to his present position of Professor of Systems and Control with the Mathematical Institute of the University of Groningen, Groningen, The Netherlands. He has held visiting appointments with the Department of Applied Mathematics and Theoretical Physics of the University of Cambridge, Cambridge, England, with the Center of Mathematical System Theory of the Department of Mathematics at the University of Florida, and with the Division of Engineering and Applied Physics of Harvard University. His area of main interest is in system theory. He is the author of *The Analysis of Feedback Systems* (M.I.T. Press, 1971) and was Associate Editor for Stability, Nonlinear Systems, and Distributed Systems of the IEEE Transactions of Automatic Control in 1971 to 1973. He is a member of the Society for Industrial and Applied Mathematics, and the Dutch General Systems Society.

Canonical Forms for the Identification of Multivariable Linear Systems

MICHAEL J. DENHAM, STUDENT MEMBER, IEEE

***Abstract*—The advantage of using a unique parameterization in a numerical procedure for the identification of a system from operating records has been well established. In this paper several sets of canonical forms are described for state space models of deterministic multivariable linear systems; the members of these sets having therefore the required uniqueness property within the equivalence classes of minimal realizations of the system. In the identification of a stochastic system, it is shown how the problem depends also upon determining a unique factorization of the spectral density matrix of the system, and the sets of canonical forms obtained for the deterministic system are extended to this case.**

I. Introduction

A BASIC requirement of any successful identification algorithm is that it should lead to consistent estimates of the parameters of the unknown system. However, systems which have weighting functions that are close in some sense can have widely differing state space representations, with the result that, in general, the maximum likelihood estimates of the parameters of the state space model are not consistent. This is even true when there is only a single output, but in that case, if the state dimension is known, a unique canonical form can be specified and consistency can be established. This paper is essentially concerned with a review of some of the canonical forms which can be specified in the multivariable case.

In Section II, the necessary definitions for an accurate description of what is meant by a canonical form are given, culminating in the basic concept of a canonical form as the unique member of an equivalence class of a given set. In Section III, these definitions are applied in the case of the equivalence classes of minimal realizations of transfer functions. In this way, we can determine what properties the state space model must satisfy to be a unique minimal realization of the given input/output map. Several such canonical forms are described and the essential properties of completeness and independence of the parameters are established. The implications of determining these canonical forms from the Hankel matrix formed from the system Markov matrices (the minimal realization problem) are also discussed.

In Section IV, the particular problems raised in the identification of stochastic systems by nonuniqueness of the model are resolved. This is done by determining a set of canonical forms for the equivalence classes of stable, minimum phase factors of the spectral density matrix $\Phi(z)$ for a discrete time system.

Finally, in Section V, some of the implications of using canonical forms in identification algorithms are discussed, including the question of consistency of the estimated parameters.

II. Notation and General Definitions

Consider any set X. We can define an equivalence relation E on X and denote the equivalence of two elements $x, y \in X$ by xEy. We shall now relate some important definitions concerning the set X and its equivalence relation E [1].

Definition 1: A function $f:X \rightarrow S$ for some set S is an *invariant* for the equivalence relation E if, for any $x,y \in X$, then

$$xEy \Rightarrow f(x) = f(y).$$

It is easy to see therefore that the equivalence relation E generates, for each $x \in X$, a disjoint set of *equivalence classes* or *orbits* in X which we will denote as

$$E(x) = \{y: yEx, \quad \text{for} \quad x,y \in X\}$$

The set of all such equivalence classes (i.e., for all $x \in X$) is called a *quotient set* or *orbit space*, and is denoted by X/E.

Manuscript received January 16, 1974; revised July 1, 1974. This work was supported in part by the United Kingdom Science Research Council.

The author is with the Department of Computing and Control, Imperial College of Science and Technology, London, England.

We can now specify some important types of invariant, the properties of which we will use to define what is meant by a set of canonical forms in X. First, we wish to define an invariant which allows us to state the reverse implication to that in Definition 1.

Definition 2: An invariant $f:X \to S$, is a *complete invariant* for equivalence E on X if, for any $x,y \in X$, then

$$f(x) = f(y) \Longrightarrow xEy.$$

Thus, it can be seen that a complete invariant defines a one-to-one correspondence between the equivalence classes $E(x)$ in X and the image of f. It is also clear that we can define the set S to be the image of f, which ensures that f is surjective, i.e., for every $s \in S$ there exists an $x \in X$ such that $f(x) = s$. This is often referred to as the property of independence of a complete invariant. We will not be concerned however with a scalar valued map f in our determination of a suitable set of canonical forms for system models, but rather with a set of maps $f_i:X \to S_i$. Thus, we will require the following definition.

Definition 3: A set of invariants $\{f_i;f_i:X \to S_i,\ i = 1,\cdots,n\}$ is a complete system (set) of invariants for E if the map f defined as

$$\begin{aligned} f:X &\to S_1 \times S_2 \times \cdots \times S_n \\ :x &\to (f_1(x),\cdots,f_n(x)) \end{aligned}$$

is a complete invariant for equivalence E on X.

We might also require that a set of invariants be defined to be independent if the corresponding invariant $f:X \to S_1 \times \cdots \times S_n$ is surjective. However, as we shall see, some sets of invariants which will be considered later will not in general satisfy such a condition. In these cases we will only require that the elements of S which are not contained in the image of f are *nongeneric* points of S, that is, an element of S picked at random will almost surely not be such a point. An intuitive notion of independence, i.e., that no f_i can be determined as a function of the others, does in fact follow under this condition [2].

It is now clear that in view of the above definitions we can uniquely characterize an equivalence class $E(x)$ by means of the set of values of the functions $f_i(x)$, $i = 1,\cdots,n$, where the $\{f_i\}$ constitute a complete set of invariants for E on X. If we now specialize the corresponding complete invariant f such that its image is itself a subset of X, we have reached our goal in specifying a set of canonical forms for equivalence E on x, according to the following definition.

Definition 4: A *set of canonical forms* for equivalence E on X is a subset C of X such that for each $x \in X$ there exists one and only one $c \in C$ for which xEc.

Clearly, if C is the image of a complete invariant f, then, for any $x \in X$ and $c_1,c_2 \in C$, xEc_1 and xEc_2 implies that $f(x) = f(c_1) = f(c_2) = c_1 = c_2$ by the invariance property. Also, by completeness, we have that, for any $c \in C$, if $f(x_1) = c$ and $f(x_2) = c$, then x_1Ex_2. Thus, $c = f(x)$ is a unique member of the equivalence class $E(x)$ for every $x \in X$. Since it is in general only possible to model a system from its input/output map to within an equivalence class of models, it is this ability to characterize the class by a unique element that makes the study of canonical forms for linear multivariable systems important in the fields of realization theory and identification.

In particular we will be concerned with the following set:

$$X_0 = \{(A,B):A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\ (A,B) \text{ controllable}\}$$

on which we define an equivalence relation in the following familiar way via the transformation group:

$$G_0 = \{T:T \in \mathbb{R}^{n\times n},\ \det\ T \neq 0\}.$$

Definition 5: The matrix pairs $(A_1,B_1),(A_2,B_2) \in X_0$ are said to be G_0-equivalent on X_0 if there exists $T \in G_0$ such that

$$\begin{aligned} A_1T &= TA_2 \\ B_1 &= TB_2. \end{aligned}$$

Other groups of transformations involving state feedback, input transformation, etc. have been studied in the literature, e.g., [3]–[6]. We will restrict our attention however to G_0-equivalence, later extending this relation to operate on the sets of matrix triples (A,B,C) and quadruples (A,B,C,D) satisfying certain properties, by adding the requirements that

$$\begin{aligned} C_1T &= C_2 \\ D_1 &= D_2 \end{aligned}$$

to those of Definition 5.

