

# Proofs regarding Cotesian numbers.

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In the following document are proofs regarding Cotesian numbers. The final result is a direct formula to calculate them. These proofs constitute appendix B of my bachelor thesis “Methods for reducing error in approximations of the Rayleigh integral”.

**Lemma 1.** *The Vandermonde matrix*

$$V = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n^2 & \cdots & n^n \end{bmatrix}$$

*is invertible.*

*Proof.* We show by contradiction that the columns of  $V$  are linearly independent.

Denote the  $j$ -column of matrix  $V$  as  $\mathbf{v}_j$ , we thus have  $\mathbf{v}_j = [0^j \ 1^j \ \cdots \ n^j]^\top \in \mathbb{R}^{n+1}$ . Assume that there exist coefficients  $a_0, a_1, \dots, a_n \in \mathbb{R}$  such that  $a_0\mathbf{v}_0 + a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$ , where  $\mathbf{0} = [0 \ 0 \ \cdots \ 0]^\top \in \mathbb{R}^{n+1}$ . Then, for each  $k \in \mathbb{N}$  with  $0 \leq k \leq n$  we get

$$a_0 + a_1k + a_2k^2 + \cdots + a_nk^n = 0.$$

Hence,  $k$  is a root of the polynomial  $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ . This means that the polynomial  $f(x)$  has  $n+1$  different roots. Since  $f(x)$  is at most an  $n$ th-order polynomial we must have  $a_0 = a_1 = \cdots = a_n = 0$ . Proving that the columns of matrix  $V$  are indeed linearly independent, therefore, the matrix is invertible.  $\square$

**Theorem 2.** *Define  $f : \mathbb{R} \rightarrow \mathbb{R}$ , and let  $x_0, x_1, \dots, x_n \in \mathbb{R}$  denote equidistant real numbers. Denote their function values as  $f_0, f_1, \dots, f_n \in \mathbb{R}$ , respectively (thus  $f_0 = f(x_0)$ ,  $f_1 = f(x_1)$ ,  $\dots$ ,  $f_n = f(x_n)$ ). Furthermore, let  $h$  denote the distance between the equidistant numbers, that is,  $h = \frac{x_n - x_0}{n}$ . Also, let  $c_0, c_1, \dots, c_n$  denote the Cotesian numbers, i.e. the numbers such that after interpolating the function values by a polynomial of degree at most  $n$  the integral over the polynomial can be approximated by*

$$\int_{x_0}^{x_n} f(x) dx \approx h(c_0f_0 + c_1f_1 + \cdots + c_nf_n).$$

*Then,  $c_0, c_1, \dots, c_n$  satisfy*

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n^2 & \cdots & n^n \end{bmatrix}^\top \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} n/1 \\ n^2/2 \\ n^3/3 \\ \vdots \\ n^{n+1}/(n+1) \end{bmatrix}.$$

*Proof.* Without loss of generality we let  $x_0 = 0$  (this can be verified by defining  $x'_0 = x_0 - x_0$ ,  $x'_1 = x_1 - x_0$ ,  $\dots$ ,  $x'_n = x_n - x_0$  and noting that  $\int_{x_0}^{x_n} f(x)dx = \int_0^{x'_n} f(x')dx'$ ). Employing this new definition allows us to write  $x_i = ih$  for  $0 \leq i \leq n$ .

All interpolating polynomials  $p(x) = p_0 + p_1x + p_2x^2 + \dots + p_nx^n$  must satisfy  $p(x_0) = f_0$ ,  $p(x_1) = f_1$ ,  $\dots$ ,  $p(x_n) = f_n$ , hence, the coefficients  $p_0, p_1, \dots, p_n$  satisfy

$$\underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}}_A \underbrace{\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}}_{\mathbf{p}} = \underbrace{\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}}_{\mathbf{f}}. \quad (1)$$

