Structural properties of multivariable linear systems

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Abstract

The design of controllers for multivariable systems, in particular large-scale systems, requires knowledge and exploitation of the structural properties of the systems. The ability to achieve certain structural properties, e.g. regulation, tracking, noninteraction, disturbance, rejection etc., can be determined on the basis of a small number of basic system concepts. In this paper, structural decomposition of a multivariable system is studied using the concept of a controllability subspace, and a review of recent work to investigate this concept is given.

1 Introduction

A great deal of the work on design of multivariable linear systems in the UK in the past decade has concentrated on the development of techniques for choosing controller parameters to achieve the desirable properties of stability, disturbance, rejection, integrity etc. A formidable set of design tools, mostly realised by comprehensive computer programs, now exists to enable an optimal or near optimal choice of controller to be made. A significant shortcoming of these methods, however, lies in their inability to cope with large systems, i.e. high order, many inputs and measurements, or with other than centralised control schemes. Also the choice of the controller structure for small scale systems remains largely *ad hoc;* for large systems very few guidelines even exist.

More recently, a greater degree of effort has been spent specifically on the qualitative structural properties of linear multivariable systems. This follows a number of remarkable achievements in this field, by several researches. The structural problem, as opposed to the parameter design problem, is concerned specifically with determining both the form of the controller, and the conditions under which the controller is able to achieve certain desirable system characteristics such as noninteraction, stability, disturbance rejection, tracking, regulation, model matching etc. The ability to achieve these properties depends upon inherent structural properties of the system itself, which can be described (as far as they are known so far) by a reasonably small number of basic system concepts. In their simplest forms, the most familiar concepts (such as those of canonical bases and cyclicity) were made early use of by control theorists to derive conditions of pole assignability[1] etc. In the following, we will consider more recent work to generalise and explore these concepts in such a way as to enable their use in design, particularly in the decentralised control of large-scale systems, which is an area of growing importance.

In Section 2, we will review some fundamental work on the concept of feedback invariants for linear multivariable systems, namely the Kronecker invariants. We describe how these invariants are closely related to a certain canonical form for the state-space equations, and how they determine bounds on the effect of feedback on the internal system structure. In Section 3, we consider further the decomposition of the state space X of the system corresponding to the system corresponding to the Brunovsky canonical form. It is demonstrated here that many decompositions of the state space into direct sums of cyclic controllability subspaces exist, the range of such decompositions being determined by the Kronecker invariants and the associated fundamental decomposition. An open problem is whether these decompositions are useful in design, if they can be chosen so that some degree of measurement space decomposition also takes place.

In Section 4, we consider the structural properties which determine whether decompositions of the measurement space of the system can be imposed, i.e. the almost classical problem of noninteracting control. We generalise this to the decentralised and partial state feedback cases. Finally, in Section 5, we consider some recent work on further exploration of the dimensions and bases for the range of controllability subspaces which can be achieved for a given system; we demonstrate that this approach can lead to new algorithms for computing such subspaces, and thus check the conditions for certain structural properties to hold, e.g. measurement space decomposition.

In the following, we use bold capitals $\mathbf{X}$, $\mathbf{Y}$ etc. to indicate vector spaces and medium face capitals to indicate linear transformations between spaces and their matrix representation. The image of a map (e.g. B) will be denoted by the bold letter (e.g. $\mathbf{B}$). For any positive integer k, we use $\underline{k}$ to denote the set of integers $1, 2, \ldots, k$. The dimension of a subspace $\mathbf{X}$ is denoted by dim $\mathbf{X}$, and the kernel subspace of a map B by ker B. We denote

$$\langle A + BF|\mathbf{R}\rangle = \mathbf{R} + (A + BF)\mathbf{R} + , \ldots , + (A + BF)^{n-1}\mathbf{R}$$

where $\mathbf{R} \subset \mathbf{X}$, the n-dimensional state space.

Finally, $R^k[s]$ is the space of k-dimensional polynomial vectors and $R^k(s)$ the space of k-dimensional rational function vectors, both defined over the field of rational functions $R(s)$.

