# Proofs regarding Cotesian numbers.

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In the following document are proofs regarding Cotesian numbers. The final result is a direct formula to calculate them. These proofs constitute appendix B of my bachelor thesis “Methods for reducing error in approximations of the Rayleigh integral”.

**Lemma 1.** *The Vandermonde matrix*

$$V = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n^2 & \cdots & n^n \end{bmatrix}$$

*is invertible.*

*Proof.* We show by contradiction that the columns of $V$ are linearly independent.

Denote the $j$-column of matrix $V$ as $\mathbf{v}_j$, we thus have $\mathbf{v}_j = \begin{bmatrix} 0^j & 1^j & \cdots & n^j \end{bmatrix}^\top \in \mathbb{R}^{n+1}$. Assume that there exist coefficients $a_0, a_1, \ldots, a_n \in \mathbb{R}$ such that $a_0\mathbf{v}_0 + a_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$, where $\mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}^\top \in \mathbb{R}^{n+1}$. Then, for each $k \in \mathbb{N}$ with $0 \leq k \leq n$ we get

$$a_0 + a_1k + a_2k^2 + \cdots + a_nk^n = 0\,.$$

Hence, $k$ is a root of the polynomial $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. This means that the polynomial $f(x)$ has $n+1$ different roots. Since $f(x)$ is at most an $n$th-order polynomial we must have $a_0 = a_1 = \cdots = a_n = 0$. Proving that the columns of matrix $V$ are indeed linearly independent, therefore, the matrix is invertible. □

**Theorem 2.** *Define $f : \mathbb{R} \to \mathbb{R}$, and let $x_0, x_1, \ldots, x_n \in \mathbb{R}$ denote equidistant real numbers. Denote their function values as $f_0, f_1, \ldots f_n \in \mathbb{R}$, respectively (thus $f_0 = f(x_0)$, $f_1 = f(x_1)$, $\ldots$, $f_n = f(x_n)$). Furthermore, let $h$ denote the distance between the equidistant numbers, that is, $h = \frac{x_n - x_0}{n}$. Also, let $c_0, c_1, \ldots c_n$ denote the Cotesian numbers, i.e. the numbers such that after interpolating the function values by a polynomial of degree at most $n$ the integral over the polynomial can be approximated by*

$$\int_{x_0}^{x_n} f(x)\mathrm{d}x \approx h(c_0f_0 + c_1f_1 + \cdots + c_nf_n)\,.$$

*Then, $c_0, c_1, \ldots, c_n$ satisfy*

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n^2 & \cdots & n^n \end{bmatrix}^\top \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} n/1 \\ n^2/2 \\ n^3/3 \\ \vdots \\ n^{n+1}/(n+1) \end{bmatrix}.$$

*Proof.* Without loss of generality we let $x_0 = 0$ (this can be verified by defining $x'_0 = x_0 - x_0$, $x'_1 = x_1 - x_0$, $\ldots$, $x'_n = x_n - x_0$ and noting that $\int_{x_0}^{x_n} f(x)\mathrm{d}x = \int_0^{x'_n} f(x')\mathrm{d}x'$). Employing this new definition allows us to write $x_i = ih$ for $0 \le i \le n$.

All interpolating polynomials $p(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n$ must satisfy $p(x_0) = f_0$, $p(x_1) = f_1$, $\ldots$, $p(x_n) = f_n$, hence, the coefficients $p_0, p_1, \ldots, p_n$ satisfy

$$\underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}}_{\mathbf{p}} = \underbrace{\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}}_{\mathbf{f}} . \tag{1}$$

Also, we know that integrating $p(x)$ yields the approximation of the integral, equalling $h(c_0 f_0 + c_1 f_1 + \cdots + c_n f_n)$, giving the relation

$$\begin{aligned} \int_{x_0}^{x_n} p(x)\mathrm{d}x = \int_0^{nh} p(x)\mathrm{d}x &= \left[ p_0 x + p_1 \frac{x^2}{2} + p_2 \frac{x^3}{3} + \cdots + p_n \frac{x^{n+1}}{n+1} \right]_0^{nh} \\ &= p_0 \frac{nh}{1} + p_1 \frac{(nh)^2}{2} + p_2 \frac{(nh)^3}{3} + \cdots + p_n \frac{(nh)^{n+1}}{n+1} = h(c_0 f_0 + c_1 f_2 + \cdots + c_n f_n) \, . \end{aligned}$$

