Student job report: Bootstrap high quantiles estimation.

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1 Introduction

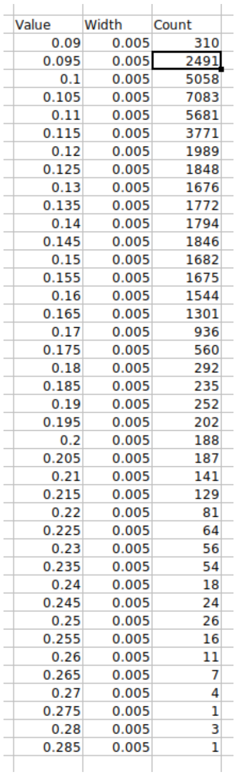
In this project we want to estimate the confidence we have when computing various quantiles. This task may be fairly straightforward when working with known distributions or arbitrary amounts of data but this seldom occurs in engineering or other real world applications. For this reason, here we try to accomplish that task on an unknown distribution and given limited amounts of data.

We define the n-th quantile as follows:

$$q_n := 1 - 10^{-n} \tag{1}$$

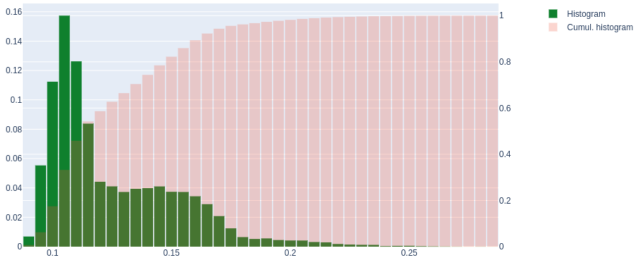
which gives $q_1 = 0.9$, $q_2 = 0.99$ ...etc. Where, simply speaking, q_n represents "0" followed by n nines. For our applications, we are mostly interested in q_3, q_4 and q_5.

The data we work with is presented in figure 1a and plotted in the histogram in figure 1b.



Value	Width	Count
0.09	0.005	310
0.095	0.005	2491
0.1	0.005	5058
0.105	0.005	7083
0.11	0.005	5681
0.115	0.005	3771
0.12	0.005	1989
0.125	0.005	1848
0.13	0.005	1676
0.135	0.005	1772
0.14	0.005	1794
0.145	0.005	1846
0.15	0.005	1682
0.155	0.005	1675
0.16	0.005	1544
0.165	0.005	1301
0.17	0.005	936
0.175	0.005	560
0.18	0.005	292
0.185	0.005	235
0.19	0.005	252
0.195	0.005	202
0.2	0.005	188
0.205	0.005	187
0.21	0.005	141
0.215	0.005	129
0.22	0.005	81
0.225	0.005	64
0.23	0.005	56
0.235	0.005	54
0.24	0.005	18
0.245	0.005	24
0.25	0.005	26
0.255	0.005	16
0.26	0.005	11
0.265	0.005	7
0.27	0.005	4
0.275	0.005	1
0.28	0.005	3
0.285	0.005	1

(a) Initial data



(b) Visual representation of the initial data

Figure 1: A look at the initial data

2 Bootstrap

We use the bootstrap to estimate the confidence in the computed quantiles, even with small to moderate sample size.

Let n be the size of the data at our disposal and let q_k be the quantile we want to compute. Here is how we proceed:

1. Do the following n times.
   (a) Draw a random sample of size n with replacement from the initial data.
   (b) Compute q_k on the sample which has just been drawn.
2. Compute the mean over the collection of values resulting from step 1b above.
3. Compute the confidence intervals (details in section 3)

The plots in figure 2 show the evolution of our estimate for the value of the quantiles as we go through runs of the bootstraps. The grey areas represent the 95% confidence intervals during that evolution. We will see how to get the confidence intervals from our histogram in section 3.