It is obvious that all the following results which are obtained for the set X_0 apply directly to the dual set

$$\bar{X}_0 = \{(A^T,C^T):A^T \in \mathbb{R}^{n\times n}, C^T \in \mathbb{R}^{n\times m},\ (A,C) \text{ observable}\}.$$

This is also true for the extensions to the matrix triples (A,B,C), etc.

III. Canonical Forms for Deterministic Multivariable Linear Systems

The earliest work in this field appears to be that of Langenhop [7]. Subsequently, Luenberger [8], [9] proposed certain sets of canonical forms. The maps from X_0 into these sets, as defined in [8], are in fact defined in such a way that their images are indeed sets of canonical forms. Luenberger omitted however a description of the image of these maps in sufficient detail to avoid the possibility of nonuniqueness, as illustrated in the following example.

Let a pair (A,B), in the form proposed by Luenberger, be given as

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

If we now examine the sequence of vectors $b_1,b_2,Ab_1,Ab_2,\cdots,A^3b_2$, for linear independence, we obtain the transformation matrix $S = [b_1\ Ab_1\ b_2\ Ab_2]^{-1}$. Applying the map $f: (A,B) \to (SAS^{-1},SB)$ into the canonical form

$$SAS^{-1} = \begin{bmatrix} 0 & -1 & 0 & -2 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 1 & 1 & 4 \end{bmatrix}, \qquad SB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

we see that this also has the proposed Luenberger structure. Further examination of (A,B) shows that the nonuniqueness arises from the presence of a nonzero number for the element a_{34} of A.

The problem of how to uniquely describe a set of canonical forms was subsequently overcome by Popov [10], Caines [11], Mayne [12], [21], and Weinert and Anton [13]. This is achieved by introducing a finite set of integers or indices $\{n_i\}$ that are uniquely defined by the map $f: X_0 \to C$ into the set of canonical forms. The structure of the canonical forms is then described uniquely in terms of these indices, which are themselves, of course, invariants, i.e., the map $g:(A,B) \to \{n_i\}$ is an invariant for G_0-equivalence on X_0.

In the following, however, we will expand on this idea by taking the more general viewpoint adopted by Kalman [14], and include within this the better known procedures for constructing sets of canonical forms in X_0.

Consider the controllability matrix $R(A,B)$ formed from a pair $(A,B) \in X_0$, i.e.,

$$R(A,B) = [B\ AB\ A^2B \cdots A^{n-1}B].$$

We will index the columns of R as $[01,02,\cdots,0m,11,12,\cdots,ij,\cdots,(n-1)m]$, i.e., the column indexed by ij is A^ib_j. A *multiindex* γ is then defined as any choice of n distinct indices ij from this set, ordered first by j and then by i, e.g., (01,11,02,12,22) for $n = 5$. The ordering constraint is necessary to prevent a reordering of the indices being regarded as producing the same multiindex. This would introduce nonuniqueness into the map which we are shortly to describe.

A multiindex is said to be *nice* if, for $jk \in \gamma$, then $ik \in \gamma$ for all $0 \leq i \leq j$. This can be made clearer by considering a "crate" diagram of indices as follows:

$$\begin{matrix} 01 & 02 & 03 & \cdots & 0m \\ 11 & 12 & 13 & \cdots & 1m \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ (n-1)1 & (n-1)2 & (n-1)3 & \cdots & (n-1)m \end{matrix}$$

corresponding to

$$\begin{matrix} b_1 & b_2 & b_3 & \cdots & b_m \\ Ab_1 & Ab_2 & Ab_3 & \cdots & Ab_m \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ A^{n-1}b_1 & A^{n-1}b_2 & A^{n-1}b_3 & \cdots & A^{n-1}b_m. \end{matrix}$$

Clearly, γ being nice implies that if any element of the diagram is in γ, then so too are all elements above it in the respective column. Let $R_\gamma(A,B)$ be the $n\times n$ matrix formed from those columns of R as specified by the multiindex γ. Then clearly if γ is nice, A^jb_k being a column of R_γ implies that the preceding columns contain A^ib_k, $0 \leq i < j$.

Now, for any nice multiindex γ, define

$$X_0{}^\gamma = \{(A,B):(A,B) \in X_0,\ \det R_\gamma(A,B) \neq 0\}.$$

Recalling our interpretation in Section II of a set of canonical forms as the image of a complete invariant, we can define a set of canonical forms C_γ for G_0-equivalence on $X_0{}^\gamma$ as the image of the complete invariant

$$\begin{aligned} \phi_\gamma &: X_0{}^\gamma \to C_\gamma \\ &: (A,B) \mapsto (R_\gamma{}^{-1}AR_\gamma, R_\gamma{}^{-1}B). \end{aligned}$$

The proof that ϕ_γ is a complete invariant is a straightforward exercise in matrix algebra. It is of interest however to consider a particular property of the set C_γ. As Kalman [14] points out, the set C_γ is *isomorphic* to the nm vector space $\mathbb{R}^{nm}$, where we assume that the elements of $(A,B) \in X_0$ belong to the field $\mathbb{R}$ of real numbers. This implies that for every nm-vector $r \in \mathbb{R}^{nm}$, there exists one, and only one, $c \in C_\gamma$ which has the nm elements of r as parameters in c, the remaining n^2 parameters being fixed as 1 or 0.

This follows directly from the fact that there are always exactly nm nonfixed parameters in any $c \in C_\gamma$. To see this, consider the matrix equation $\hat{R} = R_\gamma{}^{-1}R$, i.e.,

$$[\hat{B}\ \hat{A}\hat{B}\ \ldots\ \hat{A}^{n-1}\hat{B}] = R_\gamma{}^{-1}[B\ AB\ \ldots\ A^{n-1}B]$$

where $(\hat{A},\hat{B}) \in C_\gamma$. Let

$$\gamma = (01,\cdots,(n_1{}^\gamma-1)1,02,\cdots,(n_2{}^\gamma-1)2,03,\cdots,(n_m{}^\gamma-1)m).$$

Now, if we consider, in the controllability matrices on each side of the matrix equation, only those columns indexed by γ, and recall the construction of R_γ, it is easy to see that this set of n vector equations is trivially satisfied by columns of $\hat{R}$ corresponding to columns of the identity matrix. To specify the remaining elements of $(\hat{A},\hat{B})$, we must consider the additional set of m equations

$$\begin{aligned} \hat{A}^{n_i{}^\gamma}\hat{b}_i &= R_\gamma{}^{-1}A^{n_i{}^\gamma}b_i, & 0i \in \gamma \\ \hat{b}_i &= R_\gamma{}^{-1}b_i, & 0i \notin \gamma. \end{aligned}$$

These equations uniquely determine the nm nonfixed elements of $(\hat{A},\hat{B})$. Conversely, if we arbitrarily specify the m n-vector solutions to these equations, given R_γ, we can uniquely determine a corresponding $(A,B) \in X_0{}^\gamma$.

In the above, we have stated the conditions which specify the structure of $(\hat{A},\hat{B}) \in C_\gamma$. In the general case, the set of canonical forms C_γ correspond to pairs $(A,B) \in X_0{}^\gamma$ that have the following structure, where α_i, δ_{ij}, and β_i represent nonfixed parameters, and the $n_i{}^\gamma$ are integers determined by γ as $n_i = \max\{j : ji \in \gamma\} + 1$, $i = 1,\cdots,m$.