After removing a factor $h$ from both sides of the equal sign we use vector notation to rewrite the equation into

$$\underbrace{\begin{bmatrix} n/1 \\ n^2 h/2 \\ n^3 h^2/3 \\ \vdots \\ n^{n+1} h^n/(n+1) \end{bmatrix}^\top}_{\mathbf{n}^\top} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \underbrace{\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}^\top}_{\mathbf{c}^\top} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} . \tag{2}$$

This allows for shorter notation; equation 1 can be written as

$$A\mathbf{p} = \mathbf{f} \, ,$$

and equation 2 can be written as

$$\mathbf{n}^\top \mathbf{p} = \mathbf{c}^\top \mathbf{f} \, .$$

Using these equalities we can deduce that (note that $A$ is invertible due to Lemma 1):

$$\begin{aligned} \mathbf{p} &= A^{-1}\mathbf{f} \\ \mathbf{n}^\top \mathbf{p} = \mathbf{n}^\top A^{-1} \mathbf{f} &= \mathbf{c}^\top \mathbf{f} \\ \mathbf{f}^\top (A^{-1})^\top \mathbf{n} &= \mathbf{f}^\top \mathbf{c}, \quad \text{for arbitrary } \mathbf{f} \\ (A^{-1})^\top \mathbf{n} &= \mathbf{c} \\ \mathbf{n} &= A^\top \mathbf{c} \, . \end{aligned}$$

Writing this out and substituting $x_i = ih$ gives us:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & h & h^2 & \cdots & h^n \\ 1 & 2h & (2h)^2 & \cdots & (2h)^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & nh & (nh)^2 & \cdots & (nh)^n \end{bmatrix}^\top \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} n/1 \\ n^2h/2 \\ n^3h^2/3 \\ \vdots \\ n^{n+1}h^n/(n+1) \end{bmatrix} .$$

Removing $h$ (we can this do due to the transposition of the matrix) results in

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1^2 & \cdots & 1^n \\ 1 & 2 & 2^2 & \cdots & 2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n^2 & \cdots & n^n \end{bmatrix}^\top \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} n/1 \\ n^2/2 \\ n^3/3 \\ \vdots \\ n^{n+1}/(n+1) \end{bmatrix} ,$$

concluding our proof. □

**Definition 3.** *Let $X = \{X_1, X_2, \ldots, X_n\}$ with $X_1, X_2, \ldots, X_n \in \mathbb{R}$ and let $k \in \mathbb{N}_{\geq 0}$. The elementary symmetric polynomial is then defined as:*

$$e_k(X) = \begin{cases} 1 & \text{if } k = 0\,, \\ \displaystyle\sum_{1 \leq m_1 < \cdots < m_k \leq n} X_{m_1} \cdots X_{m_k} & \text{if } 1 \leq k \leq n\,, \\ 0 & \text{if } k > n\,. \end{cases}$$

**Lemma 4.** *Let $s(n, k)$, with $n, k \in \mathbb{N}_{\geq 0}$ denote the (un)signed Stirling numbers of the first kind, then, if $0 \leq j \leq \ell$ with $j, \ell \in \mathbb{N}$, the following equality holds:*

$$e_j(\{1, 2, \ldots, \ell\}) = |s(\ell + 1, \ell + 1 - j)|\,. \tag{3}$$

*Proof.* We use induction twice to complete the proof.

First, we use induction on $j$. For the induction base with $j = 0$ we note that by definition: $e_j(\{1, 2, \ldots, \ell\}) = 1 = |s(\ell+1, \ell+1)|$. For the induction hypothesis we assume that $e_{j-1}(\{1, 2, \ldots, \ell\}) = |s(\ell + 1, \ell + 2 - j)|$. For the induction step we then need to show that equation 3 holds.

