On a Forgotten Conjecture from a Famous Paper of Erdős

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ABSTRACT

In his paper "On sets of distances of n points", Paul Erdős conjectured that every convex curve contains a point P such that every circle centered at P intersects the curve in at most 2 points. This conjecture is false: If T is an equilateral triangle with boundary ∂T, for any point P on ∂T there is a circle centered at P that intersects ∂T at 4 points. But perhaps the number 2 in Erdős's conjecture can be replaced by some other number.

Given a convex body K in the plane, let $N(K)$ be the smallest number for which there is a point P in ∂K such that every circle centered at P intersects ∂K in at most $N(K)$ points. Erdős's original conjecture states that $N(K) \leq 2$ for every convex body K.

We give an example of a convex body K for which $N(K) = 6$ and we show that $N(K)$ is finite for every convex body K. We believe that $N(K)$ is bounded by some constant, probably by 6, but so far we have not been able to find any global upper bound. Part of the difficulty may come from the following. Given a number n, let $J(K, n)$ be the set of points P in ∂K such that there is a circle centered at P that intersects ∂K in at least n points. Then $N(K)$ is the largest N such that $J(K, N) = bd(K)$. For every $\epsilon > 0$, there is a convex body K such that $|J(K, \infty)|/|bd(K)| > 1 - \epsilon$, where $|X|$ represents the 1-dimensional Hausdorff measure of X. In the Baire category sense, for most convex bodies K the set $\bigcap J(K, n)$ contains most points of ∂K.

Categories and Subject Descriptors

G.2.0 [**Mathematics of Computing**]: Discrete Mathematics—*General*

Keywords

Geometry, Convexity, Erdős problem

1. INTRODUCTION

In his celebrated paper [1] *"On sets of distances of n points"*, Paul Erdős makes the following conjecture:

"On every convex curve there exists a point P such that every circle with center P intersects the curve in at most 2 points."

A very simple example shows that this is false: For any point P on the boundary of a regular triangle there is a circle centered at P that intersects the boundary of the triangle 4 times. In fact, any regular $(2k+1)$-gon has this property. But all is not lost. Perhaps the number 2 in Erdős's conjecture can be replaced by some larger number.

Let K be a planar convex body. We define $N = N(K) \in \mathbb{N} \cup \{\infty\}$ as the smallest number for which there is a point $P \in \partial K$ such that every circle with center P intersects ∂K in at most N points. With this notation, Erdős's original conjecture states that $N(K) \leq 2$ for every convex body K. We conjecture that $N(K)$ is indeed bounded by some finite constant independent of K, probably by 6.

THEOREM 1.1. *There exists a planar convex body K with $N(K) = 6$.*

The simplest example we found for this is a 15-gon and it is constructed in Section 3. On the other hand we can show the following.

THEOREM 1.2. *There is no planar convex body K with $N(K) = \infty$.*

A stronger version of this theorem is proved in Section 2. So far we have not been able to find a finite upper bound that works for all K. Part of the difficulty of improving this bound may come from the following two theorems.

For $n \in \mathbb{N} \cup \{\infty\}$, let $J(K, n)$ be the set of points $P \in \partial K$ such that there is a circle centered at P that intersects ∂K in at least n points. Note that, in view of Theorem 1.2, $N = N(K)$ is the largest N such that $J(K, N) = \partial K$.

We denote by $|X|$ the 1-dimensional Hausdorff measure (perimeter) of a set $X \subset \mathbb{R}^2$.

THEOREM 1.3. *Let $\epsilon > 0$, then there is a convex body K_ϵ such that*

$$\frac{|J(K_\epsilon, \infty)|}{|\partial K_\epsilon|} > 1 - \epsilon.$$

If K_0 is a segment or an acute triangle, then we can construct K_ϵ as in the previous theorem so that $\lim_{\epsilon\to 0} K_\epsilon = K_0$ in the Hausdorff metric. These examples are also constructed in Section 3.

The next theorem is in the Baire category sense (see Section 4 and Chapter 20 of [3] for notions and definitions). Let $\mathcal{K}$ be the set of planar convex bodies together with the Hausdorff metric.