Type I (A,B) *Structure*

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix}$$

where

$$A_{ii} = \left[\begin{array}{c|c} 0 & \alpha_i \\ \hline I_{n_i^\gamma - 1} & \end{array}\right]$$

an $(n_i^\gamma \times n_i^\gamma)$ matrix, where α_i is an n_i^γ-vector, and

$$A_{ij} = [0 \mid \delta_{ij}]$$

an $(n_i^\gamma \times n_j^\gamma)$ matrix where δ_{ij} is an n_i^γ-vector.

$$B = [b_1 \cdots b_m]$$

where, for $0i \in \gamma$

$$b_i = [0 \cdots 0\ 1\ 0 \cdots 0]^T$$

with the 1 in the $(1 + \sum_{j=1}^{i-1} n_j^\gamma)$th position, and for $0i \not\in \gamma$

$$b_i = [\beta_i]$$

an n-vector of nonfixed parameters.

The vectors of nonfixed parameters

$$\theta_i = \begin{bmatrix} \alpha_i \\ \delta_{2i} \\ \delta_{3i} \\ \cdot \\ \cdot \\ \cdot \\ \delta_{mi} \end{bmatrix}, \qquad i = 1, \cdots, m$$

and β_i, are the unique solutions to the equations

$$A^{n_i^\gamma} b_i = R_\gamma \theta_i, \qquad 0i \in \gamma$$
$$b_i = R_\gamma \beta_i, \qquad 0i \not\in \gamma.$$

It is not difficult to see that for every $(A,B) \in X_0$, there exists a suitable nice γ such that $(A,B) \in X_0^\gamma$. However, as we shall now see, we do not need to consider every nice γ in order to cover the whole of the orbit space X_0/G_0. Included in the set of all possible nice multiindices are those that result from two well-known procedures for selecting a set of linearly independent columns from the controllability matrix R. In the first procedure, the columns of R_γ are chosen from R in the sequence

$$b_1, b_2, \cdots, b_m, Ab_1, \cdots, Ab_m, A^2 b_1, \cdots. \tag{S1}$$

In the second procedure, they are chosen in the sequence

$$b_1, Ab_1, \cdots, A^{n-1} b_1, b_2, Ab_2, \cdots, A^{n-1} b_2, b_3, \cdots. \tag{S2}$$

It is well known that, if the procedure (S1) is used, the resulting set of integers $\{n_i,\ i = 1, \cdots, m\}$ are the Kronecker invariants [4], [10], [26]. Furthermore, if we define the set of numbers $\{\alpha_{ijk}\}$ by the set of equations for $j = 1, \cdots, m$

$$A^{n_j} b_j = \sum_{i=1}^{j-1} \sum_{k=0}^{s_{ij}} \alpha_{ijk} A^k b_i + \sum_{i=1}^{m} \sum_{k=0}^{t_{ij}} \alpha_{ijk} A^k b_i$$

where $s_{ij} = \min(n_i, n_j - 1)$, $t_{ij} = \min(n_i, n_j) - 1$ we can state the following theorem due to Popov [10].

Theorem 1: The set $\{n_i, \alpha_{ijk}\}$ is a complete set of independent invariants for G_0-equivalence on X_0.

The need to include the integers $\{n_i\}$ into the image of the complete invariant is simply because, as their definition clearly shows, the number of α_{ijk} parameters depends upon the relative values of the n_i.

As a direct result of this dependence on $\{n_i\}$, we do not have the previous property of local isomorphism between the set of canonical forms C_γ and the space $\mathcal{R}^{nm}$. This is simply because we have restricted our neighborhood in X_0 to $\{(A,B) : (A,B) \text{ have } \{n_i\} \text{ as Kronecker invariants}\}$. This is best illustrated by the following two examples, the first of which also serves to illustrate the general nature of sets of canonical forms.

Example 1: Consider $X_0 = \{a, [b_1, b_2]\}$. There are two possible nice multiindices, $\gamma_1 = (01)$, $\gamma_2 = (02)$, and the corresponding sets of canonical forms are

$$C_{\gamma_1} = \{(a, [1 \quad b])\}$$
$$C_{\gamma_2} = \{(a, [b \quad 1])\}.$$

Clearly, in each case, C_{γ_i} is isomorphic to $\mathcal{R}^2$. Consider now the sets of canonical forms obtained when the special selection procedures (S1) and (S2) are used (in this case they coincide). The possible sets of Kronecker invariants and $\{n_i\}_1 = \{1,0\}$, and, in the case when $b_1 = 0$, $\{n_i\}_2 = \{0,1\}$. The corresponding sets of canonical forms are

$$C_1 = \{(a, [1 \quad b])\}$$
$$C_2 = \{(a, [0 \quad 1])\}.$$

Fig. 1(a) and (b) illustrate these sets of canonical forms, the orbits being represented by the broken lines radiating from (but not including) the origin. Clearly, for the set C_2 there is no isomorphism with $\mathcal{R}^2$, though there is with $\mathcal{R}^1$. This is stipulated by the fact that $n_1 = 0$ in this case.

An interesting property of the sets of canonical forms C_1 and C_2 is that their union is also a set of canonical forms.[1] This is true because the structure of any $c \in C \triangleq C_1 \cup C_2$ implicitly specifies the $\{n_i\}$. As Theorem 1 implies however, the explicit inclusion of $\{n_i\}$ is necessary for the set $\{a, b_2, 1, 0\} \cup \{a, 0, 1\}$ corresponding to C, to be a complete set of independent invariants. Note that $C_{\gamma_1} \cup C_{\gamma_2}$ is *not* a set of canonical forms.

It is worthwhile mentioning a further set of canonical forms [24], which is illustrated in Fig. 1(c) by the semicircle C, and clearly this set alone covers X_0/G_0. It has the disadvantages from the identification viewpoint that con-

[1] It is true in general that $\bigcup_{\text{all } \{n_i\}_j} C_j$ is a canonical form, where $\{n_i\}_j$ result from (S1) or (S2) selection procedures. This result is due to Mayne.