2 Kronecker invariants and feedback

The modern structural theory of linear multivariable systems stems from the classical work of Kronecker,[2] who introduced the concept of the minimal column indices of a singular pencil of matrices, i.e. a matrix valued polynomial $P(s)$ of degree 1 and determinant 0. Kronecker made use of the indices to provide invariants for arbitrary pencils of matrices under the following equivalence relationship:

$P(s)$ is equivalent to $Q(s)$ if

$$P(s) = NQ(s)M \quad \det N \neq 0, \quad \det M \neq 0$$

where N, M are constant matrices independent of s [Reference 3, chapter 12].

Rosenbrock[4] made use of this concept with regard to the particular singular pencil $[sI - A\ B]$ generated from the state-space description of a linear time-invariant multivariable system

$$\dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t),\ u(t) \in R^m,\ y(t) \in R^p, x(t) \in R^n \tag{1}$$

to study the properties of such a system. Of particular interest here is his result (Reference 4, chapter 5, theorem 4.2) on the effect of state feedback on the structural properties of A (see later).

Let $p_1, \ldots, p_m$ be the ordered $(p_1 \geqslant p_2 \geqslant, \ldots, \geqslant p_m)$ controllability indices of the controllable pair (A, B). That is, the vectors

$$b_1, Ab_1, \ldots, A^{p_1-1}b_1, b_2, \ldots, A^{p_2-1}b_2, \ldots, A^{p_m-1}b_m$$

are composed of the first set of n linearly independent vectors obtained from the sequence

$$b_1, b_2, b_3, \ldots, b_m,\ Ab_1,\ Ab_2, \ldots,\ A^m b_m$$

considered in that order. (For simplicity, we assume rank $B = m$.) The ordering of the p_i is simply obtained by permutation of the columns b_i of B. It is then possible[5] to show that the set of p_i form a complete set of invariants for the pair (A, B) with respect to the equivalence classes defined by the following group of transformations on (A, B):

$$\left.\begin{array}{ll} (a) & (A, B) \to (TAT^{-1}, TB), \ \det T \neq 0 \\ (b) & (A, B) \to (A, BG), \ \det G \neq 0 \\ (c) & (A, B) \to (A + BF, B) \end{array}\right\} \tag{2}$$

Note that transformation (c) is that of state feedback, whereas (b) is that of the nonsingular input transformation and (a) simply constitutes a change of basis of the state space.

Consequently, as shown by Brunovsky,[5] there exists a canonical form for (A, B) under this set of transformations, which is completely determined by the set of k_i. This form is described by the equivalent pair $(\tilde{A}, \tilde{B})$, where

$$\tilde{A} = \text{diag}\,[A_1, A_2, \ldots, A_m]$$
$$\tilde{B} = [e_{\sigma_1},\ e_{\sigma_2}, \ldots,\ e_{\sigma_m}]$$

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where

$$A_i = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & \ddots & & \\ & & & \ddots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad e_{\sigma_i} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \leftarrow \sigma_i\text{th element} \\ \\ \\ \\ \end{array}$$

$$\sigma_i = \sum_{j=1}^{i} p_j$$

and A_i is of dimension $p_i \times p_i, i \in \underline{m}$.

The structure of the canonical matrix $\tilde{A}$ corresponds directly to a decomposition of the state space X of the system (eqn. 1) into the direct sum of a set of cyclic subspaces R_i,[6] i.e.

$$X = R_1 \oplus R_2 \oplus, \dots, \oplus R_m, \quad \dim R_i = p_i \tag{3}$$

As demonstrated in Reference 6, the R_i are independent cyclic controllability subspaces of (A, B), i.e. every state $x \in R_i$ can be reached from the origin of R_i along a controlled trajectory that is wholly contained in R_i using a scalar control.

In more concise mathematical terms

$$R_i = e_{\sigma_i} + (A + BF)e_{\sigma_i} +, \dots, + (A + BF)^{p_i - 1} e_{\sigma_i} \tag{4}$$

and $e_{\sigma_i} = B \cap R_i$. This concept of a controllability subspace was first introduced by Wonham and Morse,[7] and its use in many aspects of structural analysis of linear multivariable systems is described in References 8 and 9.