THEOREM 1.4. *For most convex bodies $K \in \mathcal{K}$, the set $\bigcap_{n=1}^{\infty} J(K,n)$ contains most points of ∂K.*

We give the proof of this theorem in Section 4.

2. THE FINITENESS OF N

First we fix some notation. We write $\mathcal{B}$ for the closed unit disk and $\mathcal{B}(Q,r)$ resp. $\mathcal{S}(Q,r)$ for disk and the circle centered at Q with radius $r > 0$.

If K is a convex body and $P \in \partial K$, then a line l is a *normal of K at P* if $P \in l$ and the line orthogonal to l through P supports K at P.

Fix a convex body K and define the set

$$\Gamma = \{(Q,l) : Q \in \partial K,\ l \text{ is a normal of } K \text{ at } Q\}.$$

The set Γ is actually a curve, this can be seen by considering the smooth convex body $K' = K + \mathcal{B}$. The set Γ is in bijective correspondence with $\partial K'$ in the following way: For every point $Q' \in \partial K'$, let l be the normal line of K' at Q' and let Q be the point in $l \cap \partial K$ at distance 1 from Q'. Then the pair $(Q,l) \in \Gamma$ corresponds to the point $Q' \in \partial K'$.

The distance between two points $Q_1', Q_2' \in \partial K'$ is the length of the shortest arc of $\partial K'$ bounded by these points. We use the above bijection to measure the distance between points in Γ and the Euclidean metric to measure distances between points in the plane.

Now we go back to the problem in question. Take $P \in \partial K$ and assume that there are two different points $Q_1, Q_2 \in S(P,r) \cap \partial K$. Let $H \subset \partial K$ be the closed arc bounded by Q_1 and Q_2 that does not contain P. Consider the function $g(Q) = \mathrm{dist}(P,Q)$ for $Q \in H$. Since $g(Q_1) = g(Q_2)$, there exists Q in the relative interior of H such that g attains either its maximum or its minimum on Q. For this Q there is a line l so that $(Q,l) \in \Gamma$, and $P \in l$.

Hence, if $P \in \partial K$ is a point such that there are exactly M pairs $(Q,l) \in \Gamma$ with $P \in l$ and $P \neq Q$, then any circle centered at P intersects ∂K in at most $M+1$ points. This implies $N(K) \le M+1$, therefore to prove Theorem 1.2 it is enough to show the following.

THEOREM 2.1. *Given a convex body K, there is a point $P \in \partial K$ such that the number M of pairs $(Q,l) \in \Gamma$ with $P \neq Q$ and $P \in l$ is finite.*

It may even be possible that M is bounded by some constant independent of K. From the proof it can be seen that M is finite in a positive fraction of the perimeter of K.

To prove this theorem we define $\Gamma_0 \subset \Gamma$ as the set of pairs (Q,l) for which $l \cap \partial K$ contains exactly one point besides Q. Let $f(Q,l)$ be this point. We shall study the function $f : \Gamma_0 \to \partial K$.

If $(Q,l) \in \Gamma \setminus \Gamma_0$ then $l \cap \partial K$ contains either one point (namely Q) or an infinite number of points (namely an edge of K with Q as an endpoint). In either case, ∂K is not smooth at Q and the internal angle formed at this point is at most $\frac{\pi}{2}$. We call such a point Q a *small angle of K*. Since there are at most 4 small angles on a closed convex curve, $\Gamma \setminus \Gamma_0$ contains at most 8 connected components.

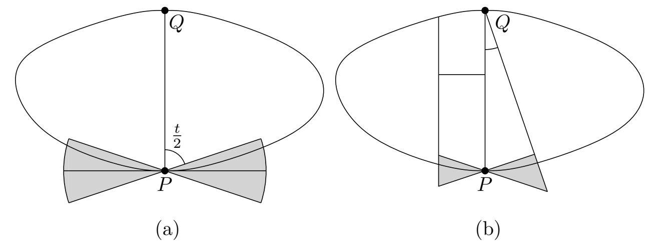

Figure 1: Lemma 2.2.