Also, we know that integrating  $p(x)$  yields the approximation of the integral, equalling  $h(c_0f_0 + c_1f_1 + \dots + c_nf_n)$ , giving the relation

$$\begin{aligned} \int_{x_0}^{x_n} p(x)dx &= \int_0^{nh} p(x)dx = \left[ p_0x + p_1\frac{x^2}{2} + p_2\frac{x^3}{3} + \dots + p_n\frac{x^{n+1}}{n+1} \right]_0^{nh} \\ &= p_0\frac{nh}{1} + p_1\frac{(nh)^2}{2} + p_2\frac{(nh)^3}{3} + \dots + p_n\frac{(nh)^{n+1}}{n+1} = h(c_0f_0 + c_1f_2 + \dots + c_nf_n). \end{aligned}$$

After removing a factor  $h$  from both sides of the equal sign we use vector notation to rewrite the equation into

$$\underbrace{\begin{bmatrix} n/1 \\ n^2h/2 \\ n^3h^2/3 \\ \vdots \\ n^{n+1}h^n/(n+1) \end{bmatrix}}_{\mathbf{n}^\top} \underbrace{\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}}_{\mathbf{p}} = \underbrace{\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}_{\mathbf{c}^\top} \underbrace{\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}}_{\mathbf{f}}. \quad (2)$$

This allows for shorter notation; equation 1 can be written as

$$A\mathbf{p} = \mathbf{f},$$

and equation 2 can be written as

$$\mathbf{n}^\top \mathbf{p} = \mathbf{c}^\top \mathbf{f}.$$

Using these equalities we can deduce that (note that  $A$  is invertible due to Lemma 1):

$$\begin{aligned} \mathbf{p} &= A^{-1}\mathbf{f} \\ \mathbf{n}^\top \mathbf{p} &= \mathbf{n}^\top A^{-1}\mathbf{f} = \mathbf{c}^\top \mathbf{f} \\ \mathbf{f}^\top (A^{-1})^\top \mathbf{n} &= \mathbf{f}^\top \mathbf{c}, \quad \text{for arbitrary } \mathbf{f} \\ (A^{-1})^\top \mathbf{n} &= \mathbf{c} \\ \mathbf{n} &= A^\top \mathbf{c}. \end{aligned}$$

Writing this out and substituting  $x_i = ih$  gives us:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & h & h^2 & \cdots & h^n \\ 1 & 2h & (2h)^2 & \cdots & (2h)^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & nh & (nh)^2 & \cdots & (nh)^n \end{bmatrix}^\top \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} n/1 \\ n^2 h/2 \\ n^3 h^2/3 \\ \vdots \\ n^{n+1} h^n/(n+1) \end{bmatrix}.$$

Removing  $h$  (we can this do due to the transposition of the matrix) results in

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1^2 & \cdots & 1^n \\ 1 & 2 & 2^2 & \cdots & 2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n^2 & \cdots & n^n \end{bmatrix}^\top \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} n/1 \\ n^2/2 \\ n^3/3 \\ \vdots \\ n^{n+1}/(n+1) \end{bmatrix},$$

concluding our proof.  $\square$

**Definition 3.** Let  $X = \{X_1, X_2, \dots, X_n\}$  with  $X_1, X_2, \dots, X_n \in \mathbb{R}$  and let  $k \in \mathbb{N}_{\geq 0}$ . The elementary symmetric polynomial is then defined as:

$$e_k(X) = \begin{cases} 1 & \text{if } k = 0, \\ \sum_{1 \leq m_1 < \dots < m_k \leq n} X_{m_1} \cdots X_{m_k} & \text{if } 1 \leq k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$

**Lemma 4.** Let  $s(n, k)$ , with  $n, k \in \mathbb{N}_{\geq 0}$  denote the (un)signed Stirling numbers of the first kind, then, if  $0 \leq j \leq \ell$  with  $j, \ell \in \mathbb{N}$ , the following equality holds:

$$e_j(\{1, 2, \dots, \ell\}) = |s(\ell + 1, \ell + 1 - j)|. \quad (3)$$

*Proof.* We use induction twice to complete the proof.

First, we use induction on  $j$ . For the induction base with  $j = 0$  we note that by definition:  $e_j(\{1, 2, \dots, \ell\}) = 1 = |s(\ell + 1, \ell + 1)|$ . For the induction hypothesis we assume that  $e_{j-1}(\{1, 2, \dots, \ell\}) = |s(\ell + 1, \ell + 2 - j)|$ . For the induction step we then need to show that equation 3 holds.