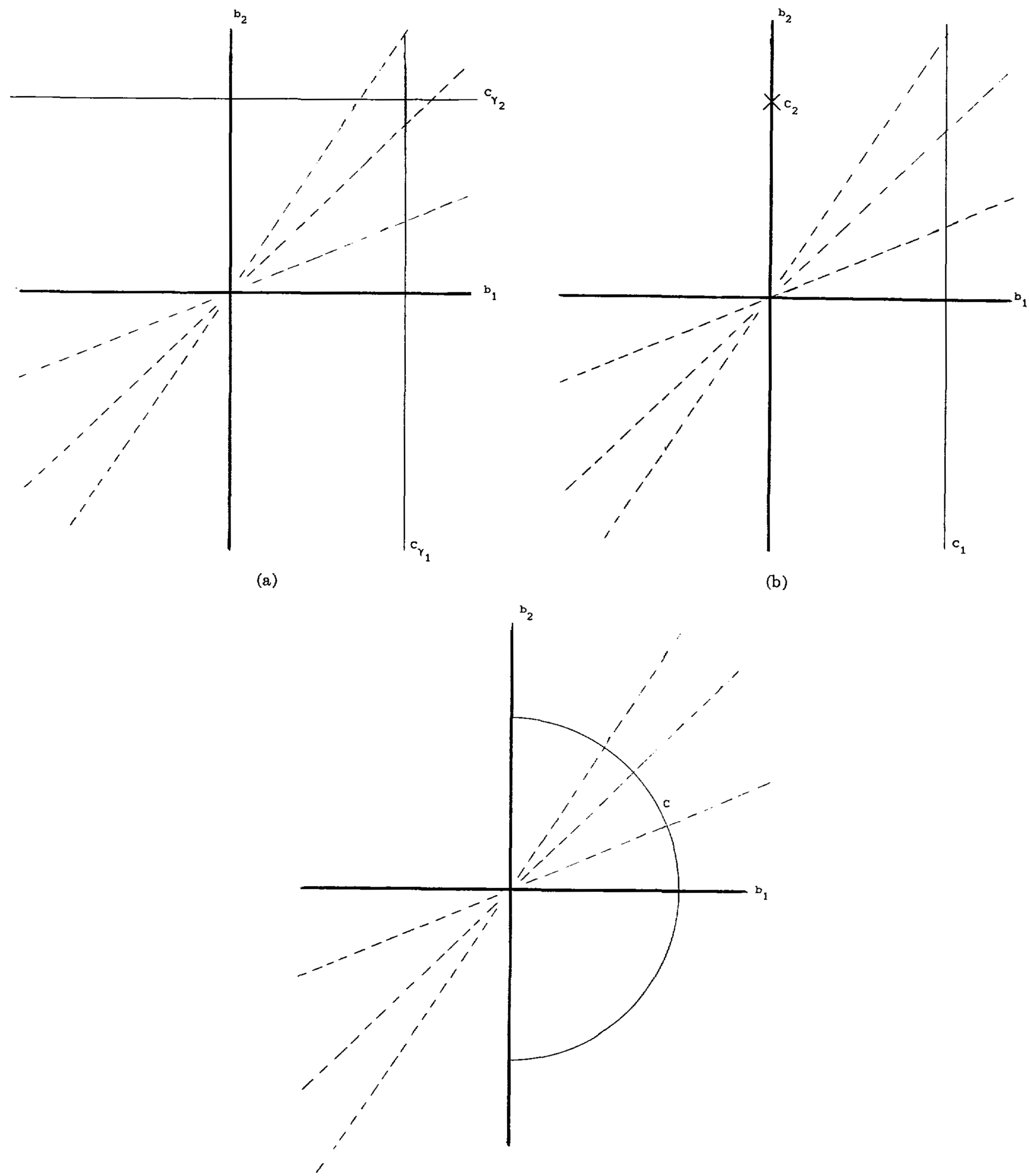

Fig. 1.

tinuous paths in X_0 are mapped into discontinuous paths in C, and the parameters of $c \in C$ are nonlinear.

Example 2: As a more general example of the loss of isomorphism between sets of canonical forms, derived from selection procedures (S1) and (S2), and $\mathcal{R}^{nm}$, consider the pair $(A,B) \in C_\gamma$, where

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and $\gamma = (01,11,21,02)$, $R_\gamma = [b_1\ Ab_1\ A^2b_1\ b_2]$. Thus, the related indices are $\{3,1\}$. However, the element $(1,2,3,1;2,3,1,2) \in \mathcal{R}^8$ does *not* correspond to any member of the set of canonical forms C where $(A,B) \in C$ is restricted to have *Kronecker* indices $\{3,1\}$.

To complete the description of the sets of canonical forms derived from selection procedures (S1) and (S2), we will explicitly state the restrictions imposed by these constraints on the general Type I (A,B) structure. In the case where (S1) is used, we have that

$$\delta_{ij} = (\alpha_{ij0}\ \alpha_{ij1} \cdots \alpha_{ijk_{ij}}\ 0 \cdots 0)^T$$

where $k_{ij} = \min(n_i,n_j) - 1$ for $i > j$, and $k_{ij} = \min(n_i,n_j - 1)$ for $i < j$. In fact δ_{ij} may have nonzero numbers for all $i \neq j$. This is true only when the first n vectors in the sequence (S1) are chosen, which is the generic situation. In this case, $n_1 \geq n_2 \geq \cdots \geq n_m > 0$ and $0 \leq |n_i - n_j| \leq 1$.

For the case where (S2) is used, we have that $\delta_{ij} = 0$ for $i > j$, and

$$\beta_i = (\epsilon_{i0}\ \epsilon_{i1} \cdots \epsilon_{ik_i}\ 0 \cdots 0)^T$$

where $k_i = (\sum_{j=1}^{i-1} n_j) - 1$. The generic case, as before, when the first n vectors of (S2) are chosen results in $n_1 = n$, $n_j = 0$ for $j > 1$. (In each generic situation, a little thought reveals that the number of nonfixed parameters is nm.) This latter set of canonical forms, in its dual form, is that studied and successfully employed for identification purposes by Mayne [12], [21], and Tse and Weinert [15].

As a final example of the sets of canonical forms we have looked at so far, consider $(A,B) \in X_0$ where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Hence, the controllability matrix R is

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

and four possible nice multiindices are

$$\begin{aligned} \gamma_1 &= (01,11,12) \\ \gamma_2 &= (01,02,12) \\ \gamma_3 &= (01,11,21) \\ \gamma_4 &= (02,12,22). \end{aligned}$$

The columns of R_{γ_1} correspond to those chosen from (S1) and of R_{γ_3} to those chosen from (S2). Both correspond to the generic case. The resulting elements in the respective sets of canonical forms C_{γ_i}, given by $A_i = R_{\gamma_i}^{-1}AR_{\gamma_i}$, $B_i = R_{\gamma_i}^{-1}B$, are

$$A_1 = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \in C_{\gamma_1}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \in C_{\gamma_2}$$

$$A_3 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \in C_{\gamma_3}$$

$$A_4 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \qquad B_4 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \in C_{\gamma_4}.$$

We shall now describe a second complete invariant for G_0-equivalence of X_0, the image of which in X_0 will hence describe a further set of canonical forms. These sets of forms were first proposed by Luenberger [8], Brunovsky [16], [17], and Tuel [18], and subsequently studied by Popov [19], Denery [20], and Weinert and Anton [13].

For any nice multiindex γ and $(A,B) \in X_0$ such that $\det R_\gamma(A,B) \neq 0$, let $\{n_i^\gamma\}$ be the set of related indices, as defined previously. Let $\{\bar{n}_j^\gamma, j = 1,\cdots,q\}$ be the set of nonzero n_i^γ, and define

$$\sigma_k = \sum_{j=1}^{k} n_j^\gamma, \qquad k = 1,\cdots,q$$

e.g., if $\{n_i^\gamma\} = \{3,0,2\}$, then $\{\bar{n}_j^\gamma\} = \{3,2\}$ and $\sigma_1 = 3$, $\sigma_2 = 5$. Now let us denote by t_k the σ_kth row of R_γ^{-1}, and then form the matrix $Q_\gamma(A,B)$ as follows:

$$Q_\gamma(A,B) = [t_1^T A^T t_1^T \cdots (A^T)^{\bar{n}_1 - 1} t_1^T t_2^T \\ A^T t_2^T \cdots (A^T)^{\bar{n}_q - 1} t_q]^T.$$

An immediate consequence of the method of constructing Q_γ is the following useful result.

Lemma 1: For any two matrices R_γ^1 and R_γ^2, and corresponding Q_γ^1 and Q_γ^2 obtained by the above construction, and some nonsingular matrix T, if

$$R_\gamma^1 = TR_\gamma^2$$

then

$$Q_\gamma^1 = Q_\gamma^2 T^{-1}.$$

The proof follows straightforwardly from the construction of Q_γ from R_γ.

We can now define a further set of canonical forms as the image of the complete invariant

$$\begin{aligned} \psi_\gamma &: X_0^\gamma \to C_\gamma \\ &: (A,B) \mapsto (Q_\gamma A Q_\gamma^{-1}, Q_\gamma B). \end{aligned}$$

The proof that ψ_γ is a complete invariant for G_0-equivalence on X_0 is straightforward using the result of Lemma 1.