Given $(Q,l) \in \Gamma_0$, define $\alpha(Q,l)$ as the smallest angle between the line l and a supporting line of K at $f(Q,l)$. Note that $\alpha(Q,l) > 0$ and that $\alpha : \Gamma_0 \to \mathbb{R}$ is a lower semicontinuous function and therefore the sets

$$\Delta_t = \{(Q,l) \in \Gamma_0 : \alpha(Q,l) > t\}$$

are open in Γ_0.

LEMMA 2.2. *For every $t > 0$, the function $f|\Delta_t : \Delta_t \to \partial K$ is locally Lipschitz. If D is the diameter of K, then $lip(f) \le \frac{\max\{1,D\}}{\sin(\frac{t}{2})}$ in any small-enough open set of Δ_t.*

PROOF. Let $(Q,l) \in \Delta_t$ and $P = f(Q,l)$. Since K is convex, there exists $\epsilon > 0$ such that any point $R \in \mathcal{B}(P,\epsilon) \cap \partial K$ satisfies $\angle QPR > \frac{t}{2}$ (see Figure 1(a)).

It is not difficult to see that there is a $\delta > 0$ such that if the pair $(Q',l') \in \Delta_t$ is at distance less than δ from (Q,l), then the point $P' = f(Q',l')$ is in $\mathcal{B}(P,\epsilon)$.

Since $\mathrm{dist}(Q,Q') \le \mathrm{dist}((Q,l),(Q',l'))$ and the angle between l and l' is at most $\mathrm{dist}((Q,l),(Q',l'))$, the region where P' is can be further bounded. If we assume in Figure 1(a) that Q' is to the left of Q, then we have $\angle P'QP \le \mathrm{dist}((Q,l),(Q',l'))$ if P' is right of P, and $\mathrm{dist}(P',l) \le \mathrm{dist}((Q,l),(Q',l'))$ if P' is to the left of P. This determines the marked region in Figure 1(b). Thus,

$$\mathrm{dist}(P,P') \le \frac{1}{\sin(\frac{t}{2})}\mathrm{dist}((Q,l),(Q',l'))$$

if P' is right of P, and

$$\mathrm{dist}(P,P') \le \frac{\mathrm{dist}(Q,P')}{\sin(\frac{t}{2})}\mathrm{dist}((Q,l),(Q',l'))$$

if P' is to the left of P. In both cases we have

$$\mathrm{dist}(P,P') \le \frac{\max\{1,D\}}{\sin(\frac{t}{2})}\mathrm{dist}((Q,l),(Q',l')).$$

This implies for the Lipschitz constant of f that

$$\mathrm{lip}(f) \le \frac{\max\{1,D\}}{\sin(\frac{t}{2})}$$

in any small-enough open set of Δ_t. $\square$

LEMMA 2.3. *If the convex body K is not a polygon with at most 6 sides, there is a set $F \subset \partial K$ with $|F| > 0$ and a number $t > 0$ such that $f^{-1}(F) \subset \Delta_t$.*

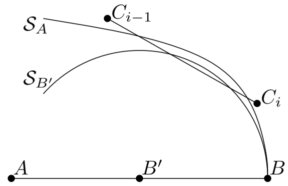

Figure 2: Lemma 3.1.

Proof. For every small angle $Q \in \partial K$ the set of pairs $(Q, l) \in \Gamma$ is a closed arc, let (Q, l_+) and (Q, l_-) be its boundary points. Define the set $L = \bigcup(l_+ \cup l_-)$, where the union is taken over all small angles of K.

Since K is not a polygon with at most 6 sides, then $\partial K \setminus L$ is open relative to ∂K and non-empty. Therefore, there is a closed set $F \subset \partial K \setminus L$ with non-empty interior relative to ∂K (and hence, with positive perimeter).

Suppose that there is a sequence of pairs $\{(Q_i, l_i)\}_{i=1}^{\infty}$ with $f(Q_i, l_i) \in F$ and satisfying $\lim_{i\to\infty} \alpha(Q_i, l_i) = 0$. Let l'_i be a supporting line of K at $f(Q_i, l_i)$ that forms an angle of $\alpha(Q_i, l_i)$ with l_i. By taking a subsequence if necessary, we may assume that (Q_i, l_i) converges to a pair (Q, l) and that l'_i converges to a line l'. Then l' must support K at (Q, l), thus $Q \in L$. This contradicts the definition of F, therefore there exists $t > 0$ such that $\alpha(Q_i, l_i) > t$ for all $(Q, l) \in F$ and hence $f^{-1}(F) \subset \Delta_t$. $\square$

Proof of Theorem 2.1. If K is a polygon with at most 6 sides, then for any $P \in \partial K$ the set $f^{-1}(P)$ contains at most 12 points, so $M \le 12$ there.

