This is the proof that we can write calculate the Cotesian numbers as such:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n^2 & \cdots & n^n \end{bmatrix}^\top \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} n/1 \\ n^2/2 \\ n^3/3 \\ \vdots \\ n^{n+1}/(n+1) \end{bmatrix}.$$

Where the integral can by approximated by $h(c_0 f_0 + c_1 f_2 + \cdots + c_n f_n)$.

Proof. First, we interpolate the points $x_0, x_1, \ldots, x_n$ with function values $f_0, f_1, \ldots, f_n$ and let $h = \frac{x_1 - x_0}{n}$. An interpolating polynomial can be written as $p(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n$, where the coefficients p_i can be computed by solving:

$$\underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}}_{\mathbf{p}} = \underbrace{\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}}_{\mathbf{f}}.$$

Furthermore, the points $x_0, x_1, \ldots, x_n$ are equidistant with spacing h, we can thus write $x_i = ih + x_0$. The integral over $p(x)$ can be written as:

$$\int_{x_0}^{x_n} p(x)\mathrm{d}x = \int_{x_0}^{x_0+nh} p(x)\mathrm{d}x = \left[p_0 x + p_1 \frac{x^2}{2} + p_2 \frac{x^3}{3} + \cdots + p_n \frac{x^{n+1}}{n+1} \right]_{x_0}^{x_0+nh}$$

However, the interpolation can also be shifted such that $x_0 = 0$, the integral would not change, thus this is the same as:

WE THUS APPROXIMATE THE INTEGRAL BY $h(c_0 f_0 + c_1 f_2 + \cdots + c_n f_n)$.

$$p_0 \frac{nh}{1} + p_1 \frac{(nh)^2}{2} + p_2 \frac{(nh)^3}{3} + \cdots + p_n \frac{(nh)^{n+1}}{n+1} = h(c_0 f_0 + c_1 f_2 + \cdots + c_n f_n)$$

Thus, in vector notation:

$$\begin{bmatrix} nh/1 \\ (nh)^2/2 \\ (nh)^3/3 \\ \vdots \\ (nh)^{n+1}/(n+1) \end{bmatrix}^\top \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} hc_0 \\ hc_1 \\ hc_2 \\ \vdots \\ hc_n \end{bmatrix}^\top \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}.$$

Removing h:

$$\underbrace{\begin{bmatrix} n/1 \\ n^2h/2 \\ n^3h^2/3 \\ \vdots \\ n^{n+1}h^n/(n+1) \end{bmatrix}^\top}_{\mathbf{n}^\top} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \underbrace{\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}^\top}_{\mathbf{c}^\top} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}.$$

Thus, in vector notation:

$$A\mathbf{p} = \mathbf{f}$$

and

$$\mathbf{n}^\top \mathbf{p} = \mathbf{c}^\top \mathbf{f}$$

We can deduce that

$$\mathbf{p} = A^{-1}\mathbf{f}$$

$$\mathbf{n}^\top \mathbf{p} = \mathbf{n}^\top A^{-1}\mathbf{f} = \mathbf{c}^\top \mathbf{f}$$

$$\mathbf{f}^\top (A^{-1})^\top \mathbf{n} = \mathbf{f}^\top \mathbf{c}$$

$$(A^{-1})^\top \mathbf{n} = \mathbf{c}$$

$$\mathbf{n} = A^\top \mathbf{c}$$

Exactly what we wanted to prove considering that $x_0 = 0$ and $x_i = ih$ (writing out gives):

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}^\top \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} n/1 \\ n^2h/2 \\ n^3h^2/3 \\ \vdots \\ n^{n+1}h^n/(n+1) \end{bmatrix}$$

Thus, since $x_0 = 0$ and $x_i = ih$ we get

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & h & h^2 & \cdots & h^n \\ 1 & 2h & (2h)^2 & \cdots & (2h)^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & nh & (nh)^2 & \cdots & (nh)^n \end{bmatrix}^\top \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} n/1 \\ n^2h/2 \\ n^3h^2/3 \\ \vdots \\ n^{n+1}h^n/(n+1) \end{bmatrix}$$

Removing the factor of h (we can do this easily since the matrix is transposed):

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1^2 & \cdots & 1^n \\ 1 & 2 & 2^2 & \cdots & 2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n^2 & \cdots & n^n \end{bmatrix}^\top \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} n/1 \\ n^2/2 \\ n^3/3 \\ \vdots \\ n^{n+1}/(n+1) \end{bmatrix}$$

Concluding our proof.