In the following, we will pursue this concept to study only the structural properties which determine how the state space and the measurement space of a system can be decomposed. We will not consider in this paper the many other structural properties, e.g. regulation, tracking etc., which this concept can be used to investigate, but refer the reader to References 8 and 9.

3 Controllability subspaces: decomposition of the state space X

Using the Kronecker invariants,* their associated Brunovsky canonical form and corresponding decomposition of X given by eqn. 3, we can determine further what other direct sum decompositons of X into cyclic invariant (i.e. such that $(A + BF) V_i \subset V_i$) subspaces can be achieved by state feedback. The first result came from Rosenbrock,[4] as already mentioned above. This states that the closed-loop system matrix $(A + BF)$ has arbitrary invariant factors $\psi_1, \dots, \psi_m$ for some F if, and only if:

(i) ψ_i divides ψ_{i-1} for $i = 2, 3, \dots, m$
(ii) $\deg \psi_m \leqslant k_1$, $\deg \psi_{m-1} \leqslant k_1 + k_2$ etc.

Since every set of invariant factors for $(A + BF)$ corresponds to a decomposition of the state space X of the system into the direct sum of a set of cyclic invariant subspaces, i.e.

$$X = V_1 \oplus V_2 \oplus, \dots, \oplus V_m$$

where

$$(A + BF) V_i \subset V_i, i \in \underline{m} \text{ and } \psi_i(A + BF)x = 0$$

for all $x \in V_i$, $i \in \underline{m}$, it follows that Rosenbrock's theorem specifies a set of upper bounds on the dimensions of the component subspaces V_i of such a decomposition.

We note here for completeness that the relationship between the transformation group (eqn. 2) and the equivalence relation of the singular pencil $[sI - A \quad -B]$ studied by Rosenbrock is given simply by

$$[sI - A \quad B] = T[sI - \tilde{A} \quad -\tilde{B}]S$$

where

$$S = \begin{bmatrix} T^{-1} & 0 \\ F & G^{-1} \end{bmatrix}, \quad \tilde{A} = T^{-1}(A + BGF)T, \quad \tilde{B} = T^{-1}BG$$

The singular pencil $[sI - A \quad B]$ and its minimal indices were also considered by Warren and Eckberg[10] to further determine the structural properties achievable by feedback, in this case the existence and uniqueness for a given system of a controllability subspace $R \subset X$ of a specified dimension. We will consider this further in Section 5 of this paper. At present we will make use of the results of Reference 10 simply to investigate the range of possible cyclic decompositions of X which are achievable using the feedback transformation group (eqn. 2). The main theorem in Reference 10 states that a controllability subspace R of dimension d exists for the controllable system (eqn. 1) if, and only if,

$$\max\{k_i | k_i \in S\} \leqslant d \leqslant \sum_{i \in S} k_i \tag{5}$$

for some subset $S = \{k_{i1}, \dots, k_{ia}\}$ of the set of k_i.

We will illustrate this result with an example. Let $m = 4$, $k_4 = 4$, $k_3 = 2$, $k_2 = k_1 = 1$. The Brunovsky canonical form is therefore given by

$$\tilde{A} = \left[\begin{array}{cccc|cc|c|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \tilde{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the corresponding decomposition of X is given by

$$X = R_1 \oplus R_2 \oplus R_3 \oplus R_4$$

where $\dim R_1 = 4$, $\dim R_2 = 2$ and $\dim R_3 = \dim R_4 = 1$. We denote this decomposition as (4, 2, 1, 1).

Consider now the decomposition (4, 3, 1). This can clearly be achieved by using state feedback to change a_{67} of $\tilde{A}$ from zero to one. Other decompositions can be achieved in a similar simple fashion, e.g. (6, 1, 1), corresponding to forming the direct sum of two or more R_i in the original decomposition.