Assume now that K is not a polygon with at most 6 sides and take F as in Lemma 2.3. By Lemma 2.2, f is Lipschitz on $f^{-1}(F)$ and by the coarea formula (see [2]) we obtain

$$\int_F \# f^{-1}(P)\, dP = \int_{f^{-1}(F)} |\nabla f(Q, l)|\, d(Q, l) \le \left|f^{-1}(F)\right| \mathrm{lip}(f).$$

Therefore, there is a point $P \in F$ which is taken only finitely many times by $f_{|f^{-1}(F)}$. Since no other pair $(Q, l) \in \Gamma$ with $Q \ne P$ can have $P \in l$, we are done. $\square$

3. EXAMPLES

In this section we give examples for Theorems 1.1 and 1.3. First we need a couple of lemmas.

Lemma 3.1. *Fix $N \in \mathbb{N} \cup \{\infty\}$. Let A, B, C, D be points in convex position ordered counter-clockwise such that the angle $\angle ABC \in (0, \frac{\pi}{2})$. For any neighborhood V of B there is a sequence of points $\{C_i\}_{i=1}^{N}$ such that:*

i) The points $D, C, C_1, C_2, \dots B, A$ are all extreme points of their convex hull and are ordered clockwise.

ii) For every $P \in [A, B]$ outside of V, there is a circle centered at P that intersects the broken line $CC_1C_2 \dots BA$ in at least $2N + 2$ points.

Proof. Given a point P on the line AB, let $\mathcal{S}_P$ be the circle centered at P that passes through B. Let $B' \in [A, B] \cap V$ so that C is outside of $\mathcal{S}_{B'}$.

We construct the points C_i inductively starting with $C_0 = C$. Once C_{i-1} is constructed, let C_i be a point such that:

- The points $D, C_0, \dots, C_i, B, A$ are all extreme points of their convex hull and are ordered clockwise,

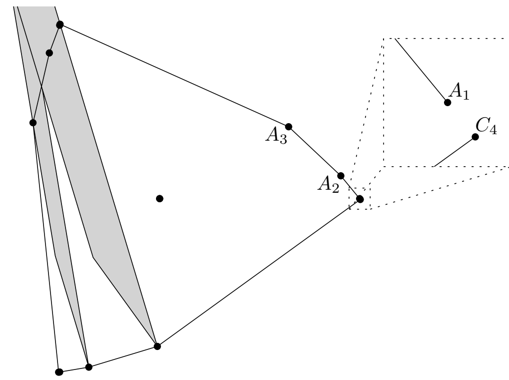

Figure 3: Construction for Theorem 1.1 with the regions from Lemma 3.2.

- C_i is outside of the circle $\mathcal{S}_A$,
- $\angle ABC_i < \frac{\pi}{2}$,
- the segment (C_{i-1}, C_i) intersects $\mathcal{S}_{B'}$ twice.

See Figure 2 for a non-realistic example of this construction. Clearly condition (i) holds.

For a given $P \in [A, B']$ the circle $\mathcal{S}_P$ is between the circles $\mathcal{S}_{B'}$ and $\mathcal{S}_A$, therefore $\mathcal{S}_P$ intersects each of the segments (C, C_1) and $(A, B]$ at least once and each of the segments (C_i, C_{i+1}) twice, giving an infinite number of intersections when $N = \infty$. If $N < \infty$, then a circle slightly smaller that $\mathcal{S}_P$ will, in addition, intersect (C_N, B) twice giving a total of $2N + 2$ intersections. $\square$

Lemma 3.2. *Let A_1, B, A_2 be points in the plane. For $i = 1, 2$ let C_i be the midpoint of A_iB and S_i be the set of points P such that the orthogonal projection of P on A_iB is contained in the segment $(B, C_i]$. Then for any point $P \in S_1 \cap S_2$ there is a circle centered at P that intersects each of the segments (A_i, B) twice.*