In the following equation:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1^2 & \cdots & 1^n \\ 1 & 2 & 2^2 & \cdots & 2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n^2 & \cdots & n^n \end{bmatrix}^\top \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} n/1 \\ n^2/2 \\ n^3/3 \\ \vdots \\ n^{n+1}/(n+1) \end{bmatrix}$$

We can calculate c_i by:

$$c_i = \begin{cases} \dfrac{1}{(n-1)!} \displaystyle\sum_{k=1}^{n+1} \frac{n^k s(n,k)}{k+1} & i = 0 \text{ or } i = n \\ \dfrac{1}{(n-1)!} \dbinom{n}{i} \displaystyle\sum_{j=1}^{i+1} \sum_{k=1}^{n-i} n^{j+k} \frac{s(i,j)s(n-i,k)}{(k+1)\binom{j+k+1}{k+1}} & \text{otherwise} \end{cases}$$

Proof. We proof this now, rewriting the equation gives

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & \cdots & n \\ 0 & 1^2 & 2^2 & \cdots & n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1^n & 2^n & \cdots & n^n \end{bmatrix}^{-1} \begin{bmatrix} n/1 \\ n^2/2 \\ n^3/3 \\ \vdots \\ n^{n+1}/(n+1) \end{bmatrix}$$

To go one step lower (up to $n-1$, for other reasons, the subscripts would get tedious) we define

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & \cdots & n \\ 0 & 1^2 & 2^2 & \cdots & n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1^n & 2^n & \cdots & n^n \end{bmatrix}^{-1}}_{(W_n^{-1})^\top}$$

By `https://proofwiki.org/wiki/Inverse_of_Vandermonde_Matrix` (which is 1-indexed, we are not, thus we do some shit ($n-1 \to n$ and $j-1 \to j$ and $x_i \to x_{i-1}$ and $i-1 \to i$)) we have:

$$\left[(W_n^{-1})^\top\right]_{ij} \text{(1-indexed)} = \begin{cases} (-1)^{j-1} \left[\dfrac{\displaystyle\sum_{\substack{1 \le m_1 < \cdots < m_{n-j} \le n \\ m_1, \ldots, m_{n-j} \ne i}} x_{m_1} \cdot x_{m_2} \cdots x_{m_{n-j-1}} \cdot x_{m_{n-j}}}{\displaystyle\prod_{\substack{1 \le m \le n \\ m \ne i}} (x_m - x_i)} \right] & 1 \le j < n \\ \dfrac{1}{\displaystyle\prod_{\substack{1 \le m \le n \\ m \ne i}} (x_i - x_m)} & j = n \end{cases}$$

We also have that the final answer can be written as:

$$c_i = \sum_{j=1}^{n} \left[(W_n^{-1})^\top \right]_{ij} n^j/j$$

Equals (still 1-indexed), also $x_i = i - 1$

$$c_i = \sum_{j=1}^{n} n^j/j \begin{cases} (-1)^{j-1} \left[\dfrac{\sum\limits_{\substack{1\le m_1<\cdots<m_{n-j}\le n \\ m_1,\ldots,m_{n-j}\neq i}} (m_1-1)\cdots(m_{n-j}-1)}{\prod\limits_{\substack{1\le m\le n \\ m\neq i}} (m-i)} \right] & 1 \le j < n \\ \dfrac{1}{\prod\limits_{\substack{1\le m\le n \\ m\neq i}} (i-m)} & j = n \end{cases}$$

For $i = 1$ (remember, 1-indexed):

$$\begin{aligned} c_0 &= \sum_{j=1}^{n} n^j/j \begin{cases} (-1)^{j-1} \left[\dfrac{\sum\limits_{\substack{1\le m_1<\cdots<m_{n-j}\le n \\ m_1,\ldots,m_{n-j}\neq 1}} (m_1-1)\cdots(m_{n-j}-1)}{\prod\limits_{\substack{1\le m\le n \\ m\neq 1}} (m-1)} \right] & 1 \le j < n \\ \dfrac{1}{\prod\limits_{\substack{1\le m\le n \\ m\neq 1}} (1-m)} & j = n \end{cases} \\ &= \sum_{j=1}^{n} n^j/j \begin{cases} (-1)^{j-1} \left[\dfrac{\sum\limits_{\substack{1\le m_1<\cdots<m_{n-j}\le n \\ m_1,\ldots,m_{n-j}\neq 1}} (m_1-1)\cdots(m_{n-j}-1)}{\prod\limits_{1\le m'\le n-1} (m')} \right] & 1 \le j < n \\ \dfrac{1}{\prod\limits_{1\le m'\le n-1} (-m')} & j = n \end{cases} \\ &= \frac{1}{(n-1)!} \sum_{j=1}^{n} n^j/j \begin{cases} (-1)^{j-1} \left[\sum\limits_{1\le m'_1<\cdots<m'_{n-j}\le n-1} m'_1 \cdots m'_{n-j} \right] & 1 \le j < n \\ (-1)^{n-1} & j = n \end{cases} \\ &= \frac{1}{(n-1)!} \sum_{j=1}^{n} n^j/j \begin{cases} (-1)^{j-1} \left[(-1)^{j-1} s(n-1, j-1) \right] & 1 \le j < n \\ (-1)^{n-1} & j = n \end{cases} \\ &= \frac{1}{(n-1)!} \sum_{j=1}^{n} n^j/j \begin{cases} s(n-1, j-1) & 1 \le j < n \\ (-1)^{n-1} & j = n \end{cases} \end{aligned}$$

