le -\sum_{j\in J} A_{pj} + \sum_{k\in\bar{K}} B_{pk} \quad (p = 1, 2, \ldots, l; \text{all } J, K) \tag{10.1}$$

$$-\sum_{i\in I}\sum_{k\in K} m_{iqk} \le -\sum_{i\in I} A_{iq} + \sum_{k\in\bar{K}} C_{qk} \quad (q = 1, 2, \ldots, m; \text{all } I, K) \tag{10.2}$$

The conditions (5) having the u_{ij}, v_{ik}, and w_{jk} specified by Table III equal to 1 and all other u_{ij}, v_{ik}, and w_{jk} equal to zero are:

$$\begin{aligned}\sum_{j\in J}\sum_{k\in\bar{K}} M_{pjk} &- \sum_{j\in J}\sum_{k\in K} m_{pjk} + \sum_{k\in\bar{K}} M^{\beta}_{pk}\\ &\le -\sum_{j\in J} A_{pj} + \sum_{k\in\bar{K}} B_{pk}\\ &\le \sum_{j\in J}\sum_{k\in\bar{K}} m_{pjk} - \sum_{j\in J}\sum_{k\in K} M_{pjk} + \sum_{k\in\bar{K}} m^{\beta}_{pk}\\ &\qquad (p = 1, 2, \ldots, l)\end{aligned} \tag{11.1}$$

$$\begin{aligned}\sum_{i\in\bar{I}}\sum_{k\in\bar{K}} M_{iqk} &- \sum_{i\in I}\sum_{k\in K} m_{iqk} + \sum_{k\in\bar{K}} M^{\gamma}_{qk}\\ &\le -\sum_{i\in I} A_{iq} + \sum_{k\in\bar{K}} C_{qk}\\ &\le \sum_{i\in\bar{I}}\sum_{k\in\bar{K}} m_{iqk} - \sum_{i\in I}\sum_{k\in K} M_{iqk} + \sum_{k\in\bar{K}} m^{\gamma}_{qk}\\ &\qquad (q = 1, 2, \ldots, m).\end{aligned} \tag{11.2}$$

The Gotlieb conditions (10.1) and (10.2) are equivalent to the first inequalities in (11.1) and (11.2) respectively when lower bounds of zero are used.

An efficient method for testing the Gotlieb conditions based on the Hungarian method for the assignment problem was reported by Lions [6].

Further Necessary Conditions

An infinite number of sets of necessary conditions can be obtained from the conditions (5) by constraining u_{ij}, v_{ik}, and w_{jk} in different ways. However, there are two sets of conditions which are of interest because they are natural extensions of the simple conditions and the Gotlieb conditions.

THEOREM 2. *If $K_1 \subseteq N$; $K_2 \subseteq N$ such that $K_1 \cap K_2 = \varnothing$; $K_3 = N - K_1 \cup K_2$; and $I, \bar{I}, J, \bar{J}$ are as defined previously, then necessary conditions for the existence of a solution to the timetable problem are:*

$$\begin{aligned}\sum_{i\in I}\sum_{j\in\bar{J}}\sum_{k\in K_1} M_{ijk} &+ \sum_{i\in\bar{I}}\sum_{j\in J}\sum_{k\in K_2} M_{ijk}\\ &- \sum_{i\in I}\sum_{j\in J}\sum_{k\in K_3} m_{ijk} + \sum_{i\in I}\sum_{k\in K_1} M^{\beta}_{ik} + \sum_{j\in J}\sum_{k\in K_2} M^{\gamma}_{jk}\\ &\le -\sum_{i\in I}\sum_{j\in J} A_{ij} + \sum_{i\in I}\sum_{k\in K_1} B_{ik} + \sum_{j\in J}\sum_{k\in K_2} C_{jk}\\ &\le \sum_{i\in I}\sum_{j\in\bar{J}}\sum_{k\in K_1} m_{ijk} + \sum_{i\in\bar{I}}\sum_{j\in J}\sum_{k\in K_2} m_{ijk}\\ &- \sum_{i\in I}\sum_{j\in J}\sum_{k\in K_3} M_{ijk} + \sum_{i\in I}\sum_{k\in K_1} m^{\beta}_{ik} + \sum_{j\in J}\sum_{k\in K_2} m^{\gamma}_{jk}\\ &\qquad \text{all } I, J, K_1, K_2.\end{aligned} \tag{12}$$

PROOF. The conditions (12) follow immediately from the conditions (5) if the value of u_{ij}, v_{ik}, w_{jk} are:

$$\begin{aligned}u_{ij} &= 1\ i \in I; j \in J,\\ &= 0 \text{ otherwise};\\ v_{ik} &= 1\ i \in I; k \in K_1,\\ &\ 0 \text{ otherwise};\\ w_{jk} &= 1\ j \in J; k \in K_2,\\ &= 0 \text{ otherwise}.\end{aligned}$$

The conditions (12) are similar to those given by Morávek and Vlach [9] and restated by Haley [4] for the multi-index problem.