Further decompositions are achievable by the sum of R_i with overlap. For example, the decomposition (5, 2, 1) can be obtained first by changing a_{46} of $\tilde{A}$ from zero to one. In this case, the vector $e_6 = b_2$ generates a subspace V_1 spanned by $\{e_6, e_5 + e_4, e_3, e_2, e_1\}$. Secondly, a_{57} is changed from zero to one, yielding the subspace V_2 spanned by $\{e_7, e_5\}$. Finally, V_3 is then spanned by $\{e_8\}$, and the decomposition takes the form

$$X = V_1 \oplus V_2 \oplus V_3$$

A similar decomposition can be achieved where

$$V_2 = \text{span}\{e_8, e_5\}, V_3 = \text{span}\{e_7\}$$

Further consideration also shows, however, that the decomposition (3, 3, 2) is not achievable. Subspaces of dimensions 2 and 3 can certainly be achieved individually, but having done so it is not possible to further obtain the other spaces independently. This observation yields a general result which states that the condition (eqn. 5) given above, for the existence of a controllability subspace of a given dimension can be applied sequentially. That is, in the case of the example, if a subspace V_3 is obtained of dimension 2 by changing a_{78} from zero to one, and a subspace of dimension $d = 3$ is now sought from the remaining blocks for which $k_4 = 4, k_3 = 2$, this violates the condition

$$\max_S \{k_i \in S\} \leqslant d \leqslant \sum_{k_i \in S} k_i$$

where S is any subset of $\{k_4, k_3\}$. Therefore the decomposition (3, 3, 2) is impossible. The same result is obtained if the subspace V_2 of dimension 3 is obtained first by changing a_{67} to one. This result enables a simple algorithm to be constructed to determine the full range of possible decompositions of X into disjoint cyclic controllability subspaces.

However, there is a further consideration to be made. Is a possible decomposition unique? For an individual controllability subspace of dimension p, necessary and sufficient conditions are derived in Reference 10 for the subspace to either not exist or be unique. Failing these conditions, it clearly exists and is nonunique. Moreover, nonuniqueness implies the existence of an uncountable number of subspaces of

* Redefined in terms of the controllability indices as $k_i = p_{m-i+1}$, $i \in \underline{m}$ (to be consistent with Rosenbrock's notation[4]) in the remainder of this paper

that dimension. We will consider this aspect in more detail in Section 5, using the results of Reference 11.

An open question exists as to whether the range of possible decompositions of X into cyclic controllability subspaces achievable via feedback can be exploited to obtain system structural properties useful in controller design for the system. Clearly this must depend on the relationship between the component subspaces V_i and the measurement map C. A system for which the measurement space is decomposed thus

$$Y = CV_1 \oplus CV_2 \oplus, \ldots, \oplus CV_m$$

for a corresponding decomposition of X, is called 'prime' by Morse[12] and possesses certain nice properties. We note that this implies a noninteracting control structure. A less stringent criterion for the relationship between the images of the subspaces V_i under the measurement map C could be devised, more closely related to practical design considerations, e.g. that the inverse image of V_i under C should have some desirable spectrum of eigenvalues with respect to $A + BF$.

Unfortunately, direct sum decompositions of the kind described above, although attractive, are in general only possible to achieve using feedback from all the system states, a significant practical drawback. In addition, it seems unreasonably restrictive to insist on the decomposition of X into a completely disjoint set of subspaces, as long as the effects of the intersections between subspaces are well defined and controlled in the measurements. For example, a general form of the noninteracting control situation has

$$X = R_1 + R_2 +, \ldots, + R_p \tag{6}$$

and

$$Y = CR_1 \oplus CR_2 \oplus, \ldots, \oplus CR_p$$

i.e. the intersection subspace between the R_i

$$R^* = \bigcap_{i \in \underline{p}} \left(\sum_{j \neq i} R_j \right)$$

is always completely contained in the kernel of the measurement map C by construction.

This additional freedom in the decomposition of X permits the investigation of partial state and decentralised feedback systems. However, there does not appear at present to be any way of characterising the set of all possible decompositions of X of the form of eqn. 6. The existence and uniqueness of individual controllability subspaces R_i are well defined,[10,11] but their 'compatibility', i.e. the existence of a common feedback matrix F such that $(A + BF)R_i \subset R_i$, for all $i \in \underline{m}$, cannot readily be determined in general. Some work is in progress in this direction.