Proof. Let $P \in S_1 \cap S_2$ and assume that $\mathrm{dist}(P, A_1B) \le \mathrm{dist}(P, A_2B)$. It is easy to see that there is a real number r larger than $\mathrm{dist}(P, A_2B)$ and smaller than $\mathrm{dist}(P, B)$, $\mathrm{dist}(P, A_1)$ and $\mathrm{dist}(P, A_2)$. Therefore, the circle centered at P with radius r intersects each of the segments (A_i, B) twice. $\square$

We note that the set of points P that satisfy the above lemma is actually larger. The regions we use are simple and enough for our purposes.

Now we are ready to construct the examples which prove Theorems 1.1 and 1.3.

Proof of Theorem 1.1. Consider the points with coordinates $A_1 = (1000, 0)$, $A_2 = (906, 114)$, $A_3 = (645, 359)$ and $A_4 = (-498, 871)$. For $i = 1, \dots 4$, let B_i and C_i be the rotation around the origin of A_i by an angle of $2\pi/3$ and $4\pi/3$, respectively. The 12 points A_i, B_i, C_i are in convex position (see Figure 3).

Using Lemma 3.2 on the triples C_1, C_2, C_3 and C_2, C_3, C_4, it can be shown by direct computation that for any point P in some neighborhood V of the broken line $A_4B_1B_2B_3$ there is a circle centered at P that intersects the broken line $C_1C_2C_3C_4$ in at least 6 points. This direct computation amounts to checking that the two shaded strips in Figure 3 together contain the broken line $B_3B_2B_1A_4$ in their interior.

The angle $\angle A_3A_4B_1$ is acute. This is again a simple computation. Lemma 3.1 implies the existence of a point $A_5 \in V$ such that for any point P on $[A_3, A_4]\backslash V$ there is a circle centered at P that intersects $A_3A_4A_5B_1$ in at least 4 points. Define B_5 and C_5 as above to obtain a 15-gon K having A_i, B_iC_i as its vertices. The radius of the circle $\mathcal{S}_P$ is close to $|PA_4|$ and therefore intersects ∂K an additional 2 times, once between C_3 and P and once again between A_5 and B_3.

By the rotational symmetry of the figure, K has the desired property.

It can also be verified that there are points $P \in \partial K$ that are not in $J(K, 7)$, for example the midpoint of $[A_3A_4]$. □

An interactive version of Figure 3 made with GeoGebra[1] can be found on-line at:
http://www.geogebratube.org/student/m33469

Proof of Theorem 1.3. As mentioned before, the convex body K_ϵ can be constructed so that it is close to any triangle or a straight line segment.

Fix a triangle $A_1A_2A_3$ and let $\epsilon > 0$. Choose points B_1, B_2 and B_3 so that $A_1B_1A_2B_2A_3B_3$ is a convex 6-gon, each B_i is ϵ-close to A_i and the angles $\angle A_iB_iA_{i+1}$ are acute. Using Lemma 3.1 with $N = \infty$ on the points $A_iB_iA_{i+1}B_{i+1}$, we obtain three families of points that together with the points A_i and B_i determine the required convex body.

For a straight segment $[A, B]$ a similar thing is done. Choose points C close to A and D close B such that $ACBD$ is a convex 4-gon and the angles $\angle ACB$ and $\angle ADB$ are acute, then Lemma 3.1 on $BCAD$ and $ADBC$ gives the required convex body. □

4. GENERIC BEHAVIOR

The set of planar convex bodies $\mathcal{K}$ with the Hausdorff metric is a complete metric space, thus, it is a Baire space.

The defining property of Baire spaces is that the intersection of countably many dense open sets is also dense. The intersection of countably many open sets is called a G_δ set. Such sets are considered large. It is said that *most points in a Baire space* satisfy a property if the set of points satisfying this property contains a dense G_δ set. These notions can be found in Chapter 20 of [3] and similar techniques are applied in [4].