3 Confidence intervals on the bootstrap

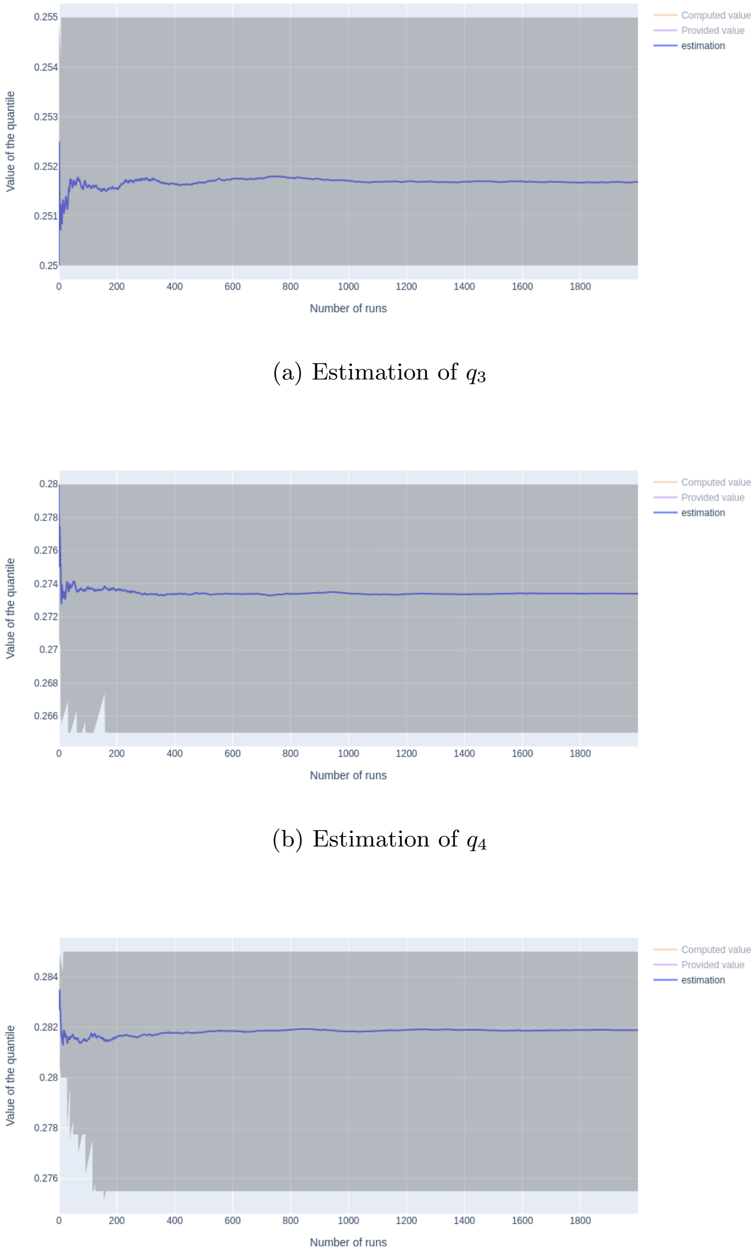
Estimating the quantile Our estimation of the quantile q for the underlying distribution is simply computed by taking the average of all the q's of each individual sample.

Computing the confidence intervals Let's say we want to compute the 95% confidence interval. To do so, we take all the q's that have been computed for each individual sample and sort them. Then we take the 0.025-th quantile as our lower bound for the confidence intervals and the 0.975-th quantile as our upper bound for the confidence intervals.
As one may expect, the values 0.025 and 0.975 are found as follows:

$$\frac{1-0.95}{2} = 0.025 \qquad \text{and} \qquad 1 - \frac{(1-0.95)}{2} = 0.975$$

where the 0.95 comes from the 95% confidence interval.
More generally, for a confidence level of γ (instead of 95%), one has that the lower and upper bounds of the confidence interval are respectively

$$\gamma_{lo} := \frac{1-\gamma}{2} \qquad \text{and} \qquad \gamma_{up} := 1 - \frac{(1-\gamma)}{2}$$



(a) Estimation of q_3

(b) Estimation of q_4

(c) Estimation of q_5

Figure 2: Estimation of the quantiles over bootstrap runs

4 Fine tuning

Let us assume that we are working with a given data set and we have no way of getting new or more reliable data. We are trying to get the quantiles as precisely as possible and with confidence intervals as small as possible.

There are two parameters of the bootstrap that we can adjust to meet our goals

1. The size of the resamples.
2. The number of resamples.

We touched on the number of resamples already in section 2, however we haven't dicussed the size of the resamples.

What impact would increasing the size of the resamples have ? The only time it comes into play is when we take the quantiles of the resamples. Then, what would happen if we took the quantiles of a resample of different size ? Taking the quantile of a smaller resample likely means that we will have less *refinement*. That is, our quantiles at each resample would likely take a smaller set of values. This is further explained by the way we proceed if we have sparse values, which is our case since we work with "high" quantiles (where we have few values available).

According to our data, it seems that there is close to no change in the estimation of the quantiles after 1000 repetitions of the bootstrap. The variations are small after 500 runs already but for safety purposes we consider that we have our final guess after 1000 runs. As for the confidence intervals, only in some edge cases do we have changes past the 1000 mark.
NB: Our estimate of 1000 runs is based on empirical evidence. It is not a theoretical result, however, we believe that it is suitable for engineering purposes.