We do this using induction on  $\ell$ . For the induction base  $\ell = 0$  and thus  $j = 0$ , we get  $e_0(\emptyset) = 1 = |s(1, 1)|$ . For the induction hypothesis we assume  $e_j(\{1, 2, \dots, \ell - 1\}) = |s(\ell, \ell - j)|$ , applying this to our previous assumption yields  $e_{j-1}(\{1, 2, \dots, \ell - 1\}) = |s(\ell, \ell + 1 - j)|$ . For the induction step we need to prove equation 3. Note that by the definition of the Stirling numbers we have [1, equation 15]:

$$|s(n, k)| = |s(n - 1, k - 1)| + (n - 1)|s(n - 1, k)|.$$

Using our induction hypotheses we now show that equation 3 holds

$$\begin{aligned}
e_j(\{1, 2, \dots, \ell\}) &= \sum_{1 \leq m_1 < \dots < m_j \leq \ell} m_1 \cdots m_j \\
&= \sum_{\substack{1 \leq m_1 < \dots < m_{j-1} \leq \ell-1 \\ m_{j-1} < m_j < \ell}} m_1 \cdots m_j + \sum_{\substack{1 \leq m_1 < \dots < m_{j-1} \leq \ell-1 \\ m_{j-1} < m_j = \ell}} m_1 \cdots m_j \\
&= \sum_{1 \leq m_1 < \dots < m_j \leq \ell-1} m_1 \cdots m_j + \ell \sum_{1 \leq m_1 < \dots < m_{j-1} \leq \ell-1} m_1 \cdots m_{j-1} \\
&= e_j(\{1, 2, \dots, \ell-1\}) + \ell e_{j-1}(\{1, 2, \dots, \ell-1\}) \\
&= |s(\ell, \ell-j)| + \ell |s(\ell, \ell+1-j)| \\
&= |s(\ell+1, \ell+1-j)|.
\end{aligned}$$

Hence, the induction step, equation 3, holds, concluding our proof.  $\square$

**Lemma 5.** Let  $0 \leq i \leq n$  with  $i \in \mathbb{N}$ , also, let  $0 \leq k \leq n$  with  $k, n \in \mathbb{N}$ , then the following relation holds:

$$e_k(\{1, \dots, n\} \setminus \{i\}) = \sum_{m=0}^k (-i)^m e_{k-m}(\{1, \dots, n\}).$$

*Proof.* This relation can easily be seen to hold for  $1 \leq i \leq n$  since

$$e_k(\{1, \dots, n\} \setminus \{i\}) = e_k(\{1, \dots, n\}) - i e_{k-1}(\{1, \dots, n\} \setminus \{i\}),$$

and  $e_0(\{1, \dots, n\} \setminus \{i\}) = 1 = e_0(\{1, \dots, n\})$ .

For  $i = 0$  the relation also holds since  $e_k(\{1, \dots, n\} \setminus \{0\}) = e_k(\{1, \dots, n\}) - 0$ .  $\square$

**Theorem 6.** We can calculate the coefficients of  $c_i$  for  $i \in \{0, \dots, n\}$  in the following equation

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1^2 & \cdots & 1^n \\ 1 & 2 & 2^2 & \cdots & 2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n^2 & \cdots & n^n \end{bmatrix}^\top \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} n/1 \\ n^2/2 \\ n^3/3 \\ \vdots \\ n^{n+1}/(n+1) \end{bmatrix} \quad (4)$$

by computing

$$c_i = \frac{1}{(n-1)!} \binom{n}{i} \sum_{j=0}^n \sum_{m=0}^{n-j} i^m n^j \frac{(-1)^{i+n} s(n+1, j+m+1)}{j+1}.$$

*Proof.* We first rewrite equation by transposing the matrix in equation 4 and bringing it to the other side, yielding