We do this using induction on $\ell$. For the induction base $\ell = 0$ and thus $j = 0$, we get $e_0(\emptyset) = 1 = |s(1, 1)|$. For the induction hypothesis we assume $e_j(\{1, 2, \ldots, \ell - 1\}) = |s(\ell, \ell - j)|$, applying this to our previous assumption yields $e_{j-1}(\{1, 2, \ldots, \ell - 1\}) = |s(\ell, \ell + 1 - j)|$. For the induction step we need to prove equation 3. Note that by the definition of the Stirling numbers we have [1, equation 15]:

$$|s(n, k)| = |s(n - 1, k - 1)| + (n - 1)|s(n - 1, k)|\,.$$

Using our induction hypotheses we now show that equation 3 holds

$$\begin{aligned}
e_j(\{1,2,\ldots,\ell\}) &= \sum_{1\le m_1<\cdots<m_j\le \ell} m_1\cdots m_j \\
&= \sum_{\substack{1\le m_1<\cdots<m_{j-1}\le \ell-1\\ m_{j-1}<m_j<\ell}} m_1\cdots m_j + \sum_{\substack{1\le m_1<\cdots<m_{j-1}\le \ell-1\\ m_{j-1}<m_j=\ell}} m_1\cdots m_j \\
&= \sum_{1\le m_1<\cdots<m_j\le \ell-1} m_1\cdots m_j + \ell \sum_{1\le m_1<\cdots<m_{j-1}\le \ell-1} m_1\cdots m_{j-1} \\
&= e_j(\{1,2,\ldots,\ell-1\}) + \ell e_{j-1}(\{1,2,\ldots,\ell-1\}) \\
&= |s(\ell,\ell-j)| + \ell|s(\ell,\ell+1-j)| \\
&= |s(\ell+1,\ell+1-j)|\,.
\end{aligned}$$

Hence, the induction step, equation 3, holds, concluding our proof. □

**Lemma 5.** *Let $0 \le i \le n$ with $i \in \mathbb{N}$, also, let $0 \le k \le n$ with $k, n \in \mathbb{N}$, then the following relation holds:*

$$e_k(\{1,\ldots,n\}\setminus\{i\}) = \sum_{m=0}^{k} (-i)^m e_{k-m}(\{1,\ldots,n\})\,.$$

*Proof.* This relation can easily be seen to hold for $1 \le i \le n$ since

$$e_k(\{1,\ldots,n\}\setminus\{i\}) = e_k(\{1,\ldots,n\}) - i e_{k-1}(\{1,\ldots,n\}\setminus\{i\})\,,$$

and $e_0(\{1,\ldots,n\}\setminus\{i\}) = 1 = e_0(\{1,\ldots,n\})$.

For $i = 0$ the relation also holds since $e_k(\{1,\ldots,n\}\setminus\{0\}) = e_k(\{1,\ldots,n\}) - 0$. □

**Theorem 6.** *We can calculate the coefficients of $c_i$ for $i \in \{0,\ldots,n\}$ in the following equation*

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1^2 & \cdots & 1^n \\ 1 & 2 & 2^2 & \cdots & 2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n^2 & \cdots & n^n \end{bmatrix}^\top \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} n/1 \\ n^2/2 \\ n^3/3 \\ \vdots \\ n^{n+1}/(n+1) \end{bmatrix} \tag{4}$$

*by computing*

$$c_i = \frac{1}{(n-1)!}\binom{n}{i}\sum_{j=0}^{n}\sum_{m=0}^{n-j} i^m n^j \frac{(-1)^{i+n} s(n+1, j+m+1)}{j+1}\,.$$

*Proof.* We first rewrite equation by transposing the matrix in equation 4 and bringing it to the other side, yielding