We prove Theorem 1.4 here, but we need some definitions and lemmas first. Let K be a convex body and assume the circle $\mathcal{S}$ intersects ∂K at Q. If for every $\epsilon > 0$ there are points $Q_1, Q_2 \in \mathcal{B}(Q, \epsilon) \cap \mathcal{S}$ such that Q_1 is in the interior of K and $Q_2 \notin K$, then we say that $\mathcal{S}$ *intersects* ∂K *transversally at* Q.

To make things simpler, we work with the set $J_0(K, n) \subset J(K, n)$ of points $P \in \partial K$ such that there is a circle centered at P that intersects ∂K transversally in at least n points. If $n < \infty$ then the sets $J_0(K, n)$ are clearly open relative to ∂K.

Remark. It can be shown that if $n < \infty$ and ∂K contains no circle-arcs (which is true for most convex bodies) then $J_0(K, n) = J(K, n)$, but we do not need this.

Instead of proving Theorem 1.4 we prove the following stronger statement.

[1]http://www.geogebra.org

Theorem 4.1. *For most convex bodies $K \in \mathcal{K}$, the set $\bigcap_{n=1}^{\infty} J_0(K, n)$ contains most points of ∂K.*

Let $\mathcal{K}_{n,m}$ be the set of convex bodies $K \in \mathcal{K}$ such that for every point $P \in \partial K$, the set $J_0(K, n) \cap \mathcal{B}(P, \frac{1}{m})$ is nonempty.

Lemma 4.2. *The set $\mathcal{K}_{n,m}$ is open and dense in $\mathcal{K}$.*

Proof. First we prove that $\mathcal{K}_{n,m}$ is open. Let $K \in \mathcal{K}_{n,m}$ and choose a finite family $\{P_i\}$ such that $\{\mathcal{B}(P_i, \frac{1}{2m})\}$ covers ∂K. From the definition of $J_0(K, n)$ and the finiteness of $\{P_i\}$, it follows that there exists $\epsilon > 0$ such that whenever $\text{dist}(K, K') < \epsilon$ the following hold:

- $\{\mathcal{B}(P_i, \frac{1}{2m})\}$ covers $\partial K'$,
- if $Q \in J_0(K, n)$ and $Q' \in \partial K' \cap \mathcal{B}(Q, \epsilon)$ then $Q' \in J_0(K', n)$.

This implies that $\mathcal{K}_{n,m}$ is open.

To show that it is dense, let $K \in \mathcal{K}$ and $\epsilon > 0$. We construct a convex body $K' \in \mathcal{K}_{n,m}$ such that $\text{dist}(K, K') < \epsilon$.

Let K_0 be a polygon such that $\text{dist}(K, K_0) < \epsilon$ and the distance between any two consecutive vertices of K_0 is less than $\frac{1}{4m}$. Let $\{P_1, \ldots, P_M\}$ be the set of midpoints of the sides of K_0. Given these points we construct new polygons $K_1, \ldots, K_M$ recursively, the following way.

Once K_{i-1} has been constructed, let Q, R, S be consecutive vertices of K_{i-1} such that R is a vertex of K_{i-1} farthest away from P_i. Now we remove the vertex R from K_{i-1} and add vertices $R_1, \ldots, R_n$ to form a new polygon K_i with the following properties:

- The points $R_1, \ldots, R_n$ are between Q and S,
- the distance between P_i and any R_j is some $r > 0$,
- the points $P_1, \ldots, P_M$ belong to ∂K_i and are not vertices of K_i,
- $\text{dist}(K, K_i) < \epsilon$.

Note that any circle centered at P_i with a radius slightly smaller than r will intersect ∂K_{i-1} transversally in at least n points.

It clear that the polygon obtained at the end of this process belongs to $\mathcal{K}_{n,m}$. □

Proof of Theorem 4.1. By Lemma 4.2 and since $\mathcal{K}$ is a Baire space, $\bigcap_{n,m=1}^{\infty} \mathcal{K}_{n,m}$ is a dense G_δ subset of $\mathcal{K}$. Let $K \in \bigcap_{n,m=1}^{\infty} \mathcal{K}_{n,m}$, then each $J_0(K, n)$ is open and dense relative to ∂K. Therefore $\bigcap_{n=1}^{\infty} J_0(K, n)$ is a dense G_δ subset of ∂K. □

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