4 Controllability subspaces: measurement decomposition and decentralised control

To take the ideas expressed at the end of the last Section a little further, let us consider the following problem.
Given a system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

can we determine a controllability subspace R such that $(A + BKC)R \subset R$ and $R \subset Z \subset X$ where Z is some specified proper subspace of X? The conditions derived in Reference 13 for output feedback decoupling provide the solution to this problem, i.e. a solution exists if, and only if, there exists a controllability subspace $R \subset Z$ such that

$$A(R \cap \ker C) \subset R$$

This is extended to the measurement decomposition situation by determining a compatible set of $R_i, i \in \underline{p}$, such that

$$R_i \subset Z_i \triangleq \bigcap_{j \neq i} \ker c_j$$

where c_j is the jth row of the measurement matrix $C, j \in \underline{p}$, and such that

$$A(R_i \cap \ker C) \subset R_i \quad i \in \underline{p}$$

and

$$R^* = \bigcap_{i \in \underline{p}} \left(\sum_{j \neq i} R_j \right) = 0$$

It has been shown[14] that is is possible to extend this idea simply to consider the problem of measurement decomposition by decentralised control. The essential idea is to formulate the constraint of decentralisation in a similar form to that of output feedback, i.e. that $F = KC$ or $F(\ker C) = 0$. For simplicity, we take only two control agents, i.e.

$$\dot{x}(t) = Ax(t) + \sum_{i=1,2} B_i u_i(t)$$

The feedback control takes the form

$$u_i(t) = F_i x(t) \quad i = 1, 2$$

where, if $X_i \subset X$ is the subspace not available to $u_i(t)$, then $F_i X_i = 0$ is the constraint on F_i. It is easy to see that this is similar to the control constraint in the output feedback case. The closed-loop system takes the form

$$\dot{x}(t) = (A + \sum_{i=1,2} B_i F_i) x(t) + \sum_{i=1,2} B_i v_i(t)$$

and we now assume, as in Reference 14, that the controllability subspaces R_i, $i = 1, 2$ are chosen generated from the columns of B_i, $i = 1, 2$, respectively, i.e. $B \cap R_i = B_i$ and

$$\left\langle \left(A + \sum_{i=1,2} B_i F_i \right) \middle| B_i \right\rangle$$

$$= \langle (A + B_j F_j) | B_i \rangle = R_i, \quad i = 1, 2, \quad j \neq i$$

Necessary and sufficient conditions for measurement decomposition by decentralised feedback then follow by analogy with the output feedback case. A solution exists if, and only if, there exist R_i, $i = 1, 2$, such that $R_i \subset Z_i \triangleq \ker c_j$,

$$\langle (A + B_j F_j) | B_i \rangle = R_i$$

and

$$A(R_i \cap X_j) \subset R_i$$

for $i, j = 1, 2; j \neq i$. The last condition ensures that, for example, in achieving R_1, the control can be chosen such that

$$u_2(t) = B_2 F_2 x(t)$$

where $F_2 X_2 = 0$, and thus $u_2(t)$ depends only on $X_1 \subset X$, X_2 not being available to it.

Many possibilities exist for extending and exploiting this concept, for example, by introducing dynamic feedback elements, as was done in the output feedback case of the measurement decomposition problem.[15]

5 Controllability subspaces: parameterisation and characterisation

To exploit those structural properties of linear multivariable systems described above, plus numerous others referred to elsewhere, it is necessary both to understand in more depth the parameterisation of controllability subspaces, and to be able to form an efficient characterisation of the subspaces to further examine their properties, e.g. the problem of compatibility of subspaces. It has been shown[11] that, by combining the work described in Reference 10 with the Wolovich and Falb 'structure theorem'[16,17] and Forney's fundamental work on minimal bases of rational vector spaces,[18] it is possible to obtain a module theoretic characterisation of the bases of all possible controllability subspaces of a given system.

Consider the polynomial matrix equation

$$[sI - A \quad -B] \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} = 0 \tag{7}$$

Then, from Reference 10, we know that a set of free generators for $\ker [sI - A \quad -B]$ (considered as contained in the rational vector space $R^{n+m}(s)$ over the field of rational functions $R(s)$) is given by a 'fundamental series' of solutions to eqn. 7.[10,4,3] If $z_i(s) = [x_i(s)^T u_i(s)^T]^T$, $i \in \underline{m}$, is such a fundamental series of solutions to eqn. 7, then $\deg z_i(s) = k_i$, $i \in \underline{m}$,[4] and the set of $z_i(s)$, $i \in \underline{m}$, constitute a minimal basis for $\ker [sI - A \quad -B]$.