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & \cdots & n \\ 0 & 1^2 & 2^2 & \cdots & n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1^n & 2^n & \cdots & n^n \end{bmatrix}^{-1}}_{(W_{n+1}^{-1})^\top} \begin{bmatrix} n/1 \\ n^2/2 \\ n^3/3 \\ \vdots \\ n^{n+1}/(n+1) \end{bmatrix},$$

where the matrix $W_{n+1}$ is the Vandermonde matrix defined according to Pr∞fWiki [2]. The inverse of the matrix (without transposition) is also given by Pr∞fWiki and turns out to be

$$[W_{n+1}^{-1}]_{ij} = \frac{(-1)^{n-i} e_{n-i}(\{0,1,\ldots,n\} \setminus \{j\})}{\prod_{m=0,m\neq j}^{n+1}(j-m)}\,,$$

note that in the citation $i, j \in \{1, \ldots, n\}$, whereas here we define $i, j \in \{0, \ldots, n\}$. Transposing this matrix and noting that we can safely omit 0 in the set yields

$$[(W_{n+1}^{-1})^\top]_{ij} = \frac{(-1)^{n-j} e_{n-j}(\{1,\ldots,n\} \setminus \{i\})}{\prod_{m=0,m\neq i}^{n}(i-m)}\,.$$

Using this matrix we can then compute the coefficients of $c_i$ by using the following relation

$$\begin{aligned} c_i &= \sum_{j=0}^{n} \left[(W_{n+1}^{-1})^\top\right]_{ij} n^{j+1}/(j+1) \\ &= \sum_{j=0}^{n} n^{j+1} \frac{(-1)^{n-j} e_{n-j}(\{1,\ldots,n\} \setminus \{i\})}{(j+1)\prod_{m'=0,m'\neq i}^{n}(i-m')}\,. \end{aligned} \tag{5}$$

Since $0 \le i \le n$ we can simplify the product in the denominator to

$$\prod_{m=0,m\neq i}^{n} (i-m) = i \cdot (i-1) \cdots 2 \cdot 1 \cdot -1 \cdot -2 \cdots (i-n-1) \cdot (i-n) = i!(n-i)!(-1)^{n-i}\,.$$

Also, from Lemma 4 in combination with Lemma 5 we have that

$$e_{n-j}(\{1,\ldots,n\} \setminus \{i\}) = \sum_{m=0}^{n-j} (-i)^m |s(n+1, n+1-(n-j-m))|\,.$$

Substituting these relations into equation 5 yields

$$\begin{aligned} c_i &= \sum_{j=0}^{n} n^{j+1} \frac{(-1)^{n-j} e_{n-j}(\{1,\ldots,n\} \setminus \{i\})}{(j+1)i!(n-i)!(-1)^{n-i}} \\ &= \sum_{j=0}^{n} n^{j+1} \frac{(-1)^{n-j} \sum_{m=0}^{n-j} (-i)^m |s(n+1, n+1-(n-j)+m)|}{(j+1)i!(n-i)!(-1)^{n-i}} \\ &= \frac{1}{(n-1)!} \binom{n}{i} \sum_{j=0}^{n} n^j \frac{(-1)^{i-j} \sum_{m=0}^{n-j} (-i)^m |s(n+1, j+m+1)|}{j+1} \\ &= \frac{1}{(n-1)!} \binom{n}{i} \sum_{j=0}^{n} \sum_{m=0}^{n-j} i^m n^j \frac{(-1)^{i-j+m}(-1)^{n-j-m} s(n+1, j+m+1)}{j+1} \\ &= \frac{1}{(n-1)!} \binom{n}{i} \sum_{j=0}^{n} \sum_{m=0}^{n-j} i^m n^j \frac{(-1)^{i+n} s(n+1, j+m+1)}{j+1}\,. \end{aligned}$$

Proving the theorem. ∎

## References

[1] E.W. Weisstein. *Stirling Number of the First Kind*. MathWorld – A Wolfram Web Resource. URL: https://mathworld.wolfram.com/StirlingNumberoftheFirstKind.html.

[2] Pr∞fWiki. [Online; accessed 10-Jun-2022]. URL: https://proofwiki.org/wiki/Inverse_of_Vandermonde_Matrix.