A further set of conditions of interest is the set of conditions (5) where u_{ij}, v_{ik}, w_{jk} are allowed to take on the values 0 or 1, but are otherwise unconstrained. These conditions will be referred to subsequently as the conditions (13).

Relationships Between the Various Sets of Necessary Conditions

The conditions (8), (11), (12), and (13) are, in the sequence stated, of increasing complexity. The conditions (11) include the conditions (8) as a subset; the conditions (12) include the conditions (11) as a subset; and the conditions (13) include the conditions (12) as a subset. This is easily seen by comparing the values of u_{ij}, v_{ik}, and w_{jk}, which give rise to the various sets of conditions.

Some indication of the scope of these sets of conditions is given by the values of u_{ij} which can occur in each of them. It is evident that each of the conditions, (8.1) for example, considers a specific class-teacher combination. Condition (11.1) considers a specific class with each subset of the teachers it meets. Conditions (12) consider each set of classes with each set of teachers, whilst conditions (13) consider each set of class-teacher combinations.

Testing the Necessary Conditions

De Werra [2] has shown that testing the simple conditions is computationally feasible, and Lions [6] has shown that the Gotlieb conditions can be tested in a reasonable amount of computer time; but the same cannot be said of the conditions (12) or the conditions (13). Indeed, the vast number of conditions in each set suggests that complete testing of all of the conditions (12) or (13) for a typical timetable problem is not feasible. This means that the conditions (12) and (13) are of little use for opportunity list maintenance, although they may be useful for testing the feasibility of the unsolved portion of the timetable.

Consider, for example, the conditions (12). If the problem is reduced so that all lower bounds are zero, the first inequality in conditions (12) becomes

$$-\sum_{i\in I}\sum_{j\in J}\sum_{k\in K_3} m_{ijk} \leq -\sum_{i\in I}\sum_{j\in J} A_{ij} + \sum_{i\in I}\sum_{k\in K_1} B_{ik} + \sum_{j\in J}\sum_{k\in K_2} C_{jk},$$

which is equivalent to

$$\sum_{k=1}^{n} \min\left\{\sum_{i\in I} B_{ik}, \sum_{j\in J} C_{jk}, \sum_{i\in I}\sum_{j\in J} m_{ijk}\right\} - \sum_{i\in I}\sum_{j\in J} A_{ij} \geq 0. \tag{14}$$

For feasibility, it is necessary to show that

$$\min_{I,J}\left\{\sum_{k=I}^{n} \min\left\{\sum_{i\in 1} B_{ik}, \sum_{j\in J} C_{jk}, \sum_{i\in I}\sum_{j\in J} m_{ijk}\right\} - \sum_{i\in I}\sum_{j\in J} A_{ij}\right\} \geq 0, \tag{15}$$

which involves less computation than testing the conditions (14) for each I, J, and may detect one or more I, J combinations for which the conditions (14) are satisfied as an equality, which would give rise to tighter bounds.

It may even be preferable to attempt the minimization heuristically, which would be far less time-consuming and may detect an infeasibility even though there would be no guarantee that the minimum was found.

Concluding Remarks

This paper has given four sets of necessary conditions for the existence of a solution to the timetable problem, together with a general framework into which they fit. These conditions could be used in a solution procedure for the timetable problem as the basis for opportunity list maintenance and/or testing the feasibility of the unsolved portion of the timetable.

The selection of a set of conditions to be used in a solution procedure is dependent on the characteristics of the timetable problems to be solved, and the assignment strategy used. At this stage in the art of timetable construction, it seems that experimenting with a representative sample of timetable problems is the only method of determining the best combination of assignment strategy and set of necessary conditions for the existence of a solution. Experimental work is currently being performed to assess the performance of various assignment strategies.

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