Equalling:

$$\left[(W_n^{-1})^\top\right]_{ij} = \begin{cases} (-1)^j \left[\dfrac{\sum\limits_{\substack{1\le m_1<\cdots<m_{n-j}\le n+1\\ m_1,\ldots,m_{n-j}\neq i+1}} x_{m_1-1}\cdot x_{m_2-1}\cdots x_{m_{n-j-1}-1}\cdot x_{m_{n-j}-1}}{\prod\limits_{\substack{1\le m\le n+1\\ m\neq i+1}}(x_{m-1}-x_i)}\right] & 0\le j<n \\ \dfrac{1}{\prod\limits_{\substack{1\le m\le n+1\\ m\neq i+1}}(x_i-x_{m-1})} & j=n \end{cases}$$

With $x_0 = 0, x_1 = 1, \ldots, x_n = n$, note that also $x_i = i$. Filling in gives (and $m_i - 1 \to m_i$):

$$\left[(W_n^{-1})^\top\right]_{ij} = \begin{cases} (-1)^j \left[\dfrac{\sum\limits_{\substack{0\le m_1<\cdots<m_{n-j}\le n\\ m_1,\ldots,m_{n-j}\neq i}} m_1\cdot m_2\cdots m_{n-j-1}\cdot m_{n-j}}{\prod\limits_{\substack{0\le m\le n\\ m\neq i}}(m-i)}\right] & 0\le j<n \\ \dfrac{1}{\prod\limits_{\substack{0\le m\le n\\ m\neq i}}(i-m)} & j=n \end{cases}$$

We also have that the final answer can be written as:

$$c_i = \sum_{j=0}^{n} \left[(W_n^{-1})^\top\right]_{ij} n^{j+1}/(j+1)$$

Thus:

$$c_i = \sum_{j=0}^{n} n^{j+1}/(j+1) \begin{cases} (-1)^j \left[\dfrac{\sum\limits_{\substack{0\le m_1<\cdots<m_{n-j}\le n\\ m_1,\ldots,m_{n-j}\neq i}} m_1\cdot m_2\cdots m_{n-j-1}\cdot m_{n-j}}{\prod\limits_{\substack{0\le m\le n\\ m\neq i}}(m-i)}\right] & 0\le j<n \\ \dfrac{1}{\prod\limits_{\substack{0\le m\le n\\ m\neq i}}(i-m)} & j=n \end{cases}$$

For the case $i = 0$, we get:

$$c_i = \sum_{j=0}^{n} n^{j+1}/(j+1) \begin{cases} (-1)^j \left[\dfrac{\sum\limits_{\substack{0 \le m_1 < \cdots < m_{n-j} \le n \\ m_1,\ldots,m_{n-j} \neq i}} m_1 \cdot m_2 \cdots m_{n-j-1} \cdot m_{n-j}}{\prod\limits_{\substack{0 \le m \le n \\ m \neq i}} (m-i)} \right] & 0 \le j < n \\ \dfrac{1}{\prod\limits_{\substack{0 \le m \le n \\ m \neq i}} (i-m)} & j = n \end{cases}$$

$$= \sum_{j=0}^{n} n^{j+1}/(j+1) \begin{cases} (-1)^j \left[\dfrac{\sum\limits_{1 \le m_1 < \cdots < m_{n-j} \le n} m_1 \cdot m_2 \cdots m_{n-j-1} \cdot m_{n-j}}{\prod\limits_{1 \le m \le n} m} \right] & 0 \le j < n \\ \dfrac{1}{\prod\limits_{1 \le m \le n} (-m)} & j = n \end{cases}$$

$$= \frac{1}{n!} \sum_{j=0}^{n} n^{j+1}/(j+1) \begin{cases} (-1)^j \left[\sum\limits_{1 \le m_1 < \cdots < m_{n-j} \le n} m_1 \cdot m_2 \cdots m_{n-j-1} \cdot m_{n-j} \right] & 0 \le j < n \\ (-1)^n & j = n \end{cases}$$

$$= \frac{1}{n!} \sum_{j=0}^{n} n^{j+1}/(j+1) \begin{cases} (-1)^j \left[(-1)^{j+1} s(n,j) \right] & 0 \le j < n \\ (-1)^n & j = n \end{cases}$$

$$= \frac{1}{(n-1)!} \sum_{j=0}^{n} n^{j}/(j+1) \begin{cases} (-1)^j \left[(-1)^{j+1} s(n,j) \right] & 0 \le j < n \\ (-1)^n & j = n \end{cases}$$

$$= \frac{1}{(n-1)!} \sum_{j=0}^{n} n^{j}/(j+1) \begin{cases} s(n,j) & 0 \le j < n \\ (-1)^n & j = n \end{cases}$$