One would expect that as a result of the one-to-one correspondence between elements of the sets $\{R_\gamma : \gamma \text{ nice}, \det R_\gamma \neq 0\}$ and $\{Q_\gamma : \gamma \text{ nice}, \det Q_\gamma \neq 0\}$, the properties which hold for the previously defined invariant ϕ_γ will also apply to ψ_γ, i.e., that the image of ψ_γ is isomorphic to $\mathcal{R}^{nm}$. In this case, however, it is not easy to describe the set of canonical forms C_γ in detail to specify exactly which are the fixed parameters. We can say that set C_γ will be contained in the set of pairs $(A,B) \in X_0^\gamma$ which have the following general structure.

Type II (A,B) Structure

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1q} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ A_{q1} & & A_{qq} \end{bmatrix}$$

where

$$A_{ii} = \begin{bmatrix} 0 & I_{n_i^\gamma - 1} \\ \hline & \alpha_i \end{bmatrix}$$

an $(n_i^\gamma \times n_i^\gamma)$ matrix, where α_i is a $(1 \times n_i^\gamma)$ row vector, and

$$A_{ij} = \begin{bmatrix} 0 \\ \hline \delta_{ij} \end{bmatrix}$$

an $(n_i^\gamma \times n_j^\gamma)$ matrix where δ_{ij} is a $(1 \times n_j^\gamma)$ row vector.

$$B = [b_1 \cdots b_m]$$

where for $0i \in \gamma$

$$b_i = [\beta_i]$$

an n-vector where $\beta_i = 0$ except for a 1 in the $(\sum_{j=1}^{i} n_j^\gamma)$th row, and arbitrary elements in the $(\sum_{j=1}^{k} n_j^\gamma)$th rows for $k = 1,\cdots,q$, and for $0i \notin \gamma$

$$b_i = [\beta_i]$$

an n-vector with arbitrary elements.

If we restrict the permissible choice of nice multiindex γ, however, to those for which the columns of R_γ corresponds to vectors chosen according to selection procedures (S1) or (S2), we can describe the resulting sets of canonical forms exactly by means of the resulting set of invariants $\{n_i\}$.

In the case where (S1) is used, we have that

$$\delta_{ij} = (\eta_{ij0}\ \eta_{ij1} \cdots \eta_{ijk_{ij}}\ 0\ 0 \cdots 0)$$

where $k_{ij} = \min(n_i,n_j) - 1$, and for $0i \in \gamma$, $\beta_i = 0$ except for $a1$ in the $(\sum_{j=1}^{k} n_j)$th row for $k = 1,\cdots,i$.

If the sequence (S2) is used to construct R_γ; and hence, Q_γ, then $\delta_{ij} = 0$ for $i > j$, and, for $0i \in \gamma$,

$$\beta_i = [0\ 0 \cdots 0\ 1\ 0 \cdots 0]$$

where the 1 is the $(\sum_{j=1}^{i} n_j)$th row. For $0i \notin \gamma$,

$$\beta_i = [\epsilon_{i0}\ \epsilon_{i1} \cdots \epsilon_{ik_i}\ 0 \cdots 0]$$

where $k_i = (\sum_{j=1}^{i-1} n_j) - 1$.

Again the generic situation occurs when the vectors chosen from (S1) or (S2) are the first n vectors of the sequence. In this case, the total number of nonfixed parameters is nm as before.

Extension of the Canonical Forms to $\{(A,B,C)\}$

Consider the sets of matrix triples, for nice γ,

$$X_1^\gamma = \{A,B,C):(A,B) \in X_0^\gamma,\ C \in \mathcal{R}^{p\times n}\}.$$

It is a simple matter to extend the maps ϕ_γ and ψ_γ to operate on X_1^γ as follows:

$$\hat{\phi}_\gamma : X_1^\gamma \to \hat{C}_\gamma$$
$$:(A,B,C) \mapsto (R_\gamma^{-1}AR_\gamma, R_\gamma^{-1}B, CR_\gamma)$$

and

$$\hat{\psi}_\gamma : X_1^\gamma \to \hat{C}_\gamma$$
$$:(A,B,C) \mapsto (Q_\gamma A Q_\gamma^{-1}, Q_\gamma B, CQ_\gamma^{-1}).$$

Then $\hat{\phi}_\gamma$ and $\hat{\psi}_\gamma$ are both complete invariants for G_0-equivalence on X_1^γ, where this is defined as in Definition 5, Section I, with the added condition that $C_1T = C_2$. Their images in X_1 therefore define sets of canonical forms as before. Moreover there clearly exists isomorphism between $\hat{C}_\gamma$ and $\mathcal{R}^{nm+np}$, or, in the case when the restricted selection procedure (S1) or (S2) is used, between $\hat{C}_\gamma$ and $\mathcal{R}^{N+np}$, $N \le nm$ dependent upon the $\{n_i\}$ as discussed above.

The extension of the sets of canonical forms to the more useful subsets of triples (A,B,C), defined, for nice γ, as

$$X_2^\gamma = \{(A,B,C):(A,B,C) \in X_1^\gamma,\ (A,C) \text{ observable}\}$$

is not so straightforward. Clearly the maps $\hat{\phi}_\gamma$ and $\hat{\psi}_\gamma$ are complete invariants as before. However, their images do not possess the nice isomorphism property. Obviously we can choose an element of $\mathcal{R}^M$ (where $M = nm + np$, or $N + np$ as above) which corresponds to the image of a triple (A,B,C) for which (A,C) is not observable. Rissanen [2] has tackled this problem in the following way.

Define a set of M invariants $\phi_{\gamma_i}: X \to \mathcal{R}$ to be independent if the complement of the image of the associated invariant

$$\hat{\phi}_\gamma : X \to \mathcal{R}^M$$

is a finite union of sets V_j, where

$$V_j = \{x \in \mathcal{R}^M : g_{ji}(x) = 0,\ i = 1,\cdots,k, \text{ finite } k\}$$

where the $g_{ji}(x)$ are polynomials. Applying this to our problem, we can say that the parameters of the set of canonical forms defined by

$$\hat{\phi}_\gamma : X_2^\gamma \to \hat{C}_\gamma$$

are independent, if any choice of $x \in \mathcal{R}^M$, which corresponds to $(A,B,C) \in X_1^\gamma$ for which (A,C) is not observable, must lie on a finite union of hypersurfaces in $\mathcal{R}^M$. Since such (A,B,C) correspond to pairs (A,C) for which

$$\operatorname{rank}[C^T A^T C^T \cdots (A^T)^{n-1} C^T] < n,$$

therefore suitable sets V_j are defined by this condition, and the definition of independence holds. As pointed out by Rissanen [2], this definition implies that no parameter of the canonical form of a triple (A,B,C) can be written as a function of the others. This satisfies the intuitive notion of independence, which was trivially satisfied when a clear isomorphism existed.

The existence of the hypersurfaces in parameter space $\mathcal{R}^M$ corresponding to nonminimal (A,B,C) raises an interesting question in the identification of linear multivariable systems parameterized by state space models. This will be discussed in Section V.