According to Forney's main theorem,[18] any $z(s) \in \ker [sI - A \quad -B]$ can be uniquely written as

$$z(s) = \sum_{i: k_i \leqslant \deg z_i(s)} z_i(s) a_i(s) \tag{8}$$

where the $a_i(s) \in R[s]$ satisfy $\deg a_i(s) \leqslant \deg z(s) - k_i$. If we now assume that the system is in controllable canonical form $[\hat{A}, \hat{B}]$, from the Wolovich-Falb 'structure theorem', we have that

$$[sI - \hat{A} \quad -\hat{B}] \begin{bmatrix} S(s) & \\ \hat{B}_m^{-1} & \delta_0(s) \end{bmatrix} = 0 \tag{9}$$

where

$$S(s) = \text{block diag}\,(\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_m)$$

$$\hat{s}_i^T = (1\, s\, s^2, \ldots, s^{k_{m-i+1}-1})$$

$$\delta_0(s) = [s^k] - \hat{A}_m S(s)$$

$$[s^k] = \text{diag}\,(s^{k_m}, s^{k_{m-1}}, \ldots, s^{k_1})$$

and $\hat{A}_m$ is the $m \times n$ matrix consisting of the m ordered σ_ith rows of $\hat{A}$, where

$$\sigma_i = \sum_{j=m}^{m-i+1} k_j, \quad i \in \underline{m}$$

and $\hat{B}_m$ is the $m \times m$ matrix consisting of the m ordered σ_ith rows of $\hat{B}$, $i \in \underline{m}$.

Eqn. 9 states that the column vectors $\tilde{z}_i(s) = (\tilde{x}_i(s)^T u_i(s)^T)^T$, $i \in \underline{m}$, of the polynomial matrix

$$\tilde{Z}(s) = [z_m(s), \ldots, z_1(s)] = \begin{bmatrix} S(s) & \\ \hat{B}_m^{-1} & \delta_0(s) \end{bmatrix} \tag{10}$$

constitute a fundamental series of solutions to eqn. 7. Furthermore, the special structure of $\tilde{Z}(s)$ qualifies it as a minimal basis *in echelon form*[18] for $\ker\,[sI - \hat{A} \quad -\hat{B}]$. This, together with eqn. 8, gives rise to the following result.[19]

Let $\tilde{x}(s) = \tilde{x}_0 + \tilde{x}_1 s +, \ldots, + \tilde{x}_{k-1} s^{k-1}$ and $u(s) = u_0 + u_1 s +, \ldots, + u_k s^k$, satisfy eqn. 7. Then there exists $\beta(s) = [\beta_m(s), \ldots, \beta_1(s)]^T$, $\deg \beta_i(s) \leqslant k - k_i$, $i \in \underline{m}$, such that

$$\tilde{x}(s) = S(s)\beta(s) \tag{11}$$

$$u(s) = \hat{B}_m^{-1} \delta_0(s) \beta(s) \tag{12}$$

If r is a possible dimension of a controllability subspace R of the system, then from Reference 10, lemma 2 there must exist $\tilde{x}(s) = \tilde{x}_0 + \tilde{x}_1 s +, \ldots, + \tilde{x}_{r-1} s^{r-1}$ with the $\tilde{x}_{i-1}$, $i \in \underline{r}$ independent, and $u(s) = u_0 + u_1 s +, \ldots, + u_r s^r$, such that $[\tilde{x}(s)^T \tilde{u}(s)^T]^T \in \ker\,[sI - \hat{A} \quad -\hat{B}]$. Then $R = \text{span}\,[\tilde{x}_{i-1}, i \in \underline{r}]$. We define such an $\tilde{x}(s)$ to be a 'proper minimal degree generating function' of R.[19]

Using the result stated above, it follows that we can write

$$\tilde{x}(s) = S(s)\beta(s) \tag{13}$$

for unique $\beta(s)$, where $\deg \beta_i(s) \leqslant r - k_i$, $i \in \underline{m}$, with equality holding for at least one $i \in \underline{m}$.