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & \cdots & n \\ 0 & 1^2 & 2^2 & \cdots & n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1^n & 2^n & \cdots & n^n \end{bmatrix}}_{(W_{n+1}^{-1})^\top}^{-1} \begin{bmatrix} n/1 \\ n^2/2 \\ n^3/3 \\ \vdots \\ n^{n+1}/(n+1) \end{bmatrix},$$

where the matrix  $W_{n+1}$  is the Vandermonde matrix defined according to ProofWiki [2]. The inverse of the matrix (without transposition) is also given by ProofWiki and turns out to be

$$[W_{n+1}^{-1}]_{ij} = \frac{(-1)^{n-i} e_{n-i}(\{0, 1, \dots, n\} \setminus \{j\})}{\prod_{m=0, m \neq j}^{n+1} (j - m)},$$

note that in the citation  $i, j \in \{1, \dots, n\}$ , whereas here we define  $i, j \in \{0, \dots, n\}$ . Transposing this matrix and noting that we can safely omit 0 in the set yields

$$[(W_{n+1}^{-1})^\top]_{ij} = \frac{(-1)^{n-j} e_{n-j}(\{1, \dots, n\} \setminus \{i\})}{\prod_{m=0, m \neq i}^n (i - m)}.$$

Using this matrix we can then compute the coefficients of  $c_i$  by using the following relation

$$\begin{aligned} c_i &= \sum_{j=0}^n [(W_{n+1}^{-1})^\top]_{ij} n^{j+1} / (j+1) \\ &= \sum_{j=0}^n n^{j+1} \frac{(-1)^{n-j} e_{n-j}(\{1, \dots, n\} \setminus \{i\})}{(j+1) \prod_{m'=0, m' \neq i}^n (i - m')}. \end{aligned} \quad (5)$$

Since  $0 \leq i \leq n$  we can simplify the product in the denominator to

$$\prod_{m=0, m \neq i}^n (i - m) = i \cdot (i-1) \cdots 2 \cdot 1 \cdot -1 \cdot -2 \cdots (i-n-1) \cdot (i-n) = i!(n-i)!(-1)^{n-i}.$$

Also, from Lemma 4 in combination with Lemma 5 we have that

$$e_{n-j}(\{1, \dots, n\} \setminus \{i\}) = \sum_{m=0}^{n-j} (-i)^m |s(n+1, n+1 - (n-j-m))|.$$

Substituting these relations into equation 5 yields

$$\begin{aligned} c_i &= \sum_{j=0}^n n^{j+1} \frac{(-1)^{n-j} e_{n-j}(\{1, \dots, n\} \setminus \{i\})}{(j+1) i! (n-i)! (-1)^{n-i}} \\ &= \sum_{j=0}^n n^{j+1} \frac{(-1)^{n-j} \sum_{m=0}^{n-j} (-i)^m |s(n+1, n+1 - (n-j-m))|}{(j+1) i! (n-i)! (-1)^{n-i}} \\ &= \frac{1}{(n-1)!} \binom{n}{i} \sum_{j=0}^n n^j \frac{(-1)^{i-j} \sum_{m=0}^{n-j} (-i)^m |s(n+1, j+m+1)|}{j+1} \\ &= \frac{1}{(n-1)!} \binom{n}{i} \sum_{j=0}^n \sum_{m=0}^{n-j} i^m n^j \frac{(-1)^{i-j+m} (-1)^{n-j-m} s(n+1, j+m+1)}{j+1} \\ &= \frac{1}{(n-1)!} \binom{n}{i} \sum_{j=0}^n \sum_{m=0}^{n-j} i^m n^j \frac{(-1)^{i+n} s(n+1, j+m+1)}{j+1}. \end{aligned}$$

Proving the theorem. □

## References

- [1] E.W. Weisstein. *Stirling Number of the First Kind*. MathWorld – A Wolfram Web Resource. URL: <https://mathworld.wolfram.com/StirlingNumberoftheFirstKind.html>.
- [2] ProofWiki. [Online; accessed 10-Jun-2022]. URL: [https://proofwiki.org/wiki/Inverse\\_of\\_Vandermonde\\_Matrix](https://proofwiki.org/wiki/Inverse_of_Vandermonde_Matrix).