Canonical Forms Derived from Hankel Matrices

It is easy to see that the G_0-equivalence classes in X_2 may be equally well defined as follows: $(A_1,B_1,C_1),(A_2,B_2,C_2) \in X_2$ are G_0-equivalent if

$$C_1A_1^{i-1}B_1 = C_2A_2^{i-1}B_2, \qquad i = 1,2,\cdots.$$

For $(A,B,C) \in X_2$, we can denote $CA^{i-1}B$ by T_i. We can then define an invariant for G_0-equivalence on X_2 as any function acting on the set of $(p \times m)$ matrices T_i. Mayne [21] and Rissanen [2] use this approach and consider row dependencies in the infinite block Hankel matrix T, where the ijth block of T is given by T_{i+j-1}, $i = 1,\cdots$; $j = 1,\cdots$. It is well known that the infinite matrix T is entirely determined by a suitably large submatrix of T. In particular, the row and column dependencies of T are defined by those of the $T(n,n)$ submatrix of T, where the ijth block of $T(n,n)$ is given by T_{i+j-1}, $i = 1,\cdots,n$; $j = 1,\cdots,n$. It is also easy to see that

$$T(n,n) = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{n-1} \end{bmatrix} [B \ AB \ \cdots \ A^{n-1}B]$$
$$\triangleq QR.$$

Thus, the row dependencies (column dependencies) of $T(n,n)$ are equivalent to row dependencies of Q (column dependencies of R). It is true therefore that a set of canonical forms, corresponding to the image of any function on the T_i's which is defined in terms of row or column dependencies of T, will correspond to one of the sets of canonical forms $\hat{C}_\gamma$, or its dual, already discussed.

Since the determination of sets of canonical forms based on the set of matrices $\{T_i\}$ is independent of the way in which the T_i come about, we can equally well use the same ideas to determine sets of canonical forms for the set defined as

$$X_3 = \{A(s),B(s): A(s) = A_0 + A_1s + \cdots + A_qs^q,$$
$$A(s) \text{ nonsingular}, B(s) = B_0 + B_1s + \cdots + B_{q-1}s^{q-1},$$
$$A_i \in \mathbb{R}^{p\times p}, B_i \in \mathbb{R}^{p\times m}\}.$$

The map from X_3 in the set $\{T_i\}$ is simply defined by

$$A^{-1}(s)B(s) = T_1s^{-1} + T_2s^{-2} + \cdots.$$

This characterization of a system in the form of a fraction of polynomial matrices has been well developed by Rosenbrock [26], Popov [19], Forney [27], and Wolovich [28], among others.

That the Kronecker indices $\{n_i\}$ associated with the set of canonical forms in X_2 have an analogous interpretation for a set of canonical forms in X_3 has been shown by Forney [27] and Rissanen [2], who, together with Popov [19], have described such sets in X_3. Dickinson, Morf, and Kailath [29] have used many of the principal ideas in this field in their recent study of the partial realization problem.

The canonical forms described in this section all have direct application to both the deterministic identification problem and the minimal realization problem. In the stochastic case, however, special difficulties arise which must be resolved by determining sets of canonical forms for equivalence defined in terms of the spectral density matrix or its expansion in positive and negative powers of s (or z), rather than the transfer function or $\{T_i(s), i = 1,2,\cdots\}$. We will now consider these additional problems.

IV. Canonical Forms for Stochastic Systems

Whereas for the deterministic identification problem one is provided with information on the system transfer function matrix directly from input/output measurements, in the stochastic case this information relates only to the spectral density matrix of the system. In order to identify a unique state space model in this case, we must specify a set of canonical forms in the orbit space defined by the following equivalence relation on the set:

$$X_4 \triangleq \{(A,B,C,D): A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n},$$
$$D \in R^{m\times m}, (A,B) \text{ controllable}, (A,C) \text{ observable}\}.$$

Definition 6: $(A_1,B_1,C_1,D_1),(A_2,B_2,C_2,D_2) \in X_4$ are Φ-equivalent on X_4 if

$$\Phi_1(z) = \Phi_2(z)$$

where $\Phi_i(z) = H_i(z^{-1})H_i^T(z)$, $H_i(z) = C_i(zI - A_i)^{-1}B_i + D_i$.

An equivalent condition would be that

$$H_1(z) = H_2(z)P(z)$$

for some $P(z)$ such that $P(z)P^T(z^{-1}) = I$.

If we restrict our attention to a subset of X_4, namely

$$X_5 \triangleq \{(A,B,C,D): (A,B,C,D) \in X_4, D \text{ nonsingular, and } A \text{ and } (A - BD^{-1}C) \text{ stable}\}$$

we can make use of the following theorem [22] to simplify our equivalence relation. This result was probably well known in the mathematical literature prior to [22]; for example see Gohberg and Krein [30].

Theorem 3: Let $H_1(z)$ and $H_2(z)$ be real, rational, stable transfer functions having regular left inverses throughout the complement of the closed unit disc in the complex plane. Then

$$H_1(z)H_1^T(z^{-1}) = H_2(z)H_2^T(z^{-1})$$

if and only if

$$H_1(z) = H_2(z)U$$

for some orthogonal matrix $U \in \mathbb{R}^{m\times m}$ (i.e., $UU^T = I$).

Choosing $(A,B,C,D) \in X_5$ clearly ensures that $H(z) = C(zI - A)^{-1}B + D$ is a real, rational, stable transfer function. The following lemma ensures that $H^{-1}(z)$ exists and is regular, for $|z| > 1$.

Lemma 2: For $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and

$D \in \mathcal{R}^{m\times m}$ with D invertible and A stable, the transfer function matrix

$$H(z) = C(zI - A)^{-1}B + D$$

has a regular left inverse for $|z| > 1$ if and only if $(A - BD^{-1}C)$ is stable.

Proof: By a well known matrix inverse equality,

$$[C(zI - A)^{-1}B + D]^{-1} = -D^{-1}C(zI - A - BD^{-1}C)^{-1}BD^{-1} + D^{-1}.$$

Hence, $H^{-1}(z)$ is regular and clearly exists, i.e., is finite, for $|z| > 1$, if and only if $(A - BD^{-1}C)$ has no poles in this region, i.e., is stable.

The restriction to the set X_5 is reasonable since it can be shown [22] that for any $\Phi(z)$ satisfying 1) $\Phi(z) = \Phi^T(z^{-1})$ for all z, 2) $\Phi(e^{-i\lambda}) \geq 0$, $0 \leq \lambda \leq 2\pi$, and 3) $\Phi(z)$ is of rank m, then there exists a spectral factor $H(z)$ of $\Phi(z)$ satisfying the conditions of Theorem 3.

Thus, we have that Φ-equivalence on the set X_5 can be defined as follows.

Definition 7: (A_1,B_1,C_1,D_1), $(A_2,B_2,C_2,D_2) \in X_5$ are Φ-equivalent on X_5 if

$$\begin{aligned} A_1T &= TA_2 \\ B_1 &= TB_2U \\ C_1T &= C_2 \\ D_1 &= D_2U. \end{aligned}$$

If we now consider the set of matrices $\mathcal{O} \triangleq \{D : D \in \mathcal{R}^{m\times m},\ D \text{ nonsingular}\}$ and define $D_1, D_2 \in \mathcal{O}$ to be U-equivalent if $D_1 = D_2U$ for some orthogonal matrix $U \in \mathcal{R}^{m\times m}$, it is an easy matter to extend the sets of canonical forms on X_2 defined in Section III to the present situation.

Lemma 3: A set of canonical forms for U-equivalence on the set $\mathcal{O}$ is given by $\mathcal{L} \triangleq \{L : L \in \mathcal{O} \text{ and } L \text{ is lower triangular}\}$.

Proof: It is well known in matrix algebra that any nonsingular matrix D admits a unique factorization into the product

$$D = LU$$

where $L \in \mathcal{L}$ and U is an orthogonal matrix. The result of the lemma follows straightforwardly bearing in mind that the product of orthogonal matrices is itself an orthogonal matrix.

Using the result of this lemma, we can now define a set of canonical forms for Φ-equivalence on X_5. (In the following, the dependence on a multiiindex γ is understood and dropped from the notation.)

Theorem 4: Let $\hat{C}$ be a set of canonical forms for G_0-equivalence on X_2. Then the set $\hat{C} \times \mathcal{L}$ is a set of canonical forms for Φ-equivalence on X_5.

Proof: The map into the set of canonical forms is defined in the following way:

$$\begin{aligned} \Omega &: X_5 \to \hat{C} \times \mathcal{L} \\ &: (A,B,C,D) \to (\tilde{A},\tilde{B},\tilde{C},\tilde{D}) \end{aligned}$$

where $\tilde{A} = R^{-1}AR$, $\tilde{B} = R^{-1}BU^T$, $\tilde{C} = CR$, $\tilde{D} = DU^T$, such that $(\tilde{A},\tilde{B},\tilde{C}) \in \hat{C}$ and $\tilde{D} \in \mathcal{L}$.

We have to show that Ω is a complete invariant. Let (A_1,B_1,C_1,D_1), $(A_2,B_2,C_2,D_2) \in X_5$ be Φ-equivalent. Thus,

$$C_1(zI - A_1)^{-1}B_1 + D_1 = [C_2(zI - A_2)^{-1}B_2 + D_2]P$$

for some orthogonal P. Perform the unique factorizations

$$D_1 = \tilde{D}_1U_1, \qquad D_2 = \tilde{D}_2U_2$$

where $\tilde{D}_1, \tilde{D}_2 \in \mathcal{L}$, U_1, U_2 orthogonal. Then clearly

$$D_1 = D_2P = \tilde{D}_2U_2P = \tilde{D}_2\bar{U} = \tilde{D}_1U_1$$

implies that $\tilde{D}_1 = \tilde{D}_2$, and $U_2P = U_1$. Hence,

$$C_1(zI - A_1)^{-1}B_1 + \tilde{D}_1U_1 = C_2(zI - A_2)^{-1}B_2P + \tilde{D}_2U_2P$$

implies that

$$C_1(zI - A)^{-1}B_1 = C_2(zI - A_2)^{-1}B_2P$$

or

$$C_1(zI - A_1)^{-1}B_1U_1^T = C_2(zI - A_2)^{-1}B_2U_2^T.$$

Substituting in this equation according to the map Ω defined above, we have that

$$\tilde{C}_1(zI - \tilde{A}_1)^{-1}\tilde{B}_1 = \tilde{C}_2(zI - \tilde{A}_2)^{-1}\tilde{B}_2$$

and since $(\tilde{A}_i,\tilde{B}_i,\tilde{C}_i) \in \hat{C}$, we have that $(\tilde{A}_1,\tilde{B}_1,\tilde{C}_1) = (\tilde{A}_2,\tilde{B}_2,\tilde{C}_2)$, proving invariance of Ω.

The proof of completeness is straightforward and will be omitted.

A factorization of the spectral density matrix $\Phi(z)$ which is often considered in the literature is given by

$$\Phi(z) = H(z)QH^T(z^{-1})$$

where $Q > 0$, and $H(z) = C(zI - A)^{-1}B + I$.

The transfer function $H(z)$ is therefore that of the state space model which corresponds to the innovations representation of the system, i.e.,

$$\begin{aligned} x_{k+1} &= Ax_k + Be_k \\ y_k &= Cx_k + e_k \end{aligned}$$

where e_k is normally distributed with zero mean and covariance Q. Since we can always factorize Q uniquely into the product of a lower triangular matrix D and its transpose, i.e.,

$$Q = DD^T, \qquad D \in \mathcal{L}$$

thus

$$\Phi(z) = H(z)DD^TH^T(z^{-1})$$

where

$$H(z)D = C(zI - A)^{-1}BD + D$$

and an immediate consequence of the result of Theorem 4 is that if $(A,B,C) \in \hat{C}$, for any set of canonical forms $\hat{C}$ for G_0-equivalence on X_2, then the innovations representation describes a unique factorization of $\Phi(z)$.

V. Application of Canonical Forms in Identification

The canonical forms described in Sections III and IV provide the required unique parameterization of the system which is essential for efficient estimation of the system parameters from input/output data. However, consistent estimation of the parameters of the canonical model requires the accurate knowledge of either a suitable nice multiindex for the system or a set of invariants $\{n_i\}$, uniquely defining the structure of the canonical form. Given this knowledge, under certain conditions, a maximum likelihood estimate of the parameters has been shown to be consistent [23].

The estimation of the $\{n_i\}$ in the stochastic case can be achieved by suitable hypothesis testing. An algorithm which uses the particular canonical form described by Mayne [12], [21], has been implemented successfully. The structure of the canonical form in this case permits a sequential determination of $\{n_i\}$. Other applications of this form have been reported [15].

It is easy to see that the nonnice multiindices may also be used to produce sets of canonical forms, though in this case it is less easy to describe exactly the structure of the elements of the sets.

A problem which was mentioned in Section III is that of the existence in parameter space of hypersurfaces corresponding to nonminimal state space representations. Clearly the set of parameters obtained in the process of any identification algorithm will most probably have to traverse such a hypersurface. This however can have little or no effect on the procedure, the tendency of any gradient technique used to improve the parameter estimates being to take the parameters through the hypersurface. Some aspects of this have been discussed by Glover [24], who has also considered the relative desirability of using different sets of canonical forms from a numerical aspect.

It is from the numerical viewpoint that it may be possible to decide on the best canonical form to use in a certain application. Moreover, it may be possible to decide in general whether to devote attention to those sets of canonical forms which restrict the choice of a basis to a particular selection scheme, e.g., (S1) or (S2), or where the linearly independent set is chosen in a more arbitrary way, e.g., to correspond to a nice (or even nonnice) multiindex. Consider the example illustrated in Figs. 1(a) and (b), and take the case when $B = [\epsilon \; b_2]$. A naive application of the selection procedure (S1) will make use of the set of canonical forms $C_1 = \{(a,[1b_2/\epsilon])\}$, which has undesirable behavior as $\epsilon \to 0$. However, if the set C_{γ_*} were chosen, then this set, i.e., $\{a,[\epsilon/b_1 1])\}$ is well-behaved as $\epsilon \to 0$.

From a more general viewpoint, assume that, for a particular set of input/output measurements, the set of canonical forms for which the parameters are to be estimated has been fixed, e.g., by specifying a $\{n_i\}$. If, in the estimation process, it becomes apparent that the wrong structure has been chosen, it would seem to be desirable that an alternative structure could be chosen in such a way that it would involve a *continuous* change in the parameters. This can clearly only be the case when the two sets of canonical forms corresponding to the different structures are close in some sense, or even intersect. This is apparently the situation when sets of canonical forms determined by arbitrary nice multiindices are used, in contrast to those which are determined by a $\{n_i\}$ resulting from a specific selection procedure and hence, are necessarily disjoint. The choice of sets of canonical forms most suitable for identification is a question which has yet to be resolved.

A major objection which has been made to the use of canonical forms in identification is that parameters of the system, some of which may well be known, bear no simple relationship in general to the parameters of the canonical model. This may be partially overcome by a judicious choice of the canonical model. However there is a great deal of interest in obtaining similar uniqueness properties for models of a more arbitrary form [24], [25].

VI. Conclusion

Several sets of canonical forms have been described, where a strict definition for such sets as the images of complete invariants has been adhered to in each case. The resultant uniqueness properties clearly have direct application in realization and identification theory. It is clear that a study of other forms which have been described in the literature as canonical would yield further useful sets of canonical forms.

The special problems associated with the identification of stochastic systems have been described, and the particular use of canonical forms in case of discrete time systems has been established. Further problems associated with the use of canonical forms in identification have been discussed, many aspects of which must represent the areas of future research in this subject.

Acknowledgment

Grateful acknowledgment is made of many stimulating discussions with Prof. D. Q. Mayne.

References

[1] G. McLane and G. Birkhoff, *Algebra*. New York: Macmillan, 1967.

[2] J. Rissanen, "Basis of invariants and canonical forms for linear dynamic systems," *Automatica*, vol. 10, pp. 175–182, 1974.

[3] S. H. Wang and E. J. Davison, "Canonical forms of linear multivariable systems," Univ. Toronto, Toronto, Canada, Control Syst. Rep. 7203, 1972.

[4] R. E. Kalman, "Kronecker invariants and feedback," in *Proc. NRL Math. Res. Center Conf. Ordinary Differential Equations*, June 1971.

[5] A. S. Morse, "Structural invariants of linear multivariable systems," *SIAM J. Contr.*, vol. 11, pp. 446-465, 1973.

[6] W. M. Wonham and A. S. Morse, "Feedback invariants of linear multivariable systems," *Automatica*, vol. 8, pp. 93–100, 1972.

[7] C. E. Langenhop, "On the stabilization of linear systems," *Proc. Amer. Math. Soc.*, vol. 15, pp. 735–742, 1964.

[8] D. G. Luenberger, "Canonical forms for linear multivariable systems," *IEEE Trans. Automat. Contr.*, vol. AC-12, pp. 290–293, June 1967.

[9] B. D. O. Anderson and D. G. Luenberger, "Design of multivariable feedback systems," *Proc. Inst. Elec. Eng.*, vol. 114, pp. 395–399, Mar. 1967.

[10] V. M. Popov, "Invariant description of linear time-invariant controllable systems," *SIAM J. Contr.*, vol. 10, pp. 252–264, May 1972.

[11] P. E. Caines, "Unique state variable models with application to the identification of linear systems," *Int. J. Contr.*, vol. 16, pp. 939–44, 1972.

[12] D. Q. Mayne, "A canonical model for identification of multivariable linear systems," *IEEE Trans. Automat. Contr.* (Tech. Notes and Corresp.), vol. AC-17, pp. 728–729, Oct. 1972.

[13] H. Weinert and J. Anton, "Canonical forms for multivariable system identification," presented at the IEEE Conf. Decision and Control, 1972.

[14] R. E. Kalman, "Global structure of classes of linear dynamical systems," presented at the NATO ASI Geometric and Algebraic Methods for Nonlinear Systems, London, England Aug. 1973.

[15] E. Tse and H. L. Weinert, "Structure determination and parameter identification for multivariable stochastic linear systems," presented at the 1973 Joint Automatic Control Conf., Columbus, Ohio.

[16] P. Brunovsky, "A classification of linear controllable systems," *Kibern.*, vol. 3, pp. 173-187, 1970.

[17] P. Brunovsky, "On stabilization of linear systems under a certain class of persistent perturbations," *Differentiatial Equations*, vol. 2, pp. 401–405, 1966.

[18] W. Tuel, "Canonical forms for linear systems—Part I," IBM Res. Lab., San Jose, Calif., IBM Res. Rep. RJ 375, 1966.

[19] V. M. Popov, "Some properties of the control systems with irreducible matrix transfer functions," in *Seminar on Differential Equations and Dynamical Systems—II.* Berlin: Springer, 1969, pp. 169–180.

[20] D. Denery, "An identification algorithm that is insensitive to initial parameter estimates," *AIAA J.*, vol. 9, pp. 371–377, 1971.

[21] D. Q. Mayne, "Parameterization and identification of linear multivariable systems," in *Lecture Notes in Mathematics.* Berlin: Springer, 1972, vol. 294, pp. 56 61.

[22] P. E. Caines, Ph.D. dissertation, Imperial College, London, England, 1970.

[23] P. E. Caines and J. Rissanen, "Maximum likelihood estimation of parameters in multivariate Gaussian stochastic processes," *IEEE Trans. Inform Theory* (Corresp.), vol. IT-20, pp. 102–104, Jan. 1974.

[24] K. Glover, "Structural aspects of system identification," Mass. Inst. Technol., Cambridge, Mass., ESL-R-516, Aug. 1973.

[25] K. Glover and J. C. Willems, "On the identifiability of linear dynamical systems," presented at the 3rd IFAC Symp., The Hague/Delft, June 1973.

[26] H. H. Rosenbrock, State Space and Multivariable Theory. New York: Wiley, 1970.

[27] G. D. Forney, "Minimal bases of rational vector spaces with applications to multivariable linear systems," *SIAM J. Contr.*, to be published.

[28] W. A. Wolovich, "The determination of state space representations for linear multivariable systems," *Automatica*, vol. 9, pp. 97–106, 1973.

[29] B. Dickinson, M. Morf, and T. Kailath, "A minimal realization algorithm for matrix sequences," *IEEE Trans. Automat. Contr.*, vol. AC-19, pp. 31–38, Feb. 1974.

Michael J. Denham (S'70) was born in London, England on April 27, 1946. He received the B.Sc. degree in electrical engineering from the University of Birmingham, Birmingham, England, in 1968 and the Ph.D. Degree in control systems from the Imperial College, London, England, in 1973.

From July 1968 to October 1969 he worked as a Development Engineer for Marconi Space and Defence Systems on radar control systems. Since October 1972 he has been employed as Research Assistant at the Imperial College where he is currently in charge of a research project investigating computer aided design of control systems. In October 1974 he was appointed Lecturer at the Imperial College. His main research interests are in algebraic and geometric system theory, identification of linear multivariable systems, and computer aided control system design.

Canonical Matrix Fraction and State-Space Descriptions for Deterministic and Stochastic Linear Systems

B. W. DICKINSON, STUDENT MEMBER, IEEE, T. KAILATH, FELLOW, IEEE AND M. MORF, STUDENT MEMBER, IEEE

Abstract—Several results exposing the interrelations between state-space and frequency-domain descriptions of multivariable linear systems are presented. Three canonical forms for constant parameter autoregressive-moving average (ARMA) models for input–output relations are described and shown to corrrespond to three particular canonical forms for the state variable realization of the model. Invariant parameters for the partial realization problem are characterized. For stochastic processes, it is shown how to construct an ARMA model, driven by white noise, whose output has a specified covariance. A two-step procedure is given, based on minimal realization and Cholesky-factorization algorithms. Though the goal is an ARMA model, it proves useful to introduce an artificial state model and to employ the recently developed Chandrasekhar-type equations for state estimation. The important case of autoregressive processes is studied and it is shown how the Chandrasekhar-type equations can be used to obtain and generalize the well known Levinson–Wiggins–Robinson (LWR) recursion for estimation of *stationary* autoregressive processes.

Manuscript received January 19, 1974; revised August 19, 1974. This work was supported in part by the Air Force Office of Scientific Research, Air Force Systems Command, under Contract AF44-620-74-C-0068, and in part by the Hasler Foundation.

T. Kailath and M. Morf are with the Department of Electrical Engineering, Stanford University, Stanford, Calif. 94304.

B. W. Dickinson was with the Department of Electrical Engineering, Stanford University, Stanford, Calif. 94304. He is now with the Department of Electrical Engineering, Princeton University, Princeton, N.J.