If we write

$$\beta_i(s) = \beta_{i0} + \beta_{i1} s +, \ldots, + \beta_{i,r-k_i} s^{r-k_i}, \quad i \in \underline{m}$$

it can be verified, after expanding eqn. 13, that the basis matrix $R = [\tilde{x}_0 \tilde{x}_1, \ldots, \tilde{x}_{r-1}]$ has the general form[11]

A polynomial vector $\beta(s)$, which gives rise via eqn. 13 to a 'proper minimal degree generating function' $\tilde{x}(s)$ of a controllability subspace R, determines and is uniquely determined by R; in Reference 19 it was called a 'decoupling vector'. A necessary condition for any $\beta(s) \in R^m[s]$ to be a decoupling vector is that the greatest common divisor of its elements $\beta_i(s)$ is equal to one. This fact gave rise to a generalisation of the classical resultant theorem concerning relative primeness of $m \geqslant 2$ polynomials.[20]

The simple stripe form of the basis matrix R of R provides explicit information on the existence and uniqueness of controllability subspaces of a given dimension k. If for a given k we define

$$\left.\begin{aligned} F^k_{m-i+1} &= 0 \quad \text{if } 1 \leqslant k < k_{m-i+1} \\ &= k_{m-i+1} \quad \text{if } k_{m-i+1} \leqslant k \leqslant n \end{aligned}\right\} i \in m$$

and

$$e^k = \sum_{i=1}^{m} F^k_{m-i+1}$$

then we can state the following result.[11]

The number of possible controllability subspaces R of dimension k is given by

(*a*) for $0 \leqslant e^k < k$ there is no controllability subspace of dimension k

(*b*) for $e^k = k$ there is a unique controllability subspace of dimension k, for which

$$R = \begin{bmatrix} 0 \\ I_k \end{bmatrix}$$

(*c*) for $e^k > k$ there is an uncountable number of controllability subspaces of dimension k.

As we have seen in Section 4, an important problem in the measurement decomposition problem is that of determining controllability subspaces contained in a given subspace Z, where usually $Z = \ker C$ for some measurement map C. The above characterisation allows a different approach to this problem than that, for example, in References 7 or 8, which requires prior computation of the maximal dimension (A, B)-invariant subspace in Z.

Let $Z = \ker C$, and rewrite eqn. 7 in the form

$$\begin{bmatrix} sI - \hat{A} & \hat{B} \\ \hat{C} & 0 \end{bmatrix} \begin{bmatrix} S(s) & \\ \hat{B}_m^{-1} & \delta_0(s) \end{bmatrix} \beta(s) = 0 \tag{14}$$

Then

$$\hat{C}S(s)\beta(s) = 0 \tag{15}$$

and hence from eqn. 11

$$\hat{C}\tilde{x}(s) = \hat{C}S(s)\beta(s) = 0 \tag{16}$$

Thus

$$R = \text{span}\,[\tilde{x}_0, \quad \tilde{x}_1, \quad \ldots, \quad \tilde{x}_r]$$

is, by the previous development, a controllability subspace contained in $Z = \ker \hat{C}$.

Define $\hat{C}(s) = \hat{C}S(s)$, and consider it as a mapping $\hat{C}(s): R^m[s] \to R^p[s]$, with rank $\hat{C}(s) = q$. Let

$$B(s) = [\beta_1(s), \ldots, \beta_{m-q}(s)]$$

$$R = \left[\begin{array}{cccccccc}
\beta_m & \beta_{m1} & \cdots\ \beta_{m,r-k_m-1} & \beta_{m,r-k_m} & 0 & \cdots & 0 & 0 \\
0 & \beta_{m0} & & \beta_{m,r-k_m-1} & \beta_{m,r-k_m} & \cdots & 0 & 0 \\
0 & 0 \ddots \beta_{m0} & \beta_{m1} \cdots & & \ddots & & \beta_{m,r-k_m} & 0 \\
0 & 0 & 0\ \ \beta_{m0}\ \ \beta_{m1} & & \cdots & & \beta_{m,r-k_m-1} & \beta_{m,r-k_m} \\
\hdashline
& & \vdots & & & & & \\
\hdashline
\beta_{10} & \beta_{11} & \cdots\ \beta_{1,r-k_1-1} & \beta_{1,r-k_1} & & \cdots & 0 & 0 \\
0 & \beta_{10} & \cdot & \beta_{1,r-k_1-1} & & \cdots & 0 & 0 \\
0 & 0 \ddots \beta_{10} & \beta_{11} & & \ddots & & \beta_{1,r-k_1} & 0 \\
0 & 0 & 0\ \ \beta_{10}\ \ \beta_{11} & & & & \beta_{1,r-k_1-1} & \beta_{1,r-k_1}
\end{array}\right]$$

be a polynomial basis[18] for ker $\hat{C}(s)$. Then from the above, it follows that

$$X(s) \triangleq S(s)B(s)$$

is a $n \times (m-q)$ polynomial matrix; the columns of which form a basis for the rational vector space $X(s)$. This has the property that every $\tilde{x}(s) \in X(s)$, written $\tilde{x}(s) = \tilde{x}_0 + \tilde{x}_1 s +, \ldots, + \tilde{x}_{k-1}s^{k-1}$, defines a controllability subspace $R = \text{span}\,[\tilde{x}_0, \ldots, \tilde{x}_{k-1}]$ contained in Z. This gives rise to the following result.[19]

There exists a controllability subspace $R \subset Z$ if and only if rank $\hat{C}(s) < m$. Moreover, if and only if rank $\hat{C}(s) = m-1$, there exists a unique R given by $\tilde{x}(s) = S(s)\beta(s)$ where $\beta(s)$ is the unique solution of

$$\hat{C}(s)\beta(s) = 0$$

Finally, if and only if rank $\hat{C}(s) < m-1$, there exists an uncountable number of controllability subspaces $R \subset Z$, given by $\tilde{x}(s) = S(s)\beta(s)$, where

$$\beta(s) = \sum_{i \in m-q} a_i(s)\beta_i(s) = B(s)a(s)$$

for arbitrary $a_i(s) \in R\,[s]$.

Furthermore, if $X(s) = S(s)B(s)$ is reduced, according to Reference 18, to a minimal order basis in echelon form

$$X^*(s) = [\tilde{x}_1^*(s), \ldots, \tilde{x}_{m-q}^*(s)]$$

then the columns $\tilde{x}_i^*(s)$ of $X^*(s)$ are 'proper minimal degree generating functions' of independent controllability subspaces R_i, $i \in m-q$. Their direct sum

$$\hat{R} = R_1 \oplus R_2 \oplus, \ldots, \oplus R_{m-q}$$

is then the *maximal* dimension controllability subspace contained in Z. The lengthy proof of this result is given in Reference 19.

6 Conclusions

In this paper, we have reviewed the basis of a structural theory for linear multivariable systems, the detailed exploitation of which is finding many new and exciting applications to control system design, in particular to the achieving of certain system structural properties. For an excellent discussion of these properties, the reader is referred to Reference 9. Here we have only used two simple structural properties to illustrate the concepts involved, namely the decomposition of the system state space into the direct sum of cyclic controllability subspaces, and the decomposition of the measurement space. In particular, we considered the latter problem in the light of decentralised control, an important area of large-scale control-system analysis and design. Finally, we considered the basic system concept of a controllability subspace in greater detail, using a recently developed characterisation of the set of all possible controllability subspaces based on polynomial matrices and the Wolovich-Falb structure theorem.

Further research in this field requires an even greater understanding of the basic system concepts, both controllability subspaces and the related concept of an (A, B)-invariant subspaces. Many important relationships have already been discovered, e.g. to the system transmission zeros.[12] A greater understanding of the relationship between the subspace decomposition of X and various system properties such as regulation, disturbance rejection etc. is now required. Perhaps the greatest challenge lies in discovering how these concepts can be used in the study of structural properties in large scale systems, where decentralised control schemes provide additional constraints over those for conventional small centralised system. The nature of this problem also implies the need for numerically efficient computational methods in the calculation of the basic system concepts. A great deal of work has already been done in this